# GAUDIN SUBALGEBRAS, MODULI SPACES AND INTEGRABLE SYSTEMS

PRODOC RESEARCH MODULE

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**Title of the project.** Gaudin subalgebras, moduli spaces and integrable systems

**ProDoc title.** This project is attached to the Training Module "Geometry, Algebra, and Mathematical Physics".

## 1. Summary of the research plan

This research project concerns some universal features of the theory of integrable systems.

Our first direction is to further study Gaudin subalgebras in the Kohno–Drinfeld Lie algebra, as well as their elliptic counterparts. We will focus most specifically on their moduli spaces, their realizations as concrete integrable systems, and their relation to representation theory.

The other direction concerns universal Knizhnik–Zamolodchikov(–Bernard) connections, defined on moduli spaces of complex curves of genus 0 and 1, and taking their values in the above mentioned Lie algebras: in particular, a very interesting relation between genus 0 and 1 will be discussed.

## 2. Research plan

### 2.1. Current state of research in the field.

The Kohno–Drinfeld algebra $\mathfrak{t}_n$ [9,24] plays a prominent role in different mathematical contexts and in this research project. It appeared in [24] as the holonomy Lie algebra of the complement $C_n$ of the union of diagonals $z_i = z_j$ in $\mathbb{C}^n$ and in [9] as the Lie algebra of values of the universal Knizhnik–Zamolodchikov connection from conformal field theory. The latter interpretation is the one important for this proposal, where both the Knizhnik–Zamolodchikov equation and its variants as its degenerate version at the critical level related to integrable models of Gaudin type are relevant.

By definition, the Kohno–Drinfeld algebra is the Lie algebra with generators $t_{ij} = t_{ji}$, $1 \leq i \neq j \leq n$ and relations

$$[t_{ij}, t_{kl}] = 0, \text{for } i, j, k, l \text{ distinct},$$

$$[t_{ij}, t_{ik} + t_{jk}] = 0, \text{for } i, j, k \text{ distinct}$$

These relations imply that the KZ connection $\nabla = d + 1/\kappa \sum_{i<j} t_{ij} d\log(z_i - z_j) \in \Omega^1(C_n) \otimes \mathfrak{t}_n$ is flat for all values of the parameter $\kappa$ and in particular that the "Gaudin Hamiltonians" $H_i = \sum_{j \neq i} t_{ij}/(z_i - z_j)$ commute for all $(z_1, \ldots, z_n) \in C_n$. These operators appear in differnent contexts, associated to different representations of $\mathfrak{t}_n$. In the context of spin chains of statistical mechanics, originally considered by Gaudin, the $t_{ij}$ are represented by the action of the tensor quadratic Casimir element of a simple Lie algebra $\mathfrak{g}$ on the $i$th and $j$th factor of a tensor product of $\mathfrak{g}$-modules, which is also the original context of the Knizhnik-Zamolodchikov equation. Via the homomorphism $U\mathfrak{t}_n \to \mathbb{C}S_n$ to the group algebra of the symmetric group sending $t_{ij}$ to the transposition $s_{ij}$, the Gaudin Hamiltonians are sent to commuting operators. As noticed in [5], in a certain limit these operators tend to the Jucys–Murphy elements, used by Vershik and Okounkov [35, 36] in their new approach to the representation theory of the symmetric group. In [37] Vinberg studied the commutative subspaces of degree 2 of the universal enveloping algebra $U\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ in relation with Poisson commutative subalgebras of the Poisson algebra $S\mathfrak{g}$ of polynomial functions on the dual of $\mathfrak{g}$. In the case of $sl_n$ these subalgebras are again spanned by Gaudin Hamiltonians, with $t_{ij}$ associated to the $sl_2$ subalgebras, and their various limiting values.

### 2.2. **Current state of your own research.**

2.2.1. *Gaudin subalgebras, moduli spaces and Bethe ansatz.* One class of results concerns the abelian Lie subalgebras of maximal dimension contained in the linear span $\mathfrak{t}_n^1$ of the generators $t_{ij}$ of the Kohno–Drinfeld algebra. Motivating examples are the algebras considered by Gaudin [17, 18] in the framework of integrable spin chains in quantum statistical mechanics and the Jucys–Murphy subalgebras spanned by $t_{12}, t_{13} + t_{23}, t_{14} + t_{24} + t_{34}, \ldots$, appearing in the representation theory of the symmetric group (see [35, 36] and references therein).

