Quantization of symplectic dynamical r-matrices

joint work with A. Alekseev (available at arXiv:math/0606438v3)

I) Twist quantization of dynamical r-matrices

I.1) Dynamical r-matrices: definition and examples

Data: $\mathfrak{h} \subset \mathfrak{g}$ inclusion of Lie algebras ($\leadsto H \subset G$ corresp. inclusion of Lie groups)

$U \subset \mathfrak{h}^*$ Ad^*-invariant open subset. $Z \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$.

Definition: a dynamical r-matrix is a H-equivariant smooth map $r: U \to \Lambda^2 \mathfrak{g}$ satisfying the (modified) classical dynamical Yang-Baxter equation

$$\frac{1}{2}[r(\lambda), r(\lambda)] - \sum_{i=1} h_i \wedge \frac{\partial r}{\partial \lambda^i}(\lambda) = Z \quad (\forall \lambda \in U)$$

Here $(h_i)_i$ and $(\lambda^i)_i$ are dual bases of $\mathfrak{h}$ and $\mathfrak{h}^*$.

examples: 1) $\mathfrak{g}$ simple Lie algebra, $\mathfrak{h}$ CSA, $\Omega \in S^2(\mathfrak{g})^{\mathfrak{g}}$.

Then the CDYBE of Roche's lecture for ρ (with coupling constant ϵ) is equivalent to the MCDYBE for $r = \rho - \rho^{2,1}$ with $Z = \epsilon^2 [\Omega^{1,2}, \Omega^{1,3}]$.

2) assume there exists $\mathfrak{m} \subset \mathfrak{g}$ s.t. $\mathfrak{h} \oplus \mathfrak{m} = \mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ ($\stackrel{\text{def}}{=}$ reductive splitting)

Assume moreover that $\exists \lambda \in \mathfrak{h}^*$ s.t. $\omega(\lambda): (x, y) \longmapsto \langle \lambda, [x,y]_{\mathfrak{h}} \rangle$ for $x, y \in \mathfrak{m}$ is a ND pairing. Then $r_{\mathfrak{h}}(\lambda) := \omega(\lambda)^{-1}$ is a dynamical r-matrix with $Z = 0$.

(Example due to Fehér-Gábor-Pusztai).

Proposition [Fehér-Gábor-Pusztai & Enriquez-Etingof]: Assume that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}$ is a ND reductive splitting.

For any dynamical r-matrix $\rho: \mathfrak{h}^* \to \Lambda^2 \mathfrak{g}$ with $Z \in \Lambda^3(\mathfrak{g})^{\mathfrak{g}}$, then

$\theta_\rho := r_\lambda^{\mathfrak{h}} + \rho|_{\mathfrak{t}^*} : \mathfrak{t}^* \longrightarrow \Lambda^2 \mathfrak{g}$ is a dynamical r-matrix (for the same Z).

I.2) Dynamical twist quantization

Let $r(\lambda)$ a dynamical r-matrix with $Z \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$.

Let $\Phi = 1 \otimes 1 \otimes 1 + \frac{\hbar^2}{6} Z + o(\hbar^3) \in \left(U(\mathfrak{g})^{\otimes 3}[[\hbar]]\right)^{\mathfrak{g}}$ an associator (existence proved by Drinfeld)

Definition: a dynamical twist quantization of $r(\lambda)$ is a H-equivariant smooth map

$J = 1 \otimes 1 + O(\hbar): U \longrightarrow \otimes^2 U(\mathfrak{g})[[\hbar]]$ satisfying the semi-classical limit condition $J(\lambda) - J(\lambda)^{2,1} = \hbar r(\lambda) + o(\hbar)$ and the dynamical twist equation (

$$J^{12,3}(\lambda) *_{PBW} J^{1,2}(\lambda + \hbar h^{(3)}) = \Phi^{-1} J^{1,23}(\lambda) *_{PBW} J^{2,3}(\lambda).$$

Explain notations : • here $J^{2,3} = 1 \otimes J$, $J^{12,3} := (\Delta \otimes id)(J)$

• $J^{1,2}(\lambda + \hbar h^{(3)}) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1 \dots i_k} \frac{\partial^k J}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}(\lambda) \otimes h_{i_1} \cdots h_{i_k}$ (Taylor expansion)

• for polynomial functions $f, g \in \mathcal{O}_{\mathfrak{h}^*} = S(\mathfrak{h})$ one defines

$f *_{PBW} g := \sigma^{-1}(\sigma(f)\sigma(g))$, where $\sigma : S(\mathfrak{h})[[\hbar]] \xrightarrow{\sim} U(\mathfrak{h}[[\hbar]], \hbar[,]_{\mathfrak{h}})$

is the PBW isomorphism.

