Three lectures on derived symplectic geometry and topological field theories

Damien Calaque

Abstract. We give an informal introduction to the new field of derived symplectic geometry, and present some applications to topological field theories. We in particular try to explain that derived symplectic geometry provides a suitable framework for the so-called AKSZ construction (after Alexandrov-Kontsevich-Schwartz-Zaboronski). We start with a brief summary of the main features of derived algebraic geometry. We then continue with the definition of n-symplectic and Lagrangian structures, after Pantev-Toën-Vaquié-Vezzosi (PTVV), and provide examples such as

- (for shifted symplectic structures) $\mathtt{BG}$, $\left[\mathfrak{g}^*/\mathtt{G}\right]$, mapping stacks with symplectic target and "compact oriented" source.
- (for Lagrangian structures) moment maps, quasi-Hamiltonian structures, mapping stacks with boundary conditions.

We finally explain how this can be used to construct (fully extended) topological field theories with values in (higher) categories of Lagrangian correspondences.

Contents

Short introduction

These lecture notes are aimed at people working in the fields of Poisson Geometry and Mathematical Physics, and that have already been exposed to some ideas from advanced homological algebra and homotopy theory (derived/dg categories, simplicial methods, higher categories,...). The reader shall have some familiarity with algebraic geometry, even though we try to provide some differential geometric intuition where this is possible.

Our first motivation comes from physics, and more precisely Quantum Field Theory. Assume that we have a space of fields X and an action functional $S : X \to \mathbf{k}$, and that we aim at computing some path integral perturbatively around classical solutions of the equations of motion. Doing this is usually problematic if the critical point of S one is looking at is degenerate. Here people seem to use a kind of trick (that is at the heart of the so-called *BV formalism* [4]) which consists in adding fields, and that we would like to understand:

- to every field x_i (i.e. coordinates on X) one associates corresponding anti-fields ξ^i.
- to every infinitesimal generator of symmetries of S (Chevalley generators) one associates a new field (called ghost). Note that there are also ghosts for ghosts, which come from higher symmetries.
- anti-ghosts (i.e. anti-fields for ghosts).

The need for derived schemes. Anti-fields naturally appear whenever one computes the derived critical locus of S. Namely the critical locus of S is nothing but the intersection of the graph of dS with the zero section in T^*X, and we will see that derived intersections are better behaved than genuine ones.

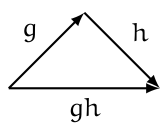
The need for stacks. Ghosts naturally appear whenever one takes the quotient by symmetries. Namely, the quotient might be very pathological and shall be replaced by a homotopy quotient (which is a stack). The presence of higher symmetries requires the use of higher stacks.

Remark 0.1. The reason for the appearance of anti-ghosts is that symmetries are Hamiltonian and thus one rather wants to apply some kind of symplectic reduction to the derived critical locus than just taking out symmetries. This is when one takes the derived zero level set of the moment map that anti-ghosts appear.

Our second motivation is to understand why some very interesting moduli spaces are symplectic (or Lagrangian). Let us give some famous examples:

- the (smooth locus of the) moduli space of flat connections (or local systems) on a closed differentiable surface Σ.
- if the above Σ is the boundary of a compact 3-fold M then the submanifold consisting of those connections that extend to M is Lagrangian.
- the (smooth locus of the) moduli space of G-bundles on a $K3$ surface S, G being a reductive group, is symplectic (see [19]).

Let us start by explaining on the first Example why introducing derived stacks might be a good idea to understand these facts. First of all observe that there is moduli stack of G-local systems on a surface Σ that can be obtained as a mapping stack. Namely, given a triangulation of Σ we get a simplicial set Σ_B; looking at the simplicial set of simplicial maps from Σ_B to the nerve of G we get (the nerve of) the groupoid of flat discrete G-connections on Σ:



In other words $\mathrm{Flat}_G(\Sigma) = \mathrm{Map}_{\mathbf{St}}(\Sigma_B, BG)$. Such a mapping stack can be naturally endowed with a derived structure by considering the *derived mapping stack* $\mathbf{Map}_{\mathbf{dSt}}(\Sigma_B, BG)$. As such it naturally carries an obstruction theory and we thus have access to nice gadgets like virtual fundamental classes.

Finally, the transgression procedure works well for derived stacks (and not for genuine ones), so that the natural 2-shifted symplectic structure on BG (see below) transgresses to a symplectic structure on $\mathbf{Map}_{\mathbf{dSt}}(\Sigma_B, BG)$... on the smooth locus it coincides with the usual one.

It is worth mentioning the survey [22] of Bertrand Toën, which is much more accurate and complete. In comparison the present set of lecture notes puts more emphasis on potential applications to Mathematical Physics.

Notation and conventions

All along the paper we work over a field $\mathbf{k}$ of zero characteristic.

We sometimes use the language of higher categories and homotopical algebra. More precisely:

- an $(\infty, 1)$-category $\mathcal{C}$ has a space of morphisms $\mathbf{Hom}_{\mathcal{C}}(X, Y)$ between two objects. As usual, now that we have a space of morphisms, the axioms (for instance the associativity of the composition of morphisms) of a category are only required to hold up to homotopy. We refer to the [16, Chapter 1] for a nice overview of higher category theory.
- all $(\infty, 1)$-categories we are dealing with appear as arising from model categories, for which we refer to [13] and references therein.
- homotopy limits and colimits, denoted holim and $\mathrm{hocolim}$, can be understood either within a given $(\infty, 1)$-categories or within a model category that models it.
- an (∞, n)-category is a category having an $(\infty, n-1)$-category of morphisms between two objects. We refer to [15] for an introduction to (∞, n)-categories and their relevance for topological field theories.
- an (∞, n)-category can be thought of as a higher category with k-morphisms being invertible for all $k > n$. Namely, we have a space of n-morphisms, in which paths, 2-cells, etc... can be interpreted as invertible $(n+1)$-morphisms, $(n+2)$-morphisms, etc...
- according to the above, $(\infty, 0)$-categories are spaces, that we interpret as ∞-groupoids.

We might also sometimes make implicit use of the following equivalences of $(\infty, 1)$-categories:

- between topological spaces and simplicial sets (using singular chains and geometric realization).
- between non-negatively graded cochain complexes of $\mathbf{k}$-modules and simplicial $\mathbf{k}$-modules (see [17, §1.2.3 & §1.2.4] for a review of the Dold-Kan correspondences and its ∞-categorical version). In particular, any dg-category can thus be viewed as an $(\infty, 1)$-category.

Given an object O equipped with a specified algebraic structure (an algebra, a group, ...), we write $O\text{-}\mathbf{mod}$ for the dg-category of complexes of linear representations of O. We also write $\mathbf{Cpx} := \mathbf{k}\text{-}\mathbf{mod}$.

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1 Lecture 1: derived algebraic geometry

We would like to warn the reader that the contents of this first Lecture is far from being accurate. It only gives the rough philosophy without providing actual definitions. The reader who really wants to learn derived algebraic geometry should consult foundational references such as [23] (see also the survey [24]).

1.1 Affine derived schemes

We denote by $\mathtt{cdga}_{\leq 0}$ the category of non-positively graded commutative differential graded algebras (or, $\mathtt{dg}_{\leq 0}$-algebras for short). For any $\mathtt{dg}_{\leq 0}$-algebra A, we write $A^\sharp$ for the underlying commutative graded algebra (i.e. we forget about the differential). The category $\mathtt{cdga}_{\leq 0}$ has the following main feature: any morphism $B \longrightarrow A$ can be factored into

$$B \longrightarrow \widetilde{A} \longrightarrow A\,, \tag{1}$$

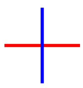
where $\widetilde{A} \longrightarrow A$ a quasi-isomorphism that is degreewise surjective in negative degrees and $B \longrightarrow \widetilde{A}$ is a *semi-free morphism*. The latter means that there is a subcomplex $V \subset \widetilde{A}$ such that $\widetilde{A}^\sharp \cong \big(B \otimes \mathtt{Sym}(V)\big)^\sharp$.

Remark 1.1. The above factorization property is the shadow of what is called a model structure on $\mathtt{cdga}_{\leq 0}$ (the projective model structure).

One shall think of a $\mathtt{dg}_{\leq 0}$-algebra A as the "ring of functions" on a geometric object, that we call an *affine derived scheme*, denoted $\mathtt{Spec}(A)$. A nice feature of (affine) derived schemes is that they behave well under fiber products (and, in particular, intersections).

Some recollection about ordinary fiber products of affine schemes

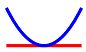
We will restrict ourselves to intersections of curves in the affine plane for simplicity. We start with the intersection of two lines in the plane.



$$\{(0,0)\} = \{x=0\} \cap \{y=0\} \subset \mathbb{A}^2_{\mathbf{k}}.$$

Algebraically: $\mathbf{k}[x,y]/(x) \underset{\mathbf{k}[x,y]}{\otimes} \mathbf{k}[x,y]/(y) = \mathbf{k}[x,y]/(x,y) = \underbrace{\mathbf{k}}_{\mathbf{k}\text{-dimension } 1}$.

Intersection of schemes actually keeps track of multiplicities:



$$\mathbf{k}[x,y]/(y-x^2) \underset{\mathbf{k}[x,y]}{\otimes} \mathbf{k}[x,y]/(y) = \mathbf{k}[x,y]/(y-x^2,y) = \mathbf{k}[x]/(x^2) = \underbrace{\mathbf{k} \oplus x\mathbf{k}}_{\mathbf{k}\text{-dimension } 2}\,.$$

The above intersections are both of zero dimension (as schemes). A problem arises when one deals with self-intersections:

$\mathbf{k}[x,y]/(y) \underset{\mathbf{k}[x,y]}{\otimes} \mathbf{k}[x,y]/(y) = \mathbf{k}[x,y]/(y) = \mathbf{k}[x]$.

