(MultiScale) Loop Vertex Expansion

Vincent Rivasseau

LPT Orsay

Montpellier, August 2015

Ordinary Model

$$d\nu = \frac{1}{Z} e^{-(\lambda/4!)\int \phi^4(x)dx} d\mu_C(\phi)$$

$$C(p) = \frac{1}{(2\pi)^2}\frac{1}{p^2+m^2}, \quad C(x,y) = \int_0^\infty d\alpha e^{-\alpha m^2}\frac{e^{-|x-y|^2/4\alpha}}{\alpha^2},$$

$$S_N(z_1,...,z_N) = \int \phi(z_1)...\phi(z_N)d\nu(\phi).$$

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Expanding in the coupling constant λ yields (bare) perturbative field theory:

$$\begin{aligned} S_N(z_1,...,z_N) &= \frac{1}{Z}\sum_{n=0}^{\infty}\frac{(-\lambda/4!)^n}{n!}\int\Big[\int\phi^4(x)dx\Big]^n\phi(z_1)...\phi(z_N)d\mu(\phi) \\ &= \sum_G A_G(z_1,\cdots,z_N) \end{aligned}$$

$$A_G(z_1,\cdots,z_N)=\int\prod_{v=1}^{n}d^dx_v\prod_{\ell}C(x_\ell,x'_\ell)$$

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Perturbative ϕ^4_d

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Two main difficulties

- Some Feynman amplitudes diverge if $d \geq 2$; problem depends on d
  - Solution: Renormalization
- Feynman graphs proliferate too fast, hence $\sum_G |A_G| = +\infty$. (ϕ^4 graphs not exponentially bounded combinatoric species); problem does NOT depend on d
  - Solution: Borel summation, constructive theory; replace Feynman graphs by trees (exponentially bounded combinatoric species)

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The Forest Formula

Let F be a smooth function of $n(n-1)/2$ line variables $x_\ell \in [0,1]$, $\ell = (i,j)$, $1 \le i < j \le n$. The forest formula states

$$F(1,...,1) = \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} \big[\int_0^1 dw_\ell \big] \Big\} \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_\ell} F \Big\} \big[x^{\mathcal{F}}(\{w\}) \big], \text{ where}$$

- the sum over $\mathcal{F}$ is over all forests over n vertices,
- the "weakening parameter" $x_\ell^{\mathcal{F}}(\{w\})$ is 0 if $\ell = (i,j)$ with i and j in different connected components with respect to $\mathcal{F}$; otherwise it is the infimum of the $w_{\ell'}$ for ℓ' running over the unique path from i to j in $\mathcal{F}$.
- Furthermore the real symmetric matrix $x_{i,j}^{\mathcal{F}}(\{w\})$ (completed by 1 on the diagonal $i = j$) is positive.

The logarithm of the forest formula is the tree formula.

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The logarithm of the forest formula is the tree formula.

Why is the Forest Formula positive?

- The set PS_n of positive n by n symmetric matrices with 1 on the diagonal and off-diagonal entries between 0 and 1 is convex.
- Order $0 = w_0 \leq w_1 \leq \cdots \leq w_n \leq 1 = w_{n+1}$.
  $x_{i,j}^{\mathcal{F}}(\{w\}) = \sum_{k=1}(w_k - w_{k-1})\Pi_k$, Π_k block matrix
- This convex combination depends on the ordering of the $w's$.

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An Example

For $n = 2$, the forest fomula is simply: $F(1) = F(0) + \int_0^1 dh\ F'(h)$. For $n = 3$ there are seven forests and the formula is:

$$F(1,1,1) = F(0,0,0) + \int_0^1 dw_1\ \partial_1 F(w_1,0,0) + \int_0^1 dw_2\ \partial_2 F(0,w_2,0)$$
$$+ \int_0^1 dw_3\ \partial_3 F(0,0,w_3) + \int_0^1\int_0^1 dw_1 dw_2\ \partial^2_{12} F(w_1,w_2,\min(w_1,w_2))$$
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Usually the logarithm of the forest formula is the tree formula.

Borel summability of a series a_n means existence of a function f with two properties

- Analyticity in a disk tangent at the origin to the imaginary axis
- plus uniform remainder estimates:

$$|f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n| \leq K^N |\lambda|^{N+1} N!$$

Given any series a_n, there is at most one such function f. When there is one, it is called the Borel sum, and it can be computed from the series to arbitrary accuracy.

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How to compute a log

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}}$$

is Borel summable. How to compute $G(\lambda) = \log F(\lambda)$ (and prove it is also Borel summable)?

- Composition of series
- With Feynman graphs (1950)
- Classical constructive theory (Glimm-Jaffe-Spencer, 1970's - => Brydges, Feldman, Slade ...)
- Loop Vertex Expansion (LVE, 2007-)

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Composition of series

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F &= 1 + H,\ H = \sum_{p \geq 1} a_p (-\lambda)^p,\ a_p = \frac{(4p)!!}{p!} \\
\log(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\
G &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H(\lambda)^n}{n} = \sum_{k \geq 1} b_k (-\lambda)^k, \\
b_k &= \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_1, \dots, p_n \geq 1 \\ p_1 + \dots + p_n = k}} \prod_j \frac{(4p_j)!!}{p_j!}
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Borel summability is unclear. Even the sign of b_k is unclear.

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A la Feynman

$$F = 1 + H, \;\; H = \sum_{p \geq 1} a_p(-\lambda)^p, \;\; a_p = \frac{1}{p!}\#\{\text{vacuum graphs on } p \text{ vertices}\}$$

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$b_1 = 3, \; b_2 = 48, \; b_3 = 1584...$

Borel summability unclear. $b_k \geq 0$ clear.

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Cluster expansion = Taylor-Lagrange expansion of the functional integral:

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$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \prod_{i=1}^{n} H_i(\lambda) \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \eta_\ell \Big\} \prod_{\ell \notin \mathcal{F}} [1 + \eta_\ell x_\ell^{\mathcal{F}}(\{w\})]$$

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where the second sum runs over trees!

- Convergence easy because each H_i contains a different "copy" $\int dx_i$ of functional integration.
- Borel summability now easy from the Borel summability of H.
- Generalizes well to lattice statistical mechanics ($d > 0$).
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Intermediate field representation

$$F = \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\sqrt{2\lambda}\sigma x^2 - x^2/2 - \sigma^2/2} \frac{dx}{\sqrt{2\pi}}\frac{d\sigma}{\sqrt{2\pi}}$$
$$= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\log[1+i2\sqrt{2\lambda}\sigma] - \sigma^2/2} \frac{d\sigma}{\sqrt{2\pi}}$$
$$= \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{V^n}{n!} d\mu(\sigma)$$

Apply the forest formula using copies ('replicas"): $V^n(\sigma) \to \prod_{i=1}^n V_i(\sigma_i)$, $d\mu(\sigma) \to d\mu_C(\{\sigma_i\})$, $C_{ij} = 1 = x_{ij}|_{x_{ij}=1}$.

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Loop Vertex Expansion II

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} \big[\int_0^1 dw_\ell \big] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{F}}}$$

where $C_{ij}^{\mathcal{F}} = x_\ell^{\mathcal{F}}(\{w\})$ if $i < j$, $C_{ii}^{\mathcal{F}} = 1$.

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Advantages

One can picture the result as a sum over trees on loops, or "cacti". Since

$$\frac{\partial^k}{\partial\sigma^k}\log[1+i2\sqrt{2\lambda}\sigma] = -(k-1)!(-i2\sqrt{2\lambda})^k[1+i2\sqrt{2\lambda}\sigma]^{-k},$$

- Convergence is easy because $|[1+i2\sqrt{2\lambda}\sigma]^{-k}| \leq 1$
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A constructive dream

Weights $w(G,T)$, for any connected graph G and spanning tree $T \subset G$, such that

$$\sum_{T \subset G} w(G,T) = 1.$$

Any sum over connected graphs G can be formally repacked as

$$S = \sum_G A_G = \sum_G \sum_{T \subset G} w(G,T) A_G = \sum_T A_T, \quad A_T = \sum_{G \supset T} w(G,T) A_G.$$

Could it be that

$$\sum_G |A_G| = +\infty, \quad \sum_T |A_T| < +\infty?$$

Then S would be well defined, and could be the Borel sum of $\sum_G A_G$! But this dream seems impossible to realize.

Any repacking at fixed order of perturbation theory cannot work. It should mix infinitely many Feynman graphs of different order!

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Indeed in the domain $0 < w_{\sigma(E)} < \cdots < w_{\sigma(1)} < 1$ the function

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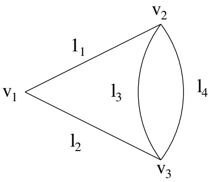
An example



Five trees. Naive weights: 1/5. Constructive weights: $w(G,(l_1,l_2)) = 1/6$ (4 leading sectors) , $w(G,(l_1,l_3)) = 5/24$ (5 leading sectors).

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\end{aligned}
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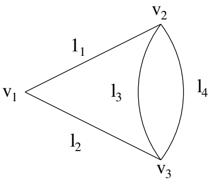

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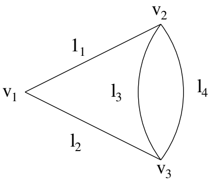

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How does the LVE repack the initial Feynman graphs?

- first step (extension): decompose each Feynman ϕ^4 graph with n vertices into 3^n combinatorial maps, with new "dashed edges" and loop vertices whose "corners" or "arcs" are the former graph edges;
- second step (collapse): contract every loop vertex to a fat black vertex $=>$ result expressed in terms of ordinary maps with dashed edges only and new black vertices
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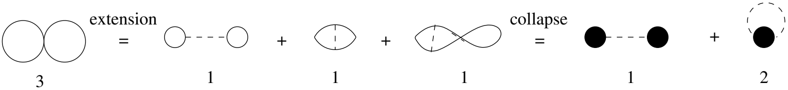
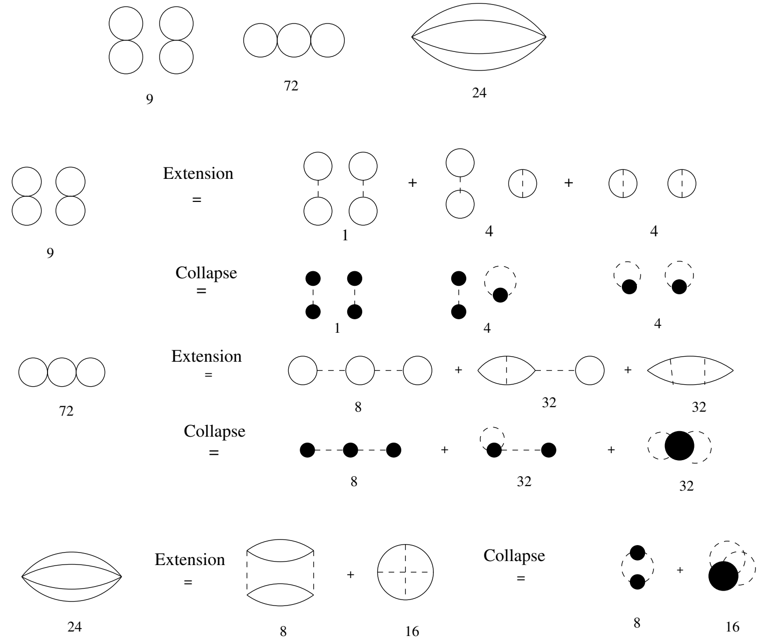


Tree structure in loop vertex expansion:


9
72
24
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Extension
=
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4
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8
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It realizes the constructive dream, but with Feynman maps M instead of Feynman graphs G

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LVE for "richer" models

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typically replace $(1+i\sqrt{\lambda}\sigma)^{-1}$ by $C^{1/2}(1+i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})^{-1}C^{1/2}$

works for any "space-time" (Riemann manifold, infinite discrete triangulation...), on which $C \geq 0$ is both bounded and trace class ...

if C not trace class: add mass renormalization; if C^2 not trace class: add coupling constant renormalization.

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hopefully: QFT of space-time, nice formalism for quantum gravity...

Let us consider a pair of complex fields $(\bar{\phi}, \phi) \in \mathcal{H}_N = \mathbb{C}^N$.

When $N \to \infty$, $\mathcal{H}_N \to \mathcal{H} = \ell_2(\mathbb{N})$, the Hilbert space.

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vector models => matrix models => tensor models

Smaller symmetry means there are more invariants available for interactions

Random vectors have exactly one connected polynomial invariant interaction, of degree 2 namely the scalar product $\bar{\phi} \cdot \phi$.

Random matrices: $N = N_1 N_2$, => $U(N_1 N_2)$ symmetry can break to $U(N_1) \otimes U(N_2)$ giving rise to infinitely many connected invariant polynomial interactions, one at every (even) degree, namely $\mathrm{Tr}\,(MM^\dagger)^p$.

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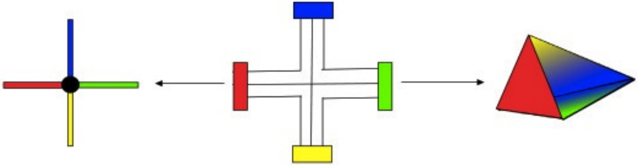

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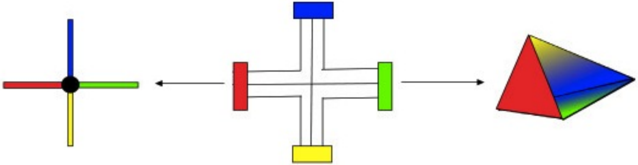

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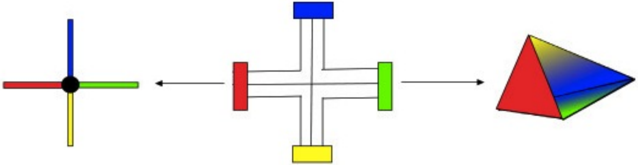

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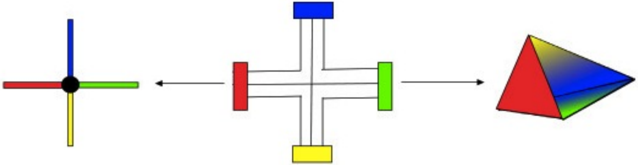

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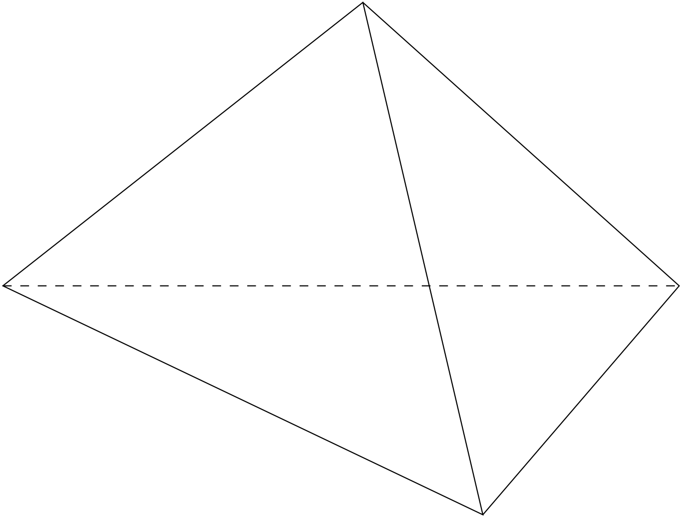
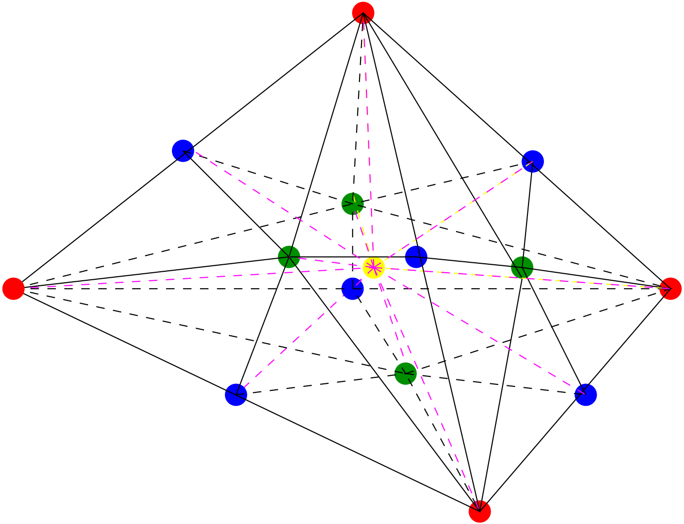
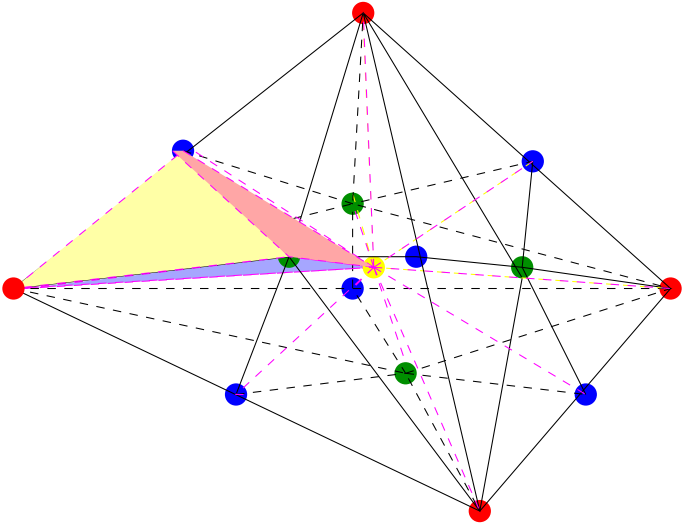


Barycentric Colored Triangulations





R. Gurau found in 2009 that crystallization theory is dual to a quantum field theory and in 2010 that this field theory admits a $1/N$ expansion.

This expansion is not topological !

Basic objects: $U(N)^{\otimes D}$ tensor invariants = regular D-edge-colored connected bipartite graphs

- are dual to colored triangulations
- are the interactions (vertices) of rank-D random tensors
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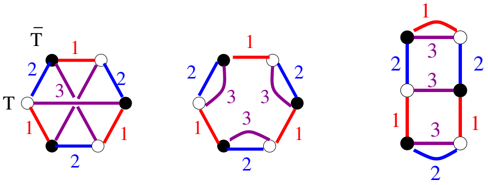

Tensor invariants can be counted as equivalence classes of permutations (J. Ben Geloun and S. Ramgoolam)

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Z_1^c(n) &= 1, 0, 0, 0, 0, \dots && \bar{\Phi} \cdot \Phi \\
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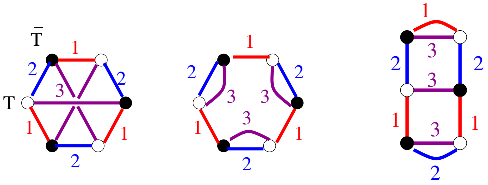

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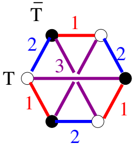

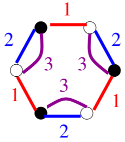

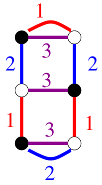

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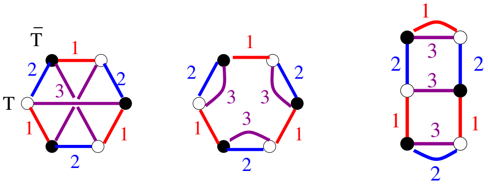

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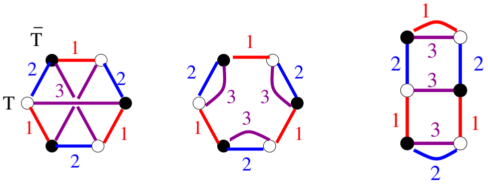

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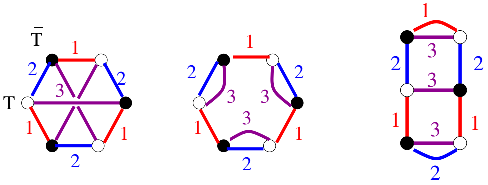

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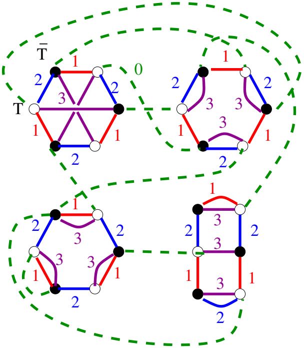

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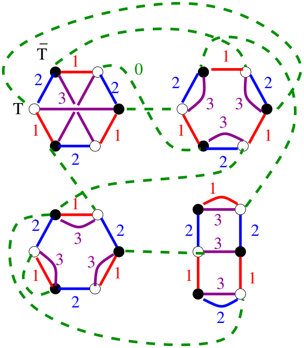

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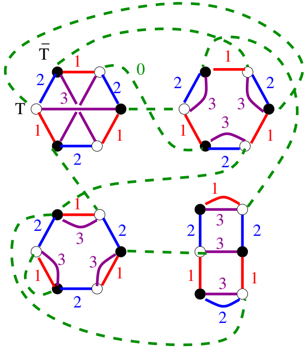
$$S(T,\bar{T}) = T \cdot \bar{T} + \sum_{\mathcal{B}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(\bar{T}, T)$$
$$Z(t_{\mathcal{B}}) = \int [d\bar{T} dT]\, e^{-N^{D-1} S(T,\bar{T})}$$

Feynman graphs: "vertices" $\mathcal{B}$. Gaussian integral: Wick contractions of T and $\bar{T} \to$ dashed edges to which we assign the index 0 (here green color).



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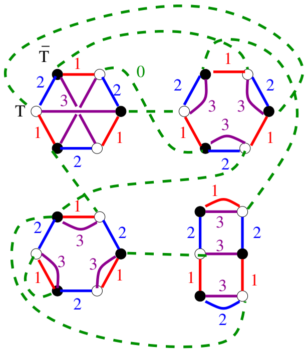

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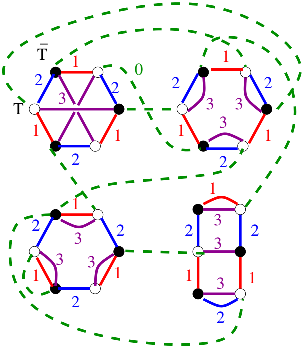
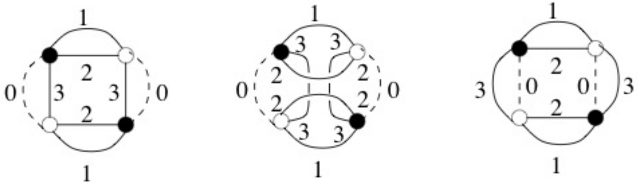


Jackets = color cycle up to orientation ($D!/2$ at rank D)
= canonical system of $D!/2$ globally defined Heegaard surfaces in the dual triangulation



Gurau's degree governs the expansion. After suitable scaling, $A(G) \propto N^{D-\omega(G)}$, where

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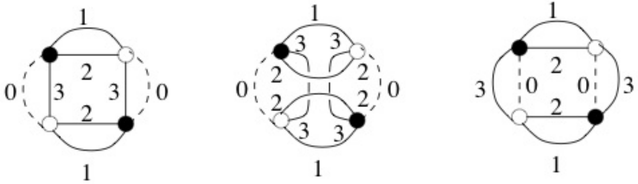

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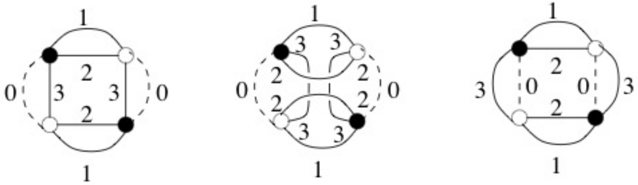

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The Feynman graphs of tensor models can be considered an equilateral version of Regge calculus (1962):

$$S_{\mathrm{Regge}} = \Lambda \sum_{\sigma_D} vol(\sigma_D) - \frac{1}{16\pi G} \sum_{\sigma_{D-2}} vol(\sigma_{D-2})\, \delta(\sigma_{D-2})$$

Discretized Einstein Hilbert action on a triangulation with Q_D equilateral D-simplices and Q_{D-2} $(D-2)$-simplices:

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On the Feynman dual graph: $Q_D \Rightarrow n$, number of vertices; $Q_{D-2} \Rightarrow F$, number of faces, hence Regge action for equilateral simplices becomes

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A Very Simple Superrenormalizable Vector Field Theory

Conjugate vector fields $\{\phi_p\}, \{\bar\phi_p\}, p = 1, \cdots, N$, with $\frac{\lambda^2}{2}(\bar\phi \cdot \phi)^2$ bare interaction.

