B1.1 Scientific Quality of the Project

Scientific quality of the proposal

This project is concerned with the study of some quantization problems, using formal deformations. A quantization of a given classical physical system is the data of a quantum physical system such that if we consider Plank's constant $\hbar$ to be negligible (mathematically speaking, one takes the limit $\hbar \to 0$) then we recover the classical one. In 1978, F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowics and D. Sternheimer found a very nice formulation of this situation in purely algebraic terms, it is usually known as deformation quantization theory. Let me breafly describe it.

On one hand in classical mechanics, states are points on a manifold and observables are functions on this manifold. These functions form a commutative algebra A together with a Poisson bracket $\{,\}$. On the other hand in quantum mechanics, states are elements of a Hilbert space and observables are self-adjoint operators on this Hilbert space. These operators form a (non commutative) associative algebra. The fundamental idea of deformation quantization is to forget Hilbert spaces and only take care of algebras (i.e. observables): starting from a commuative algebra A of observables (e.g. functions on a manifold) with a Poisson bracket $\{,\}$, one asks if there exists an associative $\hbar$-linear product $*$ on $A[[\hbar]]$ (formal power series in $\hbar$ with coefficients in A) such that $a * b = ab + \hbar\{a, b\} + o(\hbar)$. In the case $A = C^\infty(X)$ is the algebra of smooth functions on a Poisson manifold X a positive answer was given by M. Kontsevich in 1997. If the starting Poisson manifold carries an additional structure compatible (in a sens to be precised) with the Poisson bracket, then it is natural to ask if there exists a compatible quantization. Let me give a series of examples:

(1) If $X = G$ is a formal Poisson-Lie group, then its algebra of functions A carries a natural structure of a Poisson-Hopf algebra. In 1996, P. Etingof and D. Kazhdan constructed a deformation quantization of A in the category of Hopf algebras.

(2) Assume that the manifold X is equipped with a group action preserving the Poisson bracket. V. Dolgushev has recently proved (2003) that under a reasonable assumption (the existence of a torsion free invariant connexion) one can construct a deformation quantization of $A = C^\infty(X)$ such that the $*$-product is equivariant under the group action.

(3) Interesting Poisson structures are the ones arrizing from a dynamical r-matrix. The question of the existence of a deformation quantization arrizing from a quantum dynamical R-matrix remains open. A positive answer has been given in some particular cases by P. Etingof and D. Nikshych (2001), P. Xu (2002), J. Donin and A. Mudrov (2003), B. Enriquez and P. Etingof (2004), and recently by myself (see [3] below).

(4) An integrable hamiltonian system is the data of a family of independant Poisson-commuting functions $f_i \in C^\infty(X)$. It is not known if one can construct a deformation quantization together with $*$-commuting elements $P_i = f_i + O(\hbar)$ in $C^\infty(X)[[\hbar]]$.

The goal of the project is then twofold:

(a) The notion of groupoid (resp. Lie groupoid) generalizes both the notions of a space (resp. manifold) and a group (resp. Lie group). Physicists should say that it puts in the same framework internal and external symmetries. The smooth functions on a Lie groupoid form a commutative Hopf algebroid, and if one starts from a Poisson-Lie groupoid then one obtains a Poisson-Hopf algebroid. P. Xu asked if one can always construct a deformation quantization of a given Poisson-Lie groupoid **in the category of Hopf algebroids**. We will be particularly interested with the infinitesimal version of this problem: quantization of Lie bialgebroids (which is equivalent to the quantization of formal Poisson-Lie groupoids).

(b) Many integrable systems, such as Calogero-Moser and Hitchin systems, come from a dynamical r-matrix formalism. Our goal is to prove more general existence results of quantization for examples (3) and (4) before, and find explicit formulas in the most interesting situations.

These two directions **(a)** and **(b)** are actually strongly related. Dynamical r-matrices give rise to particular examples of Poisson groupoids. Deformation quantization of those very special groupoids are called dynamical quantum groups and have been intensively studied both by physicist and mathematicians in the past ten years, especially the so-called elliptic dynamical quantum group (introdued by G. Felder) which seems to be very important in the study of integrable models in statistical physics and is not completely well-understood. This could become a third direction **(c)** of the present project.

Research Methodology

In this paragraph I will outline in more technical terms the approach that I intend to adopt.

