FACTORIZATION ALGEBRAS IN DEFORMATION QUANTIZATION, AND A RELATION BETWEEN LIE THEORY AND ALGEBRAIC GEOMETRY

RESEARCH PLAN

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1. Summary of the research plan

2. Research plan

2.1. Current stage of research in the field. Factorization algebras over a topological space X have been introduced very recently in [20] (see also [37] for a slightly different and more abstract context) as a natural notion appearing in perturbative quantum field theory. They are a topological version of a notion introduced by Beilinson and Drinfeld in [3].

In [18, 19] Costello produces a translation invariant factorization algebra over $\mathbb{C}$ by means of a holomorphic Chern–Simon theory with values in a cotangent space T^*X, which shed some light on the Witten genus [44] when taking derived global sections after passing to the quotient by a lattice. This factorization algebra is conjecturally related to the algebra of chiral differential operators introduced in [28].

A more elementary (and detailed) version of Costello's result is being working out in [27] for the case of a one-dimensional real Chern–Simons theory with values in T^*X. In this situation one get a locally constant factorization algebra on $\mathbb{R}$ being equivalent to the (sheaf of) algebra of differential operator on X, the Witten genus being replaced by the Todd genus. In this work some results of Fedosov [24], Bressler, Nest and Tsygan [5] are recovered.

The first aim of the present project is to extend and precise the above mentioned pioneer works in two directions:

- higher dimensional Chern–Simons theories. In two dimensions we aim at recovering Kontsevich's formality theorem [34] for smooth manifolds as well as its more general variant [21] and their extensions to more general geometric contexts [7, 14, 13]. In dimension 2, the theory will take its values in a shifted cotangent space $T^*[1]X$ and it will give a more conceptual approach to the derivation of Kontsevich's formality from a Σ-model presented in [16]. It will actually allow to recover Tamarkin's formality [41] from the Σ-model.

- Chern–Simons theories with values in other symplectic manifolds than cotangent spaces. For real Chern–Simons in dimension 1 we expect to fully recover Fedosov's approach to deformation quantization. For complex Chern–Simons in dimension 1 we expect to discover a new object: namely, a (gerbe of) chiral analog of deformation quantization algebra. This last project will require to understand better the relation between vertex algebras and factorization algebras over $\mathbb{C}$, which can be seen as a topological analog of Lian Zuckerman conjecture on the action of chains over the (framed) little disk operad on a topological vertex operator algebra [36].

One of the main ideas of [18, 19, 27] is to re-interpret Σ-models appearing in the so-called AKSZ construction [1] as gauge theories, in the perturbative setting, for a peculiar Lie$(_\infty)$-algebra encoding the geometry of the target manifold. This Lie algebra has been introduced and studied by Kapranov [29] (see also [30]) in the holomorphic context, and allows one to treat complex manifolds (and more generally algebraic varieties, or even sheaves of Lie algebroids) "as Lie algebras".

The second aim of the present proposal is therefore to develop a dictionary between Lie theory and algebraic geometry, that would give some new insight if both domains. In addition to the recent work of the applicant and his coauthors on this subject (described in the next §), there is an amazing evidence for the existence of such a dictionary: the similarity between Lévi decomposition theorem in Lie theory (implying that any complex finite dimensional Lie algebra is a semi-direct product of a nilpotent factor, an abelian one, and a semi-simple one) and Bogomolov decomposition theorem [4] in complex geometry (stating that any compact Kahler manifold with $c_1 = 0$ has a finite unramified cover by a product of a Calabi–Yau, complex tori, and irreducible holomorphic symplectic manifolds).

2.2. **Current stage of your own research.** The research of the applicant stands at the crossroad of deformation quantization with other fields of mathematics.

After his proof with Van den Bergh [14] of a claim by Kontsevich [34] on the ring structure on Hochschild cohomology of a complex manifold (using technics from deformation quantization), Damien Calaque started to work on a conjectural relation of this claim with the Duflo isomorphism in Lie theory [22]. The book [10] written with Rossi, and which emerged after a series of lectures given by the applicant at ETH, is the first attempt at a unifying approach. In particular both the Duflo isomorphism and the description of the cohomology ring of a complex manifold are produced from a local formula valid for any graded Q-manifold.

At present a satisfying unified framework is still missing, but several progresses have been made in parallel in the two areas (complex/algebraic geometry on one side, Lie theory on the other side). In [11, 12] Calaque and Rossi proved a version of the Duflo isomorphism for coinvariants and extended it to homology (like Pevzner and Torossian [39] did from invariants to cohomology). Part of this work has been later used by the two authors together with Van den Bergh [13] to complete the proof of Caldararu's conjecture [15] about the ring and module structures on Hochschild cohomology and homology of a smooth algebraic variety.

We mention that the formalism developped in [14, 13] produces an approach to formal geometry that allows to treat the C^∞, complex analytic, and algebro-geometric settings in a unified way. This seems very relevant for the purposes of the first part of the project concerning perturbative Chern–Simons theories.

More recently, together with Caldararu and Tu [6], the applicant gave a (both necessary and sufficiant) condition for a PBW type theorem to hold for an inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. This condition is a perfect analog of a condition discovered by Arinkin and Caldararu [2] for a Hochschild-Kostant-Rosenberg (HKR) type theorem to hold in the case of a closed embedding $X \hookrightarrow Y$ of smooth algebraic varieties. PBW/HKR type results are the first step toward an attempt at proving more general Duflo type results [23] or, on the geometric side, describing the ring structure on Ext algebras of subvarieties (a-k-a "branes").

2.3. **Detailed research plan.**

2.3.1. *Deformation quantization of symplectic manifolds via Chern–Simons theory.* The starting point of any project related to deformation quantization have to be the case of symplectic manifolds. From this perspective we aim at recovering most of Fedosov's results on deformation quantization from an appropriate Chern–Simons theory in dimension 1. To be more precise, given a (curved) L_∞-algebra $\mathfrak{g}$ equipped with a degree -2 ND invariant pairing we can associate a Chern–Simons theory *via* the classical BV formalism (see e.g. [17]). Given a curve C (i.e. $\mathbb{R}$ or S^1), fields are (compactly supported) forms on C with values in $\mathfrak{g}$. The BV pairing is given by the pairing on $\mathfrak{g}$ followed by integration, the free term for the action by the de Rham differential on C, and interacting terms by the L_∞ structure maps.

Then we aim at quantizing this classical BV theory. In contrast with the situation in [27] (where they deal with formal cotangents) we can not separate variables. We will therefore have to consider $\mathbb{C}^*$-invariant quantization with a slightly different weight: we assign, like in [24], weight 2 to $\hbar$ (instead of 1 in [20]) and weight 1 to $\mathfrak{g}$.

By using very similar technics as in [27], but in contrast with their situation, we expect the quantization to be obstructed, with obstruction given by a universal expression involving structure maps of $\mathfrak{g}$. In the case where $\mathfrak{g} = \mathfrak{g}_X$ is the Atiyah Lie algebra encoding the geometry of a C^∞ manifold X (which is such that the Chevalley-Eilenberg complex of $\mathfrak{g}_X$ provides a resolution of $\mathcal{O}_X$), this obstruction will nevertheless be zero due to acyclicity of C^∞ sheaves. Notice that having a pairing of the appropriate degree here is insured by the data of a symplectic form ω on X.

Quantum observables will produce a locally constant and translation invariant factorization algebra on $\mathbb{R}$ deforming $\mathcal{O}_X$, and therefore a deformation quantization of $\mathcal{O}_X$ in the usual sens (according to the fact that locally constant factorization algebras on $\mathbb{R}$ "are" strong homotopy associative algebras, and using that we are deforming the strict associative algebra $\mathcal{O}_X$).

Notice that the Deligne class $\frac{1}{\hbar}\omega$ of the canonical Fedosov star-product is homogeneous with respect to the weight we introduced. We may obtain other classes in the affine space $\frac{1}{\hbar}\omega + H^2(X)[\![\hbar]\!]$ by allowing more flexibility than $\mathbb{C}^*$-invariance (like a preserving filtration condition). More refined results like trace formulæ will be obtained by considering the low-energy effective action on global observables on the circle (using the crucial fact that we have a translation invariant factorization algebra to descend to S^1, and that derived global sections of a locally constant factorization algebra on the circle are nothing else than Hochschild chains).

In the holomorphic and algebraic setting, we will find refinements of known conditions for the existence of quantization (see e.g. [38, 9]). Moreover, even in the presence of obstructions we expect that the obstruction class will define a gerbe that will allow us to get a twisted sheaf of algebra (or algebroid stack) quantizing $\mathcal{O}_X$, like in [9, 33, 40].

Finally, we will also study the situation with a boundary ($C = [0, +\infty[$ or $[0, 1]$). In principle it should lead to quantization of coisotropic submanifolds as modules over the quantized algebra, and their derived intersections. It would be interesting to relate this to the recent work [42] of Boris Tsygan on oscillatory modules.

2.3.2. *Formality theorems via Chern–Simons theory, and generalizations.* If we want to consider Chern–Simons theory in dimension 2 (resp. in arbitrary dimension n) then we will need the pairing on $\mathfrak{g}$ to have degree -1 (resp. $n-3$). In particular in the geometric setting we will require the presence of a symplectic structure of degree 1 (resp. $n-1$) on X. Such a symplectic structure is canonically given on any shifted cotangent bundle $T^*[1]M$ (resp. $T^*[n-1]M$).

Here we will see that the obstruction to quantization vanishes by virtue of [34] (resp. [31] in higher dimension)[1]. Applying the machinery of Costello-Gwilliam will then produce a locally constant (and translation invariant) factorization algebra $\mathcal{F}_2$ on $\mathbb{R}^2$ (resp. $\mathcal{F}_n$ on $\mathbb{R}^n$) given by quantum observables. Following [37], locally constant factorization algebras on $\mathbb{R}^n$ are E_n-algebras. We will then prove the following

[1]Actually if we are only interested by source manifolds without boundary, then easier arguments (see [32]) are sufficient.

Theorem 2.1. *$\mathcal{F}_n$ is equivalent, as an E_n-algebra, to the Hochschild cochain complex of $\mathcal{O}_M$ viewed as an E_{n-1}-algebra.*

Remember that in [37] the Hochschild cochain complex of an E_n-algebra $\mathcal{A}$ is defined as the centralizer of id : $\mathcal{A} \to \mathcal{A}$ in the $(\infty, 1)$-category of E_n-algebras, and is therefore an $E_1 \Box E_n \cong E_{n+1}$-algebra.

Moreover, there is also an e_n-algebra structure on effective global observables at scale ∞ (i.e. low energy). Explicitly we find $S(T^*[n-1]M)$ with its commutative product and degree $1-n$ bracket. We expect to be able to prove the following

Theorem 2.2. *$\mathcal{F}_n$ is equivalent as an E_n-algebra to $S(T^*[n-1]M)$.*

In the case when $n = 2$ it will give a quantum field theoretic proof of Tamarkin's G_∞-extension [41] of Kontsevich's formality theorem [34].

There are further generalizations of this picture. We can e.g. consider a version with a boundary, replacing $\mathbb{R}^n$ by $\mathbb{R}^{n-1} \times [0, +\infty[$, and starting with the shifted conormal bundle of a submanifold $N \subset M$ viewed as a Lagrangian into the shifted cotangent bundle $T^*[n-1]M$. It will produce a formality theorem for the Hochschild cochains of $\mathcal{O}_M$ acting on an appropriate E_{n-1}-algebra (functions on the shifted conormal space). In the case when $N = M$ (physicists would speak about a "space-filling brane") it boils down to the action of the Hochschild cochains on the E_{n-1}-algebra $\mathcal{O}_M$ itself.

In dimension $n = 2$, using translation invariance we can descend to an half-tube $T = S^1 \times [0, +\infty[$ (topologically, this is a pointed closed disk) and then push-forward to a quotient space $\tilde{T}$ obtained by identifying the boundary circle with a single point. I expect to prove the following result:

Theorem 2.3. *1. The $(\infty, 1)$-category of factorization algebras on $\tilde{T}$ is equivalent to the one of strong homotopy precalculus algebras.*

2. The $(\infty, 1)$-category of S^1-equivariant factorization algebras on $\tilde{T}$ is equivalent to the one of strong homotopy calculus algebras.

To do so I will use Kontsevich-Soibelman topological model for the $Calc_\infty$-operad [35], by means of tubes with disks and a marked point on the boundary. It will give a new proof of the $Calc_\infty$-formality of [21] as well as a generalization to arbitrary submanifolds $N \subset M$ (and not just M itself).

2.3.3. *From Lie theory to algebraic geometry, and back.* The previously mentioned formality theorem [21] can be used to prove both the Duflo theorem and its generalizations to (co)homology, and Caldararu's conjecture [15]; but **without** explicit formulæ. In [14, 12, 13], proofs with explicit formulæ involving essentially the square root of the Todd genus have been given.

Together with the papers [2, 6] on HKR/PBW isomorphisms for "inclusions" it suggests the following *ad hoc* dictionnary between Lie theory and algebraic geometry (that we summarize in an array):

Lie theory	Algebraic geometry
$\big(S(\mathfrak{g}^*[-1]), d_{CE}\big)$	$\mathcal{O}_X$
$\mathbf{D}(\mathfrak{g}\text{-}mod)$	$\mathbf{D}(X) := \mathbf{D}(\mathcal{O}_X\text{-}mod)$
characters of $\mathfrak{g}$	line bundles on X
L.A. object $\mathfrak{g}$	L.A. object $T_X[-1]$
algebra object $U(\mathfrak{g})$	algebra object $\mathcal{HH}^*_X$
$(-)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\mathbf{1}, -)$	$\Gamma(-) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, -)$
adjoint action $ad \in \big(\mathfrak{g}^* \otimes \mathrm{End}(\mathfrak{g})\big)^{\mathfrak{g}}$	Atiyah class $at \in H^1\big(X, \Omega^1_X \otimes \mathrm{End}(T_X)\big)$
inclusion of L.A. $\mathfrak{h} \subset \mathfrak{g}$	closed embedding $X \hookrightarrow Y$
$\mathrm{Res} : \mathfrak{g}\text{-}mod \to \mathfrak{h}\text{-}mod$	$i^* : \mathbf{D}(Y) \to \mathbf{D}(X)$
$\mathrm{Ind} : \mathfrak{h}\text{-}mod \to \mathfrak{g}\text{-}mod$	$i_! : \mathbf{D}(X) \to \mathbf{D}(Y)$
$U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h} = \mathrm{Ind}\big(\mathrm{Res}(\mathbf{1})\big)$	$i^* i_! \mathcal{O}_X$

There are many approaches to a precise dictionary. Let us sketch two of them. The first one is to deal with derived/homotopical geometry[2] and view the $\mathfrak{g}$-module $\mathfrak{g}$ as $T_X[-1]$ for $X = B\mathfrak{g}$, $B\mathfrak{g}$ being (to keep it simple) the DG spectrum of the Chevalley-Eilenberg algebra $(S(\mathfrak{g}^*[-1]), d_{CE})$. The second one is to consider a 2-category $\mathcal{V}ar\mathcal{L}ie$ with objects being pairs $(X, \mathfrak{g})$, where X is an algebraic variety and $\mathfrak{g}$ is a Lie algebra object in the derived category $\mathbf{D}(X)$ of $\mathcal{O}_X$-modules. Morphisms should be given by a suitable subcategory of kernels in $\mathbf{D}(X \times Y)$. There is an obvious forgetful functor $\mathcal{V}ar\mathcal{L}ie \to \mathcal{V}ar$ to varieties with morphisms being the category of kernels. An adjoint to this functor should be $X \mapsto (X, T_X[-1])$.

In parallel to our efforts to find a mathematically precise dictionary, we will use our *ad hoc* correspondence to attack a conjecture of Duflo [23] for homogeneous spaces. Duflo's conjecture states that for a certain character λ of $\mathfrak{h}$, the Poisson center of $\big(S(\mathfrak{g})/\langle \mathfrak{h} - \lambda \rangle\big)^{\mathfrak{h}}$ is isomorphic as an algebra to the center of the associative algebra $\big(U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{h} - \lambda)\big)^{\mathfrak{h}} = \mathrm{Hom}_{\mathfrak{g}}\big(\mathrm{Ind}(\mathbf{1}_\lambda), \mathrm{Ind}(\mathbf{1}_\lambda)\big)$. Forgetting for a moment about the character λ, and considering the analogous situation in algebraic geometry, it appears that we are looking at the derived self-intersection $X \times^R_Y X$ of X into Y. This is a derived Y-scheme groupoid with base X, and we have a conjectural description of its associated Lie algebroid. The normal bundle exact sequence provides a map $\rho : N[-1] \to T_X$ in $\mathbf{D}(X)$ (N is the normal bundle of X into Y). Moreover $N[-1]$ is the derived relative tangent sheaf (a-k-a tangent complex) of X over Y, and as such it is equipped with a Lie bracket $[,]$. This turns $N[-1]$ into a Lie algebroid, and its universal enveloping algebra $U(N[-1])$ is an object in $\mathbf{D}(\mathcal{O}_{X\times X}\text{-}mod)$ set-theoretically suppported on the diagonal. We aim at proving the following

[2] There would be too many people to cite here.

Theorem 2.4. *1) $N[-1]$ is the Lie algebroid of $X \times^R_Y X$, and as kernels we have that $U(N[-1]) \cong i^* i_!$.*

2) The obstruction class of [2] for HKR to hold is the class of

$$0 \to N[-1] \to U_+^{\leq 2}(N[-1]) \to S^2(N[-1]) \to 0\,.$$

Moreover, it is zero if and only if the filtration on $U(N[-1])$ splits.

The strategy would then be to apply the standard Duflo theorem for Lie algebroids (which the applicant knows how to prove in the non derived context, by using the methods in the book [10]) in this context, and solve an analog of Duflo's conjecture. Finally we should in principle be able to recover the original Duflo conjecture by first allowing to twist by a line bundle, and then considering the inclusion of DG schemes

$$Spec\big(S(\mathfrak{h}^*[-1]), d_{CE}\big) \to Spec\big(S(\mathfrak{g}^*[-1]), d_{CE}\big)\,.$$

I conclude this part of the project with a discussion on the possible use of the above dictionary to make significant progresses towards the classification of (compact) irreducible holomorphic symplectic manifolds. In principle they should correspond to simple Lie algebras: we know only a few examples up to deformation equivalence and they satisfy an analog of the Chevalley theorem (after [43] their even cohomology is truncated polynomial algebra).

We have already noticed that Bogomolov decomposition theorem [4] is analogous to Lévi decomposition for Lie algebras. It is interesting to notice that Bogomolov imposes the condition that $c_1 = 0$ which for Lie algebras would correspond to $tr(ad) = 0$ and would be too restrictive. But Lie algebras satisfy a weaker condition saying that the derivation of $S(\mathfrak{g})^{\mathfrak{g}}$ given by $tr(ad)$ is actually trivil. This condition does not seem to be satisfied by Lie super-algebras, for which the Lévi decomposition is known to fail. It would be interesting to see if the later condition[3] is a good criteria to have decompositions theorems.

It is commonly accepted that the analog of the Cartan-Killing form on $\mathfrak{g}$ is played by the symplectic structure ω on X, but the Cartan-Killing form being defined as $tr(ad^2)$, its analog might also be $tr(t^2)$ as well. It is therefore a natural question to ask how the 2-form ω and the second Chern character $ch_2 = tr(at^2)$ are related.

2.3.4. *Chiral deformation quantization of symplectic manifolds.* In [18, 19] Kevin Costello uses holomorphic (perturbative) Chern–Simons theory in (complex) dimension 1 to construct a (sheaf of) factorization algebra $\mathcal{F}_M$ on $\mathbb{C}$ associated to any cotangent bundle T^*M in the situation when $ch_2(M)$ vanishes. It is expected that even when $ch_2(M)$ does not vanish one should be able to construct a twisted sheaf (or factorization algebroid) as it was already mentioned above. By descending to an elliptic curve (using translation invariance) he is then able to describe global

[3]The analog in geometry would be that $tr(at)$ act by zero on $H(X, \wedge T_X)$.

derived sections on this elliptic curve by means of differential forms on T^*M in a way very similar to the Fedosov-Nest-Tsygan result (with Witten genus replacing Todd one).

Once we will have dealt with the project described in §2.3.1, it will be straightforward to adapt it to holomorphic Chern–Simons theory and associate a (twisted sheaf of) factorization algebra on $\mathbb{C}$ to any symplectic manifold X.

The most interesting feature of the factorization algebra $\mathcal{F}_M$ constructed by Costello is that it is guessed to be related to chiral differential operators [28]. Chiral differential operators form a twisted sheaf of vertex algebras (or a vertex algebroid) on M.

I plan to construct a functor from vertex algebras/algebroids to translation invariant factorization algebras/algebroids on $\mathbb{C}$ in the following way. It was observed by Beilinson and Drinfeld [3] that vertex algebras are actually the same as translation invariant chiral algebras on $\mathbb{C}$. The same authors constructed an equivalence of categories between chiral algebras over a given complex algebraic variety Y to factorizing $\mathcal{O}$-modules on the Ran space of Y (equipped with Zariski topology). Using GAGA we can then produce out of a chiral algebra a factorizing $\mathcal{O}^{an}$-module on the Ran space of Y^{an}. It is known (see again [3]) that factorizing $\mathcal{O}$- and $\mathcal{O}^{an}$-modules are naturally equipped with a flat connection. Taking compactly supported flat sections we then get a factorizing cosheaf (i.e. a factorization algebra) on Y^{an}.

