Two examples of fully extended TFTs

Damien Calaque (Université de Montpellier)

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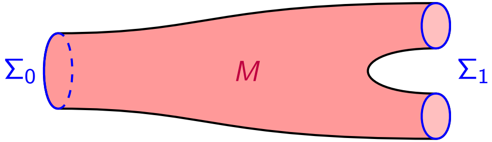

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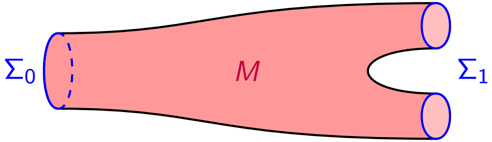

- a tensor product to a disjoint union: $\mathcal{Z}(M \sqcup N) = \mathcal{Z}(M) \otimes \mathcal{Z}(N)$.

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No classification in higher dimension.

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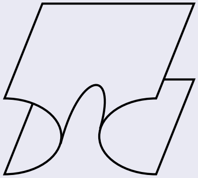
An example of a 2-morphism



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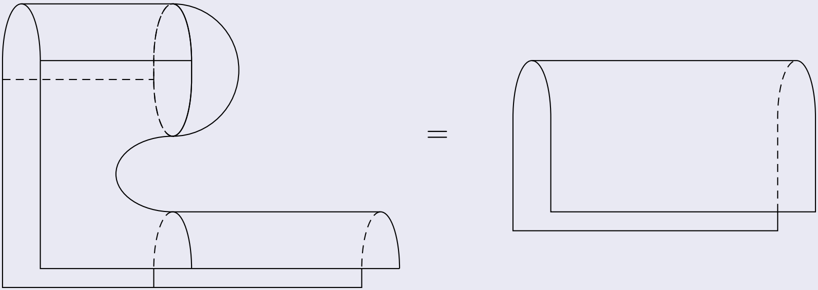
n-dualizable $\overset{def}{=}$ dualizable + (co)evaluation morphism admit adjoints + (co)unit admit adjoints + etc... up to $n-1$. In other words, $Bord_n^{fr}$ is the free symmetric monoidal (∞, n)-category generated by an n-dualizable object.

An example of an adjunction in terms of cobordisms



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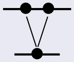

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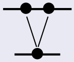

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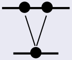

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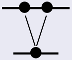

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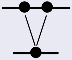

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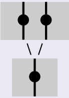

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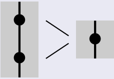

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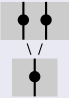

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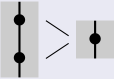

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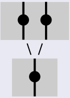

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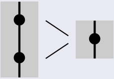

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- We will explain how to prove it, essentially by drawing pictures (in the case $n = 2$ for simplicity).
- It is sufficient to explain how that works for n-morphisms.

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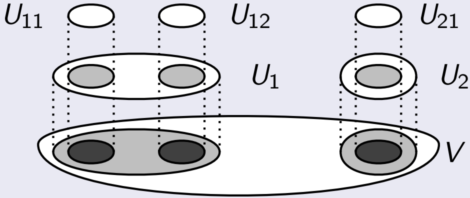

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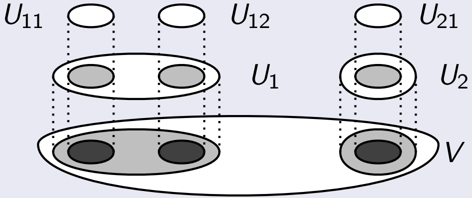

- gluing property (one can reconstruct E_U from the data of a nice enough cover $\mathcal{U}$ and $E_{\mathcal{U}}$).

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- if $f : X \to Y$ is a continuous map and E a factorization algebra on X then $f_* E : U \mapsto E_{f^{-1}(U)}$ is a factorization algebra on Y.

Examples

- The various "Disk algebras" from Hiro's talk produce factorization algebras on $\mathbb{R}^n$. They are exactly the ones that are locally constant (w.r.t. a given stratification).
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  ⚠ f_* does not preserve locally constantness (but it does if f is fiber bundle).

