

Ext algebras of subvarieties,
Formal geometry,
and Formality theorems
(Joint work with Alex Vitanov)

Introduction

Let $i: X \hookrightarrow Y$ be a closed embedding of smooth algebraic varieties (k a field of char. 0).

Aim: describe $\text{Ext}_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$ as an algebra.

can safely replace Y with $\widehat{Y}_X$.

What is known?

→ the case of the diagonal embedding $\Delta: X \hookrightarrow X \times X$

$$\text{Ext}_{X \times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) \simeq \text{HH}^*(X).$$

The HKR isomorphism $H^i(X, \Lambda^i T_X) \xrightarrow{\sim} \text{HH}^i(X)$ is **NOT**

an algebra morphism, but:

Thm [C-Van den Bergh]: there exists an isomorphism of

$$\text{algebras } H^i(X, \Lambda^i T_X) \xrightarrow{\sim} \text{HH}^i(X).$$

→ more general HKR-type isomorphisms.

[Arinkin-Caldararu] construct a class $\alpha_N \in \text{Ext}_X^2(\Lambda^2 N, N)$,

where N is the normal bundle of X into Y , such that

$$\alpha_N = 0 \Leftrightarrow \Lambda^1 N \simeq \text{RHom}_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \text{ in } \mathcal{D}(X)$$

$$(\Rightarrow H^i(X, \Lambda^i N) \simeq \text{Ext}_Y^i(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \text{ as graded } k\text{-vect}).$$

① Sketch of proof for the diagonal embedding

• $\exists$ ProAff $_X$ -space X^{coord} that is universal among such spaces

Y together with an iso $Y \times_X \widehat{X \times X}_\Delta = Y \times \widehat{A}_0^d$.

($d = \dim(X)$, and "universal" means "terminal object").

X^{coord} is in fact an $\text{Aut}(\widehat{A}_0^d)$ -torsor over X .

$$\left(\Omega_{X^{\text{coord}}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times X}_\Delta} \right)^{\text{GL}} \simeq \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times X}_\Delta} \simeq \mathcal{O}_X$$

$\rightarrow \mathcal{B}_X = \mathbb{D}_{X/X}^{\text{dR}}$ in [CPTV]

$$\Omega_{X^{\text{coord}}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times X}_\Delta} \simeq (\Omega_{X^{\text{coord}}}^1[x_1, \dots, x_d], d_{\text{dR}} + \gamma).$$

where $\gamma \in \Omega_{X^{\text{coord}}}^1 \otimes_k \text{Der}(k[x_1, \dots, x_d])$.

$$\text{[Kontsevich]} \quad A_0 = k[x_1, \dots, x_d]$$

$\exists$ L_{∞} -q-iso $T_{\text{poly}} A_0 \rightarrow C^*(A_0, A_0)$ such for

$\gamma \in C^* \otimes \text{Der}(A_0)$ a MC element the γ -twisted

L_{∞} -q-iso $T_{\text{poly}} A_\gamma \rightarrow C^*(A_\gamma, A_\gamma)$ induces an **ALGEBRA**

iso in cohomology.

② Formality theorem with a brane

$$X = \widehat{A}_0^d \hookrightarrow Y = \widehat{A}_0^{d+n} \quad x = (x_1, \dots, x_d), \quad y = (y_1, \dots, y_n)$$

$$A_0 = \mathcal{O}(Y) = k[x, y] \quad B_0 = k[x] \otimes \Lambda^1(\xi) \quad \xi = (\xi_1, \dots, \xi_n)$$

$$K_0 = \mathcal{O}(X) = k[x] : A_0 - (A_0, B_0) \text{ -bimodule. } \text{degree 1 variables}$$

Theorem [C-Felder-Ferraro-Ross]:

$\exists$ commuting square of L_{∞} -q-iso

$$\begin{array}{ccc} T_{\text{poly}}(A_0) & \xrightarrow{\quad} & C^*(A_0, A_0) \\ \downarrow \text{SI} & \searrow & \downarrow \\ T_{\text{poly}}(B_0) & \xrightarrow{\quad} & C^*(B_0, B_0) \end{array}$$

where E_0 is the A_0 -category with objects a, b and:

$$\text{Hom}(a, a) = A_0, \quad \text{Hom}(b, b) = B_0, \quad \text{Hom}(b, a) = K_0, \quad \text{and}$$

$$\text{Hom}(a, b) = 0.$$

Remark: the identification $T_{\text{poly}}(A_0) \simeq T_{\text{poly}}(B_0)$ is given by

$$y_i \leftrightarrow -\frac{\partial}{\partial \xi_i}; \quad \frac{\partial}{\partial y_i} \leftrightarrow \xi_i \quad (\text{"odd Fourier transform"}).$$

Consequence: for every Maurer-Cartan element $\gamma \in T_{\text{poly}}(A_0)$

the result remains true after twisting by γ . In particular,

there exists a Maurer-Cartan element $\tilde{\gamma} \in C^*(E_0, E_0)$

defining a deformation $(A_\gamma, K_\gamma, B_\gamma)$ of (A_0, K_0, B_0) .

