Deformation quantization with branes and coloured MZVs

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Higher structures emerging from renormalisation (ESI)

14 October 2020

Deformation quantization

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1. $A_0 = k[[x_1, \ldots, x_n]]$ (k field of char. 0);
2. $A_0 = C^\infty(M)$, M being a smooth manifold;
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Actually, (2) and (3) are obtained from (1) by globalization techniques that we are not going to discuss here. Kontsevich formula for (1) is remarkably elegant.

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  - vertex set $\{1, \ldots, n, \bar{1}, \bar{2}\}$,
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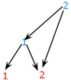

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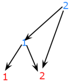

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- coefficients $c_\Gamma \in \mathbb{R}$ are of transcendental nature.

Moduli of marked disks

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These integrals converge and satisfy algebraic relations ensuring the associativity of $\star$ [Kontsevich].

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Topological invariance guaranties the associativity of $\star$

Both $\big((f \star g) \star h\big)(x)$ and $\big(f \star (g \star h)\big)(x)$ equal

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Spoiler: already with two branes, the weights (and graphs) involved are more general.

Multiple zeta values

Definition

Let $s_1, \ldots, s_\ell$ be positive integers, with $s_1 > 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}}.$$

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These numbers also have an integral representation:

$$\zeta(s_1, \ldots, s_\ell) = \int_{\Delta^k} \omega_0(t_1) \ldots \omega_0(t_{s_1 - 1}) \omega_1(t_{s_1}) \omega_0(t_{s_1 + 1}) \ldots \omega_1(t_k)$$

where

- $\omega_0(t) = dt/t$ and $\omega_1(t) = dt/(1-t)$,
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They are iterated integrals of $dlog(c.r.)$ on $\mathcal{M}_{0,4}$.

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This is essentially the same strategy as for Brown's result, with a specific difficulty for when one forgets an interior point.

Alternating MZVs

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Generalization: N-coloured MZVs

$s_1, \ldots, s_\ell \in \mathbb{N}_{>0}$ and $\xi_1, \ldots \xi_\ell \in \mu_N$, with $(s_1, \xi_1) \neq (1, 1)$:

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Consider the moduli $\mathcal{C}_{n,p+1+q}$ of marked disks: boundary marked points are given by

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We have a map $\iota : \mathcal{C}_{n,p+1+q} \hookrightarrow \mathcal{M}_{0,N(2n+p+q)+2}$ sending all coloured (blue, magenta and red) points (that we see as points in the upper half-plane) to their N-th roots and complex conjugates.

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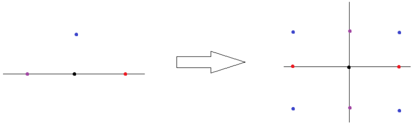

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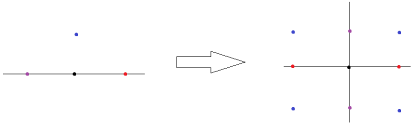

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One then defines the sheaf $\mathcal{U}^\bullet_N := \iota^* \mathcal{U}^\bullet$ of N-coloured polylogarithmic forms.

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Questions:

- Occurences of N-coloured MZVs in the Poisson σ-model?
- Nature of the weights when there are more branes?
- Higher genus version? Do eMZVs appear if one replaces the source with a genus one curve in the Poisson σ-model?