Remark 2.5. It is interesting that starting with a topological vertex operator algebra with suitable properties, we should be able to get some holonomic connection, and thus an associated factorizing cosheaf that would be constructible. According to [37] this means that the factorization algebra we get in the end is locally constant. Since locally constant factorization algebras on $\mathbb{C} = \mathbb{R}^2$ are nothing but E_2 algebra, the above construction then gives a nice conceptual explanation (and proof) of part of the Lian-Zuckerman conjecture (see e.g. [36, 26, 25]).

Conversely, it is an interesting question to ask under which condition a translation invariant factorization algebra on $\mathbb{C}$ actually comes from a vertex algebra.

All this suggests that there should exist a direct construction of a vertex algebroid on any symplectic manifold, without dealing with factorization algebras.

2.4. **Schedule and milestones.** Trying to write carefully the first steps of part 2.3.1 of the present project is a very good starting point for a PhD student. It is an excellent way of getting into both deformation quantization and the work of Costello-Gwilliam on factorization algebras in the BV formalism. There are no major difficulties, but there are a lot of technical details to be dealt with. In particular, this would be a good opportunity to re-write Fedosov's constructions using the unifying approach of [14] to formal geometry, and therefore to learn both of them. A good PhD student should not spend more than a year on this. Then he or she could turn to more difficult problems like situations with a boundary and trace formulæ, or chiral versions (see below).

I plan to ask a post-doc to work on the holomorphic and algebraic version of the story, especially on how one could adapt Costello-Gwilliam constructions to allow not only sheaves of factorization algebras but also twisted sheaves (i.e. factorization algebroids). It would be a good thing to have François Petit (who is finishing his PhD with Pierre Schapira) as a post-doc to work on this, since he is already working on deformation quantization algebroids (but from a slightly different point of view).

For the results expected in Section 2.3.2, I have already started to work informally on this with Grégory Ginot (Paris). We both have the appropriate expertise to deal with the technical issues related to this project. We have both worked on formality theorems and Hochschild (co)homology, and Grégory has a strong background in algebraic topology that might appear to be useful. Also, the part involving locally constant factorization algebras on half-spaces (and more generally quadrants) requires a systematic study of generalized swiss-cheese operads, which is something the applicant is already working on (it is related to my work with Felder, Ferrario and Rossi [8]). I consider this part being too ambitious to ask a PhD student to work on it, unless he or she would be exceptional.

Concerning the dictionary between Lie theory and algebraic geometry, there are two main directions. One is related to Hochschild (co)homology, Ext algebras of subvarieties, and Duflo's conjecture. For this, Andrei Caldararu, Junwu Tu and I have a quite clear idea (described in Section 2.3.3) of the main steps to be achieved: first prove that the kernel $i^*i_!$ is the universal enveloping algebra of the Lie algebroid $N[-1]$ (we are close to this), then extend this to DG schemes and re-interpret Duflo's conjecture as a geometric statement concerning spectra of Chevalley-Eilenberg DG algebras, and finally prove a Duflo type result for Lie algebroids in this DG context (using [10]). There are many technical issues, so it may take time (a few years) to achieve this program completely. It might therefore be a good thing to have Junwu Tu as a post-doc with me.

The second direction in the study of the relation between Lie theory and algebraic geometry concerns structure and classification results. I would like to ask a PhD student to work on this part of the project. It would be a good point for he or she to have an algebraic geometer as a coadvisor (e.g. Andrei Caldararu or Julien Grivaux). His or her two main tasks will be:

- to prove a Bogomolov decomposition theorem under a weaker condition than $c_1 = 0$ (namely when c_1 acts trivially on $H(X, \wedge T_X)$).
- to understand how the trace of squared Atiyah class $tr(at_X^2)$ and the holomorphic symplectic form ω are related for an irreducible holomorphic symplectic manifold X.

These questions, with in mind the classification of irreducible holomorphic symplectic manifolds, should be given to a very good student.

Finally, I plan to check by myself in the next few months that vertex algebras define translation invariant (but not locally constant) factorization algebras on $\mathbb{C}$. Together with the PhD student working on deformation quantization of symplectic manifolds using real Chern–Simons theory, we will then apply holomorphic Chern–Simons theory to construct sheaves of vertex algebras on symplectic manifolds that will locally look like the chiral de Rham complex. This will be one of the very last achievements of the present project.

2.5. **Importance and impact.** The present project is quite amibitious. I personnaly expect that it will produce significant advances in the field, and that it will be a source of inspiration for other researchers working in related areas.

The results will be published in peer-reviewed jourals, and PhD students and post-doc will be encouraged to present them in seminars and conferences.

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