Quantization of symplectic dynamical r-matrices

I Joint work with A. Alekseev (available at arXiv:math/0606488v3)

I.1. Dynamical r-matrices: definition and examples

Data: \( \mathfrak{g} \subset \mathfrak{g} \) inclusion of Lie algebras (\( \mathfrak{g} \subset \mathfrak{g} \) resp. inclusion of Lie groups)

\[ U \subset \mathbf{R}^* \text{ Ad }^\ast \text{ invariant open subset} \quad \mathbf{R} \subset (\mathfrak{g}^\ast)^D \]

Definition: a dynamical r-matrix is a \( H \)-equivariant smooth map \( r: U \rightarrow \mathfrak{g}^D \) satisfying

\[ \frac{1}{2} r(A), r(A) \] \[ \varepsilon h \cdot \frac{\partial r}{\partial A} (A) = Z \quad (A \in U) \]

where \( (h), (a') \) are dual bases of \( \mathfrak{g} \) and \( \mathfrak{g}^* \).

Examples: \( \mathfrak{sl}_2 \) simple Lie algebra, \( \mathfrak{g} = \mathfrak{sl}_2, \mathbf{R} \subset (\mathfrak{sl}_2)^D \).

Then the CDYBE of Rokh's lecture for \( \mathfrak{g} \) (with coupling constant \( \varepsilon \)) is equivalent to the NCDYBE for \( r = \rho - \rho'' \) with \( Z = \varepsilon^2 [Z^D, Z^D] \).

3) Assume there exists in \( \mathfrak{g} \) s.t. \( \mathfrak{h} \subset \mathfrak{g} \) and \( \mathfrak{h} \subset \mathfrak{g} \) reductive splitting

Assume moreover that \( \exists \mathfrak{a} \subset \mathfrak{g} \) s.t. \( \mathfrak{a} \leftarrow \mathfrak{g}, [\mathfrak{x}, \mathfrak{y}]_{\mathfrak{h}} \) for \( \mathfrak{x}, \mathfrak{y} \in \mathfrak{g} \) is a ND pairing. Then \( \nu(\mathfrak{a}) = \mathfrak{a} \) is a dynamical r-matrix with \( Z = 0 \).

(Example due to Fehér-Golner-Puszta et al.)

Proposition [Fehér-Golner-Puszta et al.]

Assume that \( \mathfrak{h} \subset \mathfrak{g} \) is a ND reductive splitting.

For any dynamical r-matrix \( r: \mathfrak{h} \rightarrow \mathfrak{g}^D \) with \( Z \in \mathfrak{g}^D \), then

\[ r_{\mathfrak{g}} = r_{\mathfrak{h}} + r_{\mathfrak{k}_{\mathfrak{g}}} : \mathfrak{g} \rightarrow \mathfrak{g}^D \]

is a dynamical r-matrix (for the same \( Z \)).

I.2. Dynamical twist quantization

Let \( r(\lambda) \) a dynamical r-matrix with \( Z \in \mathfrak{g}^D \).

Let \( \hat{F} = 2Id + \frac{1}{6} Z + r(\lambda) \in (U \mathfrak{g}^D) \)

an associative (existence proved by Drinfeld)

Definition: a dynamical twist quantization of \( r(\lambda) \) is a \( H \)-equivariant smooth map

\[ J = 2Id + O(\lambda) : U \rightarrow \mathfrak{g} \mathfrak{g} \]

satisfying
The semi-classical limit condition \( J(h) - J(0) = i h \alpha(h) + o(h) \) and the dynamical twist equation

\[
J^{1/2} (\lambda) \ast \text{PBW} J^{1/2} (\lambda + k h) = \Phi^{-1} J^{1/2} (\lambda) \ast \text{PBW} J^{1/2} (\lambda).
\]

**Explanation of notations:**
- \( J^{1/2} = \Lambda \circ J \), \( J^{1/2} = (\Lambda \circ \text{PBW}) (J) \)
- \( J^{1/2} (\lambda + k h) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k J(\lambda)}{d \lambda^k} \partial_k q_{\lambda - \epsilon_k} \) (Taylor expansion)
- For polynomial functions \( f, g \in C^* \), one defines \( f \ast g = \int \sigma(f) \sigma(g) \), where \( \sigma : S(k) \to \text{U}(k) \to \text{k} \)

**Categorical interpretation of the DTE**

Let \( \mathcal{E} = \text{Rep} (U(g) \otimes \mathbb{B}) \) with non trivial associativity isomorphism \( (V \otimes V_2) \otimes V_3 \xrightarrow{\Phi} V_3 \otimes (V \otimes V_2) \)

