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HOMOLOGICAL PAIRS ON SIMPLICIAL MANIFOLDS

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH ZURICH

(Dr. sc. ETH Zurich)

presented by

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2017

Abstract

In this thesis we study the relation between Chen theory of formal homology connection, Universal Knizhnik–Zamolodchikov connection and Universal Knizhnik–Zamolodchikov-Bernard connection. In the first chapter, we give a summary of some results of Chen. In the second chapter we extend the notion of formal homology connection to simplicial manifolds. In particular, this allows us to construct formal homology connection on manifolds M equipped with a smooth/holomorphic properly discontinuos group action of a discrete group G. We prove that the monodromy represetation of that connection coincides with the Malcev completion of the group M/G. In the second chapter, we use this theory to produces holomorphic flat connections and we show that the universal Universal Knizhnik–Zamolodchikov-Bernard connection on the punctured elliptic curve can be constructed as a formal homology connection. Moreover, we produce an algorithm to construct such a connection by using the homotopy transfer theorem. In the third chapter, we extend this procedure for the configuration space of points of the punctured elliptic curve. Our approach is very general and it can be used to construct flat connections on more challenging manifolds equipped with a group action. For example it can be used for the configuration space of points of a higher genus Riemann surface.

Sommario

In questa tesi studiamo la relazione tra la teoria sviluppata da Chen sulle connessioni omologiche formali, la connessione universale Knizhnik–Zamolodchikov e la connessione universale Knizhnik–Zamolodchikov-Bernard. Nel primo capitolo diamo un breve sommario dei risultati di Chen qui usati. Nel secondo capitolo estendiamo la nozione di connessioni omologiche formali alle varietà simpliciali. Tale estensione ci permette di costruire una connessione omologica formale su una varietà M sulla quale agisce un gruppo discreto G in maniera liscia e propriamente discontinua. In questa tesi viene dimostrato che la rappresentazione monodromica della connessione di cui sopra, coincide con il completamento de quoziente M/G. Nel secondo capitolo, la teoria sviluppata nel capitolo precedente viene usata per costrure connessioni olomorfe, in particolare, dimostriamo che la connessione universale Knizhnik–Zamolodchikov-Bernard sulla curve ellittica puntata puo essere costruita come una connessione omologica formale di Chen. Inoltre, utilizzando il teorema del trasferimento omotopico, viene prodotto un algoritmo per la costruzione di tale connessione. Nel terzo capitolo, questo procedimento viene esteso allo spazio dei punti della curve ellittica puntata. L approccio utilizzato in questa tesi é molto generale e può essere usato per costruire connessioni piatte su un ampia classe di varietà equipaggiate di un azione di un gruppo.Per esempio, esso puó essere utilizzato per costruire una connessione piatta sullo spazio di configurazione dei punti su una superficie di Riemann di genere più alto.

Acknowledgments

I am deeply grateful to Damien Calaque and Giovanni Felder, for their continuous support and guidance, patience, mathematical discussions, many corrections and in particular their passion for mathematics. I want to thank Benjamin Enriquez for the interest he has shown in my research, and for accepting to co-referee this thesis. A special thank goes to Daniel Robert-Nicoud, Ricardo Campos for long discussions about operads and longer discussions about signs. I wish to thank Alex, Simon, Matteo, Martino, Gino, Alberto and Simone and all my colleagues of the group 5 for all the time they spent in reading and correcting this thesis. I want to thank my family (this include dogs and cats) and friends for their help and constant support. Finally, I wish to thank Elisa, most probably without her support this work would have not been possible.

Contents

Overview

The construction of formal homology connection goes back to Chen [13]. Given a smooth manifold M, a formal homology connection consists in a differential δ^* on the free algebra of $H_+(M)$ and a formal power series C with coefficients in the differential graded algebra of smooth differential forms on M such that C is a twisting cochain. In this thesis we call (C, δ^*) a *homological pair*. In fact, it is a generalization of the notion of flat connections on a smooth manifold. It can be constructed on any connected compact smooth manifold by choosing differential graded subalgebra quasi-isomorphic to the cohomology of the manifold (1-model) and by choosing a certain vector space decomposition on that subalgebra. This objects were constructed by [13] in order to study the cohomology of the loop space of simply connected manifolds. In the non-simply connected case, a homological pair (C, δ^*) induces an ordinary flat connection $d - C_0$ on the trivial bundle on M where the fiber is the Malcev Lie algebra of the fundamental group and where monodromy representation is the Malcev completion of the fundamental group.

By using a result of Arnold ([2]), it is possible to show that the universal Knizhnik–Zamolodchikov connection (KZ connection for short) on the configuration space of points on the plane is an example of a connection induced by a homological pair. The KZ connection is a holomorphic flat connection on the (holomorphic) trivial bundle and it was constructed by Drinfeld for the study of quantum groups [18], by using some system of differential equation called Knizhnik–Zamolodchikov equation [45]. In particular, he used the holonomy of this connection to construct the first example of the so called (Drinfeld) associator. Nowadays, these objects play a crucial role in several area of mathematics like number theory, braided category, knot theory, graph complexes, Lie theory and Grothenideck-Teichmüller theory.

The elliptic version of the Knizhnik–Zamolodchikov equation is constructed in [4] (see also [5]). Using these equations, Calaque, Enriquez and Etingof constructed an elliptic version of the KZ connection in [11] called universal Knizhnik–Zamolodchikov-Bernard connection (KZB connection for short). It is a holomorphic flat connection on the configuration space of points on the punctured elliptic curve on a holomorphic bundle. Unfortunately, the ordinary Chen theory cannot be used to construct this connection, one of the reasons is that a homological pair induces only a smooth connection in general, the second is that the KZB connection is defined on a more complicated bundle (isomorphic to the trivial bundle).

In this thesis we do the following: we extend Chen theory to simplicial manifolds and we show that the KZB connection is a flat connection induced by our extended notion of homological pairs. Our methods is very general and it can be used to construct higher genus version of the KZ-connection. We use the following strategy. Assume that our manifold is the quotient of a manifold M by a holomorphic action of a discrete group G. The nerve $M_\bullet G$ is a simplicial manifold. The vector space of smooth differential forms on $M_\bullet G$ is bigger than the one of M/G, and it carries naturally the structure of a C_∞-algebra (a differential graded algebra which is commutative up to homotopy). We show that each 1-model for the above C_∞-algebra equipped with a particular vector space decomposition induces a flat connection on a (trivial) bundle over M/G where the fiber corresponds to the Malcev Lie algebra of $\pi_1(M/G)$. The construction of the connection is purely algebraic and is intimately related to the homotopy transfer theorem for C_∞-algebras, this latter fact allows us to give an explicit formula for the connection that depends only by the choice of the initial 1-model equipped with a vector space decomposition. We investigate the dependence of the connection on these choices and we conclude that they are “unique” modulo (smooth) gauge equivalences and an automorphism of the fiber. The connection is homolorphic if so is the 1-model.

We consider the the punctured elliptic curve, we construct a holomorphic 1-model. This give us a homological pair that induces precisely the KZB connection on the punctured elliptic curve. We apply a similar argument to the configuration space of points of the punctured elliptic curve and we construct

a C_∞-algebra $B_n(0)$. We get a homological pair that induces the KZB connection on the configuration space of points on the punctured elliptic curve. Our methods can be applied to other type of simplicial manifolds. In particular, it can be used to construct flat connections on the configuration space of points of punctured Riemann surfaces with higher genus which are holomorphic if we provide a holomorphic 1-model.
The thesis is subdivided as follows. In the first chapter, we give a short introduction about Chen theory and we present in a more detailed way the results of the thesis. In the second chapter, we extend Chen theory to simplicial manifolds and we investigate some properties of the obtained flat connections and its relation with ordinary Chen theory. In the third chapter, we show that the KZB connection on the punctured torus is induced by a homological pair. In the last chapter, we extend this to the configuration space of points on the punctured elliptic curve and we compare KZ and KZB connection in terms of C_∞-algebras.

Notation

Let $\Bbbk$ be a field of characteristic zero. We work in the unital monoidal tensor category of graded vector spaces $(grVect, \otimes, \Bbbk, \tau)$ where the field $\Bbbk$ is considered as a graded vector space concentrated in degree 0, the twisting map is given $\tau(v \otimes w) := (-1)^{|v||w|} w \otimes v$, and the tensor product is the ordinary graded tensor product. For a graded vector space $V^\bullet$, V^i is called the homogeneous component of V, and for $v \in V^i$ we define its degree via $|v| := i$. For a vector space $W := \oplus_{i \in I} W_i$ we denote by $pro_{W_i} : W \to W_i$ the canonical projection. For a graded vector space $V^\bullet$ we denote by $V[n]$ the n-shifted graded vector space, where $(V[n])^i = V^{n+i}$. For example, $\Bbbk[n]$ is a graded vector space concentrated in degree $-n$ (its $-n$th homogeneous component is equal to $\Bbbk$, the other homogeneous component are all equal to zero). A (homogeneous) morphism of graded vector spaces $f : V^\bullet \to W^\bullet$ of degree $|f| := r$ is a linear map such that $f(V^i) \subseteq V^{i+r}$. For two graded vector spaces V, W, the set of morphisms of degree n is denoted $\mathrm{Hom}^n_{gVect}(V, W)$. More generally, the set of maps between V and W is again a graded vector space $\mathrm{Hom}^\bullet_{gVect}(V, W)$ for which the i-homogeneous elements are those of degree i. The tensor product of homogeneous morphisms is defined through the *Koszul convention*: for two morphisms of graded vector spaces $f : V^\bullet \to W^\bullet$ and $g : V'^\bullet \to W'^\bullet$, the tensor product $f \otimes g : (V \otimes W)^\bullet \to (V' \otimes W')^\bullet$ on the homogeneous elements is given by

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|}(f(v) \otimes g(w)).$$

We denote by $s : V \to V[1]$, $s^{-1} : V[1] \to V$ the shifting morphisms that send V^n to $V[1]^{n-1} = \Bbbk \otimes V^n = V^n$ and $V[1]^n = \Bbbk \otimes V^{n+1} = V^{n+1}$ to V^{n+1}, respectively. Those maps can be extended to a map $s^n : V \to V[n]$, (the identity map shifted by n). Note that $s^n \in \mathrm{Hom}^{-n}(V, V[n])$. A graded vector space is said to be of finite type if each homogeneous component is a finite dimensional vectorspace. A graded vector space $V^\bullet$ is said to be bounded below at k if there is a k such that $V^l = 0$ for $l < k$. Analogously it is said to be bounded above at k if there is a k such that $V^l = 0$ for $l > k$. Let (V, d_V) be a differential graded vector space, then $V^{\otimes n}$ is again a differential graded vector space with differential

$$d_{V^{\otimes n}}(v_1 \otimes \cdots \otimes v_n) := \pm \sum_{i=1}^{n} v_1 \otimes \cdots \otimes d_V v_i \cdots \otimes v_n,$$

where the signs follows from the Koszul signs rule. Let (V, d_V), (W, d_W) be differential graded vector spaces, then $\mathrm{Hom}^\bullet_{gVect}(V, W)$ is a differential graded vector space with differential

$$\partial f := d_W f - (-1)^{|f|} f d_v.$$

For a (complex) smooth manifold M, we denote by $(A_{DR}(M), d, \wedge)$ the differential graded algebra of (complex) smooth differential forms.

Chapter 1

Introduction

The goal of this chapter is to present the main results of the thesis and to give a introduction about homological pairs. In Section 1.1 we give a general overview about some results of Chen. In Section 1.2 we give an overview about the results of Chapter 2, we extend the notion of homological pair to simplicial manifolds, we restrict our attention to manifolds equipped with a discrete group action and we compare it with the ordinary Chen theory. These results are contained in [50]. In Section 1.3, we give a short introduction to universal KZ and KZB connection and we present the results of Chapter 3 and 4. These results are contained in [51].

1.1 Homological pairs

In this section, we give a summary of some results of [13]. We work on the field of real numbers.

1.1.1 Differentiable spaces

Definition 1.1.1. By a n-dimensional convex set we shall mean a n dimensional convex set in $\mathbb{R}^n$. A *differentiable space* M is a set M with a family $\mathcal{U}(M)$ of maps $\alpha : U \to M$, called plots on M, where

a) each U is a convex set,

b) if $\alpha : U \to M$ is a plot, V a convex set and $\theta : V \to U_i$ a smooth map, then $\alpha \circ \theta : V \to M$ is also a plot,

c) every constant map from a convex set into M is a plot,

d) if V is a convex set, $\{V_j\}$ is a cover of convex open sets of U and there exists a map $\alpha : U \to M$ such that each restriction $\alpha|_{V_i}$ is a plot, then $\alpha : U \to M$ is also a plot.

We denote a differential space by its underlying set M. A pair $(M,\mathcal{U}(M))$ that satisfies conditions a), b),c) is called *pre-differential space.*

For each set M equipped with a family of maps $\mathcal{U}(M)$, each of whose domain is a convex set, there exists a unique maximal subfamily $\mathcal{U}'$ such that $(M,\mathcal{U}(M)')$ is a differentiable space. We call $\mathcal{U}(M)'$ the differentiable space structure of M generated by $\mathcal{U}$. In particular, for a pre-differentiable space $(M,\mathcal{U}(M)')$, the differential space structure generated by U' is the subset $U \subseteq U'$ of maps that satisfy condition d). We call $(M,\mathcal{U}(M))$ the differentiable space associated to the pre-differentiable space $(M,\mathcal{U}(M)')$.

Definition 1.1.2. A morphism $f : M \to N$ between differentiable spaces is a map $f : M \to N$ such that for any plot $\alpha : U \to M$, $f\alpha$ is a plot on N.

Each convex set V is naturally a differentiable space, where

$$\mathcal{U}(V) := \{\alpha : U \to V \mid \alpha \text{ is smooth } \}.$$

In particular, for a differentiable space M and a convex set V, a map $\alpha : V \to M$ is a plot if and only if it is a morphism of differentiable spaces.
A smooth manifold X is naturally a differentiable space, where

$$\mathcal{U}(X) := \{\alpha : U \to X \mid \alpha \text{ is smooth } \}.$$

Remark 1.1.3. If we replace the word "convex" with the word "open" in Definition 1.1.1 , we have the definition of diffeological space (see [34]). For the relationship between these two categories, see [3]. In particular the category of smooth manifolds (with corners) embeds fully and faithfully into both the categories. Hence, differentiable spaces are an extension of the category of smooth manifolds.

They have several interesting properties (see [3]). Let M, N be differentiable spaces.

1. A subset $S \subset M$ is a differentiable space where $\mathcal{U}(S)$ is defined as follows: $\alpha \in \mathcal{U}(S)$ if $\alpha \in \mathcal{U}(M)$.

2. Consider the projections $p_M : M \times N \to M$, $p_N : M \times N \to N$. Then, $M \times N$ is again a differentiable space where the plots are $\alpha' \times \alpha : V' \times U' \to M \times N$ such that $P_M (\alpha' \times \alpha)$ is a plot in M and $P_N (\alpha' \times \alpha)$ is a plot in N. In particular, the projection p_M, p_N are differentiable maps.

3. Let M be a differentiable space and let $\sim$ be an equivalence relation on M. Let $M' := M/\sim$ be the quotient and let $p : M \to M/\sim$ be the projection. Then $\alpha : V \to M'$ is a plot on M' if there exist an open cover $i_j : V_j \hookrightarrow V$ and a collection of plots $\alpha_j : V_j \to M$ such that $\alpha i_j = p\alpha_i$. This defines a differentiable structure on M'.

4. Let $\mathcal{C}^\infty (M, N)$ be the set of morphisms between M and N. Let $\mathrm{ev}_M : \mathcal{C}^\infty (M, N) \to N$ be the evaluation map. Then $\mathcal{C}^\infty (M, N)$ is a differentiable space, where the plots are given by

$$\mathcal{U}(\mathcal{C}^\infty (M, N)) := \{\alpha : U \to \mathcal{C}^\infty (M, N) \mid \mathrm{ev}_M \alpha \in \mathcal{U}(N)\}.$$

We define a de Rham functor on the category of differentiable spaces.

Definition 1.1.4. A smooth p-form w on a pre-differentiable space M is a collection (w_α) of smooth p-forms indexed by plots $\alpha : V \to M$ in $\mathcal{U}(M)$, so that w_α is an ordinary smooth p-form on V. The family (w_α) satisfies the following compatibility condition: if $\alpha : V \to M$ is a plot, V a convex set and $\theta : V' \to V$ a smooth map, then

$$(\theta^* w)_\alpha = w_{\alpha\theta}. \tag{1.1}$$

We can consider w_α being intrinsically $\alpha^* w$.
The smooth forms on M generate a differential graded algebra $A^\bullet_{DR}(M)$. Its vector space structure is given by

$$(v + w)_\alpha := v_\alpha + w_\alpha, \quad (\lambda w)_\alpha := \lambda w_\alpha,$$

the differential and wedge product are given by

$$(dw)_\alpha := dw_\alpha, \quad (w \wedge w')_\alpha = w_\alpha \wedge w'_\alpha.$$

We call $A^\bullet_{DR}(M)$ the *de Rham algebra of* M. Note that any morphism $f : M \to N$ induces a differential graded algebra map $f^* : A_{DR}(N) \to A_{DR}(M)$ via

$$(f^* w)_\alpha = w_{f\alpha}.$$

Let M' be a pre-differentiable space and M its associated differentiable space. Then $A_{DR}(M')$ is naturally isomorphic to $A_{DR}(M)$.

Definition 1.1.5. A plot $\alpha : U \to M$ is said to be compact if U is compact. For a p-form w on M, we define

$$\langle w, \alpha \rangle := \begin{cases} \int_U w_\alpha, & \text{if } \dim U = p, \\ 0, & \text{otherwise .} \end{cases}$$

1.1.2 Iterated integrals

Let M be a differentiable space and $I = [0, 1]$. We denote its path space by

$$PM := \mathcal{C}^\infty(I, M) = \{\gamma : I \to M | \, \gamma \text{ is differentiable } \}.$$

By the discussion in the previous section, a plot on PM is a map $\alpha : V \to PM$ such that $\phi_\alpha : I \times V \to M$, defined by $\phi_\alpha(t, \xi) := \alpha(\xi)(t)$ is a plot on M.

Definition 1.1.6. Let $\alpha : U \to PM$ be a plot. We say that α *starts at* $x \in M$ if $\alpha(\xi)(0) = x$ for any $\xi \in V$. We say that α *ends at* $x \in M$ if $\alpha(\xi)(1) = x$ for any $\xi \in V$. Let $\alpha : U \to PM$ be a plot ending at x and $\beta : V \to PM$ be a plot starting at x. We define $\alpha \times \beta : U \times V \to PM$ as

$$(\alpha \times \beta)(\xi, \xi', t) := \begin{cases} \alpha(\xi, 2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \beta(\xi', 2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

For each $[n] \in \mathbf{\Delta}$, we define the *geometric n-simplex* as the topological space

$$\Delta_{geo}[n] := \{0 \le t_1 \le \cdots \le t_n \le 1\} \subset \mathbb{R}^{n+1}.$$

Let $\gamma \in PM$ and let $t \in [0, 1]$. We define $\gamma^t \in PM$ as $\gamma^t(s) := \gamma(ts)$. Let $p_1 : PM \to M$ the differentiable map that sends γ to $\gamma(1)$. Let $J : A_{DR}(M) \to A_{DR}(M)$ be defined by $J(w) := (-1)^{\deg(w)} w$. We define a map scale : $\Delta_{geo}[n] \times I \to I^n$ by

$$\text{scale}\big((0 \le t_1 \le \cdots \le t_n \le 1), s\big) = (t_1 s, \ldots, t_n s).$$

For any $M \in \mathcal{C}^\infty$, this map induces a morphism rest : $\Delta^n_{geo} \times PM \to (PM)^n$

$$\text{rest}\big((0 \le t_1 \le \cdots \le t_n \le 1), \gamma\big) = \left(\gamma^{t_1}, \ldots, \gamma^{t_n}\right).$$

Let $v_1, \ldots, v_n$ be differential forms on PM of degree $d_1, \ldots, d_n$, and let $\pi_i : (PM)^n \to PM$ be the projection onto the ith coordinate. Set $\underline{v} := \pi_1^* v_1 \wedge \cdots \wedge \pi_n^* v_n \in A^*_{DR}((PM)^n)$. We define

$$I.I.(v_1, \ldots, v_n) := (-1)^l \int_{\Delta_{geo}[n]} \text{rest}^* \underline{v},$$

where l is defined by

$$l := \sum_{i=0}^{n-i} d_i (n-i) + \frac{(n-1)(n)}{2}.$$

Proposition 1.1.7. *Let $w_1, \ldots, w_n$ be differential forms on M of degree $d_1, \ldots, d_n$ and let $p_1 : PM \to M$ be the projection defined by $p_1(\gamma) = \gamma(1)$. Then,*

1. $I.I.(w_1, \ldots, w_n) = I.I.(J(I.I.(w_1, \ldots, w_{n-1})) \wedge w_n)$.
2. *Let $w_1, \ldots, w_n$ be differential forms on PM of degree $d_1, \ldots, d_n$, then*

$$\begin{aligned} dI.I.(w_1, \ldots, w_n) &= \sum_{i=1}^n (-1)^i dI.I.(Jw_1, \ldots, Jw_{i-1}, dw_i, w_{i+1}, \ldots w_n) \\ &+ \sum_{i=1}^n (-1)^i dI.I.(Jw_1, \ldots, Jw_{i-1}, Jw_i \wedge w_{i+1}, w_{i+2}, \ldots w_n) \\ &- w_1 \wedge I.I.(w_2, \ldots w_n) + J(I.I.(w_1, \ldots w_{n-1})) \wedge w_n. \end{aligned}$$

Proof. The first assertion follows by construction. The second assertion follows from the Stokes'theorem with respect to the integration $\int_{\Delta^n_{geo}}$. An inductive proof of the second assertion is given in [12](see Proposition 1.5.2.). □

Definition 1.1.8. Let $w_1, \ldots, w_r$ be forms in $A_{DR}(M)$ of degree $|r_i|$, for $i = 1, \ldots, n$. We define

$$\int w_1 \cdots w_r := I.I.\left(p_1^* w_1, \ldots, p_1^* w_n\right)$$

$\int w_1 \cdots w_r$ is a form on PM with degree $\sum_{i=1}^r (-1 + r_i)$. We call these forms *iterated integrals.*

The iterated integrals satisfy certain multiplication properties. Let Σ_n be the group of permutations on $\{1, 2, \ldots, n\}$. Consider two finite strings of natural numbers $1 \le i_1 < \cdots < i_p$, $1 \le j_1 < \cdots < j_q$ for $p, q \ge 0$. We associate to these strings a permutation $\sigma \in \Sigma_{p+q}$ via

$$\sigma(l) := \begin{cases} i_l & \text{if } l \le p, \\ j_l & \text{otherwise }. \end{cases}$$

A permutation obtained in this way is called (p, q)-*shuffle.* We denote the set of (p, q)-shuffles via $Sh(p, q)$. Let $\alpha, \beta : U \to PM$ be two plots such that $\alpha(\xi)(1) = \beta(\xi)(0)$ for any ξ in U. We define

$$\alpha\beta : U \to \{ \text{ piecewise smooth paths on } M\}$$

as

$$\alpha\beta(\xi)(t) := \begin{cases} \alpha(\xi)(2t) & \text{for } t \le \frac{1}{2}, \\ \beta(\xi)(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Proposition 1.1.9 ([12]). *Let M be a differentiable space and let $w_1, \ldots, w_n$ be differential forms on M.*

1. *Let $\alpha, \beta : U \to PM$ be two plots such that $\alpha(\xi)(1) = \beta(\xi)(0)$ for any ξ in U. Assume that $\alpha\beta$ is a plot on PM, then*

$$\left(\int w_1 \cdots w_n\right)_{\alpha\beta} = \sum_{i=1}^n \left(\int w_1 \cdots w_i\right)_\alpha \left(\int w_{i+1} \cdots w_n\right)_\beta .$$

2. *We have the shuffle product formula*

$$\left(\int w_1 \cdots w_p\right)\left(\int w_{p+1} \cdots w_{p+q}\right) = \sum_{\sigma \in Sh(p,q)} \pm \int w_{\sigma(1)} \cdots w_{\sigma(n)},$$

where the signs is given by the sign of σ and the sign rule.

1.1.3 Homological pairs and Maurer-Cartan elements

We refer to Appendix A.1 for a short introduction to coalgebras. Let (C, d_C, Δ) be a differential graded (conilpotent) coalgebra and (A, d_A, μ) be a differential graded algebra. We consider the graded vector space $\mathrm{Hom}^\bullet(C, A)$ and we define a differential

$$\partial(f) := d_A \circ f - (-1)^{|f|} f \circ d_C$$

and a product $M_2(f, g) := \mu((\alpha \otimes \alpha) \circ \Delta)$ (called convolution product).

Lemma 1.1.10. *The* $(\mathrm{Hom}^\bullet(C, A), \partial, M_2)$ *is a differential graded algebra. Let*

$$[-,-] := M_2(a,b) - (-1)^{|a||b|} M_2(b,a).$$

Then, $(\mathrm{Hom}^\bullet(C, A), \partial, [-,-])$ *is a differential graded Lie algebra.*

Proof. The proof is by direct calculation. See [39], Chapter 2. □

Definition 1.1.11. We call $(\mathrm{Hom}^\bullet(C, A), \partial, M_2)$ the *convolution algebra* and $(\mathrm{Hom}^\bullet(C, A), \partial, [-,-])$ the *convolution Lie algebra.*

Definition 1.1.12. Let $(\mathfrak{u}, \partial, [-,-])$ be a differential graded Lie algebra. An element $\alpha \in \mathfrak{u}^1$ is called *Maurer-Cartan element* if

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

Let $(\mathrm{Hom}^\bullet(C, A), \partial, [-,-])$ be as above. A Maurer-Cartan element in $(\mathrm{Hom}^\bullet(C, A), \partial, [-,-])$ is called *twisting cochain.*

Let BA be the bar construction of A (see Appendix A.1.1).

Lemma 1.1.13. *Let C, A be as above. There is a one to one correspondence between twisting cochains and and morphism of differential graded coalgebras $F : C \to BA$.*

Proof. For a proof, see theorem 2.26 of [39]. □

Definition 1.1.14. Let A be a differential graded algebra. For a $j \in \mathbb{N} \cup \{\infty\}$, a differential graded subalgebras B is a j-model for A if

1. the inclusion i induces an isomorphism up to the j-th cohomology group and it is injective on the $j+1$ cohomology group,
2. the inclusion $i^l : B^l \hookrightarrow A^l$ preserves non-exact elements for $0 \leq l \leq j+1$.

If $j = \infty$ we call B a model for A.

Let M be a connected smooth manifold such that its de Rham cohomology $H^\bullet_{DR}(M)$ is finite dimensional. Let $A \subset A_{DR}(M)$ be a differential graded subalgebra with finite type and connected cohomology (i.e. $H^0(A) \cong \mathbb{R}$ and $H^\bullet(A)$ is a finite type graded vector space). Consider a direct sum decomposition of graded vector spaces

$$A = W \oplus d\mathcal{M} \oplus \mathcal{M}, \tag{1.2}$$

where W is a graded vector subspace of closed forms on M and $\mathcal{M}$ is a graded vector subspace containing no exact forms excepts 0. Notice that $W = H(W, 0) = H(A, d)$. We fix a basis $1, w_1, \ldots, w_n$ of $W = \oplus_{p\geq 0} W^p$ and we denote by $X_1, X_2, \ldots$ the elements in $(W[1])^*$ dual to $s^{-1}(w_1), s^{-1}(w_2), \cdots \in W[1]$. The dual space $(W_+[1])^*$ is non-positively graded. In particular, the degree of each $s^{-1}(w_i)$ is $1 - |w_i|$. We consider the free algebra $\left(T(W_+[1])^*, \mu\right)$ on $(W_+[1])^*$ and we denote by I its augmentation ideal. Its powers $I^1, I^2, \ldots, I^i, \ldots$ form a filtration $I^\bullet$. We refer to Appendix A.2 for standard notions about filtrations. We denote by $\widehat{T}(W[1])^*$ the complete free algebra with respect to $I^\bullet$. We consider the complete tensor product $A\widehat{\otimes}\widehat{T}(W_+[1])^*$. Its underlying vector space is the space of formal power series connection with coefficients in A

$$\sum w_i X_i + \sum w_{ij} X_i X_j + \cdots + \sum w_{i_1 \ldots i_r} X_{i_1} \ldots X_{i_r} + \ldots .$$

Notice that $A \otimes T(W_+[1])^*$ is an associative algebra equipped with the product

$$(wY)(vZ) = (wv)\mu(Y, Z). \tag{1.3}$$

This product preserves the filtration and hence it induces an associative product on $A\widehat{\otimes}\widehat{T}(W_+[1])^*$. We denote this product by M_2 and the completion of μ by $\widehat{\mu}$. We define a Lie algebra structure on $A\widehat{\otimes}\widehat{T}(W_+[1])^*$ by defining]

$$[-,-] := M_2(a, b) - (-1)^{|a||b|} M_2(b, a) \tag{1.4}$$

on homogeneous elements.

Definition 1.1.15. A differential $\tilde{d} : \widehat{T}(W_+[1])^* \to \widehat{T}(W_+[1])^*$ is a morphism of degree $+1$ that satisfies the Leibniz-rule with respect to $\widehat{\mu}$.

Let $(T^c(W_+[1]), \Delta)$ be the tensor coalgebra on $W_+[1]$ (see Appendix A.1). There is a canonical isomorphism $(T^c(W_+[1]))^* \cong \widehat{T}\left(W_+[1]\right)^*$ (see Section A.2)

$$\Psi \,:\, \widehat{T}\left(W_+[1]\right)^* \to \mathrm{Hom}\left(T^c\left(W_+[1]\right), \Bbbk\right).$$

In particular, for each differential $\tilde{d}$, there exists a codifferential δ on $T^c\left(W_+[1]\right)$ such that[1] $\tilde{d} = \delta^*$. We prefer to use δ^* instead of $\tilde{d}$. We fix a differential δ^* and we denote by ∂ the differential

$$\partial \,:\, A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^* \to A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*$$

induced by d and δ^*,

$$\partial\left(\sum_{p\geq 0}^{\infty} w_{i_1,\dots,i_p} X_{i_1}\cdots X_{i_p}\right) := \sum_{p\geq 0}^{\infty} dw_{i_1,\dots,i_p} X_{i_1}\cdots X_{i_p} + (-1)^{|w_{i_1,\dots,i_p}|}\sum_{p\geq 0}^{\infty} w_{i_1,\dots,i_p}\delta^*\left(X_{i_1}\cdots X_{i_p}\right).$$

We have the following simple lemma.

Lemma 1.1.16. *The graded vector space* $\left(A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*, \partial, [-,-]\right)$ *is a differential graded Lie algebra and the isomorphism* Ψ *extends to an isomorphism of differential graded algebras*

$$\left(A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*, \partial, [-,-]\right) \to \left(\mathrm{Hom}^\bullet\left(\left(T^c\left(W_+[1]\right), \delta\right), A\right), \partial, [-,-]\right). \tag{1.5}$$

Consider the decomposition (1.2), W, and the basis $1, w_1, \dots w_n$ as above. The element

$$\sum_{i=1}^{n} w_i X_i \in A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^* \tag{1.6}$$

is completely determined by the choice of that decomposition and it is independent by the choice of the basis of W.

Definition 1.1.17. Let $A \subset A_{DR}(M)$ be a subalgebra and let W be defined as above. A homological pair (C, δ^*) consists of a codifferential δ of $T^c\left(W_+[1]\right)$ and a $C \in A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*$ such that C is a Maurer-Cartan element.

Theorem 1.1.18 ([12])**.** *Let M be as above. Assume that A is a model for $A_{DR}(M)$. For every direct sum decomposition* (1.2) *of A, there exists a unique homological pair (C, δ^*),*

$$C = \sum w_i X_i + \sum w_{ij} X_i X_j + \cdots + \sum w_{i_1\dots i_r} X_{i_1}\dots X_{i_r} + \cdots \in A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*$$

such that

i) *the term $\sum w_i X_i$ of C is as given in 1.6,*

ii) *the coefficients $w_{ij}, \dots, w_{i_1\dots i_p}, \dots$ belong to $\mathcal{M}$,*

iii) $\delta^*(I) \subset I^2$.

We call the pair (C, δ^*) the homological pair associated to the decomposition (1.2).

The proof in [12] is inductive. Assume $C_1 = \sum w_i X_i$ and $\delta_1^* = 0$ and set (C_2, δ_2^*), where $\delta_2^*(X_i) = \sum c_{jk}^i X_i X_j$ and $C_2 = C_1 + \sum w_{ij} X_i X_j$ such that

$$dC_2 + \delta_2^* C_2 + \frac{1}{2}\left[C_2, C_2\right] = 0 \text{ modulo } I^3. \tag{1.7}$$

[1] Recall that $\delta^* \,:\, \mathrm{Hom}\left(T^c\left(W_+[1]\right), \Bbbk\right) \to \mathrm{Hom}^{\bullet+1}\left(T^c\left(W_+[1]\right), \Bbbk\right)$ is defining by $\delta^* f = (-1)^{|f|+1} f\delta$.

Chen in [12] showed that the above equation has a unique solution where $c^i_{jk} \in \mathbb{R}$ and $w_{ij} \in \mathcal{M}$ by using some iterated integrals argument (a "geometrical argument"). This way Chen obtained a family of pairs (C_i, δ^*_i) that satisfies the above relations modulo I^{i+1}. In particular (C, δ^*) is obtained by taking the projective limit

$$C := \varprojlim C_i, \quad \delta^* := \varprojlim \delta^*_i.$$

In the literature there are some "purely algebraic" proofs (where the solvability of (1.7) is guaranteed without involving iterated integrals). A proof of Theorem 1.1.18 is contained in [32] (see the coalgebra perturbation lemma 2.1_*), in this case we can substitute A with any non negatively differential graded algebra with connected and finite type cohomology. In particular, one can show that the algorithm in the proof of Theorem 1.1.18 works as well for any subalgebra A of $A^*_{DR(M)}$ with connected and finite type cohomology. In particular, given a differential graded subalgebra A of $A^*_{DR(M)}$ with finite type and connected cohomology, for a fixed vector space decomposition as in (1.2), the above algorithm produces a pair (C, δ^*) that satisfies all the conditions of Theorem 1.1.18 with respect to (1.2). In analogy of above, we call (C, δ^*) *the homological pair associated to* (1.2).
We change our point of view.

Corollary 1.1.19. *Let M be as above. Let $A \subset A_{DR}(M)$ be a differential graded algebra with connected and finite type cohomology endowed with a decomposition* (1.2)*. For every direct sum decomposition* (1.2) *of A, the following items are equivalent.*

1. *There exists a homological pair (C, δ^*) associated to the given vector space decomposition,*

2. *There exist a codifferential δ of $T^c(W_+[1])$ and a Maurer-Cartan element $\alpha \in \mathrm{Hom}^\bullet((T^c(W_+[1]), \delta), A)$ such that*

   i) $\alpha(X_i) = w_i$,

   ii) $\alpha(X_{i_1} \cdots X_p) = w_{i_1 \ldots i_p} \in \mathcal{M}$ *for* $p > 1$,

   iii) *The dual map δ^* is a differential on $\widehat{T}(W_+[1])^*$ such that $\delta^*(I) \subset I^2$.*

Theorem 1.1.18 can be proved in terms of point 2 via algebraic methods by using the coalgebra perturbation lemma 2.1* contained in [32]. As explained in [33], this methods is equivalent to the homotopy transfer theorem for C_∞-algebras (see Section 2.1). Such a theorem gives a easier way to compute explicitly α and δ such that they depend only on the decomposition (1.2) (See for example Corollary 2.3.13 in this thesis).

Remark 1.1.20. Let $A \subset A_{DR}(M)$ be a model equipped with a decomposition as in (1.2). By the construction of δ^* in the proof of Theorem 1.1.18, Chen observed that δ^*_i is completely determined by the product and by the (higher) Massey products of $H^\bullet(M)$. Set $\delta^*_2(X_i) = \sum c^i_{jk} X_i X_j$ for some $c^i_{jk} \in \mathbb{R}$. We denote by $[w]$ the cohomology class of a closed form w on M. Consider a basis $w_1, \ldots, w_n$ of W, then

$$\sum c^i_{jk} w_i = [w_j \wedge w_k].$$

A similar results (but instead of the cup product, we have to use the Massey triple product) holds for δ^*_3.

We conclude this subsection with a technical proposition.

Definition 1.1.21. An homological pair (C, δ^*) is quadratic if $\delta^*(X_i) = \sum c^i_{jk} X_i X_j$ for some $c^i_{jk} \in \mathbb{R}$.

Proposition 1.1.22 ([13])**.** *Consider the vector space decomposition* (1.2)

$$A = W \oplus d\mathcal{M} \oplus \mathcal{M}.$$

If $d\mathcal{M} \oplus \mathcal{M}$ is a differential graded algebra ideal of A, the associated (C, δ^*) homological pair is quadratic.

1.1.4 A geometric connection on a trivial bundle

Let M be as above. We show that a homological pair induces a flat connection over a trivial bundle on M. Let $A \subset A_{DR}(M)$ be a differential graded algebra with connected and finite type cohomology endowed with a decomposition (1.2) and let (C, δ^*) be a homological pair where δ^* is a differential on $\widehat{T}(W_+[1])^*$ and $C \in A(M)\widehat{\otimes}\widehat{\mathbb{L}}(W_+[1])^*$. We denote by $\mathbb{L}(W_+[1]) \subset T(W_+[1])^*$ the free Lie algebra on $(W_+[1])^*$ and by $\widehat{\mathbb{L}}(W_+[1])^* \subset \widehat{T}(W_+[1])^*$ the completion of the Lie algebra with respect to $I^\bullet$. Clearly it is a filtration of Lie ideals where

$$I = \mathbb{L}(W_+[1])^*, \quad I^n = \left[I^{n-1}, \mathbb{L}(W_+[1])^*\right] \text{ for } n > 1.$$

Definition 1.1.23. Let A be as above equipped with a decomposition (1.2). Let (C, δ^*) be a homological pair where δ^* is a differential on $\widehat{T}(W_+[1])^*$. We call (C, δ^*) a *reduced* homological pair if

1. $\delta^* I \subset I^2$,
2. δ^* has a well-defined restriction $\delta^* : \widehat{\mathbb{L}}(W_+[1])^* \to \widehat{\mathbb{L}}(W_+[1])^*$,
3. $C \in A(M)\widehat{\otimes}\widehat{\mathbb{L}}(W_+[1])^*$.

Proposition 1.1.24 ([12])**.** *Let A be as above endowed with a decomposition* (1.2) *and let (C, δ^*) be the homological pair associated to the given decomposition. Then δ^* has a well-defined restriction $\delta^* : \widehat{\mathbb{L}}(W_+[1])^* \to \widehat{\mathbb{L}}(W_+[1])^*$ and $C \in A(M)\widehat{\otimes}\widehat{\mathbb{L}}(W_+[1])^*$. In particular, the homological pair associated to a decomposition* (1.2) *is reduced.*

Definition 1.1.25. Let $\mathfrak{u}$ be a Lie algebra. Consider the filtration given by Lie ideals $I^1 := \mathfrak{u}$, $I^{i+1} := \left[\mathfrak{u}, I^i\right]$, $\mathfrak{u}$ is said to be nilpotent if $I^{s+1} = 0$ for some s. Notice that $I^i \subset \mathfrak{u}$ is a Lie ideal. A Lie algebra $\mathfrak{u}$ is said to be *pronilpotent* if

$$\mathfrak{u} \cong \varprojlim_i \left(\mathfrak{u}/I^i\right).$$

and if $\left(\mathfrak{u}/I^i\right)$ are finite dimensional for any i.

In particular, $\widehat{\mathbb{L}}(W_+[1])^*$ is pronilpotent. We fix a reduced homological pair (C, δ^*). We denote by C^i for $i = 1, 2, \dots,$ the element

$$C^i = \sum w_i X_i + \sum w_{ij} X_i X_j + \dots + \sum w_{i_1 \dots i_r} X_{i_1} \dots X_{i_r} + \dots$$

contained in $A^i\widehat{\otimes}\widehat{\mathbb{L}}\left(W^i_+[1]\right)^*$ such that $C = \sum_{i \geq 1} C^i$.

Definition 1.1.26. For a Lie algebra $\mathfrak{u}$ we define the adjoint $\mathrm{Ad} : \mathfrak{u} \to \mathrm{End}(\mathfrak{u})$ via $\mathrm{Ad}_a(b) := [a, b]$.

We consider $\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*$ equipped with the adjoint action. This induces a map

$$\mathrm{Ad} : \widehat{T}\left(W^1_+[1]\right)^* \to \mathrm{End}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right) \tag{1.8}$$

defined by

$$\mathrm{Ad}(X_{i_1} \cdots X_{i_p}) := \mathrm{Ad}_{X_{i_1}} \circ \dots \circ \mathrm{Ad}_{X_{i_p}}$$

In particular $\mathrm{Ad}_{X_{i_1}} \circ \dots \circ \mathrm{Ad}_{X_{i_p}}$ defines a *inner* derivation on $\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*$. In particular, $d + C^1$ may be considered as a connection on the trivial bundle on M with fiber $\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*$. Notice that in general this connection is not flat, since

$$-\delta^* C^2 = dC^1 + \frac{\left[C^1, C^1\right]}{2}.$$

Let $\mathcal{R} \subset \widehat{\mathbb{L}}\left(W^1_+[1]\right)^*$ be the completion of the Lie ideal generated by $\delta^* X_i$ such that $|X_i| = 1$. Analogously, let $\mathcal{R}' \subset \widehat{T}\left(W^1_+[1]\right)^*$ be the completion of the ideal generated by $\delta^* X_i$ such that $|X_i| = 1$. The map (1.8) induces a well-defined action

$$\mathrm{Ad} : \widehat{T}\left(W^1_+[1]\right)^* / \mathcal{R}' \to \mathrm{End}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^* / \mathcal{R}\right). \tag{1.9}$$

We denote by C_0 the image of C^1 under the projection

$$A^1\widehat{\otimes}\widehat{\mathbb{L}}\left(W^1_+[1]\right)^* \to A^1\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^* / \mathcal{R}\right).$$

Corollary 1.1.27. *$d + C_0$ is a flat connection on the trivial bundle on M with fiber $\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*/\mathcal{R}$.*

Since δ^* preserves $I^\bullet$ we have that $\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*/\mathcal{R}$ is pronilpotent.

