Recollection on derived symplectic geometry

$A(B) := S_B (S^2_B [-1])$ graded dg-algebra of forms. ($B$ semi-free resolution of $B$ as $\text{a b-algebra}$)

$A : \text{dg } S_B \text{-alg} \rightarrow \text{dg } A_{\text{gr}}$ is a stack (i.e. it satisfies etale descent).

Hence it makes sense to define $A(X) := \text{holim } (A(B))_{\text{Spec}(B) \rightarrow X}$

Define the complex of $p$-forms $A^p(X) := A(X)^{(p)}[p]$, the homogeneous part of degree $p$

for the auxiliary grading shifted back by cohomological degree $p$.

Define the space of $p$-forms of degree $a$ $A^a_p(X, \lambda) := n$-cycles in $A^p(X) = \text{Maps}(k, A^p(X)[n])$

Remark: $A^a(X) = \text{IR} \backslash S_B \text{(IL} X [1-1])$

Examples: $X = BG$, $G$ group scheme. $\text{IR} : \text{Gp} \times (G\text{-mod}) \rightarrow \text{Gp}$ is given by $G$-cohomology (graded $G$-invariants)

$[IL_X = g^* [1]$. Hence we have a map $S(g^*[2]) \rightarrow A(x)$ in $\text{dg } A_{\text{gr}}$

Note that $S^2(g^*)G = S^2 (g^*[2])G [2+2].$

$\Rightarrow$ any symmetric invariant bilinear pairing on $g$ defines a 2-form of degree 2 on $BG$.

$X = [g^*/G]$. In a way similar to the previous example one has

a map $(S^2 g^*[2]) \otimes S(g^*[2]) \rightarrow A([g^*/G])$ in $\text{dg } A_{\text{gr}}$

where the differential on the source is generated by the transpose

$S^2 g^*[2] \rightarrow (g^*[2]) G$ of the infinitesimal action $g \rightarrow \Gamma(T_g)$.

When $Y = g^*$ one finds

$(S^2 g^*[2]) \otimes g^*[2] \rightarrow A^2([g^*/G], *)$

Hence the canonical element can $6 g^* g^* \text{ defines a 2-form of degree 1 on } [g^*/G].$

$C^G(X)$ (coordinates on $g^*$ and $(*)$ dual basis of $g^*$) then $C^G = C_{\text{dr}(X)} \otimes \mathbb{F}^.$
\( A^{\varphi}(B) := \left( \mathcal{S}_{\varphi}^{\mathcal{S}}( S_{\varphi}^{\mathcal{S}}( \mathcal{S}_{\varphi}^{\mathcal{S}}( \cdot)), \varphi_{B} + \varphi_{B}^{\varphi} \right) \) filtered dg-Algebra of closed forms. Note that \( gr(A^{\varphi}(B)) \cong A(B) \).

\( \mathcal{A}^{\varphi} : \text{Diff}^{op} \rightarrow \text{dg-Alg}^{\mathcal{S}_{\varphi}} \) is a stack. Hence it makes sense to define \( A^{\varphi}(X) := \text{holim}(A^{\varphi}(B)) \).

Define the complex of closed \( p \)-forms \( A^{p,\varphi}(X) := F^p(A^{\varphi}(X))[p] \), where \( F^p(-) \) mean the \( p \)-th filtration of \( - \). The space of closed \( p \)-forms of degree \( \alpha \) is

\[ A^{p,\varphi}(X, \alpha) := \text{cycles in } A^{p,\varphi}(X) = \text{holim}_{B \in \text{Diff}^{op}}(B, A^{p,\varphi}(X)[n]) \].

**Examples:**

- \( X = \mathbb{B}G \). Forms coming from invariant polynomials via the map \( S(g^*E) \rightarrow A(X) \) are canonically closed (\( S(g^*E) \) being concentrated in even degrees, there is no room for the de Rham differential to be non-trivial).

- If \( Y \) is a smooth affine scheme then a \( G \)-action then there is a map \( (S_{\varphi}^*E \otimes S(g^*E)) \rightarrow A^{\varphi}(\mathbb{B}E) \) in \( \text{dg-Alg}^{\mathcal{S}_{\varphi}} \) where the differential on the source is \( \varphi + \varphi_{\text{deRham}} \otimes \text{id} \). When \( Y = g^*E \), the 2-form of degree 1 can is canonically closed.

