Lagrangian structures in derived symplectic geometry

Recollection on derived symplectic geometry

B $dg^{\leq 0}$-algebra.

$\mathcal{A}(B) := S_{\tilde{B}}(\Omega^1_{\tilde{B}}[-1])$ graded dg-algebra of forms. ($\tilde{B}$ semi-free resolution of B as a k-algebra)

$\mathcal{A} : dg^{\leq 0}\text{-}Alg \longrightarrow dg\text{-}Alg^{gr}$ is a stack (i.e. it satisfies étale descent).
$\quad \| $
$dAff^{op}$ Hence it makes sense to define $\mathcal{A}(X) := \underset{\mathbb{R}Spec(B)\to X}{\mathrm{holim}} (\mathcal{A}(B))$

Define the <u>complex of p-forms</u> $\mathcal{A}^p(X) := \mathcal{A}(X)^{(p)}[p]$, the homogeneous part of degree p for the auxiliary grading shifted back by cohomological degree p.

Define the <u>space of p-forms of degree n</u> $\mathcal{A}^p(X,n) :=$ n-cocycles in $\mathcal{A}^p(X) = \mathrm{Map}_{Cpx}(k, \mathcal{A}^p(X)[n])$

<u>Remark</u>: $\mathcal{A}(X) = \mathbb{R}\Gamma(S_{\mathcal{O}_X}(\mathbb{L}_X[-1]))$.

<u>Examples</u>:

- $X = BG$, G group scheme. $\mathbb{R}\Gamma : Cpx(G\text{-mod}) \to Cpx$ is given by G-cohomology (derived G-invariants)
  $\mathbb{L}_X = g^*[-1]$. Hence we have a map $S(g^*[-2])^G \to \mathcal{A}(X)$ in $dg\text{-}Alg^{gr}$.
  Note that $S^2(g^*)^G = S^2(g^*[-2])^G[2+2]$.
  $\Rightarrow$ any symmetric invariant bilinear pairing on g defines a 2-form of degree 2 on BG.
- $X = [g^*/G]$. In a way similar to the previous example one has a map $(\Omega^\bullet_Y[-\bullet] \otimes S(g^*[-2]))^G \longrightarrow \mathcal{A}([Y/G])$ in $dg\text{-}Alg^{gr}$,
  where the differential on the source is generated by the transpose
  $d_g : \Omega^1_Y[-1] \to \mathcal{O}_Y \otimes g^*[-2][1]$ of the infinitesimal action $g \longrightarrow \Gamma(T_Y)$.
  When $Y = g^*$ one finds $(\Omega^1_{g^*}[-1] \otimes g^*[-2])^G[2+1] \longrightarrow \mathcal{A}^2([g^*/G], 1)$
  $$\underset{(\Omega^1_{g^*} \otimes g^*)^G \simeq (\mathbb{C}[g^*] \otimes g \otimes g^*)^G}{\wr\|}$$
  Hence the canonical element $can \in g \otimes g^*$ defines a 2-form of degree 1 on $\boxed{g^*}/G]$.

If (x_i) coordinates on g^* and (ξ^i) dual basis of g^* then $can = d_{dR}(x_i) \otimes \xi^i$.

$\mathcal{A}^{cl}(B) := \left(\widehat{S}_{\widetilde{B}}(\Omega^1_{\widetilde{B}}[-1]),\ d_{\widetilde{B}} + d_{dR}\right)$ filtered dg-Algebra of closed forms.

Note that $gr(\mathcal{A}^{cl}(B)) \simeq \mathcal{A}(B)$.

$\mathcal{A}^{cl} : dAff^{op} \longrightarrow dg\text{-}Alg^{filt}$ is a stack. Hence it makes sense to define $\mathcal{A}^{cl}(X) := \underset{\mathbb{R}Spec(B)\to X}{\mathrm{holim}}(\mathcal{A}^{cl}(B))$.

Define the <u>complex of closed p-forms</u> $\mathcal{A}^{p,cl}(X) := F^p(\mathcal{A}^{cl}(X))[p]$, where "$F^p(-)$" means the "p-th filtration of $-$". The <u>space of closed p-forms of degree n</u> is

$\mathcal{A}^{p,cl}(X,n) :=$ n-cocycles in $\mathcal{A}^{p,cl}(X) = \mathrm{Map}_{Cpx}(k, \mathcal{A}^{p,cl}(X)[n])$.

<u>Examples</u>:
- $X = BG$. Forms coming from invariant polynomials via the map $S(g^*[-2])^G \to \mathcal{A}(X)$ are canonically closed ($S(g^*[-2])$ being concentrated in even degrees, there is no room for the de Rham differential to be non-trivial).
- If Y is a smooth affine scheme then a G-action then there is a map $(\Omega^\bullet_Y[-\bullet] \otimes S(g^*[-2]))^G \longrightarrow \mathcal{A}^{cl}([Y/G])$ in $dg\text{-}Alg^{filt}$ where the differential on the source is $d_g + d_{dR,Y} \otimes id$.

  When $Y = g^*$, the 2-form of degree 1 can is canonically closed.