$\mathtt{Spec}(\mathbf{k}[x])$ doesn't have the expected dimension (its dimension is 1 instead of 0).

Derived fiber products

We have seen in the previous § that fiber products of affine schemes are nothing but tensor products of algebras. We now derive this construction. Let us consider two morphisms $B \longrightarrow A_i$, $i = 1, 2$, of $\mathtt{dg}_{\leq 0}$-algebras. Their derived tensor product is defined as

$$A_1 \overset{\mathbf{L}}{\underset{B}{\otimes}} A_2 := \widetilde{A_1} \underset{B}{\otimes} A_2\,.$$

Remark 1.2. If one permutes A_1 and A_2, then one gets a quasi-isomorphic $\mathtt{dg}_{\leq 0}$-algebra.

The derived fiber product of $X_1 = \mathsf{Spec}(A_1)$ and $X_2 = \mathsf{Spec}(A_2)$ over $Y = \mathsf{Spec}(B)$ is then

$$X_1 \overset{\mathbf{R}}{\underset{Y}{\times}} X_2 = \mathsf{Spec}\left(A_1 \overset{\mathbf{L}}{\underset{B}{\otimes}} A_2\right) .$$

Let us now compute the derived self-intersection of a line in $\mathbb{A}^2_{\mathbf{k}}$:

- we have to compute $\mathbf{k}[x] \overset{\mathbf{L}}{\underset{\mathbf{k}[x,y]}{\otimes}} \mathbf{k}[x]$.
- wet set $\widetilde{\mathbf{k}[x]} := \mathbf{k}[x, y, \xi]$, where ξ is a variable of (cohomological) degree -1 and the differential is determined by $d(x) = d(y) = 0$, $d(\xi) = y$.
- $\mathbf{k}[x] \overset{\mathbf{L}}{\underset{\mathbf{k}[x,y]}{\otimes}} \mathbf{k}[x] = \widetilde{\mathbf{k}[x]} \underset{\mathbf{k}[x,y]}{\otimes} \mathbf{k}[x] = \mathbf{k}[x, \xi]$, with differential determined by $d(x) = d(\xi) = 0$.
- the *(virtual) dimension* of the resulting affine derived scheme is 0. We obtain it as the difference between the number of even and odd generators (we will later define it as the Euler characteristic of the cotangent complex of a given derived scheme).

Sanity check. We have to check that when the usual intersection is nice enough then the derived intersection gives an equivalent result. E.g. going back to the intersection of $\{y = 0\}$ and $\{x = 0\}$ inside $\mathbb{A}^2_{\mathbf{k}}$, the derived intersection is

$$\mathbf{k}[x] \overset{\mathbf{L}}{\underset{\mathbf{k}[x,y]}{\otimes}} \mathbf{k}[y] = \widetilde{\mathbf{k}[x]} \underset{\mathbf{k}[x,y]}{\otimes} \mathbf{k}[y] = \Big(\mathbf{k}[y, \xi], d(y) = \xi\Big) \cong \mathbf{k} .$$

1.2 Derived schemes

Definition 1.3. A *derived scheme* is a pair $X = (X_0, \mathcal{O}_X)$ of a scheme X_0 and a sheaf $\mathcal{O}_X$ of $\mathsf{dg}_{\leq 0}$-algebras on X_0 such that $H^0(\mathcal{O}_X) = \mathcal{O}_{X_0}$ and $H^i(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_{X_0}$-module for every $i < 0$. We call X_0 the *underlying scheme* of X and $\mathcal{O}_X$ the *structure sheaf* of X.

Recall that on an ordinary scheme X_0, an $\mathcal{O}_{X_0}$-module $\mathcal{M}$ is quasi-coherent if its restriction $\mathcal{M}_U$ on an open affine subscheme $\mathsf{Spec}(A) = U \subset X_0$ is the $\mathcal{O}_U$-module associated with an A-module M_U. For people who are not familiar with algebraic geometry, this is fair enough to think of a derived scheme as a non-negatively graded Q-manifold (see [2]).

The category of Q-manifolds doesn't have the appropriate morphisms. One should localize it with respect to suitable weak equivalences.

Note that any affine derived scheme is a derived scheme: if A is a $\mathsf{dg}_{\leq 0}$-algebra then we have a derived scheme $\mathsf{Spec}(A)$ having underlying scheme $\mathsf{Spec}\big(H^0(A)\big)$ and structure sheaf induced by A.

The category of derived schemes satisfies a factorization property similar to (1) (this was first noticed in [10][1]). In particular we can still define derived fiber products.

Example 1.4 (Derived critical locus)**.** Let X be a smooth algebraic variety together with a function $S : X \to \mathbb{A}^1$. The derived critical locus $\mathbf{R}\mathsf{Crit}(S)$ of S is the derived intersection $X \overset{\mathbf{R}}{\underset{T^*X}{\times}} X$ of the zero section $0 : X \to T^*X$ with the graph $d_{dR}S : X \to T^*X$ of $d_{dR}S$. Let us first resolve the zero section $X \to T^*X$: we define

$$\mathcal{O}_{\widetilde{X}} := \big(\mathsf{Sym}_{\mathcal{O}_X}\big(T_X[1] \oplus T_X\big), d\big) ,$$

where d is defined as the identity on $T_X[1]$ (which is a degree one morphism from $T_X[1]$ to T_X). Picking local coordinates $x_1, \ldots, x_n$ on X, we get that $\mathcal{O}_{\widetilde{X}}$ is freely generated by ξ_i's (of degree -1) and y_i's (of degree 0) over $\mathcal{O}_X$, and $d(\xi_i) = y_i$. When restricted to the classical critical locus $\mathsf{Crit}(S)$, $\mathcal{O}_{\widetilde{X}}$ and $\mathcal{O}_X$ are $\mathcal{O}_{T^*X}$-dg-algebras, where the former is equipped with the following $\mathcal{O}_{T^*X}$-algebra structure:

$$x_i \mapsto x_i \qquad \text{and} \qquad y_i \mapsto \frac{\partial S}{\partial x_i} .$$

[1] Even though the authors consider dg-schemes rather than derived schemes.

We get that $\mathbf{R}\mathsf{Crit}(S) = \big(\mathsf{Crit}(S), \mathcal{O}_{\mathbf{R}\mathsf{Crit}(S)}\big)$, where

$$\mathcal{O}_{\mathbf{R}\mathsf{Crit}(S)} = (\mathcal{O}_{\widetilde{X}})_{|\mathsf{Crit}(S)} \underset{(\mathcal{O}_{T^*X})_{|\mathsf{Crit}(S)}}{\overset{\mathbf{L}}{\otimes}} (\mathcal{O}_X)_{|\mathsf{Crit}(S)} = \Big(\mathsf{Sym}_{\mathcal{O}_X}\big(T_X[1]\big)_{|\mathsf{Crit}(S)}, d\Big) \,,$$

with $d(\xi_i) = \frac{\partial S}{\partial x_i}$. In other words, $d = \iota_{d_{dR}S}$. We refer to [25] for more details.

There is another description of derived schemes, by means of the *functor of points* approach, that we now describe.

Recollection on the functor of points

Any scheme X defines a functor

$$\begin{aligned} \underline{X} : (\mathsf{Aff})^{\mathrm{op}} = \mathsf{Rings} &\longrightarrow \mathsf{Sets} \\ S = \mathsf{Spec}(A) &\longmapsto X(S) := \{S\text{-points in } X\} = \mathsf{Hom}_{\mathsf{Schemes}}(S, X)\,, \end{aligned}$$

where Aff is the category of affine schemes.

Example 1.5. $X = \{x^2 + 1 = 0\} \subset \mathbb{A}^1$. We have $\underline{X}\big(\mathsf{Spec}(\mathbb{C})\big) = \{\text{solutions of } x^2+1=0 \text{ in } \mathbb{C}\} = * \coprod *$, while $\underline{X}\big(\mathsf{Spec}(\mathbb{R})\big) = \{\text{solutions of } x^2+1=0 \text{ in } \mathbb{R}\} = \emptyset$.

Schemes can be characterized as those functors satisfying the following two properties (we are using that Aff is a site, i.e. a category equipped with a Grothendieck topology):

($\star$) local representability.

($\star\star$) local-to-global property (i.e the functor is a sheaf).

Remark 1.6. This is similar to the following characterization of differentiable n-manifolds as functors from the opposite category of open subsets of $\mathbb{R}^n$ to sets. Namely, any n-manifold M gives rise to such a functor $\underline{M} : U \mapsto \mathsf{Hom}_{\mathsf{Man}}(U, M)$ which satisfies the following analogs of ($\star$) and ($\star\star$):

- the restriction of $\underline{M}$ to a small enough U is naturally isomorphic to $\underline{V}$ for some open $V \subset \mathbb{R}^n$.
- $\underline{M}(U)$ can be computed from $\underline{M}(U_i)$'s if $\{U_i\}_i$ is an open cover of U.

These two properties actually characterize differentiable n-manifolds.

Derived schemes as locally representable sheaves

Any derived scheme X defines a functor $\underline{X} : (\mathsf{dAff})^{\mathrm{op}} = \mathsf{cdga}_{\leq 0} \longrightarrow \mathsf{Sets}$, which sends an affine scheme $\mathsf{Spec}(A)$ to the set of morphisms of derived schemes from $\mathsf{Spec}(A)$ to X. It can be proven that the functor $\underline{X}$ satisfies the following property:

($\varnothing$) it sends quasi-isomorphisms of $\mathsf{dg}_{\leq 0}$-algebras to bijections.