**Theorem 2.1.** [2] *Abelian Lie subalgebras of maximal dimensions contained in* $\mathfrak{t}_n^1$ *form a subvariety of the Grassmannian of* $(n-1)$*-planes in* $\mathfrak{t}_n^1$ *isomorphic to the Mumford–Knudsen compactification* $\bar{M}_{0,n+1}$ *of the moduli space of stable rational curves with* $n+1$ *marked points.*

In particular, this results gives an alternative definition of $\bar{M}_{0,n+1}$ as a subvariety of a Grassmannian of $(n-1)$-planes in an $n(n-1)/2$-dimensional space. Moreover, $\bar{M}_{0,n+1}$ comes with a sheaf of Abelian Lie subalgebras $G_n$ of $\mathfrak{t}_n^1$. The second result concerns the identification of this sheaf (which turns out to be locally free) in terms of the geometry of $\bar{M}_{0,n+1}$. Recall that the logarithmic tangent sheaf $T_X(\log D)$ along a divisor $D$ of a variety $X$ is the subsheaf of vector fields whose restriction to $D$ is tangent to $D$.

**Theorem 2.2.** [2] *The sheaf of Gaudin subalgebras $G_n$ on $X = \bar{M}_{0,n+1}$ fits in an exact sequence*

$$0 \to \mathcal{O}_X \to G_n \to T_X(\log D) \to 0$$

*where $D = \bar{M}_{0,n+1} \smallsetminus M_{0,n+1}$ is the compactification divisor.*

The sheaf $G_n$ can be more precisely described as a sheaf of first order differential operators, see [2]; the trivial subbundle $\mathcal{O}_X$ is spanned by the central element $z = \sum_{i<j} t_{ij}$.

Finally, each representation of the Kohno–Drinfeld algebra gives rise to a sheaf of commutative algebras on $\bar{M}_{0,n+1}$ of operators on the representation space. Suppose the central element $z$ acts by a scalar multiple $\alpha$ of the identity. Then the relative spectrum of the sheaf of algebras can be naturally embedded into a Poisson manifold, the *twisted logarithmic cotangent bundle* $T^*_\alpha \bar{M}_{0,n+1}(-\log D)$, see [2] (it is the logarithmic tangent bundle if $\alpha = 0$). The general result is then:

**Theorem 2.3.** [2] *The relative spectrum of the sheaf of commutative algebras generated by the the image of $G_n$ by a representation with central character $\alpha$ is a coisotropic subscheme of the twisted logarithmic cotangent bundle $T^*_\alpha \bar{M}_{0,n+1}(-\log D)$.*

2.2.2. *Universal KZB connection (genus 1).* The universal Knizhnik–Zamolodchikov-Bernard (KZB) connection, a genus 1 analogon to the universal (genus 0) Knizhnik–Zamolodchikov connection, has been introduced in [4] (see also [25]). It takes values in the holonomy Lie algebra $\mathfrak{t}_{1,n}$ of the configuration space of $n$ points on the 2-dimensional torus minus its diagonals $z_i = z_j$. This connection was used in [4] to give a geometric proof of the 1-formality of this configuration space, previously due to Bezrukavnikov [3].

Actually, this connection extends to the full moduli space $M_{1,n}$ of elliptic curves with $n$ marked points. This means that the connection can be extended to the direction of variations in moduli and that it is modular invariant. It is used in [4] to produce (universal) representations of modular mapping class groups in genus 1. It also gives a simple way of constructing an isomorphism between the rational Cherednik algebra and the double affine Hecke algebra (in type $A$) with formal parameters. This can be used to obtained interesting results for the representation theory of the double affine Hecke algebra.

2.3. **Detailed research plan.**

2.3.1. *Gaudin subalgebras, moduli spaces and Bethe ansatz.* Theorem 2.3 gives a general structural result on the spectrum of Gaudin subalgebras. A much more detailed information is provided by the Bethe ansatz [18] that was studied recently by several authors, see [13–15, 28–30] and references therein, and shown to have surprising connection with several other mathematical subjects. It will be interesting to relate these descriptions to the geometry of $\bar{M}_{0,n+1}$. The results of [2] work over any field. In particular, it holds over reals, which is important for applications. It is known that the set of real points $\bar{M}_{0,n+1}(\mathbb{R}) \subset \bar{M}_{0,n+1}$ is a smooth real manifold, which can be glued of $n!/2$ copies of the Stasheff associahedron (see [21]). This gives a very convenient geometric representation of all limiting cases of the real Gaudin subalgebras and related quantum integrable systems. In particular, the Jucys-Murphy subalgebra corresponds to one of the vertices of the associahedron. The spectrum in this case was studied in detail by Vershik and Okounkov [35, 36]. What happens at other vertices labelled by different triangulations of an $n$-gon is worth investigating further. As it was explained in [5, 6, 11] the corresponding integrable systems have a nice geometric realisation as Kapovich–Millson bending flows [22]

