Invariants for persistent homology and their stability

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Group de travail ATD, Montpellier
16 June 2021

Warning


Science seminars
Slide 1
Good morning everyone.
My lab works in this field
Slide 40
In conclusion we have shown that under semi-hypoxic conditions non canonical XHY889-2 signaling affects the cell cycle independently of XHY33-3 withought affecting the cell cycle

Motivation

Challenges for data science:

- data arising from new technologies
- not only size of data, but also complexity

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- data arising from new technologies
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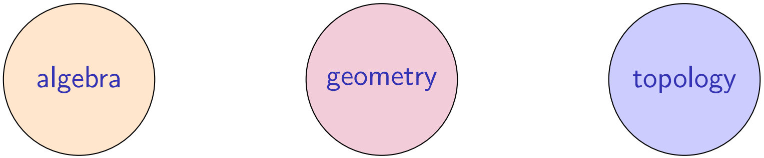
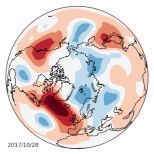
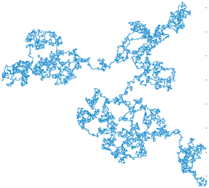
Ideas from pure mathematics can help



Topological data analysis

TDA studies the “shape” of complex data





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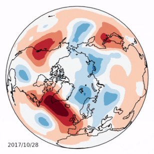

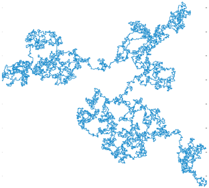

- Persistent homology
  - number of components, holes, voids, and higher dimensional holes

Topological data analysis

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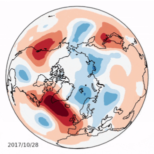
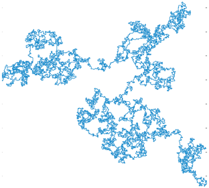




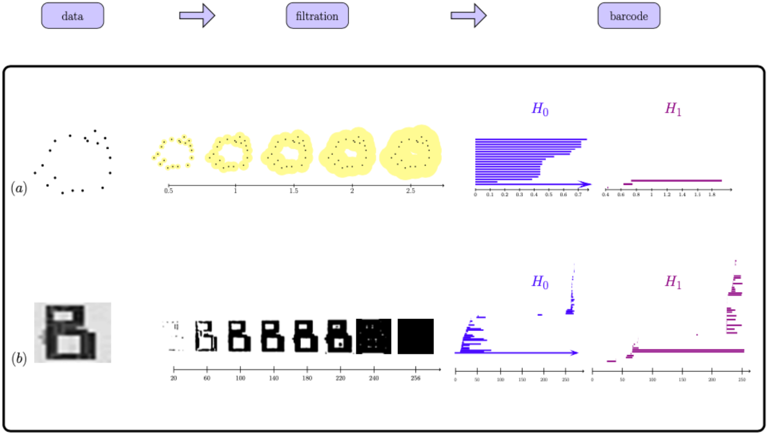
- Persistent homology
  - number of components, holes, voids, and higher dimensional holes

- Magnitude homology
  - effective number of points

A topological perspective on regimes in dynamical systems, K. Strommen, M. Chantry, J. Dorrington, N. Otter, 2021, https://arxiv.org/abs/2104.03196

(Persistent) magnitude of point-cloud data, S. Kališnik, M. O’Malley, N. Otter, in preparation

Persistent homology

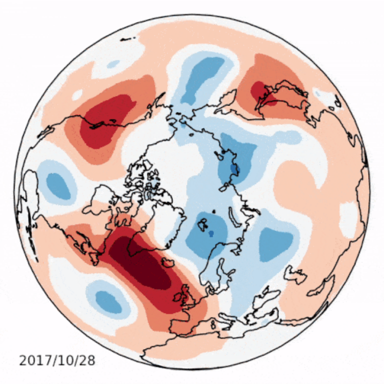


Multiparameter persistent homology (MPH)

Motivation

Data often depend on several parameters, e.g.:

- colored digital images
- large and noisy climate data sets

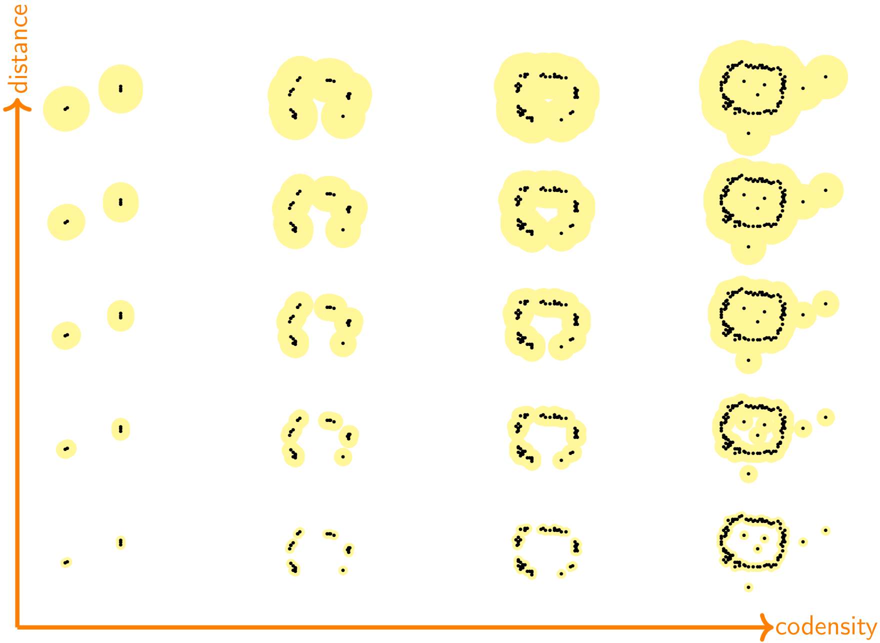


A topological perspective on regimes in dynamical systems, K. Strommen, M. Chantry, J. Dorrington, N. Otter, 2021, https://arxiv.org/abs/2104.03196

Multiparameter persistent homology (MPH)

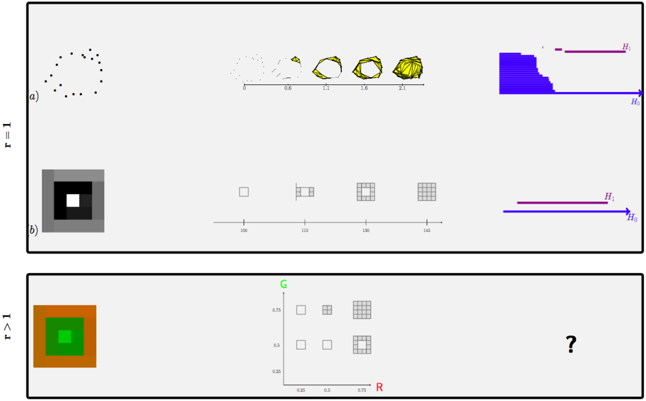
Motivation

MPH: codensity-distance bifiltration



MPH pipeline

data depending on r parameters ⇨ r-filtered space ⇨ algebraic invariants



MPH: theoretical challenges

- MPH was introduced in 2009 by Carlsson and Zomorodian[1]
- Classification problem amounts to studying isomorphism classes of multigraded modules over $\mathbb{K}[x_1, \ldots, x_r]$
- Desiderata for invariants for applications:
  - Computability
  - Stability
  - Interpretability

[1] G. Carlsson, A. Zomorodian, The theory of multidimensional persistence, Discrete & Computational Geometry, 2009

Approaches focussing on finding invariants suitable for applications

1) Efficient algorithms to compute homology of multifiltrations:

- For one-critical multifiltrations[2]
- For general multifiltrations[3]

2) Invariants of MPH modules coming from applications: Bauer and Botnan, Chachólski, Oudot

3) Restriction of 2-parameter PH to 1-parameter PH: Biasotti et al[4], Lesnick and Wright[5]

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[2] G. Carlsson, G. Singh, A. Zomorodian, Computing multidimensional persistence, ISAAC 2009, Lecture notes in computer science

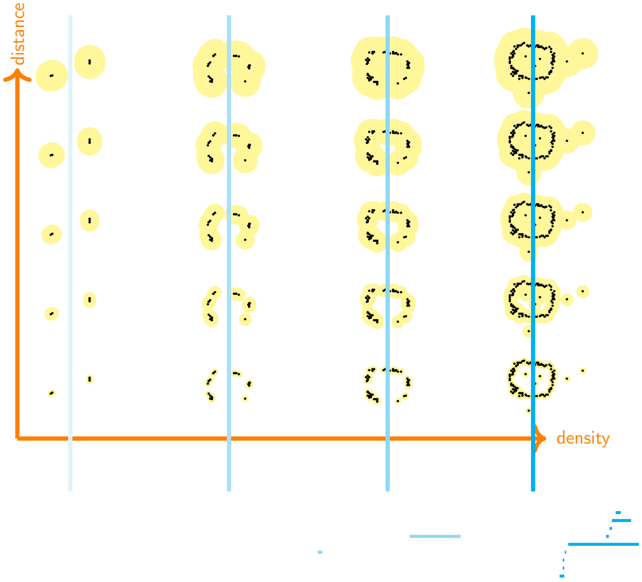
[3] W. Chachólski, M. Scolamiero, F. Vaccarino, Combinatorial presentation of multidimensional persistent homology, Journal of Pure and Applied Algebra, 2017

[4] S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, C. Landi, Multidimensional Size Functions for Shape Comparison, Journal of Mathematical Imaging and Vision, 2008

[5] M. Lesnick, M. Wright, Interactive Visualization of 2-D Persistence Modules, arxiv.org/1512.00180

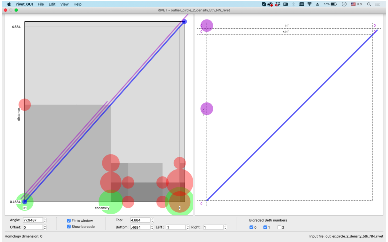
Restriction of 2-parameter PH to 1-parameter PH