Any functor satisfying the above property ($\varnothing$) descends to a functor $\mathsf{Ho}(\mathsf{cdga}_{\leq 0}) \longrightarrow \mathsf{Sets}$, where $\mathsf{Ho}(\mathsf{cdga}_{\leq 0})$ is the localization of $\mathsf{cdga}_{\leq 0}$ with respect to the class of quasi-isomorphisms (i.e. it is obtained from $\mathsf{cdga}_{\leq 0}$ by formally inverting quasi-isomorphisms). Then it has been proven (see [23, Lemma 2.2.2.13]) that $\mathsf{Ho}(\mathsf{dAff}) = \big(\mathsf{Ho}(\mathsf{cdga}_{\leq 0})\big)^{\mathrm{op}}$ is a site, so that conditions ($\star$) and ($\star\star$) still make sense for these functors that satisfy ($\varnothing$). Derived schemes can therefore be characterized as functors $(\mathsf{dAff})^{\mathrm{op}} \longrightarrow \mathsf{Sets}$ satisfying ($\varnothing$), ($\star$), and ($\star\star$).

Remark 1.7. One should note the following subtlety: while the local representability condition ($\star$) is concerned with Zariski open immersions (open submanifolds in the context of differentiable geometry), the local-to-global property ($\star\star$) uses the topology on $\mathsf{Ho}(\mathsf{dAff})$ which is defined by means of étale morphisms (local diffeomorphisms in the context of differential geometry). We refer to [23, §1.2.6, §1.2.7 and §2.2.2] for derived analogs of the standard properties (Zariski open, flat, smooth, étale) of morphisms in algebraic geometry.

1.3 Derived (∞-)stacks

1-stacks

If one replaces $\mathtt{Sets}$ by $\mathtt{Groupoids}$ then one gets the notion of 1-stacks. Namely, 1-stacks are functors[2] $\mathtt{Rings} \longrightarrow \mathtt{Groupoids}$ satisfying a weak local-to-global property (which we won't write: one should simply know that we want to glue object only up to isomorphisms) with respect to the étale topology; see [14].

Example 1.8. Let G be group scheme acting on a scheme X. We define a functor $(\mathtt{Aff})^{op} \longrightarrow \mathtt{Groupoids}$ sending S to the groupoid

$$G(S) \times X(S) \rightrightarrows X(S)$$

of the action of $G(S)$ on $X(S)$. It is not a stack, but there is a *stackification* process (very similar to the sheafification process of presheaves) that provides us with a stack $[X/G]$. In the case when $X = *$ then we have an explicit description of $BG := [*/G]$: $[*/G](S)$ is the groupoid of G-torsors over S.

There are two kinds of representability (sometimes called geometricity) assumptions one usually puts on stacks:

- *Artin stacks* are quotients stacks of smooth groupoids (recall that smooth is the algebro-geometric analog for submersive).
- *Deligne-Mumford* stacks are quotient stacks of étale groupoids (i.e. stacks having finite isotropy groups).

∞-stacks

One can actually go further and replace groupoids by higher groupoids. It has been known for a long time that simplicial sets (more precisely, Kan complexes, which are fibrant objects for a certain model structure on $\mathtt{sSets}$) are good models for weak ∞-groupoids. We then define ∞-stacks as simplicial presheaves, i.e. functors $\mathcal{X} : \mathtt{Rings} \longrightarrow \mathtt{sSets}$, satisfying the following two properties:

(a) it takes values in weak ∞-groupoids (i.e. objectwise fibrantness of the simplicial presheaf).

(b) it satisfies a homotopy local-to-global property (a.k.a. étale descent, meaning that the functor is an ∞-sheaf): for any étale cover $U_i \to U$, the map

$$\mathcal{X}(U) \longrightarrow \mathtt{holim}\left(\prod_i \mathcal{X}(U_i) \rightrightarrows \prod_{i,j} \mathcal{X}(U_i \underset{U}{\times} U_j) \cdots\right)$$

is an equivalence.

From now, we might omit ∞ and simply talk about "stacks". As for 1-stacks, there is a stackification process which sends a simplicial presheaf to its associated stack[3].

Example 1.9 (Classifying stacks)**.** Let G be an affine group scheme. We define $BG = [*/G]$ as the stack associated with the simplicial presheaf sending S to the nerve of $G(S)$. In other words, $[*/G](S)$ is the nerve of the groupoid of G-torsors over S.

Actually, any 1-stack $\mathcal{X}$ gives rise to an ∞-stack, sending an affine scheme S to the nerve of the groupoid $\mathcal{X}(S)$.

Example 1.10 (Betti stacks)**.** Let X be a topological space. We define the Betti stack X_B of X as the stack associated with the constant simplicial presheaf

$$S \longmapsto \mathtt{Sing}(X) \qquad \text{(the simplical set of singular chains on } X\text{)}.$$

[2] We shall rather call them pseudo-functors, as the target category happens to be a bicategory.

[3] This can be made very explicit using model categories. The standard model structure on $\mathtt{sSets}$ provides a model structure on simplicial presheaves, in which fibrant objects are those ones satisfying condition (a). One can perform a left Bousfield localization (see [12]) of this model structure w.r.t. Čech nerves of étale covers. Fibrant objects for the new model structure are those satisfying (a) and (b) (i.e. are stacks). The stackification functor is "simply" the left derived functor of the identity.

Derived stacks

The definition of a derived stack is in a sense a combination of the ones of a derived scheme and a stack. More precisely, derived (∞-)stacks are functors $\mathcal{X} : \mathtt{cdga}_{\leq 0} \longrightarrow \mathtt{sSets}$ satisfying properties (a), (b) and

(o) it sends quasi-isomorphisms of $\mathtt{dg}_{\leq 0}$-algebras to weak equivalences of simplicial sets.

We denote by $\mathbf{dSt}$ the $(\infty, 1)$-category of derived stacks.

Example 1.11 (Derived mapping stacks)**.** Let X, Y be derived ∞-stacks. There is a derived ∞-stack $\mathbf{Map}(X, Y)$ defined as follows: for any derived affine scheme S we have

$$\mathbf{Map}(X, Y)(S) := \mathbf{Hom}_{\mathbf{dSt}}(X \times S, Y) .$$

It is called the *derived mapping stack* from X to Y.

A (derived) stack is called n-*Artin* if it can be obtained as the quotient of a smooth groupoid in (derived) $(n-1)$-Artin stacks, with 0-Artin stacks being (derived) schemes (see [22, §2.5] for a more precise definition). A (derived) stack is called *Artin* if it is n-Artin for some n.

Note that any genuine ∞-stack X can be viewed as a derived stack in a canonical way and thus there is a derived mapping stack $\mathbf{Map}(X, Y)$ for any derived stack Y. In particular, if $X = M_B$ is the Betti stack of a finite homotopy type, or is a proper smooth algebraic variety, then one can show that the derived mapping stack $\mathbf{Map}(X, Y)$ is Artin whenever Y is (see [23]).

Remark 1.12. Note that if X and Y are genuine ∞-stacks, there is also a genuine ∞-stack $\mathrm{Map}(X, Y)$ defined in the obvious way. But $\mathbf{Map}(X, Y)$ is NOT the derived stack associated with $\mathrm{Map}(X, Y)$. For instance, if $X = \Sigma_B$ is the Betti stack of a compact surface Σ and $Y = BG$ for a reductive group G then

- the cotangent complex (defined below) of $\mathrm{Map}(X, Y) = \mathrm{Loc}_G(\Sigma)$ sits in degrees 0 and 1.
- while the cotangent complex of $\mathbf{Map}(X, Y) = \mathbf{Loc}_G(\Sigma)$ sits in degrees -1 to 1.

It has the following important consequence for our purposes: the derived stack $\mathbf{Loc}_G(\Sigma)$ of G-local systems on Σ admits a symplectic structure which induces a symplectic structure only on the smooth locus of the underived stack $\mathrm{Loc}_G(\Sigma)$.

1.4 A first excursion into TFTs

We denote by $\mathbf{Cob}_n$ the category with objects being closed $(n-1)$-manifolds and morphisms being diffeomorphism classes of n-cobordisms. Composition is provided by gluing along the boundary, and is a well-defined operation as morphisms are diffeomorphisms classes (composition happens to be associative for the very same reason). The disjoint union $\coprod$ provides $\mathbf{Cob}_n$ with the structure of a symmetric monoidal category.

Definition 1.13. A *topological field theory (TFT)* is a symmetric monoidal functor from $\mathbf{Cob}_n$ to another symmetric monoidal category.

We denote by $\mathbf{Corr}(\mathbf{dSt})$ the category with objects being derived stacks and morphisms being weak equivalence classes of correspondences: $\mathrm{Hom}_{\mathbf{Corr}(\mathbf{dSt})}(X, Y) := \{Z \to X \times Y\}/\sim$. Composition of $V \to X \times Y$ with $W \to Y \times Z$ is defined as $V \overset{\mathbf{R}}{\underset{Y}{\times}} W \to X \times Z$, which is well-defined up to weak equivalences. $\mathbf{Corr}(\mathbf{dSt})$ is symmetric monoidal, with monoidal product the product of derived stacks $\times$.

For any stack X we have a symmetric monoidal functor $\mathcal{Z}_X : \mathbf{Cob}_n \longrightarrow \mathbf{Corr}(\mathbf{dSt})$ which is defined as $\mathcal{Z}_X := \mathbf{Map}\big((-)_B, X\big)$. In other words, one can associate a TFT to every object of $\mathbf{Corr}(\mathbf{dSt})$.

