

Ext algebras of subvarieties, formal geometry, and formality theorems

Joint work (in progress) with Alex Vitanov

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An apology

“The research on the analogy between Lie theory and algebraic geometry is part of a an extremely interesting ongoing project [...] this project will surely get streamlined statements by using the full framework of derived algebraic geometry.” (from Gabriele’s report on my habilitation thesis)

During this presentation I will talk about recent thoughts on this project. Unfortunately, I failed to follow Gabriele’s advice: what I will explain does not use derived geometry at all.

Introduction

Let $i : X \hookrightarrow Y$ be a closed embedding of smooth algebraic varieties over a field k of characteristic 0.

Vague goal

Describe $\mathrm{Ext}_{\mathcal{O}_Y}^\bullet(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$ as an algebra.

In the above we can replace Y with its formal completion $\widehat{Y}_X$ along X .

What's already known?

The case of the diagonal embedding

Consider

$$\Delta : X \hookrightarrow X \times X.$$

Then

$$\mathrm{Ext}_{X \times X}^{\bullet}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \simeq HH^{\bullet}(X).$$

The HKR isomorphism

$$H^{\bullet}(X, \wedge^{\bullet} T_X) \xrightarrow{\sim} HH^{\bullet}(X)$$

is not an algebra morphism.

Theorem [C–Van den Bergh]

There exists an isomorphism of algebras

$$H^{\bullet}(X, \wedge^{\bullet} T_X) \xrightarrow{\sim} HH^{\bullet}(X).$$

What about more general HKR-type isomorphisms

Let $N = N_{X/Y}$ be the normal bundle. Arinkin and Căldăraru introduce a class $\alpha_N \in \mathrm{Ext}_X^2(\wedge^2 N, N)$ that is the obstruction for the existence of an extension of N to the first infinitesimal neighborhood $Y_X^{(1)}$.

Theorem [Arinkin–Căldăraru]

$$\alpha_N = 0 \quad \Longleftrightarrow \quad \mathrm{Sym}(N[-1]) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \quad (\text{in } D(X)).$$

Hence, when $\alpha_N = 0$,

$$H^\bullet(X, \wedge_{\mathcal{O}_X}^\bullet N) \simeq \mathrm{Ext}_Y^\bullet(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$$

as graded k -vector spaces.

1. Sketch of proof for the diagonal embedding

The formal coordinate space X^{coord}

Recall : X is a smooth algebraic variety (of pure dimension d).

There exists a pro-affine scheme relative to X , denoted X^{coord} , universal among such spaces Z equipped with an identification

$$Z \times_X \widehat{X \times X}_{\Delta} \simeq Z \times \widehat{\mathbb{A}}_0^d.$$

Here “universal” means “terminal object”.

A point of X^{coord} is a pair (x, φ) , where x is a point of X and $\varphi : k[[x_1, \dots, x_d]] \xrightarrow{\sim} \mathcal{O}(\widehat{X}_x)$ is a system of formal coordinates at that point.

The space X^{coord} is an $\text{Aut}(\widehat{\mathbb{A}}_0^d)$ -torsor over X . In particular, GL_d acts on X^{coord} . One can then consider $X^{\text{aff}} = X^{\text{coord}}/GL_d$. It has the following property:

$$\Omega_X^{\bullet} \simeq \Omega_{X^{\text{aff}}}^{\bullet} = (\Omega_{X^{\text{coord}}}^{\bullet})^{GL_d}.$$

Resolutions (after Grothendieck, Gelfand–Kazhdan, Fedosov)

We have

$$\left(\Omega_{X^{\text{coord}}}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times X_{\Delta}}} \right)^{GL_d} \simeq \underbrace{\Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times X_{\Delta}}}}_{\mathbb{D}_{X/X_{\text{dR}}} \text{ in } [\text{CPTVV}]} \simeq \mathcal{O}_X.$$

The differential on the second term is given by the action of vectors fields (on X) on the first component of $\widehat{X \times X_{\Delta}}$ (this is the Grothendieck connection).

Moreover,

$$\Omega_{X^{\text{coord}}}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times X_{\Delta}}} \simeq (\Omega_{X^{\text{coord}}}^{\bullet} [[x_1, \dots, x_d]], d_{\text{DR}} + \gamma \cdot),$$

where $\gamma \in \Omega_{X^{\text{coord}}}^1 \otimes \text{Der}(k[[x_1, \dots, x_d]])$ is a Maurer–Cartan element.

Philosophy: X is obtained by glueing formal charts. The descent data is encoded by the Maurer–Cartan 1-form γ .

Kontsevich formality and twisting

Set $A_0 = k[[x_1, \dots, x_d]]$.

Theorem [Kontsevich]

There exists an L_∞ -quasi-isomorphism $T_{\text{poly}}(A_0) \longrightarrow C^\bullet(A_0, A_0)$ such that, for a nice enough cdga B and a Maurer–Cartan element $\gamma \in B^1 \otimes \text{Der}(A_0)$, the γ -twisted L_∞ -quasi-isomorphism $B \otimes T_{\text{poly}}(A_\gamma) \longrightarrow C_B^\bullet(B \otimes A_\gamma, B \otimes A_\gamma)$ induces an algebra isomorphism in cohomology.

Remarks:

- This still works for more general MC elements (polyvector fields), but then the deformation of $B \otimes A_0$ is more complicated (possibly an A_∞ -deformation).
- Deformation quantization: $B = k[[\hbar]]$. The MC element is $\gamma = \hbar\pi$, where π is a Poisson structure. Kontsevich's theorem provides a quantization of A_0 as a Poisson algebras and identifies Poisson cohomology of A_0 with Hochschild cohomology of the deformation **as algebras**.

“The proof”

We have resolutions

$$T_{\text{poly}}(X) \longrightarrow (\Omega_{X^{\text{coord}}}^{\bullet} \otimes T_{\text{poly}}(A_0), d_{\text{DR}} + [\gamma, -]_{\text{SN}})^{GL_d}$$

$$CC(X) \longrightarrow (\Omega_{X^{\text{coord}}}^{\bullet} \otimes C(A_0, A_0), d_{\text{DR}} + [\gamma, -]_G)^{GL_d}$$

that preserves all algebraic structures one can think of.

Then applying Kontsevich's theorem with $B = (\Omega_{X^{\text{coord}}}^{\bullet}, d_{\text{DR}})$, we get an algebra isomorphism between the cohomologies of the resolutions.

2. Formality theorem with a brane

The formal local model

Consider $X = \widehat{\mathbb{A}}_0^d \hookrightarrow \widehat{\mathbb{A}}_0^{d+n} = Y$.

We then set

- $A_0 = \mathcal{O}(Y) = k[[\underline{x}, \underline{y}]]$, where $\underline{x} = (x_1, \dots, x_d)$ and $\underline{y} = (y_1, \dots, y_n)$;
- $B_0 = k[[\underline{x}]] \otimes \wedge^\bullet(\underline{\xi})$, where $\underline{\xi} = (\xi_1, \dots, \xi_n)$ and the ξ_i 's are degree 1 variables;
- $K_0 = \mathcal{O}(X) = k[[\underline{x}]]$, together with its A_∞ -(A_0, B_0)-bimodule structure.

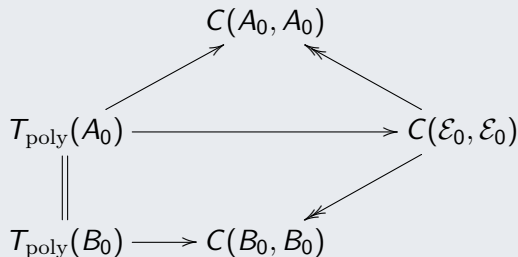
Finally, $\mathcal{E}_0$ is the A_∞ -category with two objects **a**, **b**, and

$$\mathrm{Hom}(\mathbf{a}, \mathbf{a}) = A_0, \quad \mathrm{Hom}(\mathbf{b}, \mathbf{b}) = B_0, \quad \mathrm{Hom}(\mathbf{b}, \mathbf{a}) = K_0, \quad \mathrm{Hom}(\mathbf{a}, \mathbf{b}) = 0.$$

Formality with a brane

Theorem [C–Felder–Ferrario–Rossi]

There is a commuting diagram of L_∞ -quasi-isomorphisms:



The identification

$$T_{\text{poly}}(A_0) = T_{\text{poly}}(B_0)$$

is given by the “odd Fourier transform”

$$y_i \longleftrightarrow -\partial_{\xi_i}, \quad \partial_{y_i} \longleftrightarrow \xi_i.$$

Twisting the formality theorem

For a nice enough cdga C and a Maurer–Cartan element

$$\gamma \in C \otimes T_{\text{poly}}(A_0),$$

the result remains true after twisting by γ . In particular, there exists a Maurer–Cartan element

$$\tilde{\gamma} \in C_C^\bullet(C \otimes \mathcal{E}_0, C \otimes \mathcal{E}_0)$$

defining a deformation $(A_{\tilde{\gamma}}, K_{\tilde{\gamma}}, B_{\tilde{\gamma}})$ of $(C \otimes A_0, \otimes K_0, \otimes B_0)$. Under mild conditions (saying that $\tilde{\gamma}$ has no curvature), this gives an A_∞ -quasi-isomorphism

$$B_{\tilde{\gamma}} \longrightarrow \mathrm{RHom}_{A_{\tilde{\gamma}}}(K_{\tilde{\gamma}}, K_{\tilde{\gamma}}).$$

Slogan: Koszul duality is preserved under deformation/twisting by a MC element.

Applications to deformation quantization

Let $C = k[[\hbar]]$.

In this context, the **slogan** becomes: Koszul duality is preserved under quantization.

- ① ([C–Felder–Rossi]) Let $d = 0$, and let $\gamma = \hbar\pi$ where π is a quadratic Poisson structure on A_0 . The odd Fourier transform of π is a quadratic Poisson structure on B_0 . Then the two quantizations are quadratic Koszul dual algebras.
- ② ([Nusa]) Assume that we have a formal Lie algebroid of rank n on $k[[\underline{x}]]$. This induces a Poisson structure on A_0 , whose odd Fourier transform is the Chevalley–Eilenberg differential on B_0 . Then the quantization of A_0 is isomorphic to the Rees algebra of the (completed) universal enveloping algebra of the Lie algebroid.

Global version of the formality with a brane

Theorem [C–Vitanov]

Results from [CFFR] (and the application from [Nusa]) remain true for

$$A_0 = \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(E), \quad K_0 = \mathcal{O}_X, \quad B_0 = \wedge^{\bullet}_{\mathcal{O}_X}(E^{\vee}),$$

where X is a smooth algebraic variety and $E \rightarrow X$ is a vector bundle (of finite rank).

The strategy of proof goes through X^{coord} again (there is no surprise).

Remark: from the perspective of the original problem, this corresponds to the case of the zero-section inclusion $X \hookrightarrow E^{\vee}$, which is very easy to deal with (and doesn't require to use formality).

3. General closed embeddings

Deformation to the normal bundle and formal coordinate spaces in the relative setting

We want to implement the idea that

$$X \hookrightarrow Y$$

is a deformation of

$$X \hookrightarrow N_{X/Y}$$

in a way that is compatible with formal geometry.

We introduce a pro-affine scheme relative to X , denoted Y_X^{coord} , that is universal among such spaces Z equipped with compatible identifications

$$\begin{array}{ccc} Z \times_X \widehat{X \times Y}_\Delta & \xrightarrow{\sim} & Z \times \widehat{\mathbb{A}}_0^{d+n} \\ \uparrow & & \uparrow \\ Z \times_X \widehat{X \times X}_\Delta & \xrightarrow{\sim} & Z \times \widehat{\mathbb{A}}_0^d \end{array}$$

Formal coordinate spaces in the relative setting 2

A point in Y_X^{coord} is a pair (x, φ) where x is a point of X and $\varphi : k[[\underline{x}, \underline{y}]] \xrightarrow{\sim} \mathcal{O}(\hat{Y}_x)$ is a system of formal coordinates on Y at x that restricts to a system of formal of coordinates on X (algebraically, the restriction of φ to $k[[\underline{x}]]$ factors through $\mathcal{O}(\hat{X}_x)$).

The space Y_X^{coord} is an $\text{Aut}(\hat{\mathbb{A}}_0^d \hookrightarrow \hat{\mathbb{A}}_0^{d+n})$ -torsor over X , and the corresponding linear subgroup is the parabolic subgroup

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset GL_{d+n}(k),$$

with block sizes d and n .

Grothendieck/Gelfand–Kazhdan/Fedosov resolutions in the relative setting

We have

$$\left(\Omega_{Y_X^{\text{coord}}}^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times Y_\Delta}} \right)^P \simeq \underbrace{\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times Y_\Delta}}}_{\mathbb{D}_{\widehat{Y}_X/X_{dR}} \text{ in } [\text{CPTVV}]} \simeq \mathcal{O}_{\widehat{Y}_X}.$$

Moreover, using the universal property of Y_X^{coord} :

$$\Omega_{Y_X^{\text{coord}}}^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times Y_\Delta}} \simeq \left(\Omega_{Y_X^{\text{coord}}}^\bullet [[\underline{x}, \underline{y}]], d + \gamma \cdot \right),$$

where

$$\gamma \in \Omega_{Y_X^{\text{coord}}}^1 \otimes \text{Der}(k[[\underline{x}, \underline{y}]]) .$$

Applying the formality with a brane, this seems to produce an A_∞ -quasi-isomorphism

$$\widetilde{\wedge_{\mathcal{O}_X}^\bullet N_{X/Y}} \longrightarrow \text{RHom}_{\widetilde{\mathcal{O}_{\widehat{Y}_X}}}(\widetilde{\mathcal{O}_X}, \widetilde{\mathcal{O}_X}),$$

where $\widetilde{\mathcal{O}_{\widehat{Y}_X}}$ is the above resolution of $\mathcal{O}_{\widehat{Y}_X}$.

Caveat 1

Through the odd Fourier transform $T_{\text{poly}}(A_0) \simeq T_{\text{poly}}(B_0)$ the element γ is sent to a polyvector that is not necessarily a vector field on B_0 . It is thus sent by the formality map from [CFFR] to an A_∞ -deformation of

$$\left(\Omega_{Y_X}^\bullet \otimes k[[\underline{x}]] \otimes \wedge^\bullet(\underline{\xi}) \right)^P \simeq \wedge_{\mathcal{O}_X}^\bullet N_{X/Y}.$$

The resulting A_∞ -deformation is nevertheless given by an explicit formula involving

- Feynman graphs;
- configuration-space integrals.

Caveat 2

Due to one-loop corrections (Feynman graphs having a one loop), $\tilde{\mathcal{O}}_X$ might not be a resolution of $\mathcal{O}_X$ (but rather of a line bundle on X).

We will see later that this might actually be a good thing.

A conjecture about fixed point subvarieties of involutions

Conjecture

If $X = Y^\sigma$ is the fixed-point locus of an involution $\sigma : Y \rightarrow Y$, then the A_∞ -deformation is trivial in cohomology, i.e. on

$$H^\bullet(X, \wedge_{\mathcal{O}_X}^\bullet N_{X/Y}).$$

This would imply the existence of an algebra isomorphism

$$H^\bullet(X, \wedge_{\mathcal{O}_X}^\bullet N_{X/Y}) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{O}_Y}^\bullet(i_*\mathcal{L}, i_*\mathcal{L}),$$

with $\mathcal{L}$ a line bundle on X .

Example

Consider $Y = X \times X$ with $\sigma(x_1, x_2) = (x_2, x_1)$. Then the conjecture specializes to the diagonal case discussed in the introduction.

4. An analogy: Duflo conjecture in Lie theory

Duflo conjecture for symmetric pairs

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ an involution, and set $\mathfrak{h} = \mathfrak{g}^\sigma$.

For every $h \in \mathfrak{h}$ we define $\chi(h) := \text{tr}\left(\frac{1}{2} \text{ad}(h)_{\mathfrak{g}/\mathfrak{h}}\right)$. This defines a character of $\mathfrak{h}$, and we let $\mathbf{1}_\chi$ be the corresponding one-dimensional $\mathfrak{h}$ -representation.

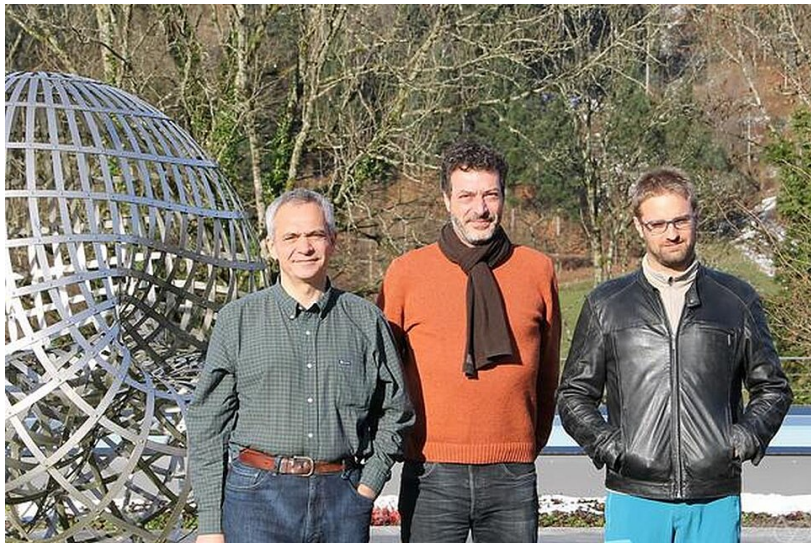
Conjecture [Duflo]

There is an algebra isomorphism

$$\text{Sym}(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}} \xrightarrow{\sim} \text{Hom}_{U(\mathfrak{g})\text{-mod}} \left(\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{1}_\chi, \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{1}_\chi \right).$$

Analogy between the two conjectures:

- Let $\mathfrak{h} = T_X[-1]$, the Lie algebra associated with the X -family of FMP $\widehat{X \times X_\Delta}$.
- Let $\mathfrak{g} = i^* T_Y[-1]$, the Lie algebra associated with the X -family of FMP $\widehat{X \times Y_\Delta}$.
- Then we have that $\text{Sym}(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}} = \text{Hom}_{\mathfrak{h}\text{-mod}}(\mathbf{1}, \text{Sym}(\mathfrak{g}/\mathfrak{h})) = H^\bullet(X, \wedge_{\mathcal{O}_X}^\bullet N_{X/Y})$.
- Finally, one can identify $\mathcal{O}_{\widehat{Y_X}}$ -modules with $\mathfrak{g}$ -representations, and i_* with $\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}$.



Happy birthday Gabriele