Another question concerns general root systems. It is natural to expect that the results in [2] are special cases of a theory for arbitrary root systems. Indeed there are several indications: for example Vinberg introduced commuting subalgebras in the degree two part of $U\mathfrak{g}$ for any simple Lie algebra $\mathfrak{g}$. A candidate for a replacement of $\bar{M}_{0,n+1}$ is then the "wonderful compactification" [7] of the quotient of the complement of root hyperplane in a Cartan subalgebras. The rather explicit description in [16] could be used as a substitute of the description of $\bar{M}_{0,n+1}$ given in [19], that was used in the proof of Theorem 2.1.

2.3.2. *Elliptic Gaudin subalgebras, universal KZB connection and a possible relation to the AGT conjecture.* Concerning the elliptic version $\mathfrak{t}_{1,n}$ of the Kohno-Drinfel Lie algebra, it seems rather natural to expect results analogous to Theorems 2.1 and 2.2, whith an avatar of the compactified moduli space $\bar{\mathcal{M}}_{1,n}$ of stable genus 1 curves with marked points . A good understanding of this case should then lead to further generalizations in two directions:

- following [34], one should be able to find a universal elliptic KZB connection associated to any root system. A candidate for the analogon of the above mentioned "wonderful compactification" would be obtained from the quotient space
$$\mathbb{C} \oplus \mathfrak{h} \oplus \mathcal{H}/\mathbb{J}\,.$$
Here $\mathbb{J}$ is the Jacobi group associated to the Weyl group of the root system and $\mathcal{H}$ is the upper half-plane.
- higher genus curves: the holonomy Lie algebra $\mathfrak{t}_{g,n}$ of the complement of diagonals in the $n$-th power of a genus $g$ complex projective curve has been explicitly described by [3], and the universal version of the KZB connection is also known ([10]).

In any of these cases (as well as the ones that appear in the previous Section), interesting Frobenius manifold structures appear. It is an interesting question to ask whether these structures are relevant to the problems we aim to study here.

An interesting relation between certain 4-points correlation functions on the sphere and 1-point correlation functions on the torus has been observed recently by theoretical physicists (see [12, 32]), in the context of Liouville theory. We propose to prove this relation in the context of the WZW model, at the universal level. Namely, it is well-known that the map $M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\} \to M_{1,1}$ that sends $\lambda$ to the cubic $y^2 = x(x-1)(x-\lambda)$ is a double-cover. Moreover, both the universal KZ connection on $M_{0,4}$ and the universal KZB connection on $M_{1,1}$ take their values in the free Lie algebra $\mathfrak{f}_2$ with two generators. We expect to be able to relate the universal KZ connection on $M_{0,4}$ with the pullback of the universal KZB connection on $M_{1,1}$.

Let us emphasize that the work of [12, 32] is motivated by the AGT conjecture [1], which describes a conjectural relation between a certain $4d$ superconformal field theory with $2d$ conformal field theories of Liouville type. It would be interesting to find a corresponding relation for the WZW model. A possible evidence for such a relation is given by the so called Mukai system [31] naturally associated to Poisson algebraic surfaces ($4d$): it degenerates to the Hitchin system [20], that is associated to algebraic curves ($2d$) and is obtained as the critical level of the KZ(B) connection in arbitrary genus (see [33]). It seems that some part of this program was guessed by Donagi and Witten in [8].

2.4. **Schedule and milestones.** Leonardo Aguirre, currently Ph. D. student of Giovanni Felder, will work on the Bethe ansatz description of the spectrum of Gaudin subalgebras associated to representations of the symmetric group. The first result will be that the relative spectrum is a smooth covering of $\bar{M}_{0,n+1}$ over a Zariski open neighbourhood of the Jucys-Murphy point. This should follow from results [30], see also [27]. The description of the Bethe ansatz as the combinatorics of Young tableaux will also appear. The extension of the results of [2] to general root system will also be attempted. It is expected that Aguirre will defend his thesis in the summer of 2012.