Digression 1. Observe that $\mathbf{Map}\big((*)_B, X\big) = X$. This is a strong evidence that $\mathcal{Z}_X$ can be actually upgraded to a fully extended TFT in the sense of Lurie [15]. This is indeed the case: one can construct an (∞, n)-category of iterated correspondences in $\mathbf{dSt}$, denoted $\mathbf{Corr}_n(\mathbf{dSt})$, and $\mathbf{Map}\big((-)_B, X\big)$ defines a symmetric monoidal (∞, n)-functor $\mathbf{Bord}_n \longrightarrow \mathbf{Corr}_n(\mathbf{dSt})$. Recall from [15] that $\mathbf{Bord}_n$ can be

informally described in the following way: objects are compact 0-dimensional manifolds, 1-morphisms are $1d$ cobordisms between these, 2-morphisms are $2d$ cobordisms between $1d$ cobordisms, $\dots$, n-morphisms are nd cobordisms between $(n-1)d$ cobordisms, $(n+1)$-morphisms are diffeomorphisms of these, $(n+2)$-morphisms are homotopies between these, $\dots$ $\mathbf{Corr}_n(\mathbf{dSt})$ admits a similar informal description: objects are stacks, 1-morphisms are correspondences, 2-morphisms are correspondences of correspondences (2-correspondences), $\dots$, n-morphisms are n-correspondences, $(n+1)$-morphisms are weak equivalences between n-correspondences, $(n+2)$-morphisms are homotopies between these, $\dots$ We refer to [7] for more details.
Note that this agrees with the philosophy of the cobordism hypothesis [3, 15], which says that fully extended TFTs are completely determined by their value on the point.

1.5 Quasi-coherent sheaves

With a $dg_{\leq 0}$-algebra A we associate two symmetric monoidal $(\infty, 1)$-categories[4]

$$\mathbf{Perf}\big(\mathsf{Spec}(A)\big) \subset \mathbf{QCoh}\big(\mathsf{Spec}(A)\big) := A\text{-}\mathbf{mod}\,,$$

where $\mathsf{Perf}\big(\mathsf{Spec}(A)\big)$ is the full subcategory of perfect (or dualizable) A-modules. This is a stack (in the sense that it satisfies étale descent) and thus the functor of points approach automatically provides us with an $(\infty, 1)$-category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves for any derived stack X,

$$\mathbf{QCoh}(X) := \underset{\mathsf{Spec}(A)\to X}{\mathrm{holim}} (A\text{-}\mathbf{mod}) \;.$$

together with a full subcategory

$$\mathbf{Perf}(X) := \underset{\mathsf{Spec}(A)\to X}{\mathrm{holim}} \Big(\mathbf{Perf}\big(\mathsf{Spec}(A)\big)\Big) \;.$$

Observe that there is a forgetful functor $A\text{-}\mathbf{mod} \longrightarrow \mathbf{k}\text{-}\mathbf{mod}$ for every A, so that we get a functor $\mathbf{QCoh}(X) \longrightarrow \mathbf{k}\text{-}\mathbf{mod}$, that we denote $\mathbf{R}\Gamma$. In the case when X is a genuine scheme $\mathbf{R}\Gamma$ coincides, on associated homotopy categories, with the usual derived global sections functor.

Example 1.14. If X is presented by a simplicial scheme $X_\bullet$, meaning that $X = \mathrm{hocolim}(X_j)$, then $\mathbf{QCoh}(X) = \mathrm{holim}\big(\mathbf{QCoh}(X_j)\big)$. In concrete terms, a quasi-coherent sheaf on X is a cosimplicial quasi-coherent sheaf $\mathcal{E}^\bullet$ on $X_\bullet$: for every n we have a quasi-coherent sheaf $\mathcal{E}^n$ on X_n and for every $f : [n] \to [m]$ we have a morphism $\mathcal{E}^f : X_f^*\mathcal{E}^n \to \mathcal{E}^m$ in $\mathbf{QCoh}(X_m)$ such that $\mathcal{E}^{f\circ g} = X_g^*(\mathcal{E}^f) \circ \mathcal{E}^g$.

Example 1.15 (Classifying stacks)**.** Let G be an affine algebraic group and $X = BG := \big[*/G\big]$, which can be presented by the simplicial scheme given by the nerve of G:

$$BG = \mathrm{hocolim}\,(* \leftleftarrows G \cdots) \;.$$

There is a functor $\mathsf{equiv} : G\text{-}\mathbf{mod} \longrightarrow \mathbf{QCoh}(BG)$ which sends a complex of G-modules V to the cosimplicial quasi-coherent sheaf $\mathcal{V}^\bullet := \mathcal{O}_{G^\bullet} \otimes V$ (the cosimplicial structure comes from the nerve of the action of G on V). One can prove that:

- equiv is an equivalence of symmetric monoidal dg-categories.
- $\mathbf{R}\Gamma \circ \mathsf{equiv}$ coincides with the functor $C(G, -)$ of G-cochains (which is the right derived functor of invariants).

Actually, if $V = \mathrm{holim}(V^j)$ for a cosimplicial object $(V^j)_j$ of $G\text{-}\mathbf{mod}$ then $\mathsf{equiv}(V) = \mathrm{holim}\big(\mathsf{equiv}(V^j)\big)$ can be described as follows:

$$\mathsf{equiv}(V)^\bullet = \mathcal{O}_{G^\bullet} \otimes V^\bullet \,.$$

Example 1.16 (Quotient stacks)**.** Let G be an affine algebraic group acting on a smooth algebraic variety X, and let $Y := \big[X/G\big]$. In a way similar to Example 1.15 one can prove that $\mathsf{QCoh}(Y)$ is equivalent to the category of complexes of G-equivariant quasi-coherent sheaves of X.

[4]They are actually symmetric monoidal dg-categories (meaning that they are enriched categories over cochain complexes).

Digression 2. There is an $(\infty, n+1)$-category $\widetilde{\mathbf{Corr}_n}(\mathbf{dSt})$ which is very similar to $\mathbf{Corr}_n(\mathbf{dSt})$ apart from the fact that $(n+1)$-morphisms are morphisms between n-correspondences (rather than just weak equivalences between those). The functor $\mathbf{QCoh}$ is actually an $(\infty, 2)$-functor

$$\widetilde{\mathbf{Corr}_1}(\mathbf{dSt}) \longrightarrow (\infty, 1)\text{-}\mathbf{Cat}\,.$$

Very roughly: derived stacks are sent to their $(\infty, 1)$-category of quasi-coherent sheaves, correspondences are sent to functors, and morphisms of correspondences to natural transformations.

Under some reasonable assumptions on a derived stack X it seems that $\mathbf{QCoh}_{\mathbf{Map}\left((-)_B, X\right)}$ gives rise to a TFT (see [5]).

2 Lecture 2: shifted symplectic structures

Most of the material in this lecture is a simplified reformulation of [20] (see also [6]).

2.1 The cotangent complex

Let A be a $\mathsf{dg}_{\leq 0}$-algebra. One can construct the A-module of differentials Ω^1_A. Recall that

$$\Omega^1_A := \ker(m)/\ker(m)^2\,, \qquad \text{where } m : A^{\otimes 2} \to A \text{ is the product.}$$

We derive this construction to get the cotangent complex:

$$\mathbb{L}_A := A \otimes_{\widetilde{A}} \Omega^1_{\widetilde{A}} \in A\text{-}\mathbf{mod}\,,$$

where $\widetilde{A}$ is a smooth resolution of A under $\mathbf{k}$. Using the functor of points approach one automatically gets an object $\mathbb{L}_X \in \mathsf{QCoh}(X)$ for any derived stack X.

Example 2.1 (Classifying stacks)**.** Let G be an affine algebraic group and $X = BG := \big[*/G\big]$, which can be presented by the simplicial scheme given by the nerve of G. Then we have seen in Example 1.15 that $\mathbf{QCoh}(X) \cong G\text{-}\mathbf{mod}$, and one can check that $\mathbb{L}_{BG}$ is just $\mathbb{L}_{G^\bullet} = \Omega^1_{G^\bullet} = \mathcal{O}_{G^\bullet} \otimes (\mathfrak{g}^*)^{\oplus \bullet}$. Hence one has that $\mathbb{L}_{BG} = \mathsf{equiv}(\mathfrak{g}^*[-1])$, where $\mathfrak{g}^*[-1] = B(\mathfrak{g}^*) = \mathsf{holim}\big((\mathfrak{g}^*)^{\oplus j}\big)$. From now we will abuse a bit the notation and write that $\mathbb{L}_{BG} \cong \mathfrak{g}^*[-1]$.

Remark 2.2. Let us provide a heuristic evidence for the fact that $\mathbb{L}_{BG} = \mathfrak{g}^*[-1]$. We rather explain that $\mathbb{T}_{BG} = \mathfrak{g}[1]$, where $\mathbb{T}_{BG} := (\mathbb{L}_{BG})^*$. Let $f : X \to BG$ be a point. Since BG classifies G-bundles then f is nothing but a G-bundle $\mathcal{P}$ on X. If $\mathcal{P}$ belongs to the smooth locus of the moduli of G-bundles on X, then recall that we have

$$T_{[\mathcal{P}]}\big(\mathsf{Bun}_G(X)\big) = H^1\big(X, \mathcal{P} \underset{G}{\times} \mathfrak{g}\big)\,.$$

A derived generalization of this statement is as follows:

$$\mathbb{T}_{[\mathcal{P}]}\big(\mathbf{Bun}_G(X)\big) = \mathbf{R}\Gamma\big(\mathcal{P} \underset{G}{\times} \mathfrak{g}[1]\big)\,.$$

Finally observe the following general nonsense:

$$\mathbb{T}_{[\mathcal{P}]}\big(\mathbf{Bun}_G(X)\big) = \mathbb{T}_f\big(\mathbf{Map}(X, BG)\big) = \mathbf{R}\Gamma\big(f^*\mathbb{T}_{BG}\big) = \mathbf{R}\Gamma\big(\mathcal{P} \underset{G}{\times} \mathbb{T}_{BG}\big)\,.$$

Hence we get that $\mathbb{T}_{BG} = \mathfrak{g}[1]$.

