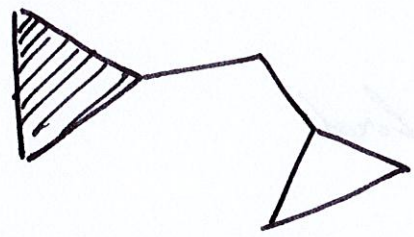


Homologie persistante.

- 1^{er} étape: complexe simplicial \rightarrow famille d'espaces vectoriels
- 2^{ème} étape: tirer une représentation graphique de ces choses-là
(diagrammes de persistence, code-barres)

Homologie d'un complexe simplicial (abstrait)

K : ensemble de simplices. $1 \leq \sigma \in K$ et $\emptyset \neq \tau \subset \sigma$, alors $\tau \in K$
+ choix d'une orientation.

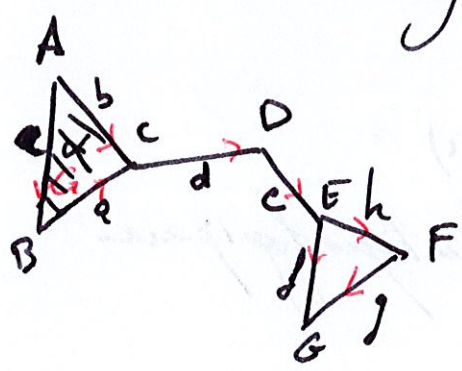


On construit un "complexe de chaînes" qui nous permet de calculer l'homologie: passage topologie \rightarrow algèbre combinatoire

$(C_k, \partial_k)_{k \in \mathbb{N}}$
 $k: \mathbb{R}, \mathbb{C}, \mathbb{Z}, \dots, \mathbb{F}_2$

$C_k := K$ -ev. engendré par les simplices de dim k .

pour tout σ de dim k
 \rightarrow vect. e_σ
 $\hookrightarrow \sum \alpha_i e_{\sigma_i}$



$C_0: kA \oplus kB \oplus kC \oplus \dots \oplus kG$
 $\cong k^7$

$C_1: ka \oplus kb \oplus \dots \oplus kg \oplus kh$
 $\cong k^8$

$C_2: kd \cong k$

puis $C_n = \{0\}$ pour $n \geq 3$.

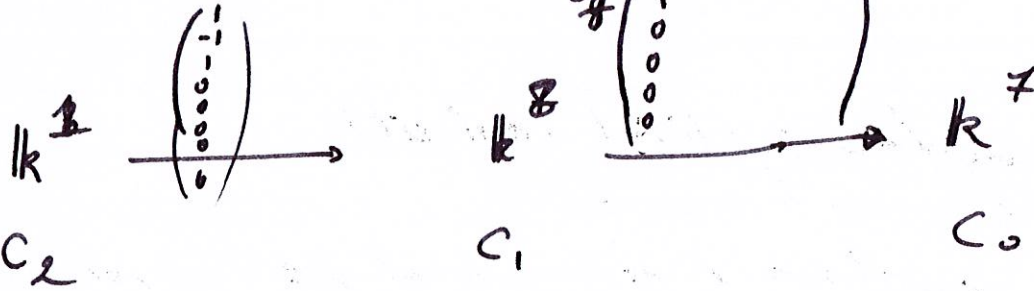
si cpt de dim 1: matrice d'incidence de Clement

$\partial_k: C_k \rightarrow C_{k-1}$ application de bord

$$\partial_2 \left(\begin{array}{c} A \\ \swarrow \quad \searrow \\ C \quad B \\ \downarrow \quad \uparrow \\ a \quad c \end{array} \right) = a - b + c.$$

$$\partial_1(B \xrightarrow{a} C) = C - B.$$

$$\partial_0(A) = 0.$$



Lemme: $\partial_{k-1} \circ \partial_k = 0$.

