From Lie theory to algebraic geometry and back

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(joint work with Andrei Caldararu and Junwu Tu)

Given a finite dimensional Lie algebra $\mathfrak{g}$ over a field k of zero characteristic, Duflo's Theorem [5] asserts that the restriction of the symmetrization map (also known as PBW isomorphism)

$$\sigma : \mathrm{S}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g})$$

to $\mathfrak{g}$-invariants, precomposed with the contraction against the series

$$\partial := \det\left(\sqrt{\frac{e^{ad}-1}{ad}}\right) = \exp\left(\sum_{k\geq 1} c_k \mathrm{tr}(ad^k)\right) \in \widehat{\mathrm{S}}(\mathfrak{g}^*)^{\mathfrak{g}}$$

is an algebra isomorphism[1].

Analogously, given a smooth algebraic variety over k, we can consider the Hochschild-Kostant-Rosenberg (HKR) isomorphism

$$\bigoplus_k \Lambda^k(\mathcal{T}_X)[-k] \xrightarrow{\sim} p_{1*}\left(\mathrm{R}\mathcal{H}om_{X\times X}(\mathcal{O}_X,\mathcal{O}_X)\right)$$

in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$. Like in the above situation the (sheaf of) algebra on the right is not (graded) commutative in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$[2], but its image under $\mathrm{R}\Gamma(-)$ is. Here again, we need to precompose with the contraction against an element

$$\partial := \det\left(\sqrt{\frac{at}{e^{at}-1}}\right) \in \bigoplus_k H^k(X,\Omega_X^k)$$

to get the following result, first guessed by Kontsevich [9]:
Theorem **([3]).** *$HKR\circ\partial\cdot$ is an algebra isomorphism.*

The element $at \in H^1\left(X,\Omega_X^1\otimes\mathcal{E}nd(\mathcal{T}_X)\right)$ is the Atiyah class of the tangent bundle. Recall that the Atiyah class of a vector bundle $E\to X$ is the obstruction against the existence of a connection on E. More abstractly it is the class of the extension

$$0\to\Omega_X^1\otimes E\to J_X^1(E)\to E\to 0\,,$$

and can be viewed as a map $\mathcal{T}_X[-1]\otimes E\to E$ in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$. One can prove (see e.g. [8]) that when E is $\mathcal{T}_X[-1]$ this turns $\mathfrak{g}=\mathcal{T}_X[-1]$ into a Lie algebra object in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$, and that $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$ is tautologically equivalent to the representation category of this Lie algebra object. Later on it was proved (see e.g. [10]) that $\mathrm{U}(\mathfrak{g})\cong p_{1*}\left(\mathrm{R}\mathcal{H}om_{X\times X}(\mathcal{O}_X,\mathcal{O}_X)\right)$. This construction actually becomes more or less tautological, and also works for singular varieties, if one considers the (co)tangent complex [7] instead.

[1] Here $ad\in\mathfrak{g}^*\otimes\mathrm{End}(\mathfrak{g})$ is the adjoint action.

[2] While it is in $\mathcal{D}(k_X-\mathrm{mod})$.

From this we observe that HKR **is** PBW, and that the above Theorem is a straightforward translation of Duflo's result. Namely,

$$\mathrm{R}\Gamma(-) = \mathrm{RHom}_X(\mathcal{O}_X, -) = \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{g})}(\mathbf{1}, -) = (-)^{\mathfrak{g}}\,.$$

Going back to Lie algebras, there are (conjectural) generalizations of Duflo's result. They concern homogeneous spaces, or (at the infinitesimal level) inclusions $\mathfrak{h} \subset \mathfrak{g}$ of finite dimensional Lie algebras. More precisely, under the assumption that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ Duflo conjectured [6] that the Poisson center of $\mathrm{S}(\mathfrak{m})^{\mathfrak{h}}$ is isomorphic (as an algebra) to the center of $\big(\mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\big)^{\mathfrak{h}}$. This conjecture seems far too much difficult for us[3]. We will therefore concentrate on an easier question:

Question. *Under what assumption do we have an isomorphism of $\mathfrak{h}$-modules*

$$\mathrm{S}\,(\mathfrak{g}/\mathfrak{h}) \stackrel{\sim}{\longrightarrow} \mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\ ?$$

For this purpose let us rewrite

$$\big(\mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\big)^{\mathfrak{h}} = \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{h})}\big(\mathbf{1}, \mathrm{Res}\circ\mathrm{Ind}(\mathbf{1})\big) = \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{g})}\big(\mathrm{Ind}(\mathbf{1}), \mathrm{Ind}(\mathbf{1})\big)\,.$$

