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Lagrangian structures in derived
Symplectic geometry

Recollection on derived symplectic geometry

B dg^{≤0}-algebra.

$A(B) := S_{\mathbb{B}}(\Omega_{\mathbb{B}}^1[-1])$ graded dg-algebra of forms. (\tilde{B} semi-free resolution of B as a k -algebra)

$A: \text{dg}^{\leq 0} \text{Alg} \rightarrow \text{dg-Alg}^{\text{gr}}$ is a stack (i.e. it satisfies étale descent).

dAff^{op} Hence it makes sense to define $A(X) := \text{holim}_{\mathbb{R}\text{Sec}(B) \rightarrow X} (A(B))$

Define the complex of p -forms $A^p(X) := A(X)^{(p)}[p]$, the homogeneous part of degree p for the auxiliary grading shifted back by cohomological degree p .

Define the space of p -forms of degree n $A^p(X, n) := n\text{-cocycles in } A^p(X) = \Gamma_{\text{Cpx}}(\mathbb{k}, A^p(X)[n])$

Remark: $A(X) = \mathbb{R}\Gamma(S_{G_X}(\mathbb{L}_X[-1]))$.

Examples: • $X = BG$, G group scheme. $\mathbb{R}\Gamma: \text{Cpx}(G\text{-mod}) \rightarrow \text{Cpx}$ is given by G -cohomology (derived G -invariants)

$\mathbb{L}_X = \mathfrak{g}^*[-1]$. Hence we have a map $S(\mathfrak{g}^*[-2])^G \rightarrow A(X)$ in $\text{dg-Alg}^{\text{gr}}$.

Note that $S^2(\mathfrak{g}^*)^G = S^2(\mathfrak{g}^*[-2])^G[2+2]$.

\Rightarrow any symmetric invariant bilinear pairing on \mathfrak{g} defines a 2-form of degree 2 on BG .

• $X = [\mathfrak{g}^*/G]$. In a way similar to the previous example one has a map $(S\Omega_Y[-\cdot] \otimes S(\mathfrak{g}^*[-2]))^G \rightarrow A([\mathfrak{g}^*/G])$ in $\text{dg-Alg}^{\text{gr}}$,

where the differential on the source is generated by the transpose

$d_Y: \Omega_Y^1[-1] \rightarrow \mathcal{O}_Y \otimes \mathfrak{g}^*[-2][1]$ of the infinitesimal action $\mathfrak{g} \rightarrow \Gamma(T_Y)$.

When $Y = \mathfrak{g}^*$ one finds $(S\Omega_{\mathfrak{g}^*}^1[-1] \otimes \mathfrak{g}^*[-2])^G[2+1] \rightarrow A^2([\mathfrak{g}^*/G], 1)$

$$(S\Omega_{\mathfrak{g}^*}^1 \otimes \mathfrak{g}^*)^G = (\mathbb{L}[\mathfrak{g}^*] \otimes \mathfrak{g} \otimes \mathfrak{g}^*)^G$$

Hence the canonical element $\text{can} \in \mathfrak{g} \otimes \mathfrak{g}^*$ defines a 2-form of degree 1 on $[\mathfrak{g}^*/G]$.

$\mathbb{I}^{\mathbb{R}}(\pi_i)$ coordinates on \mathfrak{g}^* and (ξ^i) dual basis of \mathfrak{g}^* then $\text{can} = d_{\mathbb{R}}(\pi_i) \otimes \xi^i$.

$A^{cl}(B) := \left(\hat{\bigoplus}_{\mathbb{R}} (\Omega_{\mathbb{R}}^1[-1]), d_B + d_{dR} \right)$ filtered dg-Algebra of closed forms.

Note that $gr(A^{cl}(B)) \simeq A(B)$.

$A^{cl} : dAff^{op} \rightarrow dg-Alg^{filt}$ is a stack. Hence it makes sense to define $A^{cl}(X) := \text{holim}_{\mathbb{R}Spec(B) \rightarrow X} (A^{cl}(B))$.

Define the complex of closed p-forms $A^{p,cl}(X) := F^p(A^{cl}(X))[-p]$, where " $F^p(-)$ " means the " p -th filtration of $-$ ". The space of closed p-forms of degree n is

$$A^{p,cl}(X, n) := n\text{-cocycles in } A^{p,cl}(X) = \text{Map}_{Cpx}(\mathbb{k}, A^{p,cl}(X)[n]).$$

Examples: • $X = BG$. Forms coming from invariant polynomials via the map $S(g^*[-2])^G \rightarrow A(X)$ are canonically closed ($S(g^*[-2])$ being concentrated in even degrees, there is no room for the de Rham differential to be non-trivial).

• If Y is a smooth affine scheme then a G -action then there is a map $(\Omega_Y[-1] \otimes S(g^*[-2]))^G \rightarrow A^{cl}(Y/G)$ in $dg-Alg^{filt}$ where the differential on the source is $d_Y + d_{dR, Y} \otimes id$.