Categorical interpretation of the DTE

Let $\mathcal{E} = \mathrm{Rep}(U(\mathfrak{g})[[\hbar]])$ with non trivial associativity isomorphism $(V_1 \otimes V_2) \otimes V_3 \xrightarrow{\Phi} V_1 \otimes (V_2 \otimes V_3)$

Let $\mathcal{M} = \mathrm{Rep}(\mathcal{O}_{\mathfrak{h}^*}[[\hbar]], *_{PBW})$. One has an algebra morphism $(\mathcal{O}_{\mathfrak{h}^*}[[\hbar]], *_{PBW}) \longrightarrow (U(\mathfrak{g}) \otimes \mathcal{O}_{\mathfrak{h}^*})[[$

$f(\lambda) \longmapsto f(\lambda + \hbar h^{(1)})$

Again $f(\lambda + \hbar h^{(1)}) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1 \dots i_k} \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}(\lambda) \otimes h_{i_1} \cdots h_{i_k}$.

$\Rightarrow$ bifunctor $\mathcal{E} \times \mathcal{M} \longrightarrow \mathcal{M}$; $(V, M) \longmapsto V \otimes M$ with $f(\lambda) \cdot (v \otimes m) := f(\lambda + \hbar h^{(1)}) \cdot (v \otimes m)$

Associativity isomorphism is $(V_1 \otimes V_2) \otimes M \xrightarrow{J(\lambda)^{-1}} V_1 \otimes (V_2 \otimes M)$.

The coherence condition is precisely the dynamical twist equation:

$$\begin{tikzcd}
((V_1 \otimes V_2) \otimes V_3) \otimes M \arrow[r, "\Phi"] & (V_1 \otimes (V_2 \otimes V_2)) \otimes M & V_1 \otimes ((V_2 \otimes V_3) \otimes M) \arrow[l, "J(\lambda)^{1,23}"'] \\
(V_1 \otimes V_2) \otimes (V_3 \otimes M) \arrow[u, "J(\lambda)^{12,3}"] & \circlearrowleft & V_1 \otimes (V_2 \otimes (V_3 \otimes M)) \arrow[ll, "J^{1,2}(\lambda + \hbar h^{(3)})"] \arrow[u, "J(\lambda)^{2,3}"']
\end{tikzcd}$$

II) Geometric reformulation and main result

II.1) Quasi-Poisson manifold associated to a dynamical r-matrix

Let $\mathfrak{g}$ be a Lie algebra, and $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$

Definition : a $\mathfrak{g}$-quasi-Poisson manifold is a smooth manifold X together with a $\mathfrak{g}$-action $\rho : \mathfrak{g} \to \mathfrak{X}(X)$, a $\mathfrak{g}$-invariant antisymmetric biderivation $\{,\}$ on functions such that

$$\{\{f, g\}, h\} + c.p.(f, g, h) = \langle \rho(Z), df \wedge dg \wedge dh \rangle \quad (\forall f, g, h \in \mathcal{O}_X).$$

the semi-classical limit condition $J(\lambda)-J(\lambda)^{2,1} = \hbar r(\lambda) + o(\hbar)$ and the dynamical twist equation (

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$f(\lambda) \longmapsto f(\lambda+\hbar h^{(1)})$

Again $f(\lambda+\hbar h^{(1)}) := \sum_{k\geq 0} \frac{\hbar^k}{k!} \sum_{i_1\cdots i_k} \frac{\partial^k f}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}(\lambda)\otimes h_{i_1}\cdots h_{i_k}$.