Preuve: le bord d'un simplexe n'est pas de bord.

$$\begin{aligned} \partial_0 \partial_1 \left(\begin{array}{c} A \\ \swarrow \quad \searrow \\ C \quad B \\ \downarrow \quad \uparrow \\ a \quad c \end{array} \right) &= \partial_0(a - b + c) \\ &= \partial_0(a) - \partial_0(b) + \partial_0(c) \\ &= c - b - c + a + b - a = 0 \end{aligned}$$

Conséquence: $\text{Im}(\partial_{k+1}) \subset \ker(\partial_k)$

chaînes sans bord ("cycles")

Définition: $H_k(K) := \ker(\partial_k) / \text{Im}(\partial_{k+1})$ le bord

Théorème: $H_k(K)$ ne dépend que du complexe topologique
 || non-jacent.

(3)

Exemples:

• toujours le même:

$d_2: \mathbb{k}^1 \rightarrow \mathbb{k}^8$ non nulle: $\ker d_2 = \{0\}$

$$\boxed{H_2 = \{0\}}$$

$$\text{Im}(d_2) = \begin{pmatrix} - \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

noyau de d_1 : ?

Image de d_1 :

$$\begin{aligned} & C-B \\ & C-A \\ & B-A = (C-A - (C-B)) \\ & D-C \\ & E-D \\ & G-E \\ & G-F \\ & F-E = (G-E - (G-F)) \end{aligned} \quad : \dim 6$$

$$\Rightarrow \dim \ker d_1 + \dim \text{Im } d_1 = \dim C_1 = 8$$

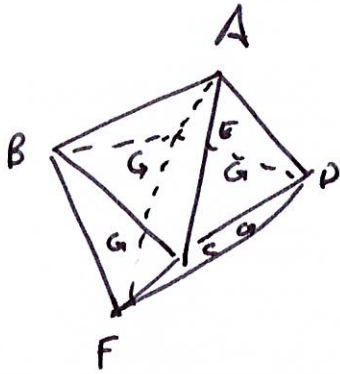
$$\Rightarrow \dim \ker d_1 = 2 : \langle a-b+c, f+g+h \rangle$$

$$\boxed{H_1 = \frac{\langle a-b+c, f+g+h \rangle}{\langle a-b+c \rangle} \simeq \mathbb{k} \langle f+g+h \rangle}$$

↳ cycle non trivial.

$$H_0: \left. \begin{array}{l} \text{Im}(d_1): \dim 6 \\ \ker(d_0): \dim 7 \end{array} \right\} \boxed{H_0 \simeq \mathbb{k}^7 / \mathbb{k}^6 \simeq \mathbb{k}} \rightarrow \underline{\text{connexe}}$$

(4)



(avec la face $\cdot B C D E$).

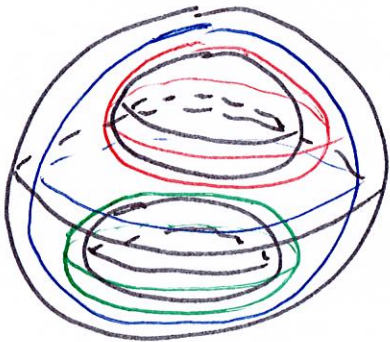
idée: $H_0 \leftarrow$ classes des points
 $H_1 \leftarrow$ classes des cercles
 $H_2 \leftarrow$ classes des sphères.

H_2 : $\partial_2 (ABC) = \vec{AB} + \vec{BC} - \vec{AC}$
 $(ACD) = \dots$

moyen: $\bullet (ABC) + (ACD) + (ADE) + (AEB) + (BCDE)$
 $\bullet - (BCDE) - (BCF) - (CDF) - (DEF) - (EBF)$
 \bullet intérieure : somme des 2 proc.

$\hookrightarrow \underline{\underline{H_2 \approx \mathbb{K}^2}}$

Une autre manière de voir ça :



ker ∂_2 : dim 3
 mais les 3 sphères sont un bord :
 $\text{Im } \partial_3 = \text{dim } 1$
 $\hookrightarrow H_2 : \text{dim } 2.$

(5)

Inclusions:

Si: $K \hookrightarrow K'$, alors $\iota_* : H_*(K) \rightarrow H_*(K')$.