(a) In order to solve the quantization problem for Lie bialgebroids (or equivalently, formal Poisson-Lie groupoids) we consider the following approachs:

- **The abstract appraoch**. To any vector space V one can associate the graded Lie algebra $\Lambda(V \oplus V^*)$, where the Lie bracket is given by $[(x,\xi),(y,\eta)] = \xi(y) - \eta(x)$. A Lie bialgebra structure on V is equivalent to an element x of bidegree $(2,1)+(1,2)$ in $\Lambda(V \oplus V^*)$ such that $[x,x]=0$. On the other hand, M. Markl discovered in a recent preprint that the Gerstenhaber-Schack bicomplex $(\mathrm{Hom}(A,A), d_{GS})$ of any bialgebra A can be equipped with a natural structure of a L_∞-algebra in such a way that axioms of bialgebras can be described via a master equation related to this L_∞-structure. Moreover in the case when $A = S(V)$ one has $H^\bullet(\mathrm{Hom}(A,A), d_{GS}) = \Lambda^\bullet(V \oplus V^*)$.

  As a first step I conjecture that one can construct a L_∞-quasi-isomorphism from $\Lambda(V \oplus V^*)$ to $\mathrm{Hom}(A,A)$, using the vanishing of a certain cohomology (like in the work of D. Tamarkin).

  The previous discussion can easily be generalized to the case of a vector bundle $V \to X$ over a manifold X. Thus the second step is to generalize the work of M. Markl to the bialgebroid case, and then globalize the first step (like in [2] below).

- **The down-to-earth approach**. The local structure of Lie bialgebroids was studied by Xu and Liu in 2002: they actually classified such structures near a generic point. I recently realized that these structures are actually in bijection with some objects that I call *classical dynamical twists* because they exactly correspond to the classical version of some natural agebraic objects arrizing in quantum group theory (quasi-comodule algebras over quasi-Hopf algebras).

  The good point is that these *classical dynamical twists* generalize both the usual classical dynamical r-matrices and the called Poisson-Lie dynamical r-matrices studied by Enriquez, Etingof and Marshall (2004) for which there are now many efficient quantization technics that could generalize to this situation.

  This approach seems to be more reasonable than the previous one. At least it will surely give quantization result very quickly for concrete examples.

(b) With respect to the problem of quantization of dynamical r-matrices we consider two approachs:

- To any classical dynamical r-matrix, associated to the inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, one can associate a triangular Lie bialgebroid structure on $T\mathfrak{h}^* \times \mathfrak{g} \to \mathfrak{h}^*$. Using the main result of [2] below, I expect to put a structure of a module category over $Rep(U\mathfrak{g})$ on $(C^\infty(\mathfrak{h}^*), *) - mod$, where $*$ is the PBW-quantization of the linear Poisson bracket on $\mathfrak{h}^*$. Then it will give a dynamical twist $J \in C^\infty(\mathfrak{h}^*, (U\mathfrak{g})^{\otimes 2})[[\hbar]]$ that satisfies the dynamical twist equation discovered by P. Xu. Finally, the quantum dynamical R-matrix is $R = (J^{\mathrm{op}})^{-1} * J$.

We also have a work in progress with A. Alekseev and B. Enriquez in order to find explicit formulae for dynamical twists, via a quantum composition formula (generalizing Etingof-Varchenko's classical composition formula) obtained using generalized Harish-Chandra maps (the categorical approach is actually hidden behind these maps).

- Hitchin and Calogero-Moser systems are algebraically completely integrable systems which can be constructed with the help of a r-matrix formalism. Trigonometric dynamical r-matrices have been recently quantized by B. Enriquez and P. Etingof using a modified version of Knizhnik-Zamolodchikov (KZ) equations. Then B. Enriquez developped a trigonometric analogue of Drinfeld's theory of associators.

  In a work in progress with B. Enriquez and P. Etingof we are studying the elliptic KZ equations from a universal viewpoint, with the help of elliptic dynamical r-matrices. We hope to obtain this way a very natural dynamical twist quantization of elliptic dynamical r-matrices. We also developped these KZ equations for curves of higher genus and discovered that their study allows us to give a new (and now constructive) proof of Bezrukavnikov's result which state that the Malcev Lie algebra of the pure braid group of n stands on a (compact and orientable) surface is isomorphic to its associated graded. After this we plan to developp an elliptic analogues of Drinfeld's theory of associators. Of course, this may also help to unterstand better Beilinson-Drinfeld's quantization of Hitchin systems (which are related to KZ equations).