Example 2.3 (Quotient stacks)**.** Let G be an affine algebraic group acting on a smooth algebraic variety X, and let $Y := \big[X/G\big]$. In a way similar to Example 2.1 one can show that the cotangent complex $\mathbb{L}_Y$ of Y, viewed as a complex of G-equivariant sheaves on X, is

$$\cdots \longrightarrow 0 \longrightarrow \overset{0}{\Omega^1_X} \longrightarrow \overset{1}{\mathcal{O}_X \otimes \mathfrak{g}^*} \longrightarrow 0 \longrightarrow \cdots\,,$$

where the middle map is the transpose of the infinitesimal action $\mathfrak{g} \longrightarrow T_X$.

2.2 Forms

Let A be a $\mathsf{dg}_{\leq 0}$-algebra. One can construct the A-module of ℓ-forms

$$\Omega^\ell_A := \mathsf{Sym}^\ell_A\big(\Omega^1_A[-1]\big)[\ell]\,.$$

We derive this construction and get

$$\mathcal{A}^\ell(A) := \Omega^\ell_{\widetilde{A}}$$

that we only consider as a complex of $\mathbf{k}$-modules. Using the functor of points approach one automatically gets a complex $\mathcal{A}^\ell(X)$ for any derived stack X:

$$\mathcal{A}^\ell(X) := \underset{\mathsf{Spec}(A)\to X}{\mathsf{holim}}\big(\mathcal{A}^\ell(A)\big) = \mathbf{R}\Gamma\left(S^\ell_{\mathcal{O}_X}(\mathbb{L}_X[-1])[\ell]\right)\,.$$

We define *the space* $\mathcal{A}^\ell(X, n)$ *of* ℓ*-forms of degree* n as the space $\mathbf{Hom}_{\mathbf{Cpx}}\big(\mathbf{k}, \mathcal{A}^\ell(X)[n]\big)$ of n-cocycles in $\mathcal{A}^\ell(X)$.

Example 2.4 (Classifying stacks)**.** Let G be an affine algebraic group. Using that there is a natural transformation from G-invariants to derived G-invariants, and borrowing the notation from Examples 1.15 and 2.1, we get that there is a morphism of complexes

$$\left(S^{\ell}(\mathfrak{g}^*)[-\ell]\right)^G \longrightarrow \mathcal{A}^{\ell}(BG)$$

(note that the differential on the left-most complex is trivial as it is concentrated in one degree). In particular, any invariant symmetric bilinear form on $\mathfrak{g}$ defines a 2-form of degree 2 on the classifying stack BG.

Example 2.5 (Quotient stacks)**.** Let G be an affine algebraic group acting on a smooth algebraic variety X, and let $Y := [X/G]$. One can show that there is a morphism of complexes

$$\left(\bigoplus_{p+q=\ell} \Omega^p(X) \otimes_k S^q(\mathfrak{g}^*)[-q]\right)^G \longrightarrow \mathcal{A}^{\ell}(Y)\,,$$

where the differential on the left-most complex can be described as follows: an element α can be viewed as G-equivariant function on $\mathfrak{g}$ with values in forms on X, and for any $x \in \mathfrak{g}$ we have $(d\alpha)(x) = -\iota_{\vec{x}}(\alpha)(x)$.

2.3 Closed forms

Let A be a $dg_{\leq 0}$-algebra. One can construct its *de Rham algebra* $DR(A)$:

$$DR(A)^{\sharp} = \prod_{n\geq 0} \Omega^n_{A^{\sharp}}[-n]$$

is equipped with the differential $d_A + d_{dR}$, where d_A is the original differential one each Ω^n_A and d_{dR} naturally extends the $A \to \Omega^1_A$ (which we view as a degree one derivation on A with values in $\Omega^1_A[-1]$). It is a complete filtered dg-algebra, with k-th filtration piece defined by

$$F^k DR(A)^{\sharp} = \prod_{n\geq k} \Omega^n_{A^{\sharp}}[-n]\,.$$

We also introduce the ℓ-th truncated de Rham complex

$$\Omega^{\ell,cl}_A := F^{\ell} DR(A)[\ell]\,.$$

Deriving this construction one gets the complex of closed ℓ-forms

$$\mathcal{A}^{\ell,cl}(A) := \Omega^{\ell,cl}_{\widetilde{A}} \in \mathbf{Cpx}\,.$$

One can prove that the functor $\mathcal{A}^{\ell,cl}(-)$ is actually a derived stack (i.e. it satisfies étale descent, which is a non-trivial result from [20]), and hence one can apply the functor of points approach to automatically get a complex $\mathcal{A}^{\ell,cl}(X)$ for any derived stack X:

$$\mathcal{A}^{\ell,cl}(X) := \operatorname*{holim}_{Spec(A)\to X} \left(\mathcal{A}^{\ell,cl}(A)\right)\,.$$

We define *the space* $\mathcal{A}^{\ell,cl}(X,n)$ *of closed* ℓ*-forms of degree* n as the space $\mathbf{Hom}_{\mathbf{Cpx}}\left(k, \mathcal{A}^{\ell,cl}(X)[n]\right)$ of n-cocycles in $\mathcal{A}^{\ell,cl}(X)$.

In very concrete terms, a closed ℓ-form of degree n on $Spec(A)$ is a sequence $\alpha = (\alpha_i)_{i\geq 0}$ such that

- α_i is a degree $n-i$ element in $\Omega^{\ell+i}_{\widetilde{A}^{\sharp}}$.
- $d_{dR}(\alpha_i) = d_A(\alpha_{i+1})$.

In particular α_0 is an n-cocycle in $\Omega^{\ell}_{\widetilde{A}}$, that is an ℓ-form of degree n which we call the *underlying* ℓ*-form* of α. Observe that this is functorial in A, so that we get an *"underlying form"* morphism of stacks $\mathcal{A}^{\ell,cl} \longrightarrow \mathcal{A}^{\ell}$.

Example 2.6 (Classifying stacks)**.** Let G be an affine algebraic group. There is a morphism of complexes

$$\left(\bigoplus_{i\geq 0} S^{\ell+i}\big(\mathfrak{g}^*\big)[-\ell-2i]\right)^G \longrightarrow \mathcal{A}^{\ell,cl}(BG)$$

(observe that the differential on the left-most complex is trivial as it sits in even degrees). Then any invariant symmetric bilinear form on $\mathfrak{g}$ defines a closed 2-form of degree 2 on the classifying stack BG.

Example 2.7 (Quotient stacks)**.** Let G be an affine algebraic group acting on a smooth algebraic variety X, and let $Y := \big[X/G\big]$. One can show that there is a morphism of complexes

$$\left(\bigoplus_{p+q\geq \ell} \Omega^p(X)\otimes_k S^q\big(\mathfrak{g}^*\big)[\ell-p-2q]\right)^G \longrightarrow \mathcal{A}^{\ell,cl}(Y)\,,$$

where the differential on the left-most complex can be described as follows, borrowing the notation from Example 2.5: the differential of an element α is $d_{dR}\big(\alpha(x)\big)+(d\alpha)(x)$.

2.4 n-symplectic structures

Let X be a derived Artin stack which is locally of finite presentation[5]. These technical assumptions are sufficient to get that $\mathbb{L}_X$ is a dualizable object in $\mathbf{QCoh}(X)$, so that we have $\mathbb{T}_X := (\mathbb{L}_X)^*$.

Definition 2.8. An n-symplectic structure on X is a point in $\mathcal{A}^{2,cl}(X,n)$ such that the underlying element of $\mathcal{A}^2(X,n)$ provides a weak equivalence $\mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[n]$.

Example 2.9 (Genuine symplectic varieties)**.** Let X be a genuine smooth algebraic variety. Then $\mathbb{L}_X \cong \Omega^1_X$ and thus a 0-symplectic structure is simply a genuine symplectic structure on X.

Example 2.10 (Shifted cotangent stacks)**.** The n-shifted cotangent stack $T^*[n]X$ of X is n-symplectic (see [20, Proposition 1.21]). In particular we get that, for any affine algebraic group G, $\big[\mathfrak{g}^*/G\big] = T^*[1]\big[*/G\big]$ is 1-symplectic. We now describe, for later purposes, this 1-symplectic structure in a very explicit way. Recall that, as complexes of G-equivariant sheaves on $\mathfrak{g}^*$

$$\mathbb{L}_{\big[\mathfrak{g}^*/G\big]} = \cdots \longrightarrow 0 \longrightarrow \underbrace{\overset{0}{\mathcal{O}_{\mathfrak{g}^*}\otimes\mathfrak{g}}}_{=\Omega^1_{\mathfrak{g}^*}} \longrightarrow \overset{1}{\mathcal{O}_{\mathfrak{g}^*}\otimes\mathfrak{g}^*} \longrightarrow 0 \longrightarrow \cdots\,.$$

Pick coordinates $(x_i)_i$ on $\mathfrak{g}^*$, denote by $(\xi^i)_i$ the dual basis of $\mathfrak{g}^*$, and define $\omega_0 := (d_{dR}x_i)\xi^i$. It defines a 2-form of degree 1 as ξ's have degree 1 and, borrowing the notation from Example 2.5, $d(\omega)(x) = -\iota_x(\omega_0)(x) = [x,x]_{\mathfrak{g}} = 0$ (viewd as a function on $\mathfrak{g}^*$). Moreover, it is trivially closed as $d_{dR}(\omega) = 0$ (because in fact, $\omega_0 = d_{dR}(x_i\xi^i)$). It is non-degenerate because that the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is (namely, $\mathfrak{g}$ is finite dimensional).