For what concerns the elliptic side of the project, the first task will be to give the appropriate definition of elliptic Gaudin subalgebras in $\mathfrak{t}_{1,n}$, and compute their dimension. It is reasonable to expect that a graduate student starting his PhD can deal with this problem. It will also be a good starting point for her/him to learn basic facts concerning elliptic integrable models: these elliptic Gaudin subalgebras should be sent to standard hamiltonian of known integrable models *via* the realizations of $\mathfrak{t}_{1,n}$ exhibited in [4]. The next task will be to identify the space of these elliptic Gaudin subalgebra with a suitable compactification either of the moduli space of elliptic curves with marked points or of the Jacobi orbit space in type $A$, and to adapt and generalize the work of [2] to this context. Generalizations to other root systems or to higher genus, and their expected relations to higher genus moduli and/or Hurwitz spaces, should be considered only if the previous program as been achieved in time.

The aim to formulate a WZW conterpart to the AGT conjecture is a very speculative project. The first (actually independant) step would be to formulate and prove a relation between universal KZ(B) connections on $M_{0,4}$ and $M_{1,1}$. It is a reasonable project that could be attacked by a student already familiar with the universal KZB connection in genus one. A second (very speculative) task to adress could be to exhibit an analog of the KZ connection on moduli spaces of sheaves on a Poisson surface, that would satisfy at least two of the following three properties:

- at some "critical level", it should gives back the Mukai system.
- correlation functions of supersymmetric Yang-Mills theory should be flat sections of this connection.
- it should degenerates in some appropriate sens to the KZ(B) connection.

2.5. **Importance and impact.** The area in which the present research project fits is very active. We expect to contribute significantly to the advances in the field.

Graduate students will have the opportunity to attack unsolved and interesting problems.

The results that will be obtained will be published in peer-reviewed journals, and we will encourage PhD students to present them in seminars and conferences.

Being a research project in mathematics physics, we expect that it will also have some positive reception in the community of theoretical physics.

## References

[1] Luis F. Alday, Davide Gaiotto, and Yuji Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. **91** (2010), no. 2, 167-197.

[2] Leonardo Aguirre, Giovanni Felder, and Alexander P. Veselov, *Gaudin subalgebras and stable rational curves*, available at `arXiv.org/abs/1004.3253v2`.

[3] Roman Bezrukavnikov, *Koszul DG-algebras arrizing from configuration spaces*, Geom. and Funct. Analysis **4** (1992), no. 2, 119–135.

[4] Damien Calaque, Benjamin Enriquez, and Pavel Etingof, *Universal KZB equations: the elliptic case*, Progress in Mathematics **269** (2010), 165-266.

[5] Alexander Chervov, Gregorio Falqui, and Leonid Rybnikov, *Limits of Gaudin algebras, quantization of bending flows, Jucys-Murphy elements and Gelfand-Tsetlin bases*, Lett. Math. Phys. **91** (2010), no. 2, 129–150, DOI 10.1007/s11005-010-0371-y. MR 2586869

[6] ———, *Limits of Gaudin systems: classical and quantum cases*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 029, 17. MR **2506183 (2010f:**82024)

[7] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), no. 3, 459–494, DOI 10.1007/BF01589496. MR **1366622 (97k:**14013)

[8] Ron Donagi and Edward Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nuclear Physics B **460** (1996), no. 2, 299–334.

[9] V. G. Drinfel′d, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* Gal($\overline{\mathbf{Q}}/\mathbf{Q}$), Algebra i Analiz **2** (1990), no. 4, 149–181 (Russian); English transl., Leningrad Math. J. **2** (1991), no. 4, 829–860. MR **1080203 (92f:**16047)

[10] Benjamin Enriquez (in progress).

[11] Gregorio Falqui and Fabio Musso, *Gaudin models and bending flows: a geometrical point of view*, J. Phys. A **36** (2003), no. 46, 11655-11676, DOI 10.1088/0305-4470/36/46/009. MR **2025867 (2004i:**37121)

[12] Vladimir A. Fateev, Alexey V. Litvinov, André Neveu, and Enrico Onofri, *Differential equation for four-point correlation function in Liouville field theory and elliptic four-point conformal blocks*, J. Physics A **42** (2009), no. 30.

[13] Boris Feigin, Edward Frenkel, and Leonid Rybnikov, *Opers with irregular singularity and spectra of the shift of argument subalgebra*, Duke Math. J. **155** (2010), no. 2, 337–363, DOI 10.1215/00127094-2010-057.