Definition 1.1.28. We call C_0 the *degree zero geometric connection associated to* (C, δ^*). If (C, δ^*) is the homological pair associated to (1.2), we call C_0 the *degree zero geometric connection associated to* (1.2).

1.1.5 Transport and holonomy

Let M be as above. Let $A \subset A_{DR}(M)$ be a differential graded algebra with connected and finite type cohomology endowed with a decomposition (1.2). Let (C, δ^*) be a reduced homological pair. Let B be a non-negatively graded differential graded algebra. Then by Lemma 1.1.10 the graded vector space $\mathrm{Hom}^\bullet\left(T^c\left(W_+[1]\right), B\right)$ carries a differential graded algebra structure, where $T^c\left(W_+[1]\right)$ is considered endowed with the codifferential δ. Moreover the isomorphism (1.5) holds as well and $B\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*$ is a differential graded Lie algebra. Let $f : A_{DR}(M) \to B$ be morphism of differential graded algebras. Then

$$f\widehat{\otimes}Id : A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^* \to B\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*.$$

is a differential graded Lie algebras morphism. Let PM be the path space of M and let $p_1 : PM \to M$ be the evaluation at 1. We define

$$C' := (p_1^*\widehat{\otimes}Id)C \in A_{DR}^*\left(PM\right)\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*.$$

and

$$\underline{C}^n := \left(\left(\pi_1^*\widehat{\otimes}Id\right)C'\right)\cdots\left(\pi_n^*\widehat{\otimes}Id)C'\right) \in A_{DR}^*\left((PM)^n\right)\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*.$$

We define

$$\int C^n := (-1)^l\int_{\Delta_{geo}^n}\left(\mathrm{rest}^*\widehat{\otimes}Id\right)\underline{C}^n.$$

Definition 1.1.29. We define the transport of C as

$$T = 1 + \sum_{n\geq 1}\int C^n \in A_{DR}^*\left(PM\right)\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*.$$

We can write

$$T = 1 + \sum T_iX_i + \sum T_{ij}X_iX_j + \cdots + \sum T_{i_1\dots i_r}X_{i_1}\dots X_{i_r} + \dots.$$

By looking at the coefficients, we have

$$T_i = \int w_i, \quad T_{ij} = \int\left(w_iw_j + w_{ji}\right),$$
$$T_{ijk} = \int\left(w_iw_jw_k + w_{ji}w_k + w_iw_{jk} + w_{ijk}\right).$$

We denote by $T^i \in A_{DR}^i\left(PM\right)\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*$ such that $T = \sum_{i\geq 0}T^i$. Let (C, δ^*) be as above. We define C^i as in the previous subsection. In particular T^0 can be written as

$$T^0 = 1 + \sum_{n\geq 1}\int\left(C^1\right)^n \in A_{DR}^0\left(PM\right)\widehat{\otimes}\widehat{T}\left(W_+^1[1]\right)^*.$$

where C^1 defines a connection form on the trivial bundle on M with fiber $\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*$ (which is considered equipped with the adjoint action). In particular, T^0 defines a map from PM to $\widehat{T}\left(W_+^1[1]\right)^*$ via the evaluation map. Let $\gamma : [0,1] \to M$ be a smooth path. For $t \in [0,1]$ we denote by γ^t the path defined by $\gamma^t(s) := \gamma(st)$. Notice that γ^t defines a path on PM. We define

$$T(\gamma(t)) := T_{\gamma^t}^0 \in A_{DR}^0\left([0,1]\right)\widehat{\otimes}\widehat{T}\left(W_+^1[1]\right)^*.$$

Let $C(t)$ be the pullback of C^1 along $\gamma : [0,1] \to M$. As noticed by Chen in [13], $T(\gamma(t))$ is the unique solution of

$$dX(t) = X(t) \wedge C(t), \quad X(0) = 1.$$

for $X : [0,1] \to \widehat{T}\left(W_+^1[1]\right)^*$. By the adjoint action T^0 defines a map from PM to $\mathrm{Aut}\left(\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*\right)$ which corresponds to the holonomy of C^1. Let (C, δ^*) be as above. Let C_0 be its associated degree zero geometric connection. We define

$$T_0 = 1 + \sum_{n \geq 1} \int (C_0)^n \in A^0_{DR}(PM) \widehat{\otimes} \left(\widehat{T}\left(W_+^1[1]\right)^* / \mathcal{R}\right).$$

The above discussion works as well for C_0 and we have the following.

Lemma 1.1.30. *The holonomy of $d + C_0$ is given by T_0.*

Since C_0 defines a flat connection on M, by standard differential geometry T_0 induces a multiplicative map

$$\Theta_0 : \pi(M,p) \to \left(\widehat{T}\left(W_+[1]\right)^* / \mathcal{R}', \widehat{\mu}\right)$$

called *monodromy representation or holonomy representation.* In order to understand this map we need to introduce a few notions related to complete Hopf algebras (we refer to [46] and [23]).
For a complete Hopf algebra H with product $\bar{\mu}$, coproduct $\bar{\Delta}$ and augmentation $\bar{\epsilon}$, the set of Lie elements $\widehat{\mathbb{P}}(H)$ are elements x in the kernel of $\bar{\epsilon}$ satisfying $\bar{\Delta}(x) = 1 \widehat{\otimes} x + x \widehat{\otimes} 1$. $\widehat{\mathbb{P}}(H)$ forms a Lie algebra where the bracket is given by the anti-symmetrization of the product $\bar{\mu}$. We define the set $\widehat{\mathbb{G}}(H) \subset H$ as the elements x such that $\overline{\Delta}(f) = f \widehat{\otimes} f$. In particular, $(\mathbb{G}(H), \bar{\mu})$ is a group. The completeness allows us to define a bijection

$$\log : \widehat{\mathbb{P}}(H) \longleftrightarrow \widehat{\mathbb{G}}(H) : \exp$$

which gives a correspondence between groups and algebras. In some cases we will denote $\exp(-)$ by $e^{(-)}$.
Let (C, δ^*) be as above. One can show that $\left(\widehat{T}\left(W_+^1[1]\right)^*, \widehat{\mu}\right)$ is a completed graded Hopf algebra where the coproduct is given by the shuffle coproduct Δ' (see Appendix A.4). Let $\mathcal{R} \subset \widehat{\mathbb{L}}\left(W_+^1[1]\right)^*$ and $\mathcal{R}' \subset \widehat{T}\left(W_+^1[1]\right)^*$ be as defined in the previous section. The quotient $\widehat{T}\left(W_+[1]\right)^* / \mathcal{R}'$ is again a Hopf algebra and it is isomorphic to the complete universal enveloping algebra of $\widehat{\mathbb{L}}\left(W_+^1[1]\right)^* / \mathcal{R}$. The group $\mathbb{G}\left(\widehat{T}\left(W_+^1[1]\right)^* / \mathcal{R}'\right)$ is given by the formal power series

$$f = 1 + \sum a_i X_i + \sum a_{ij} X_i X_j + \cdots + \sum a_{i_1 \ldots i_r} X_{i_1} \ldots X_{i_r} + \ldots$$

such that $\widehat{\Delta'}(f) = f \widehat{\otimes} f$. By the above discussion, this group corresponds to $U := \exp(\mathfrak{u}) \subset \widehat{T}\left(W_+[1]\right)^* / \mathcal{R}'$ where $\mathfrak{u} = \widehat{\mathbb{L}}\left(W_+^1[1]\right)^* / \mathcal{R}$. The group structure in terms of $\mathfrak{u}$ is given by the Baker-Campbell-Hausdorf formula $BCH(-,-) : \mathfrak{u} \times \mathfrak{u} \to \mathfrak{u}$, i.e. the unique solution of

$$\exp BCH(x,y) = \exp(x)\exp(y). \tag{1.10}$$

We have the following

Proposition 1.1.31. *The image of*

$$\Theta_0 : \pi(M,p) \to \left(\widehat{T}\left(W_+[1]\right)^* / \mathcal{R}', \widehat{\mu}\right)$$

is contained in U.

Proof. It is sufficient to consider the case where $\mathcal{R} = 0$. This is contained in [29, Proposition 4.1]. ☐

Remark 1.1.32. The monodromy or holonomy representation

$$\Theta_0 : \pi(M,p) \to U$$

gives a complete characterization of C_0 in terms of gauge theory. More precisely, this map corresponds to a flat connection in a principal U-bundle on M. The action of U on $\mathfrak{u}$ via the adjoint representation induces a flat connection on M with fiber $\mathfrak{u}$ which is gauge equivalent (see Subsection 2.4.1) to C_0 (for more details see for example the introduction of [14]).

1.1.6 Transport and generalized holonomy

Let M be a connected smooth manifold with finite type cohomology. Let $A \subset A_{DR}(M)$ be a differential graded algebra with connected and finite type cohomology endowed with a decomposition (1.2). Let W be equipped with the basis $1, w_1, \dots w_n$ as above and let (C, δ^*) be its associated homological pair. We denote by T its transport as defined in the previous section. We presents some of the main results of Chen. The material of this subsection comes from [13] and the nice introduction of [14] due to Hain and Tondeur. For a compact plot α on PM we define

$$\langle T, \alpha \rangle := \langle 1, \alpha \rangle + \sum \langle T_i, \alpha \rangle X_i + \sum \langle T_{ij}, \alpha \rangle X_i X_j + \cdots + \sum \langle T_{i_1 \dots i_r}, \alpha \rangle X_{i_1} \dots X_{i_r} + \dots .$$

in $\widehat{T}\left(W_+[1]\right)^*$.

Lemma 1.1.33. *Let α, β be two compact plots of PM such that $\alpha \times \beta$ is well-defined. Then*

$$\langle T, \alpha \times \beta \rangle := \widehat{\mu}\left(\langle T, \alpha \rangle, \langle T, \beta \rangle\right).$$

We denote by ΩM the loop space of M, i.e the set of differentiable maps $\gamma : [0,1] \to M$ such that $\gamma(0) = \gamma(1)$. We have $\Omega M \subset PM$, hence ΩM is a differentiable space. In particular, the inclusion $\Omega M \hookrightarrow PM$ is a smooth map.

Proposition 1.1.34. *Let M, C be as above.*

1. *On PM we have*
$$dT = \delta^* T \pm M_2\left(\left(p_0^* \widehat{\otimes} Id\right) C - \left(p_1^* \widehat{\otimes} Id\right) C, C\right).$$

2. *Let $T|_{\Omega M}$ be the restriction of T at ΩM along the above inclusion, then*
$$dT|_{\Omega M} = \delta^* T|_{\Omega M}.$$

Proof. This follows form the standard properties of iterated integrals and by the Maurer-Cartan equation. For a proof see [13], Theorem 3.3.1. □

Let $p \in M$. We denote by $\Omega_p M$ the based loop space and by $(C(\Omega_p M), \partial)$ its singular simplicial chain complex. An element of $C_n(\Omega_p M)$ is thus the abelian group generated by smooth maps $\alpha : \Delta_{geo}[n] \to \Omega_p M$. An element $c \in C_n(\Omega_p M)$ can be written as $c = \sum_i n_i \alpha_i$.

Corollary 1.1.35. *Let $(C(\Omega_p M), \partial)$ be the simplicial complex of the loop space. The map $\Theta : (C(\Omega_p M), \partial) \to \left(\widehat{T}\left(W_+[1]\right)^*, \delta^*\right)$ defined by*

$$\Theta(c) := \sum_i n_i \left\langle T|_{\Omega_p M}, \alpha_i \right\rangle$$

for $c = \sum_i n_i \alpha_i$ is a chain map.

Proof. By Stokes'theorem we have $\Theta(c) = \langle T|_{\Omega M}, \partial c \rangle = \langle dT|_{\Omega M}, c \rangle$ which is equal to $\langle \delta^* T|_{\Omega M}, c \rangle$ by the theorem above. Then $\langle \delta^* T|_{\Omega M}, c \rangle = \delta^* \langle T|_{\Omega M}, c \rangle$ by the definition of δ^*. □

The based loop space is an example of H-spaces. In particular, $H^\bullet(\Omega_p M)$ and $H_\bullet(\Omega_p M)$ have the structure of a Hopf algebra (see [31]). On the other hand $\widehat{T}\left(W_+[1]\right)^* \supset \widehat{\mathbb{L}}(W_+[1])$ may be viewed as the complete universal envelopping algebra of $\widehat{\mathbb{L}}(W_+[1])$, hence a Hopf algebra. Since the δ^* is a codifferential and preserves the filtration $I^\bullet$, the cohomology $H^{-\bullet}\left(\widehat{T}\left(W_+[1]\right)^*, \delta^*\right)$ is a Hopf algebra as well. For a group G and a field $\Bbbk$ of characteristic 0, we denote by $\Bbbk[G]$ its group ring. It is a Hopf algebra where the coproduct is given by $\Delta(g) := g \otimes g$. Let J be the kernel of the augmentation map $\Bbbk[G] \to \Bbbk$ that sends each element of G to 1. The powers of J (with respect to the multiplication) define a filtration J^i. Moreover the completion $\Bbbk[G]^\wedge$ is a complete Hopf algebra, i.e a complete vector space such that the structure maps are continuous (see [23] for more details).

Theorem 1.1.36. *Let M, A, C and δ^* be as above.*

1. *Assume that A is a model for $A_{DR}(M)$. The map Θ induces a morphism of Hopf algebras*

$$\Theta_\bullet \,:\, H_\bullet\left(C(\Omega M),\partial\right)\to H^{-\bullet}\left(\widehat{T}\left(W_+[1]\right)^*,\delta^*\right).$$

2. *Assume that A is a model for $A_{DR}(M)$. The above morphism is an isomorphism if M is simply connected.*

3. *Assume that A is a 1-model for $A_{DR}(M)$. Let $\mathcal{R}'\subset\widehat{T}\left(W^1_+[1]\right)^*$ be the completion of the ideal generated by $\delta^* X_i$ such that $|X_i|=1$. If M is not simply connected, Θ_0 satisfies $\Theta_0\left(J^i\right)\subset I^i$ for any i and it induces a morphism of Hopf-algebras*

$$\Theta_0\,:\,\mathbb{R}\left[\pi_1(M,p)\right]\cong H_0\left(C(\Omega M),\partial\right)\to H^0\left(\widehat{T}\left(W_+[1]\right)^*,\delta^*\right)=\widehat{T}\left(W^1_+[1]\right)^*/\mathcal{R}'.$$

which corresponds to the J-adic completion of $\mathbb{R}\left[\pi_1(M,p)\right]$ as a Hopf algebra.

Definition 1.1.37. A group H is said to be *Malcev complete* if it is isomorphic to the group like elements of $\Bbbk\left[G\right]^\wedge$ for some group G. For a group G we call the Lie elements of $\Bbbk\left[G\right]^\wedge$ *the Malcev Lie algebra.* The Malcev completion of a group consists of a Malcev complete group $\widehat{G}$ and a group homomorphism $G\to\widehat{G}$ which is universal, i.e for any Malcev complete group H, any group homomorphism $G\to H$ factors uniquely trough $\widehat{G}$.

The Malcev completion can be constructed by taking the group like elements of $\Bbbk\left[G\right]^\wedge$. The construction involves two functors. On one hand we start with a group G and we get a complete Hopf algebra $\Bbbk\left[G\right]^\wedge$, on the other hand we start with a complete Hopf algebra and we get a group using $\mathbb{G}(-)$. These two functors are adjoint (see [23]) and this induces the desired homomorphism $G\to\widehat{G}$. Let M A, C and δ^* be as above. Assume that A is a 1-model for $A_{DR}(M)$. The next corollary characterizes the monodromy representation of the flat connection r_*C_0 induced by the homological pair (C,δ^*).

Corollary 1.1.38. *Let C_0 be as above. The monodromy representation of Proposition 1.1.31*

$$\Theta_0\,:\,\pi(M,p)\to U$$

is the Malcev completion of $\pi(M,p)$. In particular $\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*/\mathcal{R}$ is the Malcev Lie algebra of $\pi_1(M,p)$.

There is a dual version of the above statements involving the cohomology of the loop space. Assume that $A\subset A_{DR}(M)$ is a differential graded subalgebra which is connected, i.e. $A^0=\mathbb{R}$. We have a map

$$BA\to A_{DR}(PM)$$

given by

$$s(w_1)\cdots s\left(w_r\right)\mapsto\int w_1\cdots w_r. \tag{1.11}$$

For $w_1,\ldots,w_r$ 1-forms the iterated integral $\int w_1\cdots w_r$ defines a function $F\,:\,PM\to\mathbb{R}$. We say that F is a *homotopy functional* if $F(\gamma)=F(\gamma')$ for any pair of homotopic paths.

Lemma 1.1.39 ([13])**.** *Let $w_1,\ldots,w_r$ be 1-forms. Then $\int w_1\cdots w_r$ is a homotopy functional if and only if $s(w_1)\cdots s_{w_r}$ is closed in BA.*

For a $p\in M$, consider the map

$$BA\to A_{DR}(\Omega_pM)$$

given by

$$s(w_1)\cdots s\left(w_r\right)\mapsto\int w_1\cdots w_r. \tag{1.12}$$

Theorem 1.1.40 ([13])**.** *Let $A\subset A_{DR}(M)$ be a connected differential graded subalgebra.*

1. *Assume that M is simply connected and A is a model. The map (1.12) restricted to Ω_xM induces an Hopf algebra isomorphismin cohomology.*

2. *Assume that A is a 1-model. If M is not simply connected and it has a finitely generated fundamental group, there is an isomorphism*

$$\left\{ \substack{\textit{elements } s(w_1)\cdots s(w_r) \textit{ in } H^0(BA) \\ \textit{of lenght } r} \right\} \mapsto \mathrm{Hom}_{\mathbb{Z}} \left(\mathbb{Z}\pi_1(M,x)/J^{r+1}, \mathbb{R} \right)$$

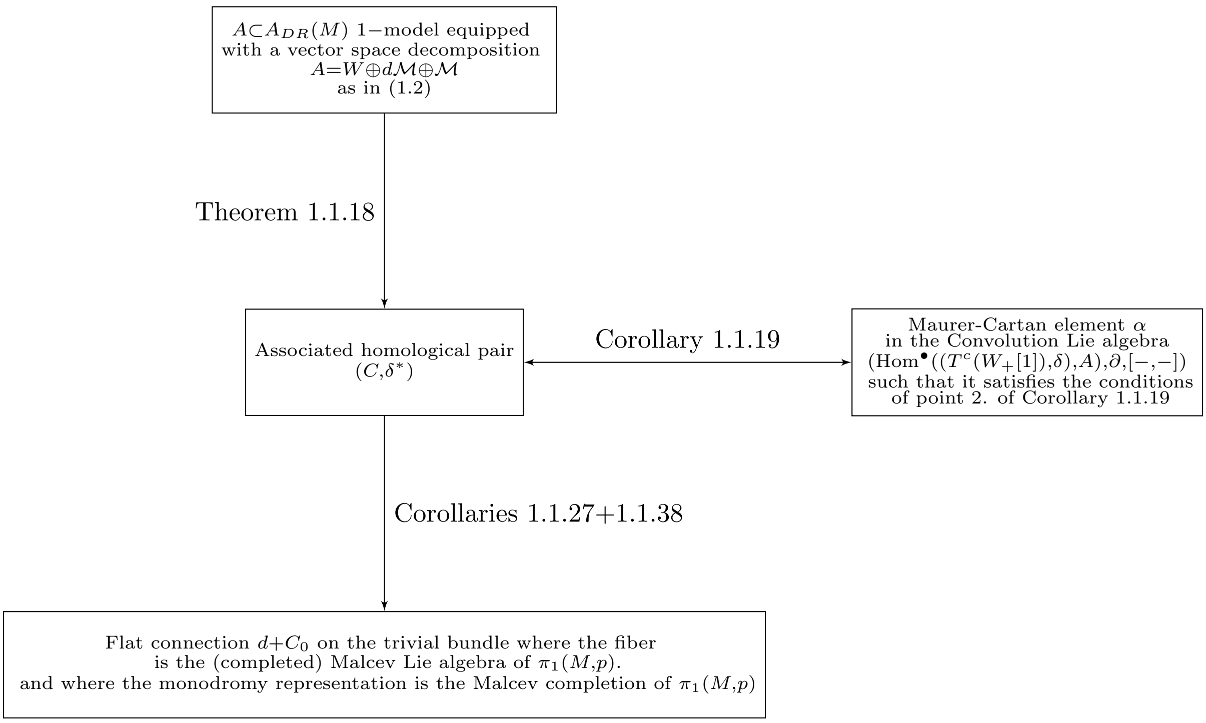
given by the map (1.12). *Moreover the fact above is true even if we replace $\pi_1(M,x)$ with a torsor $\pi_1(M;x,y)$.*

It follows that iterated integrals are good objects to describe homotopy functionals.

Remark 1.1.41. A gauge-theoretic proof of the second point of the theorem above is contained in [28].

1.1.7 A summary

We give a picture of the properties of the Homological pairs. Let M be a connected smooth manifold with finite type cohomology and $p \in M$.



As noticed by Chen in [13], the above diagrams holds as well if M is a complex smooth manifold and $A_{DR}(M)$ is the differential graded algebra of smooth complex differential forms.

1.2 Extension on simplicial manifolds

In this section we give a summary of the main results of Chapter 2. We extend the Chen formalism on simplicial manifolds. Smooth manifolds equipped with a smooth group action can be naturally turned into simplicial manifolds. In the last subsection, we apply this formalism to this class of manifolds.

1.2.1 A de Rham functor on simplicial manifolds and C_∞-algebras

We define a de Rham functor on simplicial manifolds. All the results are taken from [19] and [26]. Let $\boldsymbol{\Delta}$ be the simplex category, i.e the category where the objects are finite ordered sets

$$[n] := \{0 < 1 < 2 < 3 < \cdots < n\}$$

and the maps are order-preserving morphism. For each n we denote the coface maps by $d_n^i : [n] \to [n+1]$, $i = 0, 1, \ldots, n+1$ and by $s_n^i : [n+1] \to [n]$, $i = 0, 1, \ldots, n$ the codegenerancy maps.

A cosimplicial object in a category $\mathcal{C}$ is a functor $X : \mathbf{\Delta} \to \mathcal{C}$. We define $X^n := X([n])$ and by abuse of notation we denote $X\left(d^i_n\right)$ by d^i and $X\left(s^i_n\right)$ by s^i for all n and i. Dually, a simplicial object is a functor $X : \mathbf{\Delta}^{op} \to \mathcal{C}$. We define $X_n := X([n])$ and by abuse of notation we denote $X\left(d^i_n\right)$ with d_i and $X\left(s^i_n\right)$ by s_i. A (complex or real) simplicial smooth manifold $M_\bullet$ is a cosimplicial object in the category (complex or real) manifolds. Equivalently it consist of a family $\{M_n\}_{n\geq 0}$ equipped with smooth maps

$$d^i_n : M_{n+1} \to M_n, \text{ for } i = 1, \ldots n+1$$

and

$$s^i_n : M_n \to M_{n+1}, \text{ for } i = 1, \ldots n$$

that satisfy some relations imposed by the functoriality. We give two examples. Let M be a smooth manifold, then M gives an example of simplicial manifold $M_\bullet$ by setting

$$M_n := M \quad d^i_n, s^i_n = Id$$

for any n. We call such a simplicial manifold *the trivial simplicial manifold associated to M*.
Let G be a Lie group, and let M be a manifold equipped with a left smooth (or holomorphic) G-action. We define the simplicial manifold $M_\bullet G$ as follows:

$$M_n G = M \times G^n.$$

The face maps $d^i : M_n G \to M_{n-1} G$ for $i = 0, 1, \ldots n$ are

$$d^i(x, g_1, \ldots, g_n) := \begin{cases} (g_1 x, g_2, \ldots, g_n), \text{ if } i = 0, \\ (x, g_1, \ldots, g_i g_{i+1}, \ldots, g_n), \text{ if } 1 < i < n \\ (x, g_1, \ldots, g_{n-1}), \text{ if } i = n. \end{cases}$$

The degenerancy maps $s^i : M_n G \to M_{n+1} G$ are defined via

$$s^i(x, g_1, \ldots, g_n) := (x, g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n)$$

for $i = 1, \ldots, n$.

Definition 1.2.1. We call the simplicial manifold $M_\bullet G$ the action groupoid.

In this thesis we consider only discrete groups. We mainly work on the category of complex simplicial manifolds. Let $\mathrm{Diff}_{\mathbb{C}}$ be the category of complex smooth manifolds. For $M \in \mathrm{Diff}_{\mathbb{C}}$ we denote by $A_{DR}(M)$ the de Rham differential graded algebra of smooth complex differential forms (see Subsection 2.3.1). The functor $A_{DR}(-)$ is contravariant, hence to any simplicial manifold $M_\bullet$ it associates a cosimplicial commutative differential graded algebra $A_{DR}(M_\bullet)$. We turn this object into a cochain complex. We define the bigraded vector space $A = \oplus_{p,q} A^{p,q}$ where $A^{p,q} := A^q_{DR}(M_q)$. The alternating sum of the maps $\left(d^i_n\right)^*$ gives a differential

$$\tilde{\partial} : A^{p,q} \to A^{p+1,q},$$

and the de Rham differential d induces

$$d : A^{p,q} \to A^{p,q+1}.$$

For an element $a \in A^{p,q}$, we define $D(a) := \tilde{\partial} a + (-1)^p da$. In particular (A, D) is a chain complex. We define $\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right) \subset A$ as the sub vector space consisting of $b \in A$ such that its pullback along the degenerancy maps s^i_n vanishes for any n and i. One can show that $\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), D\right)$ is a cochain complex as well. An element w in $\left(\mathrm{Tot}^s_N\left(A_{DR}(M_\bullet)\right), D\right)$ consists of a family of differential forms $w_0, \ldots, w_s$ such that

$$w_j \in A^{s-j}_{DR}(M_j)$$

vanishes along the pull back s^i_{j-1} for any $i = 1, \ldots, j-1$. This construction is functorial and we get a functor

$$M_\bullet \mapsto \mathrm{Tot}^\bullet_N\left(A_{DR}(M_\bullet)\right)$$

from simplicial manifolds to cochain complexes. This functor is the canonical[2] extension of the de Rham functor into the category of simplicial manifolds.

[2]We use the word "canonical" because this functor can be interpreted (modulo some quasi isomorphisms) to the left Kan extension of the ordinary de Rham functor along the Yoneda embedding for a suitable sites.

Proposition 1.2.2 ([19]). *Let M be a smooth manifold.*

1. *Let $M_\bullet$ be the associated trivial simplicial manifold. There is a canonical isomorphism (of differential graded algebras)*
$$\mathrm{Tot}_N^\bullet\left(A_{DR}(M_\bullet)\right) \cong A_{DR}(M).$$

2. *Assume that M is equipped with a smooth group action of a discrete group G. If it acts discretely and properly discontinuosly, the inclusion $A_{DR}(M/G) \hookrightarrow \mathrm{Tot}_N^\bullet\left(A_{DR}(M_\bullet G)\right)$ induces a quasi-isomorphism.*

Remark 1.2.3. The above proposition is true also for smooth differential forms with logarithmic singularities (see Subsection 2.3.1). Assume that M is a smooth complex manifold equipped with a normal crossing divisor $\mathcal{D}$. Then $M - \mathcal{D}$ is again a smooth complex manifold. We denote by

$$A_{DR}(\log(\mathcal{D})) \subset A_{DR}(M - \mathcal{D})$$

the differential graded algebra of smooth differential forms with logarithmic singularities along $\mathcal{D}$. The cohomology $A_{DR}(\log(\mathcal{D}))$ corresponds to $H^\bullet(M - \mathcal{D})$. Now assume that M is equipped with a smooth group action of a group G that preserves $\mathcal{D}$. Then $(M - \mathcal{D})_\bullet G$ is an action groupoid. If G is discrete, we consider the action groupoid $\mathcal{D}_\bullet G$, in particular $\mathcal{D}_n G$ is a normal crossing divisor of $M_n G$ for any n. We denote by $A_{DR}(\log(\mathcal{D})_\bullet G)$ the unital cosimplicial commutative differential graded algebra obtained by applying the functor $A_{DR}(\log(-))$ on $(M - \mathcal{D})_\bullet$. By taking the normalized complex we get a functor

$$(M - \mathcal{D})_\bullet \mapsto \mathrm{Tot}_N^\bullet\left(\log(\mathcal{D})_\bullet G\right).$$

One can show (see Proposition 2.3.10) that the inclusion

$$\mathrm{Tot}_N^\bullet\left(\log(\mathcal{D})_\bullet G\right) \hookrightarrow \mathrm{Tot}_N^\bullet\left(A_{DR}((M - \mathcal{D})_\bullet)\right)$$

is a quasi-isomorphism.

Unfortunately, $\mathrm{Tot}_N^\bullet\left(A_{DR}(M_\bullet)\right)$ and $\mathrm{Tot}_N^\bullet\left(\log(\mathcal{D})_\bullet G\right)$ are differential graded algebras only in a weak sense: they are C_∞-algebras, i.e. a commutative version of A_∞-algebras. Next, we give an intuitive definition. A detailed summary is given in Subsection 2.1.1, for more details see [39]. An A_∞-algebra consists of cochain complex (A, m_1) equipped with a multiplication $m_2 : A \otimes A \to A$ which is associative only up to a family of homotopies
$$m_n : A^{\otimes n} \to A, \quad n > 2$$
of degree $2 - n$ such that

1. m_2 is a chain map,
2. m_3 measures the failure for m_2 to be associative,
3. m_4 measures the failure for m_3 to satisfy the pentagon equation,
4. and so on.

A differential graded associative algebra gives an example of an A_∞-algebra, where $m_n = 0$, for $n > 2$. We denote an A_∞-algebra by $(A, m_\bullet)$. The cohomology of $(A, m_\bullet)$ is defined as

$$H^\bullet(A, m_1).$$

One can show that m_2 is associative on $H^\bullet(A, m_1)$ and that the maps $m_\bullet$ induce an A_∞-structure on $H^\bullet(A, m_1)$.
A C_∞-algebra consists of an A_∞-algebra where the maps $m_n : A^{\otimes n} \to A$ satisfy additional commutativity relations. In particular, m_2 is a commutative (but not associative) bilinear map. A C_∞-subalgebra B of $(A, m_\bullet)$ is a graded subvector space closed under the operations $m_\bullet$. B is quasi-isomorphic to $(A, m_\bullet)$ if the inclusion induces a quasi-isomorphism. For $1 \leq j \leq \infty$ we define j-models for C_∞-algebras, in the same way as we did for differential graded algebras (see Definition 1.1.14). The next theorem is a special case of Theorem 17, in [26].

Theorem 1.2.4 ([26]). *The functors*

$$M_\bullet \mapsto \mathrm{Tot}_N^\bullet\left(A_{DR}(M_\bullet)\right), \quad (M-D)_\bullet \mapsto \mathrm{Tot}_N^\bullet\left(\log(D)_\bullet G\right)$$

can be upgraded to functors from the category of smooth complex simplicial manifolds to the category of C_∞*-algebras.*

In this thesis, we consider $\mathrm{Tot}_N^\bullet(A_{DR}(M_\bullet))$ equipped with $m_\bullet$ and we denote m_1 with D. In Section 2.2 we present a formula for the calculation of the higher products m_n between 1-forms.

Warning 1.2.5. In order to avoid repetitions, we use the following assumption. Let M be a smooth complex manifold equipped with a normal crossing divisor $\mathcal{D}$ and equipped with a smooth group action of a discrete group G that preserves $\mathcal{D}$. Then all the statements for $\mathrm{Tot}_N^\bullet\left(A_{DR}((M-\mathcal{D})_\bullet G)\right)$ are true for $\mathrm{Tot}_N^\bullet A_{DR}(\log(\mathcal{D})_\bullet G)$ as well (unless specifically written.)

1.2.2 Convolution L_∞-algebras

Fix a simplicial manifold $M_\bullet$. Let $A \subset \left(\mathrm{Tot}_N^\bullet(A_{DR}(M_\bullet)), m_\bullet\right)$ be a C_∞-subalgebra. Assume that its cohomology is connected and of finite type. We fix a vector space decomposition

$$A = W \oplus D\mathcal{M} \oplus \mathcal{M},$$

as in (1.2). Consider $T^c(W_+[1])$ equipped with a codifferential δ. At this point, we consider the vector space $A\widehat{\otimes}\widehat{T}(W_+[1])^*$. By Lemma 1.1.16 we should expected that it carries an algebraic structure similar to a Lie algebra and that this structure is induced by $m_\bullet$. In order to understand the next proposition we need to introduce a new object. An L_∞-algebra is a cochain complex (L, l_1) equipped with a family of maps $l_n : L^{\otimes n} \to L$ of degree $n-2$ such that

1. l_2 is a chain map and it is asymmetric,
2. l_3 measures the failure for l_2 to satisfy the Jacobi identity,
3. and so on.

We may consider L_∞-algebra as differential graded Lie algebra where the Jacobi identity for the bracket l_2 is relaxed up to a family of coherent homotopies l_n, with $n > 3$. A *Maurer-Cartan element* in an L_∞ algebra L is a $\alpha \in L^1$ such that

$$l_1(\alpha) + \sum_{k>1} \frac{l_k(\alpha, \ldots, \alpha)}{k!} = 0.$$

Notice that, in order that the equation above is well-defined we need the graded vector space underlying L to be complete. The next proposition corresponds to Lemma 2.1.15 in this thesis.

Proposition A. *The graded vector space*

$$A\widehat{\otimes}\widehat{T}(W_+[1])^* \cong \mathrm{Hom}^\bullet\left((T^c(W_+[1]), \delta), A\right),$$

carries the structure of an L_∞*-algebra where* $l_1(f) = -Df - (-1)^{|f|}C\delta$ *on homogeneous elements.*

We denote such a structure by $l'_\bullet$ and we call $\left(\mathrm{Hom}^\bullet\left((T^c(W_+[1]), \delta), A\right)\right)$ *the convolution* L_∞*-algebra.* We denote by $I^\bullet$ the filtration on $T(W_+[1])^*$ obtained by the powers of its augmentation ideal I. The proposition above motivates the following.

Definition 1.2.6. Let $A \subset \left(\mathrm{Tot}_N^\bullet(A_{DR}(M_\bullet)), m_\bullet\right)$ be a C_∞-subalgebra and let W be a connected graded vector space of finite type. A homological pair (C, δ^*) consists of a formal power series

$$C = \sum w_i X_i + \sum w_{ij} X_i X_j + \cdots + \sum w_{i_1 \ldots i_r} X_{i_1} \ldots X_{i_r} + \cdots \in A\widehat{\otimes}\widehat{T}(W_+[1])^*$$

and a codifferential δ of $T^c(W_+[1])$ such that

$$l'_1(C) + \sum \frac{l'_k(C, \ldots, C)}{k!} = 0.$$

A homological pair is said to be *reduced* if it satisfies the conditions of Definition 1.1.23.

The next Theorem is proved in this thesis, see Section 2.3.

Theorem B. *Let* $A \subset (\mathrm{Tot}_N^\bullet(A_{DR}(M_\bullet)), m_\bullet)$ *be a* C_∞*- subalgebra. Assume that its cohomology is connected and of finite type. We fix a vector space decomposition*

$$A = W \oplus D\mathcal{M} \oplus \mathcal{M}, \tag{1.13}$$

as in (1.2)*.*

1. *There exists a unique homological pair* (C, δ^*) *such that*

   i) *the term* $\sum w_i X_i$ *of* C *is as given in 1.6,*

   ii) *the coefficients* $w_{ij}, \ldots, w_{i_1 \ldots i_p}, \ldots$ *belong to* $\mathcal{M}$*,*

   iii) $\delta^*(I) \subset I^2$*.*

   We call such a homological pair the homological pair assocaited to (1.13)*.*

2. *Let* (C, δ^*) *be as above. Then* δ^* *has a well-defined restriction* $\delta^* : \widehat{\mathbb{L}}(W_+[1])^* \to \widehat{\mathbb{L}}(W_+[1])^*$ *and* $C \in A(M) \widehat{\otimes} \widehat{\mathbb{L}}(W_+[1])^*$ *where the completion is taken with respect to the filtration* $I^\bullet$ *defined in Section 1.1.4.*

3. *Let* (C, δ^*) *be as above. Let* $\mathcal{R} \subset \widehat{\mathbb{L}}\left(W_+^1[1]\right)^*$ *be the completion of the Lie ideal generated by* $\delta^* X_i$ *such that* $|X_i| = 1$*. The* L_∞*-structure* $l'_\bullet$ *restricted to* $A \widehat{\otimes} \widehat{\mathbb{L}}(W_+[1])^*$ *is well-defined. Moreover it induces an* L_∞*-structure* $l_\bullet$ *on* $A \widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*\right)/\mathcal{R}$ *such that the map* π *obtained by the concatenation*
$$A \widehat{\otimes}\left(\widehat{\mathbb{L}}(W_+[1])^*\right) \to A \widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*\right) \to A \widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*\right)/\mathcal{R}$$
   preserves Maurer-Cartan elements.

Remark 1.2.7. In the same way as for Corollary 1.1.19 the homological pair (C, δ^*) associated to the given vector space decomposition, corresponds to a codifferential δ of $T^c(W_+[1])$ and a Maurer-Cartan element $\alpha \in \mathrm{Hom}^\bullet((T^c(W_+[1]), \delta), A)$ such that

i) $\alpha(X_i) = w_i$,

ii) $\alpha(X_{i_1} \cdots X_p) = w_{i_1 \ldots i_p} \in \mathcal{M}$ for $p > 1$,

iii) $\delta^*(I) \subset I^2$.

There is an A_∞ version of Lemma 1.1.13. More precisely, consider the tensor coalgebra $T^c(A[1])$, then $m_\bullet$ corresponds to a codifferential δ' on it. One can show that α as defined above corresponds under such an equivalence to a morphism of differential graded coalgebras $F : (T^c(W_+[1]), \delta) \to (T^c(A[1]), \delta')$ such that

i) $F_1^1(w_i[1]) = w_i[1]$,

ii) $F_p^1\left((w_{i_1}[1]) \cdots (w_{i_p}[1])\right) = w_{i_1 \ldots i_p}[1]$ for $p > 1$,

where F_p^1 is as in Proposition A.1.6. Concretely: a homological pair is equivalent to a pair (F, δ) as above such that $\delta^*(I) \subset I^2$. In this thesis we first construct such a pair via the homotopy transfer theorem (see Theorem 2.1.26) and then we translate it in terms of homological pairs (see Theorem 2.1.40).

1.2.3 A geometric connection on M_0

Given $M_\bullet$ as above. Let $A \subset (\mathrm{Tot}_N^\bullet(A_{DR}(M_\bullet)), m_\bullet)$ be a C_∞-subalgebra. Assume that its cohomology is connected and of finite type. We fix a vector space decomposition as in (1.13) and a reduced homological pair (C, δ^*). Furthermore, we set $C_0 := \pi(C)$. In particular $C_0 \in A^1 \widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W_+^1[1]\right)^*\right)/\mathcal{R}$. By definition we have

$$A^1 = A^{1,0} \oplus A^{0,1}$$

where $A^{0,1} \subset A^1_{DR}(M_0)$ und $A^{1,0} \subset A^0_{DR}(M_1)$. Let $r : A \to \oplus_{p\geq 0} A^{0,p}$ be defined by

$$r(a) = \begin{cases} a & \text{if } a \in A^{0,p} \\ 0 & \text{if } a \in A^{q,p} \text{ if } q \neq 0. \end{cases}$$

We denote $r_* := r\widehat{\otimes} Id : A\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R} \to A^{0,\bullet}\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R}$. Notice that $A^{0,\bullet} \subset A^\bullet_{DR}(M_0)$ is a differential graded Lie algebra. One can show that

$$\left(A^{0,\bullet}\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R}, -d, [-,-]\right)$$

is a differential graded Lie algebra, where the bracket are the one defined in 1.4.

Proposition C. *Let A and C_0 be as above.*

1. *The map r_* sends Maurer-Cartan elements to Maurer-Cartan elements in the differential graded Lie algebra $\left(A^{0,\bullet}\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R}, -d, [-,-]\right)$. In particular*

$$-d\left(r_*C_0\right) + \frac{[r_*C_0, r_*C_0]}{2} = 0$$

2. *Consider the Lie algebra equipped with the adjoint action. Then $d - r_*C_0$ is a flat connection*[3] *on the trivial bundle on M_0 with fiber $\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R}$.*

3. *Let $M_\bullet G$ be an action groupoid where G is discrete and it acts properly and discontinuosly on M. Assume that the cohomology of M/G is connected, of finite type and that A is a 1-model for $\mathrm{Tot}^\bullet_N\left(A_{DR}(M_\bullet G)\right)$. Let (C, δ^*) be the homological pair associated to (1.13). Then $d - r_*C_0$ defines a flat connection on the trivial bundle on M with fiber $\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R}$. In particular $\left(\widehat{\mathbb{L}}\left(W^1_+[1]\right)^*\right)/\mathcal{R}$ is isomorphic to the Malcev Lie algebra of $\pi_1(M/G)$.*

1.2.4 A comparison between the two approaches

Given $M_\bullet G$ as above. Let $A, B \subset \left(\mathrm{Tot}^\bullet_N\left(A_{DR}(M_\bullet G)\right), m_\bullet\right)$ be 1-subalgebras such that B is a 1-model. Assume that their cohomology is connected and of finite type. We fix a vector space decomposition as in (1.13) on A and B via

$$A = W \oplus \mathcal{M} \oplus D\mathcal{M}, \quad B = W' \oplus \mathcal{M}' \oplus D\mathcal{M}'.$$

Let (C, δ^*) be a reduced homological pair with respect to A and let (C', δ^*) be its associated homological pair with respect to B. By Proposition C, (C, δ^*) there is a finite dimensional Lie algebra $\mathfrak{u}$ and flat connection form $r_*C_0 \in A^1_{DR}(M)\widehat{\otimes}\mathfrak{u}$ such that $d - r_*C_0$ is a flat connection on the trivial bundle on M with fiber $\mathfrak{u}$. The same arguments works for (C', δ^*) as well and we have a flat connection form $r_*{C'}_0 \in A^1_{DR}(M)\widehat{\otimes}\mathfrak{u}'$ such that $d - r_*{C'}_0$ is a flat connection on the trivial bundle on M with fiber $\mathfrak{u}'$ which corresponds to Malcev Lie algebra of $\pi_1(M)$. For any morphism of Lie algebras $K^* : \mathfrak{u}' \to \mathfrak{u}$ the map

$$k_* := Id\widehat{\otimes}K^* : A^1_{DR}(M)\widehat{\otimes}\mathfrak{u}' \to A^1_{DR}(M)\widehat{\otimes}\mathfrak{u}$$

sends Maurer-Cartan elements to Maurer-Cartan elements. We have that $\left(A_{DR}(M)\widehat{\otimes}\mathfrak{u}, -d, [-,-]\right)$ is a differential graded Lie algebra which is complete (with respect to I) if $\mathfrak{u}$ is pronilpotent. Two Maurer-Cartan elements α_0, α_1 are said to be gauge equivalent if there exists a $u \in A^0_{DR}(M)\widehat{\otimes}\mathfrak{u}$ such that

$$e^u(\alpha_0) := e^{\mathrm{Ad}_u}(\alpha) + \frac{1 - e^{\mathrm{Ad}_u}}{\mathrm{Ad}_u}(-du) = \alpha_1$$

See Subsection 2.4.1 for more details.