En effet: $\forall k, [\sigma] \in H_k(K) = \ker(\partial_k) / \text{Im}(\partial_{k+1})$
 \downarrow
 $\sigma \in \ker(\partial_k)$

alors $\iota(\sigma) \in C_k(K')$

$$\textcircled{1} \partial'_k(\iota(\sigma)) = 0 \\ \Rightarrow \iota(\sigma) \in \ker(\partial'_k)$$

et on peut définir $\iota_*([\sigma]) = [\iota(\sigma)]$.

$\textcircled{2}$ c'est indépendant du choix de relèvement

de σ : $\sigma_1, \sigma_2 : \sigma_1 - \sigma_2 \in \text{Im}(\partial_{k+1})$.

$\Rightarrow \exists$ CL de simplices de dim $k+1$ /

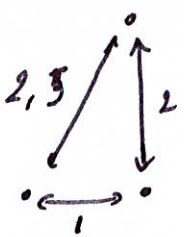
$$\partial_{k+1}(\sum_{\nu} a_{\nu} \Delta_{\nu}) = \sigma_1 - \sigma_2.$$

Les $\Delta_{\nu} \hookrightarrow K'$

$$\text{et on a } \partial_{k+1}(\sum_{\nu} a_{\nu} \iota(\Delta_{\nu})) = \iota(\sigma_1) - \iota(\sigma_2)$$

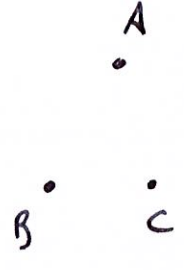
$$\Rightarrow [\iota(\sigma_1)] = [\iota(\sigma_2)]$$

Exemples et représentations:



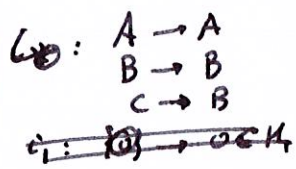
avec Victoria-Rip.

$t < 1$:

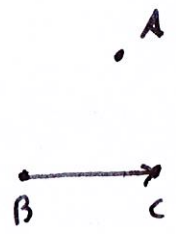


$H_0 = \mathbb{K}^3 = \langle A, B, C \rangle$

$H_i = \{0\} \quad i > 0$



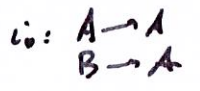
$t \in [1, 2[$



$H_0 = \mathbb{K}^2 = \langle A, B \rangle$

$H_1 = \{\mathbb{K}\} = \langle B \rangle$

$H_i = \{0\} \quad i > 1$



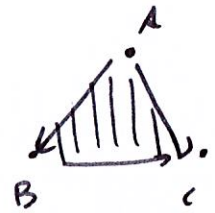
$t \in (2, 3[$



$H_0 = \mathbb{K} = \langle A \rangle$

$H_i = \{0\}$

$t \in [3, +\infty[$



$H_0 = \mathbb{K} = \langle A \rangle$

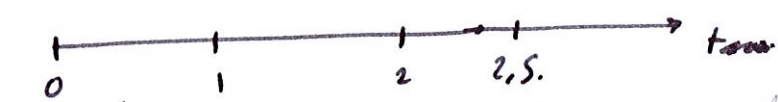
$H_i = 0$

regles P. de la distance
mesure / premier

H_0 :



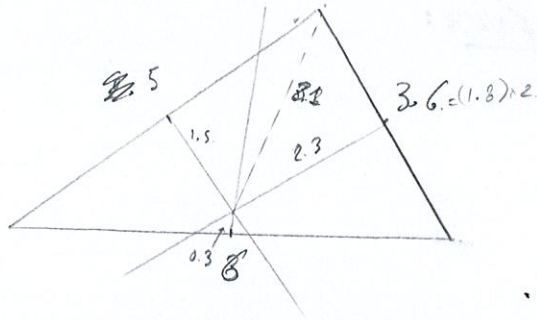
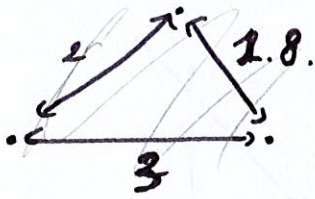
"cadré-terre"



