Critical phases for non–relativistic $2d$ interacting bosons: Renormalization Group results

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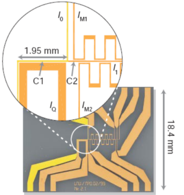
joint work with Alessandro Giuliani

Renormalization in statistical physics and lattice field theories

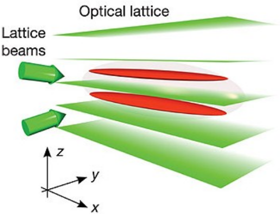
Institut Montpelliérain Alexander Grothendieck

August 24 - 28, 2015

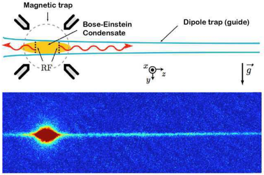
Ultracolds atoms:
an ultralow temperature laboratory for many-body physics



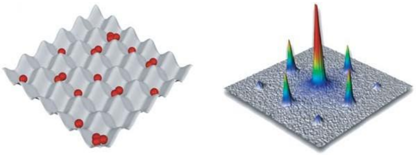
Hänsel et al., "Atom chip" (2001)



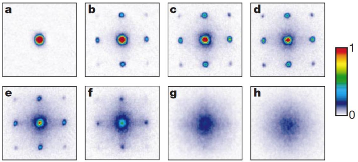
Krüger et al. (2007)



Billy et al., Guided atom laser (2008)



Nature Physics (2005)



Greiner et al., Quantum Phase Transition (2002)

The model

- Gas of non-relativistic bosons in a periodic box $\Omega_L \in \mathbb{R}^d$, $|\Omega_L| = L^d$
- weak repulsive short range two-body potential $\lambda v(\vec{x})$, $0 < \lambda \ll 1$

$$H_{N,L} = -\sum_{i=1}^{N} \Delta_{\vec{x}_i} + \lambda \sum_{1 \le i < j \le N} v\left(\vec{x}_i - \vec{x}_j\right) \qquad \text{on } \mathcal{H}_N^{sym}$$

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Goal: to construct the thermal ground state in infinite volume

$$e_0(\rho) = -\lim_{\beta\to\infty}\lim_{L\to\infty}\frac{1}{\beta L^d}\log \mathrm{Tr}_{\mathcal{H}_N^{sym}}\, e^{-\beta(H_{N,L}-\mu_{\beta,L}N)}$$

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Existence of condensation

The non-interacting case (Einstein, 1925)

	$\lambda = 0$		
$d = 3$	$0 \leq T \leq T_c^0$		
$d = 2$	T=0		
$d = 1$	T=0		

The density of the states with $\vec{k} \neq 0$ is bounded in each dimension at $T = 0$ and at finite T in $3d$ as $\mu_{L,\beta} \to 0$:

$$\lim_{|\Omega| \to +\infty} \rho_{\Omega,\beta}^{(\vec{k} \neq 0)} = \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{e^{\beta(\vec{k}^2 - \mu_{L,\beta})} - 1} \leq \rho_\beta^{critical}$$

To fix the system at $\rho = \rho_0 + \rho_\beta^{critical}$ the chemical potential has to be choosen such that $\lim_{\beta\to\infty} \lim_{L^d \to \infty}(-\mu_{\beta,L}) = 1/\rho_0$.

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The ground state wave function is simply a product of single particle wave–functions: $\Psi_0(x_1, \ldots, x_N) = \prod_i^N \psi_0(x_i)$

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Existence of condensation

The interacting case within Bogoliubov approximation (1947)

	$\lambda = 0$	$\lambda > 0$ & Bogoliubov	
$d = 3$	$0 \leq T \leq T_c^0$	$0 \leq T \leq T_c^B$	
$d = 2$	T=0	T=0	
$d = 1$	T=0	No cond.	

Total density according to Bogoliubov approximation

$$\rho \;=\; \rho_0 \;+\; \int \frac{d^d\vec{k}}{(2\pi)^d} \underbrace{\frac{F(\vec{k}) - \varepsilon(\vec{k})}{\varepsilon(\vec{k})}}_{\underset{\vec{k}\simeq 0}{\simeq}\ \frac{1}{|\vec{k}|}} \;+\; \int \frac{d^d\vec{k}}{(2\pi)^d} \underbrace{\frac{F(\vec{k})}{\varepsilon(\vec{k})} \frac{1}{e^{\beta\varepsilon(\vec{k})} - 1}}_{\underset{\vec{k}\simeq 0,\ \beta \text{ finite}}{\simeq}\ \frac{1}{\beta|\vec{k}|^2}}$$

with $F(\vec{k}) = |\vec{k}|^2 + \lambda\hat{v}(\vec{k})\rho_0$ and $\varepsilon^2(\vec{k}) = |\vec{k}|^4 + 2\,|\vec{k}|^2\lambda\hat{v}(\vec{k})\rho_0$.

Existence of condensation

The interacting case: known results on condensation

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$\Big\}$ MWH thm(1) (for $d = 2$ and $d = 1$)

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(1) Hohenberg (1967) (2) Lieb and Liniger (1963)

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- Hard–core 3d bosons on a lattice at half filling (Dyson, Lieb and Simon, 1978)
- $3d$ and $2d$ bosons in the Gross–Pitaevskii limit: $N/L = (\text{const.})$ (Lieb, Seiringer, Yngvason 2002)
- Bogoliubov's scheme has been proved to be valid in the mean field regime (Seiringer 2010, Grech-Seiringer 2012, Dereziński and Napiórkowski 2013)

[1] Hohenberg (1967) [2] Lieb and Liniger (1963)

Existence of condensation

The interacting case: which general results for homogeneous bosons?

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Constructing the theory: a great challenge. A program addressing this issue started by Balaban, Feldman, Knörrer, Trubowitz (2008–2015)

Two dimensions

- Pistolesi, Castellani, Di Castro, Strinati (1997 & 2004): RG analysis in $d = 2, 3$ by using local Ward Identities in a dimensional regularization scheme with $d = 3 - \varepsilon$. After assuming the existence of a $O(1)$ fixed point for the particle-particle effective interaction:
  - the condensate is stable in $2d$ and $T = 0$;
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(3) **the momentum cutoffs break the local gauge invariance**

$\hookrightarrow$ In low-dimensional systems of interacting fermions (Luttinger liquids) the corrections to WIs are crucial for establishing the infrared behavior of the system (Benfatto, Falco, Mastropietro, 2009)

The goal

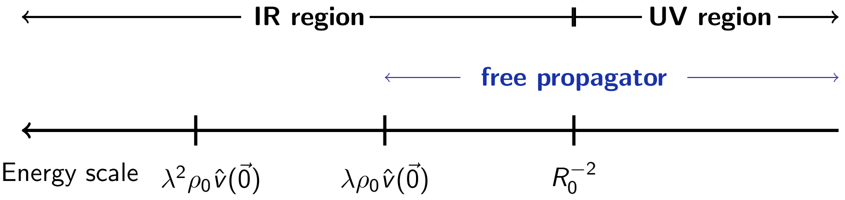
Extend the Wilsonian RG approach to the Bose gas in the $2d$ continuum, at $T = 0$, both for $\rho_0 = 0$ and $\rho_0 > 0$, in the formalism developed by Benfatto and Gallavotti.

Exact RG approach

- Explicit bounds at all orders
- Complete control of all the diagrams (irrelevant terms included)
- Momentum cutoff regularization (essential for a non perturbative construction)
- Corrections to Local Ward Identities can be studied within this scheme

Renormalization group results

S.C. and A.Giuliani, Jour. Stat. Phys. (2014)



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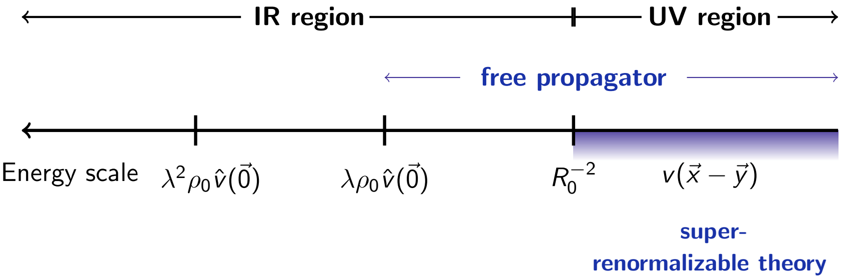

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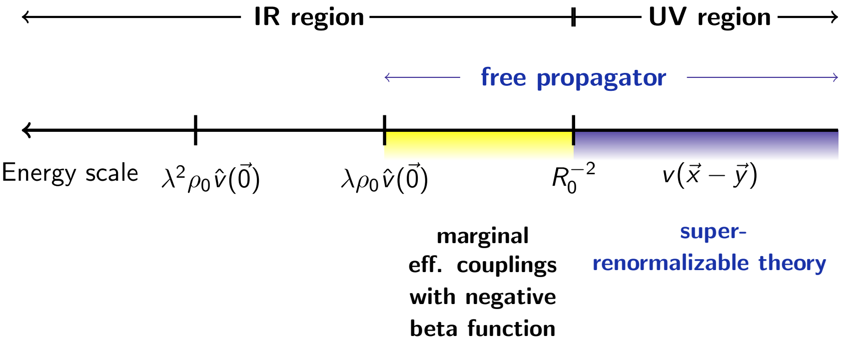

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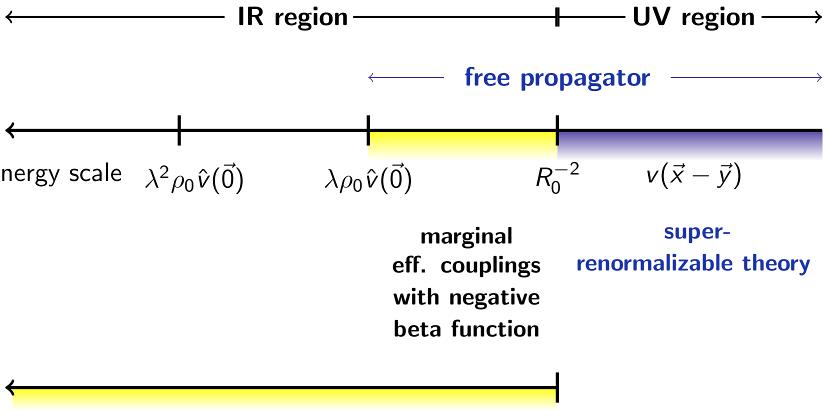

QCP ($\rho_0 = 0$) asymptotically free in the IR [1]

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[1] Fisher, Weichmann, Grinstein, Fisher (1989), Sachdev, Senthil, Shankar (1994)

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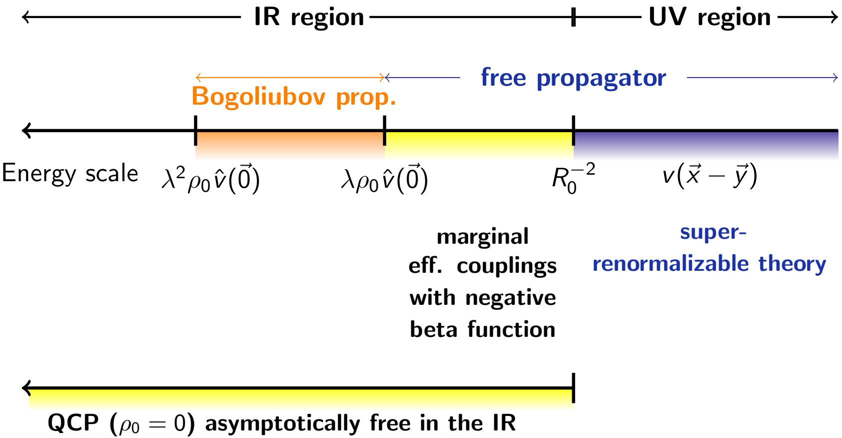

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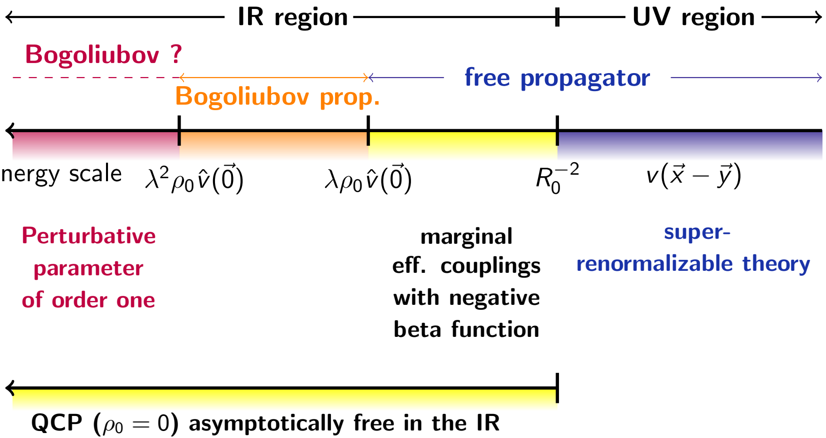

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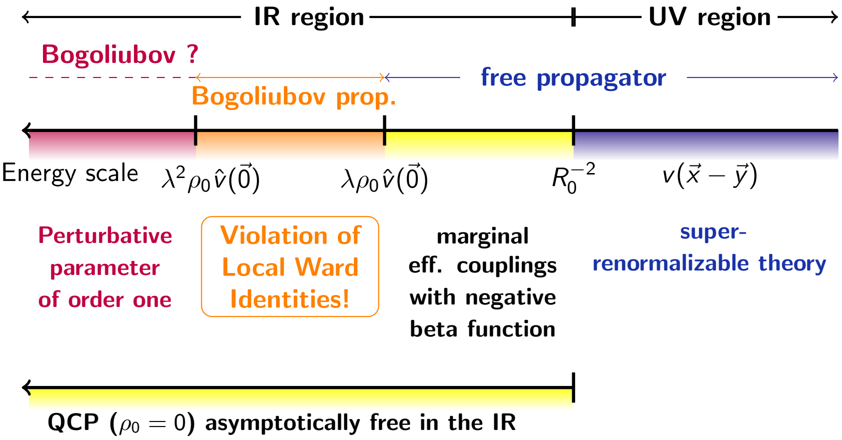

The functional integral representation

The interacting partition function can be formally expressed as a functional integral:
$$\frac{Z_\Lambda}{Z_\Lambda^0} = \int P_\Lambda^0(d\varphi)\, e^{-V_\Lambda(\varphi)}$$

- $\varphi^+_{\vec{x},t} = (\varphi^-_{\vec{x},t})^*$ complex fields (coherent states)
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$$V_\Lambda(\varphi) = \frac{\lambda}{2} \int_{L^4} d^2\vec{x} d^2\vec{y} \int_{-\beta/2}^{\beta/2} dt\, |\varphi_{\vec{x},t}|^2\, v(\vec{x}-\vec{y})\, |\varphi_{\vec{y},t}|^2 - \bar{\nu}_{\beta,L} \int_{L^2} d^2\vec{x} \int_{-\beta/2}^{\beta/2} dt\, |\varphi_{\vec{x},t}|^2$$