Let \( \mathcal{H} = \text{Rep} (G^* \otimes \mathbb{B}, \text{PBW}) \). One has an algebraic morphism \( (G^* \otimes \mathbb{B}, \text{PBW}) \to (G^* \otimes \mathbb{B}, \text{PBW}) \)

Again \( f(\lambda + k h^{\alpha}) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f}{d \lambda^k} \partial_k q_{\lambda - \epsilon_k} \)

\[ \Rightarrow \text{bimodule } \mathcal{E} \times \mathcal{H} \to \mathcal{H}, (V, M) \mapsto V \otimes M \text{ with } \sigma(A) \circ (\text{PBW}) = \sigma(A) \circ (\text{PBW}) \]

Assocativity isomorphism is \( (V \otimes V_2) \otimes V_3 \xrightarrow{\text{PBW}} \)

The coherence condition is precisely the dynamical twist equation:

\[
\xrightarrow{\Phi} \xrightarrow{\text{PBW}} \xrightarrow{\text{PBW}} \]

\[ J^{1/2} (\lambda + k h^{\alpha}) \]

\[ (V \otimes V_2) \otimes (V_2 \otimes M) \xrightarrow{\Phi} (V \otimes (V_2 \otimes V_3)) \otimes M \xrightarrow{\text{PBW}} (V \otimes (V_2 \otimes V_3)) \otimes M \xrightarrow{\Phi} (V \otimes (V_2 \otimes V_3)) \otimes M \]

\[ J^{1/2} (\lambda + k h^{\alpha}) \]

**II. Geometric reformulation and main result**

**II.1 Quasi-Poisson manifold associated to a dynamical r-matrix**

Let \( g \) be a Lie algebra, and \( \mathcal{P} \in \mathfrak{g} \)

**Definition:** A \( g \)-quasi-Poisson manifold is a smooth manifold \( X \) together with a \( g \)-action \( \rho : g \to \mathfrak{X}(X) \), a \( g \)-invariant antisymmetric bidivation \( \{, \} \) on functions such that

\[ \{ f, q^1 \lambda + q^2 \} = \{ f(q^1), \text{PBW}(q^2) \} \quad (\forall f, q^1, q^2 \in \mathbb{C}) \]
The semi-classical limit condition $J(\lambda) - J(\omega) = \mathcal{O}(1)$ and the dynamical twist equation:

$$J^{1,2}_{\lambda} \ast_{PBW} J^{2,3}_{\lambda} = J^{1,3}_{\lambda} \ast_{PBW} J^{1,2}_{\lambda}$$

**Explain notations:**
- $J^{1,2}_{\lambda} = \lambda \otimes J$, $J^{1,2}_{\lambda} = (\Delta \otimes 1)(J)$
- $J^{1,2}_{\lambda}(1 + \hbar \omega) = \sum_{k \geq 0} \frac{1}{k!} \sum_{\alpha, \beta} \Delta^{\alpha, \beta}(J) \otimes h_{\alpha, \beta} \hbar^k$ (Taylor expansion)
- For polynomial functions $f, g \in \mathfrak{o}_\lambda = \mathcal{O}(\hbar)$ one defines $f \ast_{PBW} g = \mathcal{O}(\hbar)$, where $\mathcal{O}(\hbar) : \mathcal{O}(\hbar) \to U(\mathfrak{o}_\lambda, \hbar \mathcal{L}_\hbar)$ is the PBW isomorphism.

**Categorical interpretation of the DTE**

Let $\mathcal{E} = \text{Rep}(U_\hbar \mathfrak{o}_\lambda)$ with non trivial associativity isomorphism $(V \otimes V \otimes V) \to V \otimes (V \otimes V)$. Let $\mathcal{M} = \text{Rep}(\mathfrak{o}_\lambda \otimes \mathfrak{o}_\lambda \otimes \mathfrak{o}_\lambda)$. One has an algebraic morphism $(\mathfrak{o}_\lambda \otimes \mathfrak{o}_\lambda \otimes \mathfrak{o}_\lambda) \to (\mathfrak{o}_\lambda \otimes \mathfrak{o}_\lambda \otimes \mathfrak{o}_\lambda)$.

Again $f(1 + \hbar \omega) = \sum_{k \geq 0} \frac{1}{k!} \sum_{\alpha, \beta} \Delta^{\alpha, \beta}(J) \otimes h_{\alpha, \beta} \hbar^k$.

Therefore $\mathcal{E} \otimes \mathcal{M} \to \mathcal{M}$, $(V, M) \to V \otimes M$ with $f(\lambda)_*(\omega \otimes M) = f(1 + \hbar \omega)_* (\omega \otimes M)$.