(c) I should finaly add that obtaining a dynamical twist quantization of the elliptic dynamical r-matrix might be a first step in the construction of the elliptic dynamicalquantum group I mentioned before. There is an alternative way to construct it, with the help of a precise mathematical formulation of lattice models for statistical physics. I do not know a lot about this but I plan to learn them from G. Felder who is an expert in this field.

Scientific originality and Innovation

The new idea with respect to the abstract approach to the research topic **(a)** is to use the theory of L_∞-algebras in order to solve the deformation quantization problem for Lie bialgebroids. It has been initiated in [2] with success (the paper has been accepted in a very good journal of mathematical physics). Also, the analogue of the Gersenhaber-Schack bicomplex for bialgebroids (or Hopf algebroids) has not yet been defined. It is an interesting question for itself to construct it.

The down-to-earth approach to the quantization problem for Lie bialgebroid also seem to be completely new. Moreover, it is also very interesting in the sens that it will probably produce new examples and generalizations of known dynamical quantum groups.

With respect to **(b)**, it is not new that one can interprete the quantum part of the dynamical Yang Baxter equation and the dynamical twist equation in terms of intertwiners (see the works of P. Etingof and O. Schiffmann) and module categories over tensor ones. The innovative nature of the first approach here is to go in the inverse way and use the categorical yoga to find an appropriate quantization of a given classical dynamical r-matrix (this was for example the guiding idea of P. Etingof and D. Kazhdan in their pioneer work about quantization of Lie bialgebras).

The work in progress with B. Enriquez and P. Etingof I already mentioned, even if it makes use of very standard notions in mathematical physics and quantum group theory, leads to a very ambitious program (the theory of elliptic associators) with a lot of interactions with other field like group theory and topology.

Timeliness and relevance of the Project

Defomation quantization has been a very active area of research for the past twenty years. Moreover, it has developped connexions with a large number of research topics such as representation theory, category theory, homological algebra, operads, and especially theoretical physics (classical and quantum integrable systems, and quantum field theories).
In the present project, the links are of three types :

- Physical aspects : deformation quantization theory is a part of mathematical physics. More precisely, Poisson-Lie groupoids are related to the famous Poisson-sigma model of physicists, and the existence of a deformation quantization would be a very important result. Also the KZ equations is one of the most famous equations in the recent physical litterature, and rememeber that the elliptic dynamical quantum group plays an important role in statistical physics.
- Geometrical aspects : We have already seen that classical dynamical r-matrices are strongly related to integrable systems. The second approach for their study and quantization gives rise to very interesting geometric questions, about integrable and evolution equations, $\mathcal{D}$-modules, $K3$ surfaces, etc ...
- The quantization of dynamical r-matrices in the sens of P. Xu (i.e. using dynamical twists) can also be interpreted in categorical terms, with the help of representation theory.
- KZ equations appeared to have surprising applications to group theory, topology, and also number theory (through Drinfeld's work).

BI.2 Quality of the Researcher

My past research, that was the object of the articles [2], [3] and [4] below, concerns the construction of appropriated L_∞-quasi-isomorphisms in order to obtain some results of quantization for Lie bialgebroids and dynamical r-matrices.

During the last three years I studied very advanced mathematical theories, and I developed a very good network of relations in the mathematical community, both with other young researchers and experimented ones:

- I had many interesting conversations with G. Halbout who taught me many technics I used in my papers [2] and [3]. I wrote the paper [4], which is a continuation of [2], with him and V. Dolgushev.
- I met P. Xu during the moving seminar "Geometry And Physics" GAP I. We are still discussing a lot by e-mail about Poisson structures, stacks and integrable systems.
- I also have very good relationships with G. Felder. I have been three times in ETH (Zürich) during my PhD to work with him. I spent the last months of my PhD in ETH thanks to a grant of the European Community.
- Among all conferences to which I assisted, there is in particular a series of Lectures given by P. Etingof on "Tensor Categories", which gave place to [1] below. We also have started a very ambitious project with him and my thesis advisor (B. Enriquez) on Knizhnik-Zamolodchikov equations.

- During my stay in Geneva as an assistant post-doc, I worked with A. Alekseev on a new and quite simple approach to quantization of rational dynamical r-matrices. We are currently writing a paper about it.

In the close futur From february 1st to april 30 I am invited as a visitor at the *Institut des Hautes Etudes Scientifiques* in Paris.
After this I will spend 6 months as a visitor at the *Max Planck Institut für Mathematics* in Bonn.

I have been a representative of the students of the European Doctoral College.