$\Rightarrow$ bifunctor $\mathcal{E}\times\mathcal{M} \longrightarrow \mathcal{M}$, $(V,M) \longmapsto V\otimes M$ with $f(\lambda)\cdot(v\otimes m) := f(\lambda+\hbar h^{(1)})\cdot(v\otimes m)$

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$$\{\{f,g\},h\} + c.p.(f,g,h) = \langle \rho(Z), df\wedge dg\wedge dh\rangle \quad (\forall f,g,h \in \mathcal{O}_X).$$

example: a smooth map $r : U \to \Lambda^2\mathfrak{g}$ is a dynamical r-matrix if and only if the following bracket on $M := U \times G$ is a quasi-Poisson bracket for $\rho : \mathfrak{g} \ni x \mapsto \bar{x}$.

$\forall h, h' \in \mathfrak{h}$ linear functions on $\mathfrak{h}^*$

$\forall f, g \in \mathcal{O}_G$

• $\{h, h'\} := [h, h']_{KK} = [h, h']$ • $\{h, f\} := \vec{h} \cdot f$ • $\{f, g\} := \langle \overrightarrow{r(\lambda)}, df \wedge dg \rangle$.

A dynamical r-matrix is called symplectic if this quasi-Poisson structure is (in particular $Z = 0$).

Main Theorem [Alekseev-C]: Any symplectic dynamical r-matrix admits a dynamical twist quantization (with $\Phi = 1 \otimes 1 \otimes 1$).

II.2) Compatible quantizations

A quantization of a quasi-Poisson structure is the data of an associator Φ quantizing Z together with an invariant star-product $*$ satisfying

$$m_* \circ (m_* \otimes id) = m_* \circ (id \otimes m_*) \circ \rho^{\otimes 3}(\Phi) \quad \left(\text{i.e. } (\mathcal{O}_X[[\hbar]], *) \text{ is an algebra in } Rep_\Phi(U\mathfrak{g}[[\hbar]])\right)$$

where $m_*(f \otimes g) := f * g$.

In the case of the quasi-Poisson manifold $U \times G$ arising from a dynamical r-matrix, a quantization is called compatible if it satisfies the following requirement:

$\forall u, v \in \mathcal{O}_U$, $\forall f, g \in \mathcal{O}_G$,

$$\cdot\ u * v = u *_{PBW} v, \quad \cdot\ f * u = fu, \quad \cdot\ u * f = \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1 \ldots i_k} \frac{\partial^k u}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \left(\vec{h}_{i_1} \cdots \vec{h}_{i_k} \cdot f\right)$$

Theorem [Ping Xu in case $Z = 0$ and $\Phi = 1 \otimes 1 \otimes 1$, otherwise Enriquez-Etingof]

There is a bijective correspondence between compatible quantizations of this quasi-Poisson structure and dynamical twist quantizations of the corresponding dynamical r-matrix.

Since $*$ is $\mathfrak{g}$-invariant then $\forall f, g \in \mathcal{O}_G$, $(f * g)(\lambda) = J(\lambda)(f \otimes g)$ $(\forall \lambda \in U)$.

This is the twist.

II.3) Idea of the proof of the main theorem

Observe that $U \times G \to \mathfrak{h}^*;\ (\lambda, g) \longmapsto \lambda$ is a momentum map.

Claim: it is sufficient to quantize this momentum map in order to obtain a compatible quantization (and thus to quantize a dynamical r-matrix).

More precisely, if there exists a star-product quantizing the quasi-Poisson structure on $U \times G$ which is such that $\forall h \in \mathfrak{h}$ (a linear function on $\mathfrak{h}^*$), $h * f - f * h = \hbar\{h, f\}$ for any $f \in \mathcal{O}_M$.

Then there exists a $\mathfrak{g}$-invariant gauge transformation φ such that $\varphi(h) = h$ ($\forall h \in \mathfrak{h}$) and $*' := *^{(\varphi)}$ is a compatible quantization (recall that $f *^{(\varphi)} g := \varphi^{-1}(\varphi(f) * \varphi(g))$).

In the symplectic case the existence of such quantization is ensured by a refinement of Fedosov's construction of star-products.

Remarks: • existence of the gauge transformation is simply a kind of normal reordering.

• everything works even in the Poisson regular case (i.e. $Z = 0$ and the Poisson structure has constant rank).