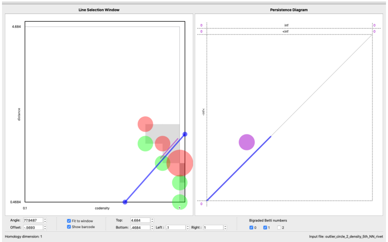
Vertical lines



Restriction of 2-parameter PH to 1-parameter PH

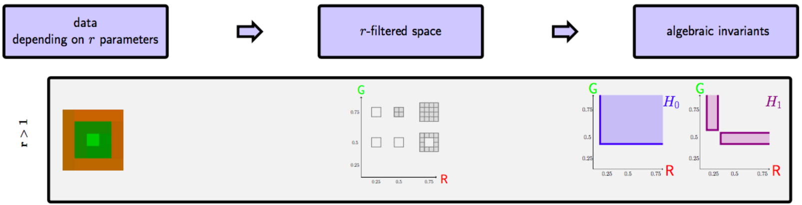
Lines with positive slope





Computations performed with RIVET by M. Lesnick and M. Wright.
M. Lesnick, M. Wright, Interactive Visualization of 2-D Persistence Modules, arxiv.org/1512.00180

4) Stratification of MPH



In HOST[6] proposal of invariants that distinguish between:

- **transient features**, support is 0-dimensional,
- **partially persistent features**, support is a subspace of dimension $1 \leq d < r$, and
- **fully persistent features**, support is a subspace of dimension r.

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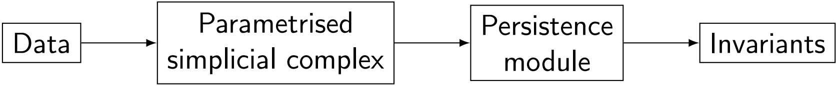
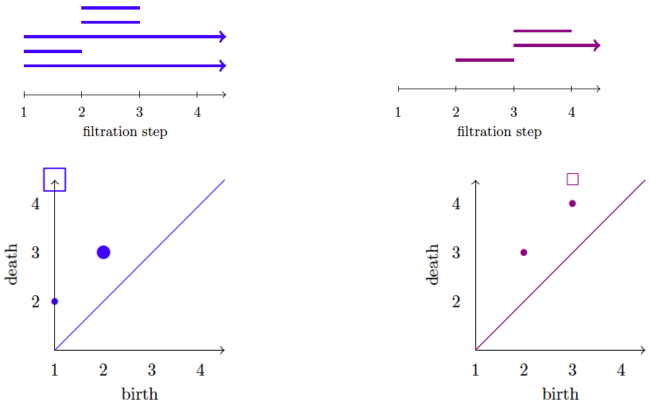
[6]H. Harrington, NO, H. Schenck, U. Tillmann, Stratifying multiparameter persistent homology, *SIAGA*, **3**(3):439–471 (2019)

Stability and distances for one-parameter PH



- Stability results for 1-parameter PH: the pipeline is Lipschitz in an appropriate sense.
- Distances on barcodes are defined using matchings between intervals in the barcode

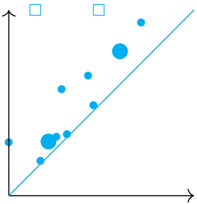
Barcodes and persistence diagrams



from N. Otter, M. A. Porter, U. Tillmann, P. Grindrod and H. A Harrington, A roadmap for the computation of persistent homology, *EPJ Data Science*, **6**: 17 (2017)

Distances on persistence diagrams

Given two persistence diagrams:



How far apart are they?

Distances on persistence diagrams

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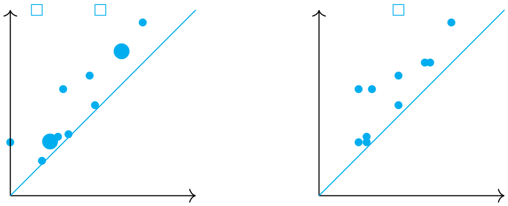


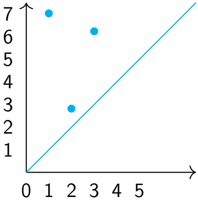
How far apart are they?

Idea: look at all possible ways to match the points in the two diagrams, and choose a way to compute distances between matched points. Then the distance is given by the optimal matching, the one minimizing these pairwise distances.

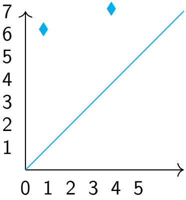
Bottleneck distance d_B

Given x, y in $\mathbb{R}^2$ choose

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$



D_1



D_2

Bottleneck distance d_B

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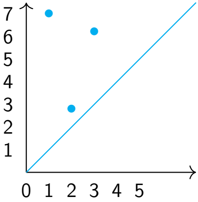
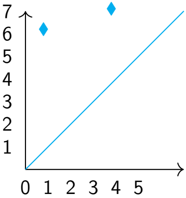


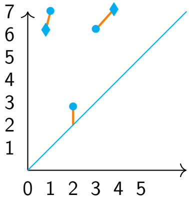
D_1



D_2

For each matching let m be the maximum of the d_∞ distances between matched points. The optimal matching is the one that minimizes this quantity, and the bottleneck distance is m for the optimal matching.