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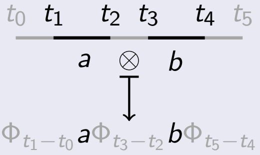

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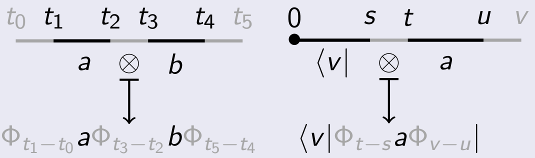

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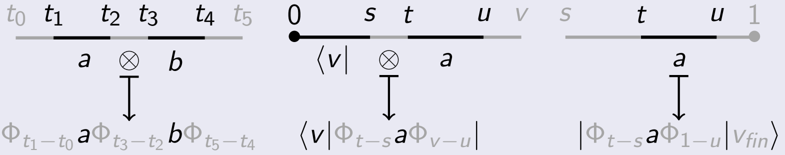

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$t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5$: $a \otimes b \longmapsto \Phi_{t_1-t_0} a \Phi_{t_3-t_2} b \Phi_{t_5-t_4}$

$0 \quad s \quad t \quad u \quad v$: $\langle v| \otimes a \longmapsto \langle v|\Phi_{t-s} a \Phi_{v-u}|$

$s \quad t \quad u \quad 1$: $a \longmapsto |\Phi_{t-s} a \Phi_{1-u}|v_{fin}\rangle$

One can show that $E_{[0,1]} = M_d \underset{A}{\otimes} M_g$ ($\mathbb{C}$ in the example).

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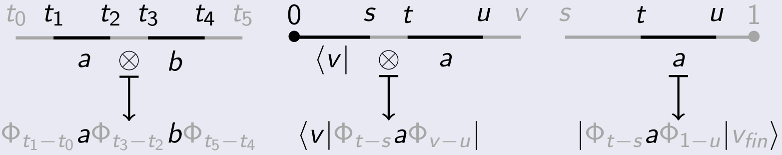

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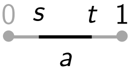

$\longmapsto \langle v_{init}|\Phi_s a \Phi_{1-t}|v_{fin}\rangle$ as a probability amplitude.

An example coming from vertex models

V vector space of states.

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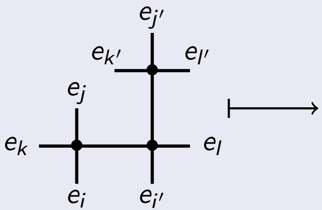

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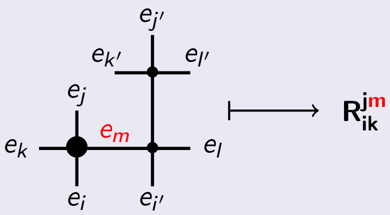

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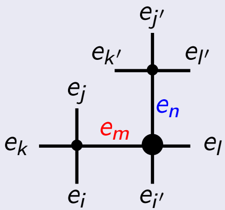

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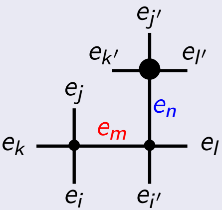

$\longmapsto \quad R_{ik}^{jm} R_{i'm}^{nl} \mathbf{R}_{\mathbf{nk'}}^{\mathbf{j'l'}}$

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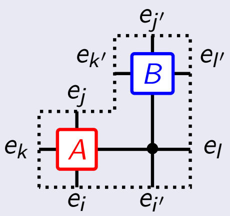

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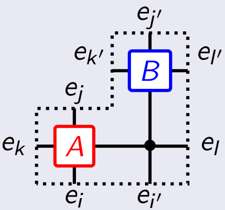

$\longmapsto \quad A \otimes B \mapsto \left(A_{ik}^{jm} R_{i'm}^{nl} B_{nk'}^{j'l'}\right)_{ii'kk'}^{jj'll'}$