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$P^0_\Lambda(d\varphi)$ is a complex Gaussian measure with covariance

$$S^0_\Lambda(x,y) = \left\langle a^+_x a_y \right\rangle\Big|_{\lambda=0} = \int P^0_\Lambda(d\varphi)\varphi^-_x \varphi^+_y$$

$$\underset{|\Omega|,\beta\to\infty}{\longrightarrow} \rho_0 + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^2\vec{k}\, dk_0\, \frac{e^{-ik\cdot x}}{-ik_0 + \vec{k}^2}$$

RG scheme for the BEC phase

$\varphi_x^{\pm} = \xi^{\pm} + \psi_x^{\pm}$ with $\xi^{\pm} = |\Lambda|^{-1} \int_\Lambda \varphi_x^{\pm} dx$, $\langle \xi^- \xi^+ \rangle = \rho_0$, $\langle \psi_x^- \psi_x^+ \rangle$ decaying

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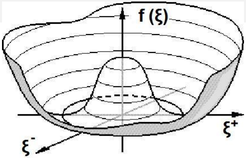

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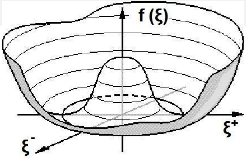

$$\varphi_x^{\pm} = \sqrt{\rho_0} + \psi_x^{\pm}$$

$$f(\rho) = f^B(\sqrt{\rho_0}) - \bar{\nu}\rho_0 + \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \int P_\Lambda^B(d\psi)\, e^{-\bar{V}_\Lambda(\psi)}$$

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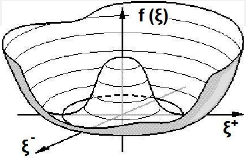

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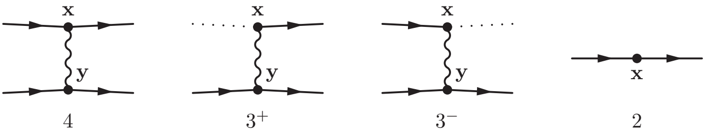

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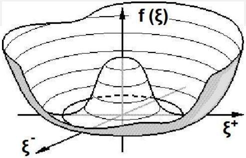

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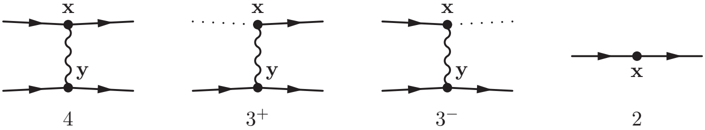


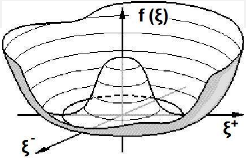
Bogoliubov propagator:

$$g^B(\mathbf{x}-\mathbf{y}) = \begin{pmatrix} g^B_{-+}(\mathbf{x}-\mathbf{y}) & g^B_{--}(\mathbf{x}-\mathbf{y}) \\ g^B_{++}(\mathbf{x}-\mathbf{y}) & g^B_{+-}(\mathbf{x}-\mathbf{y}) \end{pmatrix}$$

$$= \int \frac{dk_0 d^2\vec{k}}{(2\pi)^3} \frac{e^{-i\vec{k}(\vec{x}-\vec{y})-ik_0(x_0-y_0)}}{k_0^2+\varepsilon^2(\vec{k})} \begin{pmatrix} ik_0+|\vec{k}|^2+\lambda\hat{v}(\vec{k})\rho_0 & -\lambda\rho_0\hat{v}(\vec{k}) \\ -\lambda\rho_0\hat{v}(\vec{k}) & -ik_0+|\vec{k}|^2+\lambda\hat{v}(\vec{k})\rho_0 \end{pmatrix},$$

with $\varepsilon^2(\vec{k}) = |\vec{k}|^4 + 2\lambda\hat{v}(\vec{k})\rho_0|\vec{k}|^2$.

