Lie Theory and Algebraic Geometry

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ETH Zürich

November 28, 2011

Symmetries (Lie Theory)

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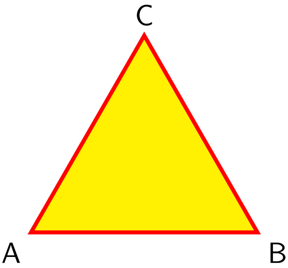

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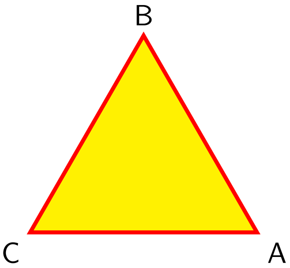

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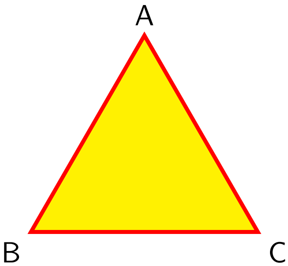

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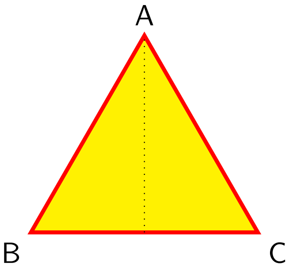

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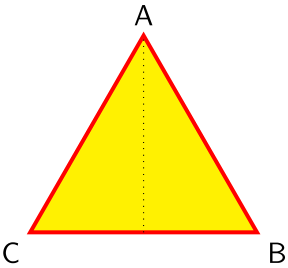

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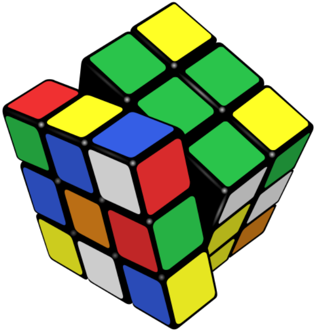

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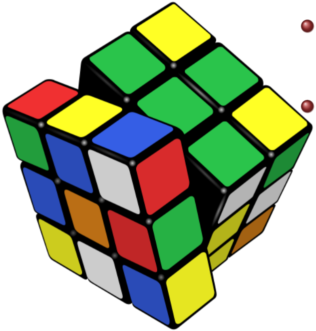

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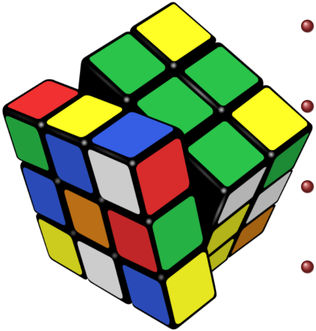

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- world record: 5,66 sec (Feliks Zemdegs)
- 43'252'003'274'489'856'000 elements
- God's number: 20

  Rokicki-Kociemba-Davidson-Dethridge (2010)

$$\begin{aligned} \text{Generators} &= \text{elementary moves} \\ \text{God's number} &= \text{diameter of the group in terms generators} \end{aligned}$$

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- Given $P(x)$, is there a criterium for the answer to be **yes**?
  - solutions: $S = \{x_1, \ldots, x_k\}$
  - Galois group: permutations of S such that any alg. eq. satisfied by $x_1, \ldots, x_k$ is still satisfied after they have been permuted.
  - solvability by radicals

$$\Leftrightarrow$$

Galois group is "solvable"

Evariste Galois
(1811 - 1832)

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$\rightsquigarrow$ Picard-Vessiot theory (a-k-a differential Galois theory)  – :

solvability by quadrature

$\Leftrightarrow$

Lie group connected and solvable

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- Lie bracket $[u, v]$ is the infinitesimal default of commutativity:

$$\exp(u)\exp(v)\big(\exp(v)\exp(u)\big)^{-1} = \exp([u,v])$$

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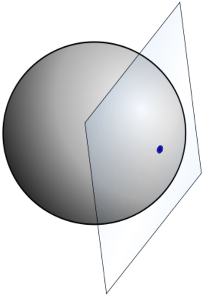

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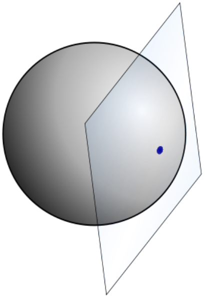

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FELIX KLEIN
(1849 - 1925)

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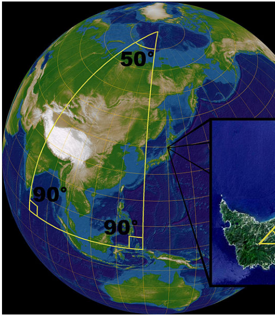

non axiomatic approach to
non-euclidan geometries

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Levi–Mal'tsev decomposition

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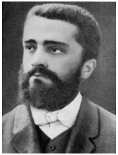
$$\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{l}$$

with $\mathfrak{r}$ solvable and $\mathfrak{l}$ semi-simple.