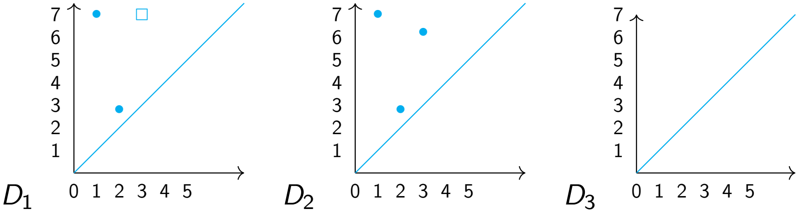
Optimal matching:



$d_B(D_1, D_2) = 0.9$

The bottleneck distance

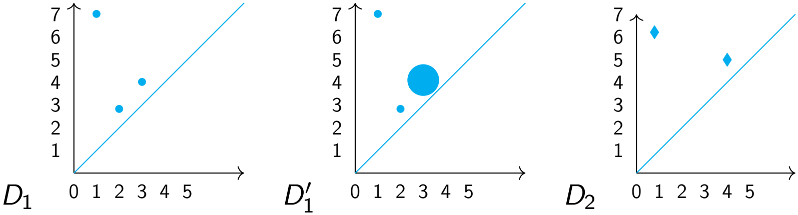
Example



We have $d_B(D_1, D_2) = d_B(D_1, D_3) = \infty$

The bottleneck distance is very coarse

Example

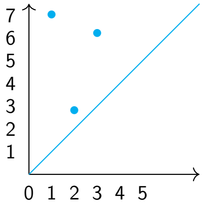


We have $d_B(D_1, D_2) = d_B(D_1', D_2) = 1$

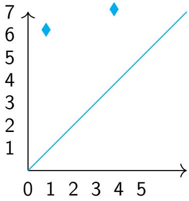
Wasserstein distance

Given x, y in $\mathbb{R}^2$ choose

$$L_p(x,y) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p \right)^{1/p}$$



D_1



D_2

Wasserstein distance

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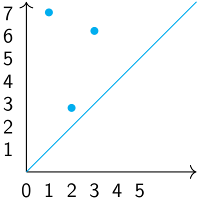
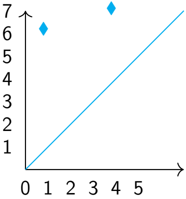


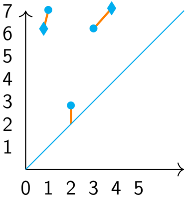
D_1



D_2

For each matching compute the the p-norm of all the L_p-distances between matched points. The optimal matching is the one that minimizes this p-norm.

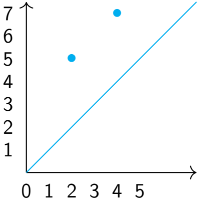
For $p = 2$: optimal matching



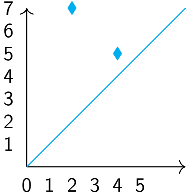
$$d_{W_2}(D_1, D_2) = \sqrt{0.82^2 + 1.2^2 + 0.8^2}$$

Geometry of the spaces of persistence diagrams

Means are not unique:

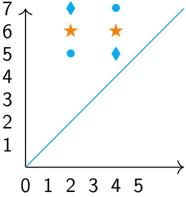


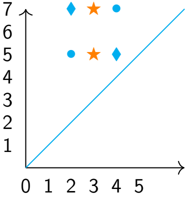
D_1



D_2

Two different optimal matchings, leading to two different means:





Geometry of the spaces of persistence diagrams

- In general: complicated

Geometry of the spaces of persistence diagrams

- In general: complicated
- For $W_2(L_2)$ is a non-negatively curved Alexandroff space: Fréchet means are not necessarily unique nor continuous (Turner et al 2014)
- Continuous Fréchet means for probability distributions of persistence diagrams for $W_2(L_2)$ (Munch et al 2015)
- Lots of work, e.g.: [Chazal et al 2013], [Turner 2020], ...

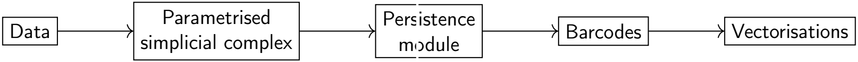
K. Turner, Y. Mileyko, S. Mukherjee, and J. Harer, Fréchet means for distributions of persistence diagrams, *Discrete & Computational Geometry*, **52**(2014)

E. Munch, K. Turner, P. Bendich, S. Mukherjee, J. Mattingly, and J. Harer, Probabilistic fréchet means for time varying persistence diagrams, *Electronic Journal of Statistics*, **9**(2015)

F. Chazal, B. Terese Fasy, F. Lecci, A. Rinaldo, A. Singh, and L. A. Wasserman. On the bootstrap for persistence diagrams and landscapes. *Modeling and Analysis of Information Systems*, **20**(6), 2013

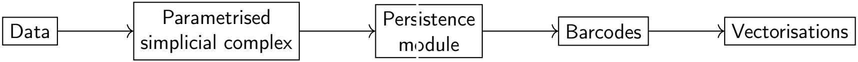
K. Turner, Medians of populations of persistence diagrams, *Homology, Homotopy and Applications* **22**(1) (2020)

Alternatives



Idea: Instead of working directly with the space of persistence diagrams, map it to another space, e.g. Banach space, in which it is easier to compute means, variances, etc.