The coherence condition is precisely the dynamical twist equation:

$$((V \otimes V \otimes V) \otimes (V \otimes V \otimes V)) \otimes V \rightleftharpoons (V \otimes (V \otimes V)) \otimes (V \otimes V)$$

**II] Geometric reformulation and main result**

**II.1] Quasi-Poisson manifold associated to a dynamical r-matrix**

Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{E} = \mathfrak{g}(\hbar \mathfrak{g})^{\mathfrak{g}}$

**Definition:** A $\mathfrak{g}$-quasi-Poisson manifold is a smooth manifold $X$ together with a $\mathfrak{g}$-action $\pi : \mathfrak{g} \to \mathcal{X}(X)$, a $\mathfrak{g}$-invariant antisymmetric bi-differential $[\cdot, \cdot]$ on functions such that

$$\langle f(\pi(\mathfrak{g}) \cdot \mathfrak{g}) + \mathfrak{g}(\pi(\mathfrak{g}) \cdot \mathfrak{g}) \rangle = \langle f(\pi(\mathfrak{g})) \rangle + \langle \mathfrak{g}(\pi(\mathfrak{g})) \rangle$$

(\forall f, \pi(\mathfrak{g}), \mathfrak{g} \in \mathfrak{g}(\hbar \mathfrak{g}))
Example: a smooth map \( \rho : U \to \mathfrak{g} \) is a dynamical r-matrix if and only if the following bracket on \( M := U \times G \) is a quasi-Poisson bracket for \( \rho : g \to x \to z \).

\[
\forall f, g \in C \quad \{ f, g \} = \{ f, g \}_{\mathfrak{g}} + \{ f, g \}_x = \{ f, g \}_x + \{ f, g \},
\]

A dynamical r-matrix is called symplectic if this quasi-Poisson structure is (in particular \( z = 0 \)).

Main Theorem [Aleksiev, C]: Any symplectic dynamical r-matrix admits a dynamical twist quantization (with \( \Phi = 1 \)).

II.2) Compatible quantizations:

A quantization of a quasi-Poisson structure is the data of an associator \( \Phi \) quantizing \( \mathfrak{g} \) together with an invariant star-product \( \star \) satisfying

\[
\mathfrak{m} \circ (\mathfrak{m} \circ \text{oid}) = \mathfrak{m} \circ (\text{id} \circ \mathfrak{m}) \circ \Phi (\mathfrak{g} \circ \mathfrak{g})
\]

(i.e., \( (\mathfrak{g} \circ \mathfrak{g}) \star \) is an algebra \( \mathfrak{m} \)).

Ref \( \Phi (U \mathfrak{g} (k)) \).

In the case of the quasi-Poisson manifold \( U \times G \) arising from a dynamical r-matrix, a quantization is called compatible if it satisfies the following requirement:

\[
\forall u, v \in G, \forall f, g \in C \&,
\]

\[
\star u \star v = \text{oid} \star (u \star v), \quad \mathfrak{m} \star u = \mathfrak{m} \star u, \quad u \star \mathfrak{m} = \Sigma \frac{1}{k!} \Sigma \frac{1}{i!} \frac{1}{k!} (u_{1}, u_{2}, \ldots, u_{k})
\]

Theorem [Peng Xu in case \( z = 0 \) and \( \Phi = \text{oid} \), otherwise Enriquez, Gong]:

There is a bijective correspondence between compatible quantizations of this quasi-Poisson structure and dynamical twist quantizations of the corresponding dynamical r-matrix.

Since \( \star \) is \( \rho \)-invariant then \( \forall f, g \in C \),

\[
(p \star f) \star g = \Phi (\mathfrak{g} \mathfrak{m} (U \mathfrak{g} (k))) \star \Phi (\mathfrak{g} \mathfrak{m} (U \mathfrak{g} (k)))
\]

This is the twist.
II.3 Idea of the proof of the main theorem

Observe that $U \times G \to \mathbb{R}$, $(v, g) \mapsto \lambda$ is a momentum map.

Claim: it is sufficient to quantize this momentum map in order to obtain a compatible quantization. (And this to quantize a dynamical r-matrix).

More precisely, if there exists a star-product quantizing the quasi-Poisson structure on $U \times G$ which is such that $\forall h \in \mathbb{R}$ (a linear function on $\mathbb{R}$), $h \ast f - f \ast h = \frac{\lambda}{\hbar} \{ h, f \}$ for any $f \in C_c^\infty(U)$.