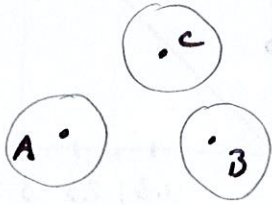
multi ∞ (+ distance)

A ou ne représente pas le point qui survit? P. 2.

avec Ech:



$t \in [0, 1.8[$:

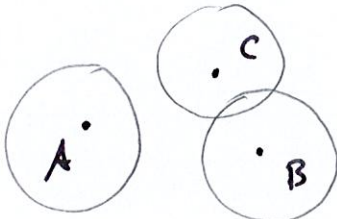


$$H_0 = \mathbb{K}^3 = \langle A, B, C \rangle$$

$$H_i = \{0\}$$

$$i: \begin{matrix} A \rightarrow A \\ B \rightarrow B \\ C \rightarrow B \end{matrix}$$

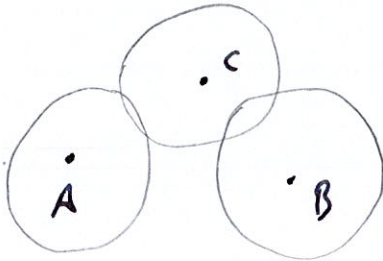
$t \in [1.8, 2.5[$:



$$H_0 = \mathbb{K}^2 = \langle A, B \rangle$$

$$H_i = \{0\}$$

$t \in [2.5, 3[$:

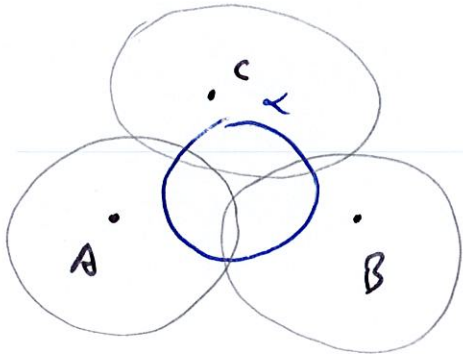


$$H_0 = \mathbb{K} = \langle A \rangle$$

$$H_i = \{0\}$$

$$i: \begin{matrix} A \rightarrow A & \mathbb{K}^0 \\ 0 \rightarrow \alpha & \mathbb{K}_1 \end{matrix}$$

$t \in [3, 3.??[$

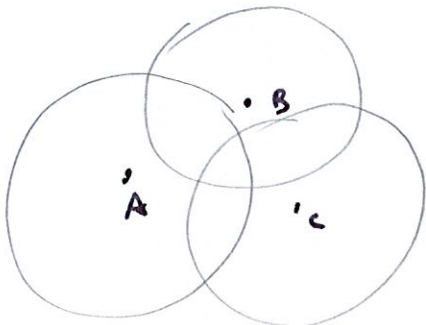


$$H_0 = \mathbb{K} = \langle A \rangle$$

$$H_1 = \mathbb{K} = \langle \alpha \rangle$$

$$H_i = \{0\}$$

$t \gg 3.?$

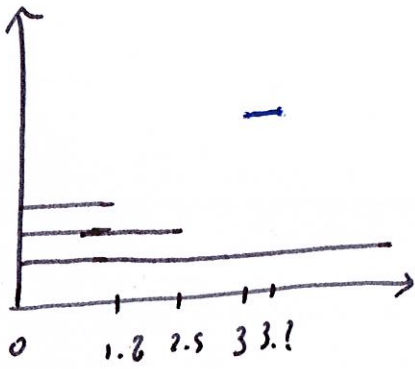


$$\begin{matrix} H_0 = \mathbb{K} \\ H_i = \{0\} \end{matrix}$$

thi

8

Code. livre:



Permittance:

