PLEASE DO NOT PRINT (THIS IS NOT THE FINAL VERSION)

1. Symplectic linear algebra

Definition 1.1: a presymplectic, resp. symplectic, vector space is a pair (V,ω) where V is a finite dimensional vector space and $\omega : \Lambda^2 V \longrightarrow \mathbb{K}$ is a skew-symmetric bilinear form, resp a non-degenerate skew symmetric bilinear form.

"non-degenerate" means that the linear map $\begin{array}{rcl} V & \longrightarrow & V^* \\ v & \longmapsto & \omega(v,-) \end{array}$ is an isomorphism.

Lemma 1.2: let (V,ω) be a presymplectic vector space. There exists a basis $(e_1,\dots,e_n,f_1,\dots,f_n,g_1,\dots,g_k)$ of V such that $\omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*$, where $(e_1^*,\dots\dots)$ is the dual basis of V^*.

Proof: • if $\omega \equiv 0$ then the property holds with $n=0$.

• if $\omega \not\equiv 0$ then $\exists\, e_1, f_1$ such that $\omega(e_1,f_1)=1$.

We consider $W := \mathbb{K}\cdot\{e_1,f_1\}$ and its (symplectic) orthogonal $W^\circ := \{v \in V \,/\, \forall y \in W,\ \omega(v,y)=0\}$.

Claim: $V = W \oplus W^\circ$

Proof: • we first prove that $V = W + W^\circ$.

$\forall v \in V$, we have
$$v = \underbrace{\omega(v,f_1)e_1 - \omega(v,e_1)f_1}_{\in W} + \underbrace{\big(v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1\big)}_{\in W^\circ}$$

• it remains to prove that $W \cap W^\circ = 0$

If $v = \alpha e_1 + \beta f_1$ is such that $\underset{-\beta}{\omega(v,e_1)} = \underset{\alpha}{\omega(v,f_1)} = 0$ then $v=0$.

Observe that $\omega = \omega_{|W} + \omega_{|W^\circ} = e_1^* \wedge f_1^* + \omega_{|W^\circ}$.

The lemma is then proved by induction (and using that $\dim(V) < +\infty$). □

Consequence 1.3 : if (V,ω) is symplectic (i.e. ω is ND) then $\dim(V) = 2n$. Moreover, any two symplectic vector spaces of the same dimension are isomorphic!

(a **morphism** $(V,\omega) \longrightarrow (V',\omega')$ of (pre)symplectic vector spaces is just a linear map $\varphi : V \longrightarrow V'$ such that $\omega'(\varphi(x),\varphi(y)) = \omega(x,y)$. Observe that in the symplectic case, φ is automatically injective.)

Examples 1.4 : a) let U be a finite dimensional vector space. $V := U \oplus U^*$ together with $\omega(x+\xi, x'+\xi') := \xi(x') - \xi'(x)$ is a symplectic vector space.

b) let (U,ω) be a presymplectic vector space. Define $\ker(\omega) := \{u \in U \mid \omega(u,-) \equiv 0\} = U^\circ$. Then ω descends to a form $\bar{\omega}$ on $V := U/\ker(\omega)$, and $(V,\bar{\omega})$ is a symplectic vector space.

(IMPORTANT) c) let V be a finite dimensional vector space $/\mathbb{C}$, together with a ND Hermitian form $\langle\,|\,\rangle$. $(V, \omega := \mathrm{Im}(\langle\,|\,\rangle))$ is a symplectic vector space $/\mathbb{R}$.

Definitions 1.5: let (V,ω) be a symplectic vector space. A subspace $W \subset V$ is

a) symplectic if $(W, \omega_{/W})$ is.

b) isotropic if $W \subset W^\circ$ (i.e. $\omega_{/W} \equiv 0$)

c) coisotropic if $W^\circ \subset W$

d) Lagrangian if $W = W^\circ$.

Examples 1.6 : a) if V is as in example 1.4.c) then any $\mathbb{C}$-subspace $W \subset V$ is symplectic.

b) any line in V is isotropic

c) any hyperplane in V is coisotropic.

d) if $V = U \oplus U^*$ is as in example 1.4.a) then U and U^* are Lagrangian.

d') if V is as in example 1.4.c) with orthogonal basis $(e_1, \dots, e_n)$, then $\mathbb{R}\cdot\{e_1, \dots, e_n\} \subset V$ is Lagrangian.

Exercise 1.7: let (V, ω) be a symplectic vector space.

- prove that for $W \subset V$, $\dim(W) + \dim(W^\circ) = \dim(V)$.
- prove that $W \subset V$ is symplectic if and only if $W \cap W^\circ = \{0\}$.

Theorem 1.8: let (V, ω) be a symplectic vectorspace.

a) Lagrangian subspaces in V are exactly the maximal isotropic ones.

b) any Lagrangian subspace $L_1 \subset V$ admits a Lagrangian Complement L_2.

Proof: a) we are going to prove that any isotropic subspace $W \subset V$ is contained in a Lagrangian one (one can then use exercise 1.7.a) to conclude).

Assume $W \subsetneq W^\circ$. Pick any $v \in W^\circ \setminus W$, and define $\tilde{W} := W \oplus \mathbb{K}v$

Observe that $\omega_{/\tilde{W}} \equiv 0$ (precisely because $W \subset W^\circ$ and $v \in W^\circ$).

Therefore $\tilde{W} \subset \tilde{W}^\circ$. Repeat the procedure in case $\tilde{W} \neq \tilde{W}^\circ$.

b) let $L \subset V$ be a Lagrangian subspace. Then pick any isotropic subspace $W \subset V$ such that $L \cap W = \{0\}$ (e.g. $W = \{0\}$).

Claim: if W is not Lagrangian, then $\exists v \in W^\circ \setminus W$ s.t. $(W \oplus \mathbb{K}v) \cap L = \emptyset$.

Proof: Assume that $W^\circ \subset L + W$. Then $L \cap W^\circ = (L + W)^\circ \subset (W^\circ)^\circ = W$.

Therefore $L \cap W^\circ \subset L \cap W = \{0\}$, therefore $\dim(W^\circ) \leq \dim(L)$ which cannot hold unless W is maximal (and thus Lagrangian).

So, if W is not Lagrangian then we pick such an element v and define $\tilde{W} := W \oplus \mathbb{K}v$. Then $\tilde{W}$ is isotropic and such that $\tilde{W} \cap L = \emptyset$.

We repeat the procedure until we reach a Lagrangian subspace.

Remark 1.9: Let (V,ω) be a symplectic vector space, and let $W\subset V$. Then $W=(W^\circ)^\circ$ (it is easy to see that $W\subset(W^\circ)^\circ$, and we know that $\dim(W)+\dim(W^\circ)=\dim(V)$).

$\Rightarrow$ most statements about isotropic subspaces can be rephrased in terms of coisotropic ones (e.g. Lagrangians are the minimal coisotropic).

Construction 1.10: Let (V,ω) be a symplectic vector space, and let $W\subset V$ be a coisotropic subspace. Then by Example 1.4.b) $\omega|_W$ descends to a ND form $\tilde\omega$ on $W^{red} := W/W^\circ$, which is then symplectic.

Moreover, if $L\subset V$ is a Lagrangian subspace then $L^{red} := L\cap W/W^\circ \subset W/W^\circ$ is Lagrangian too. Namely, $(L\cap W+W^\circ)^\circ = (L^\circ + W^\circ)\cap W = L\cap W + W^\circ$.

Example 1.11: Let $\Gamma=(V,E)$ be an electrical network. I.e. a (non-directed) graph.

Assume we are given a vector of conductances $\rho\in\mathbb{R}^E$. The corresponding energy of the electric network, expressed in terms of potentials $(x_v)_{v\in V}$, is given by $H_\rho(\underline{x}) = \sum_{e\in E}\rho_e x_e^2$, where $x_e := |x_v - x_w|$ if e joins v and w.

We now consider a set of external nodes $V_0\subset V$, and we apply to them potentials $(q_v)_{v\in V_0}\in\mathbb{R}^{V_0}$. Question: what is the response of the network? In other words, what currents do we measure at the external nodes?

Answer: apply Kirchhoff law (saying that the sum of currents at an internal node is zero) repeatedly.

There is a symplectic geometric way of doing this computation.

We consider the symplectic vector space $\mathbb{R}^V\oplus(\mathbb{R}^V)^*$ with coordinates x_v,ξ_v and symplectic form $\omega := \sum_{v\in V} x_v\wedge\xi_v$.

The quadratic form H_q (on V) may be viewed as a linear map $\mathbb{R}^V\to(\mathbb{R}^V)^*$.

We denote by L_q its graph in $\mathbb{R}^V\oplus(\mathbb{R}^V)^* =: W$, which happens to be Lagrangian.

We then define the Kirchhoff coisotropic subspace $K := \mathbb{R}^V\oplus(\mathbb{R}^{V_0})^\perp = \{\xi_v = 0 \mid v\in V\setminus V_0\}$.

Finally, $W^{red} = \mathbb{R}^{V_0}\oplus(\mathbb{R}^{V_0})^*$ and L_q^{red} is the graph of the response function.

(The response function $\mathbb{R}^{V_0}\to(\mathbb{R}^{V_0})^*$ sends the potentials $(q_v)_{v\in V_0}$ to the measured currents).

2. The Lagrangian Grassmannian & the Maslov index

Definition 2.1: the Lagrangian Grassmannian $\Lambda(V,\omega)$ of a symplectic vector space (V,ω) is the set of all Lagrangian subspaces in it.

Study of $\Lambda(V,\omega)$

$GL_W(V)=\{\varphi\in GL(V) \mid \varphi(W)\subset W\}$

First observe that $\Lambda(V,\omega)$ is a closed subset of the Grassmannian $Gr(V,n)$ of n-dimensional subspaces in V, where $n:=\frac{\dim(V)}{2}$.
Recall that $GL(V)/GL_W(V)\xrightarrow{\sim} Gr(V,n)$, given any choice of an n-dimensional $W\subset V$. The maps sends $[\varphi]$ to $\varphi(W)$.

Exercise 2.2: assume that W is Lagrangian. Then prove that the above isomorphism restricts to an isomorphism $Sp(V)/Sp_W(V)\xrightarrow{\sim}\Lambda(V,\omega)$, where $Sp(V)=\{\varphi\in GL(V) \mid \varphi \text{ symplectic morphism}\}$.
Deduce that $\Lambda(V,\omega)$ is a smooth manifold.

We now provide an explicit atlas for $\Lambda(V,\omega)$. For any Lagrangian $L\subset V$ we define an open subset $U_L:=\{L'\in\Lambda(V,\omega) \mid L'\cap L=\{0\}\}\subset\Lambda(V,\omega)$.
For any Lagrangian $\tilde{L}\in U_L$, we consider the isomorphism

$$\varphi:\mathcal{B}(\tilde{L}):=\{\text{bilinear forms on } \tilde{L}\}\longrightarrow \tilde{U}_L:=\{L'\in Gr(V,n) \mid L'\cap L=\{0\}\}$$

that sends $b\in\mathcal{B}(\tilde{L})$ to the graph of $A_b:\tilde{L}\xrightarrow{b}\tilde{L}^*\simeq L\subset V$.

Claim 2.3: b is symmetric if and only if $L_b:=\varphi(b)$ is Lagrangian.

Proof: $L_b=\{x+A_b(x) \mid x\in\tilde{L}\}$ and $b(x,y)=\omega(A_b(x),y)$.
We then compute: for any $x,y\in L_2$,

$$\begin{aligned} b(x,y)-b(y,x) &= \omega(A_b(x),y)+\omega(x,A_b(y)) \\ &= \qquad\qquad '' \qquad\qquad + \overbrace{\omega(x,y)}^{=0}+\overbrace{\omega(A_b(x),A_b(y))}^{=0} \\ &= \omega(x+A_b(x),\,y+A_b(y)) \end{aligned}$$

This last term is zero if and only if L_b is Lagrangian. $\square$

Therefore U_L is isomorphic to $Q(\tilde{L}) := \{\text{quadratic forms on } \tilde{L}\}$.
In particular we see that the dimension of $\Lambda(V,\omega)$ is $\frac{n(n+1)}{2}$.

<u>Exercise 2.4</u>: compute the transition functions for this atlas and deduce that there is a <u>canonical</u> isomorphism $T_L\Lambda(V,\omega) \xrightarrow{\sim} Q(L)$.

Yet another description of $\Lambda(V,\omega)$

We can assume without loss of generality that V admits a complex structure together with a ND Hermitian form $\langle\, | \,\rangle$ such that $\omega = \mathrm{Im}(\langle\, | \,\rangle)$. We know from Example 1.6.d') that if $(e_1,\dots,e_n)$ is a $\mathbb{C}$-basis of V, orthonormal w.r.t. $\langle\, | \,\rangle$, then $L := \mathbb{R}\cdot\{e_1,\dots,e_n\} \subset V$ is Lagrangian. Conversely:

<u>Proposition 2.5</u>: any Lagrangian subspace arises that way.

<u>Proof</u>: first observe that $(-,-) := \mathrm{Re}(\langle - | - \rangle) = \omega(J(-),-)$, where J is the multiplication by i, is positive definite. Let $L \subset V$ be a Lagrangian subspace and choose an $\mathbb{R}$-basis $(e_1,\dots,e_n)$ of L which is orthonormal w.r.t. $(-,-)$. Then $(e_1,\dots,e_n)$ is a $\mathbb{C}$-basis of V which is orthonormal w.r.t. $\langle - | - \rangle$: $\langle e_i | e_j \rangle = (e_i,e_j) + i\,\omega(e_i,e_j) = (e_i,e_j) = \delta_{ij}$. $\square$

Therefore $U(V)$ acts transitively on $\Lambda(V,\omega)$, with isotropy group being $O(L)$: $\Lambda(V,\omega) \simeq U(V)/O(L)$.

Maslov index

Let $L \subset V$ be a Lagrangian subspace, and choose a complex structure J on V together with a ND Hermitian form $\langle\, | \,\rangle$ such that $\omega = \mathrm{Im}(\langle\, | \,\rangle)$.
From above, we have an identification $\Lambda(V,\omega) \simeq U(V)/O(L)$.

We then get a differentiable map $\det^2 : \Lambda(V,\omega) \longrightarrow S^1$.
(just observe that $\det(U(V)) = S^1$ and $\det(O(L)) = \{\pm 1\}$)

Claim 2.6: the above map $\det^2$ does not depend on the choices of complex and Hermitian structures on V.

Proof: any two such choices are isomorphic: there is an $A \in GL(V)$ that intertwines the complex and Hermitian structures on both sides. In particular, A is symplectic. The induces isomorphism $U(V)/O(L) \xrightarrow{\sim} U(V)/O(L)$ is obtained via $A\cdot$.