Alternatives

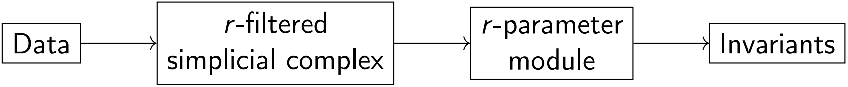


Idea: Instead of working directly with the space of persistence diagrams, map it to another space, e.g. Banach space, in which it is easier to compute means, variances, etc.

Some approaches:

- Persistence landscapes
- Persistence images
- Tropical coordinates
- Kernels
- ...

Stability and distances for MPH



- Defining distances for MPH module is a problem linked with finding suitable invariants that measure "persistence"

Interleaving distance

- Given two persistence modules $N, M\colon (\mathbb{R}^r, \leq) \to \mathrm{Vect}$, one defines a notion of "ϵ-approximate isomorphisms".
- The interleaving distance is the smallest ϵ for which there is such an approximate isomorphism.

Interleaving

Let $\epsilon \in \mathbb{R}$, and use the notation $\hat{\epsilon} = (\epsilon, \ldots, \epsilon) \in \mathbb{R}^r$.

- $T_\epsilon \colon (\mathbb{R}^r, \leq) \to (\mathbb{R}^r, \leq)$ is the functor given by $T_\epsilon(a) = a + \hat{\epsilon}$

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- $\eta_\epsilon \colon id_{(\mathbb{R}^r, \leq)} \Rightarrow T_\epsilon$ is the natural transformation given by $\eta_\epsilon(a) \colon a \leq a + \hat{\epsilon}$.

Interleaving

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- $\eta_\epsilon \colon id_{(\mathbb{R}^r, \leq)} \Rightarrow T_\epsilon$ is the natural transformation given by $\eta_\epsilon(a) \colon a \leq a + \hat{\epsilon}$.
- Given two persistence modules $N, M \colon \mathbb{R}^r \to \mathrm{Vect}$, an ϵ-**interleaving** of M and N consists of natural transformations $\varphi : M \Rightarrow NT_\epsilon$ and $\psi : N \Rightarrow MT_\epsilon$, i.e.

$$\begin{array}{ccccc}
(\mathbb{R}^r, \leq) & \xrightarrow{T_\epsilon} & (\mathbb{R}^r, \leq) & \xrightarrow{T_\epsilon} & (\mathbb{R}^r, \leq) \\
M \big\downarrow & \overset{\varphi}{\Rightarrow} & N \big\downarrow & \overset{\psi}{\Rightarrow} & M \big\downarrow \\
\mathrm{Vect} & = & \mathrm{Vect} & = & \mathrm{Vect}
\end{array}$$

such that

$$(\psi T_\epsilon)\varphi = M\eta_{2\epsilon} \text{ and } (\varphi T_\epsilon)\psi = N\eta_{2\epsilon}.$$

Interleaving distance

- Define the interleaving distance between modules M and N as

$$d_I(M,N) = \inf_{\epsilon}\{M \text{ and } N \text{ are } \epsilon\text{-interleaved}\}$$

- We set $d_I(M,N) = \infty$ if M and N are not ϵ-interleaved for any $\epsilon \in \mathbb{R}$

- In the one-parameter case, the interleaving distance is the bottleneck distance (isometry theorem)[7]

- In the multiparameter case, the computation of the interleaving distance is NP-hard[8]

[7] Categorification of Persistent Homology, P. Bubenik, J. A. Scott *Discrete & Computational Geometry* **51**:600–627 (2014).

[8] Computing the Interleaving Distance is NP-Hard, H. B. Bjerkevik, M. B. Botnan, M. Kerber, *Foundations of Computational Mathematics* **20**:1237–1271 (2020).

Generalisation of Wasserstein distance?

In Bubenik et al.[9]:

Let d be a metric on the set of indecomposable modules, let $M \cong \oplus_{a \in A} M_a$ and $N \cong \oplus_{b \in B} N_b$ where M_a and N_b are indecomposable modules. Then define:

$$W_p(d)(M,N) = \min_{\varphi} \Big(\| \left(d(M_c, N_{\varphi(c)}) \right)_{c \in C} \|_p^p + \\ + \| \left(d(M_a, 0) \right)_{a \in A-C} \|_p^p + \\ + \| \left(d(0, N_b) \right)_{b \in B-\varphi(C)} \|_p^p \Big)^{1/p}$$

where the minimum is over all partial matchings $\phi : A' \subset A \to B' \subset B$.

