Duflo isomorphism and Caldararu conjecture

1. Duflo isomorphism

$\mathfrak{g}$ finite dimensional Lie algebra.

PBW isomorphism $S(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$ is NOT an algebra isomorphism

even on invariants $S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g})^{\mathfrak{g}}$.

WANT: modify it in order to obtain an algebra isomorphism.

To do so, define $j := \det\left(\sqrt{\frac{e^{ad/2} - e^{-ad/2}}{ad}}\right) = \sum_{k \geq 0} c_k \, tr(ad^k) \in \hat{S}(\mathfrak{g}^*)^{\mathfrak{g}}$

$\left(ad \in \mathfrak{g}^* \otimes End(\mathfrak{g}) \rightsquigarrow tr(ad^k) \in S^k(\mathfrak{g}^*) \right)$.

$S(\mathfrak{g}) = \mathcal{O}_{\mathfrak{g}^*}$ algebra of polynomial functions on $\mathfrak{g}^*$

$\hat{S}(\mathfrak{g}^*) =$ diff. op. with constant coeff. on $\mathfrak{g}^*$ (possibly of infinite degree) $\rightsquigarrow$ action on $S(\mathfrak{g})$

<u>Thm</u> [Duflo]: $PBW \circ (j \cdot) : S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g})^{\mathfrak{g}}$ is an algebra isomorphism.

→ Duflo (70s): representation theoretic approach

→ Kontsevich (97): deformation quantization.

$S(\mathfrak{g})^{\mathfrak{g}}$ algebra of Casimir functions on $\mathfrak{g}^*$

$U(\mathfrak{g})^{\mathfrak{g}} = Z(U(\mathfrak{g}))$ center of the quantized algebra

general principle of deformation quantization: {Casimirs} $\xrightarrow{\sim}$ Center of quantized algebra.

→ Manchon-Pevzner (after Kontsevich)

- Torossian

Poisson cohomology $\xrightarrow{\sim}$ Hochschild cohomology of the quantized algebra. (as algebras!)

Here $PBW \circ (j \cdot) : H^{\bullet}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} H^{\bullet}(\mathfrak{g}, U(\mathfrak{g})) = HH^{\bullet}(U(\mathfrak{g}))$.

→ C-Rossi

Poisson homology $\xrightarrow{\sim}$ Hochschild homology of the quantized algebra (as modules).

As a consequence $PBW \circ (j \cdot) : H_{\bullet}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} H_{\bullet}(\mathfrak{g}, U(\mathfrak{g})) = HH_{\bullet}(U(\mathfrak{g}))$.

in particular $S(\mathfrak{g})_{\mathfrak{g}} \xrightarrow{\sim} \underset{\substack{\| \\ Ab(U(\mathfrak{g}))}}{U(\mathfrak{g})_{\mathfrak{g}}}$ iso. of modules on coinvariants.

2. Caldararu conjecture

X complex manifold. HKR q-iso $(T_{poly} X, 0) \longrightarrow (D_{poly} X, \partial)$

$T_{poly} X$ = Sheaf of polyvector fields; $D_{poly} X$ = Sheaf of poly differential operators

product $= \wedge$

HKR does NOT induce an isomorphism of algebras $H^\bullet(T_{poly}X) \xrightarrow{\sim} \mathbb{H}(\mathcal{D}_{poly}X)$.

WANT: modify it in order to obtain an algebra isomorphism.

We do it now: let ∇ be a smooth $(1,0)$ connection with values in the holomorphic tangent bundle, i.e. a connection

$$\nabla' + \nabla'' = \nabla : \Gamma^\infty(T_X) \longrightarrow \Omega^{1,0}(X, T_X) \oplus \Omega^{0,1}(X, T_X)$$

such that $\nabla'' = \bar\partial$ (locally, $\nabla = \partial + \bar\partial + \Gamma$ $\quad \Gamma \in \Omega^{1,0}(U, \mathcal{E}nd(T_X))$.)

Its curvature $\nabla^2 = R^{2,0} + R^{1,1}$ does not have a $(0,2)$-part.

$$R^{1,1} \in \Omega^{1,1}(X, \mathcal{E}nd(T_X)).$$

Locally, $R^{1,1} = \bar\partial \Gamma$ therefore $\bar\partial R^{1,1} = 0$. Define $At_X := [R^{1,1}] \in H^1(X, T_X^* \otimes \mathcal{E}nd(T_X))$

$$j := \det\left(\sqrt{\frac{At_X}{e^{At_X/2} - e^{-At_X/2}}}\right) \in \bigoplus_k H^k(X, \Omega^k_X)$$

(can be expanded formally in terms of $tr(At_X^k) \in H^k(X, \Omega^k_X)$.)

<u>Theorem</u> [C-Van den Bergh] $\quad HKR \circ (j \cdot) : H^\bullet(T_{poly}X) \xrightarrow{\sim} \mathbb{H}(\mathcal{D}_{poly}X)$ is an algebra isom.

→ the existence of such an algebra isom. was first guessed by Kontsevich.

→ it has been proved (existence, not explicit form) by Dolgushev-Tamarkin-Tsygan.

→ it is a part of the more general Căldăraru conjecture which I explain now.

We also have the HKR isomorphism on homology:

$$HKR : \Big(C^{poly}_{-\bullet} X = \widehat{\mathcal{O}}_{\underbrace{X \times \cdots \times X}_{\bullet+1}} X, b \Big) \longrightarrow (\Omega^{-\bullet}_X, 0) \qquad q\text{-isom.}$$

$$f_0 \otimes \cdots \otimes f_n \longmapsto f_0\, df_1 \wedge \cdots \wedge df_n .$$

$C^{poly}_{-\bullet} X$ is a module over the algebra $\mathcal{D}_{poly} X$

$\Omega^{-\bullet}_X$ ———— " ———— $T_{poly} X$.

<u>Theorem</u> [C-Rossi-Van den Bergh]

$$\begin{array}{ccc} H^\bullet(T_{poly}X) & \xrightarrow{HKR \circ (j \cdot)} & \mathbb{H}^\bullet(\mathcal{D}_{poly}X) \\ \circlearrowright & & \circlearrowright \\ H(\Omega^{-\bullet}_X) & \xleftarrow{j \wedge HKR} & \mathbb{H}(C^{poly}_{-\bullet}X) \end{array}$$

are isomorphisms of algebras and their modules.

→ existence has been proved by Dolgushev-Tamarkin-Tsygan as a consequence of the formality of Calculus structure on $(\mathcal{D}_{poly}X, C^{poly}_{-\bullet}X)$.

3. A unifying approach

Let A be a commutative algebra which is regular (i.e. $\mathrm{Der}(A)$ is projective of finite type / free).

In this case $HKR^\bullet : T_{poly}A \longrightarrow H^\bullet(\mathcal{D}_{poly}A, \partial)$ and $HKR_\bullet : H_{-\bullet}(C^{poly}A, b) \longrightarrow \Omega_A^{-\bullet}$ are isomorphisms of algebras and modules. <u>Question</u>: is it still true for $\left[\begin{array}{l}\text{DG algebras ?}\\ \text{sheaves of algebras ?}\end{array}\right.$

<u>Answer</u>: NO! $\left[\begin{array}{l}HKR^\bullet\\ HKR_\bullet\end{array}\right.$ is only a q-iso of complexes, not of $\left[\begin{array}{l}\text{algebras}\\ \text{modules}\end{array}\right.$

Again, one has to modify it. Let (A, Q) a DG algebra s.t. A is regular.

$d : \Omega_A^\bullet \longrightarrow \Omega_A^{\bullet+1}$ the de Rham diff on forms.

Define $dQ \in \Omega_A^1 \otimes_A \mathrm{Der}(A)$, which we view as an element in $\mathrm{End}_A(\mathrm{Der}(A)) \simeq A \otimes \mathrm{End}(V)$.

Define $\Xi = d(dQ) \in \Omega_A^1 \otimes_A \mathrm{End}_A(\mathrm{Der}(A)) \simeq \Omega_A^1 \otimes \mathrm{End}(V)$.

(If $A = \mathrm{Spec}(X)$, $\Xi \in \Omega^1(X, \mathrm{End}(T_X))$.)

Finally define $j := \det\left(\sqrt{\frac{e^{\Xi/2} - e^{-\Xi/2}}{\Xi}}\right) \in \Omega_A^\bullet$ of total degree 0.

<u>Theorem</u> [C-Rossi-Van den Bergh]: $HKR \circ (j\cdot) : H^\bullet(T_{poly}A, [Q,-]) \xrightarrow{\sim} H^\bullet(\mathcal{D}_{poly}A, \partial + [Q,-])$ is an isomorphism of algebras, and $j \wedge HKR : H^\bullet(C^{poly}A, b + Q\cdot) \xrightarrow{\sim} H^\bullet(\Omega_A, Q\cdot)$ is an isomorphism of modules.

<u>Remark</u>: Since j has now infinitely many components, it may not converge. One then might have either to complete spaces or to tensor with some nilpotent algebra $\mathfrak{m}$.

$\rightsquigarrow$ naturally leads to the case when $(\mathfrak{m}, d_{\mathfrak{m}})$ is DG nilpotent and $d_{\mathfrak{m}}(Q) + \frac{1}{2}[Q,Q] = 0$.

<u>Application</u>: let $\mathfrak{g}$ be a f.d. Lie algebra. Consider $A = \Lambda\mathfrak{g}^*$ and $Q = d_{CE}$.

Then $T_{poly}A = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}$ and $[Q,-] = d_{CE}$. (and $\Xi = \mathrm{ad}$).

$H(\mathfrak{g}, S(\mathfrak{g})) \simeq$ One then has $H(T_{poly}A, [Q,-]) \xrightarrow[HKR^\bullet]{\sim} HH^\bullet(A, d_{CE}) \underset{\text{Koszul duality}}{\simeq} HH^\bullet(U(\mathfrak{g}))$.

Similarly $\widehat{\Omega}_A = \Lambda\mathfrak{g}^* \otimes \widehat{S}\mathfrak{g}^*$ and $Q\cdot = d_{CE}$

Therefore one has $HH_\bullet(A, d_{CE})^* \simeq HH(\widehat{C}^{poly}A, b + Q\cdot) \xrightarrow[HKR_\bullet]{\sim} H^\bullet(\mathfrak{g}, \widehat{S}(\mathfrak{g}^*)) \simeq H_\bullet(\mathfrak{g}, S(\mathfrak{g}))$

|| Koszul duality

$HH_\bullet(U(\mathfrak{g}))^*$

<u>Idea of the proof</u>: it is sufficient to do it in the case $A = S(V^*)$

Kontsevich L_∞-formality map $\mathcal{U} : T_{poly}A \longrightarrow \mathcal{D}_{poly}A$,

Q being a solution of the MCE, $\mathcal{U}_Q^{(1)} : (T_{poly}A, [Q,-]) \longrightarrow (\mathcal{D}_{poly}A, \partial + [Q,-])$

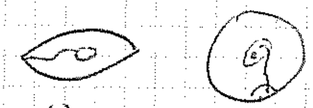
$\rightarrow$ Kontsevich: compatibility with cup-products gives the result.

Shoikhet L_∞-formality map $\mathcal{S}: C_{-\bullet}^{poly}A \longrightarrow \Omega_A^{-\bullet}$ induces

$$\mathcal{S}_Q^{(0)} : \left(C_\bullet^{poly}A, Q\cdot + b\right) \longrightarrow \left(\Omega_A^{-\bullet}, Q\cdot\right)$$

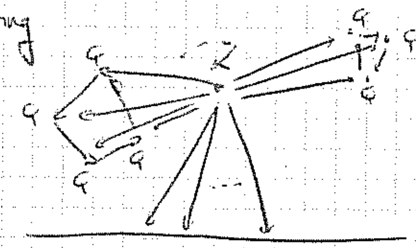
C-Rossi: Compatibility with cap-products (i.e. morphism of modules on homology).

To do so, use a homotopy inside $C_{2,0} =$  for cup-products

$\mathcal{D}_{1,1} =$  for cap-products.

Remains to prove that $\mathcal{U}_Q^{(1)} = \mathrm{HKR}\circ(j\cdot)$ and $\mathcal{S}_Q^{(0)} = j\wedge \mathrm{HKR}$.

Only graphs appearing



Computation of weights of wheels:
→ Van den Bergh.
→ Willwacher. □

4. Yet another approach (Kapranov)

The Atiyah class $At_X \in H^1(X, T_X^* \otimes \mathrm{End}(T_X))$ defines a degree 1 map $T_X \longrightarrow \mathrm{End}(T_X)$ in the derived category of $\mathcal{O}_X$-modules. I.e. a map $T_X[-1] \longrightarrow \mathrm{End}(T_X[-1])$.

Bianchi identity for ∇ $\Longleftrightarrow$ Jacobi identity for At_X

One can define $At_E \in H^1(X, T_X^* \otimes \mathrm{End}(E))$ for any holomorphic bundle, which gives a map $T_X[-1] \longrightarrow \mathrm{End}(E)$ ... i.e. a representation of $T_X[-1] = \mathfrak{g}$.

Moreover, $S(\mathfrak{g}) \simeq T_{poly}X$. One can prove that $(\mathcal{D}_{poly}X, \partial)$ is a model for $U(\mathfrak{g})$.

So Duflo isom. $\Longrightarrow$ Caldararu's conjecture.

BUT a priori, Duflo applies in the general context of symmetric abelian monoidal categories. And the derived category is only triangulated.