RG scheme for the BEC phase



$$\Xi_\Lambda = \int P_\Lambda^B(d\psi)\, e^{-\bar{V}_\Lambda(\psi)}$$

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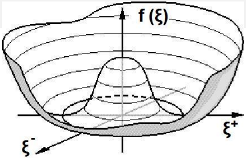

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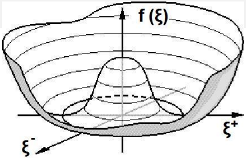

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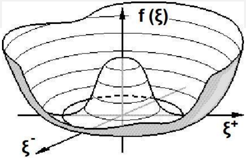

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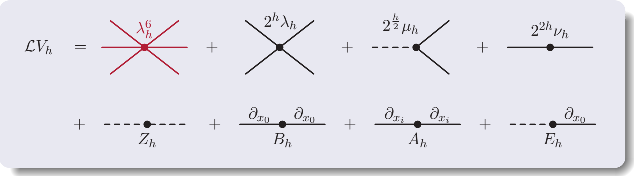

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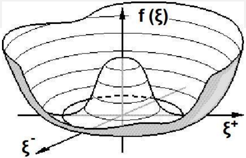

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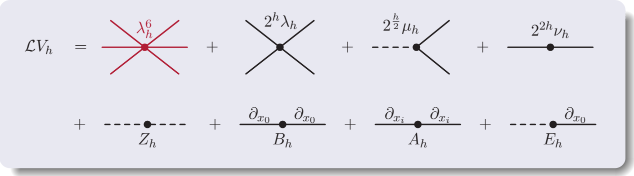

3. Using **the Gallavotti-Nicoló tree expansion** we prove that $\mathcal{R}V_h$ is well defined with explicit bounds if the terms in $\mathcal{L}V_h$ are bounded.

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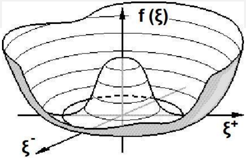

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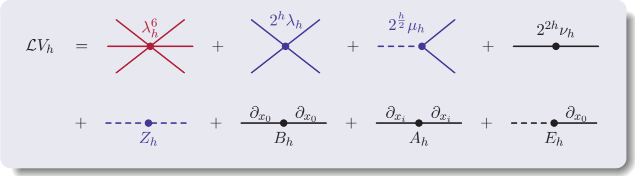

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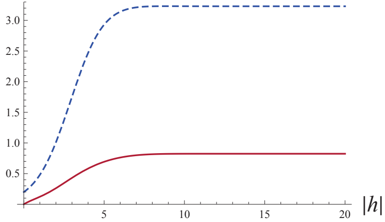
4. Cancellations in the beta function of Z_h and μ_h follow from **Global WIs**

Flow equations for the effective interactions *below* h^*

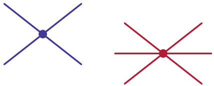
There are two effective (three and two body) interactions, whose flows are coupled among them at all orders. Under the assumptions on the propagator

$$E_h^2/(Z_h B_h) \ll 1 \qquad A_h/B_h = (\text{const.}) \qquad B_h \leq (\text{const.})$$

the flow equations for $x_h := \lambda_h$ and $y_h := \lambda_{6,h}/(\lambda_h^2)$ at leading order (in the continuum limit) are:



Numerical solutions to the leading order flows for x_h and y_h.



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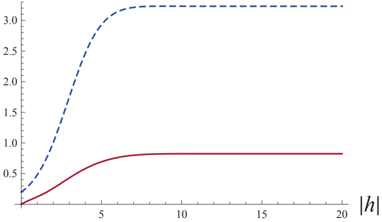

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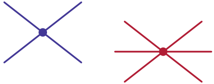

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Non perturbative fixed points?

Flow equations for the renormalized wave functions

Even assumimg the existence of the fixed points, one is left with studying the flow of A_h, E_h and B_h.