When $Y = g^*$, the 2-form of degree 1 can be canonically closed.

From now we assume that \mathbb{L}_X is a perfect \mathcal{O}_X -module.

Definition: an n -symplectic structure on X is a closed 2-form of degree n $\omega \in A^{2,cl}(X, n)$

such that the underlying 2-form $\omega_0 = \mathcal{T}(\omega)$ is non-degenerate. This means that the induced map $\mathbb{L}_X^\vee = \mathbb{T}_X \rightarrow \mathbb{L}_X$ (by ω_0) is a quasi-iso. (or a weak equiv.).

Note that \mathcal{T} denotes the "symbol map" from closed forms to forms.

Examples: • ordinary smooth schemes: symplectic structures $\Leftrightarrow \mathcal{O}$ -symplectic structures.

• $X = BG$. If \langle, \rangle is a ND symmetric invariant bilinear pairing on \mathfrak{g} then

the associated 2-form of degree 2 is ND as well: $\mathfrak{g}[1] \xrightarrow{\langle, \rangle} \mathfrak{g}^*[1] = \mathfrak{g}^*[-1][2]$

• $X = [g^*/G]$. The 2-form of degree 1 defined by can is ND: $\mathbb{T}_{BG} \xrightarrow{\langle, \rangle} \mathbb{L}_{BG}$

$$\mathbb{T}_{[g^*/G]} = \left(\mathbb{C}[g^*] \otimes \mathfrak{g}[1] \oplus \mathbb{C}[g^*] \otimes \mathfrak{g}^* \right) \xrightarrow{\mathbb{T}_{g^*}}$$

can induce an isomorphism of complexes

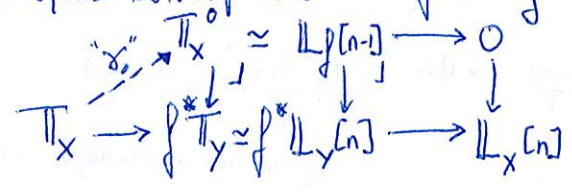
$$\mathbb{L}_{[g^*/G]}[1] = \left(\mathbb{C}[g^*] \otimes \mathfrak{g}[1] \oplus \mathbb{C}[g^*] \otimes \mathfrak{g}^*[-1][1] \right) \xrightarrow{= \Omega_{g^*}^1}$$

Lagrangian structures

$X \xrightarrow{f} Y$ morphism between stacks having perfect cotangent complexes. ω n -symplectic structure on Y .

Definition: a Lagrangian structure on f is a path γ in $A^{2,cl}(X,n)$ from $f^* \omega$ to 0 (isotropic structure) that is non-degenerate.

Non-degeneracy means that the underlying path γ_0 in $A^2(X,n)$ is such that the induced map $\pi_X \rightarrow \pi_X^0 \simeq \mathbb{L}_f[n+1]$ is a quasi-isomorphism. The following diagram explains where this map comes from:

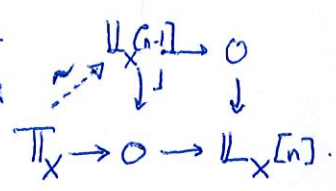


Examples: • $*_{(n)}$ point equipped with its canonical n -symplectic structure (0) .

Lagrangian structures on $X \rightarrow *_{(n)} \iff (n-1)$ -symplectic structures on X .

Why? Essentially because $\Omega_0(A^{p,cl}(X,n)) \simeq A^{p,cl}(X,n-1)$.

One sees that the notions of ND coincide on both sides:



• X smooth G -scheme together with a G -equivariant map $\mu: X \rightarrow g^*$. We get a map $[\mu]: [X/G] \rightarrow [g^*/G]$ of stacks.

Recall that $[g^*/G]$ has a 1-symplectic structure $can = d_{\text{DR}}(\pi_i) \otimes \xi^i$.

Let's see what it means to have a Lagrangian structure on $[\mu]$:

$$[\mu]^* can = d_{\text{DR}}(\mu^* \pi_i) \otimes \xi^i = d_g(w)$$

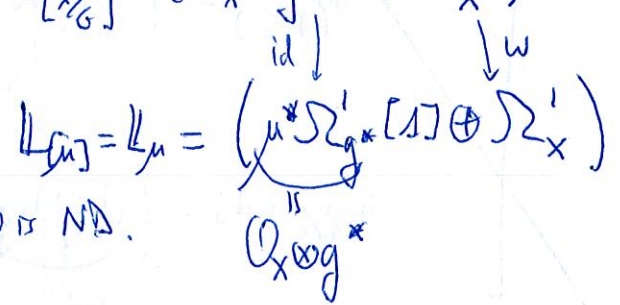
Assume that w is in the image of $(\Omega_X^2)^G \rightarrow A^{cl}(X)[2]$.

