Critical Dynamics in Driven-Dissipative Bose-Einstein Condensation

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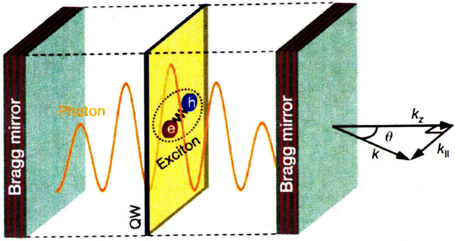
Experimental Motivation

Pumped semiconductor quantum wells in optical cavities:
driven Bose–Einstein condensation of exciton-polaritons



J. Kasprzak *et al.*, Nature **443**, 409 (2006); K.G. Lagoudakis *et al.*, Nature Physics **4**, 706 (2008)

Theoretical approach:

- nonlinear Langevin dynamics, mapped to *path integral*
- perturbatively analyze *ultraviolet divergences* ($d \geq d_c = 4$)
- scale (μ) dependence, flow equations for *running couplings*
- *emerging symmetry*: $\xi \to \infty$ induces *scale invariance*
- critical RG *fixed point* $\to$ *scale invariance*, *infrared scaling laws*
- *loop expansion* in $\epsilon = d_c - d \ll 1 \to$ *critical exponents*

Langevin Description of Critical Dynamics

Critical slowing-down as correlated regions grow ($\tau \propto T - T_c$):
$\to$ *relaxation time* $t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu}$, *dynamic exponent* z

coarse-grained description:

- *fast* modes $\to$ *random noise*
- *mesoscopic Langevin equation* for *slow* variables $S^\alpha(x,t)$

Example: purely *relaxational critical dynamics* ("model A"):

$$\frac{\partial S^\alpha(x,t)}{\partial t} = -D \, \frac{\delta H[S]}{\delta S^\alpha(x,t)} + \zeta^\alpha(x,t) \, , \ \langle \zeta^\alpha(x,t) \rangle = 0 \ ,$$

$$\langle \zeta^\alpha(x,t) \, \zeta^\beta(x',t') \rangle = 2D \, k_{\rm B} T \, \delta(x-x') \, \delta(t-t') \, \delta^{\alpha\beta}$$

Einstein relation guarantees that $\mathcal{P}[S,t] \to e^{-H[S]/k_{\rm B} T}$ as $t \to \infty$

non-conserved order parameter: $D = \text{const.}$
conserved order parameter: relaxes *diffusively*, $D \to -D \, \nabla^2$

Generally: *mode couplings* to additional *conserved*, slow fields
$\to$ various *dynamic universality classes*

Driven-Dissipative Bose–Einstein Condensation

Noisy *Gross–Pitaevskii equation* for complex bosonic field ψ:

$$i \frac{\partial \psi(\mathbf{x},t)}{\partial t} = \Big[-(A - iD)\,\nabla^2 - \mu + i\chi + (\lambda - i\kappa)\,|\psi(\mathbf{x},t)|^2 \Big]\,\psi(\mathbf{x},t) + \zeta(\mathbf{x},t)$$

$A = 1/2m_{\mathrm{eff}}$; D diffusivity (dissipative); μ chemical potential;
$\chi \sim$ pump rate - loss; $\lambda, \kappa > 0$: two-body interaction / loss
noise correlators: ($\gamma = 4D\,k_{\mathrm{B}}T$ in equilibrium)

$$\langle \zeta(\mathbf{x},t) \rangle = 0 = \langle \zeta(\mathbf{x},t)\,\zeta(\mathbf{x}',t') \rangle$$

$$\langle \zeta^*(\mathbf{x},t)\,\zeta(\mathbf{x}',t') \rangle = \gamma\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$

$$r = -\frac{\chi}{D}\,,\ r' = -\frac{\mu}{D}\,,\ u' = \frac{6\kappa}{D}\,,\ r_K = \frac{A}{D}\,,\ r_U = \frac{\lambda}{\kappa}\,,\ \zeta \to -i\zeta$$