[3] The signs $-$ appears because of point 1. If $M_\bullet = M$ is a constant simplicial manifold then $r_* = Id$ and the flat connection obtained by Chen's theory is $d - C_0$. This flat connection coincide with the one constructed in Proposition C 3.

Theorem D. *1. There exists a Lie algebra morphism $K^* : \mathfrak{u}' \to \mathfrak{u}$ such that $k_* r_* C'_0$ is gauge equivalent to $r_* C_0$. If A is a 1-model and (C, δ^*) is the homological pair associated to the given decomposition, then $K^* : \mathfrak{u}' \to \mathfrak{u}$ is an isomorphism.*

2. Assume that $M = (N - \mathcal{D})$ where $\mathcal{D}$ is a normal crossing divisor which is preserved by the group action G. If $A, B \subset \left(\mathrm{Tot}^{\bullet}_N \left(A_{DR}(\log D_\bullet G)\right), m_\bullet\right)$, the gauge e^u is in $\exp\left(A^0_{DR}(M)\widehat{\otimes}\mathfrak{u}\right)$.

Let $B \subset A_{DR}(M/G)$ be a model with a vector space decomposition as in (1.13). In particular the elements in B are G-invariant. Then C'_0 defines a flat connection on the trivial bundle on M/G with fiber $\mathfrak{u}'$. We recall the notion of factor of a automorphy from Subsection 2.4.2. A factor of automorphy is a smooth function $F : G \times M \to \mathrm{End}(\mathfrak{u})$ such that the function $g : M \times \mathfrak{u} \to M \times \mathfrak{u}$ defined by

$$(p, v) \mapsto (gp, F_g(p)v)$$

defines a group action of G on $M \times \mathfrak{u}$. In particular the quotient

$$(M \times \mathfrak{u})/G$$

is a vector bundle, where the sections are

$$s(gp) = F_g(p)s(p).$$

We denote by E_F the vector bundle induced by the factor of automorphy F. The theorem and the discussion above have the following consequence. Let B be as above. There exists a Lie algebra morphism $K^* : \mathfrak{u}' \to \mathfrak{u}$ such that $(k_* C'_0)$ is gauge equivalent to $r_* C_0$ as element in $A_{DR}(M)\widehat{\otimes}\mathfrak{u}$ via a $u \in A^0_{DR}(M)\widehat{\otimes}\mathfrak{u}$.

Theorem E. *Let A, B, K^* be as above. The map $F : G \times M \to \mathrm{End}(\mathfrak{u})$ given by*

$$F(g, p) := e^{u(p) - u(gp)}$$

defines a factor of automorphy such that $d - r_* C_0$ is a well-defined flat connection on M/G on the bundle E_F.

A flat connection ∇ on a vector bundle E on a smooth manifold M induces a representation of $\pi_1(M, p)$ for any $p \in M$ called *monodromy* or *holonomy representation of* ∇ *at* p. In Subsection 2.4.2, we show that the monodromy representation of $(d - r_* C_0, E_F)$ corresponds to $K^*\Theta'_0$, where Θ'_0 is the monodromy representation of $d - C'_0$. The gauge equivalence implies that the monodromy representation at p of $d - k_* C'_0$ is conjugate to the monodormy representation of $d - r_* C_0$ at p via $e^{u(p)} \in U = \exp(\mathfrak{u})$. If A, B are both 1-models and the two homological pair are both associated to the given decompositions, the map K^* is an isomorphism. In some cases, this map can be easily calculated. We assume that A is a 1-model and that (C, δ^*) is the homological pair associated to the chosen decomposition for A. If $H^2(M/G) = 0$, we have

$$\mathfrak{u} = \mathbb{L}(W^1_+[1]), \quad \mathfrak{u}' = \mathbb{L}({W'}^1_+[1]).$$

The map K^* can be constructed as follows. There are isomorphisms of vector spaces f and f' given by the concatenation of

$$W \to H^\bullet(A, m_1) \to H^1\left(\mathrm{Tot}^\bullet_N\left(A_{DR}(M_\bullet G)\right), m_1\right)$$

and

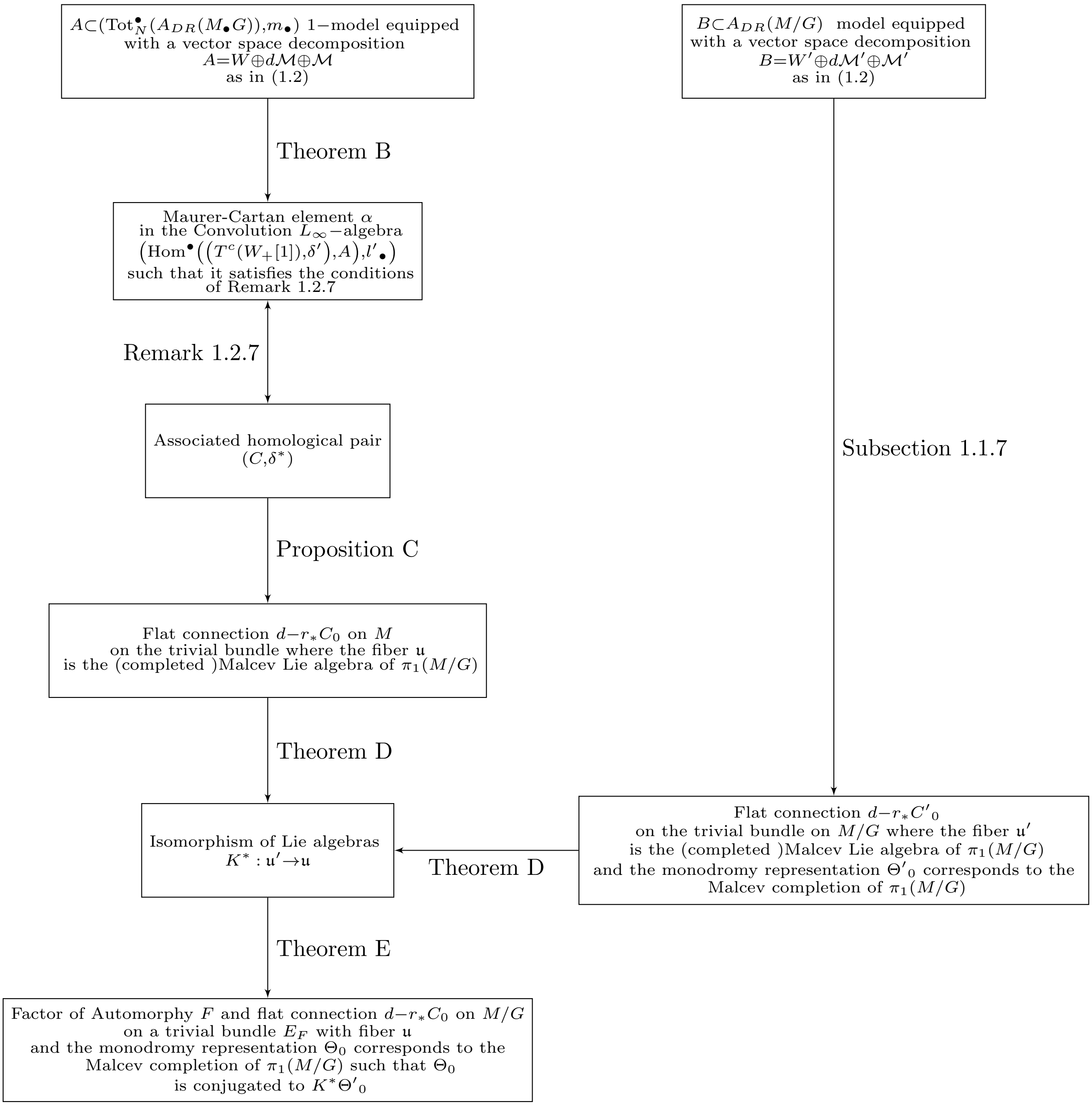
$$W' \to H^\bullet(B, m_1) \to H^1\left(\mathrm{Tot}^\bullet_N\left(A_{DR}(M_\bullet G)\right), m_1\right).$$

Notice that $f^{-1}f' : W \to W'$ defines an isomorphism of vector spaces.

Theorem F. *Assume that B is a model and that (C', δ^*) is the homological pair associated to the chosen decomposition for B. The map $K^* : \mathfrak{u}' \to \mathfrak{u}$ can be written as $K^* := \sum_i^\infty K^*_i$ where K^*_1 corresponds to the map induced by $f^{-1}f'$.*

1.2.5 A summary of Chapter 2

We give a summary of the results about the simplicial version of the notion of homological pair. This is the simplicial manifolds version of the summary given in Section 1.1.7. Let M be a complex smooth manifold and let $M_\bullet G$ be an action groupoid where G is discrete and it acts properly and discontinuosly on M. Furthermore, assume that the chomologly of M/G is connected and of finite type.



1.3 Applications: the universal KZ and KZB connection

The main application of theory developed in the previous subsection is about the comparison of two connection forms called *universal Knizhnik–Zamolodchikov connection (KZ connection for short) and universal Knizhnik–Zamolodchikov-Bernard connection (KZB connection for short)*. In the next two subsections we introduce them briefly. The Knizhnik–Zamolodchikov equation (KZ equation) and Knizhnik–Zamolodchikov-Bernard equations (KZB equation) are differential equations used in quantum field theory (see [45], resp. [4] and [5]), they can be interpretated as connection form on a trivial bundle. The universal KZ connection and the universal KZB connection (KZB connection) are universal version of these connection (see [18] and [11]).

1.3.1 The universal Knizhnik–Zamolodchikov connection

In this subsection we construct the KZ connection as a connection induced by a homological pair. For a topological space X we define its configuration spaces as

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n \ : \ x_i \neq x_j \text{ for } i \neq j\}$$

The topological space $\mathrm{Conf}_n(\mathbb{C} - \{0,1\})$ is a complex smooth manifold. Let $A_{KZ,n}$ be the unital differential graded subalgebra of $A_{DR}(\mathrm{Conf}_n(\mathbb{C} - \{0,1\}))$ generated by $\omega_{i,-1}$, $\omega_{i,0}$, and $\omega_{i,j}$ given by

$$\omega_{ij} := d\log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}$$

for $1 \leq i \neq j \leq n$ such that $z_{-1} := 1$ and $z_0 := 0$. These differential forms satisfy the so called *Arnold relations*

$$\omega_{ij}\omega_{jk} + \omega_{ki}\omega_{ij} + \omega_{jk}\omega_{ki} = 0 \tag{1.14}$$

for i, j, k, l distinct. The next result is proved in [2] (see [43] unless you don't speak Russian).

Proposition 1.3.1. *$A_{KZ,n}$ is a model for $A_{DR}(\mathrm{Conf}_n(\mathbb{C} - \{0,1\}))$.*

Consider $A_{KZ,n}$ equipped with the vector space decomposition $A_{KZ,n} = W \oplus d\mathcal{M} \oplus \mathcal{M}$ such that $W = A_{KZ,n}$ and $\mathcal{M} = 0$. The above proposition ensures that this is a decomposition as in (1.2). We define the rational Lie algebra $\mathfrak{t}_n$ with generators $T_{i,j}$ for $-1 \leq i \neq j \leq n$ with $j > 0$ or $i > 0$ such that

$$T_{ij} = T_{ji}, \quad [T_{ij}, T_{ik} + T_{jk}] = 0, \quad [T_{ij}, T_{kl}] = 0 \tag{1.15}$$

for i, j, k, l distinct. We call $\mathfrak{t}_n$ the *Kohno-Drinfeld Lie algebra.*

Corollary 1.3.2. *Consider $A_{KZ,n}$ equipped with the above decomposition. The degree zero geometric connection associated to that decomposition is*

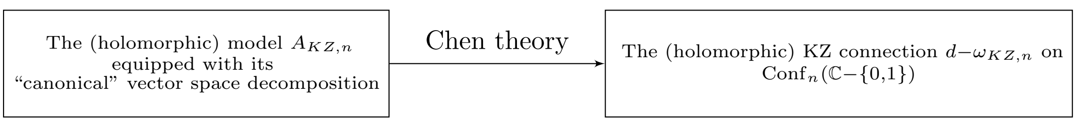
$$\omega_{KZ,n} := \sum_{1 \leq i < j \leq n} \omega_{ij} T_{ij}$$

where the fiber is given by the Lie algebra $\mathfrak{t}_n$.

Proof. Since $0 = \mathcal{M} \oplus d\mathcal{M}$, we can apply Proposition 1.1.22. The relation $T_{ij} = T_{ji}$ follows from $\omega_{ij} = \omega_{ji}$. We use the methods of Remark 1.1.20. The relations (1.14) implies the second relation (1.15). The third relations comes from the fact that there are no relations between forms w_{ij}, w_{kl} with distinct indices. Since $\mathcal{M} = 0$ we conclude that $\sum_{1 \leq i < j \leq n} \omega_{ij} T_{ij}$ is a flat connection form. □

Remark 1.3.3. This connection can be constructed for $\mathrm{Conf}_n(\mathbb{C})$ as well and for more general supbspaces of it (see [43]).

This subsection can be summarize by the following picture.



1.3.2 The universal KZB connection

In this subsection we define the universal KZB connection as in [11]. They are a genus 1 version of the universal KZ equation. Our main references are [11], [29] and [38]. In [11] the KZB connection is defined on the module space of the punctured elliptic curve $\mathcal{M}_{1,n}$. It is the universal version of the Knizhnik–Zamolodchikov-Bernard equations (KZB equation) (see [4] and [5]). In this thesis we consider its restriction on the configuration space of points of the punctured elliptic curve and we call it the *KZB*

connection (dropping the word universal).
Let ξ be the coordinate on $\mathbb{C}$. Let τ be a fixed element of the upper complex plane $\mathbb{H}$. Consider the action of $\mathbb{Z}^2$ on $\mathbb{C}$ given by

$$(l,m)\xi = \xi + l + \tau m.$$

Furthermore, let $\mathbb{Z}+\tau\mathbb{Z}$ be the lattice spanned by $1,\tau$ and let $\mathcal{E}_\tau^\times = (\mathbb{C} - \{\mathbb{Z}+\tau\mathbb{Z}\})/\mathbb{Z}^2$ be the punctured elliptic curve. We fix a $\tau \in \mathbb{H}$ and we denote $\mathcal{E}_\tau^\times$ by $\mathcal{E}^\times$. Let $\theta(\xi,\tau)$ be the "two thirds of the Jacobi triple formula":

$$\theta(\xi,\tau) = q^{1/12}\left(z^{1/2} - z^{-1/2}\right)\prod_{j=1}^{\infty}\left(1-q^j z\right)\prod_{j=1}^{\infty}\left(1-q^j z^{-1}\right) \tag{1.16}$$

where $\xi \in \mathbb{C}$ and $\tau \in \mathbb{H}$, $z := \exp(2\pi i\xi)$, $q := \exp(2\pi i\tau)$.

Definition 1.3.4. Let $\theta(0,\tau)' := \frac{\partial}{\partial\xi}\theta(0,\tau)$. The Kronecker function[4] is defined as

$$F(\xi,\eta,\tau) := \frac{\theta(0,\tau)'\,\theta(\xi+\eta,\tau)}{\theta(\xi,\tau)\,\theta(\eta,\tau)}.$$

Let τ be as above and let $n > 0$. Let $(\xi_1,\dots\xi_n)$ be the coordinates on $\mathbb{C}^n$. We define $\mathcal{D} \subset \mathbb{C}^n$ as

$$\mathcal{D} := \{(\xi_1,\dots\xi_n) \,:\, \xi_i - \xi_j \in \mathbb{Z}+\tau\mathbb{Z} \text{ for some distinct } i,j = 0,\dots n\}$$

and a $\mathbb{Z}^{2n}$-action on $\mathbb{C}^n$ via translation, i.e.

$$((l_1,m_1),\dots,(l_n,m_n))(\xi_1,\dots\xi_n) := (\xi_1 + l_1 + m_1\tau,\dots,\xi_n + l_n + m_n\tau)$$

Notice that $\mathcal{D}$ is preserved by the action of $\mathbb{Z}^{2n}$. In particular the action is properly discontinuos and there is a canonical isomorphism

$$(\mathbb{C}^n - \mathcal{D})/\left(\mathbb{Z}^{2n}\right) \cong \mathrm{Conf}_n\left(\mathcal{E}^\times\right).$$

The KZB connection is introduced in [11]. For $n \geq 0$, we define the algebra $\mathfrak{t}_{1,n}$ as the free Lie algebra with generators $X_1,\dots,X_n,Y_1,\dots,Y_n$ and $t_{i,j}$ for $1 \leq i \neq j \leq n$ modulo

$$\begin{gathered} t_{ij} = t_{ij}, \quad [t_{ij}, t_{ik}+t_{jk}] = 0, \quad [t_{ij}, t_{kl}] = 0 \\ t_{ij} = [X_i, Y_j], \quad [X_i,X_j] = [Y_i,Y_j] = 0, \quad [X_i,Y_i] = -\sum_{j|j\neq i} t_{ij} \\ [X_i, t_{jk}] = [Y_j, t_{ik}] = 0, \quad [X_i+X_j, t_{jk}] = [Y_i+Y_j, t_{ik}] = 0 \end{gathered} \tag{1.17}$$

for i,j,k,l distinct. The elements $\sum_i X_i$ and $\sum_i Y_i$ are central in $\mathfrak{t}1,n$. We denote by $\bar{\mathfrak{t}}_{1,n}$ the quotient of $\mathfrak{t}_{1,n}$ modulo

$$\sum_i X_i = \sum_i Y_i = 0 \tag{1.18}$$

We define $\overline{\mathcal{D}} \subset \mathbb{C}^{n+1}$ as

$$\overline{\mathcal{D}} := \{(\xi_1,\dots\xi_n) \,:\, \xi_i - \xi_j \in \mathbb{Z}+\tau\mathbb{Z} \text{ for some distinct } i,j = 1,\dots,n+1\}.$$

We define an action of $(\mathbb{C},+)$ on $\mathbb{C}^{n+1} - \overline{\mathcal{D}}$ via $z(\xi_1,\dots,\xi_{n+1}) := (\xi_1 - z,\dots,\xi_{n+1} - z)$. This induces an action of $\mathcal{E}$ on $\mathrm{Conf}_{n+1}(\mathcal{E})$ via $\xi'(\xi_1,\dots,\xi_{n+1}) := (\xi_1 - \xi',\dots,\xi_{n+1} - \xi')$. We get a projection $\pi_1 : \mathbb{C}^{n+1} - \overline{\mathcal{D}} \to \left(\mathbb{C}^{n+1} - \overline{\mathcal{D}}\right)/\mathbb{C}$ defined via $\pi_1(\xi_1,\dots,\xi_{n+1}) = \xi_{n+1}(\xi_1,\dots,\xi_n)$ which induces $\pi_2 : \mathrm{Conf}_{n+1}(\mathcal{E}) \to \mathrm{Conf}_{n+1}(\mathcal{E})/\mathcal{E}$. We fix a section $h_1 : \left(\mathbb{C}^{n+1} - \overline{\mathcal{D}}\right)/\mathbb{C} \to \left(\mathbb{C}^{n+1} - \overline{\mathcal{D}}\right)$ which sends $[\xi_1,\dots,\xi_n]$ to $(\xi_1,\dots,\xi_n,0)$, this induces also a section $h_2 : \mathrm{Conf}_{n+1}(\mathcal{E})/\mathcal{E} \to \mathrm{Conf}_{n+1}(\mathcal{E})$. There is an isomorphism $\chi_1 : \mathbb{C}^n - \mathcal{D} \to \left(\mathbb{C}^{n+1} - \overline{\mathcal{D}}\right)/\mathbb{C}$ given by $\chi_1(\xi_1,\dots,\xi_n) = [\xi_1,\dots,\xi_n,0]$. Its inverse is

[4] There is a conflict of notation with respect to [57]. Our function F is the one used in [9], [38],[11] and [29]. Let F^Z be the function in [57], then $F(\xi,\eta,\tau) = 2\pi i F^Z(2\pi i\xi, 2\pi i\eta,\tau)$.

$\chi_1^{-1}[\xi_1,\ldots,\xi_n,\xi_{n+1}] = (\xi_1-\xi_{n+1},\ldots,\xi_n-\xi_{n+1})$. In particular, such an isomorphism induces another isomorphism $\chi_2 : \mathrm{Conf}_n(\mathcal{E}^\times) \to \mathrm{Conf}_{n+1}(\mathcal{E})/\mathcal{E}$. For a formal variable α we define

$$k(\xi,\alpha,\tau) := F(\xi,\alpha,\tau) - \frac{1}{\alpha} = \sum_{k\geq 1} f^{(k)}(\xi)\alpha^{k-1}$$

the functions $f^{(k)}(\xi)$ are holomorphic functions on $(\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})$. For $0 \leq i,j \leq n+1$, we define $f^{(k)}_{i,j} := f^{(k)}(\xi_i-\xi_j)$ and

$$k_{ij} := \sum_k f^{(k)}_{i,j}\,\mathrm{Ad}^{(k)}_{Y_i}(X_j) \in \left(A^0_{DR}(\mathbb{C}^{n+1}-\overline{\mathcal{D}})\otimes\Omega^0(1)\right)\widehat{\otimes}\widehat{\mathfrak{t}}_{1,n+1}.$$

We set

$$K_i := -X_i + \sum_{\substack{j=1\\ j\neq i}} k_{ij}$$

and

$$\varpi := \sum_{i=1}^{n+1} K_i d\xi_i \in A^1_{DR}(\mathbb{C}^{n+1}-\overline{\mathcal{D}})\otimes\Omega^0(1)\widehat{\otimes}\widehat{\mathfrak{t}}_{1,n+1}.$$

We define the bundle $\mathcal{P}^n$ with fiber $\widehat{t}_{1,n}$ on $\mathrm{Conf}_n(\mathcal{E})$ via the following equation (see [38], [11]): each section f of $\mathcal{P}^n$ satisfies

$$\begin{aligned} f(\xi_1,\cdots,\xi_j+l,\cdots,\xi_n) &= f(\xi_1,\cdots,\xi_n),\\ f(\xi_1,\cdots,\xi_j+l\tau,\cdots,\xi_n) &= \exp(-2\pi i l Y_j)\cdot f(\xi_1,\cdots,\xi_n)\end{aligned}$$

for any integer l, where $Y_j^k \cdot a := \mathrm{Ad}^k_{Y_j}(a)$ for $a\in\widehat{t}_{1,n}$. Notice that $\mathcal{P}_1$ is trivial. Let the bundle $\tilde{\mathcal{P}}^n$ with fiber $\widehat{\bar{t}}_{1,n}$ on $\mathrm{Conf}_{n+1}(\mathcal{E})$ as the fiber quotient of $\mathcal{P}^{n+1}$ via the relation (1.18). We denote by $\overline{\mathcal{P}}^n$ the pullback of $\tilde{\mathcal{P}}^n$ along $h\chi_2$ and by $\tilde{\varpi}\in A^1_{DR}\left(\mathbb{C}^{n+1}-\overline{\mathcal{D}}\right)\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$ the image of ϖ via the quotient map $A^1_{DR}(\mathbb{C}^{n+1}-\overline{\mathcal{D}})\widehat{\otimes}\widehat{\mathfrak{t}}_{1,n+1} \to A^1_{DR}(\mathbb{C}^{n+1}-\overline{\mathcal{D}})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$. Finally let $\omega_{KZB,n}\in A^1_{DR}(\mathbb{C}^n-\mathcal{D})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$ be the element obtained by pulling back the coefficients of $\tilde{\varpi}$ along $h_2\chi_2$. We consider the fiber equipped with the adjoint action and define the KZB connection in the following way.

Definition 1.3.5 ([11])**.** The *KZB connection* is given by $d-\omega_{KZB,n}$ on $\overline{\mathcal{P}}^n$.

In particular, it is a holomorphic connection with logarithmic singularities along $\mathcal{D}$ and for $n=1$ it reduces to a holomorphic flat connection on the punctured torus

$$d-\omega_{KZB,1} = d + (\mathrm{Ad}_X)\circ(F(\xi,\mathrm{Ad}_X,\tau))\circ(\mathrm{Ad}_Y)\,d\xi$$

with logarithmic singularities at the origin. This connection appears in [38] as well. In order to understand why the above connection can be considered as the elliptic version of the KZ connection, we discuss its extension on $\mathcal{M}_{1,2}$. Consider the upper half plane $\mathbb{H}$ equipped with the action of $\mathrm{SL}(2,\mathbb{Z})$, i.e $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ defines the map

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}.$$

We denote by $\mathcal{E}_\tau$ the elliptic curve $(\mathbb{C}-\{\mathbb{Z}\tau+\mathbb{Z})\}/\mathbb{Z}^2$. The matrix above defines a map $\mathcal{E}_\tau \to \mathcal{E}_{\frac{a\tau+b}{c\tau+d}}$ via

$$\xi \mapsto \frac{\xi}{c\tau+d}.$$

Let us consider the fiber bundle $P\to\mathbb{H}$ where

$$P := \{(\xi,\tau) \mid \tau\in\mathbb{H},\,\xi\in\mathbb{C}-\{\mathbb{Z}\tau+\mathbb{Z}\}\}.$$

The moduli space of elliptic curves with two marked points is given by the quotient

$$\mathcal{M}_{1,2} := P/\left(\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^2\right).$$

Notice that the Lie algebra $\widehat{\bar{\mathfrak{t}}}_{1,2}$ is the free Lie algebra with generators X, Y. In [11] (see also [38]) it is constructed a flat connection $\nabla_{1,2}\in A_{DR}(P)\otimes\mathrm{End}\left(\widehat{\bar{\mathfrak{t}}}_{1,2}\right)$ such that

1. $\nabla_{1,2}$ is holomorphic on $P \times \operatorname{End}\left(\widehat{\bar{\mathfrak{t}}}_{1,2}\right)$,

2. there is a factor of automorphy $F : P \times \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \to \operatorname{End}\left(\widehat{\bar{\mathfrak{t}}}_{1,2}\right)$ such that $\nabla_{1,2}$ descends to a flat connection form on $\mathcal{M}_{1,2}$ with fiber $\operatorname{End}\left(\widehat{\bar{\mathfrak{t}}}_{1,2}\right)$,

3. the extension at $\overline{\mathcal{M}}_{1,2}$ has regular singularities,

4. the pullback of the connection at $\tau \in \mathbb{H}$ is a flat connection on $\mathcal{E}_\tau^\times$ whose monodromy at a point $p \in \mathcal{E}_\tau^\times$ induces an isomorphism between the Malcev completion of $\pi_1\left(\mathcal{E}_\tau^\times, p\right)$ and the free Lie algebra on two generators $\widehat{\bar{\mathfrak{t}}}_{1,2}$.

In [29], Corollary 14.4. the following proposition is proven.

Proposition 1.3.6. *There is an unique connection on $P \times \operatorname{End}\left(\widehat{\bar{\mathfrak{t}}}_{1,2}\right)$ (modulo a gauge $u \in \exp(\bar{\mathfrak{t}}_{1,2})$) that satisfies the above four properties.*

In particular the pullback of $\nabla_{1,2}$ along a $\tau \in \mathbb{H}$ is $\omega_{KZB,1}$. The KZ connection satisfies similar properties, and this gives an additional justification for $\omega_{KZB,1}$ to be the "elliptic version" of the KZ connection on the punctured torus.

Remark 1.3.7. In [11] it is constructed a flat connection $\nabla_{1,n}$ on $\mathcal{M}_{1,n}$. The above proposition is true mutatis mutandis for $\nabla_{1,n}$ as well (see Remark 14.6 in [29]).

1.4 Applications

1.4.1 A comparison between KZ and KZB connection: punctured sphere and punctured elliptic curve

We give a summary of the results of Chapter 3 and Chapter 4. Let ξ be the coordinate on $\mathbb{C}$ and let τ be a fixed element of the upper complex plane $\mathbb{H}$. Then $\mathcal{D} = \mathbb{Z} + \tau\mathbb{Z}$ is a normal crossing divisor on $\mathbb{C}$ which is preserved by $\mathbb{Z}^2$. We consider the C_∞-algebra $\operatorname{Tot}_N^\bullet A_{DR}\left(\log(D)_\bullet \mathbb{Z}^2\right)$. We consider the KZB connection on the punctured elliptic curve

$$d - \omega_{KZB,1} = d + (\mathrm{Ad}_x) \circ (F(\xi, \mathrm{Ad}_x, \tau)) \circ (\mathrm{Ad}_y)\, d\xi,$$

it is a meromorphic connection with value in $\operatorname{End}(t_{1,1})$.

Theorem G. *1. There exists a holomorphic 1-model $B \subset \operatorname{Tot}_N^\bullet A_{DR}\left(\log(\mathcal{D})_\bullet \mathbb{Z}^2\right)$ endowed with a vector space decomposition*

$$B = W \oplus \mathcal{M} \oplus D\mathcal{M}$$

as in (1.2) such that the flat connection $r_* C_0$ induced by its associated homological pair (C, δ^*) is given by $\omega_{KZB,1}$.

2. Given another 1-model $B' \subset \operatorname{Tot}_N^\bullet A_{DR}\left(\log(\mathcal{D})_\bullet \mathbb{Z}^2\right)$ endowed with a vector space decomposition

$$B' = W' \oplus \mathcal{M}' \oplus D\mathcal{M}'$$

as in (1.2). Let $r_* C'_0$ be the flat connection induced by its associated homological pair (C', δ'^*). It defines a flat connection on a bundle on $\mathcal{E}^\times$ where the fiber is the complete free Lie algebra in two generators. There exists an isomorphism K^* on the fiber as in Theorem F such that $k_* r_* C'_0$ is (smoothly) gauge equivalent to $\omega_{KZB,1}$ as a connection on $\mathbb{C} - \mathcal{D}$ on the trivial bundle with fiber $t_{1,1}$.

Remark 1.4.1. In [9] it is constructed a model for the differential graded algebra $A_{DR}(\mathcal{E}^\times)$. This model is equipped with an obvious vector space decomposition and the degree zero geometric connection induced here is gauge equivalent to $\omega_{KZB,1}$. The above 1-model B is a holomorphic version of that model, more precisely it is a holomorphic model with logarithmic singularities

$$B^{p,q} \subset A_{DR}^q\left(\log(\mathcal{D})_p\right) = A_{DR}^q\left(\log(\mathcal{D}) \times \left(\mathbb{Z}^2\right)^p\right) = \coprod_{(\mathbb{Z}^2)^p} \Omega_{DR}^q\left(\log(\mathcal{D})\right), \tag{1.19}$$

where $\Omega^q_{DR}(\log(\mathcal{D}))$ denotes the complex of holomorphic forms with logarithmic singularities along $\mathcal{D}$. Notice that it is not possible to find a holomorphic model with logarithmic singularities in $A_{DR}(\mathcal{E}^\times)$ with logarithmic singularities. In particular $d\xi$ is the only holomorphic generator of the cohomology with logarithmic singularities. The above theorem shows that such a holomorphic 1-model exists in $\mathrm{Tot}^\bullet_N A_{DR}\left(\log(\mathcal{D})_\bullet \mathbb{Z}^2\right)$, but we have to deal with C_∞-algebras. Moreover, $\omega_{KZB,1}$ is a holomorphic connection on a holomorphic bundle that is smoothly trivial but not holomorphic trivial. Hence our generalization of the Chen theory give original objects from the holomorphic point of view. The above theorem tells us that the flat connections that are induced by some associated homological pairs on the punctured torus are smoothly gauge equivalent modulo a computable automorphism K^* of the fiber. In fact k_* can be compute as in Theorem F since $H^2(\mathcal{E}^\times) = 0$. Notice that the gauge is in general smooth and not holomorphic.

Let us denote $\mathbb{Z} + \tau\mathbb{Z}$ with $\mathcal{D}_\tau$. The 1-model B is constructed by looking at the coefficents of the Kroenecker function $F(\xi, \eta, \tau)$, in particular some elements of B depend on τ (we follows the same approach of [9]). Concretely we build a 1-model for $\mathrm{Tot}^\bullet_N A_{DR}\left(\log(\mathcal{D}_\tau)_\bullet \mathbb{Z}^2\right)$, for any $\tau \in \mathbb{H}$. Such a dependence allows to construct a comparison between the KZ and the KZB connection by sending $\tau \to i\infty$. This idea was used by Hain in [29]. We denote by z the coordinate on $\mathbb{C}^*$ and we define the action of $\mathbb{Z}$ on $\mathbb{C}^*$ via

$$n \cdot z := q^n z. \tag{1.20}$$

There is a morphism $h_\bullet : \mathbb{C}_\bullet \mathbb{Z}^2 \to \mathbb{C}^*_\bullet \mathbb{Z}$ of action groupoids given by

$$h_0(\xi) = e^{2\pi i \xi}, \quad h_1(\xi, (m,n)) = \left(e^{2\pi i \xi}, n\right),$$

which induces an isomorphism on the quotient. Let $\left\{q^{\mathbb{Z}}\right\} \subset \mathbb{C}^*$, the maps above give a morphism $\left(\mathbb{C}_\bullet - \left\{\mathbb{Z}^2 + \tau\mathbb{Z}^2\right\}\right)_\bullet \mathbb{Z}^2 \to \left(\mathbb{C}^* - \left\{q^{\mathbb{Z}}\right\}\right)_\bullet \mathbb{Z}$ between action groupoids that induces an isomorphism on the punctured elliptic curve. We obtain a map

$$\mathbb{C}^*_\bullet \mathbb{Z} \to \mathbb{C} - \{0,1\} \tag{1.21}$$

by sending τ to $i\infty$. We get a morphism

$$B \to A_{KZ,1},$$

where $A_{KZ,1}$ is the Arnold's model generated by $\frac{dz}{z}$ and $\frac{dz}{z-1}$ for $A_{DR}(\mathbb{C} - \{0,1\})$. For $n = 1$ the KZ connection can be written as

$$d - \omega_0 Z_0 - \omega_1 Z_1$$

on $\mathbb{C} - \{0,1\}$ where fiber corresponds to the free Lie algebra $\widehat{\mathfrak{t}}_1$ with generators Z_0, Z_1. In particular

$$\lim_{\tau \to i\infty} \omega_{KZB,1} = \left(Id \widehat{\otimes} Q^*\right)\left(\omega_0 Z_0 + \omega_1 Z_1\right),$$

where $Q^* : \widehat{\mathfrak{t}}_1 \to \widehat{\bar{\mathfrak{t}}}_{1,2}$ is given by

$$Z_0 \mapsto \sum_{l=0}^{\infty} \frac{B'_l}{l!} \frac{dz}{z} \mathrm{Ad}^l_{2\pi i X}\left(\frac{Y}{2\pi i}\right), \quad Z_1 \mapsto -\left[2\pi i X, \frac{Y}{2\pi i}\right].$$

1.4.2 A comparison between KZ and KZB connection: the configuration space of points of the punctured elliptic curve

The aim of the Chapter 4 is to extend the results of Chapter 3 to the configuration spaces of points of the punctured elliptic curve. We define $\mathcal{D}_\tau \subset \mathbb{C}^n$ as

$$\mathcal{D}_\tau := \{(\xi_1, \dots \xi_n) : \xi_i - \xi_j \in \mathbb{Z} + \tau\mathbb{Z} \text{ for some distinct } i, j = 0, \dots n\}$$

and a $\mathbb{Z}^{2n}$-action on $\mathbb{C}^n$ via translation, i.e.

$$((l_1, m_1), \dots, (l_n, m_n))(\xi_1, \dots \xi_n) := (\xi_1 + l_1 + m_1\tau, \dots, \xi_n + l_n + m_n\tau).$$

Notice that $\mathcal{D}_\tau$ is preserved by the action of $\mathbb{Z}^{2n}$. There is a canonical isomorphism

$$(\mathbb{C}^n - \mathcal{D}_\tau) / \left(\mathbb{Z}^{2n}\right) \cong \mathrm{Conf}_n \left({\mathcal{E}_\tau}^\times\right)$$

since the action is free and properly discontinuous. We denote the action groupoid by

$$(\mathbb{C}^n - \mathcal{D}_\tau)_\bullet \left(\mathbb{Z}^{2n}\right),$$

it is a simplicial manifold equipped with a simplicial normal crossing divisor. In [9], it is constructed a smooth model A_n for the differential graded algebra $A_{DR}\left(\mathrm{Conf}_n\left(\mathcal{E}_\tau^\times\right)\right)$ for any $\tau \in \mathbb{H}$. We construct a vector space decomposition on A_n as in (1.2) and we computed the flat connection form C_0 induced by its associated homological pair. Let $\Omega(1)$ be the differential graded algebra of polynomial differential forms on $[0, 1]$. For any $\tau \in \mathbb{H}$, we construct a C_∞-algebra $B'_n \subset \mathrm{Tot}_N\left(A_{DR}\left(\log \mathcal{D}_{\tau\bullet}\mathbb{Z}^{2n}\right)\right) \otimes \Omega(1)$ such that $B'_1 = B$. More precisely, B'_n induces a family $B'_n(u) \subset \mathrm{Tot}_N\left(A_{DR}\left(\log \mathcal{D}_{\tau\bullet}\mathbb{Z}^{2n}\right)\right)$ of C_∞-algebras for $u \in [0, 1]$ such that $B'_n(1) = A_n$ and $B'_n(0)$ is holomorphic (see (1.19)). We conjecture that B'_n is a 1-model for any $\tau \in \mathbb{H}$. In order to skip this problem we work into a quotient $B_n := B'_n/J_{DR}$ for some ideal J_{DR}. We fix a $\tau \in \mathbb{H}$. The C_∞ algebra B_n is not proper a 1-model (we call it 1-*extension* of A_n). We construct a vector space decomposition as in (1.2) on B_n which induces a vector space decomposition and we calculate its associated homological pair (C, δ^*). We associate to (C, δ^*) a family of flat connections $r_*C'_0(u) \in A_{DR}(\mathbb{C}^n - \mathcal{D}_\tau)\,\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$ parametrize by u such that $r_*C'_0(1) = C_0$ and $r_*C'_0(0)$ corresponds to KZB connection. These facts hold for any $\tau \in \mathbb{H}$. We use $B_n(0)$ to give a comparison between the KZ and KZB connection on the configuration spaces at $\tau \to i\infty$. More precisely, we shows that the map (1.21) induces a map

$$B_n(0) \to A_{KZ,n}$$

of C_∞-algebras. This fact allows us to construct a Lie algebra morphism between the Kohno-Drinfeld Lie algebra $\mathfrak{t}_n$ and $\widehat{\bar{\mathfrak{t}}}_{1,n+1}$ which is the n-dimensional version of the map Q presented above.

1.5 Drinfeld Associators , Elliptic Associators and motivation

We spend some words about the role of the KZ connection and Drinfeld associators in mathematics. We begin with the KZ Drinfeld associator. For $n = 1$ the KZ connection reduces to

$$d + \omega_0 Z_0 + \omega_1 Z_1,$$

where the fiber is the free Lie algebra on the generators Z_0, Z_1 and $\omega_i = \frac{dz}{z-i_i}$ for $i_j \in \{0, 1\}$. We fix a base point x_0, for a smooth path $\gamma : [0, 1] \to \mathbb{C} - \{0, 1\}$ connecting x_0 with x_1, the parallel transport of this connection T can be considered as a multivalued function $f_{x_0}(x_1) := T(\gamma)$ with values in $\mathbb{C}\langle\langle Z_0, Z_1\rangle\rangle$. We consider the function

$$T(\gamma^t) : [0, 1] \to \mathbb{C}\langle\langle Z_0, Z_1\rangle\rangle$$

where $\gamma^t(u) = \gamma(tu)$ for $t \in [0, 1]$. The function $T(\gamma^t)$ can be written as a formal power series where the coefficients are iterated integrals

$$g(t) := \int_{\gamma^t} \omega_{i_1} \cdots \omega_{i_n}$$

for $i_j \in \{0, 1\}$. Assume that $x_0 = 0$, the functions $g : [0, 1] \to \mathbb{C}$ is not well-defined. However, since the ω_{i_1} are smooth forms with logarithmic singularities, the above expression can be regularized via a logarithmic expansion argument (see for example [10]). This idea is used in [15]. In particular, for any path starting from 0 to $x_1 \in \mathbb{C} - \{0, 1\}$, we can regularize $g(t)$ into a function $\overline{g(t)}$. Such a regularization, depends on the choice of the path **and** on the value of $\dot{\gamma}(0)$.

Remark 1.5.1. A sufficient conditions for the existence of that regularization is that the the singularities of the above differential forms are logarithmic. Given a discrete set $D \subset \mathbb{C}$, let $A_{DR}(\log(D))$ be the differential graded algebra of smooth differential forms with logarithmic singularities along D. Given a path γ such that $\gamma(0, 1] \subset \mathbb{C} - D$ and set $s := \gamma(0) \in D$. Any expression

$$g(t) := \int_{\gamma^t} a_1 \cdots a_n$$

such that $a_1, \dots, a_n \in A_{DR}(\log(D))$ can be regularized as a function $\mathrm{Reg}\int_{\gamma^t} a_1 \cdots a_n$ on $\{s\} \cup (\mathbb{C} - D)$. Assume that $\gamma(1) = s' \in D$ and $\gamma(0,1) \subset \mathbb{C} - D$. Then we can split our path in two $\gamma_1(t) := \gamma(2t)$ and $\gamma_2(t) := \gamma(\frac{t+1}{2})$. Set

$$\mathrm{Reg}\int_{\gamma^t} a_1 \cdots a_n := \begin{cases} \mathrm{Reg}\int_{\gamma_1^t} a_1 \cdots a_n, & t \leq 1/2 \\ \frac{1}{2}\left(\mathrm{Reg}\int_{\gamma_1^1} a_1 \cdots a_n - \mathrm{Reg}\int_{(\gamma_2^{-1})^t} a_1 \cdots a_n\right), & t > 1/2 \end{cases}$$

where $\gamma_2^{-1}(u) = \gamma_2(1-u)$. In order for the above expression to make sense we have to specify the initial velocity for γ_2^{-1} as well.