Example 2.11 (Classifying stack of a reductive group)**.** We now assume that G is reductive. In particular there exists a non-degenerate invariant pairing $c\in S^2(\mathfrak{g}^*)^G$. We already know (see Example 2.6) that c defines a closed 2-form of degree 2. It is easy to check that the non-degeneracy of the underlying 2-form is equivalent to the non-degeneracy of c.

2.5 Lagrangian structures

Let (X,ω) be an n-symplectic stack and L a derived Artin stack of finite presentation.

[5] A derived stack X is *locally of finite presentation* if there exists a smooth epimorphism $U\to X$ with U being a disjoint union of derived affine schemes which are finitely presented. An affine scheme $\mathrm{Spec}(A)$ is *finitely presented* if $\mathbf{Hom}_{cdga_{\leq 0}}(A,-)$ commutes with filtered homotopy colimits.

Definition 2.12. A Lagrangian structure on a morphism $f : L \longrightarrow X$ is a non-degenerate isotropic structure on it. An isotropic structure on f is a homotopy from 0 to $f^*\omega$; it is called non-degenerate if the induced map $f^*\mathbb{T}_X \longrightarrow (\mathbb{T}_X)^\circ := \mathtt{hofib}\big(f^*\mathbb{T}_X \to \mathbb{L}_L[n]\big)$ is a weak equivalence[6].

Example 2.13 (Genuine Lagrangian subvarieties)**.** Let L, X be genuine smooth algebraic varieties and let $f : L \hookrightarrow X$ be a closed embedding. We have seen that 0-symplectic structures on X are genuine symplectic structures. One can actually show that the space of 0-symplectic structures on X is the discrete space of genuine symplectic forms on X. Hence, given a symplectic form ω on X, f admits a Lagrangian structure if and only if $f^*\omega$ **equals** 0, i.e. if and only if L is Lagrangian in the usual sense.

Example 2.14 (Moment maps)**.** Let (X, ω) be a genuine smooth symplectic variety. Assume that there is an affine algebraic group G acting by symplectomorphisms on X and that there is a moment map $\mu : X \longrightarrow \mathfrak{g}^*$ for the action of G. Then we have

$$(d_{dR} + d)(\omega) = d(\omega) = -\sum_i \iota_{\overrightarrow{x_i}}(\omega)\xi^i = \mu^*(d_{dR}x_i)\xi^i = [\mu]^*\big((d_{dR}x_i)\xi^i\big)\,,$$

where $[\mu] : \big[X/G\big] \longrightarrow \big[\mathfrak{g}^*/G\big]$ is the quotient of μ (note that the above computation has been made in the subcomplex of the complex of closed 2-forms on $\big[X/G\big]$ appearing in Example 2.7).

Lie group valued moment maps (see [1]) can also be understood in terms of Lagrangian morphisms; see [6, 21] and below.

Exercise 2.15 (Symplectic is Lagrangian)*.* Denote by $*_{(n)}$ the point equipped with its canonical n-symplectic structure (this is just 0). Show that a Lagrangian structure on $X \to *_{(n)}$ is *exactly the same* as an $(n-1)$-symplectic structure on X.

[6]The *homotopy fiber* $\mathtt{hofib}(f)$ of a morphism $f : E \to F$ in a dg-category is another name for the mapping cocone of f:

$$\mathtt{hofib}(f) = \mathtt{holim}\left(E \overset{f}{\underset{0}{\rightrightarrows}} F\right)\,.$$

3 Lecture 3: (semi-)classical TFTs

3.1 Lagrangian correspondences

Let X, Y, Z be n-symplectic stacks, and let $f : L \to X \times \overline{Y}$ and $g : M \to Y \times \overline{Z}$ be Lagrangian maps. note that $\overline{Z}$ means that we consider Z equipped with the opposite n-symplectic structure: $\omega_{\overline{Z}} := -\omega_Z$.

Theorem 3.1. *There exists a natural Lagrangian structure on* $L \overset{\mathbf{R}}{\underset{Y}{\times}} M \longrightarrow X \times \overline{Z}$.

Sketch of proof. Since f is Lagrangian then we have a homotopy from $f^*\pi_X^*\omega_X$ to $f^*\pi_Y^*\omega_Y$ in $\mathcal{A}^{2,cl}(L, n)$. Since g is Lagrangian then we have a homotopy from $g^*\pi_Y^*\omega_Y$ to $g^*\pi_Z^*\omega_Z$ in $\mathcal{A}^{2,cl}(M, n)$. Moreover, we have a homotopy from $\pi_L^* f^*\pi_Y^*\omega_Y$ to $\pi_M^* g^*\pi_Y^*\omega_Y$ in $\mathcal{A}^{2,cl}\Big(L \overset{\mathbf{R}}{\underset{Y}{\times}} M, n\Big)$. Hence there is a homotopy from $\pi_L^* f^*\pi_X^*\omega_X$ to $\pi_M^* g^*\pi_Z^*\omega_Z$ in $\mathcal{A}^{2,cl}\Big(L \overset{\mathbf{R}}{\underset{Y}{\times}} M, n\Big)$, which defines an isotropic structure on the map $L \overset{\mathbf{R}}{\underset{Y}{\times}} M \longrightarrow X \times \overline{Z}$. We refer to [6, Theorem 4.4] for the proof that it is non-degenerate. $\square$

Example 3.2 (Derived Lagrangian intersections)**.** Let $X = Z = *_{(n)}$. Hence we have Lagrangian maps $L \to Y$ and $M \to Y$. The above result tells us that there is a natural Lagrangian structure on $L \underset{Y}{\times} M \to *_{(n)}$, and thus there is an $(n-1)$-symplectic structure on $L \underset{Y}{\times} M$ (recall Exercise 2.15).

We now explain two interesting applications of the above example.

Example 3.3 (Derived critical locus)**.** We let $Y = T^*N$ be the cotangent space of a genuine smooth scheme N, we set $L = M = N$, and we let f, resp. g, be the inclusion of N into T^*N as the zero section, resp. as the inclusion of the graph of dS for a given function $S : X \to \mathbb{A}^1$. L and M are thus genuine Lagrangian subvarieties inside T^*N. Their derived intersection is the derived critical locus of Example 1.4 $\mathbf{R}\mathrm{Crit}(S)$ of the function S. By the above $\mathbf{R}\mathrm{Crit}(S)$ is hence (-1)-symplectic: borrowing the notation of Example 1.4 the (-1)-symplectic structure is $d_{dR}x^i d_{dR}\xi_i$ (it is trivially closed). We refer to [25] for more details.

Example 3.4 (Derived symplectic reduction)**.** Let (X, ω) be a genuine smooth symplectic scheme, equipped with an action of a reductive group G that preserves ω. Recall from Example 2.14 that any moment map $\mu : X \to \mathfrak{g}^*$ induces a Lagrangian structure on the map $[\mu] : [X/G] \to [\mathfrak{g}^*/G]$ between the corresponding quotient stacks. For any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ we in particular have a Lagrangian structure on $[\mathcal{O}/G] \to [\mathfrak{g}^*/G]$. Therefore the derived fiber product

$$[X/G] \overset{\mathbf{R}}{\underset{[\mathfrak{g}^*/G]}{\times}} [\mathcal{O}/G] = \Big[X \overset{\mathbf{R}}{\underset{\mathfrak{g}^*}{\times}} \mathcal{O}/G\Big] = \big[\mathbf{R}\mu^{-1}(\mathcal{O})/G\big] =: [X/\!/_{\mathcal{O}} G]$$

is 0-symplectic. If G acts nicely and $\mathcal{O}$ is the coadjoint orbit of a regular value of μ, then the derived reduced stack coincides with the usual reduced variety $X_{red} := \mu^{-1}(\mathcal{O})/G$ and we recover that it inherits a symplectic structure from the one of X.

Quasi-Hamiltonian reduction (see [1]) can also be understood in a similar way; see [6, 21] and below.

Given n-symplectic stacks X, Y, Z, W and Lagrangian morphisms $L \to X \times \overline{Y}$, $M \to Y \times \overline{Z}$ and $N \to Z \times \overline{W}$, then it follows from Theorem 3.1 that the two vertical arrows in

$$\begin{tikzcd}[column sep=small]
(L \overset{\mathbf{R}}{\underset{Y}{\times}} M) \overset{\mathbf{R}}{\underset{Z}{\times}} N \arrow[rr, dash, "\sim"] \arrow[dr] & & L \overset{\mathbf{R}}{\underset{Y}{\times}} (M \overset{\mathbf{R}}{\underset{Z}{\times}} N) \arrow[dl] \\
& X \times \overline{W} &
\end{tikzcd}$$

carry Lagrangian structures. Going through the proof of the Theorem, one can easily check that these Lagrangian structures are identified *via* the horizontal equivalence.

Hence we have a genuine category $\mathbf{Symp}_n$ having n-symplectic derived stacks as objects and weak equivalence classes of Lagrangian correspondences as morphisms. Theorem 3.1 tells us that we can compose them, and the above discussion shows that the composition is associative.