$\rightarrow$ time-dependent *complex Ginzburg–Landau equation*

$$\frac{\partial \psi(\mathbf{x},t)}{\partial t} = -D\Big[r + i r' - (1 + i r_K)\,\nabla^2 + \frac{u'}{6}\,(1 + i r_U)\,|\psi(\mathbf{x},t)|^2 \Big]\,\psi(\mathbf{x},t) + \zeta(\mathbf{x},t)$$

Relationship with Equilibrium Critical Dynamics

"Model A" *relaxational kinetics* for non-conserved order parameter:

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -D \, \frac{\delta \bar{H}[\psi]}{\delta \psi^*(\mathbf{x}, t)} + \zeta(\mathbf{x}, t)$$

with non-Hermitean "Hamiltonian"

$$\bar{H}[\psi] = \int d^d x \left[(r + i r') \, |\psi(\mathbf{x}, t)|^2 + (1 + i r_K) \, |\nabla \psi(\mathbf{x}, t)|^2 \right.$$
$$\left. + \frac{u'}{12} \, (1 + i r_U) \, |\psi(\mathbf{x}, t)|^4 \right]$$

- (1) $r' = r_K = r_U = 0$: *equilibrium* model A for non-conserved two-component order parameter, GL-Hamiltonian $H[\psi]$
- (2) $r' = r_U \, r$, $r_K = r_U \neq 0$: $S_{1/2} = \mathrm{Re}/\mathrm{Im}\psi$, $\bar{H} = (1 + i r_K) \, H$

$$\frac{\partial S_\alpha(\mathbf{x}, t)}{\partial t} = -D \, \frac{\delta H[\vec{S}]}{\delta S_\alpha(\mathbf{x}, t)} + D r_K \sum_\beta \epsilon_{\alpha\beta} \frac{\delta H[\vec{S}]}{\delta S_\beta(\mathbf{x}, t)} + \eta_\alpha(\mathbf{x}, t)$$

$$\langle \eta_\alpha(\mathbf{x}, t) \rangle = 0 \, , \;\; \langle \eta_\alpha(\mathbf{x}, t) \, \eta_\beta(\mathbf{x}', t') \rangle = \frac{\gamma}{2} \, \delta_{\alpha\beta} \, \delta(\mathbf{x} - \mathbf{x}') \, \delta(t - t')$$

$\rightarrow$ *effective equilibrium* dynamics with detailed balance (FDT)

Scaling Laws and Critical Exponents

(Bi-)critical point $\tau, \tau' = r_K \tau \to 0$: correlation length $\xi(\tau) \sim |\tau|^{-\nu}$

universal scaling for dynamic response and correlation functions:

$$\chi(\mathbf{q},\omega,\tau) \propto \frac{1}{|\mathbf{q}|^{2-\eta}\,(1+ia|\mathbf{q}|^{\eta-\eta_c})}\;\hat{\chi}\Big(\frac{\omega}{|\mathbf{q}|^{z}\,(1+ia|\mathbf{q}|^{\eta-\eta_c})}, |\mathbf{q}|\xi\Big)$$

$$C(\mathbf{q},\omega,\tau) \propto \frac{1}{|\mathbf{q}|^{2+z-\eta'}}\;\hat{C}\Big(\frac{\omega}{|\mathbf{q}|^{z}}, |\mathbf{q}|\xi, a|\mathbf{q}|^{\eta-\eta_c}\Big)$$

five independent critical exponents (three in equilibrium: ν, η, z)

Non-perturbative (numerical) renormalization group study:

$$d = 3: \;\; \nu \approx 0.716,\ \eta = \eta' \approx 0.039,\ z \approx 2.121,\ \eta_c \approx -0.223$$

L.M. Sieberer, S.D. Huber, E. Altman, S. Diehl, *Phys. Rev. Lett.* **88**, 045702 (2013); *Phys. Rev. B* **89**, 134310 (2014)

Thermalization: one-loop → scenario (2); two-loop → model A (1)