Then $d_{\text{DR}}(\mu^* \pi_i) \otimes \xi^i = (\pi_i^*(w)) \otimes \xi^i$. I.e. $\mu^* d_{\text{DR}}(\pi_i) = \pi_i^*(w)$

This is the moment map condition! Next we can assume $d_{\text{DR}}(w) = 0$

and get a Lagrangian structure.

Let us check the condition for ND: $\pi_{[X/G]} = (\mathbb{Q}_X \otimes g[1] \oplus T_X)$



\Rightarrow our isotropic structure is ND iff w is ND.

Conclusion = Moment maps are Lagrangian structures.

Lagrangian correspondences

Theorem: X, Y, Z derived stacks with n -symplectic structures.

$L \xrightarrow{f} X \times \bar{Y}$ and $M \xrightarrow{g} Y \times \bar{Z}$ maps with Lagrangian structures.
 Then $L \times_Y^h M \xrightarrow{R} X \times \bar{Z}$ has a Lagrangian structure.

Corollary [PTW]: $L \xrightarrow{f} Y$ + Lagrangian structures $\Rightarrow L \times_Y^h M$ $(n-1)$ -symplectic.

Proof of Corollary: consider $X=Z=*$. \square

Application of Corollary: $X \xrightarrow{\mu} \mathfrak{g}^*$ moment map. $\mathcal{O} \subset \mathfrak{g}^*$ coadjoint orbit.

$\Rightarrow [X/\mathfrak{G}] \xrightarrow{c^*} [\mathfrak{g}^*/\mathfrak{G}]$ and $[\mathcal{O}/\mathfrak{G}] \rightarrow [\mathfrak{g}^*/\mathfrak{G}]$ Lagrangian.

Hence $[X/\mathfrak{G}] \times_{[\mathfrak{g}^*/\mathfrak{G}]}^h [\mathcal{O}/\mathfrak{G}] = [X \times_{\mathfrak{g}^*}^h \mathcal{O}/\mathfrak{G}] = [R^{j^*}(\mathcal{O})/\mathfrak{G}] =: X//\mathfrak{G}$ is \mathcal{O} -symplectic.

Application of Theorem: $\Psi: X \rightarrow Y$ symplectomorphism, i.e. $\text{graph}(\Psi): X \rightarrow X \times \bar{Y}$ Lagrangian.

Then for any Lagrangian $L \rightarrow Y$, $R\Psi^{-1}(L) := X \times_Y^h L \rightarrow X$ Lagrangian.

Proof of Theorem: Lagrangian structure on $f: f^* \pi_X^* \omega_X \xrightarrow{\sigma_f} f^* \pi_Y^* \omega_Y$

Lagrangian structure on $g: g^* \pi_Y^* \omega_Y \xrightarrow{\sigma_g} g^* \pi_Z^* \omega_Z$

$$\Rightarrow h^* \pi_X^* \omega_X = \pi_L^* f^* \pi_X^* \omega_X \sim \pi_L^* f^* \pi_Y^* \omega_Y \sim \pi_M^* g^* \pi_Y^* \omega_Y \sim \pi_M^* g^* \pi_Z^* \omega_Z = h^* \pi_Z^* \omega_Z$$

\Rightarrow We have an isotropic structure, denoted δ_R , on $R: N := L \times_Y^h M \rightarrow X \times \bar{Z}$.

Let's prove it is ND.

• We have a fiber sequence $\pi_N \rightarrow \pi_{L \oplus M} \rightarrow \pi_Y$ (all sheaves are implicitly pulled back on N for simplicity)

• $R: N \rightarrow X \times \bar{Z}$ factors into $N \rightarrow (X \times Y)_Y \times (Y \times Z) = X \times Y \times Z \rightarrow X \times \bar{Z}$.

$$\begin{array}{ccc} \mathbb{L}_R & \rightarrow & \mathbb{L}_{N/X \times Y \times Z} \rightarrow \mathbb{L}_{X \times Y \times Z / X \times Z} \quad [1] \\ & & \downarrow s_1 \\ & & \mathbb{L}_{L \times M / X \times Y \times Z} \rightarrow \mathbb{L}_Y \quad [1] \\ & & \downarrow s_1 \\ & & \mathbb{L}_f \oplus \mathbb{L}_g \end{array}$$

We therefore have a diagram

$$\begin{array}{ccccc} \pi_N & \rightarrow & \pi_{L \oplus M} & \rightarrow & \pi_Y \\ \downarrow \text{given by } \delta_R & & \downarrow s & & \downarrow s \\ \mathbb{L}_R[n-1] & \rightarrow & \mathbb{L}_f \oplus \mathbb{L}_g[n-1] & \rightarrow & \mathbb{L}_Y[n] \end{array}$$

must be a.w.e. because (lines are exact and other vertical maps are w.e.) \square

• the left-most square commutes because δ_R is the composition of σ_f and σ_g .

• the other square does because σ_f and σ_g are composable (at ω_Y)