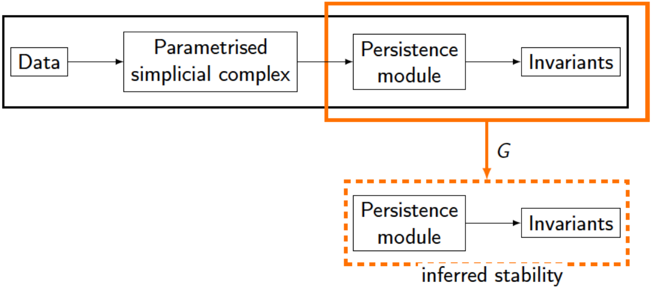
[9] P. Bubenik, J. A. Scott, and D. Stanley, An algebraic Wasserstein distance for generalized persistence modules, `https://arxiv.org/abs/1809.09654`, 2018.

In general, generalising distances to the multiparameter setting entails asking the question: what does “persistence” mean in the multiparameter case?

A general framework to study stability for MPH

We propose[10] a shift in perspective:

- a choice of invariant determines a distance
- thus, rather then asking when the pipeline is Lipzschitz, we can ask for Lipschitz changes of invariants



[10] B. Giunti, J. Nolan, N. Otter, L. Waas, Amplitudes on abelian categories, in preparation

Amplitude

Fix an abelian category $\mathcal{A}$.

Definition

A class function $\alpha\colon \operatorname{ob}\mathcal{A} \to [0,\infty]$ is called an **amplitude** if $\alpha(0) = 0$, and for every short exact sequence

$$0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$$

in $\mathcal{A}$ the following inequalities hold:

$$\text{(Monotonicity)} \qquad \begin{array}{l} \alpha(A) \le \alpha(B) \\ \alpha(C) \le \alpha(B) \end{array} \tag{1}$$

$$\text{(Subadditivity)} \qquad \alpha(B) \le \alpha(A) + \alpha(C) \tag{2}$$

Moreover, α is **additive** if (2) is an equality and **finite** if $\alpha(A) < \infty$ for all A in $\operatorname{ob}\mathcal{A}$.

Examples

- The map $M \mapsto \mathrm{rank}_R M$ gives an additive amplitude on the category of (left) R-modules;

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  - The map $V \to \dim V$ gives an additive amplitude on Vect;

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  - The map $V \to \dim V$ gives an additive amplitude on Vect;
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- The map $\delta(C) = \max\{n \in \mathbb{N} \mid C_n \neq 0\}$ gives an amplitude on **ch**;
- Given two amplitudes α and β on $\mathcal{A}$, the assignments $\max\{\alpha, \beta\}$ and $\alpha + \beta$ are two amplitudes on $\mathcal{A}$.

Relationship to other work

- Noise systems (Scolamiero, Chachólski, Lundman, Ramanujam, and Öberg)[11]: to each $\varepsilon \geq 0$ they assign a collection of objects.
- p-norms (Skraba and Turner)[12]
- Weight functions (Bubenik, Scott, and Stanley) [13]

[11] M. Scolamiero, W. Chachólski, A. Lundman, R. Ramanujam, and S. Öberg, Multidimensional persistence and noise, *Foundations of Computational Mathematics*, **17**:1367–1406, 2017

[12] P. Skraba and K. Turner, Wasserstein stability for persistence diagrams, `https://arxiv.org/2006.16824`, 2020

[13] P. Bubenik, J. A. Scott, and D. Stanley, An algebraic Wasserstein distance for generalized persistence modules, `https://arxiv.org/abs/1809.09654`, 2020.

Examples in TDA: p-norms

Definition

Let $X \cong \oplus_{i=1}^{n} \mathbb{I}[b_i, d_i) \in \mathrm{PerM}(\mathbb{R})$ and $p \in [1, \infty]$. The p-norm of X is defined by

$$\rho_p(X) = \begin{cases} \left(\sum_{i=1}^{n} |d_i - b_i|^p\right)^{1/p} & \text{if } p < \infty \\ \max_{i=1,\dots,n} |d_i - b_i| & \text{if } p = \infty \end{cases}$$

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Proposition

For all $p \in [1, \infty]$, ρ_p is an amplitude on $\mathrm{PerM}(\mathbb{R})$, additive if $p = 1$.

Proof.

Skraba and Turner, arXiv, (2020). □

Examples in TDA: Tropical coordinates

Definition (Kališnik[14])

Let $X = \oplus_{i=1}^{n} \mathbb{I}[b_i, b_i + \ell_i)$. Then $\ell\text{-trop}^k(X)$ is defined to be the sum of the length of the k-longest intervals in X, while $\partial\text{-trop}_m^p(X)$ is defined to the sum of the p largest expressions of the form $\min\{b_i + \ell_i, (m+1)\ell_i\}$, for $m \in \mathbb{N}$.

Proposition (Giunti, Nolan, **NO**, Waas)

- *The tropical coordinate* $\ell\text{-trop}^k$ *is an amplitude, for all* $k \in \mathbb{N}$ *on* $\mathrm{PerM}(\mathbb{R})$*;*
- *The tropical coordinate* $\partial\text{-trop}_m^p$ *is an amplitude on* $\mathrm{PerM}(\mathbb{R})$*, for all* $m, p \in \mathbb{N}$*;*
- *The tropical coordinate given by the sum of* l *largest expressions of the form* $\min\{b_i, m\ell_i\}$ *is not an amplitude on* $\mathrm{PerM}(\mathbb{R})$ *for any* $m, l \in \mathbb{N}$*.*

---

[14] S. Kališnik, Tropical coordinates on the space of persistence barcodes. *Foundations of Computational Mathematics*, **19**(1):101–129, 2019.

