Montpellier, April 23, 2016

Dear Chris

here is a bit more details about the strategy I was telling you about in Waterloo[1].

Let's start with a unital E_1-algebra A. It follows from Lurie's Higher Algebra (see the part on centralizers) that the pair $\big(Z(id_A), Z(1_A)\big)$ has the following structure inside the $(\infty,1)$-category of E_1-algebras: it has the structure of an E_1-algebra together with a module. Therefore, by (a stratified version of) Dunn additivity we get that $\big(Z(id_A), Z(1_A)\big)$ is an algebra over the Swiss-Cheese operad, that is to say a locally constant factorization on the stratified space $\mathbb{R}\times\mathbb{R}_+$ (with the boundary stratification).

Remark 0. $Z(id_A) = C^*(A)$ and $Z(1_A) = A$.

Let me now summarize very briefly the construction from Horel's paper[2]. Taking factorization homology of $\big(C^*(A), A\big)$ on the half-cylinder one gets an S^1-equivariant locally constant factorization algebra on the half-cylinder. Now consider the map of stratified spaces that sends the cylinder to the cone by sending the boundary circle to the vertex of the cone. Pushing forward along this map we get an S^1-equivariant locally constant factorization algebra on the cone that is nothing but the pair $\big(C^*(A), C_*(A)\big)$. Horel shows that S^1-equivariant locally constant factorization algebras on the cone, such that the restriction on the open cone is an E_2-algebra, are the same thing as algebras over the KS-operad ("KS" for Kontsevich–Soibelman).

Remark 1. This is not so much surprising since the KS-operad is a version of the Swiss-Cheese operad but on the half cylinder.

Remark 2. Note that the S^1-equivariant structure on $\big(C^*(A), A\big)$ is trivial, we could have started with an E_1-algebra with an automorphism α to get a non-trivial one. In this case we would get $C_*(A, A_\alpha)$ instead of $C_*(A) = C_*(A, A)$.

Let me make a few observations before explaining my proposal.

Observation 1. It is important to note that factorization algebras are naturally pointed (or unital). The pointing/unit being given by the inclusion of the empty open subset. As a matter of example, the pointing in $C_*(A)$ is given by the 0-cycle 1.

Observation 2. The inclusion of an open disk into the cone (the open disk not containing the vertex of the cone) gives a map $C^*(A) \to C_*(A)$. It is easily seen that this map is given by the cap-product with the pointing 1.

Observation 3. The cap-product is the operation that comes from the inclusion of the disjoint union of a disk and a small open cone into the cone.

Observation 4. $C^*(A)$ can be seen as a locally constant factorization algebra on the cone (any E_2-algebra is a module over itself). But not S^1-equivariant. Actually, Observation 2 can be upgraded to say that the map $(id, -\cap 1) : \big(C^*(A), C^*(A)\big) \to \big(C^*(A), C_*(A)\big)$ is a morphism of locally constant factorization algebras on the cone.

[1] On the occasion of the conference "Deformation Quantization of Shifted Poisson Structures", `https://pirsa.org/c16005`.

[2] "Factorization homology and calculus à la Kontsevich Soibelman", `https://arxiv.org/abs/1307.0322`.

Here is now my proposal.

a) Whenever one has a locally constant factorization algebra (E, F) on the cone, together with a map $f : k \to F[d]$, one then gets that $(E, F[d])$ is a locally constant factorization algebra on the cone. The reason is the following: operations on (E, F) "eat" either zero or one copy of F (their can be at most one open subset that contains the vertex of the cone). The shift by d obviously does not affect operations "eating" one copy of F. Same for operations not involving any copy of F at all. The problem is with those operations "eating" no copy of F and taking values in F: but these are all obtained by composing the previous ones with the pointing of F. So, as soon as we are given a pointing of $F[d]$ we're good! If (E, F) is S^1-equivariant and if the pointing is S^1-invariant (which is exactly, in our case of interest, the condition that $f : k \to C_*(A)[d]$ factors through cyclic homology) then we get that we get that $(E, F[d])$ is an S^1-equivariant factorization algebra on the cone.

b) Now assume that the d-shifted pointing of F that we consider is non-degenerate, in the sense that the induced map $E \to F[d]$ is an equivalence. Then we automatically get an S^1-equivariant structure on the locally constant factorization algebra (E, E) on the cone. So, we precisely get that E is an S^1-equivariant locally constant factorization algebra on the plane! This is exactly saying that E is a framed E_2-algebra. We are done!

If you're interested, I'd be very happy to work on this project together[3]. There's in particular one question I'd like to answer: does this approach work only for the proper case, only for the smooth case, or both? You're definitely the expert on that aspect!

All the best,
Damien

This is the latexed version of the original email, apart from a few typos that have been corrected. Footnotes have been added later and are not part of the original email.

[3] The outcome of this project is “The cyclic Deligne conjecture and Calabi-Yau structures”, `https://arxiv.org/abs/2305.10323`, authored by Chris Brav and Nick Rozenblyum.