A being symplectic, $\det(A) = 1$ and $\det^2$ is not sensitive to it. □

We therefore have continuous maps $\mathcal{L}(U(V)/O(L)) \longrightarrow \mathcal{L}(S^1) \xrightarrow{\deg} \mathbb{Z}$

- the degree map is invariant under translation at the source.
- the degree map is invariant under translation at the target. & $\det^2$ sends translations by $A \in U(V)$ to translations by $\det^2(A)$ on S^1.

In particular the composed map $\mathcal{L}(U(V)/O(L)) \longrightarrow \mathbb{Z}$ does not depend on the choice of L (two different choices differ by a translation: if $(e_1, \dots, e_n)$ and $(f_1, \dots, f_n)$ are orthonormal basis of L and L', respectively, then $U(V)/O(L) \xrightarrow{\sim} U(V)/O(L')$ via $A\cdot$ where $A(e_i) := f_i$).

Definition 2.7: for any free loop γ in $\Lambda(V, \omega)$, $\deg(\det^2 \circ \gamma)$ is called the Maslov index of γ. We denote it by $m(\gamma)$.

One can easily see that the Maslov index does only depend on the homotopy class of γ. We therefore have a surjective map $\pi_1(\Lambda(V, \omega)) \longrightarrow \mathbb{Z}$ which is independent of any choice (even the choice of a base point).

The Maslov index as an intersection number

We start by stating a very general result:

Lemma 2.8: if X is a manifold and $Z \subset X$ is a connected co-oriented hypersurface such that $X \setminus Z$ is simply connected, then $\pi_1(X, x) \simeq \mathbb{Z}$ $(x \in X)$ and an isomorphism is given by the (algebraic) intersection number of loops with Z.

Before proving this lemma we need to recall one or two things:

- Z is co-oriented means that the normal bundle $TX|_Z / TZ$ is oriented.

(in other words, a co-orientation is the choice of an external normal direction).

- the intersection number is given, when the intersection is transverse, by the following:

$$\sum_{x \in Im(\gamma) \cap Z} Or_x(\dot{\gamma}(\gamma^{-1}(x))) \qquad (Or_x : T_xX/T_xZ \longrightarrow \{\pm 1\}).$$

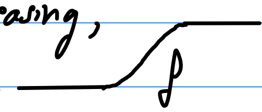
We can moreover always perturb γ (without changing its class in $\pi_1(X)$) in order to make the intersection with Z transverse.

Proof of the Lemma: • let us first prove that the above map is well-defined. I.e. that any two homotopic loops that are transverse to Z have the same intersection number with Z.

strictly speaking $\alpha = \pi_2^* df$, where $\pi_2 : Z \times \mathbb{R} \to \mathbb{R}$

There exists a neighbourhood of Z that is diffeomorphic to $Z \times \mathbb{R}$ and such that $Im(\gamma)$ crosses it finitely many times and in a monotone way w.r.t. the second coordinate (that we will denote "t", for transverse).

Define $\alpha = f'(t)dt$, where f constant outside a compact, increasing, with image being $[0,1]$. Then the intersection number is $\int_{S^1} \gamma^* \alpha$.



This is homotopy invariant after Stockes formula.

As an exercise, prove that this well-defined map is an isomorphism. □

Consider $\Lambda(n) := \Lambda(\mathbb{C}^n) = U(n)/O(n)$ and define $W_k := \{L \in \Lambda(n) \mid \dim(L \cap \mathbb{R}^n) = k\}$

Exercise 2.9: prove that W_k has codimension $\frac{k(k+1)}{2}$.

We apply the previous Lemma to the following situation: $X = \Lambda(n) \setminus \bigcup_{k \geq 2} W_k$ and $Z = W_1$. Observe that $X \setminus Z = \Lambda(n) \setminus \bigcup_{k \geq 1} W_k \simeq Q(\mathbb{R}^n)$ is simply connected.

Then Z is co-oriented by the signature of quadratic forms on the normal bundle. Namely, if $L \in Z$ then $D := L \cap \mathbb{R}^n$ has dimension 1. Remember that $T_L\Lambda(n) \simeq Q(L)$, and observe that through this identification we have $T_LZ \simeq \{q \in Q(L) \mid q|_D = 0\}$. Therefore $T_L\Lambda(n)/T_LZ \simeq Q(D) \xrightarrow{\sigma} \{\pm 1\}$.

We finally observe that $\pi_1(\Lambda(n)) = \pi_1(X)$ since $\operatorname{codim}(W_k) \geq 3$ if $k \geq 2$.

Inertia index

Let (V,ω) be a symplectic vector space, and let (L_1, L_2) be a pair of Lagrangian subspaces in it. Given any lagrangian complement $\tilde{L}$ of both L_1 and L_2 (re-read carefully the proof of Theorem 1.8 and convince yourself that it exists), we get a path $\gamma : [0,1] \to \Lambda(V,\omega)$ from L_1 to L_2. Namely:

Construction 2.10: We consider the open subset consisting of those Lagrangian that are transverse to $\tilde{L}$ and identify it with $Q(L_1) := \{\text{quadratic forms on } L_1\}$. Then L_2 corresponds to a quadratic form q_2 and we define $\gamma(t) := t q_2$.

Exercise 2.11: show that the homotopy class of the path γ from Construction 2.9 does not depend on the choice of $\tilde{L}$.

Definition 2.12: given a triple (L_0, L_1, L_2) of Lagrangian subspaces in V, we get an element $[\gamma] \in \pi_1(\Lambda(V,\omega), L_0)$ which is obtained as the class of $\gamma := \gamma_{L_3 L_1} \circ \gamma_{L_2 L_3} \circ \gamma_{L_1 L_2}$. The inertia index of (L_0, L_1, L_2) is defined to be $\tau(L_0, L_1, L_2) := m(\gamma)$.

Proposition 2.13: $\tau(L_0, L_1, L_2)$ is the signature of the quadratic form q on $L_0 \oplus L_1 \oplus L_2$ defined by $q(x_0, x_1, x_2) = \omega(x_0, x_1) + c.p.$ (where $x_i \in L_i$, $i = 0,1,2$).

We won't prove it.

References: • the main reference I used for these first two lectures is [CdV] Y. Colin de Verdière, Méthodes semi-classiques et Théorie spectrale, Chapter 1.2, available on the author's webpage.

• for the inertia index one can have a look at Appendix A.3 of [KS] M. Kashiwara & P. Schapira, Sheaves on Manifolds, Springer.

3. Symplectic manifolds

Definition 3.1: a symplectic manifold is a pair (X,ω) of a manifold X together with a non-degenerate closed 2-form ω.

I.e. $\omega \in \Omega^2(X)$ is such that $d(\omega)=0$ and $\forall x \in X$, ω_x is non-degenerate. Observe that a symplectic manifold automatically inherit an orientation, and even a canonical volume form $\Omega := \frac{1}{n!}\omega^{\wedge n}$ (where $n := \frac{\dim(X)}{2}$).

Examples 3.2: a) Symplectic vector spaces.

b) Cotangent spaces. Let M be a manifold, and consider $X := T^*M$. It carries a canonical 1-form λ, called the Liouville form, that we describe now. Let us denote by $\pi : T^*M \to M$ the projection: $\pi(x,\xi) = x$ $(x \in M, \xi \in T_x^*M)$. Then for any $v \in T_{(x,\xi)}(T^*M)$,

$$\lambda_{(x,\xi)}(v) := \xi(\pi_* v) = \pi^*\xi(v).$$

We finally define $\omega := d(\lambda)$. We obviously have $d(\omega)=0$. In order to prove that it is non-degenerate, let us express ω in local coordinates. Let $(x_1,\dots,x_n)$ be local coordinates on $U \subset M$. Assuming U is small enough we get that $T^*U \simeq U \times \mathbb{R}^n$. Let us then denote by $(\xi_1,\dots,\xi_n)$ coordinates on $\mathbb{R}^n$. In this context:

$$\lambda_{(x,\xi)}(v) = \sum_{i=1}^{n} \xi_i v^i \text{, where } v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} w^j \frac{\partial}{\partial \xi_j}.$$

In other words, $\lambda = \sum_{i=1}^{n} \xi_i dx_i$. Therefore $\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i$.

c) orientable surfaces. Let S be an orientable surface. 2-forms are automatically closed. Non-degeneracy condition boils down to being nowhere vanishing (namely, skew-symmetric bilinear forms on $\mathbb{R}^2$ form a 1-dimensional vector space). Therefore, symplectic forms on S are simply the volume forms.

d) Kähler manifolds. A Kähler manifold is a complex manifold equipped with a ND Hermitian metric of which the imaginary part is a closed 2-form. It is then symplectic.

d') $\mathbb{C}P^n$ is a Kähler manifold (exercise). We will come back to this example later on.

Theorem 3.3 [Darboux]: let (X,ω) be a symplectic manifold and let $x \in X$. There exists an open neighbourhood of x that is symplectomorphic to an open neighbourhood of 0 in $\mathbb{R}^{2n}$ equipped with its standard symplectic structure.

Proof: there exists an open $U \ni x$ that is diffeomorphic to an open $V \subset \mathbb{R}^{2n}$: $\psi : U \xrightarrow{\sim} V$. Let us denote by ω_0 the pull-back of the standard symplectic form by ψ, and write $\omega_t := \omega_0 + t(\omega - \omega_0)$ (so that $\omega_1 = \omega$).

We can assume w.l.o.g. (up to reducing U) that there exists α such that $d(\alpha) = \omega - \omega_0$ and $\alpha_x = 0$. (because $\omega_x = \omega_{0,x}$).

Then denote by X_t the inverse image of $-\alpha$ through $\begin{array}{rcl} TX & \longrightarrow & T^*X \\ v & \longmapsto & \omega_t(v,-) \end{array}$.

There exists a family of (germs of) diffeomorphisms φ_t integrating X_t: $X_{t\,\varphi_t(y)} = \frac{d}{dt}(\varphi_t(y))$ for all $y \in U$ and $\varphi_0 \equiv id$.

Lemma 3.4: $\varphi_t^*\omega_t$ is constant (and therefore is ω_0).

Proof: $\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*(\mathcal{L}_{X_t}(\omega_t) + \frac{d}{dt}\omega_t) = \varphi_t^*(\underbrace{d\iota_{X_t}(\omega_t) + d(\alpha)}_{=0})$. $\square$

Definition 3.5: Let (X,ω) be a symplectic manifold. Let $Y \subset X$ be a submanifold. We say that Y is isotropic, coisotropic, Lagrangian, symplectic whenever T_yY is so in T_yX, for any $y \in Y$, in the sense of Definition 1.5

Examples 3.6: a) a complex submanifold in a Kähler manifold is still Kähler, and therefore symplectic.

Compare with Examples 1.6

b) any 1-dimensional submanifold is isotropic.

c) any hypersurface is coisotropic.

d) the zero section in T^*M is Lagrangian.

d') $\mathbb{RP}^n$ is Lagrangian in $\mathbb{CP}^n$ (more generally, totally real submanifolds in Kähler ones are Lagrangian).

Example 3.7: let α be a closed 1-form on a manifold M.

We denote by $\Gamma_\alpha := \{(x, \alpha_x) \mid x \in M\} \subset T^*M$ the graph of $\alpha: M \to T^*M$. Γ_α is Lagrangian.

Proof: it is sufficient to prove it for $M = \mathbb{R}^n$ (and $T^*M = \mathbb{R}^{2n}$), in which case $\alpha = df$ ($f \in C^\infty(M)$): $\Gamma_\alpha = \{(\underline{x}, \underline{\xi}) \mid \xi_i = \frac{\partial f}{\partial x_i}(x)\ \forall i\}$.

Then $T\Gamma_\alpha = \ker(d\xi_i - d\frac{\partial f}{\partial x_i} \mid i \in I)$. Recall that one can write a vector in $T_{(\underline{x},\underline{\xi})}\mathbb{R}^{2n}$ as $u+v$, where $u = \sum_i u^i dx_i$ and $v = \sum_i v^i d\xi_i$.

In particular $\omega(u_1+v_1, u_2+v_2) = v_2 \cdot u_1 - v_1 \cdot u_2$. Finally, note that if $u+v \in T_{(\underline{x},\underline{\xi})}\Gamma_\alpha$ then $v^i = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}$. Therefore we get that $\omega_{/T\Gamma_\alpha}$ vanishes thanks to the symmetry of 2^{nd} order partial derivatives.

Construction 3.8: Let (X,ω) be a symplectic manifold and let $W \subset X$ be a coisotropic submanifold. Recall that $TW^\circ \subset TW$.

Let us now prove that $[TW^\circ, TW^\circ] \subset TW^\circ$.

Take $u, v \in TW^\circ$ and $w \in TW$ and compute $\omega([u,v], w)$.

Show as an exercise that this is 0 (hint: develop $d(\omega)(u,v,w) = 0$).

$\Longrightarrow$ TW° defines a regular foliation on W.

Moreover, ω is constant along the leaves ($\forall u \in TW^\circ$, $\mathcal{L}_u(\omega) = 0$).

So ω descends to a symplectic form on the leaf space.

⚠ The leaf space might be very singular (in particular, not a manifold). Therefore, the above is true only if, either the leaf space is a manifold, or understood as a local statement ($\forall x \in W$, $\exists\, U \ni x$ st. $U/_{\text{foliation}}$ is a manifold).

Example 3.9: $X = \mathbb{C}^{n+1}$ equipped with $\omega = \sum_{i=0}^{n} dz_i \wedge d\bar{z}_i$.

Consider the "energy" function $H(\underline{z}) := \sum_{i=0}^{n} |z_i^2|$. The level hypersurface $W := \{H = 1\}$ is automatically coisotropic, and one observes that TW° admits a nowhere vanishing section: $\mathcal{X}_{H}|_W$. Moreover, $\mathcal{X}_H$ is the infinitesimal generator of the action of $U(1)$ on $\mathbb{C}^{n+1}$.

Therefore $W/_{S^1} \simeq \mathbb{C}^{n+1} - \{\underline{0}\}/_{S^1 \times \mathbb{R}_{>0}} = \mathbb{C}P^n$ is symplectic

$S^{2n+1} \simeq W \longrightarrow W/_{S^1} \simeq \mathbb{C}P^n$ is called the Hopf fibration

Example 3.10: $X = T^*\mathbb{R}^n$ equipped with $\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i$.

Consider the energy function $H(\underline{x}, \underline{\xi}) := \sum_{i=1}^{n} \xi_i^2$. The level hypersurface $W := \{H = 1\}$ is automatically coisotropic and, as in the previous example, $\mathcal{X}_H|_W$ is a nowhere vanishing section of TW°. But in this case $\mathcal{X}_H$ is the infinitesimal generator of the action of $\mathbb{R}$ given by "ξ-translations in the x variables". In other words $W/_{\mathbb{R}}$ is the manifold of oriented affine lines in $\mathbb{R}^n$, and admits a symplectic structure.

We now study Lagrangian submanifolds in T^*X equipped with $\omega = d(\lambda)$.
Let $L \subset T^*X$ be Lagrangian.

Definition 3.11: the caustic of L is the locus $\Sigma_L := \{x \in L \mid \pi_L : L \to X$ is not a local diffeomorphism at $x\}$

If $x \notin \Sigma_L$ then, in a neighbourhood of x, L is the graph of a differential df, $f: X \to \mathbb{R}$. We say that f is a generating function of L.
We now describe what to do when $x \in \Sigma_L$.

Construction 3.12: Let $\varphi: X \times \mathbb{R}^N \to \mathbb{R}$ be a function. Define
$C_\varphi := \{(x,\theta) \in X \times \mathbb{R}^N \mid \frac{\partial \varphi}{\partial \theta}(x,\theta) = 0\}$. We assume that $\left(d\left(\frac{\partial \varphi}{\partial \theta_i}\right)_{(x,\theta)}\right)_{i=1,\dots,N}$ are linearly independant, so that C_φ is a submanifold.

We define $j_\varphi : C_\varphi \longrightarrow T^*X$; $(x,\theta) \longmapsto \left(x, \frac{\partial \varphi}{\partial x}(x,\theta)\right)$.

Proposition 3.13: j_φ is a Lagrangian immersion and $j_\varphi^* \lambda = d\varphi|_{C_\varphi}$.

4. The non-displaceability theorem

Let (X,ω) be a symplectic manifold.

Definitions 4.1: a) A vector field $v \in \Gamma(X,TX)$ is called symplectic if it satisfies one of the following equivalent properties:

(1) the flow of v preserves ω.

(2) $\mathcal{L}_v(\omega) = 0$.

(3) the 1-form $\omega(v,-)$ is closed.

b) A symplectic vector field v is called Hamiltonian if the 1-form $\omega(v,-)$ is exact.

c) A smooth family of symplectomorphisms is a differentiable map $\Phi : X\times\mathbb{R} \to X$ such that $\varphi_t := \Phi(-,t) : X \to X$ is a symplectomorphism for every $t\in\mathbb{R}$. We often write $\Phi = (\varphi_t)_{t\in\mathbb{R}}$.

d) An Hamiltonian isotopy is a smooth family of symplectomorphisms $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ such that the symplectic vector field $v_t : x \mapsto v_t(x) := \frac{\partial\Phi}{\partial t}(\varphi_t^{-1}(x),t) \in T_xX$ is Hamiltonian for every $t\in\mathbb{R}$.

If $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ is a Hamiltonian isotopy then there exists a smooth map $f : X\times\mathbb{R} \to X$ such that $\omega(v_t,-) = df_t$ (where $f_t := f(-,t)$). We say that f is an integral of Φ.

In general, for a smooth family of symplectomorphisms $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ the submanifold $\Lambda_t := \{(\varphi_t(x), x) \mid x\in X\} \subset X\times X^a$ is Lagrangian. ($X^a := (X,-\omega)$). Let $\Lambda' := \{(\varphi_t(x), x, t) \mid x\in X, t\in\mathbb{R}\} \subset X\times X\times\mathbb{R}$.

Proposition 4.2: the following are equivalent:

(1) there exists a Lagrangian submanifold $\Lambda \subset X\times X^a\times T^*\mathbb{R}$ such that $\pi(\Lambda) = \Lambda'$ ($\pi : X\times X\times T^*\mathbb{R} \to X\times X\times\mathbb{R}$).

(2) Φ is an Hamiltonian isotopy.

Proof: We freely use the identification $T^*\mathbb{R} \simeq \mathbb{R}^2$, with coordinates (t,τ) and symplectic form $d\tau\wedge dt$. Consider $\tau : X\times\mathbb{R} \to \mathbb{R}$ and

$$\Lambda := \{(\varphi_t(x), x, t, \tau(x,t)) \mid x\in X, t\in\mathbb{R}\}.$$

We obviously have $\pi(\Lambda)=\Lambda'$. Given $(x,t)\in X\times\mathbb{R}$, we let $p:=(\varphi_t(x), x, t, \tau(x,t))$. The tangent space $T_p\Lambda$ is generated by $w(v)=(d_x\varphi_t(v), v, 0, d_x\tau_t(v))$ and $W_0=(v_t(x), 0, 1, \frac{\partial\tau}{\partial t}(x,t))$, where $v\in T_xX$.

φ_t symplectic $\Rightarrow$ $\omega(w(v), w(v'))=0$.

Then $\omega(W_0, w(v)) = \omega(v_t(x), d_x\varphi_t(v)) - d_x\tau_t(v)$

$$= \varphi_t^*(\omega(v_t,-))_x(v) - \varphi_t^*(d(\tau_t\circ\varphi_t^{-1}))_x(v)$$

This vanishes for all v if and only if $\omega(v_t,-)=d(\tau_t\circ\varphi_t^{-1})$. $\square$.

The relation between τ_t and f_t is that $\tau_t(x)=f_t(\varphi_t(x))$ (up to a constant).

<u>Example 4.3</u>: let $X=\mathbb{R}^2$ and consider $\varphi_t(x,\xi)=(x+t,\xi)$.

Then $v_t=\frac{\partial}{\partial x}$ and $\omega(v_t,-)=-d\xi$.

$\Lambda=\{(x+t,\xi,x,\xi,t,-\xi)\mid x,t,\xi\in\mathbb{R}\}\subset\mathbb{R}^2\times(\mathbb{R}^2)^a\times\mathbb{R}^2$.

<u>Theorem 4.4.</u> [Arnold non-displaceability]: let $X=T^*M$, M compact, and let $\Phi=(\varphi_t)_{t\in\mathbb{R}}$ be a Hamiltonian isotopy which is the identity outside a compact (i.e. $\exists K\subset X$ compact such that $\varphi_t|_{X\setminus K}\equiv id$ for all $t\in\mathbb{R}$). Then for any $t\in\mathbb{R}$ $\varphi_t(M)\cap M\neq\emptyset$. Moreover, if the intersection is transversal then $\#(\varphi_t(M)\cap M)\geq\sum_j b_j(M)$ $(b_j(M):=\dim H^j(M,\mathbb{R}))$.

Observe that Example 4.3 is discarded (even if we replace $\mathbb{R}^2$ by $T^*S^1=S^1\times\mathbb{R}$). by the assumptions in Theorem 4.4.

We are now going to use some very specific features of the cotangent space. Recall that $\omega=d\lambda$ (in local coordinates $\lambda=\sum_{i=1}^n\xi_i dx_i$). The vector field corresponding to λ through the isomorphism $TX\to T^*X$; $v\mapsto\omega(v,-)$ is $eu:=-\sum_{i=1}^n\xi_i\frac{\partial}{\partial\xi_i}$. It exponentiates to a family $(\psi_s)_{s\in\mathbb{R}_{>0}}$ of diffeomorphisms $\psi_s(x,\xi)=(x,s\xi)$ $(s=e^{-u})$.

Let now $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ a family of symplectomorphisms that are homogeneous with respect to the above action of $\mathbb{R}_{>0}$: for every $t\in\mathbb{R}$, φ_t commutes with Ψ_s for all $s\in\mathbb{R}_{>0}$. This implies that $[v_t, eu]=0$. Using that $\mathcal{L}_{v_t}(\omega)=0$ and $\omega(eu,-)=\lambda$ we get that $\mathcal{L}_{v_t}(\lambda)=0$.
Therefore, $0=(d\lambda)(v_t,-)+d(\lambda(v_t)) = \omega(v_t,-)-df_t$, with $f_t := -\lambda(v_t)$.

$\Rightarrow$ Φ automatically becomes a Hamiltonian isotopy.

We say that Φ is a <u>homogeneous Hamiltonian isotopy</u> if moreover $\varphi_0 = id$.
Observe that $f_t(x,s\xi) = s f_t(x,\xi)$ (i.e. f is homogeneous of degree 1 in the fiber), and that f is actually the unique homogeneous integral for Φ.
$\Rightarrow$ the Lagrangian $\Lambda \subset X\times X^a \times T^*\mathbb{R}$ constructed above is conical.

<u>Construction 4.5</u> [homogeneization]: given a manifold M we define
$$\rho: T^*M\times \dot{T}^*\mathbb{R} \longrightarrow T^*M;\ (x,\xi,s,\sigma)\mapsto (x,\xi/\sigma)$$
If $f: T^*M\times\mathbb{R}\to\mathbb{R}$ is a smooth function then we define $\tilde{f}: T^*M\times\dot{T}^*\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $\tilde{f} := (f\circ\rho)\cdot\sigma$, which is homogeneous of degree 1.

<u>Proposition 4.6</u>: Let $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ be a Hamiltonian isotopy on $X = T^*M$ with integral $f: X\times\mathbb{R}\to\mathbb{R}$, and define $\tilde{f}$ according to Construction 4.5.

(1) there exists a homogeneous Hamiltonian isotopy $\tilde{\Phi} = (\tilde{\varphi}_t)_{t\in\mathbb{R}}$ of $\tilde{X} := T^*M\times\dot{T}^*\mathbb{R}$ such that:

(i) its homogeneous integral is $\tilde{f}$.

(ii) $\tilde{\Phi}$ lifts Φ. I.e. the following diagram commutes:
$$\begin{array}{ccc} T^*M\times\dot{T}^*\mathbb{R}\times\mathbb{R} & \xrightarrow{\ \tilde{\Phi}\ } & T^*M\times\dot{T}^*\mathbb{R} \\ \downarrow{\scriptstyle \rho\times id_{\mathbb{R}}} & \circlearrowleft & \downarrow{\scriptstyle \rho} \\ T^*M\times\mathbb{R} & \xrightarrow{\ \Phi\ } & T^*M \end{array}$$

(iii) $\exists\, u: T^*M\times\mathbb{R}\to\mathbb{R}$ such that
$$\tilde{\varphi}_t(x,\xi,s,\sigma) = (\sigma\cdot\varphi_t(x,\xi/\sigma),\ s+u_t(x,\xi/\sigma),\ \sigma).$$

(2) if M is connected and Φ is the identity outside a compact $K\subset T^*M$, then $\tilde{\Phi}$ extends to $\dot{T}^*(M\times\mathbb{R})$ and $\exists v:\mathbb{R}\to\mathbb{R}$ such that

$$\tilde{\Phi}(x,\xi,s,\sigma,t) = (x,\xi,s+v(t),0,t)$$

Proof (sketch): Fix $u: T^*M\times\mathbb{R}\to\mathbb{R}$ and define $\tilde{\varphi}_t$ according to point (iii).

Let $\tilde{v}_t(x,\xi,s,\sigma) := \frac{d\tilde{\varphi}_t}{dt}(x,\xi,s,\sigma) = \left(\sigma\cdot v_t(x,\xi), \frac{du}{dt}(x,\xi), 0\right)$.

Observe that $\tilde{f}_t := \lambda(\tilde{v}_t) = \sigma\cdot\lambda(v_t) + \sigma\frac{du}{dt} = \sigma\cdot\left(f_t + \frac{du}{dt}\right)$.

Apply $\frac{\partial}{\partial\sigma}\Big|_{\sigma=1}$ and get $eu(f_t) = f_t + \frac{du}{dt}$.

Therefore u must be a solution of $\begin{cases} \frac{du}{dt} = eu(f_t) - f_t \\ u_{|t=0} \equiv 0 \end{cases}$ $\leftarrow$ because we want $\tilde{\varphi}_0 \equiv id$

If Φ is the identity outside a compact K then f_t is constant outside K and so is u_t. Therefore $\lim_{\sigma\to 0} u_t(x,\xi/\sigma) =: v(t)$ is well-defined. □

Theorem 4.7: [homogeneous version of Arnold non-displaceability]: let $\Psi: M\to\mathbb{R}$ be a C^2 function such that $d\Psi$ never vanishes, $\tilde{\Phi} = (\tilde{\varphi}_t)_{t\in\mathbb{R}}$ be a homogeneous Hamiltonian isotopy of $\dot{T}^*M$ and $N\subset M$ be a submanifold. Then $\tilde{\varphi}_t(\dot{T}^*_N M)\cap\Lambda_\Psi \neq \emptyset$ $(\Lambda_\Psi = \Gamma_{d\Psi})$. Moreover, if the intersection is transversal then $\#(\tilde{\varphi}_t(\dot{T}^*_N M)\cap\Lambda_\Psi) \geq \sum_j b_j(N)$.

We now prove that Theorem 4.7 implies Theorem 4.4.

Let N be compact manifold. Consider $M = N\times\mathbb{R}$ and identify N with $N\times\{0\}$.

Let $\Psi: M = N\times\mathbb{R}\to\mathbb{R}$ be the second projection.

Assume $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ is an Hamiltonian isotopy of T^*N and let $\tilde{\Phi}$ be a lift to $\dot{T}^*M$ as in Proposition 4.6. We have to compare $\tilde{\varphi}_t(\dot{T}^*_N M)\cap\Lambda_\Psi$ with $\varphi_t(N)\cap N$. Namely, we will prove that these two sets are in bijection.

First observe that $\dot{T}^*_N M = N\times\{0\}\times\mathbb{R}^\times \subset T^*N\times T^*\mathbb{R} = T^*M$, and that $\Lambda_\Psi = \Gamma_{d\Psi} = \{(x,0,t,1) \mid x\in N, t\in\mathbb{R}\}\subset T^*M$. We then need the following

Lemma 4.8: Let $p: E\to X$ be a smooth map, $A,B\subset X$ and $A'\subset E$ submanifolds, and assume that $p|_{A'}: A'\xrightarrow{\sim} A$ is a diffeomorphism. If $B' := p^{-1}(B)$ then

(i) $p|_{A'\cap B'}: A'\cap B'\to A\cap B$ is a bijection.

(ii) A and B intersect transversally if and only if A' and B' do so.