Gaussian measure $d\mu(\bar\phi, \phi)$ which slightly breaks the $U(N)$ invariance of the theory. It has diagonal covariance which decreases as the inverse power of the field index:

$$d\eta(\bar\phi, \phi)\ , \qquad \int d\eta(\bar\phi, \phi)\ \bar\phi_p \phi_q = \frac{\delta_{pq}}{p}\ .$$

$$Z(\lambda, N) = \int d\eta(\bar\phi, \phi)\ e^{-\frac{\lambda^2}{2N}(\bar\phi\cdot\phi)^2}\ ,$$

Perturbative amplitudes of the model all finite in the $N \to \infty$ limit, except for a logarithmic divergence of self-loops (as in ordinary ϕ^4_2) $L_N = \sum_{p=1}^{N} \frac{1}{p} \simeq \log N$.

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Theorem

The pressure $p = \lim_{N\to\infty} \frac{1}{N} \log Z(\lambda, N)$ *is analytic and Borel summable in* $z = \lambda^2$.

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Requires MLVE Analysis (arXiv 1312.7226)

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The pressure $p = \lim_{N\to\infty} \frac{1}{N} \log Z(\lambda, N)$ *is analytic and Borel summable in* $z = \lambda^2$.

LVE alone not enough because of renormalization.

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Applying the ordinary LVE to this functional integral would express $\log Z(\lambda, N)$ as a sum over trees.

However logarithmic divergence of each leaf of the tree must be compensated;

It generates one intermediate field σ in numerator for each leaf:
$\big(1 - \imath\frac{\lambda\sigma}{p}\big)^{-1} - 1 = \imath\frac{\lambda\sigma}{p}\big(1 - \imath\frac{\lambda\sigma}{p}\big)^{-1}$

Integrated through the Gaussian measure, these numerator σ's would create divergent bounds for trees with many leaves.

The MLVE is designed to solve this problem.

We fix a number $L > 1$ (here, non-canonical choice.) and define the j-th *slice*, as made of the indices $p \in I_j \equiv [L^{j-1}, L^j - 1]$. The ultraviolet cutoff N is chosen as $N = L^{j_{max}}$, with $j_{\max}$ an integer.

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Fermionic Fields

$$Z(\lambda, N) = \int d\nu(\sigma) \prod_{j=1}^{j_{\max}} e^{-V_j} , \quad V_j = \sum_{p \in I_j} \log_2 \Big(1 - \imath \frac{\lambda \sigma}{p} \Big) .$$

The factorization of the interaction over the set of slices $\mathcal{S} = [1 \cdots j_{\max}]$ can be encoded into an integral over Grassmann numbers. Indeed,

$$a = \int d\bar{\chi} d\chi \; e^{-\bar{\chi} a \chi} = \int d\mu(\bar{\chi}, \chi) \; e^{-\bar{\chi}(a-1)\chi}$$

where $d\mu(\bar{\chi}, \chi) = d\bar{\chi} d\chi \; e^{-\bar{\chi}\chi}$ is the standard normalized Grassmann Gaussian measure with covariance 1. Hence, denoting $W_j(\sigma) = e^{-V_j} - 1$,

$$Z(\lambda, N) = \int d\nu(\sigma) \Big(\prod_{j=1}^{j_{\max}} d\mu(\bar{\chi}_j, \chi_j) \Big) \; e^{-V}$$

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m Level Jungle Formula

An *m-jungle* is a sequence $\mathcal{J} = (\mathfrak{F}_1, \dots, \mathfrak{F}_m)$ of forests on $I_n = [1, \cdots n]$ such that $\mathfrak{F}_1 \subset \dots \subset \mathfrak{F}_m$.

Given an m-jungle $\mathcal{J} = (\mathfrak{F}_1, \dots, \mathfrak{F}_m)$, we introduce the notation $\mathbf{w}$ for the vector $(w_l)_{l \in \mathfrak{F}_m}$, and $w^{\mathcal{J},k}_{\{ij\}}(\mathbf{w})$ for the functions defined by:

- if i and j are not connected by $\mathfrak{F}_k$, $w^{\mathcal{J},k}_{\{ij\}}(\mathbf{w}) = 0$.
- if i and j are connected by $\mathfrak{F}_{k-1}$, $w^{\mathcal{J},k}_{\{ij\}}(\mathbf{w}) = 1$.
- if i and j are connected by $\mathfrak{F}_k$ but not by $\mathfrak{F}_{k-1}$, $w^{\mathcal{J},k}_{\{ij\}}(\mathbf{w})$ is the infimum of the w_ℓ for ℓ in $\mathfrak{F}_k \setminus \mathfrak{F}_{k-1} \cap P^{\mathfrak{F}}_{ij}$, where $P^{\mathfrak{F}}_{ij}$ is the unique path that goes from i to j in $\mathfrak{F}_k$.

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$$F(\mathbf{1}) = \sum_{\substack{\mathcal{J}=(\mathfrak{F}_1,\dots,\mathfrak{F}_m)\\ m-\text{jungle}}} \Big(\prod_{l\in\mathfrak{F}_m}\int_0^1 dw_l\Big)\bigg(\Big(\prod_{k=1}^{m}\Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}}\frac{\partial}{\partial x_l^k}\Big)\Big)F\bigg)(X_{\mathcal{J}}^{BK}(\mathbf{h})) \quad .$$

Here $X_{\mathcal{J}}^{BK}(\mathbf{w})$ is the vector $(x_l^k)_{(l,k)}$ defined by $x_l^k = h_l^{\mathcal{J},k}(\mathbf{w})$, which is the value at which we evaluate the complicated derivative of H.