From now we assume that \( LL_X \) is a perfect \( O_X \)-module.

**Definition:** An \( n \)-symplectic structure on \( X \) is a closed 2-form of degree \( n \) \( \omega \in A^{2,\varphi}(X, \alpha) \) such that the underlying \( 2 \)-form \( \omega_0 := \varphi(\omega) \) is non-degenerate. This means that the induced map \( \mathcal{T}^*_{\varphi}X \rightarrow LL_X \) (by \( \omega_0 \)) is a quasi-isomorphism or weak equiv.

Note that \( \mathcal{T} \) denotes the "symbol map" from closed forms to forms.

**Examples:**

- Ordinary smooth schemes: symplectic structures \( \Leftrightarrow O \)-symplectic structures.

- \( X = \mathbb{B}G \). If \( \langle \cdot, \cdot \rangle \) is a \( \text{ND} \) symmetric invariant bilinear pairing on \( g^*E \), then the associated 2-form of degree 2 is \( \text{ND} \) as well: \( q^{\mathcal{S}} := \varphi(q^*E) \otimes g^*E \).

- \( X = [g^*E] \). The 2-form of degree 1 defined by \( \omega \) is \( \text{ND} \).

\[ \mathcal{T}^{[g^*E]} = (\mathcal{S}[g^*E] \otimes g^*E) \oplus (\mathcal{S}[g^*E] \otimes g^*E) \]

\[ LL_{[g^*E]} = (\mathcal{S}[g^*E] \otimes g^*E) \oplus (\mathcal{S}[g^*E] \otimes g^*E) \]

Can induces an isomorphism of complexes.
Lagrangian structures

\[ X \xrightarrow{\phi} Y \] morphism between stacks having perfect coherent complexes. \( \omega \) n-symplectic structure on \( Y \).

**Definition:** a Lagrangian structure on \( Y \) is a path \( \gamma \) in \( A^2, \ell \) \((X,\nabla)\) from \( \phi^*\omega \) to \( 0 \) (isotropic structure) that is non-degenerate.

Non-degeneracy means that the underlying path \( \gamma_0 \) in \( A^2(X,\nabla) \) is such that the induced map \( T_{X_0} \to T_{X_0}^0 \cong \mathbb{L} \cdot H^0(\nabla) \) is a quasi-isomorphism. The following diagram explains where this map comes from:

\[
\begin{array}{c}
\xrightarrow{\phi^*\omega} T_{X_0} \to \mathbb{L} \cdot H^0(\nabla) \\
\downarrow \quad \downarrow \\
\mathbb{L} \cdot H^0(\nabla) \to 0
\end{array}
\]

Examples:

- \( \gamma_0 \) point equipped with its canonical n-symplectic structure \( (0) \).

Lagrangian structures on \( X \xrightarrow{\gamma_0} Y \) \( \iff \) \((n-1)\)-symplectic structures on \( X \).

Why? Essentially because \( \mathcal{S}_\omega \mathcal{A}^\phi(X,\nabla) \cong \mathcal{A}^\phi(X,\nabla) \).

One sees that the notions of ND coincide on both sides:

\[ T_{X_0} \to \mathbb{L} \cdot H^0(\nabla) \]

- \( X \) smooth \( G \)-scheme together with a \( G \)-equivariant map \( \mu : X \to \mathbb{G}_m \).

We get a map \( [\mu] : [X/\mathbb{G}_m] \to [\mathbb{G}_m/\mathbb{G}_m] \) of stacks.

Recall that \([\mathbb{G}_m/\mathbb{G}_m]\) has a 1-symplectic structure \( \omega = d_{\text{fr}}(\mathbf{1}) \otimes \delta^i \).

Let's see what it means to have a Lagrangian structure on \([\mu]\):

\[ [\mu]^*\omega = d_{\text{fr}}(\mu^*x_i) \otimes \delta^i = \mu^*\omega. \]

Assume that \( \omega \) is in the image of \((S^2_{\mathbf{X}})^G \to \mathcal{A}^\phi(X) \).