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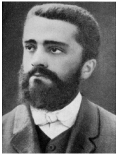

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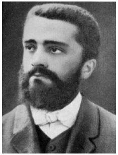

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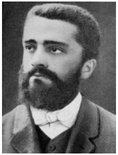

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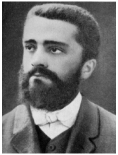

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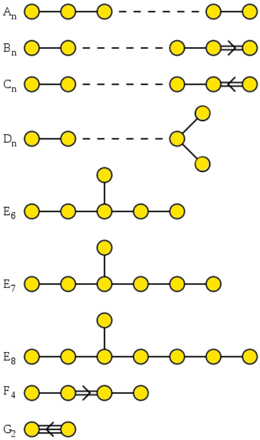

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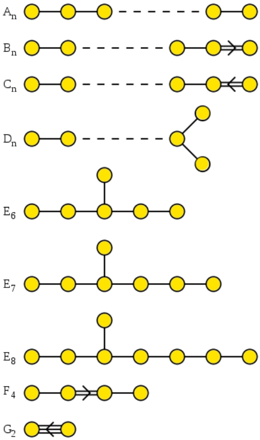

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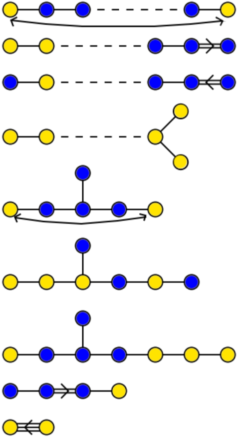
 Ichiro Satake
Dynkin diagrams $\longrightarrow$ Satake diagrams



A few Satake diagrams

Loci (Algebraic Geometry)

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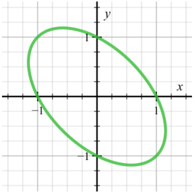

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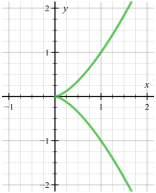

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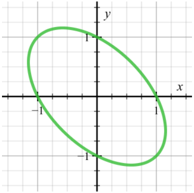

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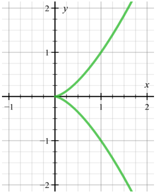

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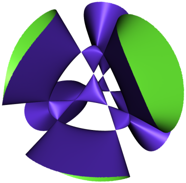
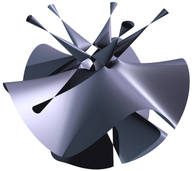
- examples of algebraic surfaces:



Kummer surface



Togliatti surface

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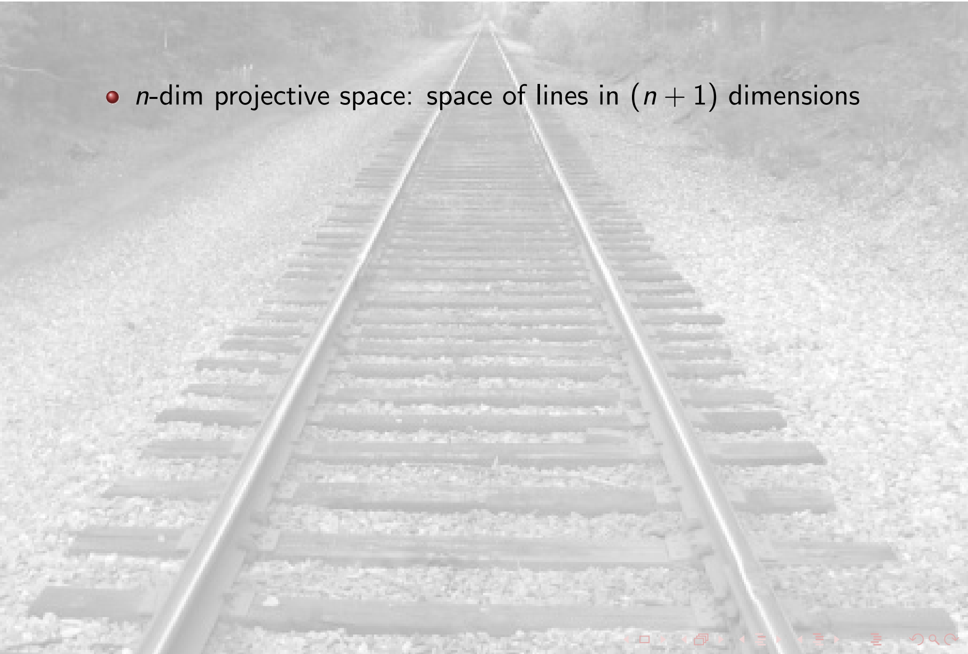