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We need only $m = 2$. Expand $e^{-V} = \sum_{n=0}^{\infty} \frac{(-V)^n}{n!}$ Set $V_n = [1, \cdots, n]$

- Introduce n copies (replicas) for the σ field of each vertex, through a Gaussian matrix with covariance 1 everywhere, then the $n(n-1)/2$ interpolation variables $x_{ij} \in [0, 1]$ for the off-diagonal elements of the covariance.
  The σ variables have no scale attached.
- Introduce in the same way $n(n-1)/2$ interpolation variables $y_{ij} \in [0, 1]$ for the for the off-diagonal elements of the Grassmann Gaussian covariance $\bar{\chi}$ and χ variables (keeping intact the fact that the Fermionic variables have scales attached, and that the measure $d\mu(\bar{\chi}, \chi) = \prod_{j=1}^{j_{\max}} d\mu(\bar{\chi}_j, \chi_j)$ is factorized over scales).

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Result

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \quad \partial_{\mathcal{J}} \Big[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \Big(W_{j_a}(\sigma^a_{j_a}) \chi^{\mathcal{B}}_{j_a} \bar{\chi}^{\mathcal{B}}_{j_a} \Big) \Big] ,$$

where

- the sum over $\mathcal{J}$ runs over all two level jungles, hence over all ordered pairs $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$ of two (each possibly empty) disjoint forests on V_n, such that $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ is still a forest on V_n. The forests $\mathcal{F}_B$ and $\mathcal{F}_F$ are the Bosonic and Fermionic components of $\mathcal{J}$. The edges of $\mathcal{J}$ are partitioned into Bosonic edges ℓ_B and Fermionic edges ℓ_F.
- $\int dw_{\mathcal{J}}$ means integration from 0 to 1 over parameters w_ℓ, one for each edge $\ell \in \bar{\mathcal{J}}$. $\int dw_{\mathcal{J}} = \prod_{\ell \in \bar{\mathcal{J}}} \int_0^1 dw_\ell$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $w_{\mathcal{J}}$ is therefore made of $|\bar{\mathcal{J}}|$ parameters $w_\ell \in [0,1]$, one for each $\ell \in \bar{\mathcal{J}}$.

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- the sum over $\mathcal{J}$ runs over all two level jungles, hence over all ordered pairs $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$ of two (each possibly empty) disjoint forests on V_n, such that $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ is still a forest on V_n. The forests $\mathcal{F}_B$ and $\mathcal{F}_F$ are the Bosonic and Fermionic components of $\mathcal{J}$. The edges of $\mathcal{J}$ are partitioned into Bosonic edges ℓ_B and Fermionic edges ℓ_F.
- $\int dw_{\mathcal{J}}$ means integration from 0 to 1 over parameters w_ℓ, one for each edge $\ell \in \bar{\mathcal{J}}$. $\int dw_{\mathcal{J}} = \prod_{\ell \in \bar{\mathcal{J}}} \int_0^1 dw_\ell$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $w_{\mathcal{J}}$ is therefore made of $|\bar{\mathcal{J}}|$ parameters $w_\ell \in [0,1]$, one for each $\ell \in \bar{\mathcal{J}}$.

Result

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \quad \partial_{\mathcal{J}} \Big[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \Big(W_{j_a}(\sigma^a_{j_a}) \chi^{\mathcal{B}}_{j_a} \bar{\chi}^{\mathcal{B}}_{j_a} \Big) \Big] \,,$$

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- $$\partial_{\mathcal{J}} = \prod_{\substack{\ell_B \in \mathcal{F}_B \\ \ell_B = (c,d)}} \Big(\frac{\partial}{\partial \sigma^c_{j_c}} \frac{\partial}{\partial \sigma^d_{j_d}} \Big) \prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \delta_{j_a j_b} \Big(\frac{\partial}{\partial \bar{\chi}^{\mathcal{B}(a)}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}(b)}_{j_b}} + \frac{\partial}{\partial \bar{\chi}^{\mathcal{B}(b)}_{j_b}} \frac{\partial}{\partial \chi^{\mathcal{B}(a)}_{j_a}} \Big) ,$$

  where $\mathcal{B}(a)$ denotes the Bosonic blocks to which a belongs.
- The measure $d\nu_{\mathcal{J}}$ has covariance $X(w_{\ell_B}) \otimes \mathbf{1}_{\mathcal{S}}$ on Bosonic variables and $Y(w_{\ell_F}) \otimes \mathbb{I}_{\mathcal{S}}$ on Fermionic variables,

  $$e^{\frac{1}{2} \sum_{a,b=1}^n X_{ab}(w_{\ell_B}) \frac{\partial}{\partial \sigma^a_{j_a}} \frac{\partial}{\partial \sigma^b_{j_b}} + \sum_{\mathcal{B},\mathcal{B}'} Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F}) \sum_{a \in \mathcal{B}, b \in \mathcal{B}'} \delta_{j_a j_b} \frac{\partial}{\partial \bar{\chi}^{\mathcal{B}}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}'}_{j_b}}} .$$