The proof of the above Lemma is left as an exercise.
We will now apply Lemma 4.8 twice successively.

$$\begin{array}{ccccc}
T^*N\times\dot{T}^*\mathbb{R}=E_2 & \xrightarrow{p_2} & X_2=T^*N\times\mathbb{R}^\times=E_1 & \xrightarrow{p_1} & X_1=T^*N \\
(x,\xi,s,\sigma) & \longmapsto & (x,\xi,\sigma) \qquad (x,\xi,\sigma) & \longmapsto & (x,\xi/\sigma) \\
\Lambda_\psi=B_2' & & B_2=N\times\{1\}=A_1' & & A_1=N \\
\tilde{\varphi}_t(\dot{T}^*_N M)=A_2' & & A_2=\Sigma_t=B_1' & & B_1=\varphi_t(N)
\end{array}$$

$$\tilde{\varphi}_t(\dot{T}^*_N M)=\left\{\begin{array}{c}(\sigma\cdot\varphi_t(x,0),\,u_t(x,0),\sigma),\\ x\in N,\sigma\in\mathbb{R}^\times\end{array}\right\} \qquad \Sigma_t=\{(\sigma\cdot\varphi_t(x,0),\sigma)\mid x\in N,\sigma\in\mathbb{R}^\times\}$$

- $p_1|_{A_1'}: A_1'=N\times\{1\}\to N=A_2'$ is obviously a diffeomorphism.
- $p_1^{-1}(B_1)=\Sigma_t=B_1'$ by definition.

- $p_2|_{A_2'}: A_2'=\tilde{\varphi}_t(\dot{T}^*_N M)\to A_2=\Sigma_t$ is a diffeomorphism thanks to the explicit description we gave above.
- $p_2^{-1}(B_2)=\{(x,0,s,1)\mid x\in N, s\in\mathbb{R}\}=\Lambda_\psi=B_2'$.

$\square$

Reference: the reference I used for this lecture is:
S. Guillermou, M. Kashiwara & P. Schapira. Sheaf quantization of Hamiltonian isotopies and applications to non-displaceability problems, arXiv:1005.1517.
Mainly §A.1, §A.3 and §4.4.

5. Sheaves

A (not so) short reminder on sheaves

Let X be a topological space and let k be a commutative ring.
A *presheaf* (of k-modules) on X is a functor $F: Op(X)^{op} \to k\text{-mod}$, where $Op(X)$ is the category of open subsets in X with morphisms being inclusions. A morphism of presheaves is a natural transformation. We denote by $PSh(X)$ the category of presheaves on X.

For any k-module M there is the constant presheaf defined by $U \mapsto M$.
If $F, G \in PSh(X)$ then $F \oplus G : U \mapsto F(U) \oplus G(U)$ is a presheaf. This turns $PSh(X)$ into an additive category (the zero object is the constant presheaf $\underline{0}_X$).

Additive category: category enriched over abelian groups with a zero object 0 such that $Hom(0,0) = 0$ and the functor $W \mapsto Hom(X,W) \times Hom(Y,W)$ is representable (by $X \oplus Y$).

If $f: F \to G$ is a morphism of presheaves then $U \mapsto \ker(f(U))$, $U \mapsto Im(f(U))$, $U \mapsto \mathrm{coker}(f(U))$ are presheaves. This turns $PSh(X)$ into an abelian category.

Abelian category: additive category having kernels and cokernels and such that for any $f: X \to Y$ the morphism $\mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f))$ is an isomorphism.

Vocabulary:
- an element $s \in F(U)$ is a *section* of F over U.
- if $V \subset U$ and $s \in F(U)$ then we denote by $s_{|_V}$ the image of s through $F(U) \to F(V)$, and call it the *restriction* of s.
- if $x \in X$ then the *stalk* of F at x is $F_x := \varinjlim_U F(U) = \coprod_{U \ni x} F(U) / \sim$ where $s \sim s'$ when $\exists W$ such that $s_{|_W} = s'_{|_{W}}$.
  The image s_x of $s \in F(U)$ in F_x is called the *germ* of s at x.
- the *restriction* $F_{|_U} \in PSh(U)$ of F to $U \subset X$ is defined by $F_{|_U}(V) := F(V)$ for any $V \subset U$.

Definition 5.1: a sheaf on X is a presheaf F on X satisfying:

(i) for any $U \in Op(X)$ and any open cover $\mathcal{U} = (U_i)_{i \in I}$ of U, if $s \in F(U)$ is such that $s_{/U_i} = 0$ for all $i \in I$ then $s = 0$.

(ii) for any $U \in Op(X)$, any open cover $\mathcal{U} = (U_i)_{i \in I}$ of U, and any family $s_i \in F(U_i)$ such that $s_{i/U_i \cap U_j} = s_{j/U_i \cap U_j}$, there exists $s \in F(U)$ such that $s_{/U_i} = s_i$.

Morphisms of sheaves are morphisms of the underlying presheaves. We denote by $Sh(X)$ the category of sheaves on X. Note that if $F, G \in Sh(X)$ then $F(\emptyset) = 0$, $F_{/U} \in Sh(U)$ and $F \oplus G \in Sh(X)$. If $f: F \to G$ is a morphism of sheaves then $U \mapsto Ker(f(U))$ and $U \mapsto Im(f(U))$ are sheaves (in general, any sub-presheaf of a sheaf is a sheaf).

BUT: $U \mapsto coker(f(U))$ is not always a sheaf.

Proposition 5.2: • a morphism $f: F \to G$ of sheaves is an isomorphism if and only if $f_x : F_x \to G_x$ is an isomorphism for all $x \in X$.

• the inclusion functor $Sh(X) \to PSh(X)$ admits a left adjoint $F \mapsto F^+$, called the sheafification functor.

Let us describe explicitely the sheafification functor. If $F \in PSh(X)$ and $U \subset X$ then

$F^+(U) := \{ s : U \to \coprod_{x \in U} F_x \mid \forall x \in X, s(x) \in F_x \text{ and } \exists V \ni x \,\&\, \exists t \in F(V) \text{ s.t. } \forall y \in V \; s(y) = t_y \in F_y \}$.

Observe that for any $x \in X$, $F^+_x = F_x$.

Definition-Proposition 5.3: let $f: F \to G$ be a morphism of sheaves. We define $coker(f) := (U \mapsto coker(f(U)))^+$. It is a cokernel in $Sh(X)$.

$\Rightarrow$ $Sh(X)$ is an abelian category. ⚠ even though $Sh(X)$ is an additive subcategory of $PSh(X)$, it is NOT an abelian subcategory of it.

Exercise 5.4: prove that $\underline{M}_X := (U \mapsto M)^+$ is such that $\underline{M}_X(U) = M^{\oplus \pi_0(U)}$.

Proposition 5.5: $Sh(X)$ has enough injectives.

• an object Z in an abelian category $\mathcal{E}$ is injective if $Hom_{\mathcal{E}}(-, Z)$ is exact.

• $\mathcal{E}$ has enough injectives if for any object Z there exists an injective object Z' and a monomorphism $Z \to Z'$.

Let $f: X \to Y$ be a continuous map between topological spaces. There is a pair of adjoint functors $f^{-1}: Sh(Y) \rightleftarrows Sh(X) : f_*$, where
$$f_* F(U) := F(f^{-1}(U)) \quad \text{and} \quad f^{-1}G(V) := \varinjlim_{U \supset f(V)} G(U).$$

as an exercise, check that $(f \circ g)_* = f_* \circ g_*$, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, $(f_* F)|_U = f_*(F|_{f^{-1}(U)})$ and $(f^{-1}F)|_{f^{-1}(U)} = f^{-1}(F|_U)$.

This means that $\mathrm{Hom}_{Sh(X)}(f^{-1}G, F) = \mathrm{Hom}_{Sh(Y)}(G, f_* F)$.

Proof of Proposition 5.5: let $\hat{X}$ be the space X endowed with the discrete topology, and let $f: \hat{X} \to X$ be the identity on the underlying set. Let $F \in Sh(X)$.

$f^{-1}F(U) = \prod_{x \in U} F_x$. For any $x \in X$ choose an injective k-module I_x with a monomorphism $0 \to F_x \to I_x$. Then $I(U) := \prod_{x \in U} I_x$ is injective in $Sh(\hat{X})$ and $0 \to f^{-1}F \to I$ is a monomorphism.

this step isn't necessary if k is a field

Apply f_* (which is left exact): one gets $0 \to f_* f^{-1} F \to f_* I$.

Claim: $f_* I$ is injective.

Proof: $\mathrm{Hom}(-, f_* I) = \mathrm{Hom}(f^{-1}(-), I)$ is exact because I is injective and f^{-1} is exact

We conclude by using the unit of the adjunction $F \to f_* f^{-1} F$. □

Definition 5.6: let $F \in Sh(X)$, $U \in Op(X)$ and $s \in F(U)$.

- the support $\mathrm{supp}(F)$ of F is defined as follows:
  $$x \notin \mathrm{supp}(F) \iff \exists V \ni x \text{ such that } F|_V = 0_V.$$
- the support $\mathrm{supp}(s)$ of s is defined as follows:
  $$x \in U \setminus \mathrm{supp}(F) \iff \exists V \ni x \text{ such that } s|_V = 0 \quad (V \subset U).$$

Observe that $\mathrm{supp}(s) = \{x \in U \,/\, s_x \neq 0\}$. ⚠ $\mathrm{supp}(F) \neq \{x \in U \,/\, F_x \neq 0\}$ (exercise).

Proposition 5.7: let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in $Sh(X)$. Then for any $i \in \{1,2,3\}$, $\mathrm{Supp}(F_i) \subset \bigcup_{j \neq i} \mathrm{supp}(F_j)$.

Proof: let $U \in Op(X)$ be such that $F_j|_U = 0$ for $j \neq i$. Then $F_i|_U = 0$. This means that $\bigcap_{j \neq i} (\mathrm{Supp}(F_j)^c) \subset \mathrm{Supp}(F_i)^c$. □

Let $Z \subset X$ be any subset and consider the sheaf $\underline{k}_Z$ defined as follows: $\underline{k}_Z(U) := k^{\oplus \pi_0(U\cap Z)}$. Then $\mathrm{supp}(\underline{k}_Z) = \overline{Z}$ (the closure of Z).

Comment 5.8: strictly speaking we should denote this sheaf $i_*\underline{k}_Z$, where $i: Z \hookrightarrow X$ is the inclusion map (this is because $\underline{k}_Z$ already denotes a constant sheaf on Z).

We are now going to explain that $Sh(X)$ is symmetric monoidal closed.

- symmetric monoidal structure: if $F, G \in Sh(X)$ then $F \otimes G := (U \mapsto F(U) \otimes_k G(U))^+$.
- enrichment: if $F, G \in Sh(X)$ then $\mathcal{H}om(F,G): U \longmapsto \mathrm{Hom}_{Sh(U)}(F|_U, G|_U)$ defines a sheaf on X.

$\Rightarrow$ The sheaf $\mathcal{H}om(H\otimes F, G)$ is canonically isomorphic to $\mathcal{H}om(F, \mathcal{H}om(H,G))$.

Proposition 5.9: the functor f^{-1} is monoidal.

Sketch of proof: $f: X \to Y$ is a continuous map between topological spaces, and we let $F, G \in Sh(Y)$. There is a morphism $f^{-1}F \otimes f^{-1}G \to f^{-1}(F\otimes G)$: namely, for any $U \in Op(X)$

$$f^{-1}F(U) \otimes f^{-1}G(U) = \left(\varinjlim_{V\supset f(U)} F(V)\right) \otimes \left(\varinjlim_{V \supset f(U)} G(V)\right) = \varinjlim_{V\supset f(U)} F(V)\otimes G(V)$$
$$\downarrow$$
$$f^{-1}(F\otimes G)(U) = \varinjlim_{V\supset f(U)} (F\otimes G)(V)$$

In order to check that it is an isomorphism we look at stalks: let $x \in X$ and $y = f(x)$. Then $(f^{-1}F \otimes f^{-1}G)_x \cong (f^{-1}F)_x \otimes (f^{-1}G)_x \cong F_y \otimes G_y \cong (F\otimes G)_y$. $\square$.

For a sheaf $F \in Sh(X)$ and a subset $Z \subset X$ we define $F_Z := F \otimes \underline{k}_Z$.
Let $i: Z \hookrightarrow X$ and $j: X \setminus Z \hookrightarrow X$ be the inclusion maps. Then $i^{-1}F_Z = i^{-1}F$ and, if Z is closed, then $j^{-1}F_Z = \underline{0}$. (Hint: use Proposition 5.9).

Proposition 5.10: the adjunction between f^{-1} and f_* is enriched:

$$\mathcal{H}om(G, f_*F) \simeq f_*\mathcal{H}om(f^{-1}G, F).$$

Proof: $f_*\mathcal{H}om(f^{-1}G, F)(U) = \mathrm{Hom}_{Sh(f^{-1}(U))}(f^{-1}(G|_U), F|_{f^{-1}(U)})$

$$= \mathrm{Hom}_{Sh(U)}(G|_U, f_*F|_U) = \mathcal{H}om(F, f_*G)(U). \square$$

For a sheaf $F \in Sh(X)$ and a subset $Z \subset X$ we define $\Gamma_Z(F) := \mathcal{H}om(\underline{k}_Z, F)$.
If Z is closed then $\Gamma_Z(F)$ is the subsheaf of F consisting of sections supported in Z.

From now X is a manifold.

Wrong definition 5.11: let $F \in Sh(X)$. The microsupport $SS(F)$ of F is the subset of T^*X defined as follows:

$p \notin SS(F) \Leftrightarrow \exists U \ni p$ such that for any $x_0 \in X$ and any $\varphi: X \to \mathbb{R}$ satisfying $(x_0, d\varphi_{x_0}) \in U$ we have

$$\Gamma_{\{x \mid \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} = 0.$$

Remark 5.12: if $x \notin supp(F)$ then for any $\xi \in T^*_x X$ $(x, \xi) \notin SS(F)$. Namely, $\Gamma_A(F)_{x_0} \subset F_{x_0}$ for any closed subset $A \subset X$.

Examples 5.13: a) let Z be a closed submanifold in X, and let $F = \underline{k}_Z$. According to the above Remark we should only consider $(x, \xi) \in T^*X$ such that $x \in Z$.

Claim: $SS(\underline{k}_Z) = T^*_Z X = \{(x, \xi) \in T^*_x X \mid x \in Z, \xi(v) = 0 \ \forall v \in T_x Z\}$.

Proof: • if $p \notin T^*_Z X$ then there exists $U \ni p$ such that $U \cap T^*_Z X = \emptyset$.
In particular, for any $x_0 \in \pi(U)$ and any $\varphi: U \to \mathbb{R}$ such that $d_{x_0}\varphi \in U$,

- either $x_0 \notin Z$ and thus $\Gamma_{\{x \mid \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} = 0$

- or $x_0 \in Z$ but $d_{x_0}\varphi$ is not conormal. In particular there is no neighborhood of x_0 into Z that is contained in $\{x \mid \varphi(x) \geq \varphi(x_0)\}$.

Therefore $\Gamma_{\{x \mid \varphi(x) \geq \varphi(x_0)\}}(\underline{k}_Z)_{x_0} = 0$.

• conversely, if $p \in T^*_Z X$ then we set $x_0 = \pi(p)$ and take $\varphi: X \to \mathbb{R}$ such that $\varphi|_Z \equiv 0$ in a neighborhood of x_0 and $(x_0, d_{x_0}\varphi) = p$.