Examples in TDA: Persistent magnitude

Definition

Let $X \cong \oplus_{i=1}^{n} \mathbb{I}[b_i, d_i) \in \mathrm{PerM}(\mathbb{R})$. The **persistent magnitude** of X is

$$\mathrm{PMagn}(X) = \sum_{i=1}^{n} e^{-b_i} - e^{-d_i} \,.$$

[15] D. Govc and R. Hepworth, Persistent magnitude, *Journal of Pure and Applied Algebra*, **225**, 2021.

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Proposition

The persistent magnitude is an additive amplitude on $\mathrm{PerM}(\mathbb{R})$.

Proof.

Govc and Hepworth[15]. □

[15] D. Govc and R. Hepworth, Persistent magnitude, *Journal of Pure and Applied Algebra*, **225**, 2021.

Additive amplitudes

Theorem (Giunti, Nolan, **NO**, Waas)

Let $\mathcal{Q}$ be a finite poset. The following map is a bijection:

$$\psi = \mathrm{Meas}(\mathcal{Q}) \longrightarrow \mathrm{AdAmp}(\mathrm{PerM}(\mathcal{Q}))$$

$$\mu \mapsto \int_{\mathcal{Q}} \mathsf{dim}(\cdot) d\mu.$$

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Theorem (Giunti, Nolan, **NO**, Waas)

Each additive amplitude on $\mathrm{PerM}(\mathbb{R}^d)$ is uniquely given by integrating the Hilbert function with respect to a content.

Amplitudes in multi-parameter persistence

- L^p-amplitudes;
- Shift amplitudes;
- Support amplitude;
- Maximal dimension amplitude;
- Rank and multirank amplitude;
- ...

Distances from amplitudes

Absolute-value distance

Definition

The **absolute-value distance**, for every $X,\ Y \in \operatorname{ob}\mathcal{A}$, is defined as

$$\mathrm{d}_{|\alpha|}(X, Y) = |\alpha(X) - \alpha(Y)|.$$

- $\mathrm{d}_{|\rho_1|}$ is used in one of the first stability results in persistent homology[16].
- $\mathrm{d}_{|\ell\text{-trop}^k|}$ and $\mathrm{d}_{|\partial\text{-trop}^p|}$ are used in the stability results for tropical coordinates[17].

---

[16] David Cohen-Steiner, Herbert Edelsbrunner, John Harer, and Yuriy Mileyko. Lipschitz functions have l_p-stable persistence, *Foundations of Computational Mathematics*, **10**(2):127–139, 2010.

[17] S. Kališnik, Tropical coordinates on the space of persistence barcodes. *Foundations of Computational Mathematics*, **19**(1):101–129, 2019.

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Path metric

Definition

Given a morphism φ in $\mathcal{A}$, its **cost** is

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$$\mathrm{d}_\alpha(X, Y) = \inf\left\{\varepsilon \in [0,\infty) \mid c_\alpha(\varphi)+c_\alpha(\psi) \leq \varepsilon \text{ with } X \xleftarrow{\varphi} Z \xrightarrow{\psi} Y\right\}.$$

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Lemma

For each $X \in \mathcal{A}$, $\alpha(X) = \mathrm{d}_\alpha(X, 0)$.

Interleaving distance as a distance from an amplitude

Let C be a cone in $\mathbb{R}^r$. A **shift amplitude** σ on $\mathrm{PerM}(\mathbb{R}^r)$ is defined for any $X \in \mathsf{ob}\,\mathrm{PerM}(\mathbb{R}^r)$ as

$$\sigma(X) = \inf \{\ ||c|| \mid c \in C \text{ and } X(t \le t + c) = 0 \text{ for all } t \in \mathbb{R}^r\}\,.$$

[18] Oliver Gäfvert and Wojciech Chachólski. Stable invariants for multidimensional persistence, arXiv:1703.03632, 2017

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Theorem

For any $X, Y \in \mathrm{PerM}(\mathbb{R}^r)$, we have:

$$\mathrm{d}_I(X, Y) \leq \mathrm{d}_\sigma(X, Y) \leq 6\mathrm{d}_I(X, Y).$$

Proof.

Prop. 12.2 in [GC17][18]. □

[18] Oliver Gäfvert and Wojciech Chachólski. Stable invariants for multidimensional persistence, arXiv:1703.03632, 2017

Categories of amplitudes

Definition

The category of amplitudes, Amp, is the category where:

1. Objects are pairs $(\mathcal{A}, \alpha)$ consisting of an abelian category $\mathcal{A}$ together with an amplitude α on it;

2. Morphisms $(\mathcal{A}, \alpha) \to (\mathcal{A}', \alpha')$ are **amplitude-bounding** functors, i.e. additive functors $F\colon \mathcal{A} \to \mathcal{A}'$ such that there exists $K \geq 0$ so that $\alpha'(FX) \leq K\alpha(X)$ for all $X \in \operatorname{ob}\mathcal{A}$.