From now we assume that $\mathbb{L}_X$ is a perfect $\mathcal{O}_X$-module.

<u>Definition</u>: an <u>n-symplectic structure</u> on X is a closed 2-form of degree n $\omega \in \mathcal{A}^{2,cl}(X,n)$ such that the underlying 2-form $\omega_0 = \sigma(\omega)$ is non-degenerate. This means that the induced map $\mathbb{L}_X^\vee = \mathbb{T}_X \longrightarrow \mathbb{L}_X$ (by ω_0) is a quasi-iso. (or a weak equiv.).

Note that σ denotes the "symbol map" from closed forms to forms.

<u>Examples</u>:
- ordinary smooth schemes: symplectic structures $\Leftrightarrow$ 0-symplectic structures.
- $X = BG$. If $\langle,\rangle$ is a ND symmetric invariant bilinear pairing on g then the associated 2-form of degree 2 is ND as well: $\underset{\mathbb{T}_{BG}}{\underset{\shortparallel}{g[1]}} \xrightarrow{\sim} g^*[1] = \underbrace{g^*[-1]}_{\mathbb{L}_{BG}}[2]$
- $X = [g^*/G]$. The 2-form of degree 1 defined by can is ND:

$$\begin{array}{c} \mathbb{T}_{[g^*/G]} = \left(\mathbb{C}[g^*]\otimes g[1] \oplus \mathbb{C}[g^*]\otimes \underset{=\mathbb{T}_{g^*}}{g^*}\right) \\ \downarrow \wr\, id \qquad\qquad \downarrow \wr\, id \\ \mathbb{L}_{[g^*/G]}[1] = \left(\underbrace{\mathbb{C}[g^*]\otimes g[1]}_{=\Omega^1_{g^*}} \oplus \mathbb{C}[g^*]\otimes g^*[-1][1]\right) \end{array}$$

can induces an isomorphism of complexes

Lagrangian structures

$X \xrightarrow{f} Y$ morphism between stacks having perfect cotangent complexes. ω n-symplectic structure on Y.

Definition: a Lagrangian structure on f is a path γ in $\mathcal{A}^{2,cl}(X,n)$ from $f^*\omega$ to 0 (isotropic structure) that is non-degenerate.

Non-degeneracy means that the underlying path γ_0 in $\mathcal{A}^2(X,n)$ is such that the induced map $\mathbb{T}_X \longrightarrow \mathbb{T}_X^{\circ} \simeq \mathbb{L}_f[n+1]$ is a quasi-isomorphism. The following diagram explains where this map comes from:

$$\begin{tikzcd}
 & \mathbb{T}_X^{\circ} \simeq \mathbb{L}_f[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
\mathbb{T}_X \arrow[r] \arrow[ur, dashed, "``\gamma_0"\!"] & f^*\mathbb{T}_Y \simeq f^*\mathbb{L}_Y[n] \arrow[r] & \mathbb{L}_X[n]
\end{tikzcd}$$

Examples:

- $*_{(n)}$ point equipped with its canonical n-symplectic structure (0).

  Lagrangian structures on $X \to *_{(n)}$ $\Longleftrightarrow$ $(n-1)$-symplectic structures on X.

  Why? Essentially because $\Omega_0(\mathcal{A}^{p,cl}(X,n)) \simeq \mathcal{A}^{p,cl}(X,n-1)$.

  One sees that the notions of ND coincide on both sides:

$$\begin{tikzcd}
 & \mathbb{L}_X[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
\mathbb{T}_X \arrow[r] \arrow[ur, dashed, "\sim"] & 0 \arrow[r] & \mathbb{L}_X[n].
\end{tikzcd}$$

- X smooth G-scheme together with a G-equivariant map $\mu: X \longrightarrow \mathfrak{g}^*$.

  We get a map $[\mu]: [X/G] \longrightarrow [\mathfrak{g}^*/G]$ of stacks.

  Recall that $[\mathfrak{g}^*/G]$ has a 1-symplectic structure $\omega_{\text{can}} = d_{dR}(x_i) \otimes \xi^i$.

  Let's see what it means to have a Lagrangian structure on $[\mu]$:

  $$[\mu]^*\omega_{\text{can}} = d_{dR}(\mu^* x_i) \otimes \xi^i = d_{\mathfrak{g}}(\omega).$$

  Assume that ω is in the image of $(\Omega_X^2)^G \longrightarrow \mathcal{A}^{cl}(X)[2]$.

  Then $d_{dR}(\mu^* x_i) \otimes \xi^i = \iota_{\vec{x}_i}(\omega) \otimes \xi^i$. I.e. $\mu^* d_{dR}(x_i) = \iota_{\vec{x}_i}(\omega)$

  This is the moment map condition! Next we can assume $d_{dR}(\omega) = 0$ and get a Lagrangian structure.