Then \( d_{\text{fr}}(\mu^*x_i) \otimes \delta^i = (\nabla_i \omega) \otimes \delta^i \). I.e. \( \mu^*d_{\text{fr}}(x_i) = \nabla_i \omega \)

This is the moment map condition! Next we can assume \( d_{\text{fr}}(\omega) = 0 \)

and get a Lagrangian structure.

Let us check the condition for ND:

\[
\begin{array}{c}
T_{[X/\mathbb{G}_m]} = (\mathcal{O}_X \otimes \mathcal{O}[\mathcal{T}_X]) \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{L}_X = \mathcal{L}_X' = (\mu^*x_i \otimes [\mathcal{T}_X] \oplus \mathcal{T}_X')
\end{array}
\]

\( \Rightarrow \) our isotropic structure is ND iff \( \omega \) is ND.

**Conclusion:** moment maps are Lagrangian structures.
Lagrangian correspondences

Theorem: $X, Y, Z$ derived stacks with $n$-symplectic structures.

$$L \xrightarrow{f} X \times Y \text{ and } M \xrightarrow{g} Y \times Z$$ maps with Lagrangian structures.

Then $L \times M \xrightarrow{f \times g} X \times Z$ has a Lagrangian structure.

Corollary [Piv]:

$$L \xrightarrow{f} Y + \text{Lagrangian structure} \Rightarrow L \times M \text{ (i-i)-symplectic.}$$

Proof of Corollary:

Consider $X = Z = \ast \ast$.

Application of corollary:

$X \xrightarrow{\psi} g^\ast$ moment map, $O \xrightarrow{\theta} g^\ast$ coadjoint orbit.

$$\Rightarrow [\psi O] \xrightarrow{\theta^\ast} [g^\ast] \text{ and } [\theta O] \xrightarrow{\psi^\ast} [g^\ast] \text{ Lagrangian.}$$

Hence $[\psi O] \times [\theta O] = \left[ X \times O \right]$ is $\ast \ast$ or $O$-symplectic.

Application of theorem:

$\Psi: X \to Y$ symplectomorphism, i.e. $\text{graph}(\Psi): X \to X \times Y$ Lagrangian.

Then for any Lagrangian $L \to Y$, $\mathcal{R} L \subset \mathcal{R} \Psi^{-1}(L) = X \times Y \to X$ Lagrangian.

Proof of Theorem:

Lagrangian structure on $f: f^\ast \pi_x^\ast \omega_x \circ f^\ast \pi_y^\ast \omega_y$.

Lagrangian structure on $g: g^\ast \pi_y^\ast \omega_y \circ g^\ast \pi_z^\ast \omega_z$.

$$\Rightarrow \pi_x^\ast \omega_x = \pi_L^\ast f^\ast \pi_x^\ast \omega_x \sim \pi_L^\ast f^\ast \pi_y^\ast \omega_y \sim \pi_M^\ast g^\ast \pi_y^\ast \omega_y \sim \pi_M^\ast g^\ast \pi_z^\ast \omega_z = \pi_T^\ast \omega_z$$

We have an isotropic structure, denoted $\mathfrak{g}$, on $R: N := L \times M \to X \times Y$.

Let's prove it is ND.

- We have a fiber sequence $T_N \to T_L \oplus T_M \to T_Y$ (all sheaves are implicitly pulled-back on $N$ for simplicity).
- $R: N \to X \times Z$ factors into $N \to (X \times Y) \times (Y \times Z) \to X \times Y \to X \times Z$.

Hence we have a fiber sequence $L_R \to L_N \to L_{X \times Y \times Z}$.

We therefore have a diagram:

$$\begin{array}{ccc} T_N & \xrightarrow{\text{given by } \delta} & T_L \oplus T_M \oplus T_Y \xrightarrow{\text{given by } \eta} \mathcal{R} \Psi \xrightarrow{\text{given by } \eta} \mathcal{R} \Psi \end{array}$$

- The left-most square commutes because $\eta$ is the composition of $\delta^\ast$ and $\eta^\ast$.
- The other square does because $\delta^\ast$ and $\eta^\ast$ are compatible (at $\mathcal{R} \Psi$)