Under mild conditions, this gives an A_{∞} -q-iso

$$B_\gamma \longrightarrow \text{RHom}_{A_\gamma}(K_\gamma, K_\gamma)$$

Theorem [C-Vitanov]: the above remain true for

$$\left| \begin{array}{l} A_0 = \widehat{\text{Sym}}_{\mathcal{O}_X}(E), \quad K_0 = \mathcal{O}_X, \quad B = \Lambda^1_{\mathcal{O}_X}(E^\vee) \\ X = \text{smooth algebraic variety}, \quad E \downarrow_X \text{ f.d. vector bundle.} \end{array} \right.$$

Strategy of proof goes through X^{coord} again.

Nice but not so interesting: this is the case $X \hookrightarrow E$.

③ What can we do for general closed embeddings?

We implement the idea that $X \hookrightarrow Y$ is a deformation of $X \hookrightarrow N_{X/Y}$

in a way that is compatible with formal geometry.

We introduce the ProAff $_X$ -space Y_X^{coord} that is universal

among such spaces Z together with identifications

$$\widehat{Z \times_X X \times Y}_\Delta \xrightarrow{\sim} \widehat{Z \times \widehat{A}_0^{d+n}}$$

$$\widehat{Z \times_X X \times X}_\Delta \xrightarrow{\sim} \widehat{Z \times \widehat{A}_0^d}$$

• Y_X^{coord} is an $\text{Aut}(\widehat{A}_0^d \hookrightarrow \widehat{A}_0^{d+n})$ -torsor over X , and

$$\left(\Omega_{Y_X^{\text{coord}}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times Y}_\Delta} \right)^{\text{P}} \simeq \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times Y}_\Delta} \simeq \mathcal{O}_{Y_X^{\text{coord}}}^1$$

→ This is $\mathbb{D}_{Y_X/X}^{\text{dR}}$ from [CPTV]

parabolic subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \text{GL}_{d+n}(k)$

$$\Omega_{Y_X^{\text{coord}}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X \times Y}_\Delta} \simeq (\Omega_{Y_X^{\text{coord}}}^1[x, y], d + \gamma),$$

$$\gamma \in \Omega_{Y_X^{\text{coord}}}^1 \otimes_k \text{Der}(k[x, y]).$$

It seems great!!! We seem to have an A_{∞} -q-iso

$$\widetilde{\Lambda_{\mathcal{O}_X}^1 N_{X/Y}} \longrightarrow \text{RHom}_{\widetilde{\mathcal{O}_X}}(\widetilde{\mathcal{O}_X}, \widetilde{\mathcal{O}_X})$$

CAVEAT: through the odd Fourier transform $T_{\text{poly}}(A_0) \simeq T_{\text{poly}}(B_0)$

γ becomes a polyvector on B_0 . It is sent by the

formality map from [CFRR] to an A_{∞} -deformation

$$\text{of } \left(\Omega_{Y_X^{\text{coord}}}^1 \otimes_k k[x, y] \otimes \Lambda^1(\xi) \right)^{\text{P}} \simeq \Lambda_{\mathcal{O}_X}^1(N_{X/Y}).$$

This A_{∞} -deformation is nevertheless given by an explicit formula

(involving Feynman graphs and configuration space integrals).

Conjecture: if $X = Y^\sigma$ is the fixed point locus of an

involution $\sigma: Y \rightarrow Y$, then this A_{∞} -deformation is trivial in

cohomology (i.e. on $H^i(X, \Lambda_{\mathcal{O}_X}^1 N_{X/Y})$).

This would imply, when $X = Y^\sigma$, that there is an algebra

$$\text{isomorphism } H^i(X, \Lambda_{\mathcal{O}_X}^1 N_{X/Y}) \xrightarrow{\sim} \text{Ext}_{\mathcal{O}_Y}^i(i_*\mathcal{O}_X, i_*\mathcal{O}_X).$$

Example: $Y = X \times X$ and $\sigma(x_1, x_2) = (x_2, x_1)$.

④ An analogy: Duflo conjecture in Lie theory

$\mathfrak{g}$ f.d. Lie algebra, $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ involution, $R = \mathfrak{g}^\sigma$.

$\chi(h) := \text{tr}(\frac{1}{2} \text{ad}(h)|_{\mathfrak{g}/R})$ defines a character on R .

Let $\mathbb{1}_R$ the corresponding 1-dimension R -representation.

Conjecture [Duflo]: there is an algebra isomorphism

$$\text{Sym}(\mathfrak{g}/R)^R \xrightarrow{\sim} \text{Hom}_{U(\mathfrak{g})\text{-mod}}(\text{Ind}_R^{\mathfrak{g}} \mathbb{1}_R, \text{Ind}_R^{\mathfrak{g}} \mathbb{1}_R)$$