  Let us check the condition for ND:

$$\begin{tikzcd}
\mathbb{T}_{[X/G]} = \big(\mathcal{O}_X \otimes \mathfrak{g}[1] \arrow[d, "\mathrm{id}"] & \oplus & \mathbb{T}_X\big) \arrow[d, "\omega"] \\
\mathbb{L}_{[\mu]} = \mathbb{L}_\mu = \big(\underbrace{\mu^*\Omega^1_{\mathfrak{g}^*}}_{\simeq\, \mathcal{O}_X \otimes \mathfrak{g}^*}[1] & \oplus & \Omega^1_X\big)
\end{tikzcd}$$

  $\Rightarrow$ our isotropic structure is ND iff ω is ND.

  Conclusion: Moment maps are Lagrangian structures.

Lagrangian correspondences

Theorem: X, Y, Z derived stacks with n-symplectic structures.
$L \xrightarrow{f} X\times\bar{Y}$ and $M \xrightarrow{g} Y\times\bar{Z}$ maps with Lagrangian structures.
Then $L\overset{h}{\underset{Y}{\times}} M \xrightarrow{h} X\times\bar{Z}$ has a Lagrangian structure.

Corollary [PTVV]: $L \xrightarrow{f} Y \xleftarrow{g} M$ + Lagrangian structures $\Rightarrow$ $L\overset{h}{\underset{Y}{\times}} M$ $(n-1)$-symplectic.

Proof of Corollary: Consider $X = Z = *_{(n)}$. □

Application of corollary: $X \xrightarrow{\mu} \mathfrak{g}^*$ moment map. $\mathcal{O}\hookrightarrow \mathfrak{g}^*$ coadjoint orbit.

$\Rightarrow$ $[X/G] \xrightarrow{[\mu]} [\mathfrak{g}^*/G]$ and $[\mathcal{O}/G] \to [\mathfrak{g}^*/G]$ Lagrangian.

Hence $[X/G] \overset{h}{\underset{[\mathfrak{g}^*/G]}{\times}} [\mathcal{O}/G] = \left[X\overset{h}{\underset{\mathfrak{g}^*}{\times}}\mathcal{O} \Big/ G\right] = \left[R\mu^{-1}(\mathcal{O})/G\right] =: X/\!\!/_{\mathcal{O}}G$ is 0-symplectic.

Application of theorem: $\varphi: X \to Y$ symplectomorphism, i.e. $\mathrm{graph}(\varphi): X \to X\times\bar{Y}$ Lagrangian.

Then for any Lagrangian $L\to Y$, $R\varphi^{-1}(L) := X\overset{h}{\underset{Y}{\times}} L \to X$ Lagrangian.

Proof of Theorem: Lagrangian structure on f: $f^*\pi_X^*\omega_X \overset{\gamma_f}{\sim} f^*\pi_Y^*\omega_Y$

Lagrangian structure on g: $g^*\pi_Y^*\omega_Y \overset{\gamma_g}{\sim} g^*\pi_Z^*\omega_Z$

$\Rightarrow$ $h^*\pi_X^*\omega_X = \pi_L^* f^*\pi_X^*\omega_X \sim \pi_L^* f^*\pi_Y^*\omega_Y \sim \pi_M^* g^*\pi_Y^*\omega_Y \sim \pi_M^* g^*\pi_Z^*\omega_Z = h^*\pi_Z^*\omega_Z$

$\Rightarrow$ we have an isotropic structure, denoted γ_h, on $h: N := L\overset{h}{\underset{Y}{\times}} M \to X\times\bar{Z}$.

Let's prove it is ND.

- We have a fiber sequence $\mathbb{T}_N \to \mathbb{T}_{L\oplus M} \to \mathbb{T}_Y$ (all sheaves are implicitly pulled-back on N for simplicity)
- $h: N \to X\times\bar{Z}$ factors into $N \to (X\times Y)\underset{Y}{\times}(Y\times Z) = X\times Y\times Z \to X\times Z$.

Hence we have a fiber sequence $\mathbb{L}_h \to \mathbb{L}_{N/X\times Y\times Z} \to \mathbb{L}_{X\times Y\times Z/X\times Z}[1]$

$$\mathbb{L}_{N/X\times Y\times Z} \simeq \mathbb{L}_{L\times M/X\times Y\times Y\times Z} \simeq \mathbb{L}_f\oplus\mathbb{L}_g, \qquad \mathbb{L}_{X\times Y\times Z/X\times Z}[1] \simeq \mathbb{L}_Y[1]$$

We therefore have a diagram

$$\begin{tikzcd}
\mathbb{T}_N \arrow[r] \arrow[d, "\text{given by } \gamma_h"'] & \mathbb{T}_L\oplus\mathbb{T}_M \arrow[r] \arrow[d, "\wr"', "s"] & \mathbb{T}_Y \arrow[d, "s\,\wr", "\text{given by } \omega_Y"'] \\
\mathbb{L}_h[n-1] \arrow[r] & (\mathbb{L}_f\oplus\mathbb{L}_g)[n-1] \arrow[r] & \mathbb{L}_Y[n]
\end{tikzcd}$$

given by $\gamma_f\oplus\gamma_g$

must be a w.e. because ((lines are exact and other vertical maps are w.e.)) □

- the left-most square commutes because γ_h is the composition of γ_f and γ_g.
- the other square does because γ_f and γ_g are composable (at ω_Y)