Amplitude-bounding functors

Lemma (Giunti, Nolan, NO, Waas)

Let $\mathrm{id}\colon (\mathrm{PerM}(\mathbb{R}), \rho_p) \to (\mathrm{PerM}(\mathbb{R}), \rho_q)$*, where* ρ_p *is the* p*-norm amplitude. Then* id *is amplitude-bounding, with* $K = 1$*, if and only if* $p \leq q \in [1, \infty]$*.*

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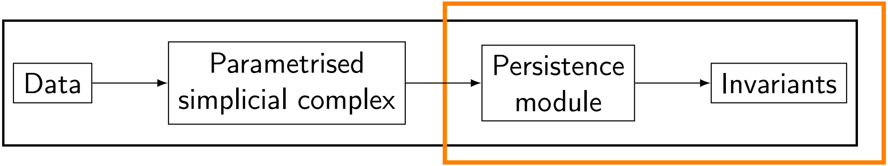
Lemma (Giunti, Nolan, NO, Waas)

Let $\mathrm{id}\colon (\mathrm{PerM}(\mathbb{R}), \rho_1) \to (\mathrm{PerM}(\mathbb{R}), \mathrm{PMagn})$*, where* PMagn *is the persistent magnitude. Then* id *is amplitude-bounding with* $K = 1$*.*

Continuous changes of invariants



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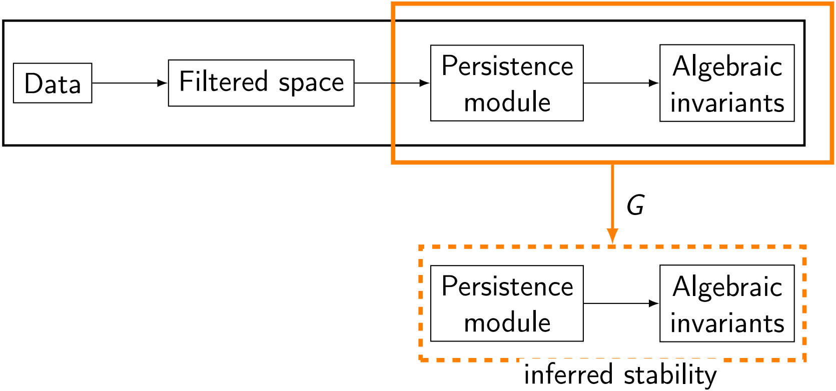

Inferred stability results

Proposition (Giunti, Nolan, NO, Waas)

Let $F\colon (\mathcal{A}, \alpha) \to (\mathcal{A}', \alpha')$ be an exact, amplitude-bounding functor with constant K. Then, for all $X, Y \in \mathsf{ob}\,\mathcal{A}$,

$$\mathrm{d}_{\alpha'}(FX, FY) \leq K \mathrm{d}_{\alpha}(X, Y)\,.$$

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Corollary (Giunti, Nolan, NO, Waas)

Given any persistence modules $X, Y \in \mathrm{PerM}(\mathbb{R})$, we have that

$$\mathrm{d}_{\mathrm{PMagn}}(X, Y) \leq W_1(\mathrm{Dgm}(X), \mathrm{Dgm}(Y))\,.$$

Future work

- Extend stability results.
- Study the computability of the distances.
- Explore the discriminating power of distances from amplitudes on a data set of tropical diseases.

Non-example: Number of indecomposables

In general, the number of indecomposables of an object in an abelian, Krull-Schmidt category is not an amplitude. Example[19]:

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[19] M. Buchet and E. G. Escolar, Realizations of Indecomposable Persistence Modules of Arbitrarily Large Dimension, *In 34th International Symposium on Computational Geometry(SoCG 2018)*, **99** 15:1–15:13, Dagstuhl, Germany, 2018

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\mathbb{F} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \mathbb{F}^2 & \xrightarrow{1} & \mathbb{F}^2 & \xrightarrow{[1\ 0]} & \mathbb{F} & \longrightarrow & 0 \\
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Non-example: c_p-rank

Let M be an $\mathbb{N}$-graded module over $\mathbb{K}[x_1, \ldots, x_r]$. Let p be an associated prime of M. Its 0**th local cohomology** with respect to p is

$$H_p^0(M) = \{m \in M \mid p^n \cdot m = 0 \text{ for all } n >> 0\}.$$

W.l.o.g. assume that $p = \langle x_1, \ldots, x_k \rangle$. The c_p-**rank** of M[20] is the rank of H_p^0 as a module over $\mathbb{K}[x_{k+1}, \ldots, x_r]$.

[20]H. Harrington, NO, H. Schenck, U. Tillmann, Stratifying multiparameter persistent homology, *SIAGA*, **3**(3):439–471 (2019)

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The c_p-rank gives a count of the ways that the module goes to infinity in the directions orthogonal to the coordinate subspace spanned by $x_1, \ldots, x_k$ [Proposition 4.20, HOST].

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In general, the c_p-rank is not an amplitude.

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