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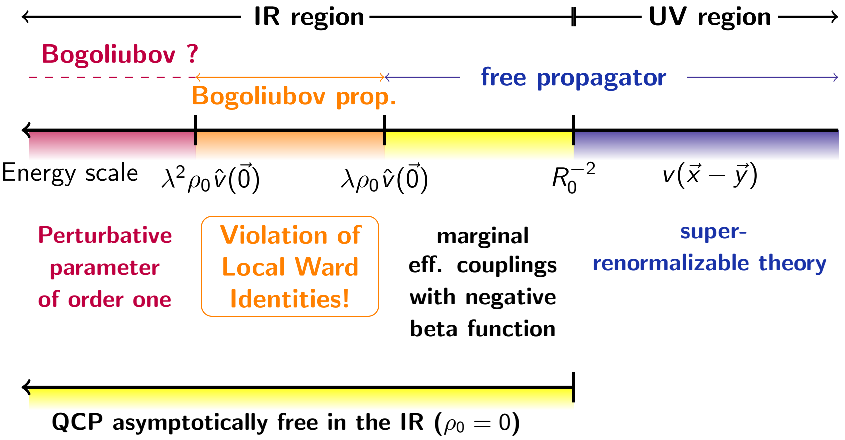
The findings based on the systematic use of local WIs, and then the **nature and existence of the** $2d$ **condensate**, should be reconsidered.

Renormalization group results

S.C. and A.Giuliani, Jour. Stat. Phys. (2014)



Perspectives

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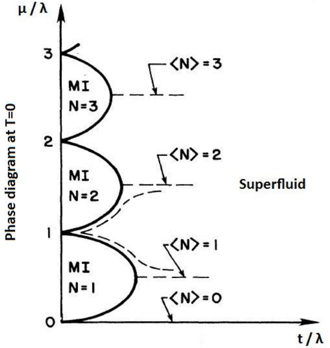
- ...
- Constructive theory (starting from the quantum critical point)

The superfluid-insulator transition in the boson Hubbard model[1]

On site interacting bosons hoppings between sites i of a lattice, $t > 0$ (hopping parameter), $\lambda > 0$ and $\mu \leq 0$ (chemical potential):

$$H_L = -t \sum_{\langle ij \rangle} (\hat{a}_i^+ \hat{a}_j + \hat{a}_j^+ \hat{a}_i) - \mu \sum_i \hat{n}_i + \frac{\lambda}{2} \sum_i \hat{n}_i (\hat{n}_i - 1)$$



The chemical potential drives a Quantum Phase transition between:

- **Mott Insulator phase** with integer boson density (no phase coherence);
- **Superfluid phase** with delocalized wave function (long-range order).

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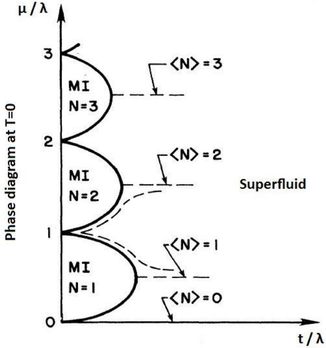

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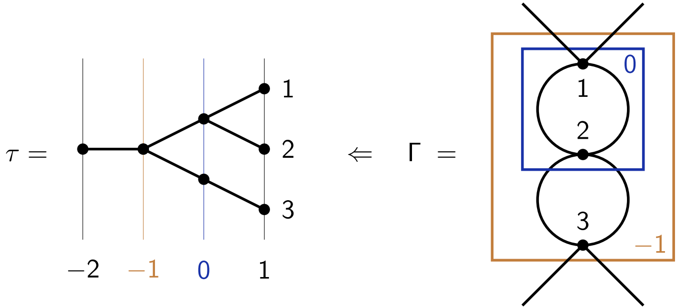
Gallavotti–Nicolò tree expansion

The $|h|$–th step of the iterative integration can be graphically represented as a sum of trees over $|h|$ scale labels. The number n of endpoints represents the order in perturbation theory.



Gallavotti–Nicolò trees are a synthetic and convenient way to isolate the divergent terms, avoiding the problem of overlapping divergences.