Superalgebra of differentiable functions and derived geometry (D. Roytenberg)

What is derived geometry?

- basic idea (Kontsevich, Kapranov, ...): "hidden smoothness". Replace singular objects with "smooth" ones in a way that retains the information about the singularities.
- singularities can be of two types:
  - quotient singularities: e.g. replace X/G by the action groupoid. I.e. we replace spaces by stacks

    $\mathrm{ComAlg} \to \mathrm{Sets}$ $\qquad$ $\mathrm{ComAlg} \to \mathrm{sSets}$

  - intersection singularities: first of all $L \times_M L'$ might not exist in smooth manifolds, but even if it exists it might be the wrong answer. E.g.

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "x \mapsto x^2"] \\ * \arrow[r] & \mathbb{R} \end{tikzcd}$$

But it should really be the double/fat point.

- BV formalism: computes the derived critical locus of a functional

BFV formalism: computes the derived reduced phase space of a constrained mechanical system.

In both cases the two types of singularities occur.

- ordinary geometric objects form an ordinary category whereas derived objects form a higher category (more precisely $(n,1)$-category, $n \in \{2,3,\dots\} \cup \{\infty\}$).

Derived objects are defined up to weak equivalences.

One can glue pieces $(U_\alpha)_\alpha$ by weak equivalences on $(U_{\alpha\beta})_{\alpha,\beta}$ + coherence conditions further up.

Spaces $\mathrm{ComAlg} \to \mathrm{Sets}$	Stacks $\mathrm{ComAlg} \to \mathrm{sSets}$
Derived spaces homotopy $\mathrm{ComAlg} \to \mathrm{Sets}$	Derived stacks homotopy $\mathrm{ComAlg} \to \mathrm{sSets}$

Models for homotopy ComAlg: $s\mathrm{ComAlg}$, E_∞-Alg, $dg_{\leq 0}$-ComAlg in characteristic 0.

The above diagram was for derived <u>algebraic</u> geometry. What about other geometries, like e.g. C^∞?.

It is known that the functor $\mathrm{Manifolds}^{op} \longrightarrow \mathrm{Com}_{\mathbb{R}}\mathrm{Alg}\,;\, M \mapsto C^\infty(M)$ is fully faithful. But contrary to what happen in algebraic geometry (where $\underbrace{\mathrm{Spec}(A) \times \mathrm{Spec}(A')}_{\mathrm{Spec}(B)} = \mathrm{Spec}(A \otimes_B A')$) this functor does NOT preserve (fiber) products. E.g. $C^\infty(M \times M') \neq C^\infty(M) \otimes C^\infty(M')$.

To fix this one needs some extra-structure: notice that if $A = C^\infty(M)$ then for any $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ we get an operation $A_f : A^n \longrightarrow A$.

This leads us to the notion of an algebraic theory:

<u>Algebraic theory</u> (W. Lawvere): category $\mathbb{T}$ with finite products.

- A <u>$\mathbb{T}$-algebra</u> is a product preserving functor $\mathbb{T} \longrightarrow \mathrm{Sets}$.
- $\mathbb{T}$ is <u>uni-sorted</u> if there is an object X such that any object is isomorphic to X^n for some $n \geq 0$ (in effect $\mathrm{Obj}(\mathbb{T}) = \mathbb{N}$, product is $+$ and 0 is the terminal object).

<u>Examples</u>: (i) $\mathrm{Com}_{\mathbb{R}} : \mathrm{Hom}(n, 1) = \mathbb{R}[x_1, \dots, x_n]$. $\mathrm{Com}_{\mathbb{R}}$-$\mathrm{Alg}$ are commutative $\mathbb{R}$-algebras.

(ii) $\mathbb{C}^\infty : \mathrm{Hom}(n, 1) = C^\infty(\mathbb{R}^n, \mathbb{R})$. A $\mathbb{C}^\infty$-Alg $\mathcal{A}$ is determined by $A_1 := \mathcal{A}(1)$ and $A_f : A_1^n = \mathcal{A}(n) \longrightarrow \mathcal{A}(1) = A_1$ for any $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$.

The functor $\mathbb{C}^\infty$-$\mathrm{Alg} \longrightarrow \mathrm{Sets}\,;\, \mathcal{A} \to \mathcal{A}(1)$ has a right adjoint (free $\mathbb{C}^\infty$-Alg functor, which e.g. sends the n-elements set to $C^\infty(\mathbb{R}^n)$).

The coproduct of $\mathbb{C}^\infty$-Alg is denoted $\otimes_\infty$ and it has the property that

$$C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m) = C^\infty(\mathbb{R}^{n+m}) \quad \left(\text{more generally } C^\infty(M) \otimes_\infty C^\infty(N) = C^\infty(M \times N)\right).$$

<u>Derived version</u>: "$\mathbb{C}^\infty$-algebras up to homotopy".

- D. Spivak (2009): "up to homotopy simplicial $\mathbb{C}^\infty$-algebras":
  $\mathbb{C}^\infty \longrightarrow \mathrm{sSets}$ product-preserving up to week equivalences.
- Borisov-Noël (2011): just strictly simplicial $\mathbb{C}^\infty$-algebras are enough.
- R-Carchedi: what about dg setting?
- D. Joyce: truncated version (forms a strict 2-category).

$\mathbb{R}[x_1,\dots,x_n]$ for $\mathrm{Com}_{\mathbb{R}}$

Observe that $\mathrm{Com}_{\mathbb{R}} \to \mathfrak{C}^\infty$. It admits a "superization":

$\stackrel{\text{def}}{=}$ even part of

$$\$C^\infty := \begin{cases} \bullet \text{ objects are pairs } (n|m). \\ \bullet\ (n|m)\times(n'|m') = (n+n'|m+m') \\ \bullet\ \mathrm{Hom}((n|m),(1|0)) = C^\infty(\mathbb{R}^n)\otimes\Lambda^m_{\underline{0}}. \\ \quad \mathrm{Hom}((n|m),(0|1)) = C^\infty(\mathbb{R}^n)\otimes\Lambda^m_{\underline{1}}. \\ \bullet \text{ Composition is given by insertion (this works because} \\ \quad C^\infty(\mathbb{R}^n)\otimes\Lambda^m_{\underline{0}} \text{ is a } \mathfrak{C}^\infty\text{-algebra).} \end{cases}$$

__Crucial fact__: if W is a nilpotent extension of $\mathbb{R}$ (i.e. $W = \mathbb{R}\oplus N$, N nilpotent) then for any $\mathfrak{C}^\infty$-algebra A, $A\otimes W$ is canonically a $\mathfrak{C}^\infty$-algebra (Taylor formula).
Note that $\Lambda^m_{\underline{0}}$ is such an extension of $\mathbb{R}$.

A __$\mathfrak{C}^\infty$-superalgebra__ is an $\$C^\infty$-algebra.

A grading on an $\$C^\infty$-algebra $\stackrel{\text{def}}{\longleftrightarrow}$ an action of the multiplicative group $\hat{\mathbb{G}}_m$ (corepresented by $\mathbb{R}[x,x^{-1}] = C^\infty(\mathbb{R}^\times)$)

A differential on an $\$C^\infty$-algebra $\stackrel{\text{def}}{\longleftrightarrow}$ an action $\hat{\mathbb{G}}^1_{ad}$ (corepresented by $\Lambda^\bullet(\mathbb{R})$).

A DG-structure on an $\$C^\infty$-algebra $\stackrel{\text{def}}{\longleftrightarrow}$ action of $\mathrm{Aut}(\hat{\mathbb{G}}^1_{ad})$.

__Theorem__ [Carchedi-Roytenberg]: there is a Quillen model structure on the category $\$C^\infty\mathrm{Alg}^{\mathrm{Aut}(\hat{\mathbb{G}}^1_{ad})}_{(\text{even})}$ with weak equivalence given by quasi-isomorphisms on the underlying cochain complexes, and fibrations are surjective on the underlying complexes.