Given a closed embedding $i : X \hookrightarrow Y$ of algebraic varieties, we are going to consider the following Lie algebras in $\mathcal{D}(\mathcal{O}_X\text{−mod})$: $\mathfrak{h} = \mathcal{T}_X[-1] \subset \mathcal{T}_X\big|_Y[-1] = \mathfrak{g}$. Then $\big(\mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\big)^{\mathfrak{h}}$ becomes

$$\mathrm{Ext}_Y(X,X) := \mathrm{RHom}_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) = \mathrm{RHom}_X(i^*i_*\mathcal{O}_X, \mathcal{O}_X)\,.$$

Therefore the above question translate into asking under what assumption do we have an isomorphism

$$\bigoplus_k \Lambda^k(\mathcal{N}_{X,Y})[-k] \stackrel{\sim}{\longrightarrow} \mathrm{R}\mathcal{H}om_X(i^*i_*\mathcal{O}_X, \mathcal{O}_X)\,.$$

in $\mathcal{D}(\mathcal{O}_X\text{−mod})$. To answer this question let us consider the normal bundle exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_Y\big|_X \to \mathcal{N}_{X,Y} \to 0\,,$$

which gives a map $\mathcal{N}_{X,Y} \to \mathcal{T}_X[1]$, by tensoring with $\mathcal{N}_{X,Y}$ and composing with the Atiyah "class" of $\mathcal{N}_{X,Y}$ we get an extension $\alpha_{X,Y} \in \mathrm{Ext}^2_X(\mathcal{N}^{\otimes 2}_{X,Y}, \mathcal{N}_{X,Y})$:

$$\mathcal{N}_{X,Y} \otimes \mathcal{N}_{X,Y} \to \mathcal{T}_X[-1] \otimes \mathcal{N}_{X,Y}[2] \to \mathcal{N}_{X,Y}[2]\,.$$

Theorem (Arinkin-Caldararu [1]**).** *The following conditions are equivalent:*

1. $\alpha_{X,Y} = 0$.
2. *$\mathcal{N}_{X,Y}$ admits an extension to the first infinitesimal neighbourhood $X^{(1)}$ of X in Y.*
3. *the answer to the question is YES.*

[3] Even in the symmetric space case, when $\mathfrak{h}$ is the fixed point subalgebra of an involution on $\mathfrak{g}$, the conjecture is not solved despite some very good improvements by Cattaneo-Torossian [4].

Going back once again to Lie algebras, it is now very natural to take a look at the exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ of $\mathfrak{h}$-modules, and the induced map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{h}[1]$ in $\mathcal{D}(\mathfrak{h}-\text{mod})$. Inspired by the geometric situation, we tensor with $\mathfrak{g}/\mathfrak{h}$ and then compose with the $\mathfrak{h}$-action to obtain a class $\alpha_{\mathfrak{h},\mathfrak{g}} \in \mathrm{Ext}^1\big((\mathfrak{g}/\mathfrak{h})^{\otimes 2}, \mathfrak{g}/\mathfrak{h}\big)$:

$$\mathfrak{g}/\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h}[1] \to \mathfrak{g}/\mathfrak{h}[1]\,.$$

By complete analogy with Arinkin-Caldararu result we can prove that:

Theorem ([2]). *The following conditions are equivalent:*

(1) $\alpha_{\mathfrak{h},\mathfrak{g}} = 0$.

(2) $\mathfrak{g}/\mathfrak{h}$ *"admits an extension to the first infinitesimal neighbourhood* $\mathfrak{h}^{(1)}$ *of* $\mathfrak{h}$ *in* $\mathfrak{g}$*".*

(3) *the answer to the question is YES.*

We now end this short note by explaining the meaning of condition (2) in the Theorem. First of all we define $\mathfrak{h}^{(1)}$ to be the Lie algebra freely generated by $\mathfrak{g}$ and subjected to the relations

$$[h,g] = [h,g]_{\mathfrak{g}}\,, \quad h \in \mathfrak{h}\,, \quad g \in \mathfrak{g}\,.$$

There is a Lie algera inclusion $\mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)}$, and we say that an $\mathfrak{h}$-module M "admits an extension to $\mathfrak{h}^{(1)}$" if there exists an $\mathfrak{h}^{(1)}$-module $M^{(1)}$ such that $\mathrm{Res}(M^{(1)}) = M$. It can be proved that $\mathcal{T}_{X^{(1)}}\big|_X[-1]$ **is truely** isomorphic to $\mathfrak{h}^{(1)}$ as a Lie algera object in $\mathcal{D}(\mathcal{O}_X\text{-mod})$ (but since $X^{(1)}$ is not smooth, we have to consider the tangent complex instead of the tangent sheaf). E.g. when $X = \{0\} \subset \mathbb{A}^n = Y$ we have that the shifted tangent complex of $X^{(1)}$ is a free Lie algebra in n odd generators.

References

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