Example 3.5 (Moore-Tachikawa's category of Hamiltonian correspondences)**.** Let us consider the full subcategory $\mathbf{MT}$ of $\mathbf{Symp}_1$ defined as follows:

- objects of $\mathbf{MT}$ are the 1-symplectic derived stacks of the form $\left[\mathfrak{g}^*/G\right]$, where G is an affine algebraic group (see Example 2.10). Hence we can simply say that that objects of $\mathbf{MT}$ are affine algebraic groups.
- morphisms from G_1 to G_2 are equivalence classes 0-symplectic derived schemes X together with a Hamiltonian action of $G_1 \times G_2$ and a moment map $X \to \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$. Note that, after Example 2.14, these are nothing but a Lagrangian structure on $\left[X/G_1 \times G_2\right] \longrightarrow \left[\mathfrak{g}_1^* \oplus \mathfrak{g}_2^*/G_1 \times G_2\right]$.

Given a morphism X from G_1 to G_2 and a morphism Y from G_2 to G_3, recall that their composition is obtained by performing the derived intersection

$$\left[X/G_1 \times G_2\right] \underset{\left[\mathfrak{g}_2^*/G_2\right]}{\overset{\mathbf{R}}{\times}} \left[Y/G_2 \times G_3\right] = \Big[\left[X \times Y/\!/_0 G_2\right]/G_1 \times G_3\Big],$$

where $\left[X \times Y/\!/_0 G_2\right]$ is the derived reduction of $X \times Y$ with respect to the difference of the moment maps $X \to \mathfrak{g}_2^*$ and $Y \to \mathfrak{g}_2^*$, borrowing the notation from Example 3.4. Apart from an additional $\mathbb{C}^*$-action that we ignore here on purpose, this gives a derived algebro-geometric approach to Moore-Tachikawa's category of "holomorphic symplectic varieties with Hamiltonian action" [18, §3.1].

Digression 3. One can construct (∞, n)-categories of iterated m-Lagrangian correspondences, denoted $\mathbf{Lag}_n^m$, which we roughly describe now. Given an m-symplectic stack X we can define an $(\infty, 0)$-category $\mathbf{Lag}_0(X)$ of maps $Y \to X$ equipped with a Lagrangian structure. Assuming we have been able to construct an (∞, n)-category $\mathbf{Lag}_n(X)$ for any m-symplectic stack X (and any m), we then define $(\infty, n+1)$-categories $\mathbf{Lag}_{n+1}(X)$ with objects being morphisms $f : Y \to X$ equipped with a Lagrangian structure and having $\mathbf{Lag}_n(Y_1 \times_X^h Y_2)$ as (∞, n)-category of morphisms from $Y_1 \to X$ to $Y_2 \to X$. Then we define $\mathbf{Lag}_n^m := \mathbf{Lag}_n(*_{(m+1)})$. Note that Symp_m is the homotopy category of $\mathbf{Lag}_1^m$. Finally observe that Lag_n^m carries a symmetric monoidal structure (given by $\times$).

There is an obvious forgetful functor $\mathbf{Lag}_n^m \longrightarrow \mathbf{Corr}_n$. Let X be a derived stack and consider the fully extended TFT

$$\mathcal{Z}_X := \mathbf{Map}\big((-)_B, X\big) : \mathbf{Bord}_n \longrightarrow \mathbf{Corr}_n$$

introduced in Digression 1. A lift of $\mathcal{Z}_X$ to an oriented fully extended TFT

$$\mathbf{Bord}_n^{or} \longrightarrow \mathbf{Lag}_n^m$$

shall be completely determined, according to the cobordism hypothesis [3, 15], by its value on the point and thus by the corresponding m-shifted symplectic structure on X. Conversely, one can show that any m-shifted symplectic structure on X determines such a lift.
Below we provide an explicit description of the oriented TFT associated with an m-symplectic derived stack.

3.2 Transgression: AKSZ-PTVV construction

Let Σ be a derived stack. There is a notion called $\mathcal{O}$-*compactness* for Σ (see [20, Definition 2.1]) that allows to get the following gadget: for any other derived stack X there exists a map

$$\mathcal{A}^{p(,cl)}(\Sigma \times X) \to \mathbf{R}\Gamma(\mathcal{O}_\Sigma) \otimes \mathcal{A}^{p(,cl)}(X).$$

Now, an m-*orientation* on an $\mathcal{O}$-compact stack Σ is the data of a map $[\Sigma] : \mathbf{R}\Gamma(\mathcal{O}_\Sigma) \to \mathbf{k}[-m]$ such that for any perfect $\mathcal{O}_\Sigma$-module E the pairing

$$\mathbf{R}\Gamma(E) \otimes \mathbf{R}\Gamma(E^*)[m] \longrightarrow \mathbf{R}\Gamma(\mathcal{O}_\Sigma)[m] \xrightarrow{[\Sigma]} \mathbf{k}$$

is non-degenerate. Hence for any other derived stack X we get a map

$$\int_{[\Sigma]} : \mathcal{A}^{p(,cl)}(\Sigma \times X) \longrightarrow \mathcal{A}^{p(,cl)}(X)[-m]\,.$$

Construction 3.6 (Transgression). Let Σ be an $\mathcal{O}$-compact m-oriented stack and let X be a stack together with a closed p-form α of degree n. Denote by ev the evaluation map $\Sigma \times \mathbf{Map}(\Sigma, X) \to X$. The following defines a closed p-form of degree $n - m$:

$$\int_{[\Sigma]} ev^*\alpha$$

Theorem 3.7 (Theorem 2.5 in [20]). *Keeping the above notation, we let* $p = 2$. *Assuming that* X *and* $\mathbf{Map}(\Sigma, X)$ *are derived Artin stacks locally of finite presentation and that* α *is non-degenerate (i.e.* α *is an* n*-symplectic structure), then* $\int_{[\Sigma]} ev^*\alpha$ *is non-degenerate (i.e. is an* $(n-m)$*-symplectic structure).*

Let us now give three Examples of applications of this Theorem (we refer to [20] for the details).

Example 3.8 (G-local systems). If $\Sigma = M_B$ with M an oriented m-dimensional closed manifold then Σ is $\mathcal{O}$-compact and m-oriented. Moreover $\mathbf{Map}(\Sigma, X)$ is Artin under the assumption that X is. In particular, recalling from Example 2.11 that BG has a 2-symplectic structure whenever G is a reductive group, we get that $\mathbf{Loc}_G(M) = \mathbf{Map}(\Sigma, BG)$ has a $(2-m)$-symplectic structure. Hence if M is an oriented surface we "recover" that the derived moduli stack of G-local systems on M is 0-symplectic (and on its smooth locus we get back an actual symplectic structure).

Remark 3.9. In the above Example, if $M = S^1$ then $\mathbf{Map}(M_B, BG) = [G/G]$ inherits a 1-symplectic structure. One can prove (see [6, 21]) that a Lie group valued moment map $\mu : Y \longrightarrow G$ in the sense of [1] induces a Lagrangian structure on the morphism $[\mu] : [Y/G] \longrightarrow [G/G]$. For any conjugacy class $C \subset G$ we in particular have a Lagrangian structure on $[C/G] \to [G/G]$. Therefore the derived fiber product

$$[Y/G] \overset{\mathbf{R}}{\underset{[G/G]}{\times}} [C/G] = \left[Y \overset{\mathbf{R}}{\underset{G}{\times}} C/G\right] = [\mathbf{R}\mu^{-1}(C)/G] =: [Y/\!/_C G]$$

is 0-symplectic. If G acts nicely and C is the conjugacy class of a regular value of μ, then the derived reduced stack coincides with the usual reduced variety $Y_{red} := \mu^{-1}(C)/G$ and we recover that it inherits a symplectic structure ([1]).

Example 3.10 (G-bundles). If Σ is a smooth Calabi-Yau variety of dimension m then it is $\mathcal{O}$-compact and m-oriented. Moreover $\mathbf{Map}(\Sigma, X)$ is Artin under the assumption that X is. Hence, if G is reductive then we get a $(2-m)$-symplectic structure on $\mathbf{Bun}_G(M) = \mathbf{Map}(\Sigma, BG)$. One can get back that way Mukai's result [19] on the symplecticity of the moduli space of G-bundles on a K3-surface or on an abelian surface.

Example 3.11 (Higgs bundles). If $\Sigma = \mathbb{T}[1]Y$ is the shifted tangent stack of a smooth proper algebraic variety Y of dimension m then Σ is $\mathcal{O}$-compact and $2m$-oriented. Moreover $\mathbf{Map}(\Sigma, X)$ is Artin under the assumption that X is. Therefore, if G is reductive then we get a $(2-2m)$-symplectic structure on $\mathbf{Higgs}_G(Y) = \mathbf{Map}(\Sigma, BG)$. One can get back that way the symplecticity of the moduli space of Higgs bundles on a smooth algebraic curve.

3.3 Boundary structures

Let $\varphi : \Sigma \to \Sigma'$ be a morphism between $\mathcal{O}$-compact stacks, and assume that Σ carries an m-orientation $[\Sigma]$. A *boundary structure* w.r.t. $[\Sigma]$ on φ is a homotopy $[\varphi]$ from 0 to $[\Sigma]$. There is a notion of non-degeneracy for boundary structures (see [6, Definition 2.10] for a precise definition). The data of an orientation on Σ and a non-degenerate boundary structures on φ is also called a *relative orientation* in [6, 22].

