Université Claude Bernard Lyon 1

Mémoire présenté pour obtenir

le diplôme d'habilitation à diriger des recherches

de l'Université Claude Bernard Lyon 1

Spécialité : Mathématiques

par

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Around Hochschild (co)homology

Higher structures in deformation quantization, Lie theory and algebraic geometry

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Soutenu le 26 mars 2013 devant le jury composé de :

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Introduction non mathématique

Remerciements

Mes premières pensées vont à Laure, Manon et Noémie, qui ne liront pas ce mémoire (pour des raisons variées). Sa rédaction m'a contraint à leur consacrer moins d'attention qu'à l'habitude, et elles ont fait preuve de beaucoup de patience (surtout Laure). Je le leur dédie.

Je souhaite ensuite et avant tout remercier Michel Van den Bergh et Carlo Rossi. C'est avec eux que j'ai obtenu certains de mes plus beaux résultats, mais aussi les moins douloureux dans le sens où notre collaboration m'a parue facile et agréable (peut-être est-ce parce que ce sont toujours eux qui vérifiaient les signes).

Je veux également remercier Giovanni Felder, non seulement pour avoir accepté de participer à mon jury mais aussi pour tout le reste: son amitiés, sa gentillesse, ses questions et remarques toujours pertinentes, ses réponses patientes et bienveillantes à mes questions récurrentes (et un peu obsessionnelles) sur la renormalisation des models de réseaux.

Il y a ensuite Andrei Căldăraru et Junwu Tu. Notre collaboration, les projets que nous avons en commun, et aussi ceux que nous n'avons pas en commun mais dont nous parlons, me passionnent (autant que nos "conference call" à 3 sur *****, qui furent parfois folkloriques - entre le décalage horaire, les obligations familiales de chacun, et les problèmes de connexion...). J'ai beaucoup appris à leur contact. J'espère juste que notre preprint sur les plongements fermés verra le jour avant la soutenance!

Un grand merci aux trois rapporteurs pour l'honneur qu'ils me font. J'ai la plus grande admiration pour chacun d'entre eux :

- Bernhard Keller - parce que ses exposés incroyables, ses articles lumineux, sa disponibilité et sa gentillesse.
- Mikhail Kapranov - parce que, tout simplement. Ses articles sont probablement, avec ceux de Kontsevich, ceux qui ont le plus influencé ma recherche.
- Gabriele Vezzosi - parce qu'à chaque fois que je jette un œil a HAG II je suis impressionné (et terrifié). Mes centres d'intérêt ont récemment dérivé vers la géométrie homotopique[1] et la découverte des trois mousquetaires du domaine (Bertand, Michel et Gabriele - il y a plusieurs candidats pour D'Artagnan) a été un vrai plaisir.

Je suis aussi reconnaissant à Dmitry Kaledin d'avoir accepté de participer au jury (c'est dans ses articles avec Bezrukavnikov que Michel et moi avons appris la géométrie formelle et les méthodes de globalisation dans le cadre algébrique), ainsi qu'à Bertrand Rémy (nos liens mathématiques sont moins évidents, mais c'est l'un des collègues les plus sympas **et** sérieux que j'ai cotoyés à Lyon[2]).

Bien que soutenue à Lyon, cette HDR a été financée par les contribuables suisses (un sorte d'exil fiscal), que je remercie. Je ne plaisante qu'à moitié ; depuis deux ans, j'ai bénéficié à l'ETH de conditions de travail ainsi que d'un environnement mathématique et humain exceptionnels.

Et puis il y a tous les autres. Je pense aux amis et collègues avec qui on parle beaucoup et on partage des idées (voire même des projets avortés), et avec qui on n'écrit pas forcément des articles[3]. Je veux

[1]Ou l'inverse, avec les directions "homotopique" et "dérivé" c'est comme pour les remplacement (co)fibrants, je ne sais jamais qui est qui.

[2]Et puis son manque de diplomatie légendaire dans les divers instances de la Fac est vraiment rafraîchissant.

[3]Parce qu'une collaboration ne se résume pas à cosigner des papiers.

mentionner en particulier Greg Ginot, Bruno Vallette, ou encore Mathieu Anel. La liste est plus longue mais il faut bien s'arrêter quelque part.

J'ai finalement une pensée pour Jean-Louis Loday, parti trop tôt, qui a un jour pris du temps pour m'expliquer tout ce qu'il faut savoir sur les algèbres B_∞.

Avertissement

Ce mémoire n'est pas un article de recherche.

La Partie II présente un condensé de la part de mes travaux qui tourne autours d'une analogie entre la théorie de Lie et la géométrie algébrique. C'est une sorte de fil rouge que j'ai commencé à suivre avec Michel Van den Bergh, puis Carlo Rossi, et enfin plus récemment avec Andrei Căldăraru et Junwu Tu.

La Partie I consiste en une tentative de synthétiser mes travaux consacrés à la quantification par déformation et de les insérer dans un cadre plus général qui a récemment émergé avec les travaux de Lurie et Costello. Il faut la lire comme un programme de recherche détaillé mais loin d'être abouti (et qui risque de m'occuper dans les mois qui viennent).

L'ensemble est assez dense, et si il a bien une cohérence globale (au final, on y parle presque toujours de cohomologie de Hochschild) il n'en reste pas moins que les deux parties sont écrites dans des styles très différents. Le tout a de surcroît été rédigé assez (peut-être trop) vite. J'espère que la lecture ne s'en trouvera pas trop malaisée, en tout cas au moins pour ceux qui savent déjà de quoi ça parle.

Ce mémoire comporte évidemment un biais dans la mesure où il met en avant les travaux de son auteur. C'est bien sûr le but de l'exercice, dont l'intérêt ne m'a de prime abord pas sauté aux yeux. Au fil de la rédaction j'ai quand même réalisé qu'outre le dépassement de soi[4], une HDR c'est aussi l'occasion d'organiser un peu ses idées (sans toutefois risquer un décollement de la rétine).

La suite se passe en anglais. Puisse L.L. me pardonner.

[4] Je pense en particulier à la privation de sommeil... alors que ma fille commence justement à faire ses nuits!

Liste des travaux postérieurs à la thèse

[C1] D. Calaque, From Lie theory to algebraic geometry and back, *in* Interactions between Algebraic Geometry and Noncommutative Algebra, Oberwolfach Report no. 22/2010, 1331-1334.

[C2] D. Calaque, A PBW theorem for inclusions of (sheaves of) Lie algebroids, à paraître dans Rendiconti del Seminario Matematico della Università di Padova. Preprint `arXiv:1205.3214`.

[CCT1] D. Calaque, A. Căldăraru & J. Tu, PBW for an inclusion of Lie algebras, Journal of Algebra **378** (2013), 64–79.

[CCT2] D. Calaque, A. Căldăraru & J. Tu, en préparation.

[CFFR] D. Calaque, G. Felder, A. Ferrario & C.A. Rossi, Bimodules and branes in deformation quantization, Compositio Mathematica **147** (2011), no. 1, 105–160.

[CFR] D. Calaque, G. Felder, A. Ferrario & C.A. Rossi, Deformation quantization with generators and relations, Journal of Algebra **337** (2011), no. 1, 1–12.

[CR1] D. Calaque & C.A. Rossi, Shoikhet's conjecture and Duflo isomorphism on (co)invariants, SIGMA **4** (2008), 060, 17 pages, Special Issue on Deformation Quantization.

[CR2] D. Calaque & C.A. Rossi, Compatibility with cap-products in Tsygan's formality and homological Duflo isomorphism, Letters in Mathematical Physics **95** (2011), no. 2, 135–209.

[CR3] D. Calaque & C.A. Rossi, Lectures on Duflo isomorphisms in Lie algebra and complex geometry, EMS Series of Lectures in Mathematics **14** (2011), EMS Publishing House.

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[WC] T. Willwacher & D. Calaque (contributeur mineur), Formality of cyclic cochains, Advances in Mathematics **231** (2012), no. 2, 624–650.

Travaux non présentés dans ce mémoire

[AC] A. Alekseev & D. Calaque, Quantization of symplectic dynamical r-matrices and the quantum composition formula, Communications in Mathematical Physics **273** (2007), no. 1, 119–136.

[CEE] D. Calaque, P. Etingof & B. Enriquez, Universal KZB equations: the elliptic case, Algebra, Arithmetic and Geometry Vol I: in honor of Yu. I. Manin, Progress in Mathematics **269** (2010), 165–266.

[CEFM] D. Calaque, K. Ebrahimi-Fard & D. Manchon, Two interacting Hopf algebras of trees, Advances in Applied Mathematics **47** (2011), no. 2, 282–308.

[CH] D. Calaque & G. Halbout, Weak quantization of Poisson structures, Journal of Geometry and Physics **61** (2011), no. 8, 1401–1414.

Contents

1. Mathematical introduction

The story that is told in this text started with deformation quantization. During my PhD thesis I had been interested about quantization problems compatible with additional structures (symmetries, complex structures, or integrability through the existence of an $\mathfrak{r}$-matix). Not very originally, all the results I obtained went through some variant of Kontsevich formality theorem [24].

After that we started some discussion with Michel Van den Bergh, based on the idea that the formality for Lie algebroids I proved in [5] was the appropriate setup for a uniform treatment of the globalization procedure ("uniform" meaning that it works equaly for differentiable, complex analytic and smooth algebraic varieties, as well as for some variants with corners or with singularities).

At some point we realized that the methods we were developping were leading to a proof of a statement Kontsevich made in [24] about the ring strutcure of the Hochschild cohomology of a complex manifold or a smooth algebraic variety [CVdB1]. As already guessed by other people the proof looked a bit like "Koszul dual" to Kontsevich's alternative proof of the Duflo isomorphism for Lie algebras. Based on this idea we wrote with Carlo Rossi yet another "Koszul dual" proof of the Duflo isomorphism using which makes more transparent the analogy between the Lie theoretic and the algebraic geometric results [CR3].

This lead me to work in three related directions:

1. attacking, together with Carlo Rossi, the problem of the compatibility with cap-products in the formality for Hochschild *chains* [CR1, CR2], with an eye towards a proof of the full Căldăraru's conjecture [7], which extends Kontsevich's statement to the module structure on chains. This has been achieved in a joint workwith Carlo Rossi and Michel Van den Bergh [CRVdB1].

2. understanding better this analogy between Lie theory and algebraic geometry. We have been working from quite some time on this project with Andrei Căldăraru and Junwu Tu. From its outcome [CCT1, CCT2] and some other work [C2] it seems very likely that the appropriate framework for this analogy is the one of higher groupoids and higher Lie algebroids in derived geometry.

3. understanding this Koszul phenomenon appearing in deformation quantization. Based on the great insight of Shoikhet [37, 38] we have been able, together with Giovanni Felder, Andrea Ferrario and Carlo Rossi, to give a clear description of the persistence of Koszul duality after quantization, based on a "two branes" version of the formality.

This mémoire is an attempt to provide a summary of this work, under a new perspective. It contains two parts: the first one focuses on formality theorems and includes some material from points 1 and 3, while the second one is more oriented towards the analogy betwen Lie theory and algebraic geometry and covers points 1 and 2.

Description of Part I

This part is a bit wird and should be considered as a report on a work-in-progress. It is based on two related observations:

1. every formality theorem appearing in deofrmation quantization admits essentially two (or maybe three) different proofs, and they always follow the same pattern.

2. Lie algebraic structures on Hochschild type complexes appearing in these formality theorems can always be upgraded to the actions of the chains of some topological operad.

In the past months I have spent some effort in trying to clarify the "big picture" and Part I presents the state of my thoughts about this. I use the formalism of factorization algebras developped by Lurie [29] and Costello-Gwilliam [10] and try to explain how they allow one to understand the action of generalized swiss-cheese operads[1] on many of the complexes appearing in these formality theorems.

I have to warn the reader that many statements are not in final form. I apologize for the possible mistakes and inaccuracies; I am quite confident there are some, but I am also confident that this project is interesting enough to be reported here.

Description of Part II

The second part is more standard. Chapter 4 contains some material about the Duflo-Kontsevich isomorphism for Q-manifolds [CR2, CR3], its application to the (co)homological Duflo isomorphism [CR1, CR2], and the globalization methods that allows one to prove the conjecture of Căldăraru [CVdB1, CRVdB1]. Chapter 5 is about the Lie theory of closed embedding and constists mainly of a hopefully consistent copy-and-paste from [C2] (where I prove some PBW theorem for an inclusion of Lie algebroids) and [CCT2] (where we study the Lie algebroid associated to a closed embedding).

Some statement are given without proof while for others I provided a sketchy one. I have tried to give enough details in order not rely to much on the faith of the reader, but not too much in order to keep the size of this mémoire within reasonable bounds. We anyway refer to the actual papers for detailed (and hopefully correct) proofs.

Appendices

There are also two appendices. The first one is made to be helpful and summarizes some basic stuff about Lie algebroids. The second one is an informal "dictionary" between Lie theory and algebraic geometry, which people might find interesting.

[1] A tribute to the country were I leave.

Part I.

Factorization algebras and formality theorems

(Abuses of) Notation for Part I

- Set is the category of sets (we ignore set-theoretical issues that can occur in category theory).
- sSet is the category of simplicial sets. More generally, for any category $\mathcal{C}$ we write $\mathsf{s}\mathcal{C}$ for the category of simplicial objects in $\mathcal{C}$.
- Top is a convenient category of topological spaces.
- unless otherwise stated $\mathbf{k}$ is a field of characteristic zero and Vect, resp. Cpx, is the category of $\mathbf{k}$-vector spaces, resp. cochain complexes of $\mathbf{k}$-vector spaces.
- An *operad*, without any other precision, means a coloured symmetric operad in Set.
- Given an operad $\mathcal{O}$ we denote by $\mathcal{O}^{\otimes}$ its monoidal envelope.
- The category $\mathcal{O}\text{-alg}_{/\mathcal{C}}$ of $\mathcal{O}$-algebras in a symmetric monoidal category $\mathcal{C}$ is the category $\mathsf{Fun}^{\otimes}(\mathcal{O}^{\otimes}, \mathcal{C})$ of symmetric monoidal functors from $\mathcal{O}^{\otimes}$ to $\mathcal{C}$.
- When $\mathcal{C}$ is clear from the context we allow ourselves to write $\mathcal{O}$-alg instead of $\mathcal{O}\text{-alg}_{/\mathcal{C}}$.
- In a coCartesian category $\mathcal{S}$ (e.g. Set, sSet, Top, Vect or Cpx) we have a symmetric monoidal product $\otimes := \coprod$ and the category $\mathcal{S}$ embbed into the category of cocommutative coalgebras in $\mathcal{S}$. Therefore, for any operad $\mathcal{O}$ in $\mathcal{S}$ and any $\mathcal{S}$-enriched symmetric monoidal category $\mathcal{C}$, the category $\mathcal{O}\text{-alg}_{/\mathcal{C}}$ inherits a symmetric monoidal structure.
- Whenever the target category $\mathcal{C}$ carries a distinguished class of weak equivalences $\mathcal{W}$ (e.g. weak homotopy equivalences in sSet and Top, or quasi-isomorphisms in Cpx), we only require symmetric monoidal functors $\mathsf{F} : \mathcal{D} \to \mathcal{C}$ to be *weak monoidal* in the following sense: the natural transformations $\mathsf{F}(\mathsf{X}) \otimes \mathsf{F}(\mathsf{Y}) \to \mathsf{F}(\mathsf{X} \otimes \mathsf{Y})$ belong to $\mathcal{W}$. This affects the definition of the category $\mathcal{O}\text{-alg}_{/\mathcal{C}}$.
- Let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched over a monoidal category $\mathcal{S}$ with a distinguished monoidal subcategory $\mathcal{W}$ of weak equivalences. Then an $\mathcal{S}$-enriched functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is a weak $\mathcal{S}$-equivalence if it indues an equivalence of the corresponding $\mathcal{S}[\mathcal{W}^{-1}]$-enriched categories. We drop the $\mathcal{S}$ from the notation whenever it is clear from the context.
- Let $\mathcal{C}$ be a simplicial category. Unless otherwise stated *weak equivalences* will be these morphisms that become isomorphism in the homotopy category $\mathsf{h}\mathcal{C}$.
- We might sometimes treat a topological space as a simplicial set without explicitly saying that we are taking singular chains.
- For a topological space X we denote by $\mathsf{Open}(\mathsf{X})$ its poset of open subsets.

2. Deligne conjecture for factorization algebras

2.1. Factorization algebras

The main references on factorization algebras are [10] and [29]. The material presented in this Section follows closely [10] and is partly extracted from (unfinished) lecture notes written after a seminar organized by Greg Ginot, Fred Paugam and the author as well as a one semester course on this topic.

2.1.1. Prefactorization algebras

Let X be a topological space; $\mathsf{PreFact}_X$ is the operad having colours the open subsets of X and

$$\mathsf{PreFact}_X(U_1,\dots,U_n;V) := \begin{cases} \{\bullet\} \text{ if } U_i\text{'s are pairwise disjoint and if } \coprod_{i\in[n]} U_i \subset V; \\ \emptyset \text{ otherwise}. \end{cases}$$

A *prefactorization algebra* on X is a $\mathsf{Prefact}_X$-algebra.

Remark 2.1.1. If $\mathcal{U} \subset \mathsf{Open}(X)$, then we can define an operad $\mathsf{PreFact}_{\mathcal{U}}$ in a similar way. Actually, observe that whenever we have a locally small category $\mathcal{C}$ we can define an operad $\mathsf{PreFact}_{\mathcal{C}}$ with colours being objects of $\mathcal{C}$ and

$$\mathsf{PreFact}_{\mathcal{C}}(X_1,\dots,X_n;Y) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}\Big(\coprod_{i\in[n]} X_i, Y\Big) \text{ if } \coprod_{i\in[n]} X_i \text{ exists and is a disjoint coproduct;} \\ \emptyset \text{ otherwise}. \end{cases}$$

Example 2.1.2 (The tautological prefactorization algebra of a space)**.** Let $\mathcal{C}$ be as above. Observe that $\mathsf{PreFact}_{\mathcal{C}}^{\otimes}$ is initial among symmetric monoidal categories $\mathcal{D}$ equipped with a functor $F:\mathcal{C}\to\mathcal{D}$ sending disjoint coproducts to tensor products. Hence any such F factors through the composition of $\mathcal{C}\to\mathsf{PreFact}_{\mathcal{C}}^{\otimes}$ with a $\mathsf{PreFact}_{\mathcal{C}}$-algebra $\mathcal{F}_F : \mathsf{PreFact}_{\mathcal{C}}^{\otimes}\to\mathcal{D}$.

Let X be a topological space and $\mathcal{U}\subset\mathsf{Open}(X)$. As an example we consider the obvious functor $F:\mathcal{U}\to\mathsf{Top}$, where the monoidal structure on Top is given by the coproduct. We therefore have a prefactorization algebra on $\mathcal{U}$ in $(\mathsf{Top},\coprod)$, which we call the *tautological prefactorization algebra* of $\mathcal{U}$ and denote it by $\mathcal{F}_{\mathcal{U}}$ (or simply $\mathcal{F}_X$ when $\mathcal{U}=\mathsf{Open}(X)$).

Observe that many examples of prefactorization algebras are obtained as variations on the following construction: composing a symmetric monoidal functor $(\mathsf{Top},\coprod)\to(\mathcal{C},\otimes)$ with the tautological prefactorization algebra $\mathcal{F}_X : \mathsf{PreFact}_X^{\otimes}\to\mathsf{Top}$ of a space X.

Example 2.1.3 (Compactly supported maps)**.** If (M,m) is a pointed topological space then the functor $C_c(-,M):(\mathsf{Top},\coprod)\to(\mathsf{Sets},\times)$ of continuous maps that are constantly equal to m outside a compact subset is a symmetric monoidal functor. Then for any topological space X we have a prefactorization algebra $X\supset U\mapsto C_c(U,M)$.

Example 2.1.4 (Prefactorization algebras associated to E_n-algebras)**.** Let E_n be the topological operad of the little n-rectangles. Observe that $E_n^{\otimes}$ is a monoidal subcategory of $(\mathsf{Top},\coprod)$. Namely, objects are iterated coproducts of the n-cube $\square^n := (-1,1)^n\subset\mathbb{R}^n$ and morphisms are those continuous maps which are rectilinear (i.e. sitting in $\mathbb{R}^n\rtimes(\mathbb{R}_{>0})^n$) embeddings on each connected component. Let now $\mathcal{B}_n$ be the basis of the topology on $\square^n$ given by images of rectilinear embeddings $\square^n\hookrightarrow\square^n$ (i.e. rectangles). Then the tautological prefactorization algebra $\mathsf{PreFact}_{\mathcal{B}_n}^{\otimes}\to\mathsf{Top}$ factors through the obvious operad morphism $\mathsf{PreFact}_{\mathcal{B}_n}\to E_n$. This induces a symmetric monoidal functor E_n-alg $\to\mathsf{PreFact}_{\mathcal{B}_n}$-alg.

The tautological factorization algebra of a space (see Example 2.1.2) is a particular case of a prefactorization algebra arising from precosheaves. Let $\mathcal{D}$ be a coCartesian category with $\otimes = \coprod$.

Proposition 2.1.5. *For any category $\mathcal{C}$, the forgetful functor $\mathsf{PreFact}_{\mathcal{C}}\text{-alg}_{/\mathcal{D}} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$, that sends a prefactoriation algebra $\mathcal{F}$ to its underlying precosheaf $\mathcal{F}^c$ is an isomorphism of categories.*

Proof. One has to prove that for any precosheaf $\mathcal{S}$ on $\mathcal{C}$ with values in $\mathcal{D}$ there exists a unique factorization algebra $\mathcal{F}$ such that $\mathcal{F}^c = \mathcal{S}$. Existence is fairly easy; the factorization product is built as follows: for any morphism $\coprod_{i\in[n]} U_i \to U$ in $\mathcal{C}$ we have a morphism $\coprod_{i\in[n]} \mathcal{S}(U_i) \to \mathcal{S}(U)$ in $\mathcal{D}$ given by the structure morphisms $\mathcal{S}(U_i) \to \mathcal{S}(U)$ of the precosheaf. Unicity follows from the Eckmann-Hilton principle. □

2.1.2. Factorization algebras and the (homotopy) gluing condition

Let X be a topological space. In what follows we express the homotopy gluing condition for prefactorization algebras, by analogy with the one for pre(co)sheaves.

Definition 2.1.6. 1. A cover $\mathfrak{U} = (U_i)_{i\in I}$ of X is *factorizing* if for any $\{x_1, \dots, x_k\} \subset X$ there exists a finite subset $\{U_{i_1}, \dots, U_{i_n}\}$ of pairwise disjoint open subsets in the cover $\mathfrak{U}$ such that $\{x_1, \dots, x_k\} \subset \coprod_\alpha U_{i_\alpha}$.
2. A *factorizing basis* of X is a basis $\mathcal{B}$ of its topology such that for any $U \in \mathsf{Open}(X)$ the cover $\mathcal{B}_U := \{V \in \mathcal{B} | V \subset U\}$ is factorizing.

Example 2.1.7. $\{X\}$ is a factorizing cover of X. For a metric space X, the set of balls with radius strictly less than a fixed $\lambda \in]0, +\infty]$ is a factorizing cover of X. More generally, any basis of the topology of a preregular space[1] X is a factorizing cover of X.

For any factorizing cover $\mathfrak{U} = (U_i)_{i\in I}$ we denote by PI the set of finite subsets $\alpha \subset I$ such that $U_i \cap U_i = \emptyset$ for any $i \neq j$ in α. Let $(\mathcal{C}, \mathcal{W}, \otimes)$ be a symmetric monoidal category with weak equivalences which have small coproducts and assume we are given an $\mathcal{F} \in \mathsf{PreFact}_X\text{-alg}_{/\mathcal{C}}$. Then we construct a simplicial object $\check{C}_\bullet(\mathfrak{U}, \mathcal{F})$ as follows:

$$\check{C}_n(\mathfrak{U}, \mathcal{F}) := \coprod_{\alpha_0,\dots,\alpha_n \in PI} \left(\bigotimes_{i_k \in \alpha_k} \mathcal{F}\left(\bigcap_{k=0}^{n} U_{i_k} \right) \right),$$

with obvious faces and degeneracies.

Definition 2.1.8. 1. A $\mathsf{PreFact}_X$-algebra $\mathcal{F}$ with values in $\mathcal{C}$ satisfies the *(homotopy) gluing condition* w.r.t. a factorizing cover $\mathfrak{U} = (U_i)_{i\in I}$ of $U \in \mathsf{Open}(X)$ if the canonical map $\check{C}_\bullet(\mathfrak{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U)$ is a weak equivalence (we could say that the map "exhibits $\mathcal{F}(U)$ as a homotopy colimit").
2. A *factorization algebra* is a prefactorization algebra satisfying the gluing condition for all factorizing covers of all open subsets.

As usual, a morphism between two factorization algebras is a morphism between their underlying prefactorization algebras. Even though there is no operad Fact_X, we denote by $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$ the category of factorization algebras over X with values in $\mathcal{C}$. If $\mathcal{C}$ is tensored over simplicial sets then so are $\mathsf{PreFact}_X\text{-alg}_{/\mathcal{C}}$ and $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$, and thus they automatically become simplicially enriched.

Let now $\mathcal{B}$ be a factorizing basis of X and $\mathcal{C}$ be a symmetric monoidal simplicial category tensored over simplicial sets admitting small coproducts and homotopy colimits. Given a $\mathsf{PreFact}_{\mathcal{B}}$-algebra, observe that it makes perfect sense to speak about the gluing condition for factorizing covers of open subsets in $\mathcal{B}$ by open subsets of $\mathcal{B}$ (factorizing $\mathcal{B}$-covers for short), and thus to speak about factorization algebras on $\mathcal{B}$ ($\mathsf{Fact}_{\mathcal{B}}$-algebras for short).

Theorem 2.1.9. *The forgetful functor $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}} \longrightarrow \mathsf{Fact}_{\mathcal{B}}\text{-alg}_{/\mathcal{C}}$, which sends a factorization algebra $\mathcal{F}$ on X with values in $\mathcal{C}$ to its restriction $\mathcal{F}_{|\mathcal{B}}$ on $\mathcal{B}$, is an equivalence of simplicial categories.*

[1]A space is preregular if topologically distinguishable points (i.e. points that have different sets of neighbourhoods) can be separated by neighbourhoods.

Proof. Let us prove that for any factorization algebra $\mathcal{G}$ on $\mathcal{B}$ with values in $\mathcal{C}$, there exists a unique factorization algebra $\mathcal{F}$ on X such that $\mathcal{F}_{|\mathcal{B}} \cong \mathcal{G}$. For any open subset $U \subset X$ we define $\mathcal{F}(U)$ as the homotopy colimit of $\check{C}_\bullet(\mathcal{B}_U, \mathcal{F})$. By definition we have that $\mathcal{F}_{|\mathcal{B}} \cong \mathcal{G}$ as $\mathsf{PreFact}_\mathcal{B}$-algebras. It remains to prove that $\mathcal{F}$ is a factorization algebra.

Let $\mathcal{U} = (U_i)_{i\in I}$ be a factorizing cover of an open subset $U \subset X$, and consider the refinement $\mathcal{B}_\mathcal{U} = \coprod_{i\in i} \mathcal{B}_{U_i}$ of $\mathcal{U}$. We then have the following commutative diagram, where vertical maps are weak equivalences:

$$\begin{array}{ccc} & & \check{C}_\bullet(\mathcal{B}_\mathcal{U}, \mathcal{F}) \\ & \swarrow & \downarrow \\ \check{C}_\bullet(\mathcal{U}, \mathcal{F}) & & \check{C}_\bullet(\mathcal{B}_U, \mathcal{F}) \\ \downarrow & \searrow & \downarrow \\ (?) & & \mathcal{F}(U) \end{array}$$

The universal property of $\mathcal{F}(U)$ and (?) then impose that they are weakly equivalent. □

Let X be a topological space and $\mathcal{B} \subset \mathsf{Open}(X)$ be such that any $\mathcal{B}$-cover is factorizing. The following result concerns $\mathsf{Fact}_\mathcal{B}$-algebras within a symmetric monoidal coCartesian category $\mathcal{C}$ with $\otimes = \coprod$.

Proposition 2.1.10. *The isomorphism of categories* $\mathsf{PreFact}_\mathcal{B}\text{-alg}_{/\mathcal{C}} \to \mathsf{Fun}(\mathcal{B}, \mathcal{C})$ *of Proposition 2.1.5 induces an isomorphism of categories between* $\mathsf{Fact}_\mathcal{B}\text{-alg}_{/\mathcal{C}}$ *and the category of cosheaves on* $\mathcal{B}$ *with values in* $\mathcal{C}$*.*

In particular, if $\mathcal{B}$ is a factorizing basis then it induces (after Theorem 2.1.9) an equivalence of simplicial categories between $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$ and the category of cosheaves on X with values in $\mathcal{C}$.

Sketch of proof. It suffices to prove that for any $U \in \mathcal{B}$, a prefactorization algebra $\mathcal{F}$ satisfies the gluing property w.r.t. the factorizing covers $\mathcal{B}_U$ if and only if $\mathcal{F}^c$ satisfies the (precosheaf) gluing condition for $\mathcal{B}_U$ (here we have implicitly used a general refinement principle, and the fact that $\mathcal{B}_U$ is the maximal refinement of any $\mathcal{B}$-cover of U and is factorizing). We refer to [10, Appendix B] for more details. □

Remark 2.1.11. The above Proposition does not hold for $\mathsf{PreFact}_X$-algebras unless $\mathsf{Open}(X)$ is a factorizing basis. This last condition is equivalent to the requirement that a factorizing basis exists. Notice that this is *a priori* a weaker condition than X being preregular (which can be rephrased as "any basis of the topology is factorizing"), that we call *weakly preregular.*

Example 2.1.12 (The tautological factorization algebra of a space)**.** Let X be a topological space and $\mathcal{B}$ be a factorizing basis of X. The tautological prefactorization algebra $\mathcal{F}_\mathcal{B}$ on $\mathcal{B}$ (see § 2.1.2), taking its values in $(\mathsf{Top}, \coprod)$, is actually a factorization algebra on $\mathcal{B}$. This follows from the fact that $\mathcal{F}_\mathcal{B}^c$ is a cosheaf[2] and Proposition 2.1.10.

Example 2.1.13 (Compactly supported maps)**.** Let X be a weakly preregular topological space and V be a vector space. Observe that $C_c(-, V)$ defines a Vect-valued cosheaf, and thus determines a factorization algebra on X with values in the symmetric monoidal category $(\mathsf{Vect}, \oplus)$. In general, if M is a pointed topological space, I don't know if the prefactorization algebra $C_c(-, M)$ defined on X and taking values in $(\mathsf{Sets}, \times)$ satisfies the gluing condition.

Example 2.1.14 (Factorization algebras associated to E_n-algebras)**.** We borrow the notation from Example 2.1.4. One can show that the functor $E_n\text{-alg} \to \mathsf{PreFact}_{\mathcal{B}_n}\text{-alg}$ actually factors through $\mathsf{Fact}_{\mathcal{B}_n}\text{-alg} \subset \mathsf{PreFact}_{\mathcal{B}_n}\text{-alg}$ and thus, after Theorem 2.1.9, any E_n-algebra determines a $\mathsf{Fact}_{\mathbb{R}^n}$-algebra.

[2] Namely, for any open cover $\mathfrak{U} = (U_i)_{i\in I}$ of a space U the map $\mathfrak{U}_\bullet \to U$ is a weak equivalence of simplicial spaces, where

$$\mathfrak{U}_n = \coprod_{(i_0,\dots i_n)\in I^{n+1}} \left(\bigcap_{k=0}^{n} U_{i_k} \right).$$

2.1.3. Direct image and Hochschild homology

Let $f : X \to Y$ be a continuous map of topological spaces, and assume we have a factorization algebra $\mathcal{F}$ over X with values in a monoidal category $\mathcal{C}$ admitting coproducts. Let us define a pre-factorization algebra $f_*\mathcal{F}$ on Y, the *direct image* of $\mathcal{F}$, in the following way: we set $f_*\mathcal{F}(U) := \mathcal{F}\big(f^{-1}(U)\big)$, observe that if U_i's are pairwise disjoint then so are $f^{-1}(U_i)$'s, and finally define the factorizing operations as

$$\bigotimes_i f_*\mathcal{F}(U_i) := \bigotimes_i \mathcal{F}\big(f^{-1}(U)\big) \to \mathcal{F}\big(f^{-1}(V)\big) = f_*\mathcal{F}(V)$$

whenever $\coprod_i U_i \subset V$. It remains to show that $f_*\mathcal{F}$ satisfies the gluing condition for any factorizing cover. This follows from the fact that $f_*\mathcal{F}$ satisfies the gluing condition w.r.t a given factorizing cover $(U_i)_{i\in I}$ of U if and only if so does $\mathcal{F}$ w.r.t. the factorizing cover $\big(f^{-1}(U_i)\big)_{i\in I}$ of $f^{-1}(U)$.

In the case when $Y = pt$ then f_* is simply the *global section functor*: $f_*\mathcal{F} = \mathcal{F}(X)$

Remark 2.1.15. It seems that for a reasonnable space X (say, preregular) there should be a model structure on $\mathsf{PreFact}_X$-alg for which the cofibrant objects are factorization algebras, and lead to the notion of derived direct image of prefactorization algebras. Accordingly, we will name *derived global sections* the functor $\mathbf{R}\Gamma : \mathsf{PreFact}_X\text{-alg}_{/\mathcal{C}} \to \mathcal{C}$ defined by $\mathbf{R}\Gamma(\mathcal{F}) := \mathrm{hocolim}\big(\check{C}_\bullet\big(\mathsf{Open}(X), \mathcal{F}\big)\big)$. It is also known as *factorization homology* or *topologiccal chiral homology*.

Example 2.1.16 (Derived tensor product)**.** Let A be an associative algebra together with a pointed right module (M_r, m_r) and a pointed left module (M_l, m_l). To this data we can associate a factorization algebra on the closed interval $[0, 1]$, of which the global sections consist of the derived tensor product $M_r \overset{\mathbf{L}}{\underset{A}{\otimes}} M_l$ together with the distinguished element $m_r \otimes m_l$ (see [3, Proposition 3.30]).

Example 2.1.17 (Hochschild homology)**.** Any associative algebra defines a factorization algebra on S^1 (defined on the factorizing basis consisting of intervals of length $< 1/2$) of which the global sections consists of $A \overset{\mathbf{L}}{\underset{A\otimes A^{op}}{\otimes}} A$ (together with the distinguished element $1 \otimes 1$). This can be seen by factoring the map $S^1 \to pt$ through a projection $S^1 \to [0, 1]$ and by using Example 2.1.16.

2.2. Dunn's theorem, centralizers and Deligne's conjecture

This Section is very much inspired by [29, §2.5] and [17, Section 7].

2.2.1. Dunn's theorem for factorization algebras

Dunn's theorem [14] roughly says that the n-fold tensor product of the associative operad is weakly equivalent to the little n-rectangles operad $\mathbf{E}_n$. In particular this implies that the homotopy category of $\mathbf{E}_k$-algebras in $\mathbf{E}_l$-algebras is equivalent to the homotopy category of $\mathbf{E}_{k+l}$-algebras.

We now prove an analogous result for factorization algebras, which appears to be somewhat easier.

Theorem 2.2.1. *Let X and Y be weakly preregular topological spaces. Then we have a symmetric monoidal equivalence between* $\mathsf{Fact}_X\text{-alg}_{/\mathsf{Fact}_Y\text{-alg}_{/\mathcal{C}}}$ *and* $\mathsf{Fact}_{X\times Y}\text{-alg}_{/\mathcal{C}}$.

Sketch of proof. Let $\mathcal{B}$ be the factorizing basis of $X \times Y$ which consists of those opens that are of the form $U \times V$, with $U \in \mathsf{Open}(X)$ and $V \in \mathsf{Open}(Y)$. Notice that we have an obvious isomorphism of symmetric monoidal simplicial categories $\mathsf{PreFact}_{\mathcal{B}}\text{-alg}_{/\mathcal{C}} \longrightarrow \mathsf{PreFact}_X\text{-alg}_{/\mathsf{PreFact}_Y\text{-alg}_{/\mathcal{C}}}$.

One can check that the gluing condition w.r.t. factorizing $\mathcal{B}$-covers is satisfied if and only if the gluing conditions w.r.t. factorizing covers of both X and Y are satisfied. This meas that the above restricts to an isomorphism of symmetric monoidal simplicial categories $\mathsf{Fact}_{\mathcal{B}}\text{-alg}_{/\mathcal{C}} \longrightarrow \mathsf{Fact}_X\text{-alg}_{/\mathsf{Fact}_Y\text{-alg}_{/\mathcal{C}}}$. One then composes this isomorphism with the equivalence $\mathsf{Fact}_{X\times Y}\text{-alg}_{/\mathcal{C}} \longrightarrow \mathsf{Fact}_{\mathcal{B}}\text{-alg}_{/\mathcal{C}}$ provided by Theorem 2.1.9. □

2.2.2. Centralizers

Let us assume for simplicity that k is a field of characteristic zero, $\mathcal{C} = \mathsf{dg}\text{-}\mathsf{k}\text{-}\mathsf{mod}$, and X is a manifold. Let $f : A \to B$ be a morphism in $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$. Following [29, §2.5] we call an object $\mathcal{Z}(f) \in \mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$ a *centralizer for* f if it is universal for the following property:

$$\begin{array}{ccc} & A \otimes C & \\ {}^{id_A \otimes 1_C} \nearrow & & \searrow \\ A & \xrightarrow{\quad f \quad} & B \,, \end{array}$$

where $1_C : \mathsf{k} \to C$ is the unit of C (all factorization algebras are unital, the unit being given by the initial map from the empty set in $\mathsf{Open}(X)$). Such an object $\mathcal{Z}(f)$ is obviously unique (up to a contractible choice of weak equivalences) if it exists, and the main goal of this Subsection is to provide a construction of it.

We define $\mathcal{Z}(f)$ to be $\mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)$, where one has to recall that an A-Fact_X-module is an object M together with a Fact_X-algebra structure on $A \oplus \epsilon M$, with $\epsilon^2 = 0$, that concides with the one on A when $\epsilon = 0$.

Remark 2.2.2. When X is connected an A-Fact_X-module is equivalent to the data of a Fact_X-algebra B together with a weak equivalence $A_{|X-\{x\}} \cong B_{|X-\{x\}}$, where $x \in X$ is fixed.

Our first task is to make $\mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)$ into an object of $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$. More precisely:

Proposition 2.2.3. *The category* A-Fact_X-mod *is enriched over* $\mathsf{Fact}_X\text{-alg}_{/\mathcal{C}}$.

Sketch of proof. Let us first restrict to the factorizing basis $\mathsf{Conv}(X)$ of X consisting of small convex open subsets. For any open $U \in \mathsf{Conv}(X)$ we set

$$\mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)(U) := \mathbf{R}\mathsf{Hom}_{A_{|U}\text{-}\mathsf{Fact}_U\text{-}\mathsf{mod}}(A_{|U}, B_{|U})$$

We consider $\coprod_{i \in I} U_i \subset V$ an inclusion of pairwise disjoint open subsets of $\mathsf{Conv}(X)$ into a bigger one, and we let $(g_i)_{i \in I}$ be such that $g_i \in \mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)(U_i)$. We have to define the image $g \in \mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)(V)$ of $(g_i)_{i \in I}$ through the factorizing operation associated to $\coprod_{i \in I} U_i \subset V$. To do so, we restrict to the factorizing basis of V consisting of small convex open subsets that intersect at most one of the U_i's. For any such $W \subset V$, we set

$$g(W) := \begin{cases} f(W) \text{ if } W \text{ does not intersect any of the } U_i\text{'s} \\ g_i(\emptyset) \otimes f(W) \text{ if } W \text{ does intersect } U_i \end{cases}$$

The gluing condition can be proved to hold. □

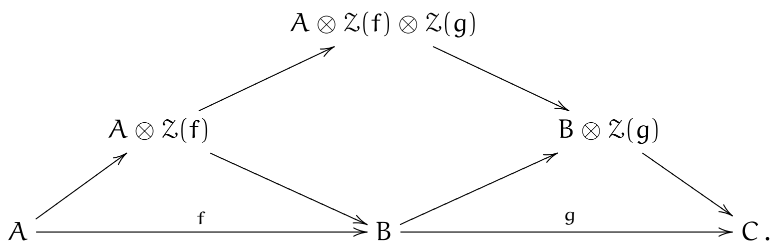
It remains to prove that $\mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)$ does have the universal property. And this is almost tautological. Let C having the above property (one says that C *centralizes* f) and denote by $\varphi : A \otimes C \to B$ the morphism which is such that for any open subset $U \subset X$ and any $a \in A(U)$, we roughly have

$$\varphi(U)(a \otimes 1_C) = f(U)(a)\,.$$

Actually, $\varphi(U)$ is completely determined by a map $C(U) \to \mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)(U)$. Conversely, any Fact_X-algebra morphism $C \to \mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)$ sends the unit 1_C to the unit (i.e. the element determined by the inclusion $\emptyset \subset U$) of $\mathbf{R}\mathsf{Hom}_{A\text{-}\mathsf{Fact}_X\text{-}\mathsf{mod}}(A, B)$, which is nothing but f.

2.2.3. Deligne's conjecture for factorization algebras

Let $f : A \to B$ and $g : B \to C$ be morphisms of factorization algebras on a manifold X. Then observe that $\mathcal{Z}(f) \otimes \mathcal{Z}(g)$ centralizes $g \circ f$:



We therefore have a composition morphism $\mathcal{Z}(f) \otimes \mathcal{Z}(g) \to \mathcal{Z}(g \circ f)$ in Fact_X-alg, which can be proven to be associative. Then we have the following:

Theorem 2.2.4 (Deligne conjecture for factorization algebras)**.** $\mathcal{Z}(A) := \mathcal{Z}(\mathrm{id}_A)$ *is an associative algebra in* Fact_X-alg*, and thus a* $\mathsf{Fact}_{\mathbb{R}\times X}$*-algebra. Moreover,* $\mathcal{Z}(1_A) = A$ *becomes a right* $\mathcal{Z}(A)$*-module and thus the pair* $\big(\mathcal{Z}(A), A\big)$ *defines a* $\mathsf{Fact}_{\mathbb{R}^+\times X}$*-algebra.*

We implicitely used that an associative algebra together with a right module produces a factorization algebra on $\mathbb{R}^+$. We discuss this fact (and generalizations) in the next Chapter.

Example 2.2.5. Let A be an associative algebra and denote by $\mathcal{F}_A$ the corresponding factorization algebra on $\mathbb{R}$. Then the pair $\big(\mathcal{Z}(\mathcal{F}_A), \mathcal{F}_A\big)$ consists of an algebra and a right module in $\mathsf{Fact}_{\mathbb{R}}$-alg. After taking global sections over $\mathbb{R}$ we get an algebra with a right module, which are $\mathsf{RHom}_{A\text{-bimod}}(A, A)$ and A, respectively. The (known) fact that the Yoneda product on the former is commutative up to homotopy comes from the $\mathsf{Fact}_{\mathbb{R}^2}$-algebra structure on $\mathcal{Z}(\mathcal{F}_A)$.

Example 2.2.6. Let A be an associative algebra and denote by $\mathcal{F}_A$ the corresponding factorization algebra on S^1. Then the pair $\big(\mathcal{Z}(\mathcal{F}_A), \mathcal{F}_A\big)$ consists of an algebra and a right module in Fact_{S^1}-alg, and gives rise to a $\mathsf{Fact}_{S^1\times\mathbb{R}^+}$-algebra. By pushing forward through the map $S^1 \times \mathbb{R}^+ \to \mathbb{C}$ that sends (θ, r) to $re^{i\theta}$ we get a factorization algebra on $\mathbb{R}^2$ that looks like $\mathcal{Z}(\mathcal{F}_A)$ away from the origin, together with $A \overset{\mathbf{L}}{\underset{A\otimes A^{\mathrm{op}}}{\otimes}} A$ being a kind of module sitting at the origin.

The above two Examples will be made a bit more precise in the next Chapter.

3. Deligne conjecture and formality theorems

3.1. Generalized swiss-cheese operads

3.1.1. Locally constant factorization algebras on $\mathbb{R}^n \times (\mathbb{R}^+)^m$

Definition 3.1.1 (Generalized from [10])**.** Let X be a stratified topological space. A (pre)factorization algebra $\mathcal{F}$ on X is said *locally constant* (w.r.t. the given stratification $\mathcal{S}$) if for any inclusion $U \subset V$ such that U is a stratified deformation retract of V the corresponding morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is a weak equivalence. We denote by $(\mathsf{Fact}_X\text{-alg})^{\mathcal{S}\text{-}l.c.}$ the full subcategory of locally constant factorization algebras. We might drop the stratification $\mathcal{S}$ from the notation when it is clear from the context.

Factorization algebras on $\mathbb{R}^n$ associated to E_n-algebras (see Examples 2.1.4 and 2.1.14) are locally constant. Similarly, the factorization algebra on S^1 associated to an E_1-algebra (see Example 2.1.17) is also locally constant. Finally, the factorization algebra of Example 2.1.16 is locally constant w.r.t to the boundary stratification on $[0,1]$.

Remark 3.1.2. If $f : X \to Y$ is a stratified continuous map such that the associated cosheaf $U \mapsto f^{-1}(U)$ is locally constant (meaning that if U is a stratified deformation retract of V then $f^{-1}(U)$ is a stratified deformation retract of $f^{-1}(V)$) then f_* preserves locally constantness.

The operad $E_{n,m}$

Let $n, m \geq 0$ be integers, $k = m + n$, $\mathcal{P}_m$ be the set of subsets of $[m] = \{1, \dots, m\}$. For any $P \in \mathcal{P}_m$ we define the partial unit ball[1] of type P:

$$B^k_P := (-1,1)^{[n+m]\setminus P} \times [0,1)^P \subset \mathbb{R}^{[n+m]} .$$

Let I be a finite $\mathcal{P}_m$-coloured set (i.e. a set equipped with a map $p : I \to \mathcal{P}_m$). We define

$$B^k_I := \coprod_{P \in \mathcal{P}_m} (B^k_P \times p^{-1}(P)) ,$$

and consider the topological space $\mathsf{Conf}(B^k_I, B^k_Q)$ of I-configurations of pairwize disjoint partial disks into B^k_Q, where $Q \in \mathcal{P}_m$. It is the topological space of open embeddings $f : B^k_I \to B^k_Q$ such that the restriction of f to each connected component is given by a rectilinear transformation of $\mathbb{R}^{[n+m]} = \mathbb{R}^k$.

Remark 3.1.3. The poset of irreducible components of B^k_P is the subposet of $(\mathcal{P}_m, \supset)$ consisting of subsets of P. For any $P' \subset P$ the corresponding irreducible component is

$$B^k_P(P') := \left(\{0\}^{P'} \times \mathbb{R}^{[n+m]\setminus P'}\right) \cap B^k_P .$$

One can easily see that the restriction of a map in $\mathsf{Conf}(B^k_I, B^k_Q)$ to a copy of B^k_P sends $B^k_P(P')$ to $B^k_Q(P')$. In particular there are no rectilinear open embeddings of B^k_P into B^k_Q if P is not contained in Q.

We are now ready to define a far-reaching generalization of the little rectangles operads E_n.

Definition 3.1.4 (Generalized swiss-cheese operads)**.** $E_{n,m}$ is the $\mathcal{P}_m$-coloured topological operad with spaces of operations being

$$E_{n,m}(I,Q) := \mathsf{Conf}(B^k_I, B^k_Q) , \quad |I| \geq 1 .$$

The composition is the obvious one, given by the composition of open embbedings.

[1] Recall that we decided, for convenience, to work with the norm $\|\cdot\|_\infty$. However, this does not matter for our construction of topological operads since any other choice of norm would lead to weakly equivalent topological operads.

Observe that $\mathsf{E}_{n,0} = \mathsf{E}_n$ is the standard little n-rectangles operad, and that $\mathsf{E}_{1,1}$ is the swiss-cheese (2-coloured) operad defined by Voronov [42] ($\mathsf{E}_{n,1}$ being its higher-dimensional generalizations).

Example 3.1.5 ($\mathsf{E}_{0,1}$-algebras *vs* associative algebras with a right module)**.** Recall that there is a weak equivalence of topological operads $\mathsf{E}_1 \to \mathbf{As}$, where $\mathbf{As}$ is the operad for associative algebras (recall that it has one colour and its operations are given by finite linear orders). It sends an ordered configuration of intervalles to the corresponding linear order. Similarly, there is a weak equivalence of topological operads $\mathsf{E}_{0,1} \to \mathbf{AsR}$, where $\mathbf{AsR}$ is the operad for associative algebras with a (pointed) right module and can be defined as follows: it has two colours $\{\bullet\!-, -\}$ and operations are given by coloured linear orders in which $\bullet\!-$ can only appear on the minimal elements and if the target colour is also $\bullet\!-$. The morphism $\mathsf{E}_{0,1} \to \mathbf{AsR}$ consists of associating the colour $-$ to the open intervalle and the colour $\bullet\!-$ to the half-closed one, and only remember about the ordering in configurations.

$\mathsf{E}_{n,m}$-algebras *vs* locally constant factorization algebras

We have the following commutative diagram of symmetric monoidal functors

$$\begin{array}{ccccc}
\mathsf{E}_{n,m}\text{-alg} & \xrightarrow{a} & (\mathsf{PreFact}_{\mathcal{B}_{n,m}}\text{-alg})^{l.c} & \xleftarrow{b} & \left(\mathsf{Fact}_{\mathbb{R}^n\times(\mathbb{R}^+)^m}\text{-alg}\right)^{l.c} \\
& \searrow^{c} & \uparrow{\scriptstyle d} & \swarrow_{e} & \\
& & (\mathsf{Fact}_{\mathcal{B}_{n,m}}\text{-alg})^{l.c} & &
\end{array}$$

where $\mathcal{B}_{n,m}$ is the factorizing basis of $(-1,1)^n \times [0,1)^m$ consisting of the images of the B_P^{n+m}, $P \in \mathcal{P}_m$, *via* rectilinear open embeddings into $(-1,1)^n \times [0,1)^m$. We now describe the functors:

- a comes from the operad morphism $\mathsf{PreFact}_{\mathcal{B}_{n,m}} \to \mathsf{E}_{n,m}$ which sends the image of a B_P^{n+m} to the colour P and a given inclusion to its associated configuration. It factors through $(\mathsf{Fact}_{\mathcal{B}_{n,m}})$ *l.c.*, which embbeds in $(\mathsf{PreFact}_{\mathcal{B}_{n,m}})$ *l.c.* as a full subcategory *via* d.
- e is given by restricting to the factorizing basis (after having pushed-forward through a homeomorphism $\mathbb{R}^n \times (\mathbb{R}^+)^m \xrightarrow{\sim} (-1,1)^n \times [0,1)^m$).

Theorem 3.1.6. *All these functors are weak equivalences.*

Sketch of proof. a is a weak equivalence thanks to [29, Theorem 3.2.7]. Then c and d are also weak equivalences. e is a weak equivalence thanks to Theorem 2.1.9, and thus so is b (this is "2 out of 3"). $\square$

Combined with Dunn's Theorem for locally constant factorization algebras this gives:

Theorem 3.1.7. *The symmetric monoidal functor* $\mathsf{E}_{n+k,m+l}\text{-alg} \longrightarrow \mathsf{E}_{n,m}\text{-alg}_{/\mathsf{E}_{k,l}\text{-alg}}$ *(which consists in restricting oneself to "product"configurations) is a weak equivalence.*

Finally observe that centralizers preserve locally constantness. Thus if A is an $\mathsf{E}_{n,m}$-algebra then the pair $\big(\mathcal{Z}(A), A\big)$ becomes an $\mathsf{E}_{0,1}$-algebra in $\mathsf{E}_{n,m}$-algebras and hence inherits an $\mathsf{E}_{n,m+1}$-algebra structure.

3.1.2. Formality of generalized swiss-cheese operads

In this Section we basically follow Kontsevich's proof of the formality [25] of E_n over $\mathbb{R}$ (see also [28]) and very shortly sketch an extension of it to $\mathsf{E}_{n,m}$.

An operad weakly equivalent to $\mathsf{E}_{n,m}$

There exists another topological operad $\overline{\mathsf{C}}_{n,m}$ which is weakly equivalent to $\mathsf{E}_{n,m}$.

Construction 3.1.8. Let $\mathsf{Conf}_{n,m}(k)$ be the configuration space of k dinstinct points in $\mathbb{R}^n \times (\mathbb{R}^+)^m$ and denote $\mathsf{C}_{n,m}(k)$ its quotient by the action of $\mathbb{R}^n \rtimes \mathbb{R}_{>0}$. One can construct a compactification $\overline{\mathsf{C}}_{n,m}(k)$ of $\mathsf{C}_{n,m}(k)$ along the lines of [24, Section 5]. It consists of nested configurations of k points in $\mathbb{R}^n \times (\mathbb{R}^+)^m$ up to translations and dilations, and has the structure of a manifold with corners. Moreover, the boundary of $\overline{C}_{n,m}(k)$ is a union of products of other configuration spaces $\overline{C}_{n',m'}(k')$, with $n'+m'=n+m$, $m'\leq m$ and $k'\leq k$.

We then define the topological operad $\overline{C}_{n,m}$. Its colours are elements of $\mathcal{P}_m$. For a $\mathcal{P}_m$-coloured set I and a colour $Q\in\mathcal{P}_m$, the space of morphism $\overline{C}_{n,m}(I,Q)$ is the boundary component of $\overline{C}_{[n+m]\setminus Q,Q}(I)$ consisting of those configurations such that for any $i\in p^{-1}(P)$, we have $P\subset Q$ and the point labelled by i belongs to $\mathbb{R}^{[n+m]\setminus P}\times\{0\}^P$. The composition is given by the inclusion of the various boundary components.

We now explain why the topological operads $\overline{C}_{n,m}$ and $\mathbf{E}_{n,m}$ are weakly equivalent .

Construction 3.1.9. Take the Boardman-Vogt W-construction $W\mathbf{E}_{n,m}$ of $\mathbf{E}_{n,m}$: it is a topological operad having the same colours as $\mathbf{E}_{n,m}$, but operations are given by planar rooted trees having their external vertices (leaves and root) labelled by colours, their internal vertices labelled by operations of $\mathbf{E}_{n,m}$, and edges labelled by $[0,1]$. There is a weak equivalence $W\mathbf{E}_{n,m}\to\mathbf{E}_{n,m}$ that consists in forgetting the label on edges and "computing" the composed operation associated to the tree. Now observe that we also have a map $W\mathbf{E}_{n,m}\to\overline{C}_{n,m}$ which sends a tree to the associated configuration of the centers of the partial balls, where every subconfiguration has been rescaled by $1-t$ if $t\in[0,1]$ is the label of the outgoing edge of the corresponding vertex. Notice that $t=1$ gives the "infinitesimal" configurations, i.e. the ones lying in the boundary.

Formality of $\overline{C}_{n,m}$ over $\mathbb{R}$

The proof goes through the following zig-zag of quasi-isomorphisms of operads in Cpx:

$$C_{-\bullet}\left(\overline{C}_{n,m},\mathbb{R}\right)\longleftarrow C^{sa}_{-\bullet}\left(\overline{C}_{n,m},\mathbb{R}\right)\longrightarrow \mathsf{Graph}_{n,m}\longleftarrow H_{-\bullet}\left(\overline{C}_{n,m},\mathbb{R}\right)$$

Below we very shortly describe the main ingredients involved (details will appear elswhere).

- all operads $\mathcal{O}$ involved have the same set of colours $\mathcal{P}_m$, and are such that $\mathcal{O}(I,Q)=0$ unless the set of colours of I lies in $\mathcal{P}_Q\subset\mathcal{P}_m$.
- $C^{sa}_{-\bullet}\left(\overline{C}_{n,m},\mathbb{R}\right)$ is the suboperad of semi-algebraic chains.
- For a colour Q and a $\mathcal{P}_Q$-coloured set I we define the cochain complex $\mathsf{Graph}_{n,m}(I,Q)$ as follows. It is the linear span of finite graphs with set of vertices $I\coprod J$, where J is any $\mathcal{P}_Q$-coloured set, such that any vertex is connected by a path to I, together with an orientation of every edge[2]. The operad structure is given by "plugging in" graphs to vertices of another graph and reconnecting the edges in all possible ways. The differential is a coloured variant of the usual graph cohomology coboundary operator.
- To any graph $\Gamma\in\mathsf{Graph}_{n,m}(I,Q)$ as above one can associate a differential form ω_Γ on $\overline{C}_{n,m}(I,Q)$. First observe that any edge e of Γ determines a map $\pi_e:\overline{C}_{n,m}(I\coprod J,Q)\to\overline{C}_{[n+m]\setminus Q,Q}(\{1,2\})$, which consists in forgetting all points except the two vertices of e. Then one uses a form ω_Q on $\overline{C}_{[n+m]\setminus Q,Q}(\{1,2\})$ extending the volume form of $S^{n+m-1}=\overline{C}_{n+m,0}(\{1,2\})$ satisfying a bunch of specific properties that we won't explain here. Finally,

$$\omega_\Gamma=(\pi_I)_*\left(\bigwedge_{e\in V(\Gamma)}\pi_e^*\omega_Q\right),$$

  where $\pi_I:\overline{C}_{n,m}(I\coprod J,Q)\to\overline{C}_{n,m}(I,Q)$ is the map that forgets the J-labelled points.

  The operad morphism $C^{sa}_{-\bullet}\left(\overline{C}_{n,m},\mathbb{R}\right)\longrightarrow\mathsf{Graph}_{n,m}$ sends a given chain c on $\overline{C}_{n,m}(I,Q)$ to

$$\sum_{\Gamma\in\mathsf{Graph}_{n,m}(I,Q)}\left(\int_c\omega_\Gamma\right)\Gamma.$$

[2]One should further mod out by some symmetries, but we are far from being at that level of detail.

- the operad morphism $\mathsf{P}_{n,m} = \mathsf{H}_{-\bullet}\left(\overline{\mathsf{C}}_{n,m}, \mathbb{R}\right) \longrightarrow \mathsf{Graph}_{n,m}$ sends the generators (all having operadic arity two) to the elementary graphs with $\#I = 2$ and $J = \emptyset$.

Examples of these configurations spaces, graphs and forms associated to them previously appeared:

- ($m = 0$) in the proof by Kontsevich [25] of the formality of $\mathbf{E}_{n,0}$.
- ($m = n = 1$) in the proof by Kontsevich [24] of the formality of the $\mathbf{E}_1$-Hochschild cochain complex of $A = \mathbf{k}[x_1, \dots, x_d]$.
- ($m = 0$ and $n = 2$) in the context of the formality in the presence of two branes in [CFFR].

3.2. Applications to deformation quantization

In this Section we let our underlying monoidal category $\mathcal{C}$ be the category of topological $\mathbf{k}$-vector spaces, where $\mathbf{k}$ is a field containing the real numbers.

3.2.1. Higher Kontsevich formality

Let A be one of the following commutative algebras in $\mathcal{C}$:

- $A = C^\infty(\mathbb{R}^d, \mathbf{k})$, with the Fréchet topology ($\mathbf{k} \in \{\mathbb{R}, \mathbb{C}\}$).
- $A = \mathbf{k}[x_1, \dots, x_d]$, with the discrete topology.
- $A = \mathbf{k}[\![x_1, \dots, x_d]\!]$, with the adic topology.

We now consider A as an $\mathbf{E}_n$-algebra in $\mathsf{Cpx}(\mathcal{C})$, and take its center $\mathcal{Z}_{\mathbf{E}_n}(A)$, which is an E_{n+1}-algebra in $\mathsf{Cpx}(\mathcal{C})$, which we see as an E_{n+1}-algebra in Cpx. One can show that $\mathsf{H}^\bullet\big(\mathcal{Z}(A)\big)$ is isomorphic, as a P_{n+1}-algebra in graded vector spaces, to $\mathrm{S}_A\big(\mathrm{Der}(A)[-n]\big)$. Notice that the P_{n+1}-structure on $\mathrm{S}_A\big(\mathrm{Der}(A)[-n]\big)$ is provided by the commutative product and the degree $-n$ Poisson bracket $\{\,,\}$ induced from the Lie bracket on $\mathrm{Der}(A)$.

Using the formality of $\mathsf{C}_{-\bullet}(\mathsf{E}_{n+1}, \mathbf{k})$ we get that $\mathcal{Z}_{\mathbf{E}_n}(A)$ inherits the structure of a strict $\mathsf{P}^{(\infty)}_{n+1}$-algebra, where $\mathsf{P}^{(\infty)}_{n+1}$ can be any cofibrant resolution of P_{n+1} in Cpx. In our case we will take the $\mathsf{P}^{(\infty)}_{n+1}$ to be the minimal model of P_{n+1} (which is a quadratic Koszul algebra). We will call *strong homotopy* P_{n+1}-algebras the strict $\mathsf{P}^{(\infty)}_{n+1}$-algebras.

By standard techniques of homotopy transfer (see e.g. [CVdB2, §A.2] and references therein) one can prove that there exists a minimal strong homotopy P_{n+1}-algebra structure on $\mathrm{S}_A\big(\mathrm{Der}(A)[-n]\big)$ which is $\mathsf{P}^{(\infty)}_{n+1}$-quasi-isomorphic to $\mathcal{Z}_{\mathbf{E}_n}(A)$, and provides in particular a deformation of the P_{n+1}-algebra structure on $\mathrm{S}_A\big(\mathrm{Der}(A)[-n]\big)$.

One can finally show, along the lines of [39, Section 3], that all such deformations are $\mathsf{P}^{(\infty)}_{n+1}$-isomorphic to the trivial one (one proves that the space where the obstruction to construct such a $\mathsf{P}^{(\infty)}_{n+1}$-isomorphism step by step lives is zero).

In particular, the above discussion would lead to a proof of the following

Conjecture 3.2.1 (Higher Kontsevich formality)**.** $\mathrm{S}_A\big(\mathrm{Der}(A)[-n]\big)[n]$ *and* $\mathcal{Z}_{\mathbf{E}_n}(A)[n]$ *are weakly equivalent in* $\mathsf{Lie}\text{-alg}_{/\mathsf{Cpx}}$.

The above Conjecture is a Theorem [24] whenever $n = 1$, in which case it admits a nice refinement known as *cyclic formality for cochains* and that we describe now.

The standard volume form on $\mathbb{R}^n$ allows one to get an S^1-action on $\mathcal{Z}_{\mathbf{E}_1}(A)$. It induces an action of $\mathsf{H}_{-\bullet}(\mathsf{S}^1, \mathbf{k})$ on $\mathrm{S}_A\big(\mathrm{Der}(A)[-1]\big)$, which can be described as follows: the above volume form allows to transport the de Rham differential from $\mathrm{S}_A\big(\Omega^1_A[1]\big)$ to a square zero degree -1 operator div on $\mathrm{S}_A\big(\mathrm{Der}(A)[-1]\big)$. It has been proved in [15] that the formality of $\mathbf{E}_2$ can be made S^1-equivariant and then upgraded to the formality of $\mathsf{F}_2 := \mathbf{E}_2 \rtimes \mathsf{S}^1$. In particular this we would lead to an alternative proof of the following

Theorem 3.2.2 (Formality of cyclic cochains, [WC]). *$S_A\big(\mathrm{Der}(A)[-1]\big)[1]^{S^1}$ and $\mathcal{Z}_{\mathbf{E}_1}(A)[1]^{S^1}$ are weakly equivalent in* $\mathsf{Lie}\text{-alg}_{/\mathsf{Cpx}}$.

Notice that the formulation of [WC] (to which we refer for more details) is a bit different: it says that there is an L_∞-quasi-isomorphism between the following *strict* dg-Lie algebras: $\Big(S_A\big(\mathrm{Der}(A)[-1]\big)[u], u\,\mathrm{div}, \{\,,\}\Big)$, where u is a variable of degree 2, and the complex of cyclic Hochschild cochains equipped with the usual Hochschild differential and the Gerstenhaber bracket.

3.2.2. Calculus formality

Let A be an associative algebra, which can viewed as a locally constant factorization algebra on S^1. We have seen that the pair $\big(\mathcal{Z}(A), A\big)$ becomes a locally constant factorization algebra on $S^1 \times \mathbb{R}^+$, where locally constantness has to be understood relatively to the boundary stratification. Let $\pi : S^1 \times \mathbb{R}^+ \to \mathbb{C}$ be defined by $\pi(\theta, r) := re^{i\theta}$, where $\mathbb{C}$ is understood as the following stratified space: $\{(0,0)\} \subset \mathbb{R}^2$. The associated cocheaf $U \mapsto \pi^{-1}(U)$ is locally constant and thus $\pi_*A = \big(\mathcal{Z}(A), A(S^1)\big)$ is a locally constant factorization algebra on $\mathbb{C}$.

Recall that $\mathcal{Z}(A) \cong \mathbf{R}\mathrm{Hom}_{A\otimes A^{op}}(A, A)$ and that $A(S^1) \cong A \overset{\mathbf{L}}{\underset{A\otimes A^{op}}{\otimes}} A$; this means that the pair (Hochschild cochains, Hochschild chains) can be naturally endowed with the structure of a locally constant factorization algebra $\mathbb{C}$. Moreover, everything can be shown to be S^1-equivariant from the very beginning of the construction.

Observe that there is a description of S^1-equivariant locally constant factorization algebras on S^1 in terms of algebras over a suitable topological operad. One defines, after [26], an operad having two colours ◯ and ⊙ and two types of operations:

- configurations of discs of the first type within the unit disc.
- configurations of discs of both types within the unit disc, satisfying the following requirements: discs of the first type should not contain the origin while the origin of a disc of the second type should match with the origin of the unit disc (this imposes that there can be at most one disc of the second type in the configuration).

It carries an S^1-action which rotates discs of the second type as well as configurations of the second type. One then defines KS to be the operad obtained from PreKS by incorporating this S^1-action in the space of operations. One can then prove the following, along the lines of Theorem 3.1.6.

Theorem 3.2.3. *The simplicial monoidal categories $(\mathsf{Fact}_{\mathbb{C}}\text{-alg})^{l.c.}$ and PreKS-alg are S^1-equivariantly weakly equivalent, leading as well to a weak equivalence between $\Big((\mathsf{Fact}_{\mathbb{C}}\text{-alg})^{l.c.}\Big)^{S^1}$ and KS-alg.*

It is worth noticing that $H_{-\bullet}(\mathsf{PreKS}, \mathbb{Q})$, resp. $H_{-\bullet}(\mathsf{KS}, \mathbb{Q})$, is the precalculus operad $\mathsf{PreCalc}$, resp. the calculus operad Calc, introduced by Tamarkin and Tsygan [40].

The outcome of the above discussion is that there is a KS-algebra structure (in Cpx) on the pair $\big(\mathcal{Z}(A), A(S^1)\big)$, which induces a Calc-algebra structure on the pair $\big(HH^\bullet(A), HH_{-\bullet}(A)\big)$ within graded vector spaces. When A is a smooth commutative algebras (meaning here that the A-module $\mathrm{Der}(A)$ is projective), this Calc-algebra can be shown to be isomorphic to $\big(S_A(\mathrm{Der}(A)[-1]), S(\Omega^1_A[1])\big)$ (on which the Calc-algebra structure is given by: the wedge product and Lie bracket of polyvector fields, themselves acting on forms by contraction and Lie derivative, together with the de Rham diferential for the circle action).

As for the topological operads $\mathbf{E}_{n,m}$, there is a configuration space version $\overline{C}_{\mathsf{KS}}$ of the operad KS, together with a operadic graph complex $\mathsf{Graph}_{\mathsf{KS}}$ and the following quasi-isomorphisms (see e.g. [43]):

$$C_{-\bullet}\left(\overline{C}_{\mathsf{KS}}, \mathbb{R}\right) \longleftarrow C^{sa}_{-\bullet}\left(\overline{C}_{\mathsf{KS}}, \mathbb{R}\right) \longrightarrow \mathsf{Graph}_{\mathsf{KS}} \longleftarrow \mathsf{Calc} \otimes_{\mathbb{Q}} \mathbb{R}$$

One could probably show the Calculus formality *via* the vanishing of some obstructions as in the previous Section, but Willwacher gave a much nicer proof based on an explicit lift of the Calc-algebra structure on $\big(S_A(\mathrm{Der}(A)[-1]), S(\Omega^1_A[1])\big)$, when A is one of the algebras considered in the previous Section, to a $\mathsf{Graph}_{\mathsf{KS}}$-algebra structure. Then he roughly proves there is a $\mathsf{Graph}_{\mathsf{KS}}$-quasi-isomorphism between $\big(S_A(\mathrm{Der}(A)[-1]), S(\Omega^1_A[1])\big)$ and $\big(\mathcal{Z}(A), A(S^1)\big)$. This makes use, again, of integrals over configuration spaces of suitable forms associated to graphs. Dolgushev-Tamarkin-Tsygan also gave a proof of the so-called "Calculus formality" using different methods [12].

Below we list a few consequences of Willwacher's result. But before that we explain the appearence of the so-called cup-product, resp. cap-product, on tangent cohomology, resp. homology. It is known that any Lie algebra $\mathfrak{g}$ in Cpx gives rise to a derived formal stack (see e.g. [19]) of which any point γ roughly provides a deformation $\mathfrak{g}_\gamma$ of $\mathfrak{g}$ (within the explicit model of L_∞-algebras that most people use this is the so-called twisting procedure by a Maurer-Cartan element). Moreover, if $\mathfrak{g}$ is the Lie structure on the shift of an E_n-algebra then so is $\mathfrak{g}_\gamma$. Therefore one gets a commutative product on $H^{\bullet+n}(\mathfrak{g}_\gamma)$, often called *cup-product*.

Similarly, if a pair $(\mathfrak{g}, \mathfrak{M})$ of a Lie algebra and a module over it actually come from the shift of an $E_{n-1,1}$-algebra then so does $(\mathfrak{g}_\gamma, \mathfrak{M}_\gamma)$, and thus one gets that the commutative algebra $H^{\bullet+n}(\mathfrak{g}_\gamma)$ acts on $H^\bullet(\mathfrak{M}_\gamma)$ *via* the so-called *cap-product*.

Theorem 3.2.4 (Compatibility with cup-poducts, [24, 30]). *There is a weak equivalence φ between $\mathfrak{g} = S_A\big(\mathrm{Der}(A)[-1]\big)[1]$ and $\mathfrak{h} = \mathcal{Z}_{E_1}(A)[1]$ in $\mathsf{Lie}\text{-alg}_{/\mathsf{Cpx}}$ such that for any $\gamma \in \mathsf{Def}(\mathfrak{g})$ and $\gamma' \in \mathsf{Def}(\mathfrak{h})$ corresponding to each other via φ, then the induced isomorphism $H^{\bullet+1}(\mathfrak{g}_\gamma)\tilde{\longrightarrow}H^{\bullet+1}(\mathfrak{h}_{\gamma'})$ is an algebra morphism.*

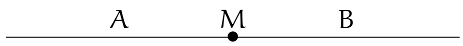
We now fix a weak equivalence between $\mathfrak{g}$ and $\mathfrak{h}$ in $\mathsf{Lie}\text{-alg}_{/\mathsf{Cpx}}$.

Theorem 3.2.5 (Tsygan formality for chains, [36]). *$\mathfrak{M} = S(\Omega^1_A[1])$ and $\mathfrak{N} = A(S^1)$ are weakly equivalent in $\mathfrak{g}\text{-}\mathsf{Lie}\text{-mod}_{/\mathsf{Cpx}}$.*

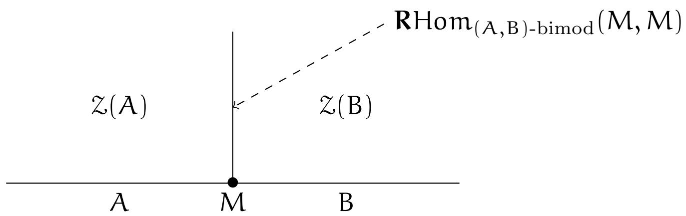
Theorem 3.2.6 (Compatibility with cap-poducts, [CR1, CR2]). *One can choose φ satisfying the requirement of Theorem 3.2.4 and the weak equivalence in Theorem 3.2.5 in such a way that the induced isomorphism $H^\bullet(\mathfrak{M}_\gamma)\tilde{\longrightarrow}H^\bullet(\mathfrak{N}_{\gamma'})$ is a morphism of $H^{\bullet+1}(\mathfrak{g}_\gamma)$-modules (with γ and γ' as above).*

3.2.3. Kontsevich formality in the presence of two branes

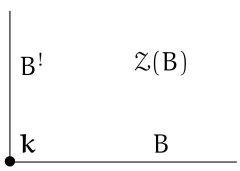
Let A and B be two associative algebras together with an A-B-bimodule M. To this data we associate a locally constant factorization algebra $\mathcal{F}$ on the stratified space $X = (\{0\} \subset \mathbb{R})$, depicted as follows:



The center $\mathcal{Z}(\mathcal{F})$ of $\mathcal{F}$ is then a locally constant factorization algebra on $X \times \mathbb{R}^+$ that we can depict in the following way:



We now restrict to the case when $A = \mathbf{k}$, with B augmented, and $M = \mathbf{k}$ with right B-module structre given by the augmentation. In this case the above factorization algebra is only non-trivial in the first quadrant. We again describe it in pictorial way (with $B^! := \mathbf{R}\mathrm{Hom}_{\text{mod-}B}(\mathbf{k}, \mathbf{k})$):



The Keller-Koszul condition [23, CFFR], which requires that B is weakly equivalent to $\mathbf{R}\mathsf{Hom}_{\mathsf{B}^!\text{-mod}}(\mathbf{k},\mathbf{k})$, immediately implies that $\mathcal{Z}(\mathsf{B})$ and $\mathcal{Z}(\mathsf{B}^!)$ are weakly equivalent as $\mathbf{E}_2$-algebras. Namely, from the above picture and by the univeral property of centers we have a morphism $\mathcal{Z}(\mathsf{B}) \to \mathcal{Z}(\mathsf{B}^!)$. Then the Keller-Koszul condition ensures that we have the very same picture but with $\mathcal{Z}(\mathsf{B}^!)$ instead of $\mathcal{Z}(\mathsf{B})$, leading in particular to a morphism of E_2-algebras $\mathcal{Z}(\mathsf{B}^!) \to \mathcal{Z}(\mathsf{B})$ which can be shown to be a weak inverse of the preceeding one.

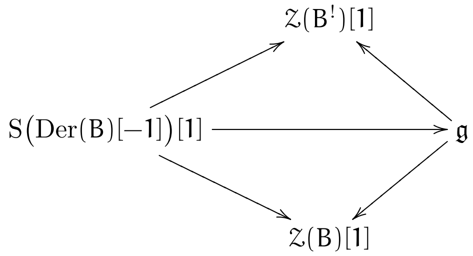
We now further restrict ourselves to the case when $\mathsf{B} = \mathsf{S}(\mathsf{V})$ and $\mathsf{B}^! = \mathsf{S}(\mathsf{V}^*[-1])$, $\mathsf{V} \in \mathsf{Vect}$. As before we expect that one can prove a formality type result for the quadruple $\big(\mathcal{Z}(\mathsf{B}),\mathsf{B},\mathsf{B}^!,\mathbf{k}\big)$ by using the formality of $\mathbf{E}_{0,2}$ together with the vanishing of some obstruction space. Such a formality theorem would have the following formal consequence: given a pair of Koszul dual first order deformations (i.e. P_0-structures) of B and $\mathsf{B}^!$ as associative algebras, and if their corresponding quantizations lead to formal deformations[3]of B and $\mathsf{B}^!$, then these are also Koszul dual.

Let us finally report on an avatar of the above formality result, which both implies the very same formal consequence and has the strong advantage of having been proven.

Theorem 3.2.7 (Formality in the presence of two branes, [CFFR])**.** *We have the following homotopy commutative square of weak equivalences in* Lie-alg$_{/\mathsf{Cpx}}$*:*

$$\begin{array}{ccc}
\mathsf{S}\big(\mathrm{Der}(\mathsf{B}^!)[-1]\big)[1] & \longrightarrow & \mathcal{Z}(\mathsf{B}^!)[1] \\
\Big\| & & \uparrow \\
\mathsf{S}\big(\mathrm{Der}(\mathsf{B})[-1]\big)[1] & \longrightarrow & \mathcal{Z}(\mathsf{B})[1]
\end{array}$$

Notice that the formulation of [CFFR] is quite different from this one and reflects better the strategy we used in the proof. First of all we work with explicit models for the centers, which are strict dg-Lie algebras, and construct another strict dg-Lie algebra $\mathfrak{g}$ together with a strictly commuting diagram of strong homotopy Lie algebra morphisms



As we have already seen this implies that Koszul duality is preserved by Kontsevich quantization, under the assumption that the quantizations we get are not curved. Moreover, we could prove in [CFR] that if one starts from a *polynomial* Poisson structure on $\mathsf{S}(\mathsf{V})$ the quantized algebra admits a presentation by generators and relations which is provided by the formality of $\mathcal{Z}\big(\mathsf{S}(\mathsf{V}^*[-1])\big)$.

Remark 3.2.8. The two branes we are refeering to are, in the example we discussed, the space-filling brane V^* and $\{0\}$. The general case of two branes $\mathsf{U},\mathsf{U}' \in \mathsf{V}^*$ is also discussed in [CFFR], and leads to some kind of relative Koszul duality.

Remark 3.2.9. The very same formality result has been obtained by Shoikhet [38]. He is also the one who raised the question of having an isomorphism between Kontsevich quantization of a polynomial Poisson structure on $\mathsf{S}(\mathsf{V})$ and the corresponding quantization by generators and relations obtained *via* the formality for $\mathcal{Z}\big(\mathsf{S}(\mathsf{V}^*[-1])\big)$ (see [37]).

[3]This is a crucial point. Notice that for an associative algebra A we have a fiber sequence $\mathsf{A} \to \mathbb{T}_\mathsf{A} \to \mathcal{Z}(\mathsf{A})[1]$ of Lie algebras, where $\mathbb{T}_\mathsf{A}$ denotes the deformation complex of A. Hence not every Maurer-Cartan element in $\mathcal{Z}(\mathsf{A})[1]$ leads to a deformation of A as an algebra. Recall that our algebras are pretty relaxed (meaning up to homotopy), so this is not at all a question of strictness. When one works with the explicit model of A_∞-algebras, those Maurer-Cartan elements not arising as actual deformations are precisely the so-called *curved* A_∞-structures.

Part II.

Lie theory *vs* Algebraic Geometry

(Abuses of) Notation for Part II

Throughout this part we let $\mathcal{C}$ be a bicomplete abelian $\mathbf{k}$-linear closed symmetric monoidal category. Most of the time $\mathcal{C}$ will be $\mathbf{k}\text{-}\mathsf{mod}$ or $\mathsf{dg}\text{-}\mathbf{k}\text{-}\mathsf{mod}$, where $\mathbf{k}$ is a fixed commutative ring. Unless otherwise specified the symbol $\otimes$ denotes the monoidal product in $\mathcal{C}$.

All algebraic structures we consider are understood as being given on objects of $\mathcal{C}$. E.g. a commutative algebra is an object A (of $\mathcal{C}$) together with a morphism $m : A \otimes A \to A$ (in $\mathcal{C}$) such that

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \quad \text{and} \quad m \circ \sigma_{A,A} = m\,, \tag{3.1}$$

where $\sigma : \otimes \to \otimes^{op}$ is the symmetry isomorphism of $\mathcal{C}$.

Notice that, from now, we will **always** make the following abuses of notation:

1. we deal with objects of $\mathcal{C}$ as if they were being $\mathbf{k}$-modules, allowing ourselves to write formulæ elementwise;
2. we forget σ from the notation.

In order to exemplify let us rewrite the second condition of (3.1) accordingly:

$$\text{“for any } a, b \in A,\ ab = ba\text{”}.$$

Dualizable objects in $\mathcal{C}$ will be sometimes called finite dimensional.

We now describe our conventions regarding tensor products. For a commutative algebra R, we write $\otimes_R$ for the tensor product of **left** R-modules. For a (possibly noncommutative) ring B, we denote by $\underset{B}{\otimes}$ the tensor product between right and left B-modules.

For a left R-module M we denote by $S_R(M)$, resp. $T_R(M)$, the symmetric, resp. tensor, algebra generated by M over R. Both are considered as graded R-algebras; however, we don't require R to be central in R-algebras. We write $S^k_R(M)$, resp. $T^k_R(M)$, for the k-the homogeneous component.

Unless otherwise stated all filtrations are ascending, indexed by non-negative integers.

Whenever we have a dg-object A in $\mathcal{C}$ we denote by $A^\sharp$ the underlying graded object.

4. Căldăraru-Duflo-Kontsevich isomorphisms

Throughout this Chapter $\mathbf{k}$ is assumed to be a field of zero characteristic.

4.1. Duflo-Kontsevich isomorphism for formal dg-manifolds

In this Section we state a general Duflo-type result for formal dg-manifolds, i.e. graded objects equipped with a formal vector field of degree 1 which squares to 0. This result implies in particular the cohomological version of Duflo's Theorem.

Throughout this Section V is a dualizable object in $\mathcal{C}^{gr}$.

4.1.1. A review of compatibility with cup- and cap-products

We introduce

- $\mathcal{O}_V := S(V^*)$, the commutative algebra of functions on V;
- $\mathfrak{X}_V := \mathrm{Der}(\mathcal{O}_V) = S(V^*)\otimes V$, the Lie algebra of vector fields on V;
- $T_{\mathrm{poly}}(V) := S(V^*\oplus V[-1]) \cong S_{\mathcal{O}_V}(\mathfrak{X}_V[-1])$, the $\mathfrak{X}_V$-module algebra of polyvector fields on V.
- the $\mathfrak{X}_V$-module algebra D_V of differential operators on V, which is the subalgebra of $\mathrm{End}(\mathcal{O}_V)$ generated by $\mathcal{O}_V$ and $\mathfrak{X}_V$;
- the $\mathfrak{X}_V$-module algebra $D_{\mathrm{poly}}(V)$ of polydifferential operators on V, which consists of multilinear maps $\mathcal{O}_V\otimes\cdots\otimes\mathcal{O}_V\to\mathcal{O}_V$ being differential operators in each argument.

It is clear that $D_{\mathrm{poly}}(V)$ is then a sub-dg-algebra of the Hochschild cochain complex[1] of the algebra $\mathcal{O}_V$ (it is obviously preserved by the Hochschild differential d_H). From now we write $\cup$ for both products on $T_{\mathrm{poly}}(V)$ and $D_{\mathrm{poly}}(V)$.

It is well-known that the cohomology of $\big(D_{\mathrm{poly}}(V), d_H\big)$ is given by $T_{\mathrm{poly}}(V)$:

Proposition 4.1.1. *The natural inclusion* $\mathsf{HKR} : \big(T_{\mathrm{poly}}(V),0\big) \hookrightarrow \big(D_{\mathrm{poly}}(V), d_H\big)$ *is a quasi-isomorphism of complexes that induces an isomorphism of algebras in cohomology.*

There are also:

- $\Omega^1_V := \Omega^1_{\mathcal{O}_V} = S(V^*)\otimes V^*$, the $\mathfrak{X}_V$-module of Kähler differentials, where the action of $\mathfrak{X}(V)$ is given by the Lie derivative;
- $\Omega(V) := S(V^*\oplus V^*[1]) \cong S_{\mathcal{O}_V}(\Omega^1_V[1])$, the $\mathfrak{X}_V$-algebra of differential forms with reversed grading, which is also a $T_{\mathrm{poly}}(V)$-module *via* contraction.
- $C^{\mathrm{poly}}(V)$, the $\mathfrak{X}_V$-algebra of formal functions near the diagonal in $V\times\cdots\times V$ (i.e. the adic-completion of $\mathcal{O}_V\otimes\cdots\otimes\mathcal{O}_V$ along the kernel of the multiplication map). We use the reverse grading convention.

$C^{\mathrm{poly}}(V)$ is a completion of the usual Hochschild chain complex and the Hochschild differential b_H extends to it[2] . It is acted on by $D_{\mathrm{poly}}(V)$ *via* contraction. From now we write $\cap$ for both actions by contraction on $\Omega(V)$ and $C^{\mathrm{poly}}(V)$.

By duality, we get:

Proposition 4.1.2. *The morphism* $\mathsf{HKR}^* : \big(\big(C^{\mathrm{poly}}(V), b_H\big) \to \big(\Omega(V),0\big)$ *that sends* $f_0\otimes f_1\otimes\cdots\otimes f_n$ *to* $f_0 df_1\dots df_n$ *is a quasi-isomorphism of complexes that induces an isomorphism* $H\big(T_{\mathrm{poly}}(V)\big)$*-modules in cohomology.*

[1]The inclusion actually being a quasi-isomorphism.
[2]And they are quasi-isomorphic.

Let Q be a cohomological vector field on V: i.e. a differential on $\mathcal{O}_V$ that turns it into a dg-algebra. Since $Q \in \mathfrak{X}_V$ then it acts as a differential $Q\cdot$ commuting with all available structures on $T_{\rm poly}(V)$, $D_{\rm poly}(V)$, $\Omega(V)$, and $C^{\rm poly}(V)$. By a spectral sequence argument one can show that HKR and HKR^* still define quasi-isomorphisms, but they no longer preserve the algebra and module structures at the level of cohomology.

We are going to modify a bit HKR and HKR^* so that the induced isomorphisms on (co)homology preserve these structures.

For later purposes we need to work in a bit more general framework. We let $\mathfrak{a}$ be a dg-commutative algebra splitting as $\mathfrak{a} = \mathfrak{m} \oplus R$, with $\mathfrak{m}$ a (pro)nilpotent ideal and R a subalgebra concentrated in degree 0, and we assume that Q is a Maurer-Cartan element in the dg-Lie algebra $\mathfrak{X}_V \otimes \mathfrak{m}$.

We then define and endomorphism valued one-form $\Xi \in \mathrm{Hom}_{\mathcal{O}_V\text{-mod}}(\mathfrak{X}_V, \Omega^1_V \otimes_{\mathcal{O}_V} \mathfrak{X}_V[1]) \otimes \mathfrak{m}$ associated with Q in the following way:

1. we observe that $\mathfrak{X}_V$ admits a canonial connection $\nabla_V : \mathfrak{X}_V \to \Omega^1_V \otimes_{\mathcal{O}_V} \mathfrak{X}_V$.
2. we then define $\Xi := [\nabla_V, Q\cdot] : \mathfrak{X}_V \to \Omega^1_V \otimes_{\mathcal{O}_V} \mathfrak{X}_V \otimes \mathfrak{m}[1]$.

Observe that for any $k \geq 1$ one can define $\mathrm{tr}_{\mathfrak{X}_V}(\Xi^k) \in S^k_{\mathcal{O}_V}(\Omega^1_V[1]) \otimes \mathfrak{m}^k$. It is a degree 0 element in $\Omega(V) \otimes \mathfrak{m}^k$ and thus the series

$$j(\Xi) := \det\left(\frac{\Xi}{1-e^{-\Xi}}\right) \in \Omega(V) \otimes \mathfrak{m}\,.$$

makes perfect sense (and has cohomological degree 0).

Theorem 4.1.3 ([CR3])**.** *There is a quasi-isomorphism*

$$\mathcal{U}_Q := \mathsf{HKR} \circ \iota_{j(\Xi)^{1/2}} \;:\; (T_{\rm poly}(V) \otimes \mathfrak{a}, d_{\mathfrak{a}} + Q\cdot) \longrightarrow (D_{\rm poly}(V) \otimes \mathfrak{a}, d_H + d_{\mathfrak{a}} + Q\cdot)$$

together with an explicit $\mathrm{Aut}(V)$*-equivariant homotopy between* $\mathcal{U}_Q \circ \cup$ *and* $\cup \circ (\mathcal{U}_Q \otimes \mathcal{U}_Q)$.

Theorem 4.1.4 ([CR2], Section 5 & Theorem 6.4)**.** *There is a quasi-isomorphism*

$$\mathcal{S}_Q := (j(\Xi)^{1/2} \wedge -) \circ \mathsf{HKR}^* \;:\; (C^{\rm poly}(V) \otimes \mathfrak{a}, b_H + d_{\mathfrak{a}} + Q\cdot) \longrightarrow (\Omega(V) \otimes \mathfrak{a}, d_{\mathfrak{a}} + Q\cdot)$$

together with an explicit $\mathrm{Aut}(V)$*-equivariant homotopy between* $\cap \circ (\mathrm{id} \otimes \mathcal{S}_Q)$ *and* $\mathcal{S}_Q \circ \cap \circ (\mathcal{U}_Q \otimes \mathrm{id})$.

One can actually replace $j(\Xi)$ by $\widetilde{j}(\Xi) := \det\left(\frac{\Xi}{e^{\Xi/2}-e^{-\Xi/2}}\right)$ because $\mathrm{tr}(\Xi)$ acts as a derivation of the whole structure.

4.1.2. Application: (co)homological Duflo isomorphism

We consider a Lie algebra $\mathfrak{g}$ which is dualizable as an object of $\mathcal{C}$ and recall the definition of the **(modified) Duflo element**

$$\mathfrak{d} := \det\left(\frac{e^{\mathrm{ad}/2} - e^{-\mathrm{ad}/2}}{\mathrm{ad}}\right) \in \widehat{S}(\mathfrak{g}^*)^{\mathfrak{g}}\,.$$

We also remind the reader that the completed algebra $\widehat{S}(\mathfrak{g}^*)$ naturally acts on $S(\mathfrak{g})$:

$$\xi^k \cdot x^n := \frac{n!}{(n-k)!}\xi(x)^k x^{n-k} \qquad (x \in \mathfrak{g}\,,\ \xi \in \mathfrak{g}^*\,,\ k > 0\,,\ n > 0)\,.$$

The following (proved in [24, 32]) is a cohomological extension of the original Duflo isomorphism [13].

Theorem 4.1.5 (Cohomological Duflo isomorphism)**.** *The morphism of* $\mathfrak{g}$*-modules*

$$\mathcal{D} := \mathrm{sym} \circ (\mathfrak{d}^{1/2}\cdot) : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$

induces an algebra isomorphism

$$H^\bullet(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, U(\mathfrak{g}))$$

at the level of Chevalley-Eilenberg cohomology.

We now observe that, if A is an algebra on which $\mathfrak{g}$ acts by derivations, the Chevalley–Eilenberg Lie algebra homology $H_{-\bullet}(\mathfrak{g}, A)$ is equipped with an $H^\bullet(\mathfrak{g}, A)$-module structure in the following way: on the level of the complexes, for any Chevalley-Eilenberg cochain $\alpha = \xi \otimes a$, resp. chain $c = x \otimes a'$, one defines

$$\alpha(c) = \iota_\xi(x) \otimes a a' ,$$

where ι denotes the usual contraction operation[3]. In what follows we will prove the following homological version of the Duflo isomorphism.

Theorem 4.1.6 (Homological Duflo isomorphism)**.** *The morphism $\mathcal{D}$ induces an isomorphism of $H^\bullet\big(\mathfrak{g}, S(\mathfrak{g})\big)$-modules*

$$H_{-\bullet}\big(\mathfrak{g}, S(\mathfrak{g})\big) \xrightarrow{\sim} H_{-\bullet}\big(\mathfrak{g}, U(\mathfrak{g})\big)$$

at the level of Chevalley-Eilenberg homology.

Considering the degree zero (co)homology, one obtains

Corollary 4.1.7. *$\mathcal{D}$ restricts to an isomorphism of algebras $S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g})^{\mathfrak{g}} = \mathcal{Z}\big(U(\mathfrak{g})\big)$ on invariants, and induces an isomorphism of $S(\mathfrak{g})^{\mathfrak{g}}$-modules $S(\mathfrak{g})_{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g})_{\mathfrak{g}} = \mathcal{A}\big(U(\mathfrak{g})\big)$ on coinvariants.*

Here $\mathcal{Z}(B)$ denotes the center of an algebra B, and $\mathcal{A}(B) = B/[B, B]$.

Proof of Theorem 4.1.5

In this Subsection, which is extracted from [CR2, §7.2.1], we follow closely [CR3, §5.2].

Let us consider $V := \mathfrak{g}[1]$. Then the graded algebra A of functions on V is $A = \wedge^\bullet(\mathfrak{g}^*)$, and hence the Chevalley-Eilenberg differential d_C defines a cohomological vector field Q on V.

On the one hand, $T^\bullet_{poly}(V)$ is naturally isomorphic to $\wedge^\bullet(\mathfrak{g}^*) \otimes S(\mathfrak{g})$ and, under this identification, $Q\cdot$ precisely gives the coboundary operator d_C of the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with values in $S(\mathfrak{g})$. On the other hand, $(D^\bullet_{poly}(V), d_H + Q\cdot)$ identifies with the complex $CC^\bullet(A, d_C)$ of Hochschild cochains of the dg-algebra (A, d_C) with values in itself.

Now we observe that we have a quasi-isomorphism of dg-algebras

$$\ell : (D^\bullet_{poly}(V), d_H + Q\cdot) \longrightarrow C^\bullet\big(\mathfrak{g}, U(\mathfrak{g})\big) ,$$

where $C^\bullet\big(\mathfrak{g}, U(\mathfrak{g})\big)$ denotes the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with values in $U(\mathfrak{g})$, given by the following composition of maps

$$D^\bullet_{poly}(V) = \wedge^\bullet(\mathfrak{g}^*) \otimes T\big(\wedge^\bullet(\mathfrak{g})\big) \twoheadrightarrow \wedge^\bullet(\mathfrak{g}^*) \otimes T(\mathfrak{g}) \twoheadrightarrow \wedge^\bullet(\mathfrak{g}^*) \otimes U(\mathfrak{g}) = C^\bullet\big(\mathfrak{g}, U(\mathfrak{g})\big) .$$

This is a manifestation of the fact that the quadratic dg-algebra $\big(\wedge^\bullet(\mathfrak{g}^*), d_C\big)$ and the quadratic-linear algebra $U(\mathfrak{g})$ are related by a Koszul-type duality (see e.g. [33]). Moreover, the following diagram of quasi-isomorphisms of complexes commutes :

$$\begin{array}{ccc} \Big(T^\bullet_{poly}(V), Q\cdot\Big) & \xrightarrow{\text{HKR}} & \Big(D^\bullet_{poly}(V), d_H + Q\cdot\Big) \\ \Big\| & & \Big\downarrow \ell \\ C^\bullet(\mathfrak{g}, S(\mathfrak{g})) & \xrightarrow{\text{sym}} & C^\bullet(\mathfrak{g}, U(\mathfrak{g})) , \end{array}$$

Finally recall the following

Lemma 4.1.8 ([CR3], Lemma 5.6)**.** *The graded endomorphism-valued 1-form Ξ identifies with the (A-linear extension of) the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}[1]$.*

[3] In fact, this defines an actual dg-module structure on the level of the complexes.

If we write ad for the usual adjoint action (on $\mathfrak{g}$) then

$$\widetilde{j}(\Xi) := \det\sqrt{\frac{\Xi}{e^{\Xi/2}-e^{-\Xi/2}}} = \mathfrak{d}^{1/2}\,,$$

and we have the following commutative diagram of algebra isomorphisms

$$\begin{array}{ccc} H^\bullet\Big(T^\bullet_{\mathrm{poly}}(V), Q\,\cdot\Big) & \xrightarrow{\ \mathcal{U}_Q\ } & H^\bullet\Big(D^\bullet_{\mathrm{poly}}(V), d_H + Q\,\cdot\Big) \\ \| & & \big\downarrow \ell \\ H^\bullet(\mathfrak{g}, S(\mathfrak{g})) & \xrightarrow{\ \mathcal{D}\ } & H^\bullet(\mathfrak{g}, U(\mathfrak{g}))\,. \end{array}$$

Hence Theorem 4.1.5 follows from Theorem 4.1.3. □

Proof of Theorem 4.1.6

This Subsection is extracted from [CR2, §7.2.2].

First of all, we observe that $\Omega^\bullet(V)$ is naturally isomorphic to $\wedge^\bullet(\mathfrak{g}^*)\otimes S(\mathfrak{g}^*)$ and that, under this identification, $Q\cdot$ precisely gives the coboundary operator of the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with values in $S(\mathfrak{g}^*)$.

Then, $\big(C^{\mathrm{poly}}_\bullet(V), b_H + Q\cdot\big)$ identifies with the complex $CC_{-\bullet}(A, d_C)$ of Hochschild chains (with reversed grading) of the dg-algebra (A, d_C) with values in itself.

We now want to apply Theorem 4.1.4 to the present situation. For $\mathcal{S}_Q = j(\Xi)\wedge \mathrm{HKR}$ to make sense and be well-defined we need a slight modification: one has to consider completed versions $\widehat{\Omega}^\bullet(V) = \wedge^\bullet(\mathfrak{g}^*)\otimes\widehat{S}(\mathfrak{g}^*)$ and $\widehat{C}^{\mathrm{poly}}_{-\bullet}(V) = \wedge^\bullet(\mathfrak{g}^*)\otimes\widehat{T}\big(\wedge^\bullet(\mathfrak{g}^*)\big)$ of the spaces involved.

Now, we recall that we have a quasi-isomorphism of complexes

$$\lambda\,:\,C^\bullet(\mathfrak{g}, U(\mathfrak{g})^*) \longrightarrow \Big(\widehat{C}^{\mathrm{poly}}_{-\bullet}(V), b + Q\,\cdot\Big)$$

given by the following composition of maps

$$C^\bullet\big(\mathfrak{g}, U(\mathfrak{g})^*\big) = \wedge^\bullet(\mathfrak{g}^*)\otimes U(\mathfrak{g})^* \hookrightarrow \wedge^\bullet(\mathfrak{g}^*)\otimes T(\mathfrak{g})^* = \wedge^\bullet(\mathfrak{g}^*)\otimes\widehat{T}(\mathfrak{g}^*) \hookrightarrow \wedge^\bullet(\mathfrak{g}^*)\otimes\widehat{T}\big(\wedge^\bullet(\mathfrak{g}^*)\big)\,,$$

which induces an isomorphism of $H^\bullet\big(\mathfrak{g}, U(\mathfrak{g})\big)$-modules on cohomology.

Moreover, the following diagram of quasi-isomorphisms of complexes commutes :

$$\begin{array}{ccc} \Big(\widehat{\Omega}(V), Q\cdot\Big) & \xleftarrow{\ \mathrm{HKR}\ } & \Big(\widehat{C}^{\mathrm{poly}}_{-\bullet}(V), b + Q\cdot\Big) \\ \| & & \big\uparrow \lambda \\ C^\bullet(\mathfrak{g}, \widehat{S}(\mathfrak{g}^*)) & \xleftarrow{\ \mathrm{sym}^*\ } & C^\bullet(\mathfrak{g}, U(\mathfrak{g})^*)\,, \end{array}$$

We observe that, for any $\mathfrak{g}$-module M, the Chevalley-Eilenberg cochain complex $C^\bullet(\mathfrak{g}, M^*)$ is naturally isomorphic to the dual of the Chevalley-Eilenberg chain complex $C_{-\bullet}(\mathfrak{g}, M)$ with reversed grading. Moreover, a direct computation shows that

Lemma 4.1.9. *For any* $\omega\in\widehat{\Omega}^\bullet(V) = C^\bullet\big(\mathfrak{g}, \widehat{S}(\mathfrak{g}^*)\big)$ *and any* $c\in C_{-\bullet}(\mathfrak{g}, S(\mathfrak{g}))$,

i) $\langle j(\Xi)\wedge\omega, c\rangle = \langle\omega, j(\Xi)\cdot c\rangle$;

ii) if $\alpha\in T^\bullet_{\mathrm{poly}}(V) = C^\bullet(\mathfrak{g}, S(\mathfrak{g}))$ *then* $\langle\iota_\alpha\omega, c\rangle = \langle\omega, \alpha(c)\rangle$.

Therefore the transpose of $\mathcal{S}_Q$ induces an isomorphism of $H^\bullet(\mathfrak{g}, S(\mathfrak{g}))$-modules

$$H_{-\bullet}(\mathfrak{g}, S(\mathfrak{g})) \longrightarrow H_{-\bullet}(\mathfrak{g}, U(\mathfrak{g}))$$

which is precisely $\mathcal{D}$, whence the proof of Theorem 4.1.6. □

Remark 4.1.10. As we already mentioned, there is a duality between the dg-algebra (A, d_C) and the quadratic-linear algebra $U(\mathfrak{g})$: in [CR1], we give a more direct proof of Corollary 4.1.7 in the same spirit of Kontsevich's approach to the original Duflo isomorphism [24], which does not make use of the aforementioned duality.

4.2. Căldăraru's conjecture

In this Section we re-formulate Căldăraru's conjecture [7] in the Lie algebroid setting. We refer to the Appendix, from which we borrow the notation, for some recollection about Lie algebroids. Notice that all constructions from the Appendix sheafify easily.

4.2.1. Hochschild (co)homology for Lie algebroids

Let X be a topological space and $(\mathcal{R},\mathcal{L})$ a sheaf of Lie algebroid on X such that $\mathcal{L}$ is locally free of finite rank d as an $\mathcal{R}$-module. The counit $J_{\mathcal{L}}\to\mathcal{R}$ endows $\mathcal{R}$ with the structure of a $J_{\mathcal{L}}$-module, and following [CRVdB2] we define the *Hochschild (co)homology* of $\mathcal{L}$ as follows:

$$\mathrm{HH}^\bullet_{\mathcal{L}}:=\mathrm{Ext}^\bullet_{J_{\mathcal{L}}}(\mathcal{R},\mathcal{R})\qquad\text{and}\qquad \mathrm{HH}^{\mathcal{L}}_{-\bullet}:=\mathrm{Ext}^{d+\bullet}_{J_{\mathcal{L}}}\big(\wedge^d_{\mathcal{R}}(\mathcal{L}),\mathcal{R}\big)\cong\mathrm{Tor}^{J_{\mathcal{L}}}_{-\bullet}(\mathcal{R},\mathcal{R})$$

Example 4.2.1. Let X be a smooth algebraic variety of dimension d, $\mathcal{R}=\mathcal{O}_X$ and $\mathcal{L}=T_X$. One can show (see [CRVdB2, Proposition 6.1]) that

$$\Big(\mathrm{HH}^\bullet_{\mathcal{L}},\mathrm{HH}^{\mathcal{L}}_{-\bullet}\Big)\qquad\text{and}\qquad\Big(\mathrm{HH}^\bullet_{X},\mathrm{HH}^{X}_{-\bullet}\Big)$$

are isomorphic as pairs (algebra,module)[4].

We now define explicit complexes computing the Hochschild (co)homology of a Lie algebroid. It goes back to [5] and was proven to compute $\mathrm{HH}^\bullet_{\mathcal{L}}$ in [CRVdB2].

Recall that $U(\mathcal{L})$ is an $\mathcal{R}$-coalgebra. We then define $D^\bullet_{\mathrm{poly},\mathcal{L}}$ as $T^\bullet_{\mathcal{R}}\big(U(\mathcal{L})\big)$ equipped with the usual Cartier-Hochschild differential: on degree n cochains it is defined as

$$d_H=1\otimes\mathrm{id}+\sum_{k=1}^{n}(-)^k\mathrm{id}^{\otimes(k-1)}\otimes\Delta\otimes\mathrm{id}^{\otimes n-k}\,.$$

One has an obvious algebra structure given by concatenation (there is actually a natural B_∞-structure on $D^\bullet_{\mathrm{poly},\mathcal{L}}$, see e.g. [CVdB2, Section 8]).

The definition of the Hochschild chains $C^{\mathrm{poly},\mathcal{L}}_{-\bullet}$ is a bit more involved. Roughly speaking it is defined as the subcomplex in the relative Hochschild chain complex of $J_{\mathcal{L}}$ w.r.t. $\mathcal{R}$ consisting of cochains that are invariant under the Grothendieck connection (we refer to [6, CRVdB1] for the details). It is naturally a module over the Hochschild cochain complex (it actually even carries two B_∞-module structures, see [CRVdB1, Appendix B] and [CR2, Section 2] for more details).

Example 4.2.2. If $X=pt$, $\mathcal{R}=\mathcal{O}_V$ and $\mathcal{L}=\mathfrak{X}_V$ (borrowing the notation from Section 4.1) then $(D_{\mathrm{poly},\mathcal{L}},C^{\mathrm{poly},\mathcal{L}})$ is isomorphic to $\big(D_{\mathrm{poly}}(V),C^{\mathrm{poly}}(V)\big)$, with all algebraic structures preserved.

Theorem 4.2.3 ([CRVdB2], Theorem 13.1)**.** *There is a natural isomorphism between*

$$\Big(\mathrm{HH}^\bullet_{\mathcal{L}},\mathrm{HH}^{\mathcal{L}}_{-\bullet}\Big)\qquad\text{and}\qquad\Big(\mathbb{H}^\bullet\big(X,D_{\mathrm{poly},\mathcal{L}}\big),\mathbb{H}^\bullet\big(X,C^{\mathrm{poly},\mathcal{L}}\big)\Big)$$

as pairs (algebra,module), where $\mathbb{H}^\bullet(X,-)=\mathbf{R}^\bullet\Gamma(X,-)$ *denotes the hypercohomology functor.*

Remark 4.2.4. Notice that this is actually an isomorphism of calculi. On the right side the above mentionned B_∞-structures induce a calculus structure on cohomology, while on the left side the calculus structure appears in a way very similar to what we have in Part I. Namely, the pair $\big(\mathrm{HH}^\bullet_{\mathcal{L}},\mathcal{R}\big)$ inherits the structure of an $H_{-\bullet}(E_{1,1},k)$-algebra because $\mathrm{HH}^\bullet_{\mathcal{L}}$ appears as the endomorphisms of the unit $\mathcal{R}$ in the monoidal category $J_{\mathcal{L}}$-mod. The calculus structure comes after passing from $\mathcal{R}$ to $\mathrm{HH}^{\mathcal{L}}_{-\bullet}$.

[4] They are actually isomorphic as calculi (see Remark 4.2.4).

4.2.2. Atiyah classes and the Todd genus

Recall that $J_{\mathcal{L}}$ carries two commuting $\mathcal{L}$-module structures (see Appendix) which we will distinguish following [CVdB1] by the subscript $i = 1, 2$. In particular $J_{\mathcal{L}}$ and all its quotients $J^n_{\mathcal{L}}$ by $(n+1)$-th powers of the augmentation ideal carry an $\mathcal{R}$-bimodule structure. We have an exact sequence of $\mathcal{R}$-bimodules

$$0 \to \mathcal{L}^* \to J^1_{\mathcal{L}} \to \mathcal{R} \to 0\,,$$

where the $\mathcal{R}$-bimodule structures on $\mathcal{L}^*$ and $\mathcal{L}$ are symmetric. Applying $-\underset{\mathcal{R}}{\otimes}\mathcal{E}$ for a given $\mathcal{R}$-module $\mathcal{E}$ one gets an exact sequence

$$0 \to \mathcal{L}^* \otimes_{\mathcal{R}} \mathcal{E} \to J^1_{\mathcal{L}}(\mathcal{E}) \to \mathcal{E} \to 0\,,$$

the class of which we denote $\mathfrak{at}_{\mathcal{L}}(\mathcal{E}) \in \mathrm{Ext}^1_{\mathcal{R}}(\mathcal{E}, \mathcal{L}^* \otimes_{\mathcal{R}} \mathcal{E})$. We call it the *Atiyah class of* $\mathcal{E}$ and derive from it the so-called *scalar Atiyah classes*: $\mathrm{tr}\Big(\wedge^i\big(\mathfrak{at}_{\mathcal{L}}(\mathcal{E})\big)\Big) \in \mathrm{H}^i\big(X, \wedge^i_{\mathcal{R}}(\mathcal{L}^*)\big)$.

We finally define the *Todd genus* of $\mathcal{L}$ as

$$\mathrm{td}_{\mathcal{L}} := \det\left(\frac{\mathrm{at}_{\mathcal{L}}(\mathcal{L})}{1-e^{-\mathrm{at}_{\mathcal{L}}(\mathcal{L})}}\right) \in \bigoplus_{i\geq 0}\mathrm{H}^i\big(X, \wedge^i_{\mathcal{R}}(\mathcal{L}^*)\big)\,.$$

As usual we have Hochschild-Kostant-Rosenberg quasi-isomorphisms

$$\mathsf{HKR} : T^{\bullet}_{\mathrm{poly},\mathcal{L}} := \wedge^{\bullet}_{\mathcal{R}}(\mathcal{L}) \longrightarrow D^{\bullet}_{\mathrm{poly},\mathcal{L}} \qquad \text{and} \qquad \mathsf{HKR}^* : C^{\mathrm{poly},\mathcal{L}}_{-\bullet} \longrightarrow \wedge^{-\bullet}_{\mathcal{R}}(\mathcal{L}^*) =: \Omega^{\mathcal{L}}_{-\bullet}\,,$$

which do not induce morphisms of algebras and their modules on hypercohomology.

Theorem 4.2.5 (Căldăraru's conjecture for Lie algebroids)**.** *We have an isomorphism of algebras*

$$\mathsf{HKR} \circ \iota_{\sqrt{\mathrm{td}_{\mathcal{L}}}} : \mathrm{H}\big(X, T_{\mathrm{poly},\mathcal{L}}\big) \xrightarrow{\sim} \mathrm{HH}_{\mathcal{L}}$$

together with an isomorphism of $\mathrm{H}\big(X, T_{\mathrm{poly},\mathcal{L}}\big)$*-modules*

$$(\sqrt{\mathrm{td}_{\mathcal{L}}} \wedge -) \circ \mathsf{HKR}^* : \mathrm{HH}^{\mathcal{L}} \xrightarrow{\sim} \mathrm{H}(X, \Omega^{\mathcal{L}})\,.$$

The proof of this statement, which was conjectured by Căldăraru [7] in the case of smooth algebraic varieties, can be found in [CRVdB1]. We roughly describe it in the next Section, but before that we give a "reason" for this result to hold.

4.2.3. Căldăraru's conjecture as an instance of the Duflo isomorphism

For simplicity, we only discuss the cohomological part of the statement.

Recall that in order to compute $\mathrm{Ext}_{J_{\mathcal{L}}}(\mathcal{R},\mathcal{R})$ one can first consider a resolution $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ by locally free $J_{\mathcal{L}}$-modules and then get $\mathrm{Ext}_{J_{\mathcal{L}}}(\mathcal{R},\mathcal{R}) = \mathbb{H}\big(X, \mathcal{H}om^{dg}_{J_{\mathcal{L}}}(\widetilde{\mathcal{R}},\widetilde{\mathcal{R}})\big)$. This is actually how the proof of Theorem 4.2.3 goes, elaborating on the so-called *Bar resolution*.

It happens that we can futher require the resolution $\widetilde{\mathcal{R}}$ to be, as a graded algebra, a free commutative $J_{\mathcal{L}}$-algebra. In the specific case we are dealing with, one has the *Koszul resolution* at hands: we define

$$\widetilde{\mathcal{R}} := \Big(S_{\mathcal{R}}(L^*[1]) \underset{\mathcal{R}}{\otimes} J_{\mathcal{L}}, d_K\Big)\,,$$

where $d_K(\xi \otimes 1) = (1 \otimes \xi)$ for any $\xi \in \mathcal{L}^* \subset J_{\mathcal{L}}$. The pair $\Big(\widetilde{\mathcal{R}}, \mathcal{D}er^{dg}_{J_{\mathcal{L}}}(\widetilde{R})\Big)$ is naturally a dg-Lie algebroid in $J_{\mathcal{L}}$-mod, with universal envelopping algebra being $\mathcal{H}om^{dg}_{J_{\mathcal{L}}}(\widetilde{\mathcal{R}},\widetilde{\mathcal{R}})$.

Moreover, on the level of the derived category $D(J_{\mathcal{L}}\text{-mod})$, $\mathcal{D}er^{dg}_{J_{\mathcal{L}}}(\widetilde{R})$ is isomorphic to $\mathcal{L}[-1]$, its Lie bracket becomes $\widetilde{\mathcal{R}}$-linear, and it coincides with the Atiyah class[5] $\mathfrak{at}_{\mathcal{L}}(\mathcal{L})$. The PBW isomorphism for the $\mathcal{R}$-Lie algebra $\mathfrak{g} := \mathcal{L}[-1]$ in $D(J_{\mathcal{L}}\text{-mod})$ then turns out to give the HKR one:

$$T_{\mathrm{poly},\mathcal{L}} \cong S_{\mathcal{R}}\big(\mathcal{L}[-1]\big) \xrightarrow{\sim} U\big(\mathcal{L}[-1]\big) \cong \mathcal{E}xt_{J_{\mathcal{L}}}(\mathcal{R},\mathcal{R})\,.$$

[5]More preciely, $\mathfrak{at}_{\mathcal{L}}(\mathcal{L})$ corresponds to the adjoint action of $L[-1]$ on L. Let us further recall that it is Kapranov [21] who first noticed that the Atiyah class **really** defines a Lie structure.

Let us now see how the Duflo isomomorphism translates in this context, even tough (and this is a crucial point) we don't have a proof of it that works for Lie algebras in triangulated categories. First of all we have to take invariants: these are morphisms from the trivial representation, which in this case is $\mathcal{R}$. If moreover the $\mathfrak{g}$-action on a complex of $\mathcal{R}$-modules $\mathcal{E}$ comes from the own Atiyah class of $\mathcal{E}$, then morphisms from $\mathcal{R}$ to $\mathcal{E}$ in $D(J_{\mathcal{L}}\text{-mod})$ are automatically $\mathfrak{g}$-equivariant and $\mathcal{E}^{\mathfrak{g}} = \mathbb{H}(X, \mathcal{E})$. Therefore $S(\mathfrak{g})^{\mathfrak{g}} = H(X, T_{\mathrm{poly},\mathcal{L}})$ and $U(\mathfrak{g})^{\mathfrak{g}} = \mathrm{Ext}_{J_{\mathcal{L}}}(\mathcal{R}, \mathcal{R})$.

The conjecture of Căldăraru then follows formally from the Duflo isomorphism for $\mathfrak{g} = \mathcal{L}[-1]$. The reason why it is the inverse series that appears in Căldăraru's conjecture finds its explanation in the fact that conjugating with the suspension sends the determinant to its inverse[6].

4.3. A proof of Călăraru's conjecture *via* globalization

In this Section we sketch a proof of Căldăraru's conjecture [7, Conjecture 5.2] on the isomorphism between the Hochschild and harmonic structures of a smooth algebraic variety. It is mainly extracted from [CVdB1, CRVdB1] and is based on globalization techniques that we shortly summarize.

The philosophy of formal geometry

The aim of this Section, which is extracted from [CRVdB1, §4] (see also [CVdB1]), is to discuss Fedosov resolutions [11]. These are needed to globalize some local results. To help the reader understand our algebraic setup we give some motivation for the definitions in the subsequent sections. For the sake of exposition we assume in this introduction that X is some kind of d-dimensional smooth space.

One of the applications of formal geometry is the globalization of local *coordinate dependent* constructions. For example using the Darboux Lemma it is trivial to quantize a symplectic manifold locally but such local quantizations are coordinate dependent and they do not globalize easily. The same is true for most of the formality morphisms of Part I, as well as for the Duflo-Kontsevich morphisms appearing previously in this Chapter.

The idea is then to replace X by a much larger infinite dimensional space $X^{\mathrm{coord}} \to X$ that parametrizes formal local coordinate systems on X. For example if X is an algebraic variety then the fiber at $x \in X$ in X^{coord} is given by the k-algebra isomorphisms $\widehat{\mathcal{O}}_{X,x} \to k[\![t_1, \ldots, t_d]\!]$. An equivalent way of saying this is that X^{coord} universally trivializes the jet bundle $(\widehat{\mathcal{O}}_{X,x})_{x \in X}$ over X.

Local constructions can be tautologically globalized to X^{coord} and this should be followed by some type of descent for X^{coord}/X. A general procedure to do this is to resolve $\mathcal{O}_X$ by a De Rham-type complex over $\mathcal{O}_{X^{\mathrm{coord}}}$ but this does not really work as the fibers of $X^{\mathrm{coord}} \to X$ are not contractible.

However in the aforementioned examples the local constructions are all compatible with *linear* coordinate changes. So if we define $X^{\mathrm{aff}} = X^{\mathrm{coord}}/\mathrm{Gl}_d$ then the constructions descend to X^{aff} and as the fibers of $X^{\mathrm{aff}} \to X$ are contractible we can descend further to X. Actually, any $\widetilde{X} \to X$ satisfying the same properties as X^{aff} would work as well (X^{aff} being universal among these).

Setup

We work over a general locally free Lie algebroid $\mathcal{L}$ rather than the tangent sheaf. This allows to uniformally treat the differentiable, holomorphic and algebraic cases, as well as other contexts (like e.g. manifolds with boundary).

As a general principle we work on the presheaf level, performing sheafification only as the very last step of the constructions. This means that we may throughout replace all spaces by rings and locally free sheaves may be treated as free modules.

We fix once and for all a Lie algebroid (R, L) such that $L \cong R \otimes V$ as an R-module, for a dualizable object V.

[6] See previous footnote.

4.3.1. Fedosov resolutions

Recall that J_L carries two commuting L-module structures (see Appendix) which we will distinguish following [CVdB1] by the subscript $i = 1, 2$. In particular the second action provides us with a morphism of Lie algebroids $(R_2, L_2) \to \big(J_L, \mathrm{Der}_{R_1}(J_L)\big)$, which happens to extend to an *isomorphism* of Lie algebroids $\big(J_L, J_L \otimes_{R_2} L_2\big) \to \big(J_L, \mathrm{Der}_{R_1}(J_L)\big)$[7]. This implies in particular that these two Lie algebroids have *isomorphic* Hochschild and harmonic structures.

Assume now that we have a dg-R-algebra $\widetilde{R}$ such that $\Omega_R \to \Omega_{\widetilde{R}}$ is a quasi-isomorphism. Then we define $A := \Omega_{\widetilde{R}} \widehat{\otimes}_{\Omega_R} C^\bullet(L)$ and consider the dg-$\widetilde{C}$-algebra

$$B := \Omega_{\widetilde{R}} \widehat{\otimes}_{\Omega_R} C^\bullet(L_1, J_L)\,.$$

It is isomorphic, as a graded algebra, to $A \widehat{\otimes}_R J_L$, but its differential is a deformation of $d_A \otimes 1$.

Lemma 4.3.1. *The algebra morphism* $R_2 \to C^\bullet(L_1, J_L)$ *is a quasi-isomorphism. Hence so is* $R_2 \to B$.

Moreover $\mathrm{Der}_A(B) = \Omega_{\widetilde{R}} \widehat{\otimes}_{\Omega_R} C^\bullet\big(L_1, \mathrm{Der}_{R_1}(J_L)\big)$ and one similarly has:

Lemma 4.3.2. *The Lie algebroid morphism* $L_2 \to \mathrm{Der}_A(B)$ *is a quasi-isomorphism.*

The main consequence of this is that we have the following results from [CVdB1, §4.3] and [CRVdB1, §4.4-4.6] (which are stated for $\widetilde{R} = R^{\mathrm{aff},L}$ being the affine coordinate ring of L):

Theorem 4.3.3 ([CRVdB1], Theorem 4.5)**.** *There is a quasi-isomorphism of calculi as in the following commutative diagram*[8]*:*

$$\begin{array}{ccc}
T_{\mathrm{poly},L} = T_{\mathrm{poly},L_2} & \hookrightarrow & T_{\mathrm{poly},\mathrm{Der}_A(B)} \\
\wr\downarrow & & \wr\downarrow \\
C^{-\bullet}(L) = C^{-\bullet}(L_2) & \hookrightarrow & C^{-\bullet}\big(\mathrm{Der}_A(B)\big) = \Omega^{-\bullet}_{B/A},
\end{array}$$

the vertical arrows denoting the contraction and Lie derivative.

Theorem 4.3.4 ([CRVdB1], Theorem 4.7)**.** *There is a quasi-isomorphism of calculi up to homotopy as in the following commutative diagram:*

$$\begin{array}{ccc}
D_{\mathrm{poly},L} = D_{\mathrm{poly},L_2} & \hookrightarrow & D_{\mathrm{poly},\mathrm{Der}_A(B)} \\
\wr\downarrow & & \wr\downarrow \\
C^{\mathrm{poly},L} = C^{\mathrm{poly},L_2} & \hookrightarrow & C^{\mathrm{poly},\mathrm{Der}_A(B)}
\end{array}$$

the vertical arrows denoting the contraction and Lie derivative.

Remark 4.3.5. These results are only stated for the precalculus structures in [CRVdB1], but the additional compatibility is immediate from the way we presented things.

4.3.2. The Maurer-Cartan form

Observe that $X^{\mathrm{aff}} \to X$ does not only have contractible fibers, but also provides a parameter space for the deformation to the normal cône of the diagonal: $X^{\mathrm{aff}} \underset{X}{\times} \widehat{X \times X} \cong X^{\mathrm{aff}} \underset{X}{\times} \widehat{TX}$.

This motivates the following additional assumption we make on $\widetilde{R}$ from now: *there is an isomorphism of* $\widetilde{R}$*-algebras* $t : \widetilde{R} \otimes_R J_L \xrightarrow{\sim} \widetilde{R} \otimes_R \widehat{S}_R(L^*)$. There is a universal solution $R^{\mathrm{aff},L}$ to this problem that has been

[7] Notice that the morphism $J_L \to J_L$ involved is NOT the identity.

[8] Ω_A, for a topological $\mathbf{k}$-algebra A, denotes the continuous De Rham complex. A similar convention holds for an extension of topological algebras B/A.

constructed in [CVdB1] (see also [41] for the case $L = \mathrm{Der}(R)$). It automatically satisfies the requirement that $\Omega_R \to \Omega_{R^{\mathrm{aff},L}}$ is a quasi-isomorphism.

The isomorphism t extends to an isomorphism $B \xrightarrow{\sim} A\widehat{\otimes}_R\widehat{S}_R(L^*)$ and hence provides us with a derivation

$$\omega := t \circ d_B \circ t^{-1} - d_A \otimes 1$$

which satisfies the Maurer-Cartan equation $[d_A \otimes 1, \omega] + \frac{1}{2}[\omega, \omega] = 0$. Any R-linear isomorphism $s : L \tilde{\to} R \otimes V$ induces an A-linear isomorphism $A\widehat{\otimes}_R\widehat{S}_R(L^*) \cong A\widehat{\otimes}\widehat{S}(V^*)$ which allows one to view ω as a Maurer-Cartan element ω_s in $A\widehat{\otimes}\mathrm{Der}\big(\widehat{S}(V^*)\big)$. As usual, the difference $\omega_s - \omega_{s'}$ lies in $A\widehat{\otimes}\mathrm{Der}_{\mathtt{lin}}\big(\widehat{S}(V^*)\big)$, where $\mathrm{Der}_{\mathtt{lin}}\big(\widehat{S}(V^*)\big) = V^* \otimes V$ is the Lie subalgebra of linear vector fields.

This allows to lift any universal construction/calculation[9] done in local coordinates on V and which is invariant by linear change of these to L.

The Maurer-Cartan form represents the Atiyah class

From what we have done in Subsection 4.3.1, we get that the following two exact sequences represent the same element in $\mathrm{Ext}^1_R(L, L^* \otimes_R L)$:

$$0 \to L^* \otimes_R L \to J^1_L(L) \to L \to 0 \qquad \text{and}$$

$$0 \to \Omega^1_{B/A} \otimes_B \mathrm{Der}_A(B) \to J^1_{\mathrm{Der}_A(B)}\big(\mathrm{Der}_A(B)\big) \to \mathrm{Der}_A(B) \to 0\,. \tag{4.1}$$

Notice that, using an isomorphism $s : B \xrightarrow{\sim} A\widehat{\otimes}\widehat{S}(V^*)$, (4.1) can be obtained by applying $A\widehat{\otimes}-$ to

$$0 \to \Omega^1_V \otimes_{\mathcal{O}_V} \mathfrak{X}_V \to J^1_{\mathfrak{X}_V}(\mathfrak{X}_V) \to \mathfrak{X}_V \to 0\,,$$

and then adding ω_s to the differential. This very last sequence splits thanks to the canonical connection $\nabla_V : \mathfrak{X}_V \to \Omega^1_V \otimes_{\mathcal{O}_V} \mathfrak{X}_V$ on the affine space V. Therefore the class of (4.1) is represented by $[\nabla_V, \omega_s\cdot]$.

To conclude, observe that Theorem 4.2.5 follows from the above discussion combined with Theorems 4.1.3 and 4.1.4 applied to $\mathfrak{a} = A$, together with Theorems 4.3.3 and 4.3.4.

Sheafification

All our constructions are functorial w.r.t. Lie algebroid morphisms $(R, L) \to (R', L')$ that are such that the induced map $R' \otimes_R L \to L'$ is an isomorphism of R'-modules. See [CRVdB1][§5.6].

The sheaf $\mathcal{L}$ being locally free we can replace X with the subsite of $\mathsf{Open}(X)$ consting of these open subsets U over which $\mathcal{L}_{|U}$ is a free $\mathcal{R}_{|U}$-module. This does not change the category of sheaves. For such $U \subset U'$ one gets that the natural map $\mathcal{R}(U') \otimes_{\mathcal{R}(U)} \mathcal{L}(U) \to \mathcal{L}(U')$ is an isomorphism. We are done.

[9] By "universal" we mean that it only makes use of the natural structures on Hochschild (co)chains, polyvectors and differential forms.

5. Lie theory of closed embeddings

In [2] Arinkin and Căldăraru gave a necessary and sufficient condition for the Ext-algebra of a closed subvariety X of an algebraic variety Y to be isomorphic, as an object of the derived category of X, to $\mathsf{S}(\mathsf{N}[-1])$, where N is the normal bundle of X into Y ; the condition is that N can be lifted to the first infinitesimal neighbourhood $\mathsf{X}^{(1)}$. This is equivalent to the vanishing of a certain class in $\mathrm{Ext}^2_{\mathcal{O}_\mathsf{X}}(\mathsf{N}^{\otimes 2}, \mathsf{N})$. This result has been translated into Lie theory in [CCT1]: for an inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, we gave a necessary and sufficient condition for $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ to be isomorphic, as an $\mathfrak{h}$-module, to $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$; the condition is that the quotient module $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$ extends to a Lie algebra $\mathfrak{h}^{(1)}$ "sitting in between" $\mathfrak{h}$ and $\mathfrak{g}$. Similarly, this condition is equivalent to the vanishing of a certain class in $\mathrm{Ext}^1_{\mathfrak{h}}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$.

It is Kapranov who observed in [21] that the shifted tangent sheaf $\mathsf{T}_\mathsf{X}[-1]$ of an algebraic variety X is a Lie algebra object in the derived category of X, with Lie bracket being given by the Atiyah class of $\mathsf{T}_\mathsf{X}[-1]$. Moreover, any object of the derived category becomes a representation of this Lie algebra *via* its own Atiyah class. In the case of a closed embedding $\mathsf{i} : \mathsf{X} \hookrightarrow \mathsf{Y}$ we then get an inclusion of Lie algebras objects $\mathsf{T}_\mathsf{X}[-1] \subset \mathsf{i}^*\mathsf{T}_\mathsf{Y}[-1]$, so that the main result of [2] can be deduced, in principle, from a version of the main result of [CCT1] that would hold in a triangulated category. We refer to the introduction of [CCT1] and to [C1] for more details on this striking analogy.

This chapter describes a tiny part of a more general project which aims at building a dictionnary between Lie theory and algebraic geometry. The first Section presents a generalization to Lie algebroids of the main results in [CCT1], while in the second Section we study a Lie algebroid associated to a closed embedding.

5.1. PBW for inclusions of Lie algebroids

The material presented in this Section is mainly extracted from [C2], to which we refer for all proofs and technicalities.

We let X be a topological space equipped with a sheaf of algebras $\mathcal{R}$, and $\mathcal{A} \subset \mathcal{L}$ be an inclusion of sheaves of Lie algebroids over $\mathcal{R}$ (we refer to Section A.1 for standard Definitions). The $\mathcal{R}$-module $\mathcal{L}/\mathcal{A}$ turns out to be naturally equipped with an action of $\mathcal{A}$ (see §5.1.1), also-known-as a flat $\mathcal{A}$-connection.

In [8] Chen, Stiénon and Xu introduce a very interesting class $\alpha_\mathcal{E} \in \mathrm{Ext}^1_\mathcal{A}\big((\mathcal{L}/\mathcal{A}) \otimes_\mathcal{R} \mathcal{E}, \mathcal{E}\big)$, for any $\mathcal{A}$-module $\mathcal{E}$, which is the obstruction to the existence of a lift of the flat $\mathcal{A}$-connection on $\mathcal{E}$ to a possibly non-flat $\mathcal{L}$-connection. They define this class in geometric terms, while we provide below a purely algebraic description of $\alpha_\mathcal{E}$ (see §5.1.1) which makes sense in a wider context.

We also introduce a new Lie algebroid $\mathcal{A}^{(1)}$, called the *first infinitesimal neighbourhood Lie algebroid*, which fits in between $\mathcal{A}$ and $\mathcal{L}$ in the sense that we have a sequence of Lie algebroid morphisms $\mathcal{A} \longrightarrow \mathcal{A}^{(1)} \longrightarrow \mathcal{L}$.

Theorem 5.1.1. *Assume that the $\mathcal{R}$-linear epimorphism $\mathrm{S}_\mathcal{R}(\mathcal{L}) \to \mathrm{gr}\big(\mathrm{U}(\mathcal{L})\big)$ is an isomorphism and that the epimorphism $\mathrm{T}_\mathcal{R}(\mathcal{L}/\mathcal{A}) \to \mathrm{S}_\mathcal{R}(\mathcal{L}/\mathcal{A})$ splits within $\mathcal{A}$-modules. Then the following statements are equivalent:*

(1) The class $\alpha_{\mathcal{L}/\mathcal{A}}$ vanishes.

(2) The $\mathcal{A}$-module structure on $\mathcal{L}/\mathcal{A}$ lifts to an $\mathcal{A}^{(1)}$-module structure.

(3) $\mathsf{U}(\mathcal{L})/\mathsf{U}(\mathcal{L})\mathcal{A}$ is isomorphic, as a filtered $\mathcal{A}$-module, to $\mathrm{S}_\mathcal{R}(\mathcal{L}/\mathcal{A})$.

We can also prove a more general version of the above result for $\mathcal{A}$-modules other than $\mathbf{1}_\mathcal{A}$:

Theorem 5.1.2. *Let $\mathcal{E}$ be an $\mathcal{A}$-module which is faithful*[1]*. Then, under the very same assumptions as in the previous Theorem, the following statements are equivalent:*

(1) The classes $\alpha_{\mathcal{L}/\mathcal{A}}$ and $\alpha_\mathcal{E}$ vanish.

(2) The $\mathcal{A}$-module structures on $\mathcal{L}/\mathcal{A}$ and $\mathcal{E}$ lift to $\mathcal{A}^{(1)}$-module structures.

(3) $\mathsf{U}(\mathcal{L}) \underset{\mathsf{U}(\mathcal{A})}{\otimes} \mathcal{E}$ is isomorphic, as a filtered $\mathcal{A}$-module, to $\mathrm{S}_\mathcal{R}(\mathcal{L}/\mathcal{A}) \otimes_\mathcal{R} \mathcal{E}$.

[1] Here we mean that $\mathcal{E}$ is faithful as an $\mathcal{R}$-module, which ensures that $- \otimes_\mathcal{R} \mathcal{E} : \mathcal{A}\text{-mod} \to \mathcal{A}\text{-mod}$ is faithful (because the forgetful functor $\mathcal{A}\text{-mod} \to \mathcal{R}\text{-mod}$ is) and thus reflects exact sequences.

5.1.1. Structures associated to inclusions of Lie algebroids

Let R be a commutative algebra and $i : A \hookrightarrow L$ an inclusion of Lie algebroids over R. All constructions and results below sheafify without problem.

The A-module L/A

It is well-known that A does not necessarily act on itself (meaning that A is not an A-module in any natural way). In this paragraph we consider the quotient R-module L/A and define an A-action on it in the following way: for any $a \in A$ and any $l \in L$, we define $a \cdot (l + A) := [a, l] + A$ (when there is no ambiguity we omit the inclusion symbol i from the notation).

From now and in the rest of the Section we make the following assumption:

The map $L \longrightarrow U(L)$ *is a monomorphism.* $\qquad (\emptyset)$

The extension class α (inspired by Chen-Stiénon-Xu)

Let E be an A-module. We define a class $\alpha_E \in \mathrm{Ext}^1_A\big((L/A)\otimes_R E, E\big)$, which generalizes the one introduced in [CCT1] for Lie algebras, *via* the following short exact sequence of A-modules:

$$0 \longrightarrow E \longrightarrow \Big(U(L) \underset{U(A)}{\otimes} E\Big)^{\leq 1} \longrightarrow L/A \otimes_R E \longrightarrow 0\,. \tag{5.1}$$

We have to explain why the middle term in (5.1) is an A-module, which is *a priori* not guaranteed. Namely, even though $U(L)$ is an A-module (*via* left multiplication) its filtered pieces $U(L)^{\leq k}$ are not (because $AU(L)^{\leq k} \subset U(L)^{\leq k+1}$). Nevertheless, $U(L) \underset{U(A)}{\otimes} E$ turns out to be a filtered A-module.

We set $\alpha := \alpha_{L/A}$.

Relation to Atiyah classes as they are defined by Chen-Stiénon-Xu in [8]

We consider the filtered subspace $J_{L/A}(E)$ of $J_L(E)$ consisting of those maps $\phi : U(L) \to E$ which are A-linear: for $Q \in U(A)$ and $P \in U(L)$,

$$\phi(QP) = Q \cdot \phi(P)\,. \tag{5.2}$$

According to Remark A.1.3 there is a residual L-module structure on $J_{L/A}(E)$: for $Q \in U(L)$, we have $(Q * \phi)(P) = \phi(PQ)$. Even though the successive quotients $J^n_L(E) := \mathrm{Hom}_R\big(U(L)^{\leq n}, E\big)$ of $J_L(E)$ are not $U(L)$-bimodules, it turns out that their subspaces $J^n_{L/A}(E)$ inherits an A-action from the above residual L-action. We then have the following exact sequence of A-modules:

$$0 \longrightarrow \mathrm{Hom}_R(L/A, E) \longrightarrow J^1_{L/A}(E) \longrightarrow E \longrightarrow 0\,. \tag{5.3}$$

This determines a class $\widetilde{\alpha}_E \in \mathrm{Ext}^1_A\big(E, \mathrm{Hom}_R(A, E)\big)$, which has been first defined in a differential geometric context by Chen-Stiénon-Xu in [8, § 2.5.1].

Proposition 5.1.3. *The images of the classes* α_E *and* $\widetilde{\alpha}_E$ *coincide in*

$$\mathrm{Hom}_{D(A)}\big((L/A) \overset{\mathbb{L}}{\otimes}_R E, E[1]\big) \cong \mathrm{Hom}_{D(A)}\big(E, \mathbb{R}\mathrm{Hom}_R(L/A, E)[1]\big)\,,$$

where $D(A)$ *denotes the bounded derived category of* A*-modules.*

The first infinitesimal neighbourhood Lie algebroid $A^{(1)}$

Being a Lie algebroid over R, L is in particular an anchored R-module. We can therefore consider the free Lie algebroid $FR(L)$ over R generated by L. Let us then consider the filtered quotient $A^{(1)}$ of $FR(L)$ by the ideal generated by[2]

$$[a, l]_{FR(L)} - [a, l]_L, \quad a \in A, l \in L\,.$$

[2]Observe that, contrary to what is suggested by the notation, $A^{(1)}$ does not only depend on A but also on L.

Observe that it is a well-defined (Lie algebroid) ideal in $\mathsf{FR}(\mathsf{L})$ as the anchor map of $\mathsf{FR}(\mathsf{L})$ coincides by definition with the one of L on generators. We call $\mathsf{A}^{(1)}$ the *first infinitesimal neighbourhood* of A (see [CCT1], where the geometric motivation behind such a denomination is given). We denote by j the Lie algebroid inclusion of A into $\mathsf{A}^{(1)}$. One can prove that, for an A-module E, the classes α_E and $\widetilde{\alpha}_\mathsf{E}$ both give the obstruction to lift E to an $\mathsf{A}^{(1)}$-module:

Proposition 5.1.4. *Let E be an A-module. Then the following statements are equivalent:*
(1) There exists an $\mathsf{A}^{(1)}$-module $\mathsf{E}^{(1)}$ such that $\mathsf{j}^*\big(\mathsf{E}^{(1)}\big) = \mathsf{E}$.
(2) $\alpha_\mathsf{E} = 0$.
(3) $\widetilde{\alpha}_\mathsf{E} = 0$.

5.1.2. Sketch of the proof of Theorem 5.1.1

Our goal is to understand the filtered A-module

$$\mathsf{i}^*\mathsf{i}_!(\mathbf{1}_\mathsf{A}) := \mathsf{U}(\mathsf{L}) \underset{\mathsf{U}(\mathsf{A})}{\otimes} \mathbf{1}_\mathsf{A} = \mathsf{U}(\mathsf{L})/\mathsf{U}(\mathsf{L})\mathsf{A}\,.$$

We write $\mathsf{G}^k := \big(\mathsf{i}^*\mathsf{i}_!(\mathbf{1}_\mathsf{A})\big)^{\leq k}$, and notice that this filtration always splits at 0-th order.

PBW for the inclusion into the first infinitesimal neighbourhood

In this § we sketch a proof of a version of the main Theorem for the inclusion $\mathsf{j}: \mathsf{A} \hookrightarrow \mathsf{A}^{(1)}$. It follows very much and hopefully simplifies the one of Darij Grinberg [18] for Lie algebras, who was himself inspired by [CCT1]. Our goal is to understand filtered the A-module

$$\mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A}) := \mathsf{U}\big(\mathsf{A}^{(1)}\big) \underset{\mathsf{U}(\mathsf{A})}{\otimes} \mathbf{1}_\mathsf{A} = \mathsf{U}\big(\mathsf{A}^{(1)}\big)/\mathsf{U}\big(\mathsf{A}^{(1)}\big)\mathsf{A}\,,$$

where the filtration comes from the one on $\mathsf{A}^{(1)}$ (see Remark A.1.5). It is worth noticing that the A-module structure on $\mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A})$ is compatible with the induced filtration $\mathsf{F}^k := \mathsf{F}^k\big(\mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A})\big)$, which always splits at 0-th order. We use the notation $\mathsf{F}^k(-)$ when we deal with filtrations induced by the one on the free Lie algebroids, as opposed to $(-)^{\leq k}$ which we keep for the ones induced by the "constant" filtration (see Appendix). According to § A.1.4 the associated graded algebra of the filtered R-algebra $\mathsf{U}\big(\mathsf{FR}(\mathsf{L})\big)$ is the tensor R-algebra $\mathsf{T}_\mathsf{R}(\mathsf{L})$. The filtered R-linear surjection $\xi: \mathsf{U}\big(\mathsf{FR}(\mathsf{L})\big) \to \mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A})$ therefore induces a graded R-linear map $\mathrm{gr}(\xi): \mathsf{T}_\mathsf{R}(\mathsf{L}) \to \mathrm{gr}\big(\mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A})\big)$. We shall also use the graded R-algebra surjection $\pi: \mathsf{T}_\mathsf{R}(\mathsf{L}) \to \mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$.

Theorem 5.1.5. *The class $\alpha = \alpha_{\mathsf{L}/\mathsf{A}}$ vanishes if and only if there exists an isomorphism of filtered A-modules $\varphi: \mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A}) \longrightarrow \mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$ such that $\mathrm{gr}(\varphi \circ \xi) = \pi$. Moreover, when this happens one can choose φ so that it is $\mathsf{A}^{(1)}$-linear.*

Very sketchy proof. For the "if" part one just has to meditate for a few seconds on the following commutative diagram of A-modules, in which all rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{L}/\mathsf{A} & \longrightarrow & \mathsf{F}^2/\mathsf{F}^0 & \longrightarrow & \mathsf{F}^2/\mathsf{F}^1 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathsf{L}/\mathsf{A} & \longrightarrow & \mathsf{F}^1\Big(\mathsf{U}\big(\mathsf{A}^{(1)}\big) \underset{\mathsf{U}(\mathsf{A})}{\otimes} (\mathsf{L}/\mathsf{A})\Big) & \longrightarrow & (\mathsf{L}/\mathsf{A}) \otimes_\mathsf{R} (\mathsf{L}/\mathsf{A}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathsf{L}/\mathsf{A} & \longrightarrow & \Big(\mathsf{U}(\mathsf{L}) \underset{\mathsf{U}(\mathsf{A})}{\otimes} (\mathsf{L}/\mathsf{A})\Big)^{\leq 1} & \longrightarrow & (\mathsf{L}/\mathsf{A}) \otimes_\mathsf{R} (\mathsf{L}/\mathsf{A}) & \longrightarrow & 0
\end{array} \tag{5.4}$$

and conclude that the filtration on $\mathsf{j}^*\mathsf{j}_!(\mathbf{1}_\mathsf{A})$ splits at 1-st order if and only if $\alpha = 0$.

For the "only if" part we now assume that $\alpha = 0$, which means that the A-action on L/A can be lifted to an $\mathsf{A}^{(1)}$-action. We therefore obtain a graded $\mathsf{A}^{(1)}$-module structure on $\mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$. We use the notation $\cdot$ for this action. For any $\mathsf{l} \in \mathsf{L}$ and any $\mathsf{P} \in \mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$ we now define $\mathsf{l} \bullet \mathsf{P} := \mathsf{l} \cdot \mathsf{P} + \bar{\mathsf{l}} \otimes \mathsf{P}$, $\bar{\mathsf{l}}$ being the class of l in L/A.

Lemma 5.1.6. *The operation $\bullet$ defines a filtered $\mathsf{FR}(\mathsf{L})$-module structure on $\mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$ such that:*
(i) It actually descends to an $\mathsf{A}^{(1)}$-module structure.
(ii) Its restriction to A is the original A-module structure on $\mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$.

We therefore obtain a filtered morphism of $\mathsf{A}^{(1)}$-modules

$$\varphi : \mathsf{U}\big(\mathsf{A}^{(1)}\big)/\mathsf{U}\big(\mathsf{A}^{(1)}\big)\mathsf{A} \longrightarrow \mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})\,, \quad \mathsf{P} \longmapsto \mathsf{P}\bullet 1\,,$$

which can be proven to be an isomorphism. □

End of the proof

Let us start with the following:

Proposition 5.1.7. *The filtration on $\mathsf{i}^*\mathsf{i}_!(\mathbf{1}_\mathsf{A})$ splits at 1-st order if and only if $\alpha = 0$.*

Proof. The Proposition again follows from some diagramatic meditation (and again, rows are exact):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{L}/\mathsf{A} & \longrightarrow & \Big(\mathsf{U}(\mathsf{L}) \underset{\mathsf{U}(\mathsf{A})}{\otimes} (\mathsf{L}/\mathsf{A})\Big)^{\leq 1} & \longrightarrow & (\mathsf{L}/\mathsf{A}) \otimes_\mathsf{R} (\mathsf{L}/\mathsf{A}) & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow & & \\
0 & \longrightarrow & \mathsf{G}^1/\mathsf{G}^0 & \longrightarrow & \mathsf{G}^2/\mathsf{G}^0 & \longrightarrow & \mathsf{G}^2/\mathsf{G}^1 & \longrightarrow & 0
\end{array}$$

Here $\psi : \mathsf{U}(\mathsf{L}) \underset{\mathsf{U}(\mathsf{A})}{\otimes} (\mathsf{L}/\mathsf{A}) \longrightarrow \mathsf{U}^+(\mathsf{L})/\mathsf{U}^+(\mathsf{L})\mathsf{A}$ is an A-module morphism which increases the filtration degree by one, where $\mathsf{U}^+(\mathsf{L}) = \ker(\epsilon) \cong \mathsf{U}(\mathsf{L})/\mathsf{R}$ is equipped with the induced filtration. □

It remains to prove that, under the assumptions of Theorem 5.1.1, if the filtration on $\mathsf{i}^*\mathsf{i}_!(\mathbf{1}_\mathsf{A})$ splits at 1-st order then it splits, which can be done (and has been done in [C2]) by using the splitting $\mathrm{S}_\mathsf{R}(\mathsf{L}/\mathsf{A}) \to \mathrm{T}_\mathsf{R}(\mathsf{L}/\mathsf{A})$ together with Theorem 5.1.5.

Remark 5.1.8. The assumption (Ø) is *a priori* weaker than asking that $\mathrm{S}_\mathsf{R}(\mathsf{L}) \to \mathrm{gr}\big(\mathsf{U}(\mathsf{L})\big)$ is an isomorphism. But I would be tempted to conjecture that they are actually equivalent.

5.1.3. Sketch of proof of Theorem 5.1.2

Let now E be an A-module, and consider the following two A-modules:

$$\mathsf{j}^*\mathsf{j}_!(\mathsf{E}) := \mathsf{U}\big(\mathsf{A}^{(1)}\big) \underset{\mathsf{U}(\mathsf{A})}{\otimes} \mathsf{E} \qquad \text{and} \qquad \mathsf{i}^*\mathsf{i}_!(\mathsf{E}) := \mathsf{U}(\mathsf{L}) \underset{\mathsf{U}(\mathsf{A})}{\otimes} \mathsf{E}\,.$$

We denote by F^n_E and G^n_E the filtration pieces on those two filtered A-modules[3]. One sees that

$$\mathsf{F}^0_\mathsf{E} = \mathsf{G}^0_\mathsf{E} = \mathsf{E}\,, \quad \mathsf{F}^1_\mathsf{E} = \mathsf{G}^1_\mathsf{E} = \Big(\mathsf{U}(\mathsf{L}) \underset{\mathsf{U}(\mathsf{A})}{\otimes} \mathsf{E}\Big)^{\leq 1}\,, \quad \text{and} \quad \mathsf{F}^1_\mathsf{E}/\mathsf{F}^0_\mathsf{E} = \mathsf{G}^1_\mathsf{E}/\mathsf{G}^0_\mathsf{E} = (\mathsf{L}/\mathsf{A}) \otimes_\mathsf{R} \mathsf{E}\,.$$

Therefore, if the filtration on either $\mathsf{j}^*\mathsf{j}_!(\mathsf{E})$ or $\mathsf{i}^*\mathsf{i}_!(\mathsf{E})$ splits then $\alpha_\mathsf{E} = 0$. We start with the following generalization of Theorem 5.1.5:

Theorem 5.1.9. *Assume that E is faithful. Then both classes α and α_E vanish if and only if there exists an isomorphism filtered A-modules $\varphi_\mathsf{E} : \mathsf{j}^*\mathsf{j}_!(\mathsf{E}) \longrightarrow \mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A}) \otimes_\mathsf{R} \mathsf{E}$ such that $\mathrm{gr}\big(\varphi_\mathsf{E} \circ \xi_\mathsf{E})\big) = \pi_\mathsf{E}$.*

Here $\xi_\mathsf{E} : \mathsf{U}\big(\mathsf{FR}(\mathsf{L})\big) \underset{\mathsf{R}}{\otimes} \mathsf{E} \overset{\xi\otimes\mathrm{id}_\mathsf{E}}{\longrightarrow} \mathsf{U}\big(\mathsf{A}^{(1)}\big) \underset{\mathsf{U}(\mathsf{A})}{\otimes} \mathsf{E}$ and $\pi_\mathsf{E} : \mathsf{T}_\mathsf{R}(\mathsf{L}) \otimes_\mathsf{R} \mathsf{E} \overset{\pi\otimes\mathrm{id}_\mathsf{E}}{\longrightarrow} \mathsf{T}_\mathsf{R}(\mathsf{L}/\mathsf{A}) \otimes_\mathsf{R} \mathsf{E}$.

[3]Even though the filtered pieces of $\mathsf{U}\big(\mathsf{A}^{(1)}\big)$ and $\mathsf{U}(\mathsf{L})$ are not A-modules, F^n_E and G^n_E are. Namely, for any $a \in \mathrm{A}$ and any $\mathsf{P} \otimes e$ in F^n_E (resp. G^n_E), $a(\mathsf{P} \otimes e) = a\mathsf{P} \otimes e = ([a, \mathsf{P}] - \mathsf{P}a) \otimes e = [a, \mathsf{P}] \otimes e - \mathsf{P} \otimes ae \in \mathsf{F}^n_\mathsf{E}$ (resp. G^n_E).

Sketch of Proof. From the above we can assume that $\alpha_E = 0$, which means that the A-module E lifts to an $A^{(1)}$-module $E^{(1)}$. This allows one to construct a surjective filtered morphism of $A^{(1)}$-modules

$$\eta_E : j_!(E) \longrightarrow j_!(1_A) \otimes_R E^{(1)}\,; \quad P \otimes e \longmapsto P \cdot (1 \otimes e)\,, \quad \big(P \in U(A^{(1)}) \text{ and } e \in E\big)\,.$$

It is well-defined: for any $a \in A$, $(Pa) \cdot (1 \otimes e) = P \cdot (a \otimes e + 1 \otimes ae) = P \cdot (1 \otimes ae)$.

We have the following commutative diagram of A-modules in which lines are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (L/A) \otimes_R E & \longrightarrow & F_E^2/F_E^0 & \longrightarrow & F_E^2/F_E^1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (L/A) \otimes_R E & \longrightarrow & (F^2/F^0) \otimes_R E & \longrightarrow & (F^2/F^1) \otimes_R E & \longrightarrow & 0 \end{array} \tag{5.5}$$

If the filtration $(F_E^n)_{n\geq 0}$ splits (in A-mod) then so does the top line in the above diagram, and thus the bottom line splits too (this is because the rightmost vertical arrow in (5.5) is surjective). Faithfulness of E ensures that $0 \to F^1/F^0 \to F^2/F^0 \to F^2/F^1 \to 0$ splits, which implies that $\alpha = 0$.

Conversely, if we assume that $\alpha = 0$ then by Theorem 5.1.5 we get a surjective morphism of filtered A-modules $\varphi_E := (\varphi \otimes \mathrm{id}_E) \circ j^*\eta_E : j^*j_!(E) \longrightarrow T_R(L/A) \otimes_R E$, which can be proven to be an isomorphism. □

We refer to [C2] for the end of the proof of Theorem 5.1.2, which is quite similar to the one of Theorem 5.1.1.

5.2. The Lie algebroid of a closed embedding

In this Section, which is extracted from [CCT2], we assume that $\mathbf{k}$ is a field of characteristic zero.

5.2.1. A short review of the relative tangent complex

We collect some definitions and facts about dg-schemes after [9], to which we refer for more details. A *dg-$\mathbf{k}$-scheme* is a pair $X = (X_0, \mathcal{O}_X)$, where X_0 is an ordinary $\mathbf{k}$-scheme and $\mathcal{O}_X$ is a sheaf of dg-algebras on X_0 which is non-positively graded and such that

1. $\mathcal{O}_X^0 = \mathcal{O}_{X_0}$;
2. $\mathcal{H}^{-i}(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_{X_0}$-modules for any $i \geq 0$.

A dg-$\mathbf{k}$-scheme X is said *quasi-projective* if it is of finite type and if X_0 is quasi-projective.

A morphism $f : X \to Y$ of dg-$\mathbf{k}$-schemes (the definition of which is obvious) is *smooth* if f_0 is smooth and $\mathcal{O}_X^\sharp$ is locally isomorphic to $S_{\mathcal{O}_{X_0}}(E) \otimes_{\mathcal{O}_{X_0}} f_0^{-1}\mathcal{O}_Y^\sharp$, where E is a finite dimensional negatively graded vector bundle on X_0. A dg-$\mathbf{k}$-scheme is said *smooth* if so is the morphism $X \to \mathrm{pt} = \mathrm{Spec}(\mathbf{k})$.

It is known (see [9, (2.7.6)]) that any morphism $f : X \to Y$ factors through a quasi-isomorphic closed embedding $j : X \to \widetilde{X}$ followed by a smooth morphism $\widetilde{f} : \widetilde{X} \to Y$.

The relative *tangent complex* $\mathbb{T}_{X/Y}$ of a morphism of quasi-projective dg-schemes $f : X \to Y$ is defined as $j^*T_{\widetilde{X}/Y}$, where j and $\widetilde{X}$ are given by the above factorization. The restriction morphism $T_{\widetilde{X}/Y} \to j_*j^*T_{\widetilde{X}/Y} = j_*\mathbb{T}_{X/Y}$ is a quasi-isomorphism.

For any sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ one has a triangle

$$\mathbb{T}_{X/Y} \longrightarrow \mathbb{T}_{X/Z} \longrightarrow \mathbf{L}f^*\mathbb{T}_{Y/Z} \xrightarrow{+1} \tag{5.6}$$

Observe that the pair $(\mathcal{O}_{\widetilde{X}}, T_{\widetilde{X}/Y})$ is naturally a dg-Lie algebroid in the category of dg-$\mathcal{O}_Y$-modules. Our main goal in this Section is to get a geometric interpretation of the natural objects associated with this Lie algebroid: its Chevalley-Eilenberg complex, its universal envelopping algebra, and its jet algebra. In particular we will argue that the jet algebra is the function algebra on the formal neighbourhood of the identity section of the derived groupoid scheme $X \overset{\mathbf{R}}{\underset{Y}{\times}} X$. We will nevertheless focus on the case of a closed embedding of ordinary smooth algebraic varieties.

The case of a closed embedding of smooth algebraic varieties

Let $i : X \hookrightarrow Y$ be a closed embedding of closed algebraic varieties, which we assume to be quasi-projective to ensure existence of resolutions by locally free sheaves on Y. In this case $\widetilde{X}_0$ can be taken to be Y. Comparing Comparing (5.6) in the case $Z = *$ with the normal bundle exact sequence

$$0 \longrightarrow T_X \longrightarrow i^*T_Y \longrightarrow N \longrightarrow 0$$

one gets that $\mathbb{T}_{X/Y} \cong N[-1]$ in $\mathbf{D}^b_{coh}(X)$. In particular, this shows that the pair $(i_*\mathcal{O}_X, i_*N[-1])$ is a Lie algebroid object in $\mathbf{D}^b_{coh}(Y)$ (which happends to be set-theoretically supported on X).

From now we will remain within this restrictive framework for simplicity (even though sone of the results sill hold in general).

5.2.2. A Lie theoretic description of $\mathcal{E}xt_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$

By the universal property of $U(T_{\widetilde{X}/Y})$ we get a morphism $U(T_{\widetilde{X}/Y}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})$ of dg-algebras in $\mathcal{O}_{\widetilde{X}}$-bimodules over $\mathcal{O}_Y$.

Remark 5.2.1. To be precise one should in principle consider $\pi_*\mathcal{O}_{\widetilde{X}}$, but in this case π_* is a fairly innocent functor, and $\pi_*\mathcal{O}_{\widetilde{X}}$ is just $\mathcal{O}_{\widetilde{X}}$ itself with its $\mathcal{O}_Y$-module structure.

Proposition 5.2.2. *The above morphism $U(T_{\widetilde{X}/Y}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})$ is a quasi-isomorphism. In particular we have an isomorphism of algebras $U(i_*N[-1]) \to \mathcal{E}xt_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$ in $\mathbf{D}^b_{coh}(Y)$.*

Main idea of the proof. In fact the map $U(T_{\widetilde{X}/Y}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})$ exists for any morphism of dg-schemes and actually gives an *isomorphism* from $U(T_{\widetilde{X}/Y})$ to the dg-algebra of relative differential operators $\mathcal{D}iff_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}) \subset \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})$. The only thing that is specific to closed embeddings is that this inclusion is a quasi-isomorphism[4]. ☐

Observe that $U(i_*N[-1])$ can be seen as an object of $\mathbf{D}^b_{coh}(X \times X)$, set-theoretically supported on the diagonal. The above Proposition tells us that it is the kernel representing the functor $i^*i_! : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(X)$. The monad structure on $i^*i_!$ coming form the product on $U(i_*N[-1])$ easily identifies with the one coming from the projection formula $i^*i_!i^*i_! \Rightarrow i^*i_!$.

Geometric interpretation of the jet algebra and Hopf monads

We have a similar (and somehow dual) picture for the jet algebra.

Since $T_{\widetilde{X}/Y}$ is a Lie algebroid of relative derivations, then its jet algebra $J_{T_{\widetilde{X}/Y}}$ is *isomorphic* to the adic-completed tensor product $\mathcal{O}_{\widetilde{X}} \widehat{\otimes}_{\mathcal{O}_Y} \mathcal{O}_{\widetilde{X}}$, where the adic completion is taken w.r.t. the kernel of the multiplication map $\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$. Notice that this is an isomorphism of *cogroupoid objects*. All this is actually true for any morphism of dg-schemes $X \to Y$. What is specific to closed embeddings comes in the following:

Proposition 5.2.3. *The morphism $\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}} \widehat{\otimes}_{\mathcal{O}_Y} \mathcal{O}_{\widetilde{X}}$ is a quasi-isomorphism.*

This means that $J_{i_*N[-1]}$, viewed as an object of $\mathbf{D}^b_{coh}(X \times X)$, is the kernel representing the functor $i^*i_* : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(X)$. Moreover, the cogroupoid structure on the jet algebra induces a Hopf comonad structure on i^*i_*. It coincides with the one coming from the fact that i^* is strong monoidal and left adjoint to i_*.

We know from A.2 that the functor associated to the kernel $U(i_*N[-1])$ is left adjoint to the functor associated to the kernel $J_{i_*N[-1]}$, and thus inherits a Hopf monad structure.

We also know that $i^*i_!$ is also left adjoint to i^*i_* and as such also inherits the structure of a Hopf monad (determined by the fact that i^* is strong monoidal and right adjoitn to $i_!$).

All in all, this tells us that the bialgebroid $U(i_*N[-1])$ is the kernel representing the Hopf monad $i^*i_!$.

[4]This follows from the following two facts: $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ vanishes for $i \gg 0$ and the filtration coincides with the (cohomological) degree in cohomology.

5.2.3. A Lie theoretic description of the formal neighbourhood

The Chevalley-Eilenbeg complex $\mathsf{C}(\mathsf{T}_{\widetilde{X}/Y})$ naturally identifies with the completed relative de Rahm complex $\widehat{\Omega}_{\widetilde{X}/Y}$.

Theorem 5.2.4. *There is a quasi-isomorphisms of sheaves of complete dg-algebras on* Y

$$\phi\,:\,\mathsf{C}(\mathsf{T}_{\widetilde{X}/Y})\longrightarrow\widehat{\mathcal{O}_Y}\,,$$

where $\widehat{\mathcal{O}_Y}$ *is the adic-completion of* $\mathcal{O}_Y$ *along the ideal sheaf* $\mathcal{I}=\ker(\mathcal{O}_Y\to i_*\mathcal{O}_X)$ *of* X.

Short explanation for why this is true. First, the (non-central) dg-$\mathcal{O}_{\widetilde{X}}$-algebra $\mathsf{U}(\mathsf{T}_{\widetilde{X}/Y})$ is Koszul dual to the complete dg-$\mathcal{O}_{\widetilde{X}}$-algebra $\mathsf{C}(\mathsf{T}_{\widetilde{X}/Y})$. Then, we have seen that in the case of a closed embedding the morphism of non-central dg-$\mathcal{O}_{\widetilde{X}}$-algebras $\mathsf{U}(\mathsf{T}_{\widetilde{X}/Y})\longrightarrow\mathcal{E}xt_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}},\mathcal{O}_{\widetilde{X}})$ is a quasi-isomorphism. Finally, one can also show that the natural morphism $\mathcal{E}xt_{\widehat{\mathcal{O}_Y}}(\mathcal{O}_{\widetilde{X}},\mathcal{O}_{\widetilde{X}})\longrightarrow\mathcal{E}xt_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}},\mathcal{O}_{\widetilde{X}})$ is a quasi-isomorphism. □

Explicit construction of ϕ. We now sketch an explicit construction of the quasi-isomorphism ϕ, following [CCT2]. One first definse an $\mathcal{O}^\sharp_{\widetilde{X}}$-linear map $\phi:\Omega^1_{\widetilde{X}_\sharp/Y}[-1]\longrightarrow\mathcal{O}_Y$ by the composition

$$\Omega^1_{\widetilde{X}_\sharp/Y}[-1]\stackrel{-\iota_Q}{\longrightarrow}\mathcal{O}^\sharp_{\widetilde{X}}=\mathrm{S}_{\mathcal{O}_Y}(\mathsf{E})\stackrel{\epsilon}{\twoheadrightarrow}\mathcal{O}_Y\,,$$

where the last map is just the canonical $\mathcal{O}_Y$-augmentation. One can then easily prove the following:

Lemma 5.2.5. ϕ *has its image inside the ideal sheaf* $\mathcal{I}$.

By the universal property of symmetric algebras, the map ϕ induces a morphism of $\mathcal{O}^\sharp_{\widetilde{X}}$-algebras

$$\Omega^*_{\widetilde{X}_\sharp/Y}=\mathrm{S}_{\mathcal{O}^\sharp_{\widetilde{X}}}\left(\Omega^1_{\widetilde{X}_\sharp/Y}[-1]\right)\longrightarrow\mathcal{O}_Y\,.$$

We shall still denote this map by ϕ. An explicit computation shows that:

Lemma 5.2.6. ϕ *is a morphism of* $\mathcal{O}_{\widetilde{X}}$*-algebras. In other words, it is a cochain map.*

By Lemma 5.2.5 the map ϕ sends the augmentation ideal (which is generated by $\Omega^1_{\widetilde{X}/Y}[-1]$) into the ideal $\mathcal{I}$. Hence, $\mathsf{C}(\mathsf{T}_{\widetilde{X}/Y})$ being completem ϕ factors through $\mathsf{C}(\mathsf{T}_{\widetilde{X}/Y})\longrightarrow\widehat{\mathcal{O}_Y}$. We shall again still denote this map by ϕ.

It remains to be shown that it is a quasi-isomorphism. This can done by proving inductively that ϕ is a quasi-isomorphism on the successive quotients of the descending filtrations. □

Below we sketch very briefly two applications of the above result.

Application 1: obstruction to splittings

One can show that the dg-Lie algebroid structure on $(\mathcal{O}_{\widetilde{X}},\mathsf{T}_{\widetilde{X}/Y})$ induces a kind of minimal L_∞-algebroid structure on $(i_*\mathcal{O}_X,i_*\mathsf{N}[-1])$. It is actually a bit more subtile than that because homotopy transfert does not work perfectly well with sheaves, and we refer to [CCT2] for the details. For example the higher anchor ρ^k and higher brackets l^k are not cocycles and thus don't define maps in $\mathbf{D}^b_{coh}(Y)$.

Nevertheless, one has the following:

Proposition 5.2.7. *Assume that there is a splitting* $s_k:X_Y^{(k)}\to X$ *of the inclusion* $X\hookrightarrow X_Y^{(k)}$ *of* X *inside its* k*-th infinitesimal neighbourhood* $X_Y^{(k)}$ *in* Y*. Then the above* L_∞*-algebroid structure can chosen so that* $\rho^i=0$ *for* $0\le i\le k$*. Moreover in this case* ρ^{k+1} *is a coycle and thus defines a class* $r_{k+1}\in\mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathrm{S}^{k+1}_{\mathcal{O}_X}(\mathsf{N}),\mathsf{T}_X\big)$*, and additionally the following are equivalent:*

(i) there exists a splitting $s_{k+1}:X_Y^{(k+1)}\to X$ *of* $X\hookrightarrow X_Y^{(k+1)}$ *lifting* s_k*;*
(ii) $r_{k+1}=0$.

The class r_{k+1} is the same as the one appearing in [1, Proposition 2.2], where the same result is proven in the complex analytic setting.

Application 2: obstruction to linearization

Let us assume that there is an isomorphism $t_{k-1} : X_N^{(k-1)} \cong X_Y^{(k-1)}$. We would like to understand when t_{k-1} lifts to an isomorphism between $X_N^{(k)}$ and $X_Y^{(k)}$.

First note that the isomorphism t_{k-1} induces a splitting $s_{k-1} : X_Y^{(k-1)} \to X$. Hence we can use Proposition 5.2.7 to analyze the lifting of s_{k-1} to a splitting s_k, which should necessarily exist if t_{k-1} lifts. Thus in the following we assume that there is a splitting $s_k : X_Y^{(k)} \to X$ compatible with t_{k-1} in the sense that the induced splitting from t_{k-1} agrees with that of s_k.

Proposition 5.2.8. *Under the above assumption on* t_{k-1} *and* s_k *we can chose the* L_∞*-algebroid structure so that* $\rho^i = 0$ *for all* $0 \leq i \leq k$ *and* $l^i = 0$ *for all* $0 \leq i \leq k-1$*. Moreover* l^k *is a cocycle and thus defines a class* $\ell_k \in \mathrm{Ext}^1_{\mathcal{O}_X}\big(S^k_{\mathcal{O}_X}(N), N\big)$*, and additionally the following are equivalent:*

(i) the isomorphism t_{k-1} *lifts to an isomorphism* $t_k : X_N^{(k)} \cong X_Y^{(k)}$*;*

(ii) $\ell_k = 0$*.*

The class ℓ_k is the same as the one appearing in [1, Corollary 3.4 & Corollary 3.6], where the same result is proven in the complex analytic setting.

The existence of these classes r_k and ℓ_k can be formulated using the language of gerbes and stacks, see for example [20, Section 4], but its relationship with an L_∞-algebroid structure seems to be new.

A. Recollection on Lie algebroids

A.1. Lie algebroids and associated structures

What we discuss in this Section, which is extracted from [C2, Section 2], is relatively standard and can be found e.g. in [35, 44, 22] and references therein (perhaps phrased in a different way).

Let R be a commutative algebra and L a *Lie algebroid* over R, which means that the pair (R, L) is a Lie-Rinehart algebra (see [35]) in $\mathcal{C}$. Namely, L is a Lie algebra equipped with an R-module structure and an R-linear Lie algebra map $\rho : L \to \mathrm{Der}(R)$ such that $[l, rl'] = r[l, l'] + \rho(l)(r)l'$ for $l, l' \in L$ and $r \in R$. The map ρ is called the *anchor map* and we usually omit its symbol from the notation: for $l \in L$ and $r \in R$, we write $l(r) := \rho(l)(r)$. In particular $R \oplus L$ inherits the structure of a Lie algebra with bracket given by $[(r, l), (r', l')] = (l(r') - l'(r), [l, l'])$, for $r, r' \in R$ and $l, l' \in L$.

A.1.1. The universal enveloping algebra of a Lie algebroid

We define the enveloping algebra $U(R, L)$ of the pair (R, L) to be the quotient of positive part of the universal enveloping algebra[1] of the Lie algebra $R \oplus L$ by the following relations: $r \otimes l = rl$ ($r \in R$, $l \in R \oplus L$). As there is no risk of confusion we simply write $U(L)$ for $U(R, L)$, which is obviously an R-algebra *via* the natural map $R \longrightarrow U(L)$. It therefore inherits an R-bimodule structure.

It turns out that $U(L)$ is also a cocommutative coring in **left** R-modules[2]. Namely, the coproduct $\Delta : U(L) \longrightarrow U(L) \otimes_R U(L)$ is the multiplicative map defined on generators by $\Delta(r) = r \otimes 1 = 1 \otimes r$ ($r \in R$) and $\Delta(l) = l \otimes 1 + 1 \otimes l$ ($l \in L$). The anchor map can be extended to an R-algebra morphism $U(L) \longrightarrow \mathrm{End}(R)$ (actually taking values in the ring $\mathrm{Diff}(R)$ of differential operators) sending $r \in R$ to the multiplication by r and $l \in L$ to $\rho(l)$. The counit $\epsilon : U(L) \longrightarrow R$ is defined by $\epsilon(P) := P(1)$.

Remark A.1.1. The above definition of Δ needs some explanation. Being the quotient of $U(L) \otimes U(L)$ by the right ideal generated by $r \otimes 1 - 1 \otimes r$ ($r \in R$), $U(L) \otimes_R U(L)$ is **not** an algebra. Nevertheless, one easily sees that $r \otimes 1$ ($r \in R$) and $l \otimes 1 + 1 \otimes l$ ($l \in L$) sit in the normalizer of that ideal, so that multiplying them together makes perfect sense.

In what follows, left $U(L)$-modules are called L-modules. We say that a given (left) R-module E is acted on by L if it is equipped with an L-module structure of which the restriction to R gives back the original R-action we started with. The abelian category $L\text{-}\mathsf{mod}$ of L-modules is monoidal, with product being $\otimes_R$ (and $U(L)$ acting on a tensor product *via* the coproduct) and unit $\mathbf{1}_L$ being R equipped with the action given by the anchor ρ.

Any morphism $f : L \longrightarrow L'$ of Lie algebroids over R automatically induces a morphism of algebras $U(L) \longrightarrow U(L')$ which preserves all the above algebraic structures. We denote the restriction (or pull-back) functor $L'\text{-}\mathsf{mod} \longrightarrow L\text{-}\mathsf{mod}$ by f^*, and by $f_! := U(L') \underset{U(L)}{\otimes} -$ its left adjoint. Notice that f^* is monoidal, while $f_!$ is not ($f_!$ is only colax-monoidal).

There is a canonical filtration on $U(L)$ obtained by assigning degree 0, resp. 1, to elements of R, resp. L. All structures we have defined so far on $U(L)$ respect this filtration. If, additionally, L is itself equipped with a filtration, then this filtration extends to $U(L)$. The canonical filtration on $U(L)$ can be seen as coming from the obvious "constant" filtration on L (the only degree 0 element is 0 and all elements in L are of degree ≤ 1).

Remark A.1.2. One can alternatively describe the functor U as a left adjoint. Namely, we consider the category of *anchored algebras*: they are defined as R-algebras B equipped with an R-algebra morphism

[1] By this we mean the subalgebra generated by $R \oplus L$ (i.e. the kernel of the natural augmentation).

[2] We would like to warn the reader that the multiplication is defined on $U(L) \underset{R}{\otimes} U(L)$ while the comultiplication takes values in $U(L) \otimes_R U(L)$, where only the left R-module structure is used.

$\rho : B \longrightarrow \mathrm{End}(R)$, where the R-algebra structure on $\mathrm{End}(R)$ is the given by $r \longmapsto (l_r : b \mapsto rb)$. There is a functor Prim from anchored algebras to Lie algebroids that sends an anchored algebra B to the sub-R-module consisting of those elements $b \in B$ such that $\rho(b) \in \mathrm{Der}(R)$. We then have an adjuntion

$$\mathcal{U} : \{\text{Lie algebroids}\} \rightleftarrows \{\text{anchored algebras}\} : \mathsf{Prim}\,.$$

A.1.2. The de Rham complex of a Lie algebroid

To any L-module E we associate the complex of graded R-modules $C^\bullet(L, E)$, consisting of $\mathrm{Hom}_R(\wedge^\bullet_R L, E)$ equipped with the differential d defined as follows: for $\omega \in C^n(L, E)$ and $l_0, \dots, l_n \in L$,

$$d(\omega)(l_0, \dots, l_n) := \sum_{i=0}^{n} (-1)^i l_i \omega\big(l_0, \dots, \widehat{l_i}, \dots, l_n\big) + \sum_{i<j} (-1)^{i+j} \omega\big([l_i, l_j], l_0, \dots, \widehat{l_i}, \dots, \widehat{l_j}, \dots, l_n\big)\,.$$

The map that associates to $l \in L$ the element $\nabla_l \in \mathrm{End}(E)$ defined by $\nabla_l(e) := d(e)(l)$ is sometimes called a flat connection. It completely determines both the differential d and the L-action on E.

We have the following functoriality property: for $f : L \to L'$ a morphism of Lie algebroids over R and $\varphi : E \to F$ a morphism of L'-modules, we have an obvious R-linear map $f^*\varphi : C^\bullet(L', E) \longrightarrow C^\bullet(L, f^*F)$ defined by $(f^*\varphi)(\omega) := \varphi \circ \omega \circ f$. We also have that for any two L-modules E and F, there is a product $C^\bullet(L, E) \otimes C^\bullet(L, F) \longrightarrow C^\bullet(L, E \otimes_R F)$. In particular, this turns $C^\bullet(L) := C^\bullet(L, \mathbb{1}_L)$ into a differential graded commutative R-algebra.

A.1.3. Lie algebroid jets

For any L-module E we define the L-module $J_L(E)$ of L-*jets*, or simply *jets*, as the internal Hom $\mathrm{Hom}_R\big(\mathcal{U}(L), E\big)$ from the universal enveloping algebra $\mathcal{U}(L)$ to E.

This requires some explanation. First of all observe that the monoidal category $L\text{-}\mathsf{mod}$ is closed. The internal Hom of two L-modules E and F is given by the R-module $\mathrm{Hom}_R(E, F)$ equipped with the following L-action: for $l \in L$, $\psi : E \to F$ and $e \in E$, $(l \cdot \psi)(e) := l \cdot \big(\psi(e)\big) - \psi(l \cdot e)$. Then $\mathcal{U}(L)$ is naturally an L-module (being a left $\mathcal{U}(L)$-module over itself).

But $\mathcal{U}(L)$ is actually an $\mathcal{U}(L)$-bimodule. Therefore, $J_L(E)$ inherits a second left $\mathcal{U}(L)$-module structure, denoted $*$, which commutes with the above one and is defined in the following way: for $\phi \in J_L(E)$ and $P, Q \in \mathcal{U}(L)$, $(P * \phi)(Q) = \phi(QP)$. When $E = \mathbb{1}_L$ the two commuting L-module structures one gets on $J_L := J_L(E)$ are precisely the ones described in [CVdB1, §4.2.5].

Remark A.1.3. This is actually true for any $\mathcal{U}(L)$-$\mathcal{U}(L')$-bimodule M: the space $\mathrm{Hom}_R(M, E)$ has an L-module and an L'-module structures that commute[3]. In particular, the space $\mathrm{Hom}_{L\text{-}\mathsf{mod}}(M, E)$ itself is naturally an L'-module.

Theorem A.1.4. *The jet algebra J_L is a cogroupoid object in the category of complete adic-rings.*

Proof. This was originally stated in [22, (A.5.10)]. A proof can be found in [27, §3.4.1] (see also [CRVdB2, Appendix A]). □

A.1.4. Free Lie algebroids (after M. Kapranov)

Let us first recall from [22] that an R-module M is *anchored* if it is equipped with an R-linear map $\rho : M \longrightarrow \mathrm{Der}(R)$, called the *anchor map*. Like for Lie algebroids we usually omit the symbol ρ from our notation: for $m \in M$ and $r \in R$, we write $m(r) := \rho(m)(r)$. There is an obvious forgetful functor which goes from the category of Lie algebroids to that of anchored modules, that forgets everything except the underlying R-module structure of the Lie algebroid and the anchor map. This functor has a left adjoint, denoted FR.

For any anchored R-module M we call $\mathsf{FR}(M)$ the *free Lie algebroid* generated by M. It can be described in the following way, as a filtered quotient of the free Lie algebra $\mathsf{FL}(M)$ generated by M.

[3] Notice that even the two underlying R-module structures are different.

First of all, by adjunction ρ naturally extends to a Lie algebra morphism $\mathsf{FL}(M) \longrightarrow \mathrm{Der}(R)$. Then we define $\mathsf{FR}(M)$ as the quotient of $\mathsf{FL}(M)$ by the following relations: for $r \in R$, $m \in \mathsf{FL}(M)$, and $m' \in \mathsf{FL}(M)$, $[m, rm'] - [rm, m'] = m(r)m' + m'(r)m$. These relations being satisfied in $\mathrm{Der}(R)$ then ρ factors through $\mathsf{FR}(M)$. Finally, we define an R-module structure on $\mathsf{FR}(M)$ in the following way: $r[m, m'] := [m, rm'] - m(r)m' = [rm, m'] + m'(r)m$.

According to Remark A.1.2 we then have a sequence of adjunctions

$$\{\text{anchored modules}\} \underset{}{\overset{\mathsf{FR}}{\rightleftarrows}} \{\text{Lie algebroids}\} \underset{\mathrm{Prim}}{\overset{\mathsf{U}}{\rightleftarrows}} \{\text{anchored algebras}\}.$$

Remark A.1.5. The above sequence of adjunctions extends to filtered versions. Unless otherwise specified, the canonical filtration we put on an anchored module M is the "constant" one we already mentioned in §A.1.1. Then the associated graded of the induced filtration on $\mathsf{FR}(M)$ is the free Lie R-algebra $\mathsf{FL}_R(M)$ generated by M, and the associated graded of the induced filtration on $\mathsf{U}\big(\mathsf{FR}(M)\big)$ is $\mathsf{U}\big(\mathsf{FL}_R(M)\big) = \mathsf{T}_R(M)$.

A.2. Monoidal (co)monads associated to Lie algebroids

This Section is extracted from [CCT2].

The Hopf monad associated with the universal enveloping algebra

Let (R, L) be a Lie algebroid. We have seen that $\mathsf{U}(L)$ is a bialgebroid. It actually has a very specific feature: source and target maps $R \to \mathsf{U}(L)$ are the same. Therefore, the forgetful functor $U : \mathsf{U}(L)\text{-mod} \longrightarrow R\text{-mod}$ is *strong* monoidal (recall that $\mathsf{U}(L)$ being a bialgebroid its category of left modules is monoidal, see e.g. [4] and references therein)[4].

Observe that U has a left adjoint: $F : \mathsf{U}(L) \underset{R}{\otimes} - : R\text{-mod} \longrightarrow \mathsf{U}(L)\text{-mod}$. Moreover, U being strong monoidal, then its left adjoint F is *colax* monoidal and hence the monad $T := UF$ is a *Hopf monad* in the sense of [31]: it is a monad in the 2-category OpMon having monoidal categories as objects, colax monoidal functors as 1-morphisms and natural transformations of those as 2-morphisms.

The dual Hopf comonad associated with the jet algebra

Notice that the strong monoidal functor U also has a right adjoint $G := \mathrm{Hom}_{R\text{-mod}}\big(\mathsf{U}(L), -\big)$, which is *lax* monoidal. Going along the same lines as above one sees that $S := UG$ (which is right adjoint to T) is a *Hopf comonad*, meaning that it is a comonad in the 2-category Mon having monoidal categories as objects, lax monoidal functors as 1-morphisms and natural transformations of those as 2-morphisms.

Finally recall that $\mathrm{Hom}_{R\text{-mod}}\big(\mathsf{U}(L), -\big) \cong \mathrm{J}_L \widehat{\underset{R}{\otimes}} -$, where the R-bimodule structure on J_L is the one described in §A.1.3. Notice that, on the one hand, the lax monoidal structure on S is given by the coproduct on $\mathsf{U}(L)$, and thus by the product on J_L:

$$\Big(\mathrm{J}_L \widehat{\underset{R}{\otimes}} -\Big) \otimes_R \Big(\mathrm{J}_L \widehat{\underset{R}{\otimes}} -\Big) \cong \big(\mathrm{J}_L \otimes_R \mathrm{J}_L\big) \widehat{\underset{R\otimes R}{\otimes}} (- \otimes -) \Longrightarrow \mathrm{J}_L \widehat{\underset{R}{\otimes}} (- \otimes_R -).$$

On the other hand, the comonad structure on S is given by the product on $\mathsf{U}(L)$, and hence by the coproduct on J_L:

$$\mathrm{J}_L \widehat{\underset{R}{\otimes}} - \Longrightarrow \Big(\mathrm{J}_L \widehat{\underset{R}{\otimes}} \mathrm{J}_L\Big) \widehat{\underset{R}{\otimes}} - \cong \mathrm{J}_L \widehat{\underset{R}{\otimes}} \Big(\mathrm{J}_L \widehat{\underset{R}{\otimes}} -\Big).$$

[4] The forgetful functor usually goes to R-bimodules, but here its essential image is the monoidal subcategory consisting of those bimodules which have the same underlying left and right module structure. I t is isomorphic to the monoidal category of R-modules.

B. Analogies

I can't refrain from giving a long list of analogies between Lie theory and algebraic geometry. It seems that the first two subsets are now quite well-understood, while the last one remains a complete mystery (at least to me).

LIE THEORY	**ALGEBRAIC GEOMETRY**
Lie algebra object $\mathfrak{g}$ in $\mathfrak{g}$-mod	Lie algebra object $T_X[-1]$ in $\mathbf{D}(X)$
adjoint action (i.e. Lie bracket)	Atiyah class of $T_X[-1]$
characters of $\mathfrak{g}$	line bundles on X
trivial representation $\mathbb{1}_{\mathfrak{g}}$	trivial line bundle $\mathcal{O}_X$
universal envelopping algebra $U(\mathfrak{g})$	Hochschild cohomology sheaf $\mathcal{HH}_X^\bullet$
symmetrization map (PBW isomorphism)	Hochschild-Kostant-Rosenberg (HKR) isomorphism
$\mathfrak{g}$-invariants of a representation	(derived) global sections of an $\mathcal{O}_X$-module
Duflo element $\mathfrak{d} \in \hat{S}(\mathfrak{g}^*)^{\mathfrak{g}}$	Todd class $\mathtt{td} \in \oplus_k H^k(X, \Omega_X^k)$
Duflo isomorphism	Kontsevich isomorphism
inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$	closed embedding of algebraic varieties $i : X \hookrightarrow Y$
Lie algebra objects $\mathfrak{h}$ and $\mathfrak{g}$ in $\mathfrak{h}$-mod	Lie algebra objects $T_X[-1]$ and $i^*T_Y[-1]$ in $\mathbf{D}(X)$
$\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$	shifted normal bundle $N[-1]$ in $\mathbf{D}(X)$
exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ in $\mathfrak{h}$-mod	normal bundle exact sequence $0 \to T_X \to i^*T_Y \to N \to 0$
$\alpha_V : \mathfrak{g}/\mathfrak{h} \otimes V \longrightarrow \mathfrak{h}[1] \otimes V \longrightarrow V[1]$ for $V \in \mathfrak{h}$-mod	$\alpha_E : N \otimes E \longrightarrow T_X[1] \otimes E \xrightarrow{at_E} E[2]$, for $E \in \mathbf{D}(X)$
Lie algebra $\mathfrak{h}^{(1)}$ of §5.1.1	$X^{(1)}$, the first infinitesimal neighborhood of X into Y
$\alpha_V = 0 \Leftrightarrow V$ extends to $\mathfrak{h}^{(1)}$	$\alpha_E = 0 \Leftrightarrow E$ extends to $X^{(1)}$
Res : $\mathfrak{g}$-mod $\to \mathfrak{h}$-mod	$i^* : \mathbf{D}(Y) \to \mathbf{D}(X)$
Ind : $\mathfrak{h}$-mod $\to \mathfrak{g}$-mod	$i_! : \mathbf{D}(X) \to \mathbf{D}(Y)$
$U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h} = \mathrm{Res}(\mathrm{Ind}(\mathbb{1}_{\mathfrak{h}}))$ in $\mathfrak{h}$-mod	$i^*i_!(\mathcal{O}_X)$ in $\mathbf{D}(X)$
Calaque-Căldăraru-Tu Theorem [CCT1] (PBW type)	Arinkin-Căldăraru Theorem [2] (HKR type)
solvable Lie algebra	Calabi-Yau manifold
nilpotent Lie algebra	pure Calabi-Yau variety
abelian Lie algebra	complex torus
simple Lie algebra	irreducible holomorphic symplectic manifold
rank one simple Lie algebra ($\mathfrak{sl}_2$)	K3 surface
Lévi decomposition theorem	Bogomolov decomposition theorem

Notice that by *Calabi-Yau* I mean compact Kähler with holonomy in $SU(n)$ for $n \geq 3$, while a *pure Calabi-Yau variety* will be a complex projective X with $\dim(X) \geq 3$, trivial canonical bundle and $h^{2,0}(X) = 0$.

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