Hochschild cohomology of smooth algebraic varieties

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Let X be a smooth algebraic variety (over a field of zero characteristic). We define its *Hochschild cohomology ring* to be

$$HH^{\cdot}(X) := Ext^{\cdot}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)\,,$$

where $\Delta : X \to X \times X$ is the diagonal map.

1. Hochschild cohomology as the (hyper)cohomology of poly-differential operators

1.1. **Local Hochschild cochains.** We have the following sequence of ring isomorphisms:

$$\begin{aligned} Ext^{\cdot}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) &\cong \mathbb{R}\Gamma\big(X\times X, \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X\times X}}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)\big) \\ &\cong \mathbb{R}\Gamma\big(X, \mathbb{R}\mathcal{H}om_{(\pi_1)_*\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X)\big) \\ &\cong \mathbb{R}\Gamma\big(X, \mathbb{R}\mathcal{H}om_{(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}}(\mathcal{O}_X, \mathcal{O}_X)\big)\,, \end{aligned}$$

where π_1 is the first projection and $\widehat{X\times X}$ is the formal neighborhood of the diagonal in $X \times X$. The last identification comes from the fact that $(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}$ is flat over $(\pi_1)_*\mathcal{O}_{X\times X}$.

Below we provide an explicit description of the algebra

$$\mathbb{R}\mathcal{H}om_{(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}}(\mathcal{O}_X, \mathcal{O}_X)\big)$$

of *local Hochschild cochains*, as an algebra object in $\mathrm{D}(\mathcal{O}_X\!-\!mod)$.

1.2. **Local Hochschild cochains as Lie algebroid Hochschild cochains.** Let $\mathcal{L}$ be a Lie algebroid over X which is locally free of finite rank as an $\mathcal{O}_X$-module. As an example to keep in mind, one can consider the tangent Lie algebroid $\mathcal{L} = T_X$. There are several algebraic objects one can associate to $\mathcal{L}$, such as:

- its *universal envelopping algebra* $U(\mathcal{L})$, which is a filtered Hopf algebroid. Whenever $\mathcal{L} = T_X$, $U(\mathcal{L})$ is the algebra of differential operators on X.
- its *jet algebra* $J(\mathcal{L})$, defined as the $\mathcal{O}_X$-linear dual to $U(\mathcal{L})$, and that one can view as the algebra on the formal groupoid integrating $\mathcal{L}$.
  Whenever $\mathcal{L} = T_X$, $J(\mathcal{L})$ is isomorphic to $(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}$.

  Sketch of proof of this fact. The isomorphism sends a section f of $(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}$ to the jet j_f defined as follows: j_f sends a differential operator P to $(id\otimes P)(f)$, which is a section of $\mathcal{O}_X$ because P has finite order. □

- its *Hochschild cohomology ring* $HH^{\cdot}_{\mathcal{L}} := Ext_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X)$.

The upshot is that we can describe the algebra of local Hochschild cochains as

$$\mathbb{R}\mathcal{H}om_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X)\,,$$

with $\mathcal{L} = T_X$.

1.3. **An explicit description of Lie algebroid Hochschild cochains.** Borrowing the notation from above, we have the following:

Proposition 1.1 ([2]). *There is an isomorphism of algebras*

$$\mathbb{R}\mathcal{H}om_{J(\mathcal{L})}(\mathcal{O}_X,\mathcal{O}_X)\cong(\mathcal{D}^{poly,\cdot}_{\mathcal{L},X})^{op}$$

in $\mathrm{D}(\mathcal{O}_X{-}mod)$. *Here* $\mathcal{D}^{poly,n}_{\mathcal{L},X}:=U(\mathcal{L})^{\otimes_{\mathcal{O}_X}n}$, *the product is the concatenation, and the differential is the Cartier (a-k-a co-Hochschild) differential for the coalgebra* $U(\mathcal{L})$.

Whenever $\mathcal{L}=T_X$, $\mathcal{D}^{poly,n}_{\mathcal{L},X}(U)$ is the subcomplex of the Hochschild complex of $\mathcal{O}_X(U)$ consisting of these cochains that are differential operators in each argument.

Sketch of proof of the Proposition. Note that $J(\mathcal{L})$ is a topological algebra, and that the morphism

$$\mathbb{R}\mathcal{H}om^{cont.}_{J(\mathcal{L})}(\mathcal{O}_X,\mathcal{O}_X)\to\mathbb{R}\mathcal{H}om_{J(\mathcal{L})}(\mathcal{O}_X,\mathcal{O}_X)$$

is an isomorphism in $\mathrm{D}(\mathcal{O}_X{-}mod)$. Let us now give an explicit resolution $\mathcal{B}^{\cdot}J(\mathcal{L})$ of $\mathcal{O}_X$ as a topological $J(\mathcal{L})$-module:

$$\mathcal{B}^nJ(\mathcal{L})=J(\mathcal{L})^{\hat{\otimes}(n+1)}$$

and the differential sends $j_0\otimes\cdots\otimes j_n$ to

$$j_0j_1\otimes\cdots\otimes j_n+\cdots+(-1)^nj_0\otimes\cdots\otimes j_{n-1}j_n+(-1)^{n+1}j_0\otimes\cdots\otimes j_{n-1}j_n(1)\,.$$

We conclude by noting there is a (right) action of $\mathcal{D}^{poly,\cdot}_{\mathcal{L},X}$ on $\mathcal{B}^{\cdot}J(\mathcal{L})$. □

2. Hochschild cohomology as the cohomology of poly-vector fields

2.1. **The Hochschild–Kostant–Rosenberg (HKR) theorem.** Let $\mathcal{L}$ be a Lie algebroid as above. The skew-symmetrization map

$$\begin{array}{rcl}\wedge^{\cdot}_{\mathcal{O}_X}\mathcal{L} & \longrightarrow & \mathcal{D}^{poly,\cdot}_{\mathcal{L},X}\\ u_1\wedge\cdots\wedge u_m & \longmapsto & \dfrac{1}{m!}\displaystyle\sum_{\sigma\in\mathfrak{S}_m}\varepsilon^{\sigma}u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(m)}\end{array}$$

is a quasi-isomorphism of sheaves, known as the *Hochschild–Kostant–Rosenberg (or, HKR) morphism.* It induces in particular as isomorphism *of graded vector spaces*

$$\mathrm{HKR}\,:\,H^{\cdot}(X,\wedge^{\cdot}\mathcal{L})\xrightarrow{\sim}HH^{\cdot}_{\mathcal{L}}(X)\,.$$

2.2. **A multiplicative version of the HKR morphism.** Consider the short exact sequence

$$0 \to \mathcal{L} \to U(\mathcal{L})_+^{\leq 2} \to S^2_{\mathcal{O}_X}\mathcal{L} \to 0\,,$$

where $U(\mathcal{L})_+$ denotes the augmentation ideal of $U(\mathcal{L})$, i.e. $\mathcal{L}$-differential operators vanishing on constants. This extension defines the *Atiyah class of* $\mathcal{L}$:

$$\mathrm{At}_{\mathcal{L}} \in Ext^1_X\big(S^2(T_X),T_X\big) \to Ext^1_X\big(T_X^{\otimes 2},T_X\big) \cong Ext^1_X\big(T_X,End(T_X)\big) \cong H^1\big(X,\Omega^1_X \otimes End(T_X)\big)\,.$$

We derived from it the *Todd genus of* $\mathcal{L}$:

$$\mathrm{Td}_{\mathcal{L}} := \det\sqrt{\frac{\mathrm{At}_{\mathcal{L}}}{1-exp(-\mathrm{At}_{\mathcal{L}})}} \in \oplus_k H^k(X,\Omega^k_X)\,.$$

It is given by a formal expression involving sums of products of $c_k = \mathrm{tr}(\mathrm{At}^k_{\mathcal{L}})$'s.

Theorem 2.1 ([1])**.** *Composing the HKR morphism together with the contraction against the Todd genus leads to a ring isomorphism*

$$\mathrm{HKR}\circ(\mathrm{Td}_{\mathcal{L}}\llcorner -)\;:\;H^{\cdot}(X,\wedge^{\cdot}\mathcal{L}) \xrightarrow{\sim} HH^{\cdot}_{\mathcal{L}}(X)\,.$$

2.3. **Sanity check: the original HKR morphism is not multiplicative.** Let us show that when X is a K3 surface and $\mathcal{L}=T_X$ the HKR morphism is not a ring isomorphism in cohomology. Using Theorem 2.1 above this is equivalent to show that the contraction $\mathrm{Td}_{T_X}\llcorner -$ against the Todd genus is not a ring isomorphism. Note that, for degree reasons, in the case of a K3 surface the Todd genus takes the form $exp(ac_1+bc_2)$, with a and b non-zero. Since the contraction $c_1\llcorner -$ with c_1 is known to be a derivation, we are left to show that the contraction with c_2 is not a derivation.

Sketch of proof that $c_2\llcorner -$ is not a derivation. Let ω be the symplectic form on X and Π be the corresponding Poisson bivector. Observe that c_2 is proportional to $[\omega\wedge\bar\omega] \in H^2(X,\Omega^2_X)$.

One the one hand, we have that $c_2\llcorner(\Pi\wedge\Pi)=0$ ($\Pi\wedge\Pi=0$ because of dimension). On the other hand, $(c_2\llcorner\Pi)\wedge\Pi = \Pi\wedge(c_2\llcorner\Pi)$ is proportional to $[\bar\omega\wedge\Pi]$, which is non-zero in $H^2(X,\wedge^2 T_X)$. Hence $c_2\llcorner -$ is not a derivation. □

References

[1] D. Calaque & M. Van den Bergh, *Hochschild cohomology and Atiyah classes*, Advances in Mathematics **224** (2010), no. 5, 1839–1889

[2] D. Calaque, C.A. Rossi & M. Van den Bergh, *Hochschild cohomology for Lie algebroids*, International Mathematical Research Notices **2010** (2010), no. 21, 4098–4136.