[14] B. Feigin, E. Frenkel, and V. Toledano Laredo, *Gaudin models with irregular singularities*, Adv. Math. **223** (2010), no. 3, 873–948, DOI 10.1016/j.aim.2009.09.007. MR 2565552

[15] Edward Frenkel, *Gaudin model and opers*, Infinite dimensional algebras and quantum integrable systems, Progr. Math., vol. 237, Birkhäuser, Basel, 2005, pp. 1–58. MR **2160841 (2007e:**17022)

[16] Giovanni Gaiffi, *Blowups and cohomology bases for De Concini-Procesi models of subspace arrangements*, Selecta Math. (N.S.) **3** (1997), no. 3, 315–333, DOI 10.1007/s000290050013. MR **1481132 (99d:**52009)

[17] M. Gaudin, *Diagonalisation d'une classe d'Hamiltoniens de spin*, J. Physique **37** (1976), no. 10, 1089–1098 (French, with English summary). MR 0421442 (54 #9446)

[18] Michel Gaudin, *La fonction d'onde de Bethe*, Collection du Commissariat à l'Énergie Atomique: Série Scientifique. [Collection of the Atomic Energy Commission: Science Series], Masson, Paris, 1983 (French). MR **693905 (85h:**82001)

[19] L. Gerritzen, F. Herrlich, and M. van der Put, *Stable $n$-pointed trees of projective lines*, Indag. Math. (Proceedings) **91** (1988), no. 2, 131–163. MR **952512 (89i:**14005)

[20] Nigel Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), 91–114.

[21] Mikhail M. Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J. Pure Appl. Algebra **85** (1993), no. 2, 119–142, DOI 10.1016/0022-4049(93)90049-Y. MR **1207505 (94b:**52017)

[22] Michael Kapovich and John Millson, *The symplectic geometry of polygons in Euclidean space*, J. Differential Geom. **44** (1996), no. 3, 479-513. MR **1816048 (2001m:**53159)

[23] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199. MR **702953 (85d:**14038a)

[24] Toshitake Kohno, *Série de Poincaré-Koszul associée aux groupes de tresses pures*, Invent. Math. **82** (1985), no. 1, 57–75, DOI 10.1007/BF01394779 (French). MR **808109 (87c:**32015a)

[25] Andrey Levin and Georges Racinet, *Towards elliptic multiples polylogarithms*, available at `arXiv:math/0703237v1`.

[26] S. V. Manakov, *A remark on the integration of the Eulerian equations of the dynamics of an $n$-dimensional rigid body*, Funkcional. Anal. i Priložen. **10** (1976), no. 4, 93–94 (Russian). MR 0455031 (56 #13272)

[27] Aaron Marcus, *A Bethe Ansatz for Symmetric Groups*, available at `arXiv.org/abs/1003.0490v1`.

[28] E. Mukhin, V. Tarasov, and A. Varchenko, *Schubert calculus and representations of the general linear group*, J. Amer. Math. Soc. **22** (2009), no. 4, 909–940, DOI 10.1090/S0894-0347-09-00640-7. MR 2525775

[29] ———, *Three sides of the geometric Langlands correspondence for $gl_N$ Gaudin model and Bethe vector averaging maps*, available at `arxiv.org/abs/0907.3266`.

[30] ______, *Bethe algebra of the $gl_{N+1}$ Gaudin model and algebra of functions on the critical set of the master function*, available at `arxiv.org/abs/0910.4690`.

[31] Shigeru Mukai, *Symplectic Structure of the Moduli Space of Sheaves on an Abelian or $K3$ Surface*, Invent. Math. **77** (1984), 101-116.

[32] Rubik Poghossian, *Recursion relations in CFT and N=2 SYM theory*, JHEP **12** (2009), no. 038.

[33] Vladimir Rubtsov, *Hitchin and Beauville-Mukai System: Classical and Quantum Correspondence*, Acta Appl. Math. **99** (2007), no. 3, 283-292.

[34] Ian A.B. Strachan, *Weyl groups and elliptic solutions of the WDVV equations*, Advances in Mathematics **224** (2010), no. 5, 1801-1838.

[35] Andrei Okounkov and Anatoly Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math. (N.S.) **2** (1996), no. 4, 581–605, DOI 10.1007/PL00001384. MR **1443185 (99g:**20024)

[36] A. M. Vershik and A. Yu. Okun′kov, *A new approach to representation theory of symmetric groups. II*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **307** (2004), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10, 57–98, 281 (Russian, with English and Russian summaries); English transl., J. Math. Sci. (N. Y.) **131** (2005), no. 2, 5471–5494. MR **2050688 (2005c:**20024)

[37] E. B. Vinberg, *Some commutative subalgebras of a universal enveloping algebra*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 3–25, 221 (Russian); English transl., Math. USSR-Izv. **36** (1991), no. 1, 1–22. MR **1044045 (91b:**17015)