Let γ be the path form 0 to 1 with initial velocity 1. Let $T_r(\gamma^t)$ be the regularized holonomy of the KZ connection with respect to the paths with initial velocity 1. We set

$$\Phi_{KZ}(Z_0, Z_1) := T_r(\gamma).$$

The above formal power series is called KZ-Drinfeld associator. Let us consider$\Bbbk\langle\langle Z_0, Z_1\rangle\rangle$ equipped with its Hopf algebra structure (it is the complete universal envelopping algebra on the complete free Lie algebra on two generators).

Definition 1.5.2. Let $\Bbbk$ be a filed of charachteristic zero. A *Drinfeld associator* is a pair $(\mu, \Phi(Z_0, Z_1))$ with $\mu \in \Bbbk^\times$ and group-like element $\Phi(Z_0, Z_1) \in \Bbbk\langle\langle Z_0, Z_1\rangle\rangle$ such that

$$\Phi(Z_0, Z_1) = \Phi(Z_1, Z_0)^{-1} \tag{1.22}$$

$$1 = e^{\frac{\mu}{2}Z_\infty}\Phi(Z_0, Z_1)e^{\frac{\mu}{2}Z_0}\Phi(Z_1, Z_\infty)e^{\frac{\mu}{2}Z_1}\Phi(Z_\infty, Z_1) \tag{1.23}$$

$$\Phi(T_{12}, T_{23} + T_{24})\Phi(T_{13} + T_{23}, T_{34}) = \Phi(T_{23}, T_{34})\Phi(T_{12} + T_{13}, T_{24} + T_{34})\Phi(T_{12}, T_{23}) \tag{1.24}$$

where for the second equation we assume $Z_1 + Z_0 + Z_\infty = 0$ and the third equation has to be understood in the complete universal enveloping algebra of the Lie subalgebra $\widehat{\mathfrak{t}}'_4 \subset \widehat{\mathfrak{t}}_4$ generated by $T_{i,j}$ for $0 \leq i, j \leq 4$. Let GRT_1 be the set of solutions of the above tree equations providing $\mu = 0$. We define the *Grothenideck-Teichmüller group* as $GRT := \Bbbk^\times \ltimes GRT_1$, where the action of $\Bbbk^\times$ on GRT_1 is given by $\lambda \cdot \Phi(Z_0, Z_1) := \Phi(\lambda Z_0, \lambda Z_1)$.

In particular, $(1, \Phi_{KZ}(Z_0, Z_1))$ is a Drinfeld associator over $\Bbbk$. Drinfeld's original motivation to define such associator was to construct quasi-Hopf algebras, i.e. a generalization of the notion of coalgebras obtained by weakening the coassociativity of the coproduct. The name "associator" comes from that those facts. In particular, the Drinfeld associator is used for the construction of the quantum enveloping algebra. The Drinfeld associator allows to prove that the group GRT is isomorphic to a more complicated group GT, called the *pro-unipotent Grothenideck-Teichmüller group*. This group has a pro-finite cousin called the *pro-finite Grothenideck-Teichmüller group*. It is proved that there is an injection between $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ and the pro-finite Grothenideck-Teichmüller group and it is conjectured that this injection is an isomorphism. Another application of Drinfeld associators occurs in Lie theory (see [1]), in particular they give a solution of the Kashivara-Vergne conjecture which is essentially the possibility to construct anautomorphism ϕ on the free Lie algebra on two generators that trivializes the Baker-Campbell-Hausdorff formula, i.e. $\phi\left(BCH(Z_0, Z_1)\right) = Z_0 + Z_1$ such that ϕ satisfying other suitable conditions. The consequence of such a trivialization is one of the few theorems that hold for any Lie algebra. It is conjectured that there is a one to one correspondence between solutions and Drinfeld associators. Drinfeld associators are also related with number theory. Consider $\Phi_{KZ}(Z_0, Z_1)$. One can write

$$\Phi_{KZ}(Z_0, Z_1) := \sum_{v \in \mathbb{C}\langle Z_0, Z_1\rangle} \zeta(v)\, v$$

where $\zeta : \mathbb{Q}\langle\langle Z_0, Z_1\rangle\rangle \to \mathbb{R}$ is the unique function such that:

1. for $v = Z_0^{n_1-1} \cdots Z_1 Z_0^{n_r-1} Z_1$ is defined as

$$\zeta(v) = \mathrm{Li}_{n_1,\dots,n_r}(1) := \frac{(-1)^r}{(2\pi i)^n} \sum_{0 \leq k_1 \leq \cdots \leq k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}},$$

2. $\zeta(v) = 0$ for $v = Z_0 = Z_1$, $n \geq 0$ and $\zeta(\mu'(v,w)) = \zeta(v)\zeta(w)$, where $\mu'(v,w)$ is the shuffle product.

The numbers defined at point 1. are called multiple zeta values. In particular, it is possible to write $\Phi_{KZ}(X_0, X_1)$ as a formal series of Lie elements where the coefficents are Multiple Zeta values (this fact was noticed by Kontsevich). The multiple zeta values are crucial in number theory and in the theory of Motives (see [7] and [53]). They are related with quantum field theory, since they appear as value of some Feynman path integrals (see [8]).

The elliptic version of the associators is constructed in [22], they are essentially the reguralized monodromy of the KZB equation constructed along a path connecting the two singularities. As for the genus zero case, the elliptic story is an active field of research. For the elliptic aspect of Grothendieck-Teichmüller theory see [48]. See also [30] for a Motivic approach to elliptic zeta values. In particular, notice that an extension of the notion of KZ connection on higher genus curves implies[5] a possible extension on the notion of higher genus Drinfeld associators. The Chen's theory developed in this thesis reduces the problem of the construction of the KZB connection to the construction of a holomorphic 1-model with logarithmic singularities. This is in perfect analogy with the genus 0 case by using the Arnold's relations and ordinary Chen's theory.

[5]Of course there will be a lot of work to do, see for example [22].

Chapter 2

Chen theory on simplicial manifolds

In this chapter we extend the notion of homological pair to simplicial manifolds. The differential graded vector space of differential forms on a simplicial manifold carries a C_∞-structure. In the first section we extend the notion of convolution algebra to C_∞-algebras. This section is purely algebraic. In Section 2.2 we study the aforementioned C_∞-structure and we give some formulas. In Section 2.3 we construct homological pairs on simplicial manifolds by applying the results of the previous sections. The geometric application are presented in Section 2.4.

2.1 A_∞, C_∞-structures and homological pairs

In this section we present a proof of the existence of associated homological pair (Theorem 1.1.18) in terms of the homotopy transfer theorem (see [33]). This point of view is essential for the construction of a geometric connection on simplicial manifolds. This section is purely algebraic and there is no geometry involved. In this settings, a connection is merely a Maurer-Cartan element. In the final part we study how the homotopy transfer theorem imposes a " homotopy equivalence" between Maurer-Cartan elements (see in particular Proposition 2.1.50).

2.1.1 A_∞, C_∞, L_∞-structures

We introduce A_∞, C_∞, L_∞-structures. Our reference is [39]. For a graded vector space V, let $T^c(V[1])$ be the (graded) tensor coalgebra on shift of V. Let $NTS(V) \subset T^c(V[1])$ be the subspace of non-trivial shuffles, i.e the vector space generated by $\mu'(a,b)$ such that μ' is the graded shuffle product and $a, b \notin \Bbbk \subset T^c(V[1])$.

Definition 2.1.1. Let $V^\bullet$ be a graded vector space. An A_∞-*algebra structure* on $V^\bullet$ is a coderivation $\delta : T^c(V[1]) \to T^c(V[1])$ of degree $+1$ such that $\delta^2 = 0$. A C_∞-*algebra structure* on $V^\bullet$ is an A_∞-structure such that $\delta(NTS(V[1])) = 0$.

Each coderivation is uniquely determined by the maps of degree 1

$$\delta_n : A[1]^{\otimes n} \hookrightarrow T^c(A[1]) \xrightarrow{\delta} T^c(A[1]) \xrightarrow{pro_{A[1]}} A[1].$$

For $n > 0$ we define maps $m_n : A^{\otimes n} \to A$ of degree $2-n$ via

$$A^{\otimes n} \xrightarrow{(s)^{\otimes n}} A[1]^{\otimes n} \xrightarrow{\delta_n} A[1] \xrightarrow{s^{-1}} A.$$

The condition $\delta^2 = 0$ for the maps $m_n : A^{\otimes n} \to A$ implies $m_1^2 = 0$ and the relations

$$\sum_{\substack{p+q+r=n \\ k=p+1+r \\ k,q>1}} (-1)^{p+qr} m_k \circ \left(\mathrm{Id}^{\otimes p} \otimes m_q \otimes \mathrm{Id}^{\otimes r}\right) = \partial m_n, \quad n > 0. \tag{2.1}$$

Conversely, starting from maps $m_\bullet := \{m_n\}_{n\geq 0}$ that satisfy the above relations, we get a sequence of maps $\delta_n : A[1]^{\otimes n} \to A[1]$, for $n \geq 0$ defined via

$$\delta_n : (A[1])^{\otimes n} \xrightarrow{(s^{-1})^{\otimes n}} A^{\otimes n} \xrightarrow{m_n} A \xrightarrow{s} A[1].$$

In particular we have $m_1^2 = 0$. These maps can be viewed as the restriction of a coderivation δ, which is a differential by (2.1). We denote an A_∞- (C_∞-) algebra $(T^c(V[1]), \delta)$ by $(A^\bullet, m_\bullet)$ as well.

Definition 2.1.2. An A_∞-algebra $(A, m_\bullet)$ is said to be *unital* if there exists a m_1-closed element 1 of degree zero such that $m_2(1,a) = 1 = m_2(a,1)$ and $m_k(a_1 \ldots, 1, \ldots, a_k) = 0$ for $k \geq 3$. $(A, m_\bullet)$ is said to be *connected* if $A^0 = \Bbbk 1$ and A is a non-negatively graded vector space. Let $A^\bullet$, $B^\bullet$ be two A_∞-algebras. A *morphism between A_∞-algebras* is a morphism of differential graded coalgebras

$$F : (T^c(A[1]), \Delta, \delta_A) \to (T^c(B[1]), \Delta, \delta_B).$$

Each morphism is completely determined by the degree zero maps, i.e.

$$F_n : A[1]^{\otimes n} \hookrightarrow T^c(A[1]) \xrightarrow{F} T^c(B[1]) \xrightarrow{pro_{B[1]}} B[1], \quad n > 0.$$

and $F_0(1) := F(1) = 1$. F is said to be a *morphism of C_∞-algebras* if A and B are C_∞-algebras and $F_n(NTS(V[1]) \cap V[1]^{\otimes n}) = 0$.

We denote by $f_\bullet := \{f_n\}_{n\in\mathbb{N}}$ the family of maps of degree $1-n$ given by

$$f_n : A^{\otimes n} \xrightarrow{(s)^{\otimes n}} A[1]^{\otimes n} \xrightarrow{pro_{B[1]} \circ F_n} B[1] \xrightarrow{s^{-1}} B.$$

Let m_n^A be the degree $2-n$ maps obtained from δ_A, and let m_n^B be the ones from δ_B. The condition $F \circ \delta_A = \delta_B \circ F$ implies the following equations $f_1 m_1^A = m_1^B f_1$ and

$$\begin{aligned} &\sum_{\substack{p+q+r=n \\ k=p+1+r}} (-1)^{p+qr} f_k(Id^{\otimes p} \otimes m_q^A \otimes Id^{\otimes r}) \\ &- \sum_{k\geq 2,\, i_1+\cdots+i_k=n} (-1)^s m_k^B (f_{i_1} \otimes \cdots \otimes f_{i_k}) = \partial f_n, \quad n > 1, \end{aligned} \tag{2.2}$$

where $s = (k-1)(i_1-1) + (k-2)(i_2-1) + \cdots + 2(i_{k-2}-1) + (i_{k-1}-1)$. In particular f_1 is an ordinary cochain map. A family of maps $f_\bullet$ satisfying condition (2.2) induces also a morphism of A_∞-algebras $F : (T^c(A[1]), \Delta, \delta_A) \to (T^c(B[1]), \Delta, \delta_B)$.

Remark 2.1.3. We will adopt the following notation. We will use small dotted letters (e.g. $f_\bullet$) to denote morphisms of A_∞, C_∞-algebras. On the other hand, we will use capital letters (e.g. F) to denote morphism of A_∞, C_∞-algebras as morphism of quasi-free coalgebras.

Definition 2.1.4. A morphism between A_∞ (C_∞)-algebras is a *quasi-isomorphism* if the cochain map

$$f_1 : (A, m_1^A) \to (B, m_1^B)$$

is a quasi-isomorphism. A morphism between A_∞ (C_∞) -algebras $F : T^c(A[1]) \to T^c(B[1])$ is an isomorphism if f_1 is an isomorphism. A morphism $f_\bullet$ is called strict if $f_n = 0$ for $n > 1$.

Each isomorphism of A_∞-algebras has a unique inverse (for an explicit construction, see [39], section 10.4.1). Let $F : T^c(A[1]) \to T^c(B[1])$ be a quasi-isomorphism. There exists a quasi-isomorphism $G : T^c(B[1]) \to T^c(A[1])$ such that $[f_1]^{-1} = [g_1]$ in cohomology (see [39], section 10.4.3).
We denote by $\Omega(1)$ the free commutative graded algebra generated by $1, t_0$, t_1 of degree zero and by dt_0, dt_1 of degree 1 such that

$$t_0 + t_1 = 1, \quad dt_0 + dt_1 = 0.$$

We put a differential d on $\Omega(1)$ by $d1 = 0$ and $d(t_j) := dt_j$, for $j = 0, 1$ such that $\Omega(1)$ is a differential free commutative graded algebra. Equivalently, we may define $\Omega(1)$ as the commutative free graded algebra generated by $1, t$ of degree 0 and dt in degree 1. The differential here is given by $d(1) := 0, d(t) := dt$. The two presentations are isomorphic via the map $t \mapsto t_0$. We denote by $i_j : \Omega(1) \to \Bbbk$ the dg algebra map sending t_j to 1 and dt_j to 0 for $j = 0, 1$.

Lemma 2.1.5. *Let $f_\bullet : (A, m^A_\bullet) \to (B, m^B_\bullet)$ be a morphism of A_∞-algebras and $g : \Omega(1) \to \Omega(1)$ be a morphism of differential graded algebras.*

1. $\left(\Omega(1) \otimes A, m^{\Omega(1)\otimes A}_\bullet\right)$ *is an A_∞-algebra via*

$$m^{\Omega(1)\otimes A}_n(p_1 \otimes a_1, \dots, p_n \otimes a_n) := \pm (p_1 \cdots p_n) \otimes m^A_n(a_1, \otimes, a_n),$$

where the sign $\pm$ follows by the signs rule. In particular $i_j \otimes Id : \Omega(1) \otimes A \to A$ is a well-defined strict A_∞-morphism.

2. *The map $(g \otimes f)_\bullet : \left(\Omega(1) \otimes A, m^{\Omega(1)\otimes A}_\bullet\right) \to \left(\Omega(1) \otimes B, m^{\Omega(1)\otimes B}_\bullet\right)$ defined by*

$$(g \otimes f)_n(p_1 \otimes a_1, \dots, p_n \otimes a_n) := \pm g (p_1 \cdots p_n) \otimes f_n(a_1, \dots, a_n),$$

where the signs $\pm$ follows by the signs rule, is a morphism of A_∞-algebras. If $g = id$ we have $(i_j \otimes Id)(Id \otimes f)_\bullet = f_\bullet (i_j \otimes Id)$ for $j = 0, 1$.

Proof. Straightforward calculation. □

Definition 2.1.6. Let $f_\bullet, g_\bullet : A \to B$ be A_∞-morphisms (resp. C_∞-morphism). A *homotopy* between $f_\bullet$ and $g_\bullet$ is an A_∞ (resp. C_∞) map $H_\bullet : A \to \Omega(1) \otimes B$ such that

$$(i_0 \otimes Id)\, H_\bullet = f_\bullet, \quad (i_1 \otimes Id)\, H_\bullet = g_\bullet,$$

two morphisms are homotopy equivalent if there exists a finite sequence of homotopy maps connecting them.

By Lemma 2.1.5 we have the following. Let $f_\bullet, g_\bullet : A \to B$ be A_∞-morphisms. Let $H_\bullet : A \to \Omega(1)\otimes B$ be a homotopy between $f_\bullet$ and $g_\bullet$, let $p^1_\bullet : A_1 \to A$ and $p^2_\bullet : B \to A_2$ be A_∞-maps. Then $H_\bullet(Id \otimes p^2)_\bullet$ is a homotopy between $f_\bullet p^2_\bullet$ and $g_\bullet p^2_\bullet$ and $(Id \otimes p^1)_\bullet H_\bullet$ is a homotopy between $p^1_\bullet f_\bullet$ and $p^1_\bullet g_\bullet$.

Proposition 2.1.7. *Let $\mathcal{P}_\infty$ be A_∞ or C_∞. Then any $\mathcal{P}_\infty$-quasi-isomorphism $f_\bullet : A \to B$ has a $\mathcal{P}_\infty$-homotopical inverse, i.e. there exists a $\mathcal{P}_\infty$-map $g_\bullet : B \to A$ such that $g_\bullet \circ f_\bullet \cong Id_A$ and $f_\bullet \circ g_\bullet \cong Id_B$.*

Proof. This is Theorem 3.6 of [54]. The $\mathcal{P}_\infty$ objects enjoy this property because they are fibrant-cofibrant objects in a certain model category structure where $\Omega(1) \otimes (-)$ is a functorial cylinder object. □

Definition 2.1.8. Let $(A, m^A_\bullet)$ and $(B, m^B_\bullet)$ be two non-negatively graded A_∞-algebras. A 1-A_∞-*algebra morphism* consists in a graded coalgebra maps

$$F : \oplus_{i\leq 1} \left(T^c(A[1])\right)^i \to \oplus_{i\leq 1} \left(T^c(B[1])\right)^i$$

such that $(F \otimes F)\,\Delta = \Delta F$ and $F\delta^A = \delta^B F$. Note that F corresponds to a family degree $n-1$ maps $f_n : \oplus_{i\leq n+1} \left(A^{\otimes n}\right)^i \to A \to B$ that satisfies (2.2) on $\oplus_{i\leq n} \left(A^{\otimes n}\right)^i$ for any $n > 0$. We denote the $1 - A_\infty$-morphism F by $f_\bullet : (A, m^A_\bullet) \to (B, m^B_\bullet)$. If the two algebras are $1 - C_\infty$ then F is said to be $1 - C_\infty$ if it vanishes on non-trivial shuffles of total degree smaller than 2.

Notice that a 1-A_∞-morphism $f_\bullet : (A, m^A_\bullet) \to (A, m^A_\bullet)$ induces a graded map $F : \left(T^c(A[1])\right)^1 \to T^c(B[1])$ such that $(F \otimes F)\,\Delta = \Delta F$ and such that its restriction $F : \left(T^c(A[1])\right)^0 \to T^c(B[1])$ commutes with the differentials. A 1-A_∞ (resp. C_∞)-morphism is said to be a 1-A_∞ (resp. C_∞) -isomorphism if f_1 is an isomorphism. A 1-A_∞ (resp. C_∞)-morphism is said to be a 1-A_∞ (resp. C_∞) -quasi-isomorphism if f_1 induces an isomorphism $H^1(A) \to H^1(B)$ and an injection $H^2(A) \hookrightarrow H^2(B)$. A strict 1-$A_\infty$-morphism is a 1-$A_\infty$-morphism $f_\bullet$ such that $f_n = 0$ for $n > 1$.

Definition 2.1.9. Let $f_\bullet, g_\bullet : A \to B$ be 1-A_∞-morphisms. A *homotopy* between $f_\bullet$ and $g_\bullet$ is a 1-A_∞ map $H_\bullet : A \to \Omega(1) \otimes B$ such that

$$(i_0 \otimes Id)\, H_\bullet = f_\bullet, \quad (i_1 \otimes Id)_\bullet\, H_\bullet = g_\bullet.$$

Two morphisms are homotopy equivalent if there exists a finite sequence of homotopy maps connecting them.

Notice that the composition of 1-A_∞-morphisms is again a 1-A_∞-morphism and the same is true for 1-C_∞-morphisms. In particular A_∞, C_∞-maps give examples of 1-A_∞, C_∞-maps (when restricted to $A_{1,n}$).
We conclude this subsection by defining L_∞-structures. Let $S(V)$ be the symmetric algebra, then we denote by $S^c(V)$ the graded coalgebra given by the graded vector space $S(V)$ and the concatenation coproduct Δ, i.e. $S^c(V)$ is the cocommutative cofree conilpotent coalgebra generated by V.

Definition 2.1.10. An L_∞*-structure* on a graded vector space V is a coderivation $\delta : S^c(V[1]) \to S^c(V[1])$ of degree $+1$ such that $\delta^2 = 0$.

As for the A_∞ case, the above definition is equivalent to family of maps $l_n : V^{\otimes n} \to V$ of degree $2-n$. These maps are skew symmetric and satisfy the relations

$$\sum_{p+q=n+1,\, p,q>1} \sum_{\sigma^{-1}\in Sh^{-1}(p,q)} \operatorname{sgn}(\sigma)(-1)^{(p-1)q} l_p \left(l_q \otimes Id^{\otimes(p-1)}\right)^{\sigma} = \partial l_n,$$

for $n \geq 1$. The morphisms, quasi-isomorphisms and isomorphisms between L_∞-algebras are defined in the same way, as for the A_∞ case.
Given an associative algebra A, then it carries a Lie algebra structure where the bracket is obtained by anti-symmetrizing the product. The same is true between A_∞ and L_∞-algebras.

Theorem 2.1.11. *Let $(V, m_\bullet)$ be an A_∞-algebra. The anti-symmetrized map $l_n : V^{\otimes} \to V$, given by*

$$l_n := \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) m_n^{\sigma},$$

define an L_∞-algebra structure on V.

Definition 2.1.12. Given a L_∞-algebra $\mathfrak{g}$ with structure maps $l_\bullet$.

1. An L_∞*-ideal* $I \subset \mathfrak{g}$ is a subgraded vector space such that $l_k(a_1, \dots, a_k) \in I$ if one of the a_i lies in I (in particular $(\mathfrak{g}/I, l_\bullet)$ is an L_∞-algebra).

2. An L_∞-algebra is said to be *filtered* if it is equipped with a filtration $F^\bullet$ of L_∞-ideals such that for $a_i \in F^{n_i}(\mathfrak{g})$, we have $l_k(a_1, \dots, a_k) \in F^{n_1+\cdots+n_k}(\mathfrak{g})$.

3. A filtered L_∞-algebra is said to be *complete* if $\mathfrak{g} \cong \lim_i \mathfrak{g}/F^i(\mathfrak{g})$ as a graded vector space.

4. A *Maurer-Cartan element* in a complete L_∞-algebra $\mathfrak{g}$ is a $\alpha \in \mathfrak{g}^1$ such that

$$\partial(\alpha) + \sum_{k\geq 2}^{\infty} \frac{l_k(\alpha, \dots, \alpha)}{k!} = 0.$$

   We denote by $MC(\mathfrak{g})$ the set of Maurer-Cartan elements[1].

5. Let $\mathfrak{g}$ be a complete L_∞-algebra. Then $\Omega(1)\widehat{\otimes}\mathfrak{g}$ is again a complete L_∞-algebra. An *homotopy* between two Maurer-Cartan elements $\alpha_0, \alpha_1 \in MC(\mathfrak{g})$ is a Maurer-Cartan element $\alpha(t) \in MC(\Omega(1)\widehat{\otimes}\mathfrak{g})$ such that $\alpha(0) = \alpha_0$ and $\alpha(1) = \alpha_1$. Two Maurer-Cartan elements are said to be *homotopy equivalent* or *homotopic* if they are connected by a finite sequence of homotopies.

Definition 2.1.13. Let $\mathcal{P}_\infty$ be A_∞ or C_∞. We denote by $\mathcal{P}_\infty - \mathrm{ALG}$ the category of bounded below P_∞-algebras. We denote by $(\mathcal{P}_\infty - \mathrm{ALG})_1$ the category whose objects are P_∞-algebras and the arrows are 1-$\mathcal{P}_\infty$-morphisms between them. We denote by $(\mathcal{L}_\infty - \mathrm{ALG})_p$ be the category whose objects are L_∞-algebras and the arrows are defined as follows: an arrow $\mathfrak{g} \to \mathfrak{g}'$ is a set map $f : MC(\mathfrak{g}) \to MC(\mathfrak{g})$. We denote by $(\mathcal{P}_\infty - \mathrm{ALG}_{\geq 0})_1$ and $(\mathcal{P}_\infty - \mathrm{ALG}_{>0})_1$ the two full subcategories whose objects are non-negatively graded $\mathcal{P}_\infty$-algebras and positively graded $\mathcal{P}_\infty$-algebras respectively.

[1]Notice that the above sum is well-defined in $\mathfrak{g}$ since it is complete.

Remark 2.1.14. There are several sign conventions to define A_∞, C_∞, L_∞-algebras and the related objects. We use the convention of [39], but for example in [42], the sign $(-1)^{p+qr}$ in (2.1) and in (2.2) is replaced with $(-1)^{qp+q+p}$. In particular, the signs in the above definitions are unique up the action of the infinity group $(\mathbb{Z}_2)^\infty$. This means that: let $m_\bullet$ be an A_∞-algebra as in (2.1) and let $m'_\bullet$ be an A_∞-algebra with respect to different convention. There exists a $(\epsilon_1, \dots, \epsilon_i, \dots) \in (\mathbb{Z}_2)^\infty$ such that $m_1 = m'_1$ and for $i > 1$

$$m_i = (-1)^{\epsilon_i} m'_i, \quad m'_i = (-1)^{\epsilon_i} m_i.$$

The same rule works for morphism as well.

2.1.2 Convolution L_∞-algebras

Let C be a coalgebra and let A be a differential graded algebra. The space of morphisms between graded vector spaces $\mathrm{Hom}^\bullet(C, A)$ is equipped with a differential graded Lie algebra structure called convolution algebra (see [39], chapter 1).
Let $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$ be A_∞-algebras and assume they are both bounded below. Let δ be the codifferential of $T^c(V[1])$ and δ^A be the codifferential on $T^c(A[1])$. Consider $A^\bullet$ as a graded cochain vector space equipped with m_1^A as differential. We have a differential graded vector space of morphisms between graded vector spaces

$$\mathrm{Hom}^\bullet(T^c(V[1]), A).$$

If $m_n^A = 0$ for $n > 2$, there is a one to one correspondence between coalgebras morphism $F : T^c(V[1]) \to T^c(A[1])$ and twisting cochains. For general A_∞-structure, there is a similar property. For each F, we associate a graded map $\alpha \in \mathrm{Hom}^1(T^c(V[1]), A)$ defined as

$$T^c(V[1]) \xrightarrow{F} T^c(A[1]) \xrightarrow{\mathrm{proj}_{A[1]}} A[1] \xrightarrow{s^{-1}} A. \tag{2.3}$$

with $\alpha(1) = 0$. The condition $F \circ \delta|_{V[1]^{\otimes n}} = \delta^A \circ F|_{V[1]^{\otimes n}}$ reads

$$\left(\alpha \circ \delta^V\right)_{V[1]^{\otimes n}} = \sum_{k \geq 1,\, i_1 + \dots i_k = n} (-1)^{k+1} m_k^A\left(\alpha_{i_1}, \dots, \alpha_{i_k}\right) \circ \Delta^{k-1}$$

where $\alpha_{i_j} := \alpha|_{V[1]^{\otimes i_j}}$ and Δ^k is the iterated coproduct in the tensor coalgebra $T^c(V[1])$. It is an easy exercise to show that

$$\tilde{m}_k^A := (-1)^k m_k^A \;:\; A^{\otimes k} \to A$$

is again an A_∞-structure on A. We conclude

$$\alpha \circ \delta = -\sum_{k \geq 1} \tilde{m}_k^A(\alpha, \dots, \alpha) \circ \Delta^{k-1}. \tag{2.4}$$

The above equation can be interpreted as the A_∞-version of the twisting cochain condition. For $n > 1$, we define the maps $M_n : \left(\mathrm{Hom}^\bullet(T^c(V[1]), A)\right)^{\otimes n} \to \mathrm{Hom}^\bullet(T^c(V[1]), A)$ via

$$M_n(f_1, \dots, f_n) := \tilde{m}_n^A(f_1, \dots, f_n) \circ \Delta^{n-1},$$

the map $M_1 : \mathrm{Hom}^\bullet(T^c(V[1]), A) \to \mathrm{Hom}^\bullet(T^c(V[1]), A)$ as $M_1(f) := \tilde{m}_1^A(f)$ and the map ∂ : $\mathrm{Hom}^\bullet(T^c(V[1]), A) \to \mathrm{Hom}^\bullet(T^c(V[1]), A)$ as

$$\partial(f) := \tilde{m}_1^A f - (-1)^{|f|} f \circ \delta.$$

We define $L_{V[1]^*}(A)$ be the set of morphisms $\mathrm{Hom}^\bullet(T^c(V[1]), A)$ whose kernel contains the set of non-trivial shuffles $NTS(V[1])$ (see Section A.2).

Lemma 2.1.15. *Let V, A as above.*

1. $(M_\bullet, \mathrm{Hom}^\bullet(T^c(V[1]), A))$ *is an A_∞-algebra,*

2. $\left(\partial, \{M_n\}_{n \geq 2}, \mathrm{Hom}^\bullet(T^c(V[1]), A)\right)$ *is an A_∞-algebra,*

3. *We denote by $l'_\bullet$ the maps on* $\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)^{\otimes n}$ *induced by the anti-symmetrization of the maps* $\left(\partial,\{M_n\}_{n\geq 2}\right)$ *via Theorem 2.1.11 and by $l_\bullet$ the maps induced by the anti-symmetrization of the maps $(\partial, M_2, M_3, \dots)$. Assume that $\delta\mu'(a,b)=0$ for any non-trivial shuffle $\mu'(a,b)\in T^c(V[1])$ and that the product m_2^A is graded commutative (but not necessarily associative). Then we have two L_∞-subalgebras*

$$\left(l'_\bullet, L_{V[1]^*}(A)\right)\subset\left(l'_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)\right),\quad \left(l_\bullet, L_{V[1]^*}(A)\right)\subset\left(l_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)\right).$$

For the proof see the appendix, Section A.2.1.

Definition 2.1.16. Let $\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)$ be A_∞-algebras.

1. We call $\left(\partial,\{M_n\}_{n>1}\,\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)\right)$ the *convolution A_∞-algebra* associated to $\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)$.
2. We call
$$\mathrm{Conv}\left(\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)\right):=\left(l'_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)\right)$$
the *convolution L_∞-algebra* associated to $\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)$.
3. Let $\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)$ be C_∞-algebras. We call
$$\mathrm{Conv}_r\left(\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)\right):=\left(l'_\bullet, L_{V[1]^*}(A)\right)\subset\left(l'_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)\right)$$
the *reduced convolution L_∞-algebra* associated to $\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)$.

The next proposition is a consequence of the discussion due at the beginning of this subsection.

Proposition 2.1.17. *Let $\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)$ be A_∞-algebras (resp. C_∞-algebra). There exists a one to one correspondence between*

1. *A_∞ (resp. C_∞)-morphisms $f_\bullet:\left(V,m_\bullet^V\right)\to\left(A,m_\bullet^A\right)$.*
2. *Morphisms of differential graded quasi-free coalgebras $F: T^c(V[1])\to T^c(A[1])$ (resp. such that $F(NTS(V[1]))=0$).*
3. *Maurer-Cartan elements $\alpha\in\mathrm{Conv}\left(\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)\right)$ (resp. $\mathrm{Conv}_r\left(\left(V,m_\bullet^V\right),\left(A,m_\bullet^A\right)\right)$) such that $\alpha(1)=0$.*

Let $V[1]^0$ be the degree 0 part of $V[1]$, we consider $\mathrm{Hom}^\bullet(T^c(V[1]^0),A)\subset\mathrm{Hom}^\bullet(T^c(V[1]),A)$ as the graded vector subspace of morphisms with support in $T^c(V[1]^0)$. We define $L^0_{V[1]^*}(A):=L_{V[1]^*}(A)\cap\mathrm{Hom}^\bullet(T^c(V[1]^0),A)$ and we denote $L^0_{V[1]^*}(\Bbbk)$ by $L^0_{V[1]^*}$. The restriction of the dual δ^* : $\mathrm{Hom}^\bullet((T^c(V[1]))^1\oplus T^c(V[1]^0),A)\to\mathrm{Hom}^\bullet(T^c(V[1]^0),A)$ vanishes on $\mathrm{Hom}^\bullet(T^c(V[1]^0),A)$ if $V[1]$ is a non-negatively graded vector space (equivalently, if V is positively graded).

Corollary 2.1.18. *Let $\left(M_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]),A\right)\right)$ be as above. Assume that V is a positively graded vector space.*

1. *$\left(M_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]^0),A\right)\right)$ is an A_∞-algebra,*
2. *Let $f_1,\dots,f_n\in\mathrm{Hom}^\bullet\left(T^c(V[1]^0),A\right)$ and assume that there is a g with $\delta^*g=f_i$ for some i, then $M_n(f_1,\dots,f_n)\in\mathrm{Im}\,(\delta^*)$ for $n>1$. In particular*
$$\left(M_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]^0),A\right)/\,\mathrm{Im}(\delta^*)\right)$$
is an A_∞-algebra.
3. *Consider $\mathrm{Hom}^\bullet\left(T^c(V[1]^0),A\right)/\mathrm{Im}(\delta^*)$ equipped with the L_∞-structure $l_\bullet$ induced by the maps $M_\bullet$ via Theorem 2.1.11. Assume that $\delta\mu'(a,b)=0$ for any non-trivial shuffle $\mu'(a,b)\in T^c(V[1]^0)$ and that the product m_2^A is graded commutative. The subgraded vector space $L^0_{V[1]^*}(A)/\mathrm{Im}(\delta^*)$ equipped with $l_\bullet$ is a L_∞-subalgebra of*
$$\left(l_\bullet,\mathrm{Hom}^\bullet\left(T^c(V[1]^0),A\right)/\mathrm{Im}(\delta^*)\right).$$

For the proof see the Appendix, Section A.2.1.

Definition 2.1.19. Let $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$ be A_∞-algebras. Assume that V is positively graded.

1. We call
$$\mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) := \left(l_\bullet, \mathrm{Hom}^\bullet\left(T^c(V[1]^0), A\right)/\mathrm{Im}(\delta^*)\right)$$
the *degree zero convolution L_∞-algebra* associated to $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$.

2. Let $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$ be C_∞-algebras. We call
$$\mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) := \left(l_\bullet, L_{V[1]^*}(A)/\mathrm{Im}(\delta^*)\right) \subset \left(l_\bullet, \mathrm{Hom}^\bullet\left(T^c(V[1]^0), A\right)/\mathrm{Im}(\delta^*)\right)$$
the *degree zero reduced convolution L_∞-algebra* associated to $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$.

Proposition 2.1.20. *Let $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$ be A_∞-algebras, (resp. C_∞-algebras) where V is a positively graded vector space. There exists a one to one correspondence between*

1. *1-A_∞ (resp. C_∞)-morphisms $f_\bullet : \left(V, m_\bullet^V\right) \to \left(A, m_\bullet^A\right)$.*
2. *Graded maps $F : \left(T^c(A[1])\right)^1 \to T^c(B[1])$ such that $(F \otimes F)\Delta = \Delta F$ and such that its restriction $F : \left(T^c(A[1])\right)^0 \to T^c(V[1])$ commutes with the differentials (resp. such that $F(NTS(V[1])^0 \oplus NTS(V[1])^1) = 0$).*
3. *Maurer-Cartan elements $\alpha \in \mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ (resp. in $\mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$) such that $\alpha(1) = 0$.*

Let $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$, and $\left(W, m_\bullet^W\right)$ be A_∞-algebras. Let $g_\bullet : \left(W, m_\bullet^W\right) \to \left(V, m_\bullet^V\right)$ be an A_∞-morphism. The by Proposition 2.1.17 it corresponds to a morphsim $G : T^c(W[1]) \to T^c(V[1])$ of differential graded coalgebras. Then G induces a map (bye pre-composition)

$$g^* : \mathrm{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right).$$

Assume that V, W are positively graded. In the same way, for a 1-A_∞-map $f_\bullet : \left(W, m_\bullet^W\right) \to \left(V, m_\bullet^V\right)$ we have a well-defined map

$$f^* : \mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_0\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right).$$

Consider $\pi := p \circ r$ where

$$r : \mathrm{Hom}^\bullet\left(T^c(V[1]), A\right) \to \mathrm{Hom}^\bullet\left(T^c(V[1]^0), A\right) \tag{2.5}$$

is the restriction map and $p : \mathrm{Hom}^\bullet\left(T^c(V[1]^0), A\right) \to \mathrm{Hom}^\bullet\left(T^c(V[1]^0), A\right)/\mathrm{Im}(\delta^*)$ is the quotient map. Assume that V, W are positively graded vector spaces. The map π gives two maps

$$\begin{aligned} &\mathrm{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right), \\ &\mathrm{Conv}_r\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right). \end{aligned} \tag{2.6}$$

Proposition 2.1.21. *Let $g_\bullet, f_\bullet$ as above.*

1. *π is a strict morphism of L_∞-algebras.*
2. *The map*
$$g^* : \mathrm{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right)$$
is a strict morphism of L_∞-algebras. Assume that $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$, $\left(W, m_\bullet^W\right)$ and $g_\bullet$ are C_∞, then
$$g^* : \mathrm{Conv}_r\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_r\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right)$$
is a strict morphism of L_∞-algebras.

3. *Assume that V, W are positively graded vector spaces. The map*

$$f^* : \mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_0\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right)$$

is a strict morphism of L_∞-algebras. Assume that $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$, $\left(W, m_\bullet^W\right)$ and $f_\bullet$ are C_∞-algebras, then

$$f^* : \mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_{r,0}\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right)$$

is a strict morphism of L_∞-algebras.

4. *Since $g_\bullet$ is a 1-A_∞-morphism. We have $g^*\pi = \pi g^*$.*

In particular all the maps listed above send Maurer-Cartan elements to Maurer-Cartan elements.

Proof. Direct verification. □

Let $\left(B, m_\bullet^B\right)$ be a an A_∞-structure. Let $g_\bullet : A \to B$ be an A_∞-morphism. Then $g_\bullet$ induces a morphism of graded vector spaces from $\mathrm{Hom}^\bullet\left(T^c(V[1]), A\right)$ to $\mathrm{Hom}^\bullet\left(T^c(V[1]), B\right)$ in the following way. Let $\alpha \in \mathrm{Hom}^\bullet\left(T^c(V[1]), A\right)$, this corresponds to a morphism of differential graded coalgebras

$$G' : T^c\left(V[1]\right) \to T^c\left(A[1]\right).$$

On the other hand $g_\bullet$ corresponds to a morphism of differential graded coalgebras

$$G : \left(T^c\left(A[1]\right), \Delta, \delta_A\right) \to \left(T^c\left(B[1]\right), \Delta, \delta_B\right).$$

We consider the composition $G \circ G'$. It is again a morphism of differential graded coalgebras and corresponds to a Maurer-Cartan element $g_*(\alpha) \in \mathrm{Hom}^\bullet\left(T^c(V[1]), B\right)$, explicitly it is given by

$$g_*(\alpha)|_{V[1]^{\otimes n}} = \sum_{l=1}^{n} \pm g_l\left(\sum_{i_1+\ldots i_l=n} \alpha|_{V[1]^{\otimes i_1}} \otimes \cdots \otimes \alpha|_{V[1]^{\otimes i_l}}\right)$$

where the signs are a consequence of the Koszul convention. In particular if $g_\bullet$ is strict we have $g_*(\alpha) = g \circ \alpha$. Assume that V is a positively graded vector spaces. In the same way, for a 1-A_∞-map $f_\bullet : A \to B$ we have a well-defined map

$$f^* : \mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \to \mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(B, m_\bullet^B\right)\right).$$

Proposition 2.1.22. *Let $g_\bullet$ and $f_\bullet$ be as above.*

1. *The map*

$$g_* : MC\left(\mathrm{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)\right) \to MC\left(\mathrm{Conv}\left(\left(V, m_\bullet^V\right), \left(B, m_\bullet^B\right)\right)\right)$$

is well-defined. Assume that all the data are C_∞, then

$$g_* : MC\left(\mathrm{Conv}_r\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)\right) \to MC\left(\mathrm{Conv}_r\left(\left(V, m_\bullet^V\right), \left(B, m_\bullet^B\right)\right)\right)$$

is well-defined.

2. *Assume that V is positively graded. The map*

$$f_* : MC\left(\mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)\right) \to MC\left(\mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(B, m_\bullet^B\right)\right)\right)$$

is well-defined. Assume that all the data are C_∞, then

$$f_* : MC\left(\mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)\right) \to MC\left(\mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(B, m_\bullet^B\right)\right)\right)$$

is well-defined.

3. *Since $g_\bullet$ is a 1-A_∞-morphism. We have $g_*\pi = \pi g_*$.*

Proof. Direct verification. ☐

The propositions above and corollaries can be summarized as follows.