Then $\Gamma_{\underset{\text{locally around } x_0}{\{x \mid \varphi(x) \geq 0\} \supset Z}}(\underline{k}_Z) = k$.

b) Let $Z = \mathbb{R}_{\geq 0} \subset \mathbb{R} = X$ and consider $F = \underline{k}_Z$.

Over $\mathbb{R}_{>0}$ it will be just as in a): if $x > 0$ then $(x,\xi) \in SS(F) \Leftrightarrow \xi = 0$.

Let $p = (0,\xi)$. • if $\xi \geq 0$ then we set $x_0 = 0$ and take $\varphi(x) = \xi x$.

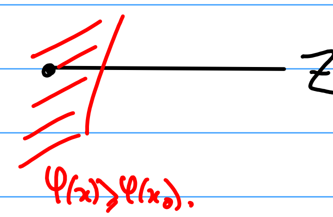
Then $\Gamma_{\{x \mid \varphi(x) \geq 0\}}(\underline{k}_Z)_0 = \Gamma_Z(\underline{k}_Z)_0 = k$.

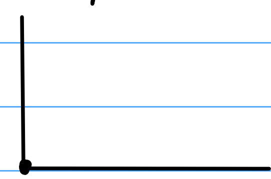
or $\mathbb{R}$ if $\xi = 0$.

• Conversely if $\xi < 0$ then $\exists U$ open neighborhood of p such that $U \subset \mathbb{R} \times \mathbb{R}_{<0}$. Then $\forall x_0 \in \pi(U)$ $\forall \varphi : U \to \mathbb{R}$ s.t. $d_{x_0}\varphi < 0$

- either $x_0 \notin Z$ ✓.
- or $x_0 \in Z$ but φ is decreasing around x_0.

$\Rightarrow \nexists$ neighborhood of x_0 in Z that is contained in $\{x \mid \varphi(x) \geq \varphi(x_0)\}$.



Thus $SS(F) =$ 

Why is this a wrong definition? Because if $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence in $Sh(X)$, then we don't have that $SS(F_i) \subset \bigcup_{j \neq i} SS(F_j)$.

This is because Γ_A is not an exact functor: $\Gamma_A(F_1)_x = \Gamma_A(F_2)_x = 0 \not\Rightarrow \Gamma_A(F_3)_x = 0$.

6. Homological algebra

Let $\mathcal{A}$ be an abelian category and $F : Sh(X) \to \mathcal{A}$ an additive functor. Assume that F is left exact: if $0 \to U \to V \to W$ is an exact sequence then $0 \to F(U) \to F(V) \to F(W)$ is an exact sequence.

Let $G \in Sh(X)$. We know that there exists an injective sheaf G_0 and an exact sequence $0 \to G \to G_0$. Let G_0' be the cokernel of $G \to G_0$. We know that there exists an injective sheaf G_1 and a monomorphism $G_0' \to G_1$. Therefore $0 \to G \to G_0 \to G_1$ is exact. By induction we get a semi-infinite sequence $0 \to G \to G_0 \to G_1 \to G_2 \to G_3 \to \cdots$.

This can be rephrased by saying that $G_0 \to G_1 \to G_2 \to \cdots$ is an injective resolution of G. We then define $\mathbb{R}^i F(G) := \dfrac{\mathrm{Ker}\,(F(G_i) \to F(G_{i+1}))}{\mathrm{Im}\,(F(G_{i-1}) \to F(G_i))}$.

If $0 \to G \to G' \to G'' \to 0$ is an exact sequence of sheaves, then we have a long exact sequence $0 \to F(G) \to F(G') \to F(G'') \to \mathbb{R}^1 F(G) \to \mathbb{R}^1 F(G') \to \cdots$

Correct Definition 6.1: let $F \in Sh(X)$. The microsupport $SS(F)$ is defined as follows:

$$p \notin SS(F) \iff \exists\, U \ni p \text{ s.t. } \forall (x_0, d\varphi_{x_0}) \in U,\ \mathbb{R}\Gamma_{\{x \mid \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} = 0.$$

We are going to axiomatize this.

Let $\mathcal{E}$ be an additive category.

Definition 6.2: a complex C in $\mathcal{E}$ consists of the data $(C^n, d^n)_{n \in \mathbb{Z}}$ such that

for any $n \in \mathbb{Z}$: C^n is an object of $\mathcal{E}$, $d^n \in \mathrm{Hom}_{\mathcal{E}}(C^n, C^{n+1})$ and $d^{n+1} \circ d^n = 0$.

We will abbreviate $C := (C^n)_{n \in \mathbb{Z}}$, $d := (d^n)_{n \in \mathbb{Z}}$ and $d \circ d = 0$.
A morphism of complexes is a sequence $f = (f^n)_{n \in \mathbb{Z}}$ of morphisms $f^n : C^n \to D^n$

such that
$$\begin{tikzcd} C^n \arrow[r, "d_C^n"] \arrow[d, "f^n"'] & C^{n+1} \arrow[d, "f^{n+1}"] \\ D^n \arrow[r, "d_D^n"] & D^{n+1} \end{tikzcd}$$
commutes.

We denote by $C(\mathcal{E})$ the category of complexes in $\mathcal{E}$. It is additive (and even abelian whenever $\mathcal{E}$ is). We identify $\mathcal{E}$ with the full subcategory of $C(\mathcal{E})$ consisting of those complexes C^n "concentrated in degree zero": i.e. $C^n = 0$ for $n \neq 0$.

We denote by $\begin{cases} C^b(\mathcal{E}) \\ C^+(\mathcal{E}) \\ C^-(\mathcal{E}) \end{cases}$ the full subcategory of complexes that are $\begin{cases} \text{bounded} \\ \text{bounded below} \\ \text{bounded above} \end{cases}$

which means that $C^n = 0$ for $\begin{cases} |n| \gg 0 \\ n \ll 0 \\ n \gg 0 \end{cases}$

For any $k \in \mathbb{Z}$ and any $C \in C(\mathcal{E})$ we define $C[k]^n := C^{n+k}$.
$d^n_{C[k]} := (-1)^k d_C^{n+k}$.

For a morphism $f \in \mathrm{Hom}_{C(\mathcal{E})}(C, D)$ we define $f[k]^n := f^{n+k}$.

I.e. $(-)[k] : C(\mathcal{E}) \to C(\mathcal{E})$ is a functor.

Definition 6.3: a morphism $f \in \mathrm{Hom}_{C(\mathcal{E})}(C, D)$ is null homotopic if there exists morphisms $h^n \in \mathrm{Hom}_{\mathcal{E}}(C^n, D^{n-1})$ such that for any n:
$$f^n = h^{n+1} d^n + d^{n-1} h^n$$

$\mathrm{Null}(C, D) :=$ morphisms in $\mathrm{Hom}_{C(\mathcal{E})}(C, D)$ that are null-homotopic.
Composition of morphisms descends to quotient $\overline{\mathrm{Hom}}_{C(\mathcal{E})}(C, D) := \mathrm{Hom}_{C(\mathcal{E})}(C, D) / \mathrm{Null}(C, D)$

$\Rightarrow K(\mathcal{E})$ is the category having complexes in $\mathcal{E}$ as objects, and
$$\mathrm{Hom}_{K(\mathcal{E})}(C, D) := \overline{\mathrm{Hom}}_{C(\mathcal{E})}(C, D).$$

$K(\mathcal{E})$ is called the homotopy category of complexes in $\mathcal{E}$. We still have
$$\mathcal{E}, K^+(\mathcal{E}), K^-(\mathcal{E}), K^b(\mathcal{E}) \subset \mathcal{E}.$$

From now we assume that $\mathcal{E}$ is an abelian category.

Notation 6.4: let C be a complex in $\mathcal{E}$.

$Z^k(C) := \mathrm{Ker}(d_C^k)$, $B^k(C) := \mathrm{Im}(d_C^{k-1})$, $H^k(C) := \mathrm{CoKer}(B^k(C) \to Z^k(C))$.

One calls $H^k(C)$ the k-th cohomology of C.

Observe that $H^k(-) = H^0(-[k])$ defines an additive functor $C(\mathcal{E}) \to \mathcal{E}$.

Exercise 6.5: show that if $f: C \to D$ is null-homotopic then

$H^k(f) = 0$ for all $k \in \mathbb{Z}$.

$\Rightarrow H^k(-)$ factors through $K(\mathcal{E}) \to \mathcal{E}$.

We have the following exact sequences:

- $C^{k-1} \xrightarrow{d} Z^k(C) \to H^k(C) \to 0$.
- $0 \to H^k(C) \to \mathrm{CoKer}(d^{k-1}) \xrightarrow{d} C^{k+1}$.
- $0 \to Z^{k-1}(C) \to C^{k-1} \xrightarrow{d} B^k(C) \to 0$.
- $0 \to B^k(C) \to C^k \to \mathrm{CoKer}(d^{k-1}) \to 0$.
- $0 \to B^k(C) \to C^k \xrightarrow{d} Z^{k+1}(C) \to H^{k+1}(C) \to 0$

Proposition 6.6: let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an exact sequence in $C(\mathcal{E})$.

Then there exists a canonical long exact sequence in $\mathcal{E}$:

$$\cdots \to H^n(C_1) \to H^n(C_2) \to H^n(C_3) \to H^{n+1}(C_1) \to \cdots$$

Moreover, this assignement is functorial.

Proof: exercise.

Let $f: C \to D$ a morphism in $C(\mathcal{E})$.

Definition 6.7: the mapping cone of f, denoted by $M(f)$, is the following object of $C(\mathcal{E})$ defined as follows: $M(f)^n := C^{n+1} \oplus D^n$

$$d_{M(f)}^n := \begin{pmatrix} -d_C^{n+1} & 0 \\ f^{n+1} & d_D^n \end{pmatrix}$$

In other words $M(f) := C[1] \oplus D$ and $d_{M(f)} := \begin{pmatrix} d_{C[1]} & 0 \\ f[1] & d_D \end{pmatrix}$.

We have obvious morphisms $D \xrightarrow{\begin{pmatrix} 0 \\ id_D \end{pmatrix}} M(f)$ and $M(f) \xrightarrow{(id_{C[1]} \ 0)} C[1]$

Definition 6.8: 1) a triangle is a sequence of morphisms $C \to D \to E \to C[1]$.

2) an exact/distinguished triangle is a triangle $C \to D \to E \to C[1]$ isomorphic to a triangle $C' \xrightarrow{f} D' \to M(f) \to C'[1]$.

Theorem 6.9: the following properties are satisfied:

(TR1) a triangle isomorphic to an exact one is exact.

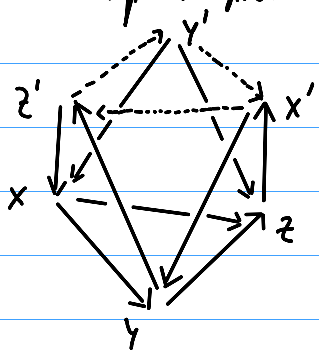
(TR2) $C \xrightarrow[id_C]{} C \to 0 \to C[1]$ is exact.

(TR3) any morphism $C \xrightarrow{f} D$ can be completed by an exact triangle $C \to D \to E \to C[1]$.

(TR4) $C \xrightarrow[f]{} D \to E \to C[1]$ is exact if and only if $D \to E \to C[1] \xrightarrow[-f[1]]{} D[1]$ is.

(TR5) the completion procedure in (TR3) is "functorial."

(TR6)



We summarize these properties by saying that $K(\mathcal{E})$ is a triangulated category.

Definition 6.10: An additive functor $F: \mathcal{T} \to \mathcal{A}$ from a triangulated category $\mathcal{T}$ to an additive category $\mathcal{A}$ is called cohomological if for any exact triangle $C \to D \to E \to C[1]$ the sequence $F(C) \to F(D) \to F(E)$ is exact.

Writing $F^k := F \circ (-)[k]$ we get a long exact sequence

$$\cdots \to F^{k-1}(E) \to F^k(C) \to F^k(D) \to F^k(E) \to F^{k+1}(C) \to \cdots$$

Proposition 6.11: 1) if $C \xrightarrow{f} D \xrightarrow{g} E \to C[1]$ is an exact triangle then $g \circ f = 0$

2) $\forall C \in \mathcal{T}$, $Hom_{\mathcal{T}}(C,-)$ and $Hom_{\mathcal{T}}(-,C)$ are cohomological

Proof: 1)

$$\begin{tikzcd} C \arrow[r, "id"] \arrow[d, "id"'] & C \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed, "\exists \phi"] & C[1] \arrow[d, "id"] \\ C \arrow[r, "f"] & D \arrow[r, "g"] & E \arrow[r] & C[1] \end{tikzcd}$$

2)

$$\begin{tikzcd} C \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C[1] \\ C' \arrow[r, "f"] & D' \arrow[r, "g"] & E' \arrow[r] & C'[1] \end{tikzcd}$$

$\square$.

The functor $H^n : K(\mathcal{E}) \to \mathcal{E}$ is cohomological.

Localization

$\mathcal{T}$ triangulated. $\mathcal{N}$ family of objects in $\mathcal{E}$ such that:

1) $0 \in \mathcal{N}$
2) $X \in \mathcal{N} \iff X[1] \in \mathcal{N}$
3) $X \to Y \to Z \to X[1]$ $X, Y \in \mathcal{N} \Rightarrow Z \in \mathcal{N}$.

this is called a null system.

$\exists\ \mathcal{T}/\mathcal{N}$ such that

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d] & \mathcal{T}' \\ \mathcal{T}/\mathcal{N} \arrow[ur, dashed] & \end{tikzcd}$$

for $F(X) \simeq 0$ if $X \in \mathcal{N}$.

$\mathcal{T} = K(\mathcal{E})$ $\mathcal{N} = \{ X \in K(\mathcal{E}) \mid H^n(X) = 0\ \forall n \}$.

$\mathcal{D}(\mathcal{E}) := K(\mathcal{E})/\mathcal{N}$. Derived category.

If $\mathcal{E}$ has enough injectives then $K^+(\mathrm{Inj}(\mathcal{E})) \to \mathcal{D}^+(\mathcal{E})$ is an equiv. of categories

Definition 6.12: let $F: \mathcal{E} \to \mathcal{E}'$ be an additive functor between abelian categories.
A right derived functor of F is a triangulated functor $T: D^+(\mathcal{E}) \to D^+(\mathcal{E}')$ together with a natural transformation $s: Q \circ K^+(F) \Rightarrow T \circ Q$, where $Q: K^+(\mathcal{E}) \to D^+(\mathcal{E})$, such that for any triangulated functor $G: D^+(\mathcal{E}) \to D^+(\mathcal{E}')$

$$\mathrm{Hom}(T, G) \xrightarrow{s} \mathrm{Hom}(Q \circ K^+(F), G \circ Q) \text{ is an isomorphism.}$$

When it exists (T, s) is unique up to a unique isomorphism. We then denote it by $\mathbb{R}F$, and $\mathbb{R}^n F := H^n \circ \mathbb{R}F$ is the n-th derived functor.