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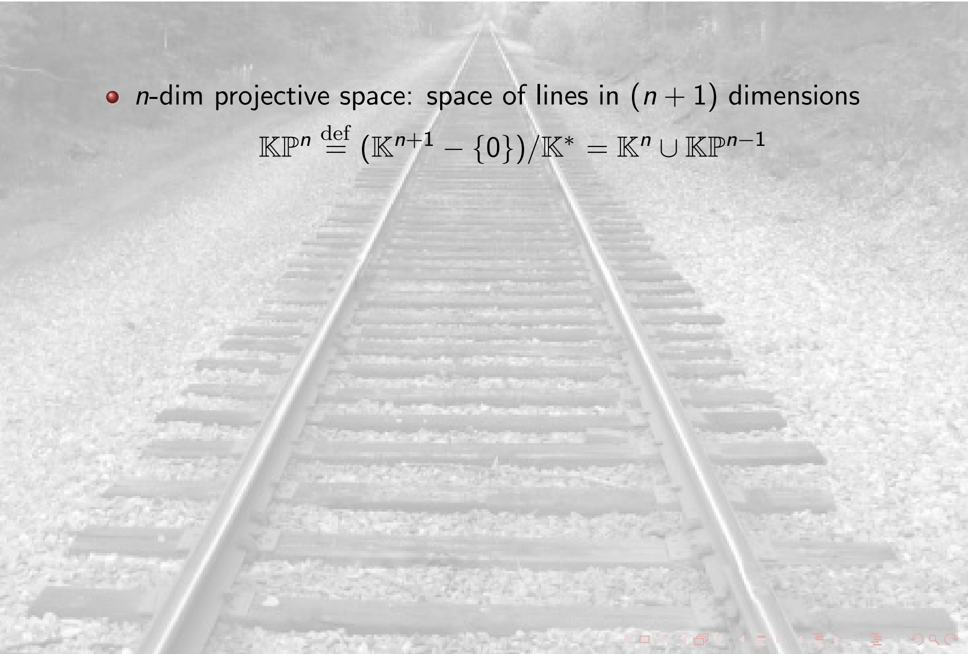

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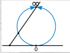
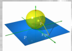
    $\mathbb{RP}^1$:  $\mathbb{CP}^1$: 

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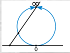
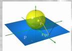
    $\mathbb{RP}^1$: 

    $\mathbb{CP}^1$: 

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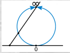
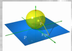
    $\mathbb{RP}^1$: 

    $\mathbb{CP}^1$: 

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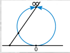
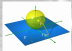
    $\mathbb{RP}^1$:  $\mathbb{CP}^1$: 

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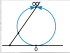
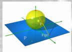
    $\mathbb{R}\mathbb{P}^1$: 

    $\mathbb{C}\mathbb{P}^1$: 

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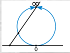
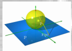
    $\mathbb{RP}^1$:  $\mathbb{CP}^1$: 

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- **varieties $\longrightarrow$ schemes**
- invented by Alexander Grothendieck
- solve problem of tangential points

Solving the third problem... derived algebraic geometry

Recent ($\geq$ 2000) extension of algebraic geometry

- Bertrand Toën & Grabriele Vezzosi
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Calaque-Caldararu-Tu: analogy between closed embeddings $X \hookrightarrow Y$ of algebraic varieties and inclusions of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$.

Towards a dictionary

Bogomolov decomposition

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$$\mathcal{N} \times \mathcal{T} \times (\mathcal{L}_1 \times \cdots \times \mathcal{L}_k)$$

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We would like to understand this analogy in more rigourous terms

THANK YOU