Critical exponents in $\epsilon = 4 - d$ expansion:

$$\nu = \frac{1}{2} + \frac{\epsilon}{10} + O(\epsilon^2)\,,\ \eta = \frac{\epsilon^2}{50} + O(\epsilon^3)$$

$$z = 2 + c\eta\,,\ c = 6\ln\frac{4}{3} - 1 + O(\epsilon)$$

as for equilibrium model A; in addition, *novel critical exponent*:

$$\eta_c = c'\eta\,,\ c' = -\left(4\ln\frac{4}{3} - 1\right) + O(\epsilon)\,,\ \text{but FDT} \to \eta' = \eta$$

$$\epsilon = 1: \;\; \nu \approx 0.625,\ \eta = \eta' \approx 0.02,\ z \approx 2.01452,\ \eta_c \approx -0.0030146$$

Onsager–Machlup Functional

Coupled *Langevin equations* for mesoscopic stochastic variables:

$$\frac{\partial S^\alpha(x,t)}{\partial t} = F^\alpha[S](x,t) + \zeta^\alpha(x,t)\,, \ \langle \zeta^\alpha(x,t) \rangle = 0 \ ,$$
$$\langle \zeta^\alpha(x,t) \zeta^\beta(x',t') \rangle = 2 L^\alpha \, \delta(x-x') \, \delta(t-t') \, \delta^{\alpha\beta}$$

- *systematic* forces $F^\alpha[S]$, *stochastic forces (noise)* ζ^α
- noise *correlator* L^α: can be operator, functional of S^α

Assume *Gaussian stochastic process* $\rightarrow$ probability distribution:

$$\mathcal{W}[\zeta] \propto \exp\left[-\frac{1}{4} \int d^d x \int_0^{t_f} dt \sum_\alpha \zeta^\alpha(x,t) \left[(L^\alpha)^{-1} \zeta^\alpha(x,t) \right] \right]$$

switch variables $\zeta^\alpha \rightarrow S^\alpha$: $\mathcal{W}[\zeta] \, \mathcal{D}[\zeta] = \mathcal{P}[S] \mathcal{D}[S] \propto e^{-\mathcal{G}[S]} \mathcal{D}[S]$,
with *Onsager-Machlup functional* providing field theory action:

$$\mathcal{G}[S] = \frac{1}{4} \int d^d x \int dt \sum_\alpha (\partial_t S^\alpha - F^\alpha[S]) \left[(L^\alpha)^{-1} (\partial_t S^\alpha - F^\alpha[S]) \right]$$

- functional determinant $= 1$ with *forward* (Itô) discrectization
- normalization: $\int \mathcal{D}[\zeta] W[\zeta] = 1 \ \rightarrow$ "partition function" $= 1$
- problems: $(L^\alpha)^{-1}$, high non-linearities $F^\alpha[S] \, (L^\alpha)^{-1} \, F^\alpha[S]$

Janssen–De Dominicis Response Functional

Average over *noise "histories"*: $\langle A[S] \rangle_\zeta \propto \int \mathcal{D}[\zeta] \, A[S(\zeta)] \, W[\zeta]$:

use $1 = \int \mathcal{D}[S] \prod_\alpha \prod_{(x,t)} \delta\big(\partial_t S^\alpha(x,t) - F^\alpha[S](x,t) - \zeta^\alpha(x,t)\big)$

$= \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp\left[- \int d^d x \int dt \sum_\alpha \widetilde{S}^\alpha (\partial_t S^\alpha - F^\alpha[S] - \zeta^\alpha)\right]$

$$\langle A[S] \rangle_\zeta \propto \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp\left[- \int d^d x \int dt \sum_\alpha \widetilde{S}^\alpha (\partial_t S^\alpha - F^\alpha[S])\right]$$

$$\times A[S] \int \mathcal{D}[\zeta] \exp\left(- \int d^d x \int dt \sum_\alpha \left[\frac{1}{4} \zeta^\alpha (L^\alpha)^{-1} \zeta^\alpha - \widetilde{S}^\alpha \zeta^\alpha \right] \right)$$

perform *Gaussian integral* over noise ζ^α:

$$\langle A[S] \rangle_\zeta = \int \mathcal{D}[S] \, A[S] \, \mathcal{P}[S] \, , \; \mathcal{P}[S] \propto \int \mathcal{D}[i\widetilde{S}] \, e^{-\mathcal{A}[\widetilde{S},S]} \; ,$$

with *Janssen–De Dominicis response functional*

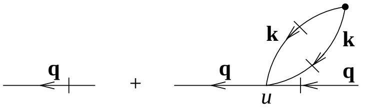
$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int_0^{t_f} dt \sum_\alpha \left[\widetilde{S}^\alpha \left(\partial_t S^\alpha - F^\alpha[S] \right) - \widetilde{S}^\alpha L^\alpha \widetilde{S}^\alpha \right]$$

$\int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \, e^{-\mathcal{A}[\widetilde{S},S]} = 1$; integrate out $\widetilde{S}^\alpha \to$ Onsager–Machlup

One-Loop Renormalization Group Analysis

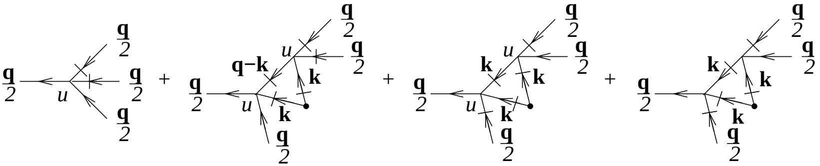
Causality: *propagator* → directed line, *noise* → two-point vertex

two-point vertex function with
$\gamma = 4 D T$, $r' = r_U\, r$, $u = u' T$:



$$\Gamma_{\tilde{\psi}\psi^*}(\mathbf{q}, \omega) = -i\omega + D \left[r(1+i r_U) + (1+i r_K)\mathbf{q}^2 + \tfrac{2}{3} u (1+i r_U) \int_k \frac{1}{r + k^2} \right]$$

→ *fluctuation-induced shift* of critical point: $\tau = r - r_c$, $\tau' = r_U\, \tau$



$\Delta = r_U - r_K$: $\quad \beta_\Delta = \Delta_R \left(1 + \frac{2 r_{KR}\, \Delta_R + \Delta_R{}^2}{1 + r_{KR}^2} \right) \frac{u_R}{3} \;\Rightarrow\; \Delta_R \to 0$

$$\beta_u = u_R \left[-\epsilon + \frac{5}{3}\, u_R - \frac{\Delta_R{}^2}{3(1 + r_{KR}^2)}\, u_R \right] \;\Rightarrow\; u_R \to u^*$$

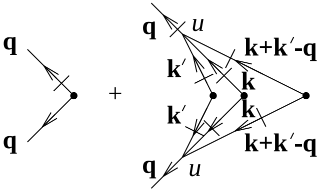
→ *thermalization*; to $O(\epsilon)$: $\nu^{-1} = 2 - \frac{2}{5}\,\epsilon$, $\eta = \eta_c = \eta' = 0$, $z = 2$

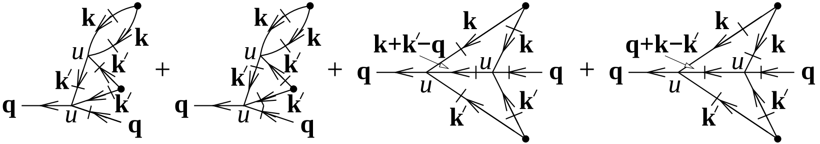
Two-Loop Renormalization Group Analysis

two-loop Feynman graphs for two-point vertex functions $\Gamma_{\tilde{\psi}\tilde{\psi}^*}(\mathbf{q},\omega)$, $\Gamma_{\tilde{\psi}\psi^*}(\mathbf{q},\omega)$

special case $r' = 0$:

T. Risler, J. Prost, and F. Jülicher, *Phys. Rev. E* **72**, 016130 (2005)





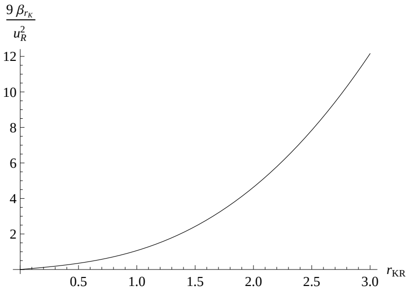
RG beta function β_{r_K}:

$\Rightarrow r_{KR} \to 0$, hence to $O(\epsilon^2)$:

$$\eta = \eta' = \frac{\epsilon^2}{50} + O(\epsilon^3)$$

$$z = 2 + \frac{\epsilon^2}{50}\left(6\ln\frac{4}{3} - 1\right)$$

$$\eta_c = -\frac{\epsilon^2}{50}\left(4\ln\frac{4}{3} - 1\right)$$



subleading scaling exponent

Conserved Dynamics Variant

Complex "model B" variant for *conserved* order parameter:

$$\partial_t \psi(\mathbf{x}, t) = D \nabla^2 \frac{\delta \bar{H}[\psi]}{\delta \psi^*(\mathbf{x}, t)} + \zeta(\mathbf{x}, t) \, , \; \langle \zeta(\mathbf{x}, t) \rangle = 0$$

$$\langle \zeta^*(\mathbf{x}, t) \, \zeta(\mathbf{x}', t') \rangle = -4 T D \nabla^2 \delta(\mathbf{x} - \mathbf{x}') \, \delta(t - t')$$

non-linear vertex $\sim \mathbf{q}^2 \to$ *to all orders*:

$$\Gamma_{\tilde{\psi}\psi^*}(\mathbf{q} = 0, \omega) = -i\omega \, , \; \partial_{q^2} \Gamma_{\tilde{\psi}\tilde{\psi}^*}(\mathbf{q}, \omega = 0)|_{\mathbf{q}=0} = -2 D T$$

$\to$ *exact* scaling relations: $\eta' = \eta \, , \; z = 4 - \eta$

dynamic scaling laws:

$$\chi(\mathbf{q}, \omega, \tau) \propto \frac{1}{|\mathbf{q}|^{2-\eta} \left(1 + i a |\mathbf{q}|^{\eta - \eta_c}\right)} \, \hat{\chi}\Big(\frac{\omega}{|\mathbf{q}|^{4-\eta} \left(1 + i a |\mathbf{q}|^{\eta - \eta_c}\right)}, |\mathbf{q}| \xi \Big)$$

$$C(\mathbf{q}, \omega, \tau) \propto \frac{1}{|\mathbf{q}|^{6-2\eta}} \; \hat{C}\Big(\frac{\omega}{|\mathbf{q}|^{4-\eta}}, |\mathbf{q}| \xi, a |\mathbf{q}|^{\eta - \eta_c} \Big)$$

to two-loop order: $\eta_c = \eta + O(\epsilon^3) = \frac{\epsilon^2}{50} + O(\epsilon^3)$

$\to$ non-equilibrium drive induces no independent critical exponent

Critical Aging and Outlook

"Quench" from random initial conditions onto critical point:

- *time translation invariance broken*
- $t_c \to \infty$: system always "remembers" disordered initial state
- *critical aging scaling* in limit $t'/t \to 0$:

$$\chi(\mathbf{q}, t, t', \tau) \propto \left(\frac{t}{t'}\right)^{\theta} \frac{|\mathbf{q}|^{z-2+\eta}}{1 + ia|\mathbf{q}|^{\eta-\eta_c}} \hat{\chi}\Big(|\mathbf{q}|^{z} \left(1 + ia|\mathbf{q}|^{\eta-\eta_c}\right) t, |\mathbf{q}|\xi\Big)$$

model A: $\theta = \gamma_0/2z$, $\gamma_0 = \frac{2}{5}\epsilon + O(\epsilon^2)$ *as in equilibrium kinetics*
model B: $\theta = 0$ *exactly*

Current and future projects:

- critical quench of driven complex Model A: two-loop analysis
- coarsening, critical aging in suitable "spherical model" limit ?
- thermalization and emerging Model E dynamics ?

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