III) Classification

Two dynamical twist quantizations J_1 and J_2 of $r(\lambda)$ are said equivalent if there exists a H-equivariant map $T : U \longrightarrow U(\mathfrak{g})[[\hbar]]$ such that $T = 1 + O(\hbar)$ and

$$T^{12}(\lambda) *_{PBW} J_1(\lambda) = J_2(\lambda) *_{PBW} T^1(\lambda + \hbar h^{(2)}) *_{PBW} T^2(\lambda)$$

Remark: this is equivalent to say that the identity functor $\mathcal{M} \to \mathcal{M}$ is an equivalence between the two corresponding module categories $/\mathcal{E} = \mathrm{Rep}_{\Phi}(U(\mathfrak{g})[[\hbar]])$.

Theorem [Alekseev-C]: the set $\{\text{dyn. twist quant. of } r(\lambda)\}/_{\text{equivalences}}$ is an affine space modelled on $H^2_r(U, \mathfrak{g})[[\hbar]]$.

Here $C^n_r(U, \mathfrak{g}) := C^\infty(U, \wedge^n \mathfrak{g})^H$, differential is given by $d(f)(\lambda) = \sum h_i \wedge \frac{\partial f}{\partial \lambda^i}(\lambda) + [r(\lambda), f(\lambda)]$.

IV] The quantum composition formula

<u>Data</u>: $\mathfrak{t}\subset\mathfrak{h}\subset\mathfrak{g}$ inclusions of Lie algebras s.t. $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{m}$ is a ND reductive splitting.

$Z\in(\Lambda^3\mathfrak{g})^{\mathfrak{g}}$. $\rho:\mathfrak{h}^*\supset U\longrightarrow\Lambda^2\mathfrak{g}$ a dynamical r-matrix with Z as before.

Consider the dynamical r-matrix $r_{\mathfrak{t}}^{\mathfrak{m}}:\mathfrak{t}^*\supset V\longrightarrow\Lambda^2\mathfrak{h}$ associated with the ND reductive splitting $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{m}$.

Now observe that $r_{\mathfrak{t}}^{\mathfrak{m}}+\rho:(\mathfrak{t}\oplus\mathfrak{h})^*\supset V\times U\longrightarrow\Lambda^2(\mathfrak{h}\oplus\mathfrak{g})$ is a dynamical r-matrix with Z.

Moreover one can check that there is a momentum map

$$\begin{aligned}\mu: V\times U\times H\times G &\longrightarrow \mathfrak{h}^*\\ (\tau,\lambda,x,y)&\longmapsto \lambda-\mathrm{Ad}^*_{x^{-1}}(p^*\tau)\end{aligned}$$

(where $p:\mathfrak{h}\to\mathfrak{t}$ is the projection along $\mathfrak{m}$.)

on the quasi-Poisson manifold associated to the dynamical r-matrix $r_{\mathfrak{t}}^{\mathfrak{m}}+\rho$.

Then observe that the smooth map

$$\begin{aligned}\Psi: V\times U\times H\times G &\longrightarrow U\cap V\times G\\ (\tau,\lambda,x,y)&\longmapsto(\tau,yx)\end{aligned}$$

induces a diffeomorphism from the reduced space $\mu^{-1}(0)/H$ to $U\cap V\times G$.

A straightforward computation shows that the induced Poisson (or better, quasi-Poisson) bracket on $U\cap V\times G$ has the desired form, with $r=r_{\mathfrak{t}}^{\mathfrak{m}}+\rho_{|\mathfrak{t}^*}$. This proves in particular the Fehér-Gábor-Pusztai result mentioned above.

<u>Quantization</u> (ideas)

Assume we are given a quantization $J(\lambda)$ of $\rho(\lambda)$.

Fedosov methods $\Rightarrow$ one can find a quantization $J_{\mathfrak{t}}^{\mathfrak{m}}(\tau)$ of $r_{\mathfrak{t}}^{\mathfrak{m}}(\tau)$ satisfying a nice condition.

Then $\mathbb{J}(\tau,\lambda)=J_{\mathfrak{t}}^{\mathfrak{m}}(\tau)J(\lambda)$ is a quantization of $r_{\mathfrak{t}}^{\mathfrak{m}}+\rho$

The nice condition for $J_{\mathfrak{t}}^{\mathfrak{m}}$ ensure us that we have a quantization of the momentum map μ.

One then get the desired result by performing a quantum reduction procedure. The result is then that we have a set map:

$$\{\text{quantizations of } \rho\}\longrightarrow\{\text{quantizations of } r_{\mathfrak{t}}^{\mathfrak{m}}+\rho_{|\mathfrak{t}^*}\}.$$