Then there exists a $G$-invariant gauge transformation $\varphi$ such that $\varphi(h) = h$ $(\forall h \in \mathbb{R})$ and $\ast' := \ast \circ \varphi$ is a compatible quantization. (Recall that $\varphi(f) := \varphi'(\varphi(f) \ast \varphi(g))$).

In the symplectic case the existence of such quantization is assured by a refinement of Fedosov's construction of star-products.

Remarks:
- existence of the gauge transformation is simply a kind of normal ordering.
- everything works even in the Poisson regular case (i.e. $\mathbb{R} = 0$ and the Poisson structure has constant rank).

III Classification

$J_1$ and $J_2$

Two dynamical twist quantizations of $\mathcal{M}$ are said equivalent if there exists an $H$-equivalent map $T : U \to U \otimes \mathbb{F}$ such that $T = 1 + \Theta(v)$ and

$$T^r(v) \ast_{\mathcal{M}} J_1(v) = J_2(v) \ast_{\mathcal{M}} T^r(v) \ast_{\mathcal{M}} T^i(v)$$

Remark: this is equivalent to say that the identity functor $\mathcal{M} \to \mathcal{M}$ is an equivalence between the two corresponding module categories $\mathcal{E}_T$ over $\mathcal{M}$.

Theorem [Aleksievich-C]: the set of dynamical quantizations $\mathcal{M} / \text{equivalences}$ is an affine space modelled on $H^*(U,G)[[\hbar]]$.

Here $C^r(U,G) := C_c^\infty(U, \Lambda^* G)^H$, differential is given by $d(f)_\hbar = \sum \frac{\partial f}{\partial \hbar^i} \delta^i \otimes [\mathcal{M}, \mathcal{M}]$. :}
The quantum composition formula

Data: $\mathfrak{g}$ includes of Lie algebra $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{m}$ is a ND reductive splitting.

$Z\mathfrak{g}\subset(\mathfrak{a}^*\mathfrak{g})^\mathfrak{g}$, $\rho: h^* U \to \Lambda^* \mathfrak{g}$ a dynamical $\mathfrak{m}$-matrix with $Z$ as before.

Consider the dynamical $\mathfrak{m}$-matrix $\gamma_\mathfrak{m}: \Lambda^* \mathfrak{g} \to \Lambda^* \mathfrak{g}$ associated with the ND reductive splitting $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{m}$.

Now observe the $\gamma_\mathfrak{m}+\rho: (\mathfrak{a}^*\mathfrak{g})^* \to VxU \to \Lambda^2(R\otimes \mathfrak{g})$ is a dynamical $\mathfrak{m}$-matrix with $Z$.

Moreover one can check that there is a momentum map

$$
\mu: VxU \times H \times G \to \mathfrak{g}^*^\mathfrak{m}_G
$$

where $\rho_\mathfrak{m}: \mathfrak{g} \to \mathfrak{m}$ is the
projection along $\mathfrak{a}$.

on the quasi-Poisson manifold associated to the dynamical $\mathfrak{m}$-matrix $\gamma_\mathfrak{m}+\rho$.

Then observe that the smooth map

$$
\Psi: VxU \times H \times G \to UNxG
$$

induces a diffeomorphism from the reduced space $\mu^* (\mathfrak{g}/H)$ to $UNxG$.

A straightforward computation shows that the induced Poisson (or better, quasi-Poisson) bracket on $UNxG$ has the desired form, with $\mathcal{R}=\gamma_\mathfrak{m}+\rho_\mathfrak{m}$. This proves in particular the Fedorov-Galer-Pukhtin result mentioned above.

Quantization (ideas)

Assume we are given a quantization $\mathcal{J}(\mathfrak{g})$ of $\mathfrak{g}$.

Fedorov methods $\Rightarrow$ one can find a quantization $\mathcal{J}_\mathfrak{m}(\mathfrak{g})$ of $\mathfrak{g}_\mathfrak{m}(\mathfrak{g})$ satisfying a nice condition.

Then $\mathcal{J}(\mathfrak{g})=\mathcal{J}_\mathfrak{m}(\mathfrak{g})\mathcal{J}(\mathfrak{g})$ is a quantization of $\gamma_\mathfrak{m}+\rho$.

The nice condition for $\mathcal{J}_\mathfrak{m}$ ensure us that we have a quantization of the momentum map $\mu$.

One then get the desired result by performing a quantum reduction procedure. The result is then that we have a set map:

$$
\left\{ \text{quantizations of } \gamma_\mathfrak{m} \right\} \rightarrow \left\{ \text{quantizations of } \gamma_\mathfrak{m}+\rho_\mathfrak{m} \right\}.
$$