Construction 3.12 (Transgression with boundary[7]). Let $\varphi : \Sigma \to \Sigma'$ as above and let X be a stack together with a closed p-form α of degree n. Denote by ev, resp. ev' the evaluation map $\Sigma \times \mathbf{Map}(\Sigma, X) \to X$, resp. $\Sigma' \times \mathbf{Map}(\Sigma', X) \to X$. Recall from Construction3.6 $\int_{[\Sigma]} ev^*\alpha$ defines a closed p-form of degree $n - m$ on $\mathbf{Map}(\Sigma, X)$. Therefore,

$$\mathrm{rest}^* \int_{[\Sigma]} ev^*\alpha = \int_{[\Sigma]} (\varphi \times \mathrm{id})^* ev'^*\alpha = \int_{\varphi_*[\Sigma]} ev'^*\alpha \,,$$

where $\mathrm{rest} := (\cdots \circ \varphi) : \mathbf{Map}(\Sigma', X) \to \mathbf{Map}(\Sigma, X)$, defines a closed p-form of degree $n - m$ on $\mathbf{Map}(\Sigma', X)$. The following defines a homotopy from 0 to $\mathrm{rest}^* \int_{[\Sigma]} ev^*\alpha$ in $\mathcal{A}^{p,cl}\big(\mathbf{Map}(\Sigma', X), n-m\big)$:

$$\int_{[\varphi]} ev'^*\alpha \,.$$

Theorem 3.13 (Theorem 2.11 in [6])**.** *Keeping the above notation, we let $p = 2$. Assuming that X, $\mathbf{Map}(\Sigma, X)$ and $\mathbf{Map}(\Sigma', X)$ are derived Artin stacks locally of finite presentation and that α is non-degenerate (i.e. α is an n-symplectic structure), then $\int_{[\varphi]} ev'^*\alpha$ is non-degenerate (i.e. it is a Lagrangian structure on rest).*

Example 3.14 (G-local systems)**.** We borrow the notation from Example 3.8; if $\Sigma' = N_B$, where N is an oriented $(m+1)$-dimensional compact manifold with $\partial N = M$, then Σ' is $\mathcal{O}$-compact and the map $\iota : \Sigma \to \Sigma'$ given by the inclusion $\partial N \subset N$ carries a non-degenerate boundary structure [6, §3.1]. Moreover $\mathbf{Map}(\Sigma', X)$ is also Artin under the assumption that X is. Hence, if G is reductive then we get that $\mathbf{Loc}_G(N) \to \mathbf{Loc}_G(M)$ has a Lagrangian structure.

Remark 3.15. If one takes $m = 1$ in the above Example, then M is a disjoint union of k circles ($k \in \mathbb{N}$) and the resulting Lagrangian structure on the morphism $\mathbf{Loc}_G(N) \longrightarrow \big[G^k/G^k\big]$ is precisely the one induced by the quasi-Hamiltonian structure[8] on $\mathbf{Loc}_G(N)$. In order to get a symplectic moduli stack on must constraint the monodromy on each boundary circle to lie in fixed conjugacy classes $C_1, \ldots, C_k$, which can be done by performing quasi-Hamiltonian reduction (see Remark 3.9 and [6, 21]).

Example 3.16 (G-bundles)**.** We borrow the notation from Example 3.10; if Σ' is a smooth proper algebraic variety of dimension $m+1$ admitting Σ has a divisor which has anticanonical class, then the map $\iota : \Sigma \to \Sigma'$ carry a non-degenerate boundary structure [6, §3.2]. Hence, if G is reductive then we get that $\mathbf{Bun}_G(\Sigma') \to \mathbf{Bun}_G(\Sigma)$ has a Lagrangian structure.

The oriented (semi-)classical TFT associated with an m-symplectic stack

According to Digression 3 there is an oriented fully extended TFT

$$\mathbf{Bord}_n^{or} \longrightarrow \mathbf{Lag}_n^m$$

associated with every m-symplectic stack (X, ω). In this § we only describe the induced oriented TFT

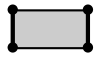
$$\mathcal{Z}_{X,\omega} : \mathbf{Cob}_n^{or} \longrightarrow \mathbf{Symp}_{m-n+1} \,.$$

Claim 3.17. $\mathcal{Z}_{X,\omega} = \mathbf{Map}\big((-)_B, X\big)$

More precisely, it sends an oriented cobordism N with $\partial N = M_+ \coprod M_-^{op}$ to the Lagrangian correspondence $\mathbf{Map}\big((N)_B, X\big) \longrightarrow \mathbf{Map}\big((M_+)_B, X\big) \times \overline{\mathbf{Map}\big((M_-)_B, X\big)}$ given by Theorem 3.13.

[7] Note that this has nothing to do with Sokal's hoax!
[8] This quasi-Hamiltonian structure has been described in [1]

3.4 TFTs with boundary conditions

Note that there is a variation $\mathbf{Cob}_n^{bc}$ on $\mathbf{Cob}_n$ where objects are now manifolds with boundary and morphisms are given by cobordisms with boundary. E.g. the strip  is a self-cobordism (with boundary) of the closed interval .

We claim that any morphism $f : L \to X$ together with an m-symplectic structure ω on X and a Lagrangian structure γ on f gives rise to an oriented TFT with boundary

$$\mathcal{Z} : \mathbf{Cob}_n^{or,bc} \longrightarrow \mathbf{Symp}_{m-n+1} .$$

Without going too much into details, this relies on the following.

Let $\varphi : \Sigma \to \Sigma'$ be a morphism of $\mathcal{O}$-compact derived stacks together with a relative m-orientation $([\Sigma], [\varphi])$. Assume that all mapping stacks involved are derived Artin stacks locally of finite presentation. Then recall (notation shall be clear from the context):

- from Theorem 3.7 we get that $\int_{[\Sigma]} ev_X^* \omega$ is an $(m-n+1)$-symplectic structure on $\mathbf{Map}(\Sigma, X)$.
- from Theorem 3.13 we get that $\int_{[\varphi]} ev_X'^* \omega$ is a Lagrangian structure on $rest$, i.e. a non-degenerate homotopy from 0 to $rest^* \int_{[\Sigma]} ev_X^* \omega = \int_{f_*[\Sigma]} ev_X'^* \omega$.

On the other hand $\int_{[\Sigma]} ev_L^* \gamma$ defines a homotopy 0 to $\int_{[\Sigma]} ev_L^* f^* \omega = f^* \int_{[\Sigma]} ev_X^* \omega$, which can be proven to be non-degenerate. In other words, it defines a Lagrangian structure on $f : \mathbf{Map}(\Sigma, L) \to \mathbf{Map}(\Sigma, X)$.

According to Theorem 3.1 and Example 3.2, composing $\int_{[\varphi]} ev_X'^* \omega$ with $\int_{[\Sigma]} ev_L^* \gamma$ provides us with a non-degenerate self-homotopy η of 0, which can be understood as an $(m-n)$-symplectic structure on

$$\mathbf{Map}(\varphi, f) := \mathbf{Map}(\Sigma', X) \overset{\mathbf{R}}{\underset{\mathbf{Map}(\Sigma, X)}{\times}} \mathbf{Map}(\Sigma, L) .$$

Now observe that we have a map $\iota : \mathbf{Map}(\Sigma', L) \longrightarrow \mathbf{Map}(\varphi, f)$. When pull-backed along this map the self-homotopy η becomes the composition of $\int_{[\varphi]} ev_L'^* \omega$ with $\int_{[\Sigma]} ev_L'^* \gamma$. This composed homotopy is homotopic to the zero one *via* $\int_{[\varphi]} ev_L'^* \gamma$, which can be proven to be non-degenerate. Hence we get a Lagrangian structure on ι (the existence of which is stated in [22, §4.3]).

3.5 Towards a derived description of the Poisson sigma-model

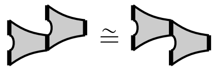
Let (X, π) be a smooth Poisson variety[9] and consider the π-twisted and 1-shifted derived cotangent stack $Y = \mathbb{T}^*[1]_\pi X$ of X. It is affine over X and can be defined as $\mathrm{Spec}\big(\mathrm{Sym}_{\mathcal{O}_X}(T_X[-1]), [\pi, -]_S\big)$, where $[-,-]_S$ is the Schouten bracket on the sheaf $\mathrm{Sym}_{\mathcal{O}_X}(T_X[-1])$ of poly-vector fields.

The derived stack Y, resp. the zero section morphism $X \longrightarrow Y$, can be shown to carry a 1-symplectic structure, resp. a Lagrangian structure[10]. Hence we get a $2d$ oriented TFT with boundary conditions

$$\mathcal{Z}_{(X,\pi)} : \mathbf{Cob}_2^{or,bc} \longrightarrow \mathbf{Symp}_0 ,$$

which sends the object  of $\mathbf{Cob}_2^{or,bc}$ (a 1-dimensional manifold with boundary) to the relative derived mapping stack from $\big(($  $)_B \to ($  $)_B\big)$ to $(X \to Y)$, which is nothing but the derived self-intersection $\mathcal{G} := X \overset{\mathbf{R}}{\times}_Y X$ (and is indeed 0-symplectic).

Note that the cobordism with boundary  then gives us a Lagrangian correspondence between $\mathcal{G} \times \mathcal{G}$ and $\mathcal{G}$. This turns $\mathcal{G}$ into an algebra object in $\mathbf{Symp}_0$. Namely, associativity of composition is given by the following diffeomorphism:



[9] Note that one can make all this work in the differentiable setting as well.

[10] Poisson structures can actually be defined as Lagrangian morphisms in the derived setting ([8]).

In [11] Contreras and Scheimbauer prove that $\mathcal{G}$ is actually a *Calabi-Yau algebra* in Lag_1^0 (in the sense of [15, §4.2]). This in particular provides a very nice interpretation of the rather mysterious axioms of a relational symplectic groupoid introduced in [9].

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