Claim 6.13: if $\mathcal{E}$ has enough injectives then any left exact functor $F: \mathcal{E} \to \mathcal{E}'$ admits a right derived functor.

One easily sees that $\mathbb{R}(F \circ F') = \mathbb{R}F \circ \mathbb{R}F'$ whenever this makes sense.

Proposition 6.14: let $F, F', F'': \mathcal{E} \to \mathcal{E}'$ be left exact functors together with natural transformations $\lambda: F' \to F$ and $\mu: F \to F''$. Assume that for any injective object U the sequence

$$0 \to F'(U) \xrightarrow[\lambda_U]{} F(U) \xrightarrow[\mu_U]{} F''(U) \to 0 \quad \text{is exact.}$$

Then there exists a natural transformation $\nu: \mathbb{R}F'' \to \mathbb{R}F'[1]$ such that for any $V \in D^+(\mathcal{E})$ we have an exact triangle

$$\mathbb{R}F'(V) \xrightarrow[\mathbb{R}\lambda_V]{} \mathbb{R}F(V) \xrightarrow[\mathbb{R}\mu_V]{} \mathbb{R}F''(V) \xrightarrow[\nu_V]{} \mathbb{R}F'(V)[1].$$

7. More on sheaves and cohomology

Derived functors (continued)

We want to compute derived functors using other types of resolutions than injective ones.

Definition 7.1: let $F: \mathcal{E} \to \mathcal{E}'$ be a left exact additive functor between abelian categories. A full additive subcategory $\mathcal{J}$ of $\mathcal{E}$ is called <u>F-injective</u> if

(i) $\forall C \in \mathcal{E}, \exists C' \in \mathcal{E}$ and an exact sequence $0 \to C \to C'$.

(ii) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence and $C', C'' \in \mathcal{J}$ then $C \in \mathcal{J}$.

(iii) F is exact on $\mathcal{J}$.

Let $\mathcal{J}$ be an F-injective subcategory. Then for any bounded below complex $C \in K^+(\mathcal{J})$ of objects in $\mathcal{J}$ such that $H^\bullet(C) = 0$ we have that $H^\bullet(F(C)) = 0$.
(Hint: prove it by induction on the cohomological degree).
Observe that $N_{\mathcal{J}}^+ := \{C \in K^+(\mathcal{J}) \mid H^n(C) = 0 \ \forall n\} = N \cap Ob(K^+(\mathcal{J}))$ is a null-system.
Therefore the composed functor $K^+(\mathcal{J}) \longrightarrow K^+(\mathcal{E}') \longrightarrow D^+(\mathcal{E}')$ factors through $K^+(\mathcal{J})/N_{\mathcal{J}}^+$.

$$\begin{tikzcd} K^+(\mathcal{J}) \arrow[r] \arrow[dr] & K^+(\mathcal{E}') \arrow[r] & D^+(\mathcal{E}') \\ & K^+(\mathcal{J})/N_{\mathcal{J}}^+ \arrow[ur, dashed] & \end{tikzcd}$$

Proposition 7.2: the triangulated functor $K^+(\mathcal{J})/N_{\mathcal{J}}^+ \to D^+(\mathcal{E})$ is an equivalence of categories.

Sketch of proof: any object $C \in \mathcal{E}$ admits a resolution by objects of $\mathcal{J}$ (this is property (i) in Definition 7.1). Therefore any $C \in K^+(\mathcal{E})$ admits such a resolution.
This means that every triangulated functor $G: K^+(\mathcal{E}) \longrightarrow \mathcal{T}'$ such that $G(C) \simeq 0$ for any $C \in N$ factors through $K^+(\mathcal{J})$, and even further through the localization.

$$\begin{tikzcd} K^+(\mathcal{E}) \arrow[r] \arrow[dr] & \mathcal{T}' \\ & K^+(\mathcal{J}) \arrow[u, dashed] \arrow[dr] \\ & & K^+(\mathcal{J})/N_{\mathcal{J}}^+ \arrow[uul, dashed] \end{tikzcd}$$

We now apply this to the localization functor $K^+(\mathcal{E}) \longrightarrow D^+(\mathcal{E})$ and get $K^+(\mathcal{E}) \longrightarrow K^+(\mathcal{J})/N_{\mathcal{J}}^+ \longrightarrow D^+(\mathcal{E})$. The first arrow factors through $D^+(\mathcal{E})$ by definition. This gives an equivalence. $\square$

$$\begin{tikzcd} K^+(\mathcal{E}) \arrow[dr] \arrow[rr] & & K^+(\mathcal{J})/N_{\mathcal{J}}^+ \\ & D^+(\mathcal{E}) \arrow[ur] & \end{tikzcd}$$

The conclusion is that, whenever an F-injective subcategory $\mathcal{J}$ exists then the right derived functor $RF: D^+(\mathcal{E}) \to D^+(\mathcal{E}')$ exists.

Flabby sheaves

We now present a full additive subcategory of $Sh(X)$ which is F-injective for

- $F = \Gamma(X,-) : Sh(X) \to k\text{-mod}$ being the "global sections" functor.
- $F = \Gamma_Z : Sh(X) \to Sh(X)$ being the "sections supported on Z" functor.

Definition 7.3: a sheaf F on X is flabby if the restriction morphism $F(X) \to F(U)$ is surjective for any $U \in Op(X)$.

Proposition 7.4: let F be a flabby sheaf on X.

(i) $\forall U \in Op(X)$, $F|_U$ is flabby

(ii) if $f : X \to Y$ is a continuous map then $f_* F$ is flabby.

(iii) if $Z \subset X$ is locally closed then $\Gamma_Z(F)$ is flabby.

(iv) if $Z \subset X$ is locally closed and $Z' \subset Z$ is closed then
$$0 \to \Gamma_{Z'}(F) \to \Gamma_Z(F) \to \Gamma_{Z \setminus Z'}(F) \to 0 \text{ is exact.}$$

(v) if $U_1, U_2 \in Op(X)$ then
$$0 \to \Gamma_{U_1 \cup U_2}(F) \to \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \to \Gamma_{U_1 \cap U_2}(F) \to 0 \text{ is exact.}$$

(vi) if $Z_1, Z_2 \subset X$ are closed then
$$0 \to \Gamma_{Z_1 \cap Z_2}(F) \to \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \to \Gamma_{Z_1 \cup Z_2}(F) \to 0 \text{ is exact.}$$

(vii) if $G, H \in Sh(X)$ with H injective then $\mathcal{H}om(G,H)$ is flabby.

Proof:

- (i) and (ii) are easy.
- Z is closed in an open V, therefore we can assume w.l.o.g. that Z is closed.

Then we have the following commutative diagram, where rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_Z(X,F) \arrow[r] \arrow[d, two heads] & F(X) \arrow[r] \arrow[d, two heads] & F(X \setminus Z) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Gamma_Z(U \cup X \setminus Z, F) \arrow[r] \arrow[d, two heads] & F(U \cup X \setminus Z) \arrow[r] & F(X \setminus Z) \arrow[r] & 0 \\
& \Gamma_Z(U,Z) \arrow[u, dashed, bend right] & & &
\end{tikzcd}$$

This gives (iii).

- $Z' \subset Z$ closed $\Rightarrow \Gamma_{Z'}(F) \subset \Gamma_Z(F)$.

(iii) $\Rightarrow \Gamma_Z(U,F) \to \Gamma_{Z \setminus Z'}(U,F) = \Gamma_Z(U \setminus Z', F)$ is surjective for any U.

This gives (iv).

- $\forall U, V \in Op(X)$, $\Gamma_U(V, F) = \Gamma(U\cap V, F)$. The only non-trivial thing to prove is then that $\Gamma(U_1\cap V, F) \oplus \Gamma(U_2 \cap V, F) \longrightarrow \Gamma(U_1 \cap U_2 \cap V, F)$ is surjective.
  F flabby $\Rightarrow$ both components are surjective.
- prove (vi) as an exercise.
- let $U \in Op(X)$. H injective $\Rightarrow$ $Hom(-, H)$ injective.

$$\Rightarrow 0 \to Hom(G_{X\setminus U}, H) \to \underset{\substack{\| \\ \mathcal{H}om(G,H)(X)}}{Hom(G_X, H)} \to \underset{\substack{\| \\ \mathcal{H}om(G,H)(U)}}{Hom(G_U, H)} \to 0 \text{ exact}$$

$\square$.

Proposition 7.5: let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $Sh(X)$. If F' is flabby the sequence $0 \to F'(X) \to F(X) \to F''(X) \to 0$ is exact.

Proof: we only need to prove that $F(X) \to F''(X)$ is surjective. Let $s'' \in F''(X)$.
$\exists (U, s)$ with $U \in Op(X)$ and $s \in F(U)$ such that $s \mapsto s''|_U$.
(this is exactness of $F \to F'' \to 0$). Take such a pair with U maximal, $U \neq X$.
Pick $x \in X\setminus U$. then $\exists (V, \tilde{s})$, $x \in V$, $\tilde{s} \mapsto s''|_V$. Gluing $\tilde{s}$ with s we get a new pair $(U\cup V, \tilde{\tilde{s}})$. $\dot{X}$.

we can do so because $(\tilde{s} - s)|_{U\cap V} \in F'(U\cap V)$
$\Rightarrow \exists r \in F'(X)$ s.t. $r|_{U\cap V} = (\tilde{s}-s)|_{U\cap V}$.

Corollary 7.6: $Z \subset X$ locally closed, $0 \to F' \to F \to F'' \to 0$ exact, with F' flabby.
Then the sequence $0 \to \Gamma_Z(X, F') \to \Gamma_Z(X, F) \to \Gamma_Z(X, F'') \to 0$ and
$0 \to \Gamma_Z(F') \to \Gamma_Z(F) \to \Gamma_Z(F'') \to 0$ are exact.

Proof: we have the following diagram in which all columns are exact and the last two rows are exact too (for any U such that $Z \cap U \subset U$ closed).

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_Z(U, F') \arrow[r] \arrow[d] & \Gamma_Z(U, F) \arrow[r] \arrow[d] & \Gamma_Z(U, F'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'(U) \arrow[r] \arrow[d] & F(U) \arrow[r] \arrow[d] & F''(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'(U\setminus Z) \arrow[r] \arrow[d] & F(U\setminus Z) \arrow[r] & F''(U\setminus Z) \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Therefore the first row is exact.

$\square$

Corollary 7.7: if $0 \to F' \to F \to F'' \to 0$ is exact and F', F are flabby, then F'' is flabby.

Proof: F' flabby $\Rightarrow$ $F(U) \to F''(U)$ surjective for any $U \in Op(X)$.

F flabby $\Rightarrow$ $F(X) \to F(U)$ ———— " ————.

Therefore in
$$\begin{array}{ccc} F(X) & \twoheadrightarrow & F''(X) \\ \downarrow & & \downarrow \\ F(U) & \twoheadrightarrow & F''(U) \end{array}$$
the rightmost arrow is surjective. $\square$.

The outcome of all this is that the full subcategory of flabby sheaves is injective for $\Gamma(X,-)$, $\Gamma_Z(-)$ (Z locally closed) and f_*.

In particular, k_Z being flabby we get that the wrong and the correct definition of the microsupport coincide.

Exercise 7.8: Let $F \in Sh(X)$ and $\mathcal{U} = (U_i)_{i\in I}$ an open cover of X.

(i) prove that F flabby $\Longleftrightarrow$ $F|_{U_i}$ flabby $\forall i \in I$.

(ii) prove that $F|_{U_i}$ injective $\forall i \in I \Rightarrow F$ injective.

Soft sheaves

We will be very sketchy here.

Let X be a topological space, $Z \subset X$ and $F \in Sh(X)$. We have a canonical morphism $\Psi: \varinjlim_{U \supset Z} F(U) \longrightarrow \Gamma(Z,F) := F|_Z(Z)$ ($F|_Z := j^{-1}F$, where $j: Z \hookrightarrow X$).

Proposition 7.9: Ψ is injective and it is an isomorphism in the two following cases:

1) X Hausdorff and Z compact.

2) X paracompact and Z closed.

Recall that paracompact $\overset{def}{=}$ Hausdorff + any open cover can be refined to a locally finite one.

Let $f: X \to Y$ be a continuous map.

Recall that proper $\overset{def}{=}$ closed + fibers are relatively Hausdorff and compact.

(when X, Y are locally compact this is equivalent to f^{-1}(compact) is compact).

Definition 7.10: for any $U \in Op(Y)$ we define $f_! F(U) := \{ s \in F(f^{-1}(U)) \,/\, f: supp(s) \to U \text{ proper} \}$.

This defines a subsheaf $f_! F$ of $f_* F$ that we call the <u>direct image with proper support</u>.

We also define $\Gamma_c(X,F) := a_! F = \{ s \in F(X) \,/\, supp(s) \text{ Hausdorff and compact} \}$ ($a: X \to pt$).

In general we have that $f_! \circ g_! = (f \circ g)_!$.

Proposition 7.11: if X, Y are locally compact then for any $x \in X$ the canonical map $(f_! F)_x \to \Gamma_c(f^{-1}(x), F|_{f^{-1}(x)})$ is an isomorphism.

Compare with: no assumption on X and Y but $f: X \to Y$ proper on $\mathrm{supp}(F)$.
Then $(f_* F)_x \longrightarrow \Gamma(f^{-1}(x), F|_{f^{-1}(x)})$ is an isomorphism.

From now, all spaces are assumed to be locally compact.

$f_* F \otimes G \longrightarrow f_*(F \otimes f^{-1} G)$ descends to an iso $f_! F \otimes G \xrightarrow{\sim} f_!(F \otimes f^{-1} G)$.

$F = k_Z$ $f = i: Z \hookrightarrow X \Rightarrow G_Z = i_! i^{-1} G$.

Definition: soft sheaf if $F(X) \to \Gamma(Z, F)$ surjective for any compact Z.

Soft = injective for $\Gamma_c(X, -)$ and $f_!$.

If X locally compact + countable at infinity $\Rightarrow$ soft = inj. for $\Gamma(X, -)$.

Application. $0 \to k_X \to (\Omega^\bullet_X, d_{dR})$ is exact.

$$\begin{array}{c} (\Omega^\bullet_X, d_{dR}) \\ \uparrow \\ \text{soft.} \end{array}$$

$\Rightarrow R\Gamma(X, k_X) \underset{\text{q-iso}}{\xrightarrow{\sim}} \Gamma(X, \Omega^\bullet_X), d_{dR})$.