- $X_{ab}(w_{\ell_B})$ is the infimum of the w_{ℓ_B} parameters for all the Bosonic edges ℓ_B in the unique path $P^{\mathcal{F}_B}_{a \to b}$ from a to b in $\mathcal{F}_B$. The infimum is set to zero if such a path does not exists and to 1 if $a = b$.
- $Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F})$ is the infimum of the w_{ℓ_F} parameters for all the Fermionic edges ℓ_F in any of the paths $P^{\mathcal{F}_B \cup \mathcal{F}_F}_{a \to b}$ from some vertex $a \in \mathcal{B}$ to some vertex $b \in \mathcal{B}'$. The infimum is set to 0 if there are no such paths, and to 1 if such paths exist but do not contain any Fermionic edges.

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- $Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F})$ is the infimum of the w_{ℓ_F} parameters for all the Fermionic edges ℓ_F in any of the paths $P^{\mathcal{F}_B\cup\mathcal{F}_F}_{a\to b}$ from some vertex $a\in\mathcal{B}$ to some vertex $b\in\mathcal{B}'$. The infimum is set to 0 if there are no such paths, and to 1 if such paths exist but do not contain any Fermionic edges.

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MLVE Analysis

- $$\partial_{\mathcal{J}} = \prod_{\substack{\ell_B \in \mathcal{F}_B \\ \ell_B=(c,d)}} \Big(\frac{\partial}{\partial \sigma^c_{j_c}} \frac{\partial}{\partial \sigma^d_{j_d}} \Big) \prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F=(a,b)}} \delta_{j_a j_b} \Big(\frac{\partial}{\partial \bar\chi^{\mathcal{B}(a)}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}(b)}_{j_b}} + \frac{\partial}{\partial \bar\chi^{\mathcal{B}(b)}_{j_b}} \frac{\partial}{\partial \chi^{\mathcal{B}(a)}_{j_a}} \Big) ,$$

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The log is then easily computed!

$$Z(\lambda, N) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ jungle}} \cdots \quad =>$$

$$\log Z(\lambda, N) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}}\, \partial_{\mathcal{J}} \Big[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \Big(W_{j_a}(\sigma^a_{j_a}) \chi^{\mathcal{B}}_{j_a} \bar{\chi}^{\mathcal{B}}_{j_a} \Big) \Big],$$

where the sum is the same but conditioned on $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ being a spanning tree on $V = [1, \cdots, n]$.

Theorem

This series is absolutely convergent uniformly in $j_{\max}$, in a Borel domain for $z = \lambda^2$.

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Why does it work

1) The Grassmann variable integration can be bounded by 1. Here positivity of the jungle formula is essential!

Any positive real matrix R admits a square root $R = Z^2$, and by the Hadamard inequality,

$$\det R = (\det Z)^2 \leq \Big[\prod_{i=1}^{n} (\sum_k z_{ik} z_{ki})\Big] = \Big[\prod_{i=1}^{n} R_{ii}\Big]^2 .$$

In the case at hand, the diagonal R_{ii} terms are all equal to 1.

2) Some factorials cannot be avoided. Each renormalized leaf at scale j gives a σ numerator but also a convergent L^{-j} factor and a small λ coupling.

3) Bosonic blocks have distinct scales (ensured by the Grassmann variables; there is a single Grassmann variable per scale in each block). Hence inside a Bosonic block $\mathcal{B}$ the scales $j_1, \cdot j_p$ must be distinct.

4) Convergence follows because for any fixed $\alpha > 0$ and λ small, $\sum_p \sum_{j_1 \neq \cdots \neq j_p} \lambda^p [p!]^\alpha \prod_{k=1}^p L^{-j_k} << 1$ since $\prod_{k=1}^p L^{-j_k} \leq L^{-p(p-1)/2}$.

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General Conclusion, LVE

- For ϕ^4 models with no renormalization required, the LVE provides an explicit and canonical convergent repacking of perturbation theory, based on the forest formula.
- For superrenormalizable models incorporating (mass) renormalization, the MLVE with Bosons, Fermions and a 2-level jungle formula provides a convergent series, based on a Wilsonian slice decomposition (with a non-canonical parameter L).
- Open Question 1: Can one remove this slice decomposition?
- Simple superrenormalizable tensor models can be controlled non perturbatively by the MLVE (T. Delepouve, R., 2014), (although the σ field in that case is a large random matrix).
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- Simple superrenormalizable tensor models can be controlled non perturbatively by the MLVE (T. Delepouve, R., 2014), (although the σ field in that case is a large random matrix).
- Open question 2: could more general models, in particular just renormalizable asymptotically free tensor models incorporating coupling constant renormalization be also treated by this MLVE. do they require more than a 2-level jungle formula?

General Conclusion, Tensor Field Theories

Tensor field theories have promising features for quantum gravity and random geometry in dimension $d \geq 3$:

- they perform background independent sums over all geometries (including topology change)
- pondered by discretized EH action
- when renormalizable, they are typically asymptotically free (at least in the case of simple quartic interactions)
- at least in some cases they can be controlled non-perturbatively with an MLVE.

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