Theorem 2.1.23. *We have four functors*

$$\begin{aligned}
\mathrm{Conv} &: (A_\infty - \mathrm{ALG})^{op} \times (A_\infty - \mathrm{ALG}) \to (\mathcal{L}_\infty - \mathrm{ALG})_p\,, \\
\mathrm{Conv}_r &: (C_\infty - \mathrm{ALG})^{op} \times (C_\infty - \mathrm{ALG}) \to (\mathcal{L}_\infty - \mathrm{ALG})_p\,, \\
\mathrm{Conv}_0 &: (A_\infty - \mathrm{ALG}_{>0})_1^{op} \times (A_\infty - \mathrm{ALG})_1 \to (\mathcal{L}_\infty - \mathrm{ALG})_p\,, \text{ and} \\
\mathrm{Conv}_{r,0} &: (C_\infty - \mathrm{ALG}_{>0})_1^{op} \times (C_\infty - \mathrm{ALG})_1 \to (\mathcal{L}_\infty - \mathrm{ALG})_p\,.
\end{aligned}$$

We want to express convolution L_∞-algebras in terms of formal power series. Let V be of finite type. In Appendix A.2 we define two filtrations $I^\bullet$ and $G^\bullet$ on $T\left((V[1])^*\right)$ such that for a graded vector space W there exists a canonical isomorphism (see (A.8)) of completed graded vector spaces

$$\Psi \,:\, \mathrm{Hom}\left(T^c\left(V[1]\right), W\right) \to \widehat{T}\left((V[1])^*\right)\widehat{\otimes} W$$

where $\widehat{T}\left((V[1])^*\right)$ is the completion of $T\left((V[1])^*\right)$ with respect to the two aforementioned filtrations. In particular $I^\bullet$ and $G^\bullet$ induce a filtration on the Lie algebra of primitive elements of $T\left((V[1])^*\right)$ as well (viewed as a Hopf algebra), since this is precisely the free graded Lie algebra $\mathbb{L}\left((V[1])^*\right)$ on $(V[1])^*$, we can show that Ψ restricts to an isomorphism

$$\Psi \,:\, L_{V[1]^*}\left(W\right) \to \widehat{\mathbb{L}}\left((V[1])^*\right)\widehat{\otimes} W.$$

For the proof see Lemma A.2.5. We define $I^\bullet$ (see the appendix A.2 for a definition of $G^\bullet$). Consider the augmentation ideal I of the free algebra $T(V)$. Hence the powers of I define a filtration $I^\bullet$ on $T^c(V)$. We define $\mathcal{I}^i$ as the graded vector subspace of morphisms in $\mathrm{Hom}^\bullet\left(T^c(V[1])^{\otimes n}, W\right)$ such that $f|_{I^i} \cong 0$. For each $n > 0$, we consider $\mathrm{Hom}^\bullet\left(T^c(V[1]), A\right)^{\otimes n}$ equipped with the tensor product filtration $\mathcal{I}^{\otimes n}$ (see the appendix, Section A.2). If the dual $\delta^* \,:\, \mathrm{Hom}^\bullet\left(T^c(V[1]), \Bbbk\right) \to \mathrm{Hom}^\bullet\left(T^c(V[1]), \Bbbk\right)$ preserves the filtration $\mathcal{I}$ then

$$\left(\mathrm{Hom}^\bullet\left(T^c(V[1]), A\right), l_\bullet, \mathcal{I}^\bullet\right)$$

is a complete filtered L_∞-algebra and the maps in (2.6) are strict morphisms between completed filtered L_∞-algebras. For V of finite type, we denote the restricted differential by $\delta_r^* \,:\, \widehat{T}\left(\left(V[1]^0 \oplus V[1]^1\right)^*\right) \to \widehat{T}\left(\left(V[1]^0\right)^*\right)$. Let p, r and π as above and let $\mathbb{L}(V)$ be the graded free Lie algebra on V.

Corollary 2.1.24. *Let* $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$ *be* A_∞*-algebras, both assumed to be bounded below and that* V *is of finite type. The isomorphism* Ψ *induces the following isomorphism between complete graded vector spaces.*

1. $\mathrm{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \cong A\widehat{\otimes}\left(\widehat{T}((V[1])^*)\right)$.

2. *If* $\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)$ *are* C_∞ *we have* $\mathrm{Conv}_r\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \cong A\widehat{\otimes}\left(\widehat{\mathbb{L}}((V[1])^*)\right)$.

3. *Assume that* $\left(A, m_\bullet^A\right)$ *is unital and* V *positively graded. Let* $\langle \mathrm{Im}\left(\delta_r^*\right)\rangle \subset \widehat{T}(\left(V[1]^0\right)^*)$ *be the ideal generated by* $\mathrm{Im}\left(\delta_r^*\right)$ *(with respect to the concatenation product), then*

$$\mathrm{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \cong A\widehat{\otimes}\left(\widehat{T}(\left(V[1]^0\right)^*)/\left\langle \mathrm{Im}\left(\delta_r^*\right)\right\rangle\right).$$

4. *If* $m_\bullet^V$ *is* C_∞ *then* $\mathrm{Im}\left(\delta_r^*\right) \subset \widehat{\mathbb{L}}(\left(V[1]^0\right)^*)$. *If* $m_\bullet^A$ *is* C_∞ *we have*

$$\mathrm{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right) \cong A\widehat{\otimes}\left(\widehat{\mathbb{L}}(\left(V[1]^0\right)^*)/\left\langle \mathrm{Im}\left(\delta_r^*\right)\right\rangle\right),$$

where $\langle \mathrm{Im}\left(\delta_r^*\right)\rangle$ *is the Lie ideal generated by* $\mathrm{Im}\left(\delta_r^*\right)$*. Moreover if* δ^* *preserves the filtration* $\mathcal{I}^\bullet$*, the* L_∞*-algebras are complete.*

5. *Let $C \in \operatorname{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ (resp. $\operatorname{Conv}_r\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$) be a Maurer-Cartan element. Then $C_0 := \pi(C)$ is a Maurer-Cartan element in the L_∞-algebra $\operatorname{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ (resp. $\operatorname{Conv}_{r,0}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$.*

Proof. The first isomorphism is a special case of Ψ, where $W = A$. We prove the second. We consider $T\left((V[1])^*\right)$ equipped with the filtrations $I^\bullet$ (i.e. the augmentation ideal filtration) and $G^\bullet$. In particular $G^\bullet$ induces a trivial filtration on $T\left(\left(V[1]^0\right)^*\right) \subset T\left((V[1])^*\right)$, i.e. the completion of $T\left(\left(V[1]^0\right)^*\right) \subset T\left((V[1])^*\right)$ with respect to $I^\bullet$ and $G^\bullet$ corresponds to the completion of $T\left(\left(V[1]^0\right)^*\right)$ with respect to the $I^\bullet$ and the morphism Ψ (with $W = A$) implies an isomorphsim

$$\Psi \,:\, \operatorname{Hom}\left(T^c\left(V[1]^0\right), A\right) \to \widehat{T}\left(\left(V[1]^0\right)^*\right) \widehat{\otimes} A,$$

where $\mathcal{I}$ above is mapped to the filtration induced by I in $\widehat{T}\left(\left(V[1]^0\right)^*\right)$. The vector space of degree zero elements in $\operatorname{Im}(\delta^*)$ is isomorphic via Ψ to

$$\operatorname{Im}(\delta_r^*) \widehat{\otimes} A.$$

Since $(A, m_\bullet)$ is unital, let $\Bbbk 1 \subset A^0$. Then

$$(\partial, M_2, \operatorname{Hom}^\bullet(T^c(V[1]), \Bbbk 1) \subset \operatorname{Hom}^\bullet(T^c(V[1]), A))$$

is a sub differential graded algebra (compare with Lemma A.2.5), hence

$$\operatorname{Im}(\delta_r^*) \subset \widehat{T}\left(\left(V[1]^0\right)^*\right) \subset A \widehat{\otimes} \widehat{T}\left(\left(V[1]^0\right)^*\right).$$

If V is C_∞, by Lemma A.2.4 we have $\operatorname{Im}(\delta_r^*) \subset \left(l_0, \widehat{\mathbb{L}}\left(\left(V[1]^0\right)^*\right)\right)$. If $m_\bullet^A$ is C_∞ by Lemma A.2.4 we have the fourth point. The last point is immediate. □

Let $\left(V, m_\bullet^V\right)$ positively graded and of finite type, let $\left(A, m_\bullet^A\right)$ be a unital A_∞-algebras and let $\alpha^0, \alpha^1 \in \operatorname{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ be Maurer-Cartan elements. By Proposition 2.1.17, there are two A_∞-morphisms $f_\bullet^0, f_\bullet^1 \,:\, \left(V, m_\bullet^V\right) \to \left(A, m_\bullet^A\right)$ associated to them.

Proposition 2.1.25. 1. *let $\alpha^0, \alpha^1 \in \operatorname{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ be Maurer-Cartan elements. Let $f_\bullet^0, f_\bullet^1 \,:\, \left(V, m_\bullet^V\right) \to \left(A, m_\bullet^A\right)$ be the two corresponding A_∞-map. Then if they are homotopic, so are α^0, α^1.*

2. *Let $\left(V, m_\bullet^V\right)$ be a finite type positively graded A_∞-algebra. Let $\alpha^0, \alpha^1 \in \operatorname{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ be Maurer-Cartan elements. Let $f_\bullet^0, f_\bullet^1 \,:\, \left(V, m_\bullet^V\right) \to \left(A, m_\bullet^A\right)$ be the two corresponding A_∞-map. Then if the latter are 1-homotopic, so are α^0, α^1.*

3. *Let $(\mathcal{A}_\infty - \mathrm{ALG}_{un})$ be the full subcategory of unital A_∞-algebras, let $f_\bullet, g_\bullet \in (\mathcal{A}_\infty - \mathrm{ALG}_{un})$ with source $\left(A, m_\bullet^A\right)$. Assume that they are homotopic, then $f_*\alpha$ is homotopic to $g_*\alpha$ for any Maurer-Cartan $\alpha \in \operatorname{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$.*

4. *Let $\left(V, m_\bullet^V\right)$ be a finite type positively graded A_∞-algebra. Let $(\mathcal{A}_\infty - \mathrm{ALG}_{un})_1$ be the full subcategory of unital A_∞-algebras, let $f_\bullet, g_\bullet \in (\mathcal{A}_\infty - \mathrm{ALG}_{un})$ with source $\left(A, m_\bullet^A\right)$. Assume that they are 1-homotopic, then $f_*\alpha$ is homotopic to $g_*\alpha$ for any Maurer-Cartan $\alpha \in \operatorname{Conv}_0\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$.*

Proof. We prove the first assertion. Let $H_\bullet$ be a homotopy between the two maps. We apply (2.1.17) to $H_\bullet$. This gives a Maurer-Cartan element

$$\alpha(t) \in \operatorname{Conv}\left(\left(V, m_\bullet^V\right), \left(\Omega(1) \otimes A, m_\bullet^{\Omega(1)\otimes A}\right)\right) \cong \Omega(1) \widehat{\otimes} \operatorname{Conv}\left(\left(V, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right),$$

where the last isomorphism is given by Ψ (see Lemma A.2.5). On the left hand side we have the desired homotopy between α^0 and α^1 the sense of definition (2.1.12). The other assertion follows similarly by using the map Ψ as well and point 2 of Lemma 2.1.5. □

2.1.3 Transferring A_∞, C_∞-structures

The construction of the associated homological pair is a special case of the *homotopy transfer theorem* (Theorem 2.1.26 below). This theorem appears in [35]. We recall a more general version contained in [42] (see also [17]). In the first part of this subsection we use the sign convention of [42] (see remark (2.1.14)) where the sign $(-1)^{p+qr}$ in (2.1) and in (2.2) is replaced with $(-1)^{qp+q+p}$. Consider the following situation: let $(V^\bullet, d_V)$, $(W^\bullet, d_W)$ be two cochain differential graded vector spaces. Assume that there are two cochain maps

$$f \,:\, (V^\bullet, d_V) \overset{\longleftarrow}{\longrightarrow} (W^\bullet, d_W) \,:\, g \tag{2.7}$$

such that gf is homotopic to Id_V via a cochain homotopy h.

Theorem 2.1.26. *Let*

$$f \,:\, (V^\bullet, d_V) \overset{\longleftarrow}{\longrightarrow} (W^\bullet, d_W) \,:\, g$$

as above and assume that $(V^\bullet, d_V)$ is equipped with an A_∞-algebra structures $m^V_\bullet$, such that $m^V_1 = d_V$. There exist

1. *an A_∞-algebra structures $m^W_\bullet$ on $W^\bullet$, such that $m^V_1 = d_W$;*
2. *an A_∞-morphism $g_\bullet \,:\, \big(W, m^W_\bullet\big) \to \big(V, m^V_\bullet\big)$, such that $g_1 = g$;*

Explicit formulas for the maps above are contained in [42]. The formulas for $g_\bullet$ and $m^W_\bullet$ are the same as in [35] (without explicit signs) and are given by a finite summation over some set of trees. The strategy is first to build a family of degree $2-n$ maps

$$p_n \,:\, V^{\otimes n} \to V,$$

for $n > 1$ called the *p-kernels* and second to define the maps $m^W_\bullet, g_\bullet$ via

$$m^W_n := f \circ p_n \circ g^{\otimes n}, \quad g_n := h \circ p_n \circ g^{\otimes n}$$

Explicit formulas for the maps above are contained in [42] and a similar approach (for more general operads is contained in [17]).

Theorem 2.1.27 ([26], [17])**.** *The output formulas of Theorem 2.1.26 are C_∞ if so is V.*

Remark 2.1.28. Consider Theorem 2.1.26. In [42] it is proved the following. There exist

1. an A_∞-morphism $f_\bullet \,:\, \big(V, m^V_\bullet\big) \to \big(W, m^W_\bullet\big)$, such that $f_1 = f$;
2. an A_∞-homotopy $h_\bullet$ between gf and Id_V, such that $h_1 = h$, where $h_\bullet$ is a chain homotopy in the sense of [42] (i.e. with respect to a different cylinder object),

morevoer these map are explicitly constructed. For more details about the cylinder object used in [42] see Remark 1.11 in [49]. In the recent paper [17], a map $f_\bullet$ is constructed for any Koszul operad assuming that the diagram (2.7) satisfies (C2), (C4) and (C6) (see below) In particular it holds $\tilde{f}_\bullet \circ \tilde{g}_\bullet = id_W$. It is constructed a candidate for the homotopy $h_\bullet$ as well. The approach used in this paper is very closed to the one in [42] but much more general.

Remark 2.1.29. The output maps $m_\bullet$, $g_\bullet$ are not unique in general. There may be more solutions. However, when we refer to Theorem 2.1.26 we consider the maps $m^W_\bullet$, $g_\bullet$ as the used for Theorem 2.1.27 and explained below.

Definition 2.1.30. An oriented planar rooted tree T is a connected oriented planar graph that contains no loops, such that the orientation goes toward one marked external vertex (the root). Let $V(T)$ be the set of vertices, $E(T)$ the set of (oriented) edges.
Given an edge e, between two vertices v_1, v_2, if the orientation goes from v_1 to v_2 we call v_1 the *source* and v_2 the *target* of e, respectively. A edge e is *internal* if its source is the target of another edge. A non internal edge is called *leaf*. The root is the only one vertex which is not the source of any other edges. The *arity* of a vertex is the number of incoming edges.

We denote by $\mathcal{P}$ the set of finite oriented planar rooted trees where the arity of each internal vertex is ≥ 2. We denote by $\mathcal{P}_n$ the trees in $\mathcal{P}$ with exactly n leaves and with $\mathcal{P}_l^2$ the set oriented planar rooted trees where the arity of each internal vertex is less then or equal to l. Now fix a diagram of the type (2.7). Each tree can be decorated as follows: we associate to each internal edges the map h and to each internal vertex of arity k the map m_k^V. Figure 2.1 is an example of decorated tree T' where the root is the lower vertex.

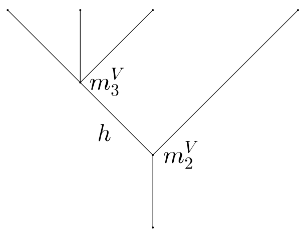


Figure 2.1: T' decorated

The three above gives a map $\mathcal{P}_{T'} : V^{\otimes 4} \to V$ via $\mathcal{P}_{T'} := m_2^V \circ \left(m_3^V \otimes Id\right)$; hence to each tree $T \in \mathcal{P}_n$ we associate a linear map $\mathcal{P}_T : V^{\otimes n} \to V$ as above. To each trees we can associated a values $\theta(T)$ as follows. For a vertex v in T with arity k, consider the edges $e_1, \ldots, e_k$. For each e_i let n_i be the numbers of all the paths that connect the root of T with a leaf, passing trough e_i. Define $\theta_T(v) := \theta(n_1, \ldots, n_k)$, and $\theta(T) := \sum_v \theta_T(v)$, where the sum is taken over the internal edges.

Proposition 2.1.31. *For each $n \geq 2$, the p-kernel (with respect to the convention of [42]) are*

$$p_n := \sum_{T \in \mathcal{P}_n} (-1)^{\theta(T)} P_T.$$

This is Proposition 6. in [42].

Remark 2.1.32. In order to get a formula with respect to different convention we do as follows: assume the same situation as in Theorem 2.1.26 where $m_\bullet^V$ is an A_∞-algebra defined with respect to different sign conventions. Then (see remark (2.1.14)) there exists a $(\epsilon_2, \ldots, \epsilon_i, \ldots) \in (\mathbb{Z}_2)^\infty$ such that $\underline{m}_1^V := m_1^V$ and

$$\underline{m}_i^V := (-1)^\epsilon m_i^V,$$

is an A_∞-algebra with respect to the convention above. Let $\underline{p}_n$ be the p-kernels with respect to $\underline{m}_i^V$, hence $p_n := (-1)^{\epsilon_n} \underline{p}_n$ are the p-kernels with respect to $m_\bullet^V$.

In order to have simpler formulas we consider some conditions on the diagram

(C1) (2.7) is a homotopy retract, i.e g is a quasi-isomorphism,

(C2) $f \circ g = 1_W$,

(C3) g is injective,

(C4) $d_W = 0$, and

(C5) $\left(V, m_\bullet^V\right)$ is a differential graded algebra ($m_n^V = 0$ for $n \geq 3$).

Notice that a diagram of the type (2.7) that satisfies (C2) automatically satisfies (C1) and (C3).

Lemma 2.1.33. *Let $(V^\bullet, d_v)$ be a cochain graded vector space. Assume that there is a decomposition*

$$V^\bullet = W \oplus d\mathcal{M} \oplus \mathcal{M}, \tag{2.8}$$

where $W = \oplus_{p \geq 0} W^p \cong H^\bullet(V)$ is a graded vector subspace of closed elements, and $\mathcal{M}$ is a graded vector subspace containing no exact elements except 0. *There exist maps f, g, h and a diagram of the type* (2.7), *that satisfies (C1), ..., (C4). Moreover the maps satisfy the* side conditions

(C6) $f \circ h = 0$*,* $h \circ g = 0$ *and* $h \circ h = 0$

Proof. Let W be a graded vector isomorphic to the cohomology of $V^\bullet$. We consider $H^\bullet$ as a differential graded vector space, with the zero differential. Note that the inclusion $g : (W^\bullet, 0) \hookrightarrow (V, d_V)$ is a quasi-isomorphism. We define $f : W \oplus d\mathcal{M} \oplus \mathcal{M} \to W$ to be the projection on the first coordinate. Let $h : W \oplus d\mathcal{M} \oplus \mathcal{M} \to W \oplus d\mathcal{M} \oplus \mathcal{M}$ defined by

$$h(a_1, da_2, a_3) := h(0, 0, a_2).$$

It is a degree -1 map between graded vector spaces. A short calculation show that h defines a homotopy between $f \circ g$ and 1_W and fulfills the conditions. □

Definition 2.1.34. We call the decomposition (2.8) *Hodge type decomposition.*

The inverse of the lemma above is also true.

Lemma 2.1.35. *Let* $V^\bullet, d_v$ *be a cochain complex. Assume that there is a diagram of the type* (2.7) *that satisfies (C2), (C4) and (C6). There is a Hodge type decomposition*

$$V^\bullet = W \oplus d\mathcal{M} \oplus \mathcal{M}.$$

Proof. This follows by setting $\mathcal{M} := \mathrm{Im}(h)$, $W := \mathrm{Im}(g)$. □

Definition 2.1.36. Given a diagram of the type (2.7) that satisfies (C2), (C4) and (C6). We call the Hodge type decomposition above obtained by Lemma 2.1.35 *the Hodge type decomposition associated to the diagram.*

Remark 2.1.37. A proof of Theorem 2.1.26 under the condition (C6), is contained in [32] (see the coalgebra perturbation lemma 2.1_*).

Lemma 2.1.38. *Let* (V, d_V) *be a chain complex. There exists a diagram of the type* (2.7) *that satisfies conditions (C1)-(C4) and (C6).*

Proof. Let $B^n := \{w \in V^n \mid \text{there exists } w' \in V^{n-1} \text{ such that } dw' = w\}$. Let $H^\bullet$ be a graded vector space isomorphic the cohomology of $V^\bullet$. We consider $H^\bullet$ as a differential graded vector space, with the zero differential equipped with an inclusion $(H^\bullet, 0) \hookrightarrow (V, d_V)$ which is a quasi-isomorphism. We have a short exact sequence

$$0 \longrightarrow B^n \oplus H^n \hookrightarrow V^n \xrightarrow{\ d\ } B^{n+1} \longrightarrow 0. \tag{2.9}$$

Since we are working on a field $\Bbbk$, for each $n \geq 0$, there exists a (non-canonical) split $\tilde{h}_{n+1} : B^{n+1} \to V^n$. Equivalently $d\tilde{h}_{n+1} = Id_{B^{n+1}}$ implies

$$V^n = H^n \oplus \mathrm{Im}(\tilde{h}_{n+1}) \oplus B^n.$$

Let $v \in V$, using the decomposition above we write $v = (v_1, v_2, v_3)$. We define the cochain homotopy $h^n : V^n \to V^{n-1}$ via $h^0 := 0$ and for $n > 0$ as

$$h^n(v) := h^n((v_1, v_2, v_3)) = \Big(0, \tilde{h}_{n+1}(v_3), 0\Big)$$

we define $f : H^n \oplus \mathrm{Im}(\tilde{h_{n+1}}) \oplus B^n \to H^n$ as the projection on the first coordinates and $g : H^n \to H^n \oplus \mathrm{Im}(\tilde{h}_{n+1}) \oplus B^n$ as the obvious inclusion. An easy calculation shows that this diagram is of the type (2.7) that satisfies conditions (C1)-(C4) and (C6). □

The proof of the next is contained in [37].

Lemma 2.1.39. *Consider a diagram of the type* (2.7)*, that satisfies (C2). There exists a cochain homotopy between* gf *and* 1_V *such that the diagram satisfies (C6).*

2.1.4 Homotopy transfer theorem and L_∞-convolution algebra

From now $W^\bullet$ will be always a non-negatively graded vector space. Let $m_\bullet^W$ be an A_∞-structure on $W^\bullet$. For $n>1$, let $m_n^{V_+}$ be the maps given by the composition

$$W_+^{\otimes n} \hookrightarrow W^{\otimes n} \xrightarrow{m_n^W} W_+.$$

they are well-defined since m_n^W are maps of degree $2-n$ and $(W_+^\bullet, m_\bullet^{W_+})$ is an A_∞-algebra. Assume that $m_1^W=0$. Let $g_\bullet : (W, m^W) \to (V, m^V)$ a morphism of A_∞-algebras. We denote by $g_\bullet^+$ the restriction of $g_\bullet$ to $\left(W_+^\bullet, m_\bullet^{W_+}\right)$. Then $g_\bullet^+ : (W_+, m^{W_+}) \to (V, m^V)$ is a well-defined morphism of A_∞-algebras.

Theorem 2.1.40. *1. Let $\left(A, m_\bullet^A\right)$ be a non-negatively graded A_∞-algebra. Fix a diagram*

$$f : (A^\bullet, m_1^A) \underset{\longrightarrow}{\longleftarrow} (W^\bullet, d_W) : g \tag{2.10}$$

and a homotopy h that satisfies (C2), (C4) and (C6) and let

$$A = W \oplus d\mathcal{M} \oplus \mathcal{M}, \tag{2.11}$$

be the associated Hodge type decomposition. There exists a unique pair $(m_\bullet^{W_+}, \alpha)$ where $m_\bullet^{W_+}$ is an A_∞-structure on W_+ and α is a Maurer-Cartan elements in

$$\mathrm{Conv}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right),$$

such that

(a) $m_1^{W_+}=0$,

(b) $\alpha|_{W^+[1]} = g$, and

(c) $\alpha|_{(W_+[1])^{\otimes n}} \subset \mathcal{M}$ for $n \geq 2$.

2. If A is C_∞ then $m_\bullet^{W_+}$ is C_∞ and $\alpha \in \mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right)$.

Proof. We construct a pair $\left(m_\bullet^W, g_\bullet\right)$ by using the p-kernel (see Proposition 2.1.31) and the maps in (2.10). By construction we have

i) an A_∞-algebra structure $\left(m_1^W, m_2^W, \dots\right)$ on $W^\bullet$, such that $m_1^W=0$ by condition (C4);

ii) an A_∞-morphism $g_\bullet : \left(W, m_\bullet^W\right) \to \left(A, m_\bullet^A\right)$, such that $g_1 = g$.

We define α as Maurer-Cartan element in

$$\mathrm{Conv}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right)$$

associated to the morphism $g_\bullet^+$. It fulfills the conditions 2 and 3. We prove the uniqueness of $\left(m_\bullet^W, \alpha\right)$. Assume that $(\tilde{m}_\bullet, \alpha')$ is another pair satisfying the conditions above. We have $\tilde{m}_1 = 0 = m_1^{W_+}$ and ${g'}_1^+ = g_1^+$. Assume that $\tilde{m}_i = m_i^W$, ${g'}_i^+ = g_i^+$ for $i<n$. We have By (2.2) we have

$$g_1^+ m_n^W - m_1^W g_n^+ = {g'}_1^+ \tilde{m}_n - \tilde{m}_1^W {g'}_n^+ \in A$$

Let $w_1, \dots, w_n \in W$, then by (2.11) we have

$$g_1^+ m_n^W (w_1, \dots, w_n) = {g'}_1^+ \tilde{m}_n (w_1, \dots, w_n)$$
$$m_1^W g_n^+ (w_1, \dots, w_n) = \tilde{m}_1^W {g'}_n^+ (w_1, \dots, w_n)$$

By point 2 we conclude $m_n^W (w_1, \dots, w_n) = \tilde{m}_n (w_1, \dots, w_n)$ and by point 3 $g_n^+ (w_1, \dots, w_n) = {g'}_n^+ (w_1, \dots, w_n)$. The last statement follows from Theorem 2.1.27. ☐

Remark 2.1.41. The uniqueness implies that $\left(m_\bullet^W, \alpha\right)$ is independent of the choice of maps $\left(m_\bullet^W, g_\bullet\right)$ that satisfy condition (1) and (3) in Theorem 2.1.26. Moreover, quasi-isomorphic A_∞-algebras $\left(A, m_\bullet^A\right)$ give isomorphic differential graded coalgebras $\left(\delta_W^+, T^c W_+[1]\right)$.

Corollary 2.1.42. $\mathrm{Conv}_0\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right)$ *is a complete L_∞-algebra with respect to the filtration $\mathcal{I}^\bullet$ (see Section A.2).*

Proof. Let δ_W^+ be the codifferential of the quasi-free coalgebra $T^c\left(W_+[1]\right)$ corresponding to $m_\bullet^{W_+}$. The underlying vector space of

$$\mathrm{Conv}_0\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right)$$

is a quotient of the complete vector space $A^\bullet \widehat{\otimes} \widehat{T}\left(\left(W^1[1]\right)^*\right)$ by the image of $\left(\delta_W^+\right)^*$. Condition a) corresponds to $\left(\mathrm{pro}_{W^+} \delta_W^+\right)|_{W^+} = 0$. Since $\left(\delta_W^+\right)^*$ satisfies the Leibniz rule, this condition implies that $\left(\delta_W^+\right)^* \mathcal{I}^\bullet \subset \mathcal{I}^{\bullet+1}$ and hence the statement. □

Note that if A is associative (condition (C5)), then $\left(T\left(A[1]\right), \Delta, \delta_A\right)$ is the Bar construction BA of $\left(A, m_\bullet^A\right)$. This gives an algebraic proof of the existence of the associated homological pair if A is the differential graded algebra of smooth differential forms on some smooth manifold M (see [12]).

Definition 2.1.43. We call the pair $\left(\left(\delta_W^+\right)^*, \alpha\right)$ *the (generalized) homological pair with respect to the decomposition* (2.11). Notice that if W is of finite type, then α can be written as a formal power series C on $T\left(W_+[1]^*\right)$ with coefficients in A.

Definition 2.1.44. Let $\left(\mathcal{A}_\infty - ALG_{\geq 0,\mathrm{dec}}\right)$ be the category where the objects are non-negatively graded A_∞-algebras $\left(\left(A, m_\bullet\right), W, \mathcal{M}\right)$ equipped with a Hodge type decomposition and the maps are morphism of A_∞-algebras. We define

$$\mathrm{Conv}\left(\left(A, m_\bullet\right), W, \mathcal{M}\right) := \mathrm{Conv}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right).$$

We call the $\alpha \in \mathrm{Conv}\left(\left(A, m_\bullet\right), W, \mathcal{M}\right)$ above, as obtained from Theorem 2.1.40, *the Maurer-Cartan elements associated to* $(W, \mathcal{M})$. Analogously we define

$$\mathrm{Conv}_r\left(\left(A, m_\bullet\right), W, \mathcal{M}\right) := \mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right).$$

We call the above $\alpha \in \mathrm{Conv}_r\left(\left(A, m_\bullet\right), W, \mathcal{M}\right)$, as obtained from Theorem 2.1.40, *the Maurer-Cartan elements associated to* $(W, \mathcal{M})$. Let $\left(\mathcal{A}_\infty - ALG_{\geq 0,\mathrm{dec}}\right)_1$ be the category where the objects are non-negatively graded A_∞-algebras $\left(\left(A, m_\bullet\right), W, \mathcal{M}\right)$ equipped with a Hodge type decomposition and the maps are 1-morphism of A_∞-algebras. We define

$$\mathrm{Conv}\left(\left(A, m_\bullet\right), W, \mathcal{M}\right) := \mathrm{Conv}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right).$$

We call the $\alpha \in \mathrm{Conv}_0\left(\left(A, m_\bullet\right), W, \mathcal{M}\right)$ above, as obtained from Theorem 2.1.40, *the Maurer-Cartan elements associated to* $(W, \mathcal{M})$. Analogously we define

$$\mathrm{Conv}_{r,0}\left(\left(A, m_\bullet\right), W, \mathcal{M}\right) := \mathrm{Conv}_{r,0}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A, m_\bullet^A\right)\right).$$

We call the $\alpha \in \mathrm{Conv}_{r,0}\left(\left(A, m_\bullet\right), W, \mathcal{M}\right)$ above, as obtained from Theorem 2.1.40 *the Maurer-Cartan elements associated to* $(W, \mathcal{M})$.

2.1.5 Homotopies between homological pairs

Let $\left(A, m_\bullet^A\right), \left(B, m_\bullet^B\right)$ and $\left(W, m_\bullet^A\right)$ be C_∞-algebras. Let $\alpha \in \mathrm{Conv}_r\left(\left(W, m_\bullet^V\right), \left(A, m_\bullet^A\right)\right)$ be a Maurer-Cartan element and consider a diagram of the type (2.7)

$$f' \,:\, \left(B^\bullet, d_B\right) \underset{}{\overset{}{\longleftrightarrow}} \left(W'^\bullet, d_{W'}\right) \,:\, g'$$

where $m_1^B = d_B$. By using Theorem 2.1.26 and Proposition 2.1.7, we turn the situation above into a diagram of C_∞-algebras

$$f'_\bullet \,:\, (B^\bullet, m_B) \longleftrightarrow (W'^\bullet, m_{W'}) \,:\, g'_\bullet \tag{2.12}$$

where $f'_\bullet$ is a homotopical inverse $g'_\bullet$, i.e. there exist a homotopy $H_\bullet$ between $(g'f')_\bullet$ and the identity. Notice that in general $f'_1 \neq f'$. Let $g_\bullet$ be the morphism corresponding to α. Let $p_\bullet \,:\, A \to B$ be a C_∞-morphism. We have the following diagram of C_∞-algebras

$$\begin{array}{ccc} \left(A, m_\bullet^A\right) & \xrightarrow{p_\bullet} & \left(B, m_\bullet^B\right) \\ {\scriptstyle g_\bullet}\uparrow & & {\scriptstyle f'_\bullet}\downarrow\uparrow{\scriptstyle g'_\bullet} \\ \left(W, m_\bullet^W\right) & & \left(W', m_\bullet^{W'}\right). \end{array}$$

We set $q_\bullet := f'_\bullet \circ p_\bullet \circ g_\bullet$.

Proposition 2.1.45. *Consider the diagram in* $\left(\mathcal{L}_\infty - \mathrm{ALG}\right)_p$

$$\begin{array}{ccc} \mathrm{Conv}_r\left(\left(W, m_\bullet^W\right), \left(A, m_\bullet^A\right)\right) & & \mathrm{Conv}_r\left(\left(W', m_\bullet^{W'}\right), \left(B, m_\bullet^B\right)\right). \\ & \searrow^{p_*} & \downarrow{\scriptstyle q^*} \\ & & \mathrm{Conv}_r\left(\left(W, m_\bullet^W\right), \left(B, m_\bullet^B\right)\right) \end{array}$$

Let α' be the Maurer-Cartan element corresponding to $g'_\bullet$. Then $p_*(\alpha)$ and $q^*(\alpha')$ are homotopic.

Proof. $p_*(\alpha)$ and $q^*(\alpha)$ are the Maurer-Cartan elements corresponding to $p_\bullet \circ g_\bullet$ and $g'_\bullet \circ f'_\bullet \circ p_\bullet \circ g_\bullet$, respectively. These two morphisms are homotopic via $H_\bullet \circ \left(Id \otimes (p_\bullet \circ f_\bullet)\right)_\bullet$, where $\left(Id \otimes (p_\bullet \circ f_\bullet)\right)_\bullet$ is the tensor product of morphisms defined in Lemma 2.1.5. □

Remark 2.1.46. We have $p_*(\alpha) = q^*(\alpha)$ if $g'_\bullet \circ f'_\bullet = \mathrm{Id}$. This situation is rare in general. For example if our diagram satisfies condition $(C2)$, $g'_\bullet \circ f'_\bullet = \mathrm{Id}$ is equivalent to $f'_1 = g'^{-1}_1$. This implies that $(B, m_\bullet)$ is *formal* i.e.

$$[f'_1] \,:\, B \to H(B, m_1^B)$$

is a strict isomorphism of C_∞-algebras.

2.1.6 1-model for C_∞-algebras and uniqueness

Let $\left(A, m_\bullet^A\right)$ be equipped with a Hodge type decomposition $(W, \mathcal{M})$. By Proposition 2.1.22, we know that

$$\pi \,:\, \mathrm{Conv}\left((A, m_\bullet), W, \mathcal{M}\right) \to \mathrm{Conv}_0\left((A, m_\bullet), W, \mathcal{M}\right)$$

is a strict morphism of L_∞-algebra and hence preserves Maurer-Cartan elements.

Definition 2.1.47. Let $\left(A, m_\bullet^A\right)$ be an A_∞-algebra (resp. C_∞-algebra). An A_∞-sub algebra $\left(B, m_\bullet^B\right)$ is an A_∞-algebra such that the inclusion is a strict morphism $i \,:\, B \hookrightarrow A$ of A_∞-algebras (resp. C_∞-algebras). Hence $m_\bullet^A = m_\bullet^B$. Let $1 \leq j \leq \infty$. An A_∞-sub algebras B is a j-model for $\left(A, m_\bullet^A\right)$ if

1. i induces an isomorphism up to the j-th cohomology group and is injective on the $j+1$ cohomology group.
2. the inclusion $i^l \,:\, B^l \hookrightarrow A^l$ preserves non-exact elements for $0 \leq l \leq j+1$.

If $j = \infty$ we call B a model for $\left(A, m_\bullet^A\right)$.

Proposition 2.1.48. *Let $\left(A, m_\bullet^A\right)$ be equipped with a Hodge type decomposition $(W, \mathcal{M})$. Let $i \,:\, B \hookrightarrow A$ be a 1-model of A equipped with a decomposition $(W', \mathcal{M}')$ such that $W' \subset W$ and $\mathcal{M}' \subset \mathcal{M}$. Let $\alpha \in \mathrm{Conv}\left((A, m_\bullet), W, \mathcal{M}\right)$, $\alpha' \in \mathrm{Conv}\left((B, m_\bullet), W', \mathcal{M}'\right)$ be the Maurer-Cartan elements associated to the decomposition $(W, \mathcal{M})$ and $(W', \mathcal{M}')$. Then $\pi(\alpha) = i_*\pi(\alpha')$. If $\left(A, m_\bullet^A\right)$ is C_∞, the statement is true for $\mathrm{Conv}_r\left((A, m_\bullet), W, \mathcal{M}\right)$ as well.*

Proof. By definition we have $W^1 = W'^1$. We show that the two induced C_∞-structures $m_\bullet^{W_+}$ and $m_\bullet^{W'_+}$ coincide on W^1. We have $m_1^{W_+} = 0 = m_1^{W'_+}$. Assume that $m_i^{W_+} = m_i^{W'_+}$ for $i < n$ on W^1. We have

$$\begin{aligned} m_1^{W_+} m_i^{W'_+} &= \sum_{\substack{p+q+r=n \\ k=p+1+r \\ k,q>1}} (-1)^{p+qr} m_k^{W'_+} \left(1^{\otimes p} \otimes m_q^{W'_+} \otimes 1^{\otimes r}\right) \\ &= \sum_{\substack{p+q+r=n \\ k=p+1+r \\ k,q>1}} (-1)^{p+qr} m_k^{W_+} \left(1^{\otimes p} \otimes m_q^{W_+} \otimes 1^{\otimes r}\right) \\ &= m_1^{W_+} m_n^{W_+} \end{aligned}$$

Hence $m_n^{W_+}|_{(W^1)^{\otimes n}} = m_n^{W'_+}|_{(W^1)^{\otimes n}}$, since W contains no exact form except zero. Let $g'_\bullet$ be the morphism corresponding to $i_*\pi(\alpha')$ and let $g_\bullet$ be the morphism corresponding to $\pi(\alpha)$, we have $m_1^{W_+} g'_n|_{W^{1\otimes n}} = m_1^{W_+} g_n|_{W^{1\otimes n}}$. The codomain of $g'_n|_{W^{1\otimes n}}, g_n|_{W^{1\otimes n}}$ is $\mathcal{M}^1$, then $g'_n|_{W^{1\otimes n}} = g_n|_{W^{1\otimes n}}$. □

Lemma 2.1.49. *Let B be a 1-model for a connected C_∞-algebra $\left(A, m_\bullet^A\right)$. Assume that B is equipped with a Hodge type decomposition*

$$B = W \oplus \mathcal{M} \oplus d\mathcal{M}.$$

1. *There exists a model $i_C : C \hookrightarrow A$ such that $i_B(B) \subseteq i_C(C) \subseteq A$ is equipped with a Hodge type decompositions $(W_C, \mathcal{M}_C)$ such that*
   (a) $W_C^i = W^i$, $i = 0, 1$ and $W^2 \subseteq W_C^2$
   (b) $\mathcal{M}^i \subseteq \mathcal{M}_C^i$ for $i = 0, 1, 2$.
2. *Let C be as above. There exists a Hodge type decomposition $(W_A, \mathcal{M}_A)$ on A such that*
   (a) $W_C^i = W_A^i$, $i \geq 0$
   (b) $\mathcal{M}_C^i \subseteq \mathcal{M}_A^i$ for $i \geq 0$.

In particular, there exists a cochain map $p_C : C \hookrightarrow A$ and a cochain homotopy such that $p_C i_C = Id_C$ and $m_1^A h + h m_1^A = i_C \circ p_C - Id_C$. Moreover, the homotopy h satisfies (C6).

Proof. Since the inclusion $B \hookrightarrow A$ preserves non-exact elements, we have $\mathcal{M}^i$ as a vector subspace of A doesn't contains closed forms for $i = 0, 1, 2$ except 0. We prove 1. by setting $C = A$, then, thanks to Lemma 2.1.38, the Hodge type decomposition extends to an Hodge type decomposition for A as well. This proves 2. automatically. The last part is similar to 2.1.33. □

We study the relationship between homological pairs and the choice of 1-models. Let $\left(A, m_\bullet^A\right)$ be a C_∞-algebra. Let $\bar{B}_1$, $\bar{B}_2$ be two 1-models for A equipped with distinct Hodge type decompositions $(W_j, \mathcal{M}_j)$ for $j = 1, 2$. Let $C \subseteq A$ be a model for A that contains $\bar{B}_1$, $\bar{B}_2$. By the lemma above, we can construct two models $i^1 : B_1 \to C, i^2 : B_2 \to C$ and there are maps i^j, p^j and h^j satisfying (C2) and (C6) for $j = 1, 2$. For $j = 0, 1$, by Lemma 2.1.33, the Hodge type decomposition in B_j corresponds to a diagram of the type (2.7)

$$g^j : (W_j, 0) \overset{\longleftarrow}{\longrightarrow} (B_j, d) : f^j,$$

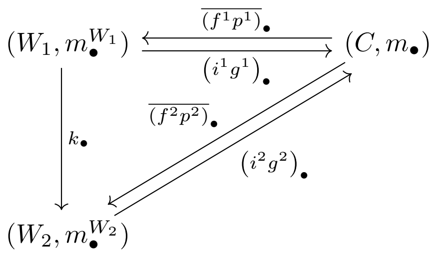
where the homotopy is denoted by $\bar{h}^j$. It is easy to see that the diagram

$$i^j g^j : (W_j, 0) \overset{\longleftarrow}{\longrightarrow} (C, d) : f^j p^j,$$

equipped with the homotopy $h(j) := i^j \bar{h}^j p^j + h^j$ is a diagram that satisfies (C2), (C4) and (C6). By using Theorem 2.1.26 and Proposition 2.1.7, we turn the two diagrams above in terms of C_∞-algebras.

$$\left(g^j\right)_\bullet : (W_j, m_\bullet^{W_j}) \overset{\longleftarrow}{\longrightarrow} (B_j, m_\bullet) : \overline{(f^j)}_\bullet \qquad \left(i^j g^j\right)_\bullet : (W_j, m_\bullet^{W_j}) \overset{\longleftarrow}{\longrightarrow} (C, m_\bullet) : \overline{(f^j p^j)}_\bullet$$

such that $\overline{(f^j p^j)}_\bullet \left(i^j g^j\right)_\bullet$ and Id_C are homotopic, and $\overline{(f^j)}_\bullet g^j_\bullet$ is homotopic to Id_{B_j} for any $j = 1, 2$. On the other hand, the uniqueness property in Theorem 2.1.40 implies that $\left(i^j g^j\right)_\bullet = i^j \left(g^j\right)_\bullet$. Consider the diagram



where $k_\bullet := \left(f^2 p^2\right)_\bullet \left(i^1 g^1\right)_\bullet$.

Proposition 2.1.50. *1. For $j = 1, 2$ let α_j be the Maurer-Cartan elements corresponding to $g^j_\bullet$. Then $i^1_* (\alpha_1)$ is homotopic to $k^* (\alpha_2)$. Moreover, $k_\bullet$ is an isomorphism.*

2. Let $\tilde{B} \subset A$ be a C_∞-subalgebra. Consider a C_∞-morphism

$$\tilde{g} \,:\, \left(\tilde{W}, \tilde{m}_\bullet\right) \to \left(\tilde{B}, m_\bullet\right)$$

Let $\tilde{C} \subset A$ be a model such that $\tilde{B} \subset \tilde{C}$ and $B_2 \subset \tilde{C}$. Point 1. is true if we replace g^1 with $\tilde{g}$.

Proof. We prove point 1. Proposition 2.1.7, there exists a homotopy $H_\bullet$ between $\left(i^2 g^2\right)_\bullet \overline{(f^2 p^2)}_\bullet$ and Id_C. Then

$$\left(i^2 g^2\right)_\bullet k_\bullet = \left(i^2 g^2\right)_\bullet \overline{(f^2 p^2)}_\bullet \left(i^1 g^1\right)_\bullet$$

is homotopic to $\left(i^1 g^1\right)_\bullet$ via $\left(i^1 g^1\right)_\bullet H_\bullet$. Notice that $k_1 = p^2 i^1 \,:\, W_1 \to W_2$ is a quasi-isomorphism. Since $m_1^{W_j} = 0$ for $j = 1, 2$, it is an isomorphism. We prove point 2. Since the above proof does not involve f^1, it carry over this general situation. ☐

Remark 2.1.51. An explicit formula for $\overline{(f^j)}_\bullet$ and $\overline{(f^j p^j)}_\bullet$ is constructed in [17], theorem 5. There is an explicit candidate for the homotopy H used in the above proof as well.

2.2 Cosimplicial commutative algebras

We give a very short introduction about the Dupont contraction and the results of [26]. We introduce a C_∞-structure that corresponds to the natural algebraic structure on the differential forms of a smooth complex simplicial manifold (see Theorem (2.2.6)). This C_∞-structure is in general hard to calculate. In the last we use a result of [26] to present an almost complete formula on degree 1-elements (see Theorems 2.2.7 and 2.2.9).

In this section we work on a field $\Bbbk$ of charactersitic zero. We denote by $sSet$ the category of simplicial sets and by $\Delta \,:\, \boldsymbol{\Delta} \to sSet$ the Yoneda embedding.

For each $[n] \in \boldsymbol{\Delta}$ we define the n-gemetric simplex

$$\Delta_{geo}[n] := \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^n \mid t_0 + t_1 + \cdots + t_n = 1 \right\}.$$

For each $[n] \in \boldsymbol{\Delta}, i = 0, \dots, n+1$ we define the smooth maps $d^i \,:\, \Delta_{geo}[n] \to \Delta_{geo}[n+1]$

$$d^i (t_0, t_1, \dots, t_n) = (t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots t_n)$$

and $s^i \,:\, \Delta_{geo}[n+1] \to \Delta_{geo}[n], i = 0, \dots, n$ via

$$s^i (t_0, t_1, \dots, t_{n+1}) = (t_0, t_1, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_{n+1}).$$

In particular, $\Delta[\bullet]_{geo}$ is a cosimplicial topological space. For each $[n]$ let $\Omega^\bullet(n)$ be the symmetric graded algebra (over $\Bbbk$) generated in degree 0 by the variables $t_0, \dots, t_n$ and in degree 1 by $dt_0, \dots, dt_n$ such that

$$t_0 + \cdots + t_n = 1, \quad dt_0 + \cdots + dt_n = 0.$$

We equip $\Omega(n)$ with a differential $d : \Omega^\bullet(n) \to \Omega^{\bullet+1}(n)$ via $d(t_i) := dt_i$. $\Omega(n)$ is the differential graded algebra of polynomial differential forms on $\Delta[n]_{geo}$. It follows that $\Omega(\bullet)$ is a simplicial commutative differential graded algebra, where the face maps d_i are obtained via the pullback along d^i, and the codegenerancy maps are obtained via the pullback along s^i.
For a set X we denote by $\Bbbk\langle X\rangle$ the module generated by X and by $X^{\Bbbk}$ the module $\mathrm{Hom}_{Set}(X, \Bbbk)$. Thus for a simplicial set $X_\bullet$ we denote by $\Bbbk\langle X_\bullet\rangle$ the simplicial module $\Bbbk\langle X\rangle_n := \Bbbk\langle X_n\rangle$ and by $X_\bullet^{\Bbbk}$ the cosimplicial module $\left(X^{\Bbbk}\right)_n := X_n^{\Bbbk}$. Both of these constructions are functorials and $(-)^{\Bbbk} : sSet \to cMod$ is contravariant.
Let $\mathbf{\Delta}$ be the simplex category and let $\Delta : \mathbf{\Delta} \to sSet$ be the Yoneda embedding. Then $\Delta[\bullet]$ is a cosimplicial object in the category of simplicial sets and $C_\bullet := (\Delta[\bullet])^{\Bbbk}$ is a simplicial cosimplicial module. We get that $NC_\bullet$ is a simplicial differential graded module. Explicitly, for a fixed n we have

$$(NC_n)^p := \begin{cases} \Bbbk\left\langle \mathrm{Hom}_{Set}\left(\Delta[n]_p^+, \Bbbk\right)\right\rangle, \text{ if } & p \leq n, \\ 0, \text{ if } & p > n \end{cases}$$

where $\Delta[n]_p^+$ is the set of inclusions $[p] \hookrightarrow [n]$. A cosimplicial differential graded module is a cosimplicial object in the category $dgMod$ of differential graded modules. Explicitly, we denote this objects by $A^{\bullet,\bullet}$ where the first slot denote the cosimplicial degree and the second slot denotes the differential degree. It defines a functor $A^{\bullet,\bullet} : \mathbf{\Delta} \to dgMod$ and we get a bifunctor $NC_\bullet \otimes A^{\bullet,\bullet} : \mathbf{\Delta}^{op} \times \mathbf{\Delta} \to dgMod$. We consider the coend

$$\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet} \in dgMod.$$

An element v of degree k in $\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ is a sequence $v := (v_n)_{n\in\mathbb{N}}$ where $v_n \in (NC_n \otimes A^{n,\bullet})^k$ such that for any map $\theta : [n] \to [m]$ in $\mathbf{\Delta}$ we have

$$(1 \otimes \theta^*)\, w_n = (\theta_* \otimes 1)\, w_m,$$

where $\theta^* := A^{\bullet,\bullet}(\theta)$, and $\theta_* := NC_\bullet(\theta)$. Since

$$(NC_n \otimes A^{n,\bullet})^k = \oplus_{p+q=k} NC_n^p \otimes A^{n,q}$$

we say that v has bidegree (p, q) if v has degree $p+q$ and each $v_n \in NC_n^p \otimes B^{n,q}$ for each n. Let (V, d_V) and (W, d_w) be two cochain complexes, $(V \otimes W)^\bullet$ is again a cochain complexes where the differential is

$$d_{V\otimes W}(v \otimes w) := d_V(v) \otimes w + (-1)^p v \otimes d_w(w)$$

for $v \otimes w \in V^p \otimes W^q$. The differential on $\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ is defined via

$$(dv)_n := dv_n$$

where d is the induced differential on $(NC_n \otimes A^{n,\bullet})^\bullet$. Consider the differential graded module (called Thom-Whitney normalization, see [26])

$$\mathrm{Tot}_{TW}(A) := \int^{[n]\in\mathbf{\Delta}} \Omega(n) \otimes A^{n,\bullet} \in dgMod.$$

Explicitly, an element $v \in \mathrm{Tot}_{TW}(A)^k$ is a collection $v = (v_n)$ of $v_n \in (\Omega(n) \otimes A^{n,\bullet})^k$ such that for any map $\theta : [n] \to [m]$ in $\mathbf{\Delta}$ we have

$$(1 \otimes \theta_*)\, v_n = (\theta^* \otimes 1)\, v_m,$$

where $\theta_* := A^{\bullet,\bullet}(\theta)$, and $\theta^* := \Omega(n)(\theta)$. Since

$$(\Omega(n) \otimes A^{n,\bullet})^k = \oplus_{p+q=k} \Omega^p(n) \otimes A^{n,q},$$

we say that v has bidegree (p, q) if v has degree $p+q$ and each v_n is contained in $\Omega^p[n] \otimes A^{n,q}$. We denote by $\mathrm{Tot}_{TW}(A)^{p,q}$ the set of elements of bidegree (p, q). If $A^{\bullet,\bullet}$ is a cosimplicial unital differential graded commutative algebra, then $(\mathrm{Tot}_{TW}(A), d_{\bullet,A}, d_{\bullet,poly})$ is a differential graded commutative algebra as well where the multiplication and the differential are

$$(v \wedge w)_n := (v)_n \wedge (w)_n, \quad (dv)_n := d(v_n).$$

2.2.1 The Dupont retraction

We give a short summary of the results of [26], where a C_∞-structure is induced on $\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ (and hence on $\mathrm{Tot}_N(A)$) from $\mathrm{Tot}_{TW}(A)$ via the homotopy transfer theorem.

Theorem 2.2.1 ([25],[20]). *Let $\Omega(\bullet)$, $NC_\bullet$ be the two simplicial differential graded modules defined above. We denote the differential of $\Omega(\bullet)$ by $d_{\bullet,poly}$. There is a diagram between simplicial graded modules*

$$E_\bullet \,:\, NC_\bullet \longleftrightarrow \Omega(\bullet) \,:\, \int_\bullet \tag{2.13}$$

and a simplicial homotopy operator $s_\bullet : \Omega(\bullet) \to \Omega^{\bullet-1}(\bullet)$ between $\left(E_\bullet \int_\bullet\right)$ and the identity, i.e., $s_\bullet$ is a map between simplicial differential graded modules such that for each $n \geq 0$

$$d_{n,poly}s_n + s_n d_{n,poly} = E_n \int_n -Id.$$

In particular, the diagram

$$E_\bullet \,:\, NC_\bullet \longleftrightarrow \Omega(\bullet) \,:\, \int_\bullet \tag{2.14}$$

together with the simplicial homotopy $s_\bullet$ satisfies the properties $(C1)$, $(C2)$, $(C3)$ and $(C6)$.

Proof. The first statement is originally contained in [20]. The second part of the theorem is proved in [25]. □

See Section A.4 for more details about the above maps. Let $m_\bullet^{[n]}$ be the C_∞-algebra structure induced on the differential graded modules NC_n by the above diagram. Let $A^{\bullet,\bullet}$ be a cosimplicial commutative algebra. The differential graded algebra $\Omega(n)\otimes A^{n,\bullet}$ is commutative as well. For any $n, m \geq 0$, we denote the C_∞-structure induced along the diagram

$$E_n \otimes Id \,:\, NC_n^\bullet \otimes A^{n,\bullet} \longleftrightarrow \Omega(n) \otimes A^{n,\bullet} \,:\, \int_n \otimes Id \tag{2.15}$$

by $m_\bullet^{n,m}$. This structure depends only on $m_\bullet^{[n]}$. Let $w_1, \ldots, w_l \in NC_n \otimes A^{n,\bullet}$ be such that $w_i = f_i \otimes a_i$ for $i = 1, \ldots, l$. Then

$$m_l^{n,m}(w_1, \ldots, w_l) = (-1)^{\sum_{i<j}|f_i||a_j|} m_l^{[n]}(f_1, \ldots, f_l) \otimes (a_1 \wedge \cdots \wedge a_l)\,. \tag{2.16}$$

The above structure defines a well-defined C_∞-structure $m_\bullet$ on $\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$.The maps $m_\bullet$ can be obtained in another way. We apply the coend functor to the simplicial diagram (2.15). Since $s_\bullet$, $E_\bullet$, $\int_\bullet$ are all simplicial maps, they induce degree zero maps E, $\int$ between the coends

$$\int \,:\, \mathrm{Tot}_{TW}(A) \to \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}, \quad E \,:\, \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet} \to \mathrm{Tot}_{TW}(A)$$

and a degree -1 map

$$s \,:\, \mathrm{Tot}_{TW}(A) \to \mathrm{Tot}_{TW}(A)\,.$$

In particular,

$$E \,:\, \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet} \longleftrightarrow \mathrm{Tot}_{TW}(A) \,:\, \int \tag{2.17}$$

together with the homotopy s is a diagram of type (2.7) that satisfies the properties $(C1)$, $(C2)$, $(C3)$ and $(C6)$. This gives a unital C_∞ algebra structure on $\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ induced from $\mathrm{Tot}_{TW}(A)$ along the diagram (2.17). In particular, this structure coincides with $m_\bullet$. Notice that the construction of $\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ gives a functor form the category of cosimplicial unital non-negatively graded commutative differential graded algebra (cdgA for short) toward the category of cochain complexes. We have a correspondence

$$A^{\bullet,\bullet} \mapsto \left(\int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}, m_\bullet \right). \tag{2.18}$$

For a field $\Bbbk$ of characteristic zero we denote by $(C_\infty - \mathrm{Alg})_{\Bbbk,str}$ the category of C_∞-algebras on $\Bbbk$ and strict morphisms.

Theorem 2.2.2. *The correspondence* (2.18) *gives a functor* cdgA $\to (C_\infty - \mathrm{Alg})_{\Bbbk,str}$.

Proof. Let $f : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ be a morphism.The correspondence (2.18) is a functor toward the category of chain complexes. We denote its image with f_1. Since f is a differential graded algebra map on each degree, the same argument of Lemma 2.1.5 shows that f_1 induces a strict morphism. □

We give an explicit formula for $m_\bullet$. Fix a n and a $p \le n$. Notice that each inclusion $[p] \hookrightarrow [n]$ is equivalent to an ordered string $0 \le i_0 < i_i < \cdots < i_p \le n$ contained in $\{0, 1, \ldots, n\}$. For each string $0 \le i_0 < i_i < \cdots < i_p \le n$, we denote the associated inclusion by $\sigma_{i_0,\ldots i_p} : [p] \hookrightarrow [n]$, and we define the maps $\lambda_{i_0,\ldots,i_p} : \Delta[n]_p^+ \to \Bbbk$, via

$$\lambda_{i_0,\ldots,i_p}(\phi) := \begin{cases} 1 \text{ if } \sigma_{i_0,\ldots i_p} = \phi, \\ 0. \text{ otherwhise} \end{cases}$$

Let $v_1, \ldots, v_n \in \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ be elements of bidegree (p_i, q_i). Then $m_n(v_1, \ldots, v_n)$ is an element of bidegree $(\sum_i pi + 2 - n, \sum q_i)$. Let $l := \sum_i pi + 2 - n$, then Lemma A.3.1 implies that $m_n(v_1, \ldots, v_n)$ is completely determined by $m_n((v_1)_l, \ldots, (v_n)_l)_l$. We write $(v_i)_{p_i} = \lambda_{0,\ldots,p_i} \otimes a_i \in NC_{p_i}^{p_i} \otimes A^{p_i,q_i}$ for all the i. We denote by I the subsets $\{i_0, \ldots, i_p\} \subseteq \{0, \ldots, l\}$, for $I = \{i_0, \ldots, i_p\}$ we define $|I| := p$ and we write λ_I instead of $\lambda_{0,\ldots,p}$. Each I corresponds to an inclusion in $\mathbf{\Delta}$; we denote by $\sigma_I : [p] \to [l]$ the map induced by I. We have

$$\begin{aligned} m_n(v_1, \ldots, v_n)_l &= m_n^{l,l}((v_1)_l, \ldots, (v_n)_l)_l \\ &= m_n^{l,l}\left(\sum_{|I_1|=p_1} \lambda_{I_1} \otimes (\sigma_{I_1})_* a_1, \ldots, \sum_{|I_n|=p_n} \lambda_{I_n} \otimes (\sigma_{I_n})_* a_n \right) \\ &= \sum_{|I_1|=p_1,\ldots,|I_n|=p_n} (-1)^{\sum_{i<j}|p_i||q_j|} m_n^{[l]}(\lambda_{I_1}, \ldots, \lambda_{I_n}) \otimes ((\sigma_{I_1})_* a_1 \wedge \cdots \wedge (\sigma_{I_n})_* a_n). \end{aligned} \tag{2.19}$$

In particular, the above formula implies that if $v_1, \ldots, v_n$ were all of degree 1, then $m_n(v_1, \ldots, v_n)$ would only depend on

- the restriction of $m_\bullet^{[2]}$ on the elements of degree 1, if all the v_i are of bidegree $(1, 0)$;
- $m_\bullet^{[0]}$, if all the v_i are of bidegree $(0, 1)$;
- $m_\bullet^{[1]}$ in the other cases.

Lemma 2.2.3. *Consider $A^{0,\bullet}$ equipped with its differential graded algebra structure. There is a canonical inclusion $i : A^{0,\bullet} \hookrightarrow \mathrm{Tot}_N(A)$ which is a strict C_∞-algebra map.*

Proof. The map i is clearly a cochain map. The structure $m_\bullet^{[0]}$ is trivial, and by setting $l = 0$ in (2.19) we obtain that i is strict. □

The $m_\bullet^{[1]}$ is given in [26]. We first set a convenient basis for NC_1. Notice that the maps E_n are all injective. This allows us to interpret NC_1 as a submodule of $\Omega^\bullet(1)$. Recall that $\Omega^\bullet(1)$ is the free differential graded commutative algebra generated by the degree-zero variables t_0, t_1 modulo the relations

$$t_0 + t_1 = 1, \quad dt_0 + dt_1 = 0.$$

NC_1^0 is a two-dimensional vector space generated by λ_0 and λ_1 and NC_1^1 is one-dimensional generated by $\lambda_{0,1}$. We have

$$E_1(\lambda_0) = t_0, \quad E_1(\lambda_1) = t_1, \quad E_1(\lambda_{0,1}) = t_0 dt_1 - t_1 dt_0. \tag{2.20}$$

Let $t := t_0$, hence $t_1 = 1 - t$. Then $\Omega^\bullet(1)$ may be considered as the free differential graded commutative algebra generated by t in degree zero and NC_1 is the subgraded module generated by $1, t, dt$. In particular 1 is the unit of the C_∞-structure.

Proposition 2.2.4 ([26]). *The structure $m_\bullet^{[1]}$ on NC_1 is defined as follows:*

1. $m_2^{[1]}(t,t) = t,$
2. $m_{n+1}^{[1]}(dt^{\otimes i}, t, dt^{\otimes n-i}) = (-1)^{n-i}\binom{n}{i} m_{n+1}^{[1]}(t, dt, \dots, dt),$
3. $m_{n+1}^{[1]}(t, dt, \dots, dt) = \frac{B_n}{n!}dt,$ *where B_n are the second Bernoulli numbers,*

and all remaining products vanishes.

It remains to find a formula for $m_n^{[2]}|_{(NC_2^1)^{\otimes n}}$. For $n > 2$ we are not aware of an explicit formula.

Proposition 2.2.5. *Consider $NC_2^\bullet$ equipped with $m_\bullet^{[2]}$. We have*

$$m_2^{[2]}(\lambda_{01}, \lambda_{02}) = m_2^{[2]}(\lambda_{01}, \lambda_{12}) = m_2^{[2]}(\lambda_{02}, \lambda_{12}) = \frac{1}{6}\lambda_{012}.$$

Proof. By explicit calculation. The details are given in Appendix A.4. □

Consider $A^{\bullet,\bullet}$ as above. We denote by $N(A)^{\bullet,\bullet}$ its bigraded bidifferential module (see Appendix A.3 for a definition) and by $\mathrm{Tot}_N(A) \in dgMod$ its associated total complex. It is well known that there is a natural isomorphism $\psi : \mathrm{Tot}_N(A) \to \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ of differential graded modules (see Lemma A.3.1 for a proof). With an abuse of notation we denote again by $m_\bullet$ the C_∞-structure induced on $\mathrm{Tot}_N(A)$ via the isomorphism ψ. We have the following.

Theorem 2.2.6. *The association*

$$A^{\bullet,\bullet} \mapsto (\mathrm{Tot}_N(A), m_\bullet) \tag{2.21}$$

is part of a functor $\mathrm{cdgA} \to (C_\infty - \mathrm{Alg})_{\Bbbk,str}$.

We give an explicit formula for $m_\bullet$ in $\mathrm{Tot}_N(A)$ for elements of degree $0, 1$. We denote by $\tilde{\partial} : A^{\bullet,\bullet} \to A^{\bullet+1,\bullet}$ the differential given by the alternating sum of coface maps, in particular $\tilde{\partial} = d^0 - d^1$ on $A^{0,\bullet}$.

Theorem 2.2.7. *Let $l > 2$.*

1. *Let $a_1, \dots, a_l \in \mathrm{Tot}_N^1(A)$ and let $b_i \in A^{1,0}$, $c_i \in A^{0,1}$ be such that $a_i = b_i + c_i$ for every $i = 1, \dots, l$. Then*
$$m_l(a_1, \dots, a_l) = \sum_{i=1}^{l} (-1)^{l-1}\binom{l-1}{i-1}\frac{B_{l-1}}{(l-1)!} b_1 \cdots \widehat{b_i} \cdots b_l \tilde{\partial} c_i + m_l(c_1, \dots, c_n).$$
2. *Let $x \in A^{0,0}$. Then*
$$m_l(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_l) = (-1)^i \binom{l-1}{i-1}\frac{B_{l-1}}{(l-1)!} b_1 \cdots b_l \left(\tilde{\partial} x\right),$$

   and if we replace some a_i by an element in $A^{0,0}$, the above expression vanishes.

Proof. For two subsets $\mathcal{B}, \mathcal{C} \subseteq \{1, \ldots, l\}$ such that $\mathcal{B} \cup \mathcal{C} = \{1, \ldots, l\}$, we denote by $m_l(b_{\mathcal{B}}, c_{\mathcal{C}})$ the expression $m_l(y_1, \ldots, y_l)$ such that $y_i = b_i$ for $i \in \mathcal{B}$ and $y_i = c_i$ for $i \in \mathcal{B}$. In particular, we have $|m_l(b_{\mathcal{B}}, c_{\mathcal{C}})| = (|\mathcal{B}| + 2 - l, |\mathcal{C}|)$. It follows that $|\mathcal{B}| \geq l - 2$ and hence $|\mathcal{C}| \leq 2$. We have

$$\begin{aligned} m_l(a_1, \ldots, a_l) &= \sum_{\mathcal{B}, \mathcal{C}} m_l(b_{\mathcal{B}}, c_{\mathcal{C}}) \\ &= \sum_{|\mathcal{B}|=l-2, |\mathcal{C}|=2} m_l(b_{\mathcal{B}}, c_{\mathcal{C}}) + \sum_{|\mathcal{B}|=l-1, |\mathcal{C}|=1} m_l(b_{\mathcal{B}}, c_{\mathcal{C}}) + \sum_{|\mathcal{B}|=l, |\mathcal{C}|=0} m_l(b_{\mathcal{B}}, c_{\mathcal{C}}). \end{aligned}$$

The first summand vanishes by (2.19). We conclude

$$m_l(a_1, \ldots, a_l) = \sum_{i=1}^{l} m_l(b_1, \ldots, c_i, \ldots, b_l) + m_l(b_1, \ldots, b_l). \tag{2.22}$$

We calculate explicitly $m_l(b_1, \ldots, b_l, c_1) \in \mathrm{Tot}_N^2(A)$. We work in $\int^{[n] \in \mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ and we use the isomorphism ψ (see Appendix A.3). In particular, for $c \in \mathrm{Tot}_N^{0,1}(A)$ we have a $(0,1)$ element $\psi(b) \in \int^{[n] \in \mathbf{\Delta}} NC_n \otimes A^{\bullet,n}$. Its projection at $NC_1^0 \otimes A^{1,1} \subset NC_1^\bullet \otimes A^{1,\bullet}$ is

$$\lambda_0 \otimes \sigma_0^*(c) + \lambda_1 \otimes \sigma_1^*(c).$$

Let d^0, d^1 be the coface maps of A. By definition, σ_0 corresponds to the coface map d^1 and σ_1 corresponds to the coface map d^0. Hence we can write $NC_1^\bullet \otimes A^{1,\bullet}$ as $t \otimes d^1 c + (1-t) \otimes d^0 c$ (see (2.20)). Similarly, an element b of bidegree $(0,1)$ can be written as $-dt \otimes b \in NC_1^1 \otimes A^{1,0}$. We have

$$\begin{aligned} m_l^{1,1}(\phi(b)_1, \ldots, \phi(b)_{l-1}, \phi(c)_l)_1 &= m_{l+1}^{1,1}\left(-dt \otimes b_1, \ldots, -dt \otimes b_{l-1}, t \otimes d^1 c_l + (1-t) \otimes d^0 c_l\right) \\ &= (-1)^l m_{l+1}^{[1]}(dt, \ldots, dt, t) \otimes b_1 \cdots b_{l-1}\left(d^1 c_l - d^0 c_l\right) \\ &= (-1)^l \frac{B_{l-1}}{(l-1)!} dt \otimes (b_1 \cdots b_{l-1}) \tilde{\partial} c_l \in NC_1^1 \otimes A^{1,1} \end{aligned}$$

where $b_1 \cdots b_{l-1}$ has to be understood as a multiplication inside $A^{1,0}$. The above expression defines an element of bidegree $(1,1)$ inside $\int^{[n] \in \mathbf{\Delta}} NC_n \otimes A^{\bullet,n}$. By applying ψ^1 we get

$$m_l(b_1, \ldots, b_{l-1}, c_l) = (-1)^{l-1} \frac{B_{l-1}}{(l-1)!} b_1 \cdots b_{l-1} \tilde{\partial} c_l.$$

On the other hand, thanks to point 2 in Proposition 2.2.4, we have

$$m_l(b_1, \ldots, b_{i-1}, c_i, b_{i+1}, \ldots, b_l) = (-1)^{l-1} \binom{l-1}{i-1} \frac{B_{l-1}}{(l-1)!} b_1 \cdots b_l \tilde{\partial} c_i.$$

By (2.22), we have

$$m_l(a_1, \ldots, a_l) = \sum_{i=1}^{l} (-1)^{l-1} \binom{l-1}{i-1} \frac{B_{l-1}}{(l-1)!} b_1 \cdots \widehat{b_i} \cdots b_l \tilde{\partial} c_l + m_l(b_1, \ldots, b_l). \tag{2.23}$$

Now let $x \in A^{0,0}$. Then, the above computations give

$$\begin{aligned} m_l(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_l) &= m_l(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_l) \\ &= (-1)^{l-1}(-1)^{l-i-1} \binom{l-1}{i-1} \frac{B_{l-1}}{(l-1)!} b_1 \cdots b_l \left(\tilde{\partial} x\right) \\ &= (-1)^i \binom{l-1}{i-1} \frac{B_{l-1}}{(l-1)!} b_1 \cdots b_l \left(\tilde{\partial} x\right). \end{aligned} \tag{2.24}$$

□

The following corollary follows directly from the previous proof.

Corollary 2.2.8. *Let $n > 2$ and and let $a_1, \dots, a_n \in \mathrm{Tot}^1_N(A)$. We have*

$$m_n(a_1, \dots, a_n) = d_1 + d_2 + d_3,$$

where $d_1 \in \mathrm{Tot}^{0,2}_N(A)$, $d_2 \in \mathrm{Tot}^{1,1}_N(A)$ and $d_3\, \mathrm{Tot}^{0,2}_N(A)$. Then $d_1 = 0$, $d_3 = m_l(b_1, \dots, b_3)$.

Theorem 2.2.9. *Let $a_1, a_2 \in \mathrm{Tot}^1_N(A)$ and let $b_i \in \mathrm{Tot}^{1,0}_N(A)$, $c_i \in \mathrm{Tot}^{0,1}_N(A)$ such that $a_i = b_i + c_i$ for $i = 1, 2$. Then*

$$m_2(a_1, a_2) = m_2(c_1, c_2) + m_2(b_1, c_2) + m_2(c_1, b_2) + m_2(b_1, b_2),$$

where

1. $m_2(c_1, c_2) = c_2 c_2 \in A^{0,2}$,
2. $m_2(b_1, c_2) = -\frac{1}{2} b_1 \tilde{\partial} c_2 + b_1 d^0 c_2$,
3. $m_2(b_1, b_2) = \frac{1}{6}\left(-\left(d^0 b_1\left(d^1 b_2 + d^2 b_2\right)\right) + \left(d^1 b_1\left(d^0 b_2 - d^2 b_2\right)\right) + \left(d^2 b_1\left(d^0 b_2 + d^1 b_2\right)\right)\right).$

Let $x, y \in \mathrm{Tot}^0_N(A)$. Then, $m_2(c_1, x) = c_1 x \in \mathrm{Tot}^{0,1}_N(A)$, $m_2(x, y) = xy \in A^{0,1}$, and $m_2(b_1, x) = -\frac{1}{2} b_1 \tilde{\partial} x + b_1 d^0 x \in \mathrm{Tot}^{1,0}_N(A)$.

Proof. The second and the last terms of $m_2(a_1, a_2)$ can be calculated by a computation similar to the proof above. We have

$$\begin{aligned} m_2^{1,1}(\psi(b)_1, \psi(c)_2)_1 &= m_2^{1,1}\left(-dt \otimes b_1, t \otimes d^1 c_2 + (1-t) \otimes d^0 c_2\right) \\ &= -m_2^{[1]}(dt, t) \otimes b_1 c_2 + m_2^{[1]}(dt, 1) \otimes \left(-b_1 d^0 c_2\right) \\ &+ m_2^{[1]}(dt, t) \otimes \left(b_1 d^0 c_2\right) \\ &= dt \otimes \left(B_1 b_1 \tilde{\partial} c_2 - b_1 d^0 c_2\right), \end{aligned}$$

and thus $m_2(b_1, c_2) = -B_1 b_1 \tilde{\partial} c_2 + b_1 d^0 c_2$. It remains to add $m_2(b_1, b_2)$ where $B_1 = \frac{1}{2}$. The expression for $m_2(b_1, x)$ can be computed in the same way. Recall that NC_2 is the graded vector space generated by $\lambda_0, \lambda_1, \lambda_2$ in degree 0, $\lambda_{12}, \lambda_{02}, \lambda_{01}$ in degree 1 and λ_{012} in degree 2. In particular, for $b \in \mathrm{Tot}^{1,0}_N(A)$ we have a $(1,0)$ element $\psi(b) \in \int^{[n] \in \boldsymbol{\Delta}} NC_n \otimes A^{\bullet, n}$. Its projection at $NC_2^{\bullet} \otimes A^{2,\bullet}$ is

$$\lambda_{12} \otimes d^0(b) + \lambda_{02} \otimes d^1(b) + \lambda_{01} \otimes d^2(b).$$

Then, by Proposition 2.2.5 we have

$$\begin{aligned} &m_2^{2,2}(\psi(b_1), \psi(b_2))_2 = \\ &= m_2^{2,2}\left(\lambda_{12} \otimes d^0(b_1) + \lambda_{02} \otimes d^1(b_1) + \lambda_{01} \otimes d^2(b_1), \lambda_{12} \otimes d^0(b_2) + \lambda_{02} \otimes d^1(b_2) + \lambda_{01} \otimes d^2(b_2)\right)_2 \\ &= \frac{1}{6}\lambda_{012} \otimes \left(-\left(d^0 b_1\left(d^1 b_2 + d^2 b_2\right)\right) + \left(d^1 b_1\left(d^0 b_2 - d^2 b_2\right)\right) + \left(d^2 b_1\left(d^0 b_2 + d^1 b_2\right)\right)\right). \end{aligned}$$

Then

$$m_2(b_1, b_2) = \frac{1}{6}\left(-\left(d^0 b_1\left(d^1 b_2 + d^2 b_2\right)\right) + \left(d^1 b_1\left(d^0 b_2 - d^2 b_2\right)\right) + \left(d^2 b_1\left(d^0 b_2 + d^1 b_2\right)\right)\right).$$

□

2.3 Geometric connections

We put the C_∞-structure of Section 2.2 on the cosimplicial module of differential forms on a simplicial manifold $M_\bullet$. We use the results of section 2.1 to compute associated homological pairs on a simplicial manifold. We show that this connection induces a flat connection form on M_0 (Corollary 2.3.19). Let G be a discrete group acting properly and discontinuosly via diffeomorphisms on M. We study associated homological pair on the action groupoid $M_\bullet G$ and its relation with the "ordinary" associated homological pair on M/G and we show that they are gauge equivalent modulo an endomorphism of the fiber (see Theorem 2.4.9). We use this relation to construct a flat connection on M/G.

2.3.1 Simplicial De Rham theory

We recall some basic notions about simplicial manifolds. Our main reference is [19] and [20]. Let $\mathrm{Diff}_{\mathbb{C}}$ be the category of smooth manifolds over $\mathbb{C}$. We denote with dgA the category of complex commutative non-negatively graded differential graded algebra.

Definition 2.3.1. A *simplicial manifold* $M_\bullet$ is a simplicial object in $\mathrm{Diff}_{\mathbb{C}}$.

Each smooth complex manifold M can be viewed as a constant simplicial manifold $M_\bullet$, where $M_n := M$ and all the degenerancies and faces maps are equal to the identity.
Let $A_{DR} : \mathrm{Diff}_{\mathbb{C}} \to \mathbb{C} - dg_{\geq}Comm$ be the smooth complex De Rham functor, i.e., $A_{DR}(M)$ is the differential graded algebra of smooth complex valued differential forms on M. This is the total complex associated to a bicomplex $A_{DR}^{\bullet;\bullet}(M)$, where a differential forms w of type (p,q) is an element in $A_{DR}^{p+q}(M)$ such for any given holomorphic coordinates $z = (z_1, \dots, z_n) : \mathbb{C}^n \to M$, it can be written as

$$\sum f dz_I \wedge d\overline{z_J} = \sum f(z_1, \dots z_n) dz_{i_1} \wedge \dots \wedge dzi_p \wedge d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}j_q$$

where f is a smooth function over $\mathbb{C}^n$. We denote with $A_{DR}^{p,q}(M)$ the set of differential forms of type (p,q). The differential $d : A_{DR}^{\bullet}(M) \to A_{DR}^{\bullet+1}(M)$ is is given by $d = \partial + \overline{\partial}$. A_{DR} is contravariant and $A_{DR}(M_\bullet)$ is a cosimplicial complex commutative differential graded algebra. As explained in the previous section, $\mathrm{Tot}_{TW}(A_{DR}(M_\bullet))$ is a differential graded commutative algebra over $\mathbb{C}$. We obtain a contravariant functor

$$\mathrm{Tot}_{TW}(A_{DR}(-)) : \mathrm{Diff}_{\mathbb{C}}^{\mathbf{\Delta}^{op}} \to \mathrm{dgA}.$$

A smooth map between simplicial manifolds $f_\bullet : M_\bullet \to N_\bullet$ induces a morphism of differential graded algebras via $(f^*(w))_n := f_n^*(w_n)$. If $M_\bullet$ is a constant simplicial manifold then $\mathrm{Tot}_{TW}(A_{DR}(M_\bullet))$ is naturally isomorphic to $A_{DR}(M_\bullet)$. Given a simplicial manifold $M_\bullet$, consider the cosimplicial complex commutative differential graded algebra $A_{DR}(M_\bullet)$. By Theorem 2.2.6, we have a functor

$$\mathrm{Tot}_N(A_{DR}(-)) : \mathrm{Diff}_{\mathbb{C}}^{\mathbf{\Delta}^{op}} \to (C_\infty - \mathrm{Alg})_{\mathbb{C},str}.$$

There is a De Rham theorem for simplicial manifolds. This is very useful because it allows us to determine the cohomology of $\mathrm{Tot}_N(A_{DR}(M_\bullet))$. Let Top be the category of topological spaces. Let $T_\bullet \in \mathrm{Top}^{\mathbf{\Delta}^{op}}$ be a simplicial topological space. Let $\mathbf{\Delta}_+$ be the subcategory of $\mathbf{\Delta}$ with the same objects but only injective maps. By restriction, we have a functor $T_\bullet : \mathbf{\Delta}_+^{op} \to \mathrm{Top}$.

Definition 2.3.2. The *fat realization* of simplicial topological space $T_\bullet$ is the coend

$$||T_\bullet|| := \int^{[n]\in\mathbf{\Delta}_+} T_n \times \Delta_{geo}[n].$$

The *geometric realization* of $T_\bullet$ is the topological space

$$|T_\bullet| := \int^{[n]\in\mathbf{\Delta}} T_n \times \Delta_{geo}[n].$$

Remark 2.3.3. The natural quotient map $||T_\bullet|| \to |T_\bullet|$ is not a weak equivalence in general. However, if the simplicial topological space is "good" (see the appendix of [52]), then it is a weak equivalence. In particular for a simplicial manifold $M_\bullet$, this is true when its degeneracy s^i maps are embeddings. We call this class of simplicial manifolds *good* simplicial manifolds.

For a topological space T we denote by $C_\bullet(T)$ its singular chain complex and by $C^\bullet(T)$ its singular cochain complex. In particular $C_\bullet(-)$ extends to a functor from simplicial topological spaces to simplicial chain complexes; on the other hand $C^\bullet$ (since is contravariant) extends to a contravariant functor from simplicial topological spaces to cosimplicial cochain complexes. We associate to each simplicial topological space $T_\bullet$ the bicomplex $C^{\bullet,\bullet}(T_\bullet)$ as follows

$$C^{p,q}(T_\bullet) := C^p(T_q),$$

where the differential $\delta' : C^p(T_q) \to C^{p+1}(T_q)$ is the ordinary differential on $C^\bullet(T_q)$. The faces maps $d_i : T_{q+1} \to T_q$ induce maps

$$(d_i)^* : C^\bullet(T_q) \to C^\bullet(T_{q+1}),$$

for $i = 0, \dots, q$. We define $\delta'' : C^p(T_q) \to C^{p+1}(T_q)$ as $\delta'' := \sum_{i=0}^q (-1)^i (d_i)^*$. We denote by δ the total differential and by $(C^\bullet(T_\bullet), \delta)$ the total complex.

Proposition 2.3.4 ([19], Proposition 5.15). *Let $T_\bullet$ be a topological space, then*

$$H^\bullet(||T_\bullet||, \mathbb{R}) \cong H^\bullet\left(\mathrm{Tot}_N\left(C^\bullet_{smooth}(T_\bullet)\right), \delta\right)$$

For a smooth manifold M, we denote by $C_\bullet(M)_{\text{smooth}}$ its smooth singular chain complex, i.e., $C_p(M)_{\text{smooth}}$ is the free module generated by the smooth maps $\sigma : \Delta[p]_{geo} \to M$. The inclusions

$$C_\bullet(M)_{\text{smooth}} \hookrightarrow C_\bullet(M), \quad C^\bullet(M)_{\text{smooth}} \hookrightarrow C^\bullet(M)$$

are quasi-isomorphisms. The integration map

$$\int : A^p_{DR}(M) \to C^p_{\text{smooth}}(M)$$

defined by $\int(w)(\sigma) := \int_{\Delta_{geo}[p]} \sigma^* w$ induces a multiplicative quasi-isomorphism between $H^\bullet_{DR}(M)$ and $H^\bullet(M) := H^\bullet(C^\bullet(M))$, where $H^\bullet(M)$ is equipped with the cup product. We can generalize such a theorem in simplicial settings as follows. The map $\int$ can be extended to a map between bigraded vector spaces

$$\int : A^p_{DR}(M_q) \to C^p_{\text{smooth}}(M_q),$$

and the Stokes'theorem implies that $\int$ is a cochain map. For a double complex $(A^{\bullet,\bullet}, d, d')$, we define a new double complex $\left(\left(E_1^A\right)^{\bullet,\bullet}, d_1, d'_1\right)$, where $\left(E_1^A\right)^{p,q} := H^p\left(A^{\bullet,q}, d\right)$, $d_1 = d$ and $d'_1 = 0$.

Lemma 2.3.5 ([19], Lemma 1.19). *Let $f : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ be a morphism of double complex. Assume that $A^{p,q} = B^{p,q} = 0$ if p or q are negative and that $f : E_1^A \to E_1^B$ is an isomorphism between chain complexes. Then f induces an isomorphism in the cohomology of the total complex.*

An application of the above lemma shows that

$$\int : \mathrm{Tot}_N\left(A_{DR}\left(M_\bullet\right)\right) \to \mathrm{Tot}_N\, C^\bullet_{\text{smooth}}(M_\bullet),$$

is a quasi-isomorphism. Combining the above lemma with the results of Section 2.2 we have a de Rham theorem for simplicial manifolds.

Proposition 2.3.6. *Let $M_\bullet$ be a simplicial manifold, we have a sequence of multiplicative isomorphisms:*

$$H^\bullet\left(\mathrm{Tot}_{TW}\left(A_{DR}\left(M_\bullet\right)\right)\right) \cong H^\bullet\left(\mathrm{Tot}_N\left(A_{DR}\left(M_\bullet\right)\right)\right) \cong H^\bullet\left(||M_\bullet||, \mathbb{C}\right).$$

This is Theorem 6.10 of [19]. We introduce the complex of smooth logarithmic differential forms. Let M be a complex manifold.

Definition 2.3.7. A normal crossing divisor $D \subset M$ is given by $\cup_i D_i$, where each D_i is a non-singular divisor (a codimension 1 object) and for each $p \in D_{i_1} \cap \cdots \cap D_{i_l}$ there exist local coordinates $z = (z_1, \dots, z_n) : \mathbb{C}^n \to U \subset M$ near p such that $U \cap D_{i_1} \cap \cdots \cap D_{i_l}$ is given by the equation $\prod_{i=1}^q z_{j_i} = 0$.

A differential form w in $A^\bullet_{DR}(\log(D))$ is a smooth complex valued differential form on $M - D$ whose extension on M admits some singularites along D of degree 1. More precisely it can be viewed as a non smooth complex differential form on M such that

1. for any given holomorphic coordinates $z = (z_1, \dots, z_n) : \mathbb{C}^n \to U \subset M$ such that $U \cap D = \emptyset$, w can be written as an ordinary smooth complex valued differential forms

$$\sum f(z_1, \dots z_n) dz_{i_1} \wedge \cdots \wedge dzi_p \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}j_q$$

where f is a smooth function over U.

2. For any given holomorphic coordinates $z = (z_1, \ldots, z_n) : \mathbb{C}^n \to U \subset M$ near $p \in D$, such that $U \cap D_{i_1} \cap \cdots \cap D_{i_l}$ is given by the equation $\prod_{i=1}^{q} z_i = 0$; w can be written as

$$\sum w_J \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_l}}{z_{j_l}}$$

for $j_i \in \{1, \ldots, p\}$ and where w_J is a smooth complex valued differential forms on U.

The graded vector spce $A^{\bullet}_{DR}(\log(D))$ equipped with the differential d and wedge product $\wedge$ is a commutative differential graded algebra. Its cohomology gives the cohomology of $M - D$.

Proposition 2.3.8 ([16]). *The inclusion $M - D \hookrightarrow M$ induces a map*

$$A^{\bullet}_{DR}(\log(D)) \to A^{\bullet}_{DR}(M - D)$$

which is a quasi-isomorphism of differential graded algebras.

Definition 2.3.9. Let Diff_{Div} be the category of complex manifolds equipped with a normal crossing divisors, i.e the objects are pairs $(M, \mathcal{D})$ where $\mathcal{D}$ is a normal crossing divisor of M and the maps are holomorphic maps $f : (M, \mathcal{D}) \to (N, \mathcal{D}')$ such that

$$f^{-1}(\mathcal{D}') \subset \mathcal{D}.$$

A *simplicial complex manifold $M_\bullet$ with a simplicial normal crossing divisor $\mathcal{D}_\bullet$* is a simplicial objects in Diff_{Div}.

So given a simplicial complex manifold with divisor $(M_\bullet, \mathcal{D}_\bullet)$, then $(M - D)_\bullet$ defines by $(M - \mathcal{D})_n = M_n - \mathcal{D}_n$ is again a simplicial complex manifolds. Let $(M_\bullet, \mathcal{D}_\bullet)$ be a simplicial complex manifold with simplicial normal crossing divisor. Then $A_{DR}((M - \mathcal{D})_\bullet)$ and $A_{DR}(\log(\mathcal{D}_\bullet))$ are cosimplicial commutative dg algebras.

Proposition 2.3.10. *The inclusion $(M - \mathcal{D})_\bullet \hookrightarrow M_\bullet$ induces a map between chain complexes*

$$\mathrm{Tot}_N(A_{DR}(\log(\mathcal{D}_\bullet))) \to \mathrm{Tot}_N(A_{DR}((M - \mathcal{D})_\bullet))$$

which is a quasi-isomorphism.

Proof. This is a direct consequence of Proposition (2.3.8) and Lemma (2.3.5). □

We conclude this section with a standard example of simplicial manifold. Let G be a Lie group, and let M be a manifold equipped with a left smooth (or holomorphic) G action. We define the simplicial manifold $M_\bullet G$ as follows:

$$M_n G = M \times G^n,$$

The face maps $d^i : M_n G \to M_{n-1} G$ for $i = 0, 1, \ldots n$ are

$$d^i(x, g_1, \ldots, g_n) := \begin{cases} (g_1 x, g_2, \ldots, g_n), & \text{if } i = 0, \\ (x, g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & \text{if } 1 < i < n \\ (x, g_1, \ldots, g_{n-1}), & \text{if } i = n. \end{cases}$$

The degenerancy maps $s^i : M_n G \to M_{n+1} G$ are defined via

$$s^i(x, g_1, \ldots, g_n) := (x, g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n)$$

for $i = 1, \ldots, n$.

Definition 2.3.11. We call the simplicial manifold $M_\bullet G$ the action groupoid.

In particular the geometric realization of $M_\bullet G$ is weakly equivalent to the Borel construction

$$EG \times_G M$$

and if the action of G is free, the projection $\pi : EG \times_G M \to M/G$ is a homotopy equivalence.

2.3.2 Geometric connections

Let $M_\bullet$ be a real or complex simplicial manifold with connected cohomology. Then $A_{DR}(M_\bullet)$ is a cosimplicial unital commutative differential graded algebra. By Theorem 2.2.6 $\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right)$ is an unital C_∞-algebras. We denote its structure with $m_\bullet$ where $m_1 = D$. Here D is the differential on $\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right)$ defined on elements of bidegree (p,q) by

$$D(a) = \tilde{\partial}a + (-1)^p da,$$

where $\tilde{\partial}$ is differential obtained by the alternating sum of the pullback of the cofaces maps of the simplicial manifold. We denote the unit by 1. It corresponds to the constant function at 1 inside $A^0_{DR}(M_0)$.

Definition 2.3.12. Let $A \subset \left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right)$ be a C_∞-subalgebra and let W be a positively graded vector space of finite type. A *reduced homological pair* (C, δ^*) consists of a codifferential δ of $T^c\left(W_+[1]\right)$ and a $C \in A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*$ such that C is a Maurer-Cartan element in the reduced convolution L_∞-algebra $\left(A\widehat{\otimes}\widehat{T}\left(W_+[1]\right)^*, l'_\bullet\right)$ and δ^* preserves the filtration $\mathcal{I}$.

Assume that the cohomology of $\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right)$ is of finite type. For any Hodge type decomposition $(W, \mathcal{M})$, let

$$w_1, \ldots, w_i, \ldots \tag{2.25}$$

be a basis of W_+. Let $X_1, \ldots X_n, \ldots$ be the basis of $(W[1])^*$ dual to $s(w_1), \ldots s(w_n), \dots \in W_+[1]$. We write Theorem 2.1.40 in terms of formal power series.

Corollary 2.3.13. *Assume that the cohomology of* $\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right)$ *is of finite type. Let* $B \subseteq \mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right)$ *be a 1-model. For any Hodge type decomposition* $(W, \mathcal{M})$ *of* $(B, m_\bullet)$ *the associated homological pair* $(\left(\delta^+_W\right)^*, C)$ *consists in a codifferential* δ^+_W *so that* $pr_{W_+}\left(\delta^+_W\right)|_{W_+[1]} = 0$ *and a Maurer-Cartan element*

$$C \in \mathrm{Conv}_r\left((B, m_\bullet), W, \mathcal{M}\right)$$

that can be written uniquely as a formal power series

$$C = \sum w_i X_i + \sum w_{ij} X_i X_j + \cdots + \sum w_{i_1 \ldots i_r} X_{i_1} \ldots X_{i_r} + \cdots \in B\widehat{\otimes}\widehat{\mathbb{L}}((W_+[1])^*).$$

such that the coefficients $w_{ij}, \ldots, w_{i_1 \ldots i_p}, \ldots$ *belong to* $\mathcal{M}$ *for* $p > 1$. *In particular* $(\left(\delta^+_W\right)^*, C)$ *is a reduced homological pair.*

Let $M_\bullet$ and $(W, \mathcal{M})$ be as above. Consider the strict L_∞-algebra morphism

$$\pi \,:\, \mathrm{Conv}_r\left((B, m_\bullet), W, \mathcal{M}\right) \to \mathrm{Conv}_{r,0}\left((B, m_\bullet), W, \mathcal{M}\right)$$

defined in Proposition 2.1.22.

Definition 2.3.14. We call $C_0 := \pi(C)$ the *degree zero geometric connection of* $M_\bullet$ *associated to the decomposition* $(W, \mathcal{M})$ of B.

In order to find a degree zero geometric connection of $M_\bullet$ for a certain model B, we may start with a simpler 1-model $B' \subset B$ equipped with a Hodge decomposition compatible with the one given in B. Then Proposition 2.1.48 ensures that we have the same degree zero geometric connection. Let $M_\bullet$ be as above. For $j = 1, 2$, let B_j be C_∞-subalgebras of $\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right)$ equipped with a Hodge decomposition

$$B_j = W_j \oplus \mathcal{M}_j \oplus d\mathcal{M}_j.$$

We assume that B_1 is a model and that B_2 has a connected and finite type cohomology. We denote by $\left(C^j, \delta^+_{W_j}\right)$ the homological pairs obtained and by C^0_j the degree zero geometric connection. Hence Proposition 2.1.50 implies the following.

Corollary 2.3.15. *Let* $k_\bullet$ *be as defined in Section 2.1.6. The pushforward along* $k_\bullet$ *inherits a strict morphism of* L_∞*-algebras*

$$k^* \,:\, \mathrm{Conv}_{r,0}\left((B_1, m_\bullet), W_1, \mathcal{M}_1\right) \to \mathrm{Conv}_{r,0}\left((B_2, m_\bullet), W_2, \mathcal{M}_2\right)$$

such that $k^*\left(C^1_0\right)$ *is homotopic to* C^2_0.

Here $k^*\left(C_0^1\right)$ is the precomposition of C_0^1 with the the dual of the differential graded coalgebras map

$$K \,:\, T^c\left((W_2)_+[1]\right) \to T^c\left((W_1)_+[1]\right) \tag{2.26}$$

induced by $k_\bullet$. Its dual gives

$$K^* \,:\, \widehat{T}\left(\left(W_2^1\right)_+[1]\right)^*/\mathcal{R}_0^2 \to \widehat{T}\left(\left(W_1^1\right)_+[1]\right)^*/\mathcal{R}_0^1. \tag{2.27}$$

In particular, K^* is (by construction) a morphism of complete Hopf algebras. In some cases, the map above depends only by W_1 and W_2. For $j = 1, 2$, we denote by $i^j \,:\, W_j \to \mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right)$ the inclusion and by $p^j \,:\, \mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right) \to W_j$ the projection associated to the chosen decomposition. By construction,

$$[i^1][p^2] \,:\, H^1(W_1, 0) \to H^1(W_2, 0)$$

is an isomorphism.

Corollary 2.3.16. *Let $M_\bullet$ B_1, B_2 be as above. Then $k_\bullet = [i^1][p^2]$ is strict and*

$$K^* = \sum_i^\infty K_i^* \,:\, \widehat{T}\left(\left(W_2^1\right)_+[1]\right)^*/\mathcal{R}_0^2 \to \widehat{T}\left(\left(W_1^1\right)_+[1]\right)^*/\mathcal{R}_0^1$$

where K_1^* is the map induced by (the dual of) $[i^1][p^2]$.

2.3.3 Restriction to ordinary flat connections on M_0

Consider $M_\bullet$, B and $(W, \mathcal{M})$ as in corollary 2.3.13. Consider the projection

$$r \,:\, B \to B^{0,1} \subset A_{DR}(M_0)$$

that sends all the forms of type (p, q) to 0 if $p \neq 0$ and preserves the forms of type $(0, q)$.
Consider $A_{DR}(M_0)$ equipped with its differential graded algebra structure. Then

$$\mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}(M_0)\right)\right)$$

is a differential graded Lie algebra. In particular, it is a ordinary convolution Lie algebra (compare with [39]).

Proposition 2.3.17. *Let $M_\bullet$, B and $(W, \mathcal{M})$ as above. The pushforward along r induces two maps*

$$\underline{r}_* \,:\, \mathrm{Conv}_r\left((B, m_\bullet), W, \mathcal{M}\right) \to \mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}(M_0)\right)\right)$$

and

$$r_* \,:\, \mathrm{Conv}_{r,0}\left((B, m_\bullet), W, \mathcal{M}\right) \to \mathrm{Conv}_{r,0}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}(M_0)\right)\right)$$

such that

$$\begin{array}{ccc}
\mathrm{Conv}_r\left(\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right), W, \mathcal{M}\right) & \xrightarrow{\underline{r}_*} & \mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}(M_0)\right)\right) \\
\big\downarrow \pi & & \big\downarrow \pi \\
\mathrm{Conv}_{r,0}\left(\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right), W, \mathcal{M}\right) & \xrightarrow{r_*} & \mathrm{Conv}_{r,0}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}(M_0)\right)\right))
\end{array}$$

commute as diagram of differential graded vector space and $r_* \in (L_\infty - ALG)_p$.

Proof. Let $l'_\bullet$ be the L_∞ structure on $\mathrm{Conv}_r\left(\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right), W, \mathcal{M}\right)$. Since

$$\underline{r}_* l'_1 = \underline{r}_*\left(\delta_{W_+} \pm D\right) = \left(\delta_{W_+} \pm D\right)\underline{r}_*,$$

we get that $\underline{r}_*$ is a cochain map and induces a well-defined cochain map r_* between the degree zero convolution L_∞-algebras. It remains to prove that r_* preserves Maurer-Cartan elements. By Corollary 2.2.8, we have

$$rm_n(a_1,\dots,a_n) = m_n(ra_1,\dots,ra_n)$$

for $n \geq 1$ and $a_1,\dots,a_n \in B^1$. Let

$$C \in \mathrm{Conv}_{r,0}\left(\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet)\right), m_\bullet\right), W, \mathcal{M}\right)$$

be a Maurer-Cartan element, let

$$C' \in \left(A^1_{DR}(M_0)\right)\widehat{\otimes}\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right), \quad \text{resp. } \tilde{C} \in \left(A^0_{DR}(M_1)\right)\widehat{\otimes}\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)$$

be such that

$$C = C' + \tilde{C}.$$

In particular, $r_*C = C'$ and

$$\begin{aligned} 0 &= r_*\left(\partial C + \sum_{k>1}\frac{l_k\left(C,\dots,C\right)}{k!}\right) \\ &= \left(\partial r_*C + \sum_{k>1}\frac{l_k\left(r_*C,\dots,r_*C\right)}{k!}\right) \\ &= \left(\partial r_*C + \frac{l_2\left(r_*C, r_*C\right)}{2}\right), \end{aligned}$$

i.e., r_*C is a Maurer-Cartan element in the convolution Lie algebra $\mathrm{Conv}_{r,0}\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}\left(M_0\right)\right)\right)$. ☐

Remark 2.3.18. Let $M_\bullet = M_\bullet G$ for some smooth manifold M and discrete group G. The morphism of simplicial manifold $M_\bullet\{e\} \to M_\bullet G$ given by the inclusion gives gives the map

$$r : \mathrm{Tot}_N\left(A_{DR}\left(M_\bullet G\right)\right) \to A_{DR}\left(M\right) \tag{2.28}$$

which is a strict morphism of C_∞-algebras.

Corollary 2.3.19. *Let $\mathcal{R}_0 \subset \widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)$ be the Lie ideal generated by $\delta^*\left(\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)\right)$. Then*

1. $\mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}\left(M_0\right)\right)\right) = \left(l_\bullet, A_{DR}(M_0)\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0\right)\right)$.
   In particular,
   $$l_1 = -d, \quad l_2 = [-,-], \quad l_n = 0, \text{ for } n > 2,$$
   where $[-,-]$ is the obvious Lie bracket on $A^\bullet_{DR}(M_0)\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0\right)$. In particular
   $$A^\bullet_{DR}(M_0)\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0\right), \widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0$$
   are complete Lie algebras and $\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0$ is pronilpotent.
2. *r_*C_0 is a Maurer-Cartan element in* $\mathrm{Conv}_r\left(\left(W_+, m_\bullet^{W_+}\right), \left(A_{DR}\left(M_0\right)\right)\right)$.

Proof. The lie algebras $A^\bullet_{DR}(M_0)\widehat{\otimes}\left(\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0\right)$, $\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0$ are complete because δ_r^* preserves the filtration given by the power of the augmented ideal I in $\mathbb{L}\left(\left(W^1_+[1]\right)^*\right)$, hence $\widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right)/\mathcal{R}_0$ can be written as a projective limit of finite dimensional nilpotent Lie algebras. ☐

For a Lie algebra $\mathfrak{u}$ we define the action ad $: \mathfrak{u} \to \mathrm{End}(\mathfrak{u})$ via $\mathrm{ad}_v(w) := [v, w]$. By Section 2.1.2, the $\left(\widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)/\mathcal{R}_0\right)$ is a Lie algebra. We call such a Lie algebra the *fiber Lie algebra of the simplicial manifold* $M_\bullet$, this Lie algebra corresponds in some cases to the Malcev Lie algebra of the geometric realization of $M_\bullet$.

Proposition 2.3.20. *Let $M_\bullet$ $N_\bullet$ be simplicial manifolds with finite type connected cohomology. Assume that there is a map $f_\bullet : N_\bullet \to M_\bullet$ that induces a quasi-isomorphism in cohomology.*

1. *The fiber Lie algebra of $M_\bullet$ is isomorphic to the fiber Lie algebra of $N_\bullet$.*
2. *Let G be a discrete group acting properly and discontinuously on a smooth manifold M. In this case the quotient M/G is again a smooth manifold. Consider the action groupoid $M_\bullet G$. Assume that the cohomology of M/G is of finite type. The fiber Lie algebra of $M_\bullet G$ is the Malcev Lie algebra of $\pi_1(M/G)$ (see Definition 1.1.37).*

In particular the fiber Lie algebra is independent (up to isomorphism) by the choice of Hodge type decomposition $(W, \mathcal{W})$).

Proof. We prove 1. We fix a Hodge type decomposition $(W, \mathcal{M})$ of $\mathrm{Tot}_N(A_{DR}(M_\bullet))$ and a Hodge decomposition $(W', \mathcal{M}')$ of $\mathrm{Tot}_N(A_{DR}(N_\bullet))$. In particular the inclusion

$$f^* : \mathrm{Tot}_N(A_{DR}(N_\bullet)) \to \mathrm{Tot}_N(A_{DR}(M_\bullet))$$

is a quasi-isomorphism and a strict C_∞-map. We get a diagram

$$\begin{array}{ccc} \mathrm{Tot}_N(A_{DR}(M_\bullet)) & \xrightarrow{f^*} & \mathrm{Tot}_N(A_{DR}(N_\bullet)) \\ f_\bullet \Big\downarrow\Big\uparrow g_\bullet & & f'_\bullet \Big\downarrow\Big\uparrow g'_\bullet \\ (W, m_\bullet^W) & & (W', m_\bullet^{W'}). \end{array}$$

of C_∞ -quasi isomorphism. In particular the map $(g'if)_\bullet$ is an isomorphism. It follows that the two fiber Lie algebras are isomorphic. We prove 2. We set $N_\bullet = M_\bullet G$ and $M_\bullet$ is the constant simplicial manifold M/G. We replace f by the inclusion

$$i : A_{DR}(M/G) \to \mathrm{Tot}_N(A_{DR}(M_\bullet G)).$$

In particular the fiber Lie algebra of $M_\bullet G$ is the fiber Lie algebra of M/G. We use Theorem 1.1.36, the the fiber Lie algebra is (the completion of) the Malcev Lie algebra of $\pi_1(M/G)$. ☐

The next theorem follows from point 2. of the previous corollary.

Theorem 2.3.21. *Consider the adjoint action* ad *of* $\widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)/\mathcal{R}_0$ *on itself. Then $d - r_*C_0$ is a flat connection on the trivial bundle over M_0 with fiber* $\widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)/\mathcal{R}_0$.

Remark 2.3.22. For the construction of (C, δ^*) we have to choose a basis of W (see (2.25)). However (C, δ^*) is independent by the choice of that basis. Conversely r_*C_0 depends by this choice. This is because in the first case we consider C as a map (a twisting cochain) and in the second case r_*C_0 is a formal power series. The dependence of the degree zero connection from the basis of W is given as follows: let $\{w_i\}_{i\in I}$ and $\{\overline{w}_i\}_{i\in I}$ be two basis of W and let $d - r_*C_0$ and $d - \overline{r_*C_0}$ be the two resulting degree zero geometric connections with fiber Lie algebra $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ respectively. There is an aoutomorphism of W that sends $\{w_i\}_{i\in I}$ to $\{\overline{w}_i\}_{i\in I}$ which induces a Lie algebra automorphism $\theta : \overline{\mathfrak{u}} \mapsto \mathfrak{u}$. The image of $\overline{r_*C_0}$ via θ corresponds to r_*C_0.

2.3.4 1-Extensions

Let G be a discrete group acting properly and discontinuously on a smooth manifold M. Consider the action groupoid $M_\bullet G$ and assume that the cohomology of M/G is of finite type. Recall the map r defined at Remark 2.3.18. By abuse of notation, we denote again with $m_\bullet$ the C_∞-structure on

$\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)$. Let $J\subset\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1),m_\bullet\right)$ be a C_∞-ideal, i.e for any $k>1$ we have

$$m_k\left(b_1,\ldots,b_k\right)\in J$$

if some $b_i\in J$. Then $\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\otimes\Omega(1)\right)/J,m_\bullet\right)$ is a C_∞-algebra as well and the projection

$$\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\otimes\Omega(1)\right)\to\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\otimes\Omega(1)\right)/J$$

is a strict C_∞-morphism. Assume that $(r\otimes Id)(J)=0$, then we have a well-defined strict C_∞-map

$$r\otimes Id\,:\,\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)/J\to A_{DR}(M)\otimes\Omega(1).$$

Let $Id\otimes i_1\,:\,\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)\to\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)$. We assume that $(Id\otimes i_1)\,J=0$.

Definition 2.3.23. Let J be a C_∞-ideal as above such that $(r\otimes Id)(J)=0=(Id\otimes i_1)\,J$. Let A be a 1-model for $\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)$. A C_∞-algebra $\left(B,m_\bullet^B\right)$ is *a 1-extension for* A if there exist strict C_∞-morphisms $f\,:\,B\to\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)/J$ and $g\,:\,B\to A$ such that

1. g induces an isomorphism in H^i for $i=0,1$ and is injective for $i=2$,
2. the diagram

$$\begin{array}{ccccc} B & \xrightarrow{\;g\;} & A & \xleftarrow{\;Id\otimes i_1\;} & A_{DR}(M)\otimes\Omega(1) \\ & {\scriptstyle f}\searrow & \uparrow{\scriptstyle Id\otimes i_1} & \nearrow{\scriptstyle r\otimes Id} & \\ & & \left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)\right)/J. & & \end{array}$$

is commutative. A *compatible Hodge type decomposition for* B is a Hodge type decomposition $(W,\mathcal{M})$ such that $(g(W),g(\mathcal{M}))$ is a Hodge type decomposition for A.

Let $\left(B,m_\bullet^B\right)$ be a 1-extension for A, let $(W,\mathcal{M})$ be a compatible Hodge type decomposition for B. Then we have a strict morphism of L_∞-algebras

$$\mathrm{Conv}_{r,0}\left(\left(B,m_\bullet^B\right),W,\mathcal{M}\right)\xrightarrow{\;f_*\;}\mathrm{Conv}_{r,0}\left(\left(W_+,m_\bullet^{W_+}\right),\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)/J,m_\bullet\right)\right)$$

Let ${C'}_0\in\mathrm{Conv}_{r,0}\left(\left(B,m_\bullet^B\right),W,\mathcal{M}\right)$ be the canonical Maurer-Cartan element associated to $(W,\mathcal{M})$.

Definition 2.3.24. We call $C_0:=f_*{C'}_0$ the *degree zero geometric connection associated to the Hodge type decomposition* $(W,\mathcal{M})$.

Since the map

$$(r\otimes Id)_*:\mathrm{Conv}_{r,0}\left(\left(W_+,m_\bullet^{W_+}\right),\left(\mathrm{Tot}_N\left(A_{DR}(M_\bullet G)\right)\otimes\Omega(1)/J,m_\bullet\right)\right)\to\mathrm{Conv}_{r,0}\left(\left(W_+,m_\bullet^{W_+}\right),A_{DR}(M)\otimes\Omega(1)\right)$$

preserves Maurer-Cartan elements, the commutativity of the above diagram implies that

$$(Id\otimes i_1)_*\,(r\otimes Id)_*\,f_*{C'}_0=(Id\otimes i_1)_*\,f_*{C'}_0=g_*{C'}_0$$

where $g_*{C'}_0$ is the canonical Maurer-Cartan element corresponding to $(g(W),g(\mathcal{M}))$. In fact $(r\otimes Id)_*\,{C'}_0$ defines a homotopy between Maurer-Cartan elements.

2.4 Connections and bundles

Let M be a complex smooth manifold. Let $M_\bullet G$ be an action groupoid where G is discrete and it acts properly and discontinuosly on M. Assume that the chomology of M/G is connected and of finite type. The results of the previous section allow the construction of a flat connection r^*C_0 on M where the fiber is the Malcev completion of $\pi_1\left(M/G\right)$. In the next section, we show that r^*C_0 induces a flat connection on M/G as well.

2.4.1 Gauge equivalences

Let $\mathfrak{u}$ be a pronilpotent Lie algebra, i.e., the projective limit of finite dimensional Lie algebras

$$\mathfrak{u} \cong \varprojlim_i \left(\mathfrak{u}/I^i\right),$$

where the filtration $I^\bullet$ is defined via

$$I^0 = \mathfrak{u}, \quad I^i := \left[I^{i-1}, \mathfrak{u}\right] \text{ for } i \geq 1.$$

Lemma 2.4.1. *Let $\mathfrak{u}$ be a pronilpotent graded Lie algebra concentrated in degree 0 and let $(A, d, \wedge)$ be a non-negatively graded differential graded algebra.*

1. *The vector space $A\widehat{\otimes}\mathfrak{u}$ is a differential graded Lie algebra where the differential is given by the tensor product and the brackets are defined via*

$$[a \otimes v, b \otimes w] := \pm (a \wedge b) \otimes [v, w],$$

   where the signs follow from the signs rule.

2. *The Lie algebra $A^0\widehat{\otimes}\mathfrak{u}$ is complete with respect to the filtration induced by I.*

3. *Let $\Omega(1)$ be the differential graded algebra of polynomials forms on the interval $[0, 1]$. Consider the differential graded Lie algebra $\Omega(1)\widehat{\otimes}\left(A\widehat{\otimes}\mathfrak{u}\right)$ obtained as in point 1. Then, there is a canonical isomorphism*

$$\Omega(1)\widehat{\otimes}\left(A\widehat{\otimes}\mathfrak{u}\right) \cong (\Omega(1) \otimes A)\,\widehat{\otimes}\mathfrak{u}.$$

Proof. We refer to the Appendix A.2 for general facts about filtered vector spaces. The first statement is straightforward. The second follows from

$$I^i\left(A\widehat{\otimes}\mathfrak{u}\right) \cong A\widehat{\otimes}I^i(\mathfrak{u})$$

for every $i \geq 0$. Analogously, point 3 follows from

$$I^i\left((\Omega(1) \otimes A)\,\widehat{\otimes}\mathfrak{u}\right) \cong (\Omega(1) \otimes A)\,\widehat{\otimes}I^i(\mathfrak{u}).$$

□

Let $\mathfrak{h}$ be a pronilpotent differential graded Lie algebra. The pronilpotency guarantees that the (complete) universal enveloping algebra $\mathbb{U}\left(\mathfrak{h}^0\right)$ is a complete Hopf algebra and its group-like elements can be visualized as $H := \exp\left(\mathfrak{h}^0\right)$. Moreover there is group action of H on $MC(\mathfrak{h})$ given by

$$e^h(\alpha) := e^{\mathrm{Ad}_h}(\alpha) + \frac{1 - e^{\mathrm{Ad}_h}}{\mathrm{Ad}_h}(du)$$

where $h \in \mathfrak{h}^0$. Assume that $\mathfrak{h}$ is concentrated in degree zero and that it is equipped with the trivial differential. Let A be a positively graded differential graded commutative algebra. We consider the complete differential graded algebra $A\widehat{\otimes}\mathfrak{h}$. The universal enveloping algebra of $\mathfrak{h}$ is equipped with the filtration induced by $I^\bullet$. In particular H is complete with respect to such a filtration. The map exp can be extended to a map

$$Id\widehat{\otimes}\exp\,:\, A^0\widehat{\otimes}\mathfrak{h} \to A^0\widehat{\otimes}H \subset A^0\widehat{\otimes}\mathbb{U}\left(\mathfrak{h}^0\right)$$

By abuse of notation we denote the above map again by exp. The completed tensor product gives to $A^0\widehat{\otimes}\mathbb{U}\left(\mathfrak{h}^0\right)$ the structure of an associative algebra. The image of $Id\widehat{\otimes}\exp$ is again a group where the inverse of e^u is given by e^{-u} for a $u \in A^0\widehat{\otimes}\mathfrak{h}$. This group acts on the set of Maurer-Cartan elements $MC\left(A^0\widehat{\otimes}H\right)$ as above, i.e.

$$e^u(\alpha) := e^{\mathrm{Ad}_u}(\alpha) + \frac{1 - e^{\mathrm{Ad}_u}}{\mathrm{Ad}_u}(du).$$

We call this action the *gauge-action* and the above group the *gauge group.*

Definition 2.4.2. Two Maurer-Cartan elements α_0, α_1 are said to be *gauge equivalent* in $A\widehat{\otimes}\mathfrak{h}$ if there is a u in $A^0\widehat{\otimes}\mathfrak{h}$ such that $e^u(\alpha_0) = \alpha_1$.

There is another equivalence relation between Maurer-Cartan elements.

Definition 2.4.3. A *homotopy* between two Maurer-Cartan elements $\alpha_0, \alpha_1 \in MC(\mathfrak{h})$ is a Maurer-Cartan element $\alpha(t) \in MC\left((\Omega(1) \otimes A)\widehat{\otimes}\mathfrak{h}\right)$ such that $\alpha(0) = \alpha_0$ and $\alpha(1) = \alpha_1$. Two Maurer-Cartan elements are said to be *homotopy equivalent* if they are connected by a finite sequence of homotopies.

Proposition 2.4.4. *Let $A\widehat{\otimes}\mathfrak{h}$ be as above. Two Maurer-Cartan elements are gauge equivalent if and only if they are homotopy equivalent.*

Proof. We have that $A\widehat{\otimes}\mathfrak{h}$ is a complete Lie algebra with respect to $I^\bullet$. The result follows from [56, Section 2.3]. ☐

Remark 2.4.5. The gauge action has a geometrical interpretation. Let V be a finite dimensional vector space concentrated in degree zero and let $\widehat{T}(V)$ be the complete free algebra with respect to the filtration given by the power of its augmentation ideal I. Let M be a smooth manifold. Then $A_{DR}(M)\widehat{\otimes}\widehat{T}(V)$ is a differential graded algebra where the product is given as in (1.3) and the differential is the one induced by the tensor product (we tacitly assume that $\widehat{T}(V)$ is equipped with the zero differential). We consider two cases.

1. Let $\widehat{\mathbb{L}}(V)$ be the complete free Lie algebra on V, (note that it is pronilpotent). Then

$$\left(A_{DR}(M)\widehat{\otimes}\widehat{\mathbb{L}}(V), -d, [-,-]\right)$$

   is a complete differential graded Lie algebra. We consider $\widehat{\mathbb{L}}(V)$ equipped with the adjoint action. A Maurer-Cartan elements α gives a flat connection

$$d - \alpha$$

   on the trivial bundle E on M with fiber $\widehat{\mathbb{L}}(V)$ (equipped with the adjoint action). Let $h \in A_{DR}(M)\widehat{\otimes}\widehat{\mathbb{L}}(V)$ be a degree zero element, then e^h can be considered as an element of $A_{DR}(M)\widehat{\otimes}\widehat{T}(V)$ and it defines a bundle map. We have the following

$$\begin{aligned} e^h\,(d-\alpha)\,e^{-u} &= d + e^u\left(de^{-u}\right) + e^u\,(-\alpha)\,e^{-h} \\ &= d + e^h\left(de^{-u}\right) - e^{\mathrm{Ad}_u}\,(\alpha) \\ &= d + \frac{1-e^{\mathrm{Ad}_u}}{\mathrm{Ad}_u}(du) - e^{\mathrm{Ad}_u}\,(\alpha)\,, \end{aligned}$$

   where the second equality is a standard result about representation of Lie algebras, and the third equality is proved for examples in [47] (see Section 3.4).

2. The same reasoning can be repeated replacing $\widehat{\mathbb{L}}(V)$ with a quotient $\mathfrak{h} = \widehat{\mathbb{L}}(V)/\mathcal{R}$, where $\mathcal{R}$ is a complete Lie ideal of $\widehat{\mathbb{L}}(V)$.

2.4.2 Factors of automorphy

We show how to construct a flat connection on the quotient. We use the same point of view of [29]. Let X be a set and G a group acting on it from the left. Let V be a module. A *factor of automorphy* is a map $F : G \times X \to \mathrm{Aut}(V)$ such that $g : X \times V \to X \times V$ defined by $(x, v) \mapsto (gx, F_g(x)v)$ gives a group action of G on $X \times V$. This is equivalent to

$$F_{gh}(x)(v) = F_g(hx)F_h(x).$$

Let M be a smooth manifold equipped with a smooth properly discontinuous action of a discrete group G (from the left). Let V be a vector space and let $F : G \times M \to \mathrm{Aut}(V)$ be a factor of automorphy. Then F induces a G-action on $M \times V$. In particular, the quotient

$$(M \times V)\,/G$$

is a vector bundle where the sections satisfy

$$s(gx) = F_g(x)s(x).$$

We denote by E_F the vector bundle induced by the factor of automorphy F.

Proposition 2.4.6. *Let M be a smooth manifold equipped with a smooth properly discontinuous action of a group G. Let V be a complete vector space and let $F : G \times M \to \mathrm{Aut}(V)$ be a factor of automorphy. Let $\alpha \in A^1_{DR}(M)\widehat{\otimes}\,\mathrm{End}(V)$ be a 1-form with vaules in $\mathrm{End}(V)$.*

1. *The connection $d + \alpha$ is a well-defined connection on E_F if*

$$d - g^*\alpha = F_g\,(d - \alpha)\,F_g^{-1}$$

for any $g \in G$.

2. *Let $T(\gamma)$ be the parallel transport along a smooth path $\gamma : [0, 1] \to M$ with respect to α. Assume that α is well-defined on E_F, Then*

$$T(g\gamma) = F_g(\gamma(0))T(\gamma)F_g(\gamma(1))^{-1}.$$

Proof. The first statement is proved in [29], Proposition 5.1. The second statement is Proposition 5.7. □

Let M, G be as above. We fix a pronilpotent Lie algebra $\mathfrak{u}$ concentrated in degree zero as in Remark 2.4.5 and we set $U = \exp(\mathfrak{u})$. Let $\alpha \in A^1_{DR}(M)\widehat{\otimes}\,\mathrm{End}(\mathfrak{u})$ be a well-defined connection form on E_F, where E_F is a bundle over M with respect to some factor of automorphy $F : G \times M \to \mathrm{End}\,(\mathfrak{u})$. We fix a $p \in M$ and we denote its class in M/G by $\overline{p}$. By covering space theory, this choice induces a group homomorphism

$$\rho : \pi_1\,(M/G, \overline{p}) \to G,$$

which is constructed via the homotopy lifting property. For a path starting at $\overline{p}$, we denote its (unique) lift starting at p with c_γ. In particular, $c_\gamma(1) = \rho(\gamma)p$. Given two loops γ_1, γ_2 starting at $\overline{p}$ on M/G, we have $c_{\gamma_1\cdot\gamma_2} = c_{\gamma_1} \cdot (\rho(\gamma_1)\gamma_2)$ (see [29], lemma 5.8).

Proposition 2.4.7. *Let $U = \exp(\mathfrak{u})$ and let p be as above. The holonomy of α on M induces a group homomorphism*

$$\Theta_0 : \pi_1\,(M/G, \overline{p}) \to U$$

given by $\Theta_0\,(\gamma) := T(c_\gamma)F_{\rho(\gamma)}(p)$, where T is the parallel transport with respect to α on M.

Proof. See Proposition 5.9 in [29]. □

We call the above group homomorphism the *holonomy representation* of α on M/G.

Proposition 2.4.8. *Let M, G be as above. We fix a pronilpotent Lie algebra $\mathfrak{u}$ concentrated in degree zero as in Remark 2.4.5. Let α' be a Maurer-Cartan elements in $A_{DR}(M)\widehat{\otimes}\mathfrak{u}$ such that it defines a well-defined flat connection on the bundle $E_{F'}$ with fiber $\mathfrak{u}$ and factor of automorphy*

$$F'_g(p) = Id.$$

Let α be a Maurer-Cartan element in $A_{DR}(M)\widehat{\otimes}\mathfrak{u}$. Assume that it is gauge equivalent to α via some $h \in A^0_{DR}(M)\widehat{\otimes}\mathfrak{u}$. Then α is a well-defined connection form on the bundle E_F, where F is given by

$$F_g(p) := e^{-h(gp)}e^{h(p)}.$$

Proof. $F_g(p)$ is clearly a factor of automorphy. We have

$$\begin{aligned} e^{-g^*h}e^{h}\,(d+\alpha)\,e^{-h}e^{g^*h} &= e^{-g^*h}\,(d+\alpha')\,e^{g^*h} \\ &= e^{-g^*h}\,(d+g^*\alpha')\,e^{g^*h} \\ &= d + g^*\alpha. \end{aligned}$$

□

Notice that there is canonical bundle map $E_{F'} \to E_F$ defined by

$$(p, v) \mapsto (p, e^{\mathrm{Ad}_{-h(gp)}} e^{\mathrm{Ad}_{h(p)}} v).$$

Consider the action groupoid $M_\bullet G$ and assume that the cohomology of M/G is connected and of finite type. Let $A, B \subset (\mathrm{Tot}_N^\bullet (A_{DR}(M_\bullet G)), m_\bullet)$ be C_∞-sub algebras. We assume that B is a 1-model and that the cohomology of A is connected and of finite type. We fix a Hodge type decomposition on A and B via

$$A = W \oplus \mathcal{M} \oplus D\mathcal{M}, \quad B = W' \oplus \mathcal{M}' \oplus D\mathcal{M}',$$

respectively. Let (C, δ^*) be a reduced homological pair with respect to A and let (C', δ^*) be the associated homological pair with respect to the decomposition given in B. By the results of the previous section, (C, δ^*) induces a flat connection form $r^*C_0 \in A^1_{DR}(M)\widehat{\otimes}\mathfrak{u}$ such that $d - r^*C_0$ is a flat connection on the trivial bundle on M with fiber $\mathfrak{u}$. Repeating the same reasoning for (C', δ^*) and we have a flat connection form $r^*C'_0 \in A^1_{DR}(M)\widehat{\otimes}\mathfrak{u}'$ such that $d - r^*C'_0$ is a flat connection on the trivial bundle on M with fiber Lie algebra $\mathfrak{u}'$. Moreover, $\mathfrak{u}'$ corresponds to the Malcev Lie algebra of $\pi_1(M/G)$.

Theorem 2.4.9. *Consider $r_*C'_0, r_*C_0$ as above.*

1. *There exists a morphism of Lie algebras $K^* : \mathfrak{u}' \to \mathfrak{u}$ such that*

$$r_*k^*C'_0, r_*C_0 \in A_{DR}(M)\widehat{\otimes}\mathfrak{u}$$

   are gauge equivalent.

2. *Assume that $M = (N - \mathcal{D})$, where $\mathcal{D}$ is a normal crossing divisors that is preserved by the group action G. If $A, B \subset (\mathrm{Tot}_N^\bullet (A_{DR}(\log D_\bullet G)), m_\bullet)$ the gauge h is in $\exp\left(A^0_{DR}(M)\widehat{\otimes}\mathfrak{u}\right)$.*

Proof. By Corollary 2.3.15, there exists a morphism of Lie algebras $K^* : \mathfrak{u}' \to \mathfrak{u}$ such that $r_*k^*C'_0, r_*C_0$ are homotopy equivalent as Maurer-Cartan elements in the L_∞-algebra $\mathrm{Tot}_N^\bullet (A_{DR}(M_\bullet G))\,\widehat{\otimes}\mathfrak{u}$. The homotopy is given by a Maurer-Cartan element C''_0 in $(\Omega(1) \otimes \mathrm{Tot}_N^\bullet (A_{DR}(M_\bullet G)))\,\widehat{\otimes}\mathfrak{u}$. Similarly to Proposition 2.3.17, $r_*C''_0$ is a Maurer-Cartan element in $(\Omega(1) \otimes A_{DR}(M))\,\widehat{\otimes}\mathfrak{u}$. In particular, it defines a homotopy between $r_*k^*C'_0$ and r_*C_0. Since W is of finite type, $\mathfrak{u}$ is pronilpotent. Then we can apply Proposition 2.4.4, and the results follow. ☐

The above theorem and the explanation given in the previous section imply the following.

Corollary 2.4.10. *Assume that $B \subset A_{DR}(M/G)$ is a 1-model with a Hodge type decomposition. Then C'_0 defines a flat connection on the trivial bundle on M/G with fiber $\mathfrak{u}$, and there is a Lie algebra morphism $K^* : \mathfrak{u}' \to \mathfrak{u}$ and a factor of automorphy $F : G \times M \to \mathrm{End}(\mathfrak{u})$ such that $d - r_*C_0$ is a well-defined flat connection on M/G on the bundle E_F.*

Proof. We have $k^*C'_0$ is G-invariant since C'_0 is G-invariant. The proof follows from Proposition 2.4.8. ☐

Let $k^*C'_0$, C_0 be as in Corollary 2.4.10. We calculate its holonomy representations. We first relate the holonomy representation of $k^*C'_0$ with C'_0. We denote by T, T' and T^K the parallel transport of r_*C_0, C'_0 and $k^*C'_0$ on M respectively.

Lemma 2.4.11. 1. *Consider a smooth path $\gamma : [0,1] \to M$. Then*

$$T^K(\gamma) = K^*(T'(\gamma))$$

2. *Let $p \in M$ and let $\Theta_0^K : \pi_1(M/G, \overline{p}) \to \widehat{T}\left(\left(\left((W)_+[1]\right)^0\right)^*\right)/\mathcal{R}_0$ be the monodromy representation of the flat connection $d - k^*C'_0$. Then for any loop γ on M we have*

$$\Theta_0^K(\gamma) \subset U' = \exp(\mathfrak{u}') \subset \widehat{T}\left(\left(\left((W')_+[1]\right)^0\right)^*\right)/\mathcal{R}'_0$$

   and $\Theta_0^K(\gamma) = K^*\Theta'_0(\gamma)$, where Θ'_0 is the holonomy representation of $d - C'_0$.

Proof. We prove 1. The parallel transport of $d - k^*C'_0$ is given by iterated integral via the formula given in Lemma 1.1.30. The map $K^* : \mathfrak{u}' \to \mathfrak{u}$ is the restriction of a multiplication preserving map

$$K^* : \widehat{T}\left(\left(\left((W')_+[1]\right)^0\right)^*\right)/\mathcal{R}'_0 \to \widehat{T}\left(\left(\left((W)_+[1]\right)^0\right)^*\right)/\mathcal{R}_0.$$

Let $\alpha : \{0\} \to PM$ be the plot with image γ, then

$$\begin{aligned} T^K(\gamma) &= 1 + \sum_{n\geq 1} \int \left(r_* k^* C'_0\right)^n_\alpha \\ &= 1 + \sum_{n\geq 1} \int K^* \left(r_* C'_0\right)^n_\alpha \\ &= K^* \left(1 + \sum_{n\geq 1} \int \left(r_* C'_0\right)^n_\alpha\right) \\ &= K^* \left(T'(\gamma)\right). \end{aligned}$$

We prove 2. The argument of [29, Proposition 4.1] carry over this situation and we have $\Theta_0^K(\gamma) \subset U'$, the second part follows by point 1. □

We consider $k^*C'_0$ and r_*C_0. Since they are gauge equivalent as elements in $A_{DR}(M)\widehat{\otimes}\mathfrak{u}$ by a gauge h, we have

$$T(\gamma) = e^{-h(\gamma(0))} T^K(\gamma) e^{h(\gamma(1))}$$

for a path $\gamma : [0,1] \to M$. We fix a $p \in M$ and we denote its class in M/G by $\overline{p}$. We denote with ρ its induce group homomorphism

$$\rho : \pi_1(M/G, \overline{p}) \to G.$$

We have already defined a formula for the holonomy representation (see Proposition 2.4.7).

Lemma 2.4.12. *Fix a $p \in M$. The monodromy representation*

$$\Theta_0^K : \pi_1(M/G, \overline{p}) \to U$$

of $d - k^*C'_0$ is coniugate to the monodromy representation of $d - r_*C_0$ via $e^{h(p)}$.

Proof. Let γ be a loop based at $\overline{p}$ and let c_γ be its unique lift starting at p. We denote with Θ_0 the holonomy representation of $d - C_0$. Then

$$\begin{aligned} \Theta'_0(\gamma) &= T^K(c_\gamma) e^{-h(\rho(\gamma)p)} e^{h(p)} \\ &= e^{-h(c_\gamma(0))} T(c_\gamma) e^{h(c_\gamma(1))} e^{-h(\rho(\gamma)p)} e^{h(p)} \\ &= e^{-h(p)} T(c_\gamma) e^{h(p)} \\ &= e^{-h(p)} \Theta_0(\gamma) e^{h(p)}. \end{aligned}$$

□

Corollary 2.4.13. *Let A, B be as above. Assume that A is a 1-model and that (C, δ^*) is the homological pair associated to the given vector space decomposition. The holonomy representation of $d - r_*C_0$ corresponds to the Malcev completion of $\pi_1(M/G, \overline{p})$.*

Proof. If A is a 1-model, the map K^* is an isomorphism. The result follows. □

2.4.3 Some observations about the formality of $\pi_1(M/G)$

Let $\mathfrak{u}$ be a pronilpotent Lie algebra concentrated in degree zero. We denote its associated graded (with respect to the filtration $I^\bullet$) by $\mathrm{gr}(\mathfrak{u}) = \oplus_{i\geq 0} I^i\ I^{i+1}$ and the completed associated graded by $\widehat{\mathrm{gr}}(\mathfrak{u}) = \widehat{\oplus}_{i\geq 0} I^i\ I^{i+1}$. Notice that they are both filtered by the grading.

Definition 2.4.14 ([11])**.** The Lie algebra $\mathfrak{u}$ is said to be *formal* if there is an isomorphism of filtered Lie algebra $\mathfrak{g} \to \widehat{\mathrm{gr}}\,(\mathfrak{g})$ whose associated graded is the identity. The group $U = \exp(\mathfrak{u})$ is *formal* if so is $\mathfrak{u}$. A group G is *formal* if so is its Malcev completion.

In particular, if there exists a positively graded Lie algebra $\mathfrak{t}$ and an isomorphism of filtered Lie algebra $\mathfrak{u} \to \widehat{\mathfrak{t}}$, then $\mathfrak{u}$ is formal and the map induced via its associated graded is an isomorhism of graded Lie algebra $\mathrm{gr}\,(\mathfrak{u}) \to \mathfrak{t}$. Let M be a complex smooth manifold. Let $M_\bullet G$ be an action groupoid where G is discrete that acts properly and discontinuously on M. Assume that the cohomology of M/G is connected and of finite type. Let $A \subset \left(\mathrm{Tot}_N^\bullet\left(A_{DR}(M_\bullet G)\right), m_\bullet\right)$ be a 1-model. We fix a Hodge type decomposition on A via

$$A = W \oplus \mathcal{M} \oplus D\mathcal{M}.$$

Let (C, δ^*) be the associated homological pair with respect to the above decomposition. Let $\mathcal{R}_0 \subset \widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)$ be as above. We consider $\widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)$ equipped with the filtration $I^\bullet$. We set

$$\mathbb{L}^j\left(\left(W_+^1[1]\right)^*\right) := \left(I^{j+1} \cap \mathbb{L}\left(\left(W_+^1[1]\right)^*\right)^*\right)/I^j \subset \mathbb{L}\left(\left(W_+^1[1]\right)^*\right).$$

We say that $\mathcal{R}_0$ is graded if

1. $\delta^*\mathbb{L}\left(\left(W_+^2[1]\right)^*\right) \subset \mathbb{L}\left(\left(W_+^1[1]\right)^*\right)$, i.e. $\mathcal{R}_0$ is the completion of the Lie ideal R in $\mathbb{L}\left(\left(W_+^1[1]\right)^*\right)$ generated by $\delta^*\mathbb{L}\left(\left(W_+^2[1]\right)^*\right)$, and
2. $R = \oplus_{j>0}\mathbb{L}^j\left(\left(W_+^1[1]\right)^*\right) \cap R$.

Proposition 2.4.15. *Let $p \in M$ and let $\overline{p} \in M/G$ be its class. Assume that $\mathcal{R}_0$ is graded, then $\pi\left(M, \overline{p}\right)$ is formal.*

Proof. By Corollary 2.4.13 we know that the holonomy representation induces an isomorphism between the (completed) Malcev Lie algebra and $\mathfrak{u} := \widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)/\mathcal{R}_0$. Since $\mathcal{R}_0$ is graded, by point 1. we can consider $\mathfrak{u}$ as the completion of $\mathbb{L}\left(\left(W_+^1[1]\right)^*\right)/R$ with respect to the induced filtration. Moreover

$$\mathbb{L}\left(\left(W_+^1[1]\right)^*\right)/R = \bigoplus\nolimits_{j>0}\left(\mathbb{L}^j\left(\left(W_+^1[1]\right)^*\right) + R\right)/R,$$

and hence $\mathfrak{u}$ is isomorphic as a filtered Lie algebra to the completion of a positively graded Lie algebra. It follows that $\mathfrak{u}$ is formal. Hence so is $\pi\left(M/G, \overline{p}\right)$. □

Chapter 3

The universal KZB connection on the punctured elliptic curve

We construct a family of 1-models for the differential graded algebra of smooth differential forms on punctured elliptic curves. The family is parametrized by the holomorphic structure τ in the complex upper half plane and is a holomorphic C_∞ version of the smooth models given in [9]. We apply the theory developed in the previous section to build a flat connection $d-\alpha_\tau$ that happens to be equal to the universal KZB connection on the punctured elliptic curve (see [11], [38] and [29]). A comparison between KZB and KZ connection is contained in [29], by sending τ to $i\infty$. We describe this comparison in terms of pushforwards and pullbacks of Maurer-Cartan elements (see Section 3.3). In the last section we present another holomorphic connection gauge equivalent to the KZB connection on the punctured elliptic curve.

3.1 The complex punctured elliptic curve

Let τ be a fixed element of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$. Let $\mathbb{Z} + \tau\mathbb{Z}$ be the lattice spanned by $1, \tau$. Let ξ be the coordinate on $\mathbb{C}$. We define the action of $\mathbb{Z}^2$ on $\mathbb{C}$ via translations by

$$(m,n)(\xi) = \xi + m + \tau n.$$

In particular, the action is holomorphic. Moreover $\mathbb{Z}+\tau\mathbb{Z}$ is a normal crossing divisor of $\mathbb{C}$ and it is preserved by the action of $\mathbb{Z}^2$. Hence the action groupoid $\mathbb{C}_\bullet\mathbb{Z}^2$ is a simplicial manifold equipped with a simplicial normal crossing divisor $(\mathbb{Z}+\tau\mathbb{Z})_\bullet\mathbb{Z}^2$ (see Subsection 2.3.1). Since the action of $\mathbb{Z}^2$ is free and properly discontinuos we have

$$H^\bullet\left(\mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})_\bullet\mathbb{Z}^2\right)\right)\right) \cong H^\bullet\left(\mathcal{E}^\times, \mathbb{C}\right).$$

Let $\gamma : \mathbb{Z}^2 \to \mathbb{C}$ be the group homomorphism defined by $\gamma(m,n) = n2\pi i$. Then $d\xi$ and γ are closed forms in $\mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})_\bullet\mathbb{Z}^2\right)\right)$. They are of type $(0,1)$ resp. $(1,0)$ and they generate the cohomology. We construct a 1-model for $\mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})_\bullet\mathbb{Z}^2\right)\right)$.
We fix a family of holomorphic functions $f^{(i)} : \mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\} \to \mathbb{C}$ indexed by $i \in \mathbb{N}$ such that

$$f^{(0)} = 1, \quad f^{(n)}(\xi+l) = f^{(n)}(\xi), \quad f^{(n)}(\xi+l\tau) = \sum_{j=0}^{n} \frac{f^{(n-j)}(\xi)(-2\pi i l)^j}{j!} \tag{3.1}$$

for all $l, n \in \mathbb{Z}$. We denote the total differential of $\mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})_\bullet\mathbb{Z}^2\right)\right)$ by D. In particular for an element a of bidegree (p,q) we have

$$D(a) = \partial_G a + (-1)^p da,$$

where ∂_G is differential obtained by the alternating sum of the pullback of the cofaces maps of the action groupoid.

Lemma 3.1.1. *Let $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ be a family of holomorphic functions indexed by $\mathbb{N}$ that satisfy* (3.1). *Then they are linearly independent.*

Proof. Assume that there is a non trivial relation $\sum_{i \in I} \lambda_i f^{(i)} = 0$. Let p be the maximal integer of I such that $\lambda_p \neq 0$. Notice that $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ are elements of $\mathrm{Tot}_N\left(A_{DR}\left(\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z})_{\bullet}\, \mathbb{Z}^2\right)\right)$ of bidegree $(0,0)$. Let V be the complex vector space generated by $\left\{f^{(i)}\right\}_{i \in \mathbb{N}}$. Then each $f^{(i)}$ defines a map $\partial_G f^{(i)} : \mathbb{Z}^2 \to V$ via

$$((m,n)) \mapsto \partial_G f(m,n) = \sum_{j=0}^{i} \frac{f^{(i-j)}(\xi)(-2\pi i n)^j}{j!}.$$

Set $x := n$, then $\sum_{i \in I} \lambda_i \partial_G f^{(i)}$ can be seen as a polynomial $p(x)$ in variable x and coefficients in V of the form

$$p(x) = \sum h^a x^a$$

where $h^a \in V$. In particular $h^p = \lambda_p f^{(0)}(-2\pi i)^p = \lambda_p(-2\pi i)^p$. Since $\sum_{i \in I} \lambda_i \partial_G f^{(i)} = 0$ we get that $p(x) = 0$ for any $x \in \mathbb{N}$. This implies that $h^p = 0$ and then $\lambda_p = 0$. ☐

For $i \in \mathbb{N}$, we set $\phi^{(i)} := f^{(i)} d\xi$ then

$$\phi^{(i)} \in \mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_{\bullet}\, \mathbb{Z}^2\right)\right)$$

are linearly indipendent elements of bidegree $(0,1)$. Note that $d\phi^{(i)} = 0$ since they are holomorphic 1-forms. We consider $\mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_{\bullet}\, \mathbb{Z}^2\right)\right)$ equipped with the unital C_∞-structure $m_\bullet$ induced by Theorem 2.2.1.

Lemma 3.1.2. *1. $D\left(-\phi^{(n)}\right) = \sum_{l=1}^{n} m_{l+1}(\gamma, \ldots, \gamma, \phi^{(p-l)})$, for any n,*

2. $m_{l+1}(\gamma^{\otimes i}, \phi^{(k)}, \gamma^{\otimes l+1-i}) = (-1)^l \binom{l}{i} m_{l+1}(\phi^{(k)}, \gamma, \ldots, \gamma)$,

3. Let $B^\bullet \subset \mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_{\bullet}\, \mathbb{Z}^2\right)\right)$ be C_∞-sub algebra generated by

$$1, \gamma, \left\{\phi^{(i)}\right\}_{i \in \mathbb{N}}.$$

Then

$$(m_\bullet, B^\bullet) \subset \left(m_\bullet, \mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_{\bullet}\, \mathbb{Z}^2\right)\right)\right)$$

is a 1-model.

Proof. We use the same notation as in theorems (2.2.7) and (2.2.9). Since the elements $m_{l+1}\left(\gamma, \ldots, \gamma, \phi^{(p-l)}\right)$ are of bidegree $(1,1)$ we have to prove the statement in the C_∞-algebra

$$\left(m_\bullet^{1,1}, NC_1^\bullet \otimes A_{DR}^\bullet\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_1\, \mathbb{Z}^2\right)\right).$$

Fix a $(m,n) \in \mathbb{Z}^2$.

1. If $l > 1$,

$$\begin{aligned}
& m_{l+1}\left(\gamma, \ldots, \gamma, \phi^{(p-l)}\right)_1 (m,n) \\
&= m_{l+1}^{1,1}\left(-dt \otimes \gamma(m,n), \ldots, -dt \otimes \gamma(m,n), t \otimes \phi^{(p-l)} + (1-t) \otimes \left((m,n) \cdot \phi^{(p-l)}\right)\right) \\
&= m_{l+1}^{[1]}(dt, \ldots, dt, t) \otimes (-\gamma(m,n))^l \phi^{(p-l)} - m_{l+1}^{[1]}(dt, \ldots, dt, t) \otimes ((m,n)) \cdot \phi^{(p-l)} \\
&= -dt \otimes \left(\frac{B_l}{l!} \sum_{i \geq 1}^{p-l} \frac{\phi^{(p-l-i)}(-2\pi i n)^{i+l}}{i!}\right) \qquad (3.2)
\end{aligned}$$

2. If $l = 1$,

$$
\begin{aligned}
& m_2\left(\gamma, \phi^{(p-1)}\right)(m,n) \\
&= m_2^{1,1}\left(-dt \otimes \gamma(m,n), t \otimes \phi^{(p-1)} + (1-t) \otimes \left((m,n) \cdot \phi^{(p-1)}\right)\right) \\
&= m_2^{[1]}(dt,t) \otimes (-\gamma(m,n))\, \phi^{(p-1)} + m_2^{[1]}(dt,1) \otimes (-\gamma(m,n)) \left((m,n) \cdot \phi^{(p-1)}\right) \\
&- m_2^{[1]}(dt,t) \otimes (-\gamma(m,n)) \left((m,n) \cdot \phi^{(p-1)}\right) \\
&= dt \otimes \left((1-B_1) \sum_{i \geq 1}^{p-l} \frac{\phi^{(p-1-i)}(-2\pi i n)^{i+1}}{i!} + (-2\pi i n)\, \phi^{(p-1)}\right) \qquad (3.3)
\end{aligned}
$$

On the other hand we have

$$
\left(D\left(-\phi^{(p)}\right)\right)_1 (m,n) = \sum_{i \geq 1}^{p} (dt) \otimes \frac{\phi^{(p-i)}(-2\pi i n)^i}{i!} \qquad (3.4)
$$

We prove that (3.2) + (3.3) = (3.4) by comparing the coefficients. For $p = 1, 2$ the two expressions agree. For $n > 2$ (3.2) + (3.3) = (3.4) is equivalent to

$$
\frac{1 - B_1}{(i-1)!} - \sum_{j=2}^{i-1} \frac{B_j}{j!} \frac{1}{(i-j)!} = \frac{1}{i!}.
$$

The first Bernoulli numbers ${B'}_j$ satisfy $\sum_{j=0}^{\infty} \frac{{B'}_j}{j!} = \frac{t}{e^t - 1}$. Since $B_j = {B'}_j$ for $j \neq 1$ and $B_1 = \frac{1}{2} = -{B'}_1$, the condition above is equivalent to

$$
\left(1 - \frac{t}{1 - e^t}\right)\left(e^t - 1\right) = e^t - 1 - t
$$

This proves the first statement. The second statement follows from Proposition 2.2.4. We prove point 3. Let $i : B \to \mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_\bullet \mathbb{Z}^2\right)\right)$ be the inclusion. We have to show that the induced map $[i]$ in cohomology is an isomorphism between elements of degree 1 and an is injective for the elements of degree 2. Assume that there is an element $\sum_{i \in I} \lambda_i \phi^{(i)}$ such that $\sum_{i \in I} \lambda_i D\phi^{(i)} = 0$, we assume that $0 \notin I$. Let p be the maximal integer of I such that $\lambda_p \neq 0$. Let V be the complex vector space generated by $\left\{\phi^{(i)}\right\}_{i \in \mathbb{N}}$. Then each $\phi^{(i)}$ defines a map $D\phi^{(i)} : \mathbb{Z}^2 \to V$ via

$$
((m,n)) \mapsto (Df)(m,n) = \sum_{j=0}^{i} \frac{\phi^{(i-j)}(\xi)(-2\pi i n)^j}{j!}.
$$

Set $x = n$, hence $\sum_{i \in I} \lambda_i D\phi^{(i)}$ can be written as a polynomial $p(x)$ with coefficients in V of the forms

$$
p(x) = \sum h^a x^a
$$

where $h^a \in V$. In particular the monomial that multiplies $\phi^{(p-1)}$ is x, i.e we get

$$
h^1 = \lambda_p \phi^{(p-1)} + h
$$

where h lies in the subvector space of V generated by $\phi^{(0)}, \ldots, \phi^{(p-2)}$. Since $\sum_{i \in I} \lambda_i D\phi^{(i)} = 0$ we get $p(x) = 0$ for any $x \in \mathbb{N}$. By Lemma 3.1.1 implies that $h^1 = 0$ and hence $\lambda_p = 0$. This shows that the vector space of closed forms in B^1 is generated by ϕ^0 and γ.
The second condition is equivalent to show that $H^2(B)$ vanishes. Suppose that there is a closed element

$$
f := \sum_{i,j}^{n} \lambda_{i,j} m_i \left(\gamma, \ldots, \gamma, \phi^{(j)}\right),
$$

for $i > 1$ and $j \geq 0$. By part 1. we can assume that $\lambda_{2,j} = 0$. Moreover since $m_i\left(\gamma, \ldots, \gamma, \phi^{(0)}\right) = 0$ for $i > 2$ we conclude that $j > 0$. We define $f^{[j]} := \sum_i^n \lambda_{i,j} m_i\left(\gamma, \ldots, \gamma, \phi^{(j)}\right)$. Let p be the maximal integer such that $\lambda_{i,p} \neq 0$ for some i and let l be the maximal integer such that $\lambda_{l,p} \neq 0$. The element $Dm_l\left(\gamma, \ldots, \gamma, \phi^p\right)$ is of type $(2,1)$, then for $((m,n),(m',n')) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ we have

$$\begin{aligned}(Dm_l\left(\gamma, \ldots, \gamma, \phi^p\right))\left((m,n),(m',n')\right) &= \frac{B_{l-1}}{(l-1)!} \sum_{i \geq 1}^{p} \frac{(m,n) \cdot \phi^{(p-i)}\left(-2\pi i n'\right)^{i+l-1}}{i!} \\ &+ \frac{B_{l-1}}{(l-1)!} \sum_{i \geq 1}^{p} \frac{\phi^{(p-i)}\left(-2\pi i(n+n')\right)^{i+l-1}}{i!} \\ &+ \frac{B_{l-1}}{(l-1)!} \sum_{i \geq 1}^{p} \frac{\phi^{(p-i)}\left(-2\pi i n\right)^{i+l-1}}{i!}\end{aligned}$$

Then Df defines a map $Df : \mathbb{Z}^2 \times \mathbb{Z}^2 \to V$ via

$$((m,n),(m',n')) \mapsto (Df)\left((m,n)(m',n')\right).$$

Now set $x = n$, $y = n'$, then Df can be written as a polynomial $p(x,y)$ with coefficients in V of form

$$p(x,y) = \sum h^{a,b} x^a y^b.$$

The monomials that multiply $\phi^{(p-1)}$ are

$$\frac{\lambda_{l,p} B_{l-1}(-2\pi i)^l}{(l-1)!}\left(x^l - (x+y)^l + y^l\right) + r(x,y)$$

where $r(x,y)$ is a polynomial containing monomials of total degree smaller than l. We conclude that

$$h^{l-1,1} = \frac{\lambda_{l,p} B_{l-1}(-2\pi i)^l}{(l-1)!}\phi^{(p-1)} + h, \quad h^{1,l-1} = \frac{\lambda_{l,p} B_{l-1}(-2\pi i)^l}{(l-1)!}\phi^{(p-1)} + h'$$

where $h, h' \in W$. Since $p(x,y)$ vanishes on $\mathbb{Z}^2 \times \mathbb{Z}^2$ we conclude that $h^{a,b} = 0$ for any a, b. We get relations

$$\frac{\lambda_{l,p} B_{l-1}(-2\pi i)^l}{(l-1)!}\phi^{(p-1)} + h = 0, \quad \frac{\lambda_{l,p} B_{l-1}(-2\pi i)^l}{(l-1)!}\phi^{(p-1)} + h = 0$$

which implies $\lambda_{l,p} = 0$. Hence the only closed form in B^2 is

$$\lambda m_2\left(\gamma, \phi^{(0)}\right)$$

and the second cohomology group of (B, m_1) vanishes. □

The proof of Lemma 2.1.38, Lemma 2.1.35 and the proposition above imply the following.

Corollary 3.1.3. *Let $(B, m_\bullet)$ be as above. There exists a Hodge decomposition*

$$B = W^\bullet \oplus D\mathcal{M} \oplus \mathcal{M} \tag{3.5}$$

such that

1. *W^1 is generated by $\phi^{(0)}$ and γ, $\mathcal{M}^1$ is generated by $\phi^{(i)}$ for $i > 0$, and $(D\mathcal{M})^0 = 0$,*
2. *$W^2 = 0$, $\mathcal{M}^2$ is generated by $m_l\left(\phi^{(i)}, \gamma, \ldots, \gamma\right)$ for $i > 0, l > 2$, and $(D\mathcal{M})^2$ is generated by $D\left(\phi^{(i)}\right)$ for $i > 1$.*

Notice that the model B is completely determined by the choiche of holomorphic functions $f^{(i)}$: $\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ that satisfy (3.1). The above facts are true for any $\tau \in \mathbb{H}$ as well. Hence a consequence of Lemma 3.1.2 and Corollary 3.1.3 we have the following.

Corollary 3.1.4. *For any $\tau \in \mathbb{H}$ there exists a holomorphic model B for* $\mathrm{Tot}_N\left(A_{DR}\left((\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})_{\bullet}\mathbb{Z}^2\right)\right)$ *equipped with a Hodge type decomposition.*

We fix a τ and a family of holomorphic functions $f^{(i)} : \mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\} \to \mathbb{C}$ that satisfy (3.1). We consider $(B, m_{\bullet})$ equipped with the Hodge type decomposition above. We calculate the degree zero geometric connection of the action groupoid $(\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})_{\bullet}\mathbb{Z}^2$ associated to the decomposition (3.5). We apply Lemma 2.1.33 to the decomposition (3.5) and get chain maps f, g and a chain homotopy h. Let $p_n = \sum_{T\in\mathcal{P}_n}(-1)^{\theta(T)}P_T$ be the p-kernel of Proposition 2.1.31. There is a C_∞-structure $m^W_{\bullet}$ on W given by

$$m_n := f \circ p_n \circ g^{\otimes n}, \quad n \geq 1$$

and a C_∞-quasi-isomorphism $g_n : W^{\otimes n} \to W$ is given by

$$g_n := h \circ p_n \circ g^{\otimes n}, \quad n \geq 1.$$

The corresponding L_∞ Maurer-Cartan element $\alpha \in \mathrm{Conv}\left((B, m_{\bullet}), W, \mathcal{M}\right)$ is given by

$$\alpha_n : (W[1])^{\otimes n} \xrightarrow{\left(s^{-1}\right)^{\otimes n}} W^{\otimes n} \xrightarrow{g_n} B.$$

Since $H^2(B, D) = 0$, the C_∞-structure m^W_n vanishes on W^1.

Lemma 3.1.5. *Let $g_{\bullet}$ be the above quasi-isomorphism. Let $v_{i_1}, \ldots, v_{i_n} \in \left\{\gamma, \phi^{(0)}\right\}$ for $l = 1, \ldots, n$.*

1. *$g_n(v_1, \ldots, v_n) = 0$ if $(v_1, \ldots, v_n)$ is not of the following form: there exists exactly one j such that $v_j = \phi^{(0)}$ and $v_s = \gamma$, for $s \neq j$.*
2. *$g_n\left(\phi^{(0)}, \gamma, \ldots, \gamma\right) = \phi^{(n)}$.*

Proof. For each $n \geq 2$, the p-kernels are

$$p_n := \sum_{T\in\mathcal{P}_n}(-1)^{\theta(T)}P_T.$$

Since m_l is C_∞ we have $m_l(\gamma, \ldots, \gamma) = 0$ for $l > 1$. Consider $m_l(w_{i_1}, \ldots, w_{i_l})$ such that $w_{i_l} \in \left\{\gamma, \phi^{(0)}, \phi^{(1)}, \ldots\right\}$. Assume that there exists exactly $j_1, \ldots j_r$ where $l > r > 1$ such that $w_{j_i} = \phi^{(k)}$ for $i = 1, \ldots, r$ for some $k \geq 0$. Since this expression depends only by $m^{[1]}_{\bullet}$ (see Proposition 2.2.4) we have $m_l(w_{i_1}, \ldots, w_{i_l}) = 0$. We assume $r = l$, then our expression depends only by $m^{[0]}_l$, which vanishes if $l > 2$. For $l = 2$ we get $m_2(w_{i_1}, w_{i_2})$ which vanishes for dimensional reasons. Let $T \in \mathcal{P}_n$ be an oriented planar tree and consider the induced map P_T. Lemma 3.1.2 and the decomposition (3.5) imply

$$h\left(m_2(\gamma, \phi^{(k-1)})\right) = h\left(m_1\left(\phi^{(k)}\right) - \sum_{l=2}^{k}(-1)^{l+1}m_{l+1}(\gamma, \ldots, \gamma, \phi^{(k-l)})\right) = \phi^{(k)} \tag{3.6}$$

and

$$h\left(m_l(\gamma, \ldots, \gamma, \phi^{(k)}, \gamma, \ldots, \gamma)\right) = 0 \quad l > 2$$

for any $k \geq 1$. This proves point 1. The only binary tree T such that $h \circ P_T \circ g^{\otimes n}\left(\phi^{(0)}, \gamma, \ldots, \gamma\right) \neq 0$ is the one in Figure 3.1. Here we have $m_k = 0$ for $k > 2$ even. A direct calculation shows $h \circ p_T \circ g^{\otimes n}\left(\phi^{(0)}, \gamma, \ldots, \gamma\right) = \phi^{(n)}$. ☐

We consider W equipped with the basis $-\gamma$, $-\phi^{(0)}$ (see Remark 2.3.22). We consider $\left(W^1[1]\right)^*$ as the vector space generated by the degree zero elements X_0, X_1, where X_1, and resp. X_0 denote the dual of $-s\phi^{(0)}$ and of $-s\gamma$ in $W[1]$ respectively. Let π, r, p be as in (2.5). Since $W^2 = 0$, we have $\mathcal{R}_0 = 0$ and the fiber Lie algebra of the punctured elliptic curve is the free Lie algebra on two generators, $\pi(\alpha) \in \mathrm{Hom}^1\left(T(W^1_+[1]), B\right)$ can be written as a formal power series

$$C_0 \in B\widehat{\otimes}\widehat{\mathbb{L}}\left(X_0, X_1\right).$$

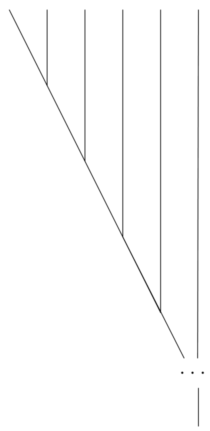


Figure 3.1: Three with non-trivial induced map

Hence C_0 is the degree zero geometric connection associated to the decomposition (3.5). Let $L \subset \widehat{\mathbb{L}}(X_0, X_1)$ be the complete vector subspace spanned by X_0 and by all the Lie monomials P (see [40], chapter 5) such that its associated monomial is

$$\text{Mon}(P) = X_1^{\otimes s} \otimes X_0 \otimes X_1^{\otimes r} \tag{3.7}$$

for some $s, r \geq 0$. In particular X_0 and the set of Lie monomials of the forms $[[X_1, X_0], \ldots, X_0]$ form a basis of L. Let $\text{Ad}^0_{X_0}(X_1) := X_1$ and $\text{Ad}^p_{X_0}(X_1) := \left[X_0, \text{Ad}^{p-1}_{X_0}(X_1)\right]$ for $p > 0$. We conclude

$$\begin{aligned} C_0 &= -\gamma X_0 - \sum_{p\geq 0}(-1)^p \phi^{(p)} [[X_1, X_0], \ldots, X_0] \\ &= -\gamma X_0 - \sum_{p\geq 0} \phi^{(p)} \text{Ad}^p_{X_0}(X_1) \end{aligned}$$

Theorem 3.1.6. *Consider a family of holomorphic functions $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ indexed by $i \in \mathbb{N}$ that satisfy (3.1). Let $(B, m_\bullet)$ be the 1-model constructed in Lemma 3.1.2 and equipped with the decomposition $(W, \mathcal{M})$ as in (3.5). We consider W equipped with the basis $-\gamma$, $\phi^{(0)}$.*

1. *The degree zero geometric connection is given by*

$$C_0 = -\gamma X_0 - \sum_{p\geq 0} \phi^{(p)} \text{Ad}^p_{X_0}(X_1)$$

2. *Let $(A, m_\bullet)$ be a 1-model equipped with an Hodge type decomposition $(W', \mathcal{M})$. Let ${C'}_0$ be the associated degree zero geometric connection. Let $(B', m_\bullet)$ be the 1-model constructed in Lemma 3.1.2 and equipped with the decomposition (3.5). Let ${C'}_0$ be the associated degree zero geometric connection. There exist a Lie algebra isomorphism*

$$K^* : \widehat{\mathbb{L}}\left(\left(W^1_+[1]\right)^*\right) \to \widehat{\mathbb{L}}\left(\left({W'}^1_+[1]\right)^*\right)$$

We denote by $k^*{C'}_0$ the precomposition of ${C'}_0$ with K. Then $r_* k^* {C'}_0$ is gauge equivalent to $r_* C_0$.

Proof. Point 1. is proved above. We first prove point 3. Since the second cohomology group of A and B vanishes, we can use the second part of Corollary 2.3.16 and this conclude the proof. ☐

The above theorem tells us that the degree zero geometric connection on the punctured torus is independent by the choice of the model and by the choiche of the Hodge type decomposition (modulo gauge equivalence and automorphisms of the fiber Lie algebra). Notice that the same argument works for punctured Riemann surfaces as well.

Remark 3.1.7. Note that all the statements above are true even by assuming that $f^0(z)$ is a $\mathbb{Z}^2$-invariant holomorphic function (an elliptic function with pole in 0). In this case we can construct a new model in the same spirit as above. We construct $\bar{\phi}^{(i)}$ by

$$\bar{\phi}^{(i)} := m_2(\phi^{(i)}, f^0(z)), \quad i \geq 0$$

This gives a family of 1-forms $\bar{\phi}^{(i)}$ indexed by $i \in \mathbb{N}$ that satisfies the relation (3.4). Hence $\left(\bar{\phi}^{(i)}, \gamma\right)$ generates a new holomorphic model.

3.2 A smooth gauge equivalent connection

Consider a family of holomorphic functions $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ indexed by $i \in \mathbb{N}$ that satisfy (3.1). Let $(B, m_\bullet)$ be the 1-model constructed in Lemma 3.1.2 and equipped with the decomposition (3.5) and let C be its associated Maurer-Cartan element. By Theorem 2.3.21, we know that

$$r^* C_0 = \sum_{p \geq 0} \phi^{(p)} \operatorname{Ad}^p_{X_0}(X_1)$$

is a flat connection form on $\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\}$ on the trivial bundle with fiber $\widehat{\mathbb{L}}\,(X_0, X_1)$, where the Lie algebra is considered to be equipped with the adjoint action.

$$\widehat{T}\,(X_0, X_1) \to \operatorname{End}\left(\widehat{\mathbb{L}}\,(X_0, X_1)\right).$$

We define the factor of automorphy $F : \mathbb{Z}^2 \times \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to$ via $F((n, m), \xi) = \exp(-2\pi i m X_0)$. We denote by $\mathcal{P}$ the bundle $E_F = (\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\}) \times \widehat{\mathbb{L}}\,(X_0, X_1)/\mathbb{Z}^2$ where the $\mathbb{Z}^2$ action is induced by F, i.e.

$$(n, m)(\xi, v) = (\xi + n + m\tau, \exp(-2\pi i m X_0) v).$$

The section of $\mathcal{P}$ satisfies:

$$s(\xi + l) = s(\xi), \quad s(\xi + l\tau) = \exp(-2\pi i l X_0) \cdot s(\xi) \tag{3.8}$$

where $X_0^p \cdot a := \operatorname{Ad}^p_{X_0}(a)$ for $a \in \widehat{\mathbb{L}}\,(X_0, X_1)$. The conditions (3.1) implies that

$$\sum_{p \geq 0} \phi^{(p)}(\xi + l) \operatorname{Ad}^p_{X_0}(X_1) = \sum_{p \geq 0} \phi^{(p)}(\xi) \operatorname{Ad}^p_{X_0}(X_1), \tag{3.9}$$

$$\sum_{p \geq 0} \phi^{(p)}(\xi + l\tau) \operatorname{Ad}^p_{X_0}(X_1) = \exp(-2\pi i X_0) \cdot \sum_{p \geq 0} \phi^{(p)}(\xi) \operatorname{Ad}^p_{X_0}(X_1) \tag{3.10}$$

for $l \in \mathbb{Z}$.

Lemma 3.2.1. *We consider $\mathbb{L}\,(X_0, X_1)$ acting on itself via the adjoint action* ad.

1. *Then $r^* C_0 \in A^1_{DR}\,(\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})\,\widehat{\otimes}\widehat{\mathbb{L}}\,(X_0, X_1)$ defines a flat connection form on the trivial bundle $(\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\}) \times \widehat{\mathbb{L}}\,(X_0, X_1)$.*
2. *$d - \sum_{p \geq 0} \phi^{(p)} \operatorname{Ad}^p_{X_0}(X_1)$ is a holomorphic flat connection on the bundle $\mathcal{P}$.*

Proof. It follows from the construction. □

Notice that the factor of automorphy is holomorphic. The bundle $\mathcal{P}$ is holomorphically non-trivial on the torus and trivial on the punctured torus. We conclude this section by introducing a smooth version of our model. We denote by $\xi = s + \tau r$ the coordinates on the punctured elliptic curve. We define the smooth differential forms $\omega^{(i)} \in A^1_{DR}\,(\mathcal{E}^\times_\tau)$ via

$$\sum_{k \geq 0} \omega^{(k)} \alpha^k := (\exp(2\pi i r \alpha)) \left(\sum_{k \geq 0} \phi^{(k)} \alpha^k\right)$$

where α is a formal variable. Let $\nu := 2\pi i dr$. It is easy see that the forms above satisfiy the following relations:

$$D\omega^{(k)} = d\omega^{(k)} = \nu \wedge \omega^{(k-1)}, \quad \omega^{(0)} = d\xi$$

Let $\bar{B} \subset (A_{DR}(\mathcal{E}_\tau^\times), D, \wedge)$ be the differential graded algebra generated by 1, ν and the $\omega^{(i)}$ for $i \geq 0$. Notice that $\bar{B}$ depends by the choice of functions $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ indexed by $i \in \mathbb{N}$ that satisfy (3.1).

Proposition 3.2.2. *Let $\bar{B} \subset A_{DR}(\mathcal{E}_\tau^\times)$ as above.*

1. *$\bar{B}$ is a model for $A_{DR}(\mathcal{E}_\tau^\times)$.*
2. *There exists a Hodge decomposition*

$$\bar{B} = \bar{W} \oplus D\bar{\mathcal{M}} \oplus \bar{\mathcal{M}} \tag{3.11}$$

such that

(a) $\bar{W}^1$ is generated by $\omega^{(0)}$ and ν, $\bar{\mathcal{M}}^1$ is generated by $\omega^{(i)}$ for $i > 0$, and $\left(D\bar{\mathcal{M}}\right)^0 = 0$,

(b) $\bar{W}^2 = 0$, $\bar{\mathcal{M}}^1$ is generated by $\omega^{(i)}\nu$ for $i > 0$,, and $\left(D\bar{\mathcal{M}}\right)^1$ is generated by $D\left(\omega^{(i)}\right)$ for $i > 1$.

Proof. The point 1 is proved in [9]. In [9], it is proved that $\omega^{(0)}$, $\omega^{(1)}$, ..., ν are linearly independent and that $\omega^{(0)}\nu$, $\omega^{(1)}\nu$, ... are linearly independent. This proves the second point. □

We consider $\left(\bar{W}^1[1]\right)^*$ as the vector space generated by the degree zero elements $\bar{X}_0, \bar{X}_1$, where $\bar{X}_1$, and resp. $\bar{X}_0$ denote the dual of $-s\omega^{(0)}$ and resp. of $s\nu$. By using the same calculation of the previous section, we have the following.

Proposition 3.2.3. *Consider a family of holomorphic functions $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ indexed by $i \in \mathbb{N}$ that satisfy (3.1) and let $\bar{B}$ equipped with the decomposition (3.11). The degree zero geometric connection is given by*

$$\bar{C}_0 = \nu\bar{X}_0 - \sum_{p\geq 0} \omega^{(p)} \operatorname{Ad}^p_{\bar{X}_0}\left(\bar{X}_1\right)$$

In particular, $d - \bar{C}_0$ is a smooth flat connection on the trivial bundle $\mathcal{E}_\tau^\times \times \widehat{\mathbb{L}}\left(\bar{X}_0, \bar{X}_1\right)$

Proposition 3.2.4. *Consider a family of holomorphic functions $f^{(i)} : \mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\} \to \mathbb{C}$ indexed by $i \in \mathbb{N}$ that satisfy (3.1) and let $\bar{B}$ and B as above equipped with their aforementioned Hodge decompositions. The two connection forms r_*C_0 and $\bar{C}_0$ are gauge equivalent via*

$$2\pi i r\bar{X}_0 \in \operatorname{Tot}_N\left(A_{DR}\left((\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_\bullet \mathbb{Z}^2\right)\right)\widehat{\otimes}\widehat{\mathbb{L}}\left(\bar{X}_0, \bar{X}_1\right).$$

Proof. The results follows by a direct calculation. □

3.3 Kronecker function and KZ connection

We show that the KZB connection can be constructed as degree zero geometric connection. In the second part of this section we study some results of Section 12 of [29], where a link between universal KZ and KZB connection is given by considering the restriction of the connection to the first order Tate curve. We give an interpretation of such a result in terms of C_∞-algebras over $\mathbb{Q}(2\pi i)$.
Let $\theta(\xi, \tau)$ denote the "two thirds of the Jacobi triple formula" (see Definition 1.16). Let $\theta(0,\tau)' := \frac{\partial}{\partial\xi}\theta(0,\tau)$. The Kronecker function is defined as

$$F(\xi, \eta, \tau) := \frac{\theta(0,\tau)'\,\theta(\xi+\eta,\tau)}{\theta(\xi,\tau)\,\theta(\eta,\tau)}.$$

Proposition 3.3.1.

i) F is a meromorphic function with simple pole at (ξ, η, τ) where $\xi \in \mathbb{Z} + \tau\mathbb{Z}$ and $\eta \in \mathbb{Z} + \tau\mathbb{Z}$, *ii) It satisfies the quasi-periodicity*

$$F(\xi+1,\eta,\tau)=F(\xi,\eta,\tau)\,,\quad F(\xi+\tau,\eta,\tau)=\exp(-2\pi i\eta)F(\xi,\eta,\tau)\,. \tag{3.12}$$

Proof. See [57], Theorem 3. □

In [57], there is a Fourier expansion for F

$$F(\xi,\eta,\tau)=\pi i\left(\coth(\pi i\xi)+\coth(\pi i\eta)\right)+4\pi\sum_{n=1}^{\infty}\left(\sum_{d|n}\sin\left(2\pi\left(\frac{n}{d}\xi+d\eta\right)\right)\right)q^n \tag{3.13}$$

We fix a $\tau\in\mathbb{H}$ and consider F restricted at τ. We consider η as a formal variable and we define the function $g^{(i)}$, $i\geq 0$ as the coefficients of

$$\eta F(\xi,\eta,\tau)=\sum_{i\geq 0}g^{(i)}\eta^i.$$

In particular the functions $g^{(i)}$ are meromorphic. Formula (3.13) give a way to describe the functions $g^{(i)}$ explicitly. First notice that

$$\frac{1}{2}\coth\left(\frac{t}{2}\right)=\frac{1}{2}+\frac{1}{e^t-1}=\sum_{m=0}^{\infty}\frac{B_{2m}}{(2m)!}t^{2m-1}.$$

Hence

$$\pi i\eta\left(\coth(\pi i\xi)+\coth(\pi i\eta)\right)=\pi i\eta+\frac{2\pi i\eta}{e^{2\pi i\xi}-1}+\sum_{m=0}^{\infty}\frac{B_{2m}}{(2m)!}(2\pi i\eta)^{2m},$$

by the de Moivre formula

$$\begin{aligned}&4\pi\sum_{n=1}^{\infty}\left(\sum_{d|n}\sin\left(2\pi\left(\frac{n}{d}\xi+d\eta\right)\right)\right)q^n\\&=-2\pi i\sum_{n=1}^{\infty}\sum_{d|n}\left(e^{2\pi i\frac{n}{d}\xi}\left(\sum_{l=0}^{\infty}\frac{(2\pi i d\eta)^l}{l!}\right)-e^{-2\pi i\frac{n}{d}\xi}\left(\sum_{l=0}^{\infty}\frac{(-2\pi i d\eta)^l}{l!}\right)\right)q^n.\end{aligned}$$

Hence

$$\begin{aligned}g^{(0)}(\xi)&=1,\\g^{(1)}(\xi)&=\pi i+\frac{2\pi i}{e^{2\pi i\xi}-1}-2\pi i\sum_{n=1}^{\infty}\sum_{n|d}\left(e^{2\pi i\frac{n}{d}\xi}-e^{-2\pi i\frac{n}{d}\xi}\right)q^n,\\g^{(l)}(\xi)&=-\frac{(2\pi i)^l}{l!}\sum_{n=1}^{\infty}\left(\sum_{n|d}ld^l\left(e^{2\pi i\frac{n}{d}\xi}+(-1)^le^{-2\pi i\frac{n}{d}\xi}\right)\right)q^n+\frac{(2\pi i)^l B_l}{l!},\ \text{for } l>1.\end{aligned}$$

In particular, for any τ, $g^{(0)}=1$ and $g^{(1)}$ is a meromorphic function with simple poles at $\xi\in\mathbb{Z}+\tau\mathbb{Z}$ and $g^{(i)}$ for $i>1$ is holomorphic (see [57]). The quasi-periodicity of F implies that the functions $\left\{g^{(i)}\right\}_{i\in\mathbb{N}}$ satisfies (3.1). Recall that (see Definition 1.3.5) that KZB connection is a flat connection form

$$d_{DR}-\omega_{KZB,n}$$

defined on $\overline{\mathcal{P}}^n$. We analyze the case $