Reference

The canonical reference for Lectures 5, 6, 7 are the first two chapters of [KS].

8. The microsupport and its properties

Recall the correct definition 6.1: let $F \in D^b(Sh(X))$. Its microsupport $SS(F)$ is defined as follows:

$$p \notin SS(F) \iff \exists U \ni p \text{ such that } \forall (x_0, d\varphi_{x_0}) \in U,\ \mathbb{R}\Gamma_{\{x \mid \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} = 0.$$

Observe that for a closed subset $Z \subset X$ we have the following exact sequence:

$0 \to \Gamma_Z(F) \to F \to \Gamma_{X \setminus Z}(F) \to 0$, which is natural in $F \in Sh(X)$

Then, following Theorem 6.14, we get an exact triangle

$$\mathbb{R}\Gamma_Z(F) \to F \to \mathbb{R}\Gamma_{X\setminus Z}(F) \xrightarrow{+1}.$$

In particular, the condition $\mathbb{R}\Gamma_{\{x \mid \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} = 0$ is equivalent to the requirement that $F_{x_0} \to \mathbb{R}\Gamma_{\{x \mid \varphi(x) < \varphi(x_0)\}}(F)_{x_0}$ is a quasi-isomorphism.

This is what allows one to understand the microsupport as codirections of non-propagation of sections.

<u>Properties 8.1</u>: (i) $SS(F)$ is closed and conic (i.e. $\mathbb{R}_{>0}$-invariant)

(ii) $SS(F) \cap X = \pi(SS(F)) = \mathrm{Supp}(F)$.

(iii) $SS(F) = SS(F[1])$

(iv) let $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ be an exact triangle and $\{i,j,k\} = \{1,2,3\}$. Then

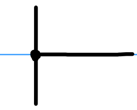
$$SS(F_i) \subset SS(F_j) \cup SS(F_k)$$

and $(SS(F_i) \setminus SS(F_j)) \cup (SS(F_j) \setminus SS(F_i)) \subset SS(F_k)$.

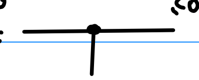
Before proving it, let us give one application:

<u>Example 8.2</u>: let $Z = \mathbb{R}_{\geq 0}$ and $F = k_Z$. By (ii) we have that $SS(F) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}$

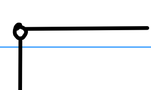
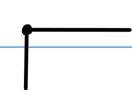
(we use the identification $T^*\mathbb{R} \to \mathbb{R}^2$; $(x, \xi dx) \mapsto (x, \xi)$).

Locally around $x > 0$ F is isomorphic to the constant sheaf $\underline{k}$, therefore we have already seen that $(x,\xi) \in SS(F)$ if and only if $\xi \neq 0$. Therefore $SS(F) \subset$  $\cup$ 

We now consider the exact sequence $0 \to k_{\mathbb{R}_{<0}} \to \underline{k} \to k_{\mathbb{R}_{<0}} \to 0$.

By (iv) $SS(k_{\mathbb{R}_{<0}}) \subset SS(k) \cup SS(k_{\mathbb{R}_{<0}}) =$  and

$\cup$

$$\begin{matrix}(SS(k) \setminus SS(k_{\mathbb{R}_{<0}})) \\ \cup \\ (SS(k_{\mathbb{R}_{<0}}) \setminus SS(k))\end{matrix} =$$  $\Rightarrow SS(k_{\mathbb{R}_{<0}}) =$ 

Proof: (i) the complement of $SS(F)$ is defined as the union of all opens $U \subset T^*X$ such that a property holds. It is therefore open.

Observe also that $p \notin SS(F) \Leftrightarrow \lambda p \notin SS(F)$ $(\lambda > 0)$.

(ii) assume $p = (x, 0)$

- if $x \in Supp(F)$ then take $\varphi \equiv 0$ and $x_0 = x$, in which case
$$\mathbb{R}\Gamma_{\{y \mid \varphi(y) \geq 0\}}(F)_{x_0} = F_x \neq 0. \text{ Thus } (x,0) \in SS(F)$$

- if $x \notin Supp(F)$ then $\exists V$ such that $F_{|V} \simeq 0$. Then for any $(x_0, d\varphi_{x_0})$ with $x_0 \in V$
$$\mathbb{R}\Gamma_{\{y \mid \varphi(y) \geq \varphi(x_0)\}}(F)_{x_0} = 0. \text{ Thus } (x,0) \notin SS(F).$$

Therefore $SS(F) \cap X = Supp(F)$.

$SS(F)$ being closed and conic $(x, \xi) \in SS(F) \Rightarrow (x, 0) \in SS(F) \Rightarrow x \in Supp(F)$.

$x \in Supp(F) \Rightarrow (x, 0) \in SS(F)$.

(iii) obvious.

(iv) exercise.

If $F \in D^b(Sh(X))$ is a local system, i.e. F is isomorphic to a locally constant sheaf, then $SS(F) = X \subset T^*X$ (as soon as it is non-zero, in which case $SS(F) = \emptyset$).

Conversely:

Theorem 8.3: let $F \in D^b(Sh(X))$ be such that $SS(F) \subset X$. Then F is a local system.

Proof: • we start by proving the result for $X = \mathbb{R}$.

Let us fix $s \in \mathbb{R}$. For any $\varepsilon > 0$ and any $G \in C(Sh(X))$ we have
$$0 \to \Gamma_{\{t \mid t \geq s\}}(G)(-\infty, s+\varepsilon) \to G(-\infty, s+\varepsilon) \to G(-\infty, s) \to 0$$
$$\Gamma_{\{t \mid t \geq s\}}(G)(-\infty, s+\varepsilon) = \Gamma_{\{t \mid t \geq s\}}(G)(s-\varepsilon, s+\varepsilon)$$

if G is flabby.

* or injective

Moreover, if G is a flabby* resolution of F (i.e. $0 \to F \to G^0 \to G^1 \to \cdots$ exact) then $\Gamma_{\{t \mid t \geq s\}}(G)(s-\varepsilon, s+\varepsilon)$ represents $\mathbb{R}\Gamma_{\{t \mid t \geq s\}}(F)(s-\varepsilon, s+\varepsilon)$.

When $\varepsilon \to 0$ this converges to $\mathbb{R}\Gamma_{\{t \mid t \geq s\}}(F)_s$, which is zero by assumption.

Therefore, by exactness of $\varinjlim$, we get that $\varinjlim_{\varepsilon} \mathbb{R}\Gamma((-\infty, s+\varepsilon), F) \to \mathbb{R}\Gamma((-\infty, s), F)$

is an isomorphism. Therefore we have the following

<u>Lemma 8.4</u>: for any $s_1 > s_0$ we have that $\mathbb{R}\Gamma((-\infty,s_1),F) \to \mathbb{R}\Gamma((-\infty,s_0),F)$ is an isomorphism.

<u>Proof of the lemma</u>: • from the previous isomorphism we get that for s_1 and s_0 close enough the map must be surjective.

• we now prove that it is injective. Assume there exists $s_1 > s_0$ for which it is not. Fix $s_1 \in \mathbb{R}$ and a non-zero $u \in \mathbb{R}\Gamma((-\infty,s_1),F)$ which restricts to zero for some $s_0 < s_1$. Let $s_0 := \sup\{ s < s_1 \mid u \text{ restricts to zero on } \mathbb{R}\Gamma((-\infty,s),F)\}$.

The map $\varinjlim_{\varepsilon} \mathbb{R}\Gamma((-\infty,s_0+\varepsilon),F) \to \mathbb{R}\Gamma((-\infty,s_0),F)$ being injective, if u restricts to zero on $(-\infty,s_0)$ then $\exists \varepsilon > 0$ such that it restricts to zero on $(-\infty,s_0+\varepsilon)$. Therefore s_0 must be s_1, and u restricts to zero on all $(-\infty,s)$, $s < s_1$.

Finally, observe that $\mathbb{R}\Gamma(U,F) = \varprojlim_{V \subset U} \mathbb{R}\Gamma(V,F)$. Therefore $u = 0$ $\divideontimes$.

Let us now turn to the general case. It is actually sufficient to prove it for $X = \mathbb{R}^n$, around $x_0 = \underline{0}$. Let $r : \mathbb{R}^n \to \mathbb{R};\ x \mapsto r(x) := d(x,\underline{0})$.

<u>Claim 8.5</u>: $SS(\mathbb{R}r_* F) = \mathbb{R} \subset T^*\mathbb{R}$.

<u>Proof</u>: it is a consequence of more general results that we state and prove below.

Therefore, $\mathbb{R}\Gamma(B(0,R),F) = \mathbb{R}\Gamma((-\infty,R),\mathbb{R}r_* F) \xrightarrow{\sim} \mathbb{R}\Gamma((-\infty,\varepsilon),\mathbb{R}r_* F) = \mathbb{R}\Gamma(B(0,\varepsilon),F)$. $\square$

<u>Remark 8.6</u>: if $f : X \to Y$ is a continuous map then f_* sends injectives to injectives (<u>Hint</u>: use the adjunction between f^{-1} and f_*). Therefore, $\mathbb{R}(f \circ g)_* = \mathbb{R}f_* \circ \mathbb{R}g_*$.

<u>Lagrangian correspondences</u>

Let M and N be two symplectic manifolds, and $\Lambda \subset M^a \times N$ a Lagrangian submanifold. The diagonal $\Delta_M \subset M \times M^a$ is Lagrangian. For any $C \subset M$ we can therefore define $C \circ \Lambda := ((C \times \Lambda) \cap (\Delta_M \times N))_{red} \subset (\Delta_M \times N)_{red} \cong N$ ($\Delta_M \times N$ is coisotropic in $M \times M^a \times N$).

- if C is a submanifold, then $C \circ \Lambda$ is a submanifold provided $C \times N$ is transverse to Λ.
- if C is isotropic / coisotropic / Lagrangian then $C \circ \Lambda$ is isotropic / coisotropic / Lagrangian.

Observe that the identification $(\Delta_\Pi \times N)_{red} \cong N$ is given by the projection $\Pi \times \Pi \times N \to N$. Therefore $C \circ \Lambda$ is the image, through this projection, of $(C \times \Lambda) \cap (\Delta_\Pi \times N)$. I.e. it is the image of $(C \times N) \cap \Lambda$ through the second projection $\Pi \times N \to N$.

Proposition 8.7: let $f: X \to Y$ be a smooth map which is proper on $\operatorname{supp}(F)$ $(F \in Sh(X))$.

Then $SS(Rf_* F) \subset SS(F) \circ \Lambda_f$, where $\Lambda_f := \{(x,\xi,y,\eta) \mid y = f(x),\ \xi = \eta \circ df_x\}$.

Before proving the Proposition let us observe that for $C \subset T^*X$ we have that

$$C \circ \Lambda_f = \{(y,\eta) \in T^*Y \mid \exists (x,\xi) \in C \text{ s.t. } y = f(x) \text{ and } \xi = \eta \circ df_x\}.$$

In other words: $T^*X \xleftarrow{T^*f} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y$. $\Lambda_f \circ L = f_\pi((T^*f)^{-1}(C))$.

Proof of Proposition 8.7: let $y \in Y$ and $\varphi: Y \to \mathbb{R}$ such that $\varphi(y) = 0$.

Then $R\Gamma_{\{\varphi \geq 0\}}(Rf_* F)_y \simeq Rf_*(R\Gamma_{\{\varphi \circ f \geq 0\}}(F))_y$ (always true)

$$\simeq R\Gamma(f^{-1}(y), R\Gamma_{\{\varphi \circ f \geq 0\}}(F)) \quad (f \text{ proper on } \operatorname{Supp}(F)).$$

If $(y,\eta) \notin SS(F) \circ \Lambda_f$ then for any $x \in f^{-1}(y)$, $(x, \eta \circ df_x) \notin SS(F)$.

I.e. $\exists U_x \ni (x, \eta \circ df_x)$ s.t. $\forall (x_0, d\psi_{x_0}) \in U$ with $\psi(x_0) = 0$,

$$R\Gamma_{\{\psi \geq 0\}}(F)_{x_0} = 0.$$

Let $U = \bigcup_{x \in f^{-1}(y)} U_x$, and set V the biggest open subset such that $f^{-1}(V) \subset U$.

Take any $(y_0, d\varphi_{y_0}) \in V$ with $\varphi(y_0) = 0$ and set $\psi = \varphi \circ f$.

$$R\Gamma_{\{\varphi \geq 0\}}(Rf_* F)_{y_0} \simeq R\Gamma(f^{-1}(y_0), R\Gamma_{\{\psi \geq 0\}}(F)) = 0.$$

Therefore $(y,\eta) \notin SS(Rf_* F)$. $\square$.

Proof of Claim 8.5: the radius map $r: \mathbb{R}^n \to \mathbb{R}$ is proper. Therefore

$SS(Rr_* F) \subset SS(F) \circ \Lambda_r$. But $SS(F) \subset \mathbb{R}^n \subset T^*\mathbb{R}^n$ (by assumption) and r has no positive critical values, thus $SS(F) \circ \Lambda_r \subset \mathbb{R} \subset T^*\mathbb{R}$. $\square$.

Sheaf theoretic Morse Lemma and Morse inequalities

Proposition 8.8: let $f: X \to \mathbb{R}$ be proper on $\mathrm{Supp}(F)$, $F \in Sh(X)$, and $a<b$.
If the graph of df doesn't intersect $SS(F)$ over $[a,b]$ then the restriction map $\mathbb{R}\Gamma(f^{-1}(-\infty,b), F) \to \mathbb{R}\Gamma(f^{-1}(-\infty,a), F)$ is an isomorphism.

Proof: We prove that $\mathbb{R}\Gamma((-\infty,b), \mathbb{R}f_* F) \to \mathbb{R}\Gamma((-\infty,a), \mathbb{R}f_* F)$ is an isomorphism.
This follows from the proof of Theorem 8.3 and the fact that $SS(\mathbb{R}f_*) \subset SS(F) \circ \Lambda_f \subset$ zero section over $[a,b]$. $\square$.

Corollary 8.9: $\mathrm{Supp}(F)$ compact $+ \mathbb{R}\Gamma(\Pi, f) \neq 0 \Rightarrow \mathrm{graph}(df) \cap SS(F) \neq \phi$.

Let F be a bounded complex of sheaves with finite dimensional cohomologies:
$b_j := \dim(H^j(X,F)) = \dim(\mathbb{R}\Gamma^j(X,F)) < +\infty$.
Let $f: X \to \mathbb{R}$ be proper on $SS(F)$ and assume that $SS(F) \circ \Lambda_f = \{p_1, \dots, p_n\}$
(recall that $\Lambda_f \circ SS(F) = \{p \in SS(F) \mid p = (x, df_x) \text{ for some } x \in X\}$)
We set $x_i = \pi(p_i)$ and $V_i = \mathbb{R}\Gamma_{\{x \mid f(x) \geq f(x_i)\}}(F)_{x_i}$
We assume that $f^{-1}((-\infty,t])$ is compact for every $t \in \mathbb{R}$.
We also assume that for all $i \in \{1, \dots, n\}$ and $j \in \mathbb{Z}$ the spaces $H^j(V_i)$ are finite dimensional.

Proposition 8.10: under the above assumptions we have that for any $j \in \mathbb{Z}$,
$$b_j(F) \leq \sum_{i=1}^{n} \dim(H^j(V_i)).$$

Proof: let $f(\{x_1, \dots, x_n\}) = \{t_1, \dots, t_m\}$ with $t_1 < \dots < t_m$ and set $G := \mathbb{R}f_* F$.
We have: (a) $(-\infty,t] \cap \mathrm{Supp}(G)$ is compact.

(b) $SS(G) \cap \{(t, dt) \mid t \in \mathbb{R}\} \subset \bigcup_{i=1}^{m} \{(t_i, dt)\}$. $\Leftarrow$ Proposition 8.7.

(c) $(\mathbb{R}\Gamma_{\{t \geq t_i\}}(G))_{t_i} = \bigoplus_{j \mid f(x_j) = t_i} V_j =: W_i$

We set by convention $t_0 = -\infty$, $t_{m+1} = +\infty$, $I_t = (-\infty, t)$, $Z_t = (-\infty, t]$,
$I_i = I_{t_i}$ and $Z_i = Z_{t_i}$.

$\mathbb{R}\Gamma^k(I_{i+1}, G) \simeq \mathbb{R}\Gamma^k(Z_i, G)$

$\Uparrow$

Recall that for $t_i < t \leq t_{i+1}$, $\mathbb{R}\Gamma(I_{i+1}, G) \xrightarrow{\sim} \mathbb{R}\Gamma(I_t, G)$.

In particular $\mathbb{R}\Gamma(I_1, G) = 0$. Consider then the exact triangle

$$W_i := (\mathbb{R}\Gamma_{\{t \geq t_i\}}(G))_{t_i} \longrightarrow \mathbb{R}\Gamma(Z_i, G) \longrightarrow \mathbb{R}\Gamma(I_i, G) \xrightarrow{+1}$$

We have that $\chi(\mathbb{R}\Gamma(Z_i, G)) = \chi(W_i) + \chi(\mathbb{R}\Gamma(I_i, G))$

and $b_j^*(\mathbb{R}\Gamma(Z_i, G)) \leq b_j^*(W_i) + b_j^*(\mathbb{R}\Gamma(I_i, G))$, where $b_j^*(V) = (-1)^j \sum_{k \leq j} (-1)^k b_k(V)$.

Observe that $b_j(F) = b_j(G) = \sum_{i=1}^{m} b_j(\mathbb{R}\Gamma(Z_i, G)) - b_j(\mathbb{R}\Gamma(I_i, G))$.

Conclude as an exercise.

9. Kernels and correspondences

Recall that if $f : X \to Y$ is a smooth map that is proper on $\mathrm{Supp}(F)$, with $F \in D^b(Sh(X))$, then $SS(\mathbb{R}f_* F) \subset SS(F) \circ \Lambda_f$

Observe that we also have $SS(\mathbb{R}f_! F) \subset SS(F) \circ \Lambda_f$

The proof is exactly the same, except that we use

$$\mathbb{R}f_! \left(\mathbb{R}\Gamma_{\{\varphi \circ f \geq 0\}}(F)\right)_y \simeq \mathbb{R}\Gamma_{\cancel{*}}\left(f^{-1}(y), \mathbb{R}\Gamma_{\{\varphi \circ f \geq 0\}}(F)\right)$$

because of properness assumption.

We want similar results for the inverse image, the external product, the tensor product, as well as for the composition of kernels. For this we need to introduce some terminology.

Lagrangian correspondences (continued)

Let M, N be symplectic manifolds. Notice that $\Lambda \subset M \times N$ is Lagrangian for the symplectic structure on $M^a \times N$ if and only if it is so for the one on $M \times N^a$. From we will assume that the above holds.

- The contravariant correspondence (recap from Ch. 8)

To any subset $C \subset M$ we associate $\Phi_\Lambda(C) := C \circ \Lambda = \pi_N((C \times N) \cap \Lambda)$.

Alternatively, observe that $\Delta_M \times N$ is coisotropic in $M \times M^a \times N$, with reduced symplectic manifold being N. Then $C \circ \Lambda$ is the reduction of $(C \times \Lambda) \cap (\Delta_M \times N)$.

If C is isotropic / coisotropic / Lagrangian in M, then so is $C \times \Lambda$ in $M \times M^a \times N$, and thus so is $C \circ \Lambda$ in N (when it makes sense).

- The covariant correspondence

To any subset $D \subset N$ we associate $\Phi_\Lambda(D) := \Lambda \circ D = \pi_M(\Lambda \cap (M \times D))$

Alternatively, observe that $M \times \Delta_N$ is coisotropic in $M \times N^a \times N$, with reduced symplectic manifold being M. Then $\Lambda \circ D$ is the reduction of $(M \times \Delta_N) \cap (\Lambda \times D)$.

If D is isotropic / coisotropic / Lagrangian in N, then so is $\Lambda \times D$ is $M \times N^a \times N$, and thus so is $\Lambda \circ D$ in M (when it makes sense).

• <u>Composing correspondences</u>

Let W be a third symplectic manifold and let Λ' be Lagrangian in $N^a \times W$ (equivalently, in $N \times W^a$). We define $\Lambda \circ \Lambda' := \pi_{MW}((\Lambda \times W) \cap (M \times \Lambda'))$.
Alternatively, observe that $M \times \Delta_N \times W$ is coisotropic in $M^a \times N \times N^a \times W$ (equivalently, in $M \times N^a \times N \times W^a$). Then $\Lambda \circ \Lambda'$ is the reduction of $(\Lambda \times \Lambda') \cap (M \times \Delta_N \times W)$. Observe that $\Lambda \times \Lambda'$ is Lagrangian, so that $\Lambda \circ \Lambda'$ will be Lagrangian in $M^a \times W$ (or $M \times W^a$) whenever it makes sense.

One easily see that $\Phi_{\Lambda \circ \Lambda'} = \Phi_\Lambda \circ \Phi_{\Lambda'}$ and $\overline{\Phi}_{\Lambda \circ \Lambda'} = \overline{\Phi}_{\Lambda'} \circ \overline{\Phi}_\Lambda$.

Let $f: X \to Y$ be a smooth map. We defined $\Lambda_f := \{(x,\xi,y,\eta) \mid y = f(x),\ \xi = \eta \circ df_x\} \subset T^*X \times T^*Y$.
Notice that Λ_f is Lagrangian for the difference of the symplectic structures on T^*X and T^*Y.
Namely, $\left(\sum_i d\xi^i \wedge dx^i + \sum_i dy^i \wedge d\eta^i\right)_{|\Lambda_f} = \sum_i d(\eta \circ df)^i \wedge dx^i + \sum_i df^i \wedge d\eta^i$
$$= \sum_{i,j} \frac{\partial f^j}{\partial x^i} d\eta^j \wedge dx^i + \sum_{i,j} \frac{\partial f^i}{\partial x^j} dx^j \wedge d\eta^i = 0.$$

For $C \subset T^*X$ we get that $C \circ \Lambda_f = q_Y((C \times T^*Y) \cap \Lambda_f) = f_\pi((T^*f)^{-1}(C))$.
For $D \subset T^*Y$ we get that $\Lambda_f \circ D = q_X(\Lambda_f \cap (T^*X \times D)) = T^*f(f_\pi^{-1}(D))$.

$\left(\text{Recall} \quad T^*X \xleftarrow{T^*f} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y\right)$

Here and below we use the notation $q_X := \pi_{T^*X}$.

Kernels

Let X, Y be manifolds and $K \in D^b(Sh(X \times Y))$.

• <u>the contravariant correspondence:</u>

We define a functor $\overline{\Phi}_K := R\pi_{Y!}(\pi_X^{-1}(-) \otimes K) : D^b(Sh(X)) \to D^b(Sh(Y))$.
We also write $F \circ K := \overline{\Phi}_K(F)$ for $F \in D^b(Sh(X))$.

• <u>the covariant correspondence:</u>

We define a functor $\Phi_K := R\pi_{X!}(K \otimes \pi_Y^{-1}(-)) : D^b(Sh(Y)) \to D^b(Sh(X))$.
We also write $K \circ G := \Phi_K(G)$ for $G \in D^b(Sh(Y))$.

• <u>Composing correspondences:</u>

Let Z be a third manifold and let $K' \in D^b(Sh(Y \times Z))$. We define
$$K \circ K' := R\pi_{XZ!}(\pi_{XY}^{-1} K \otimes \pi_{YZ}^{-1} K') = R\pi_{XZ!}(\Delta_Y^{-1}(K \boxtimes K'))$$

Observe that when $Z=pt$ then $K\circ = \Phi_K$ (so that our notation is consistent).
Similarly, if $X=pt$ then $\circ K' = \overline{\Phi}_{K'}$.

Finally, one can prove that $(K\circ K')\circ K'' \simeq K\circ(K'\circ K'')$ (if $K''\in D^b(Sh(Z\times W))$).

One actually should prove that both are isomorphic to $K\circ K'\circ K'' := R\pi_{XW!}(\pi_{XY}^{-1}K\otimes\pi_{YZ}^{-1}K'\otimes\pi_{ZW}^{-1}K'')$.

In particular, one gets that $\Phi_{K\circ K'}\simeq \Phi_K\circ\Phi_{K'}$ and $\overline{\Phi}_{K\circ K'}\simeq\overline{\Phi}_{K'}\circ\overline{\Phi}_K$.

<u>Proposition 9.1</u>: (i) if $f:X\to Y$ is submersive and $G\in D^b(Sh(Y))$ then $SS(f^{-1}G)=\Lambda_f\circ SS(G)$

(ii) if $F\in D^b(Sh(X))$ and $G\in D^b(Sh(Y))$ then $SS(F\boxtimes G)\subset SS(F)\times SS(G)$.

(Recall that $F\boxtimes G := \pi_X^{-1}F\otimes\pi_Y^{-1}G$)

We won't prove it, but we would actually like to have a refinement of (i).

<u>Definition 9.2</u>: we say that a smooth map $f:X\to Y$ is <u>non-characteristic</u> with respect to a closed conic subset $A\subset T^*Y$ if $\{(y,\eta)\in A \mid \exists x\in X \text{ s.t. } y=f(x) \text{ and } \eta\circ df_x=0\}$ sits inside the zero section of T^*Y.

<u>Proposition 9.3</u>: let $G\in D^b(Sh(Y))$ and $f:X\to Y$ be non-characteristic with respect to $SS(G)$. Then $SS(f^{-1}G)\subset\Lambda_f\circ SS(G)$.

We won't prove this one either, but the following consequence instead.

<u>Corollary 9.4</u>: let $K\in D^b(Sh(X\times Y))$ and $K'\in D^b(Sh(Y\times Z))$ be good kernels.

This means, e.g. for K, that the projection $q_X: T^*(X\times Y)\to T^*X$ is proper on $SS(K)$.

Then $SS(K\circ K')\subset SS(K)^{a_Y}\circ SS(K')$, where $(x,\xi,y,\eta)^{a_Y}=(x,\xi,y,-\eta)$.

<u>Proof</u>: • first of all we prove that $id\times\Delta\times id: T^*X\times T^*Y\times T^*Z\to T^*X\times T^*Y\times T^*Y\times T^*Z$ is non-characteristic with respect to $SS(K)\times SS(K')$. Namely, let us fix $(x,y,y,z)\in X\times\Delta_Y(Y)\times Z$ and assume that $(\xi,\eta,\eta',\zeta)\in T_x^*X\times T_y^*Y\times T_y^*Y\times T_z^*Z$ is such that $\xi=\eta+\eta'=\zeta=0$. The map $SS(K)\to T^*X$ being proper and the set $SS(K)$ being conic, we get that $\eta=0$. Therefore $\eta'=0$.

• $SS(K\circ K')\subset SS(\Delta_Y^{-1}(K\boxtimes K'))\circ\Lambda_{\pi_{XZ}}\subset\Lambda_{\Delta_Y}\circ(SS(K)\times SS(K'))\circ\Lambda_{\pi_{XZ}}$.

<u>Lemma 9.5</u>: let W be a manifold, $\Delta:W\to W\times W$ the diagonal map and $L\subset T^*(W\times W)$.

Then $\Lambda_\Delta\circ L=\{(x,\xi'+\xi'')\mid(x,x,\xi',\xi'')\in L\}$.

<u>Proof of the Lemma</u>: $\Lambda_\Delta=\{(x,x',x'',\xi,\xi',\xi'')\mid(x',x'')=\Delta(x)\text{ and }\xi=(\xi',\xi'')\circ d\Delta_x\}$

$=\{(x,x',x'',\xi,\xi',\xi'')\mid x=x'=x'' \text{ and } \xi=\xi'+\xi''\}$. □

Therefore $\Lambda_{\Delta_Y}\circ(SS(K)\circ SS(K'))=\{(x,\xi,y,\eta'+\eta'',z,\zeta)\mid (x,\xi,y,\eta')\in SS(K) \text{ and } (y,\eta'',z,\zeta)\in SS(K')\}$

Finally observe that $\Lambda_{\pi_{XZ}} = \{(x,y,z,\xi,\eta,\zeta,x',z',\xi',\zeta') \mid x=x', z=z', \xi=\xi', \zeta=\zeta', \eta=0\}$.
Thus for any $C \subset T^*(X\times Y\times Z)$, $C\circ\Lambda_{\pi_{XZ}} = \{(x,z,\xi,\zeta) \mid \exists y\in Y \text{ s.t. } (x,y,z,\xi,0,\zeta)\in C\}$.
As a consequence we get that:

$$\Lambda_{\Delta_Y}\circ(SS(K)\times SS(K'))\circ\Lambda_{\pi_{XZ}} = \left\{(x,z,\xi,\zeta) \;\middle|\; \begin{array}{l} y\in Y, \eta,\eta'\in T_y^*Y \text{ s.t. } \eta+\eta'=0 \text{ and} \\ (x,\xi,y,\eta)\in SS(K),\ (y,\eta',z,\zeta)\in SS(K') \end{array}\right\}$$

$$= q_{XZ}\left(\left(SS(K)^{a_Y}\times T^*Z\right)\cap\left(T^*X\times SS(K')\right)\right)$$

$$= SS(K)^{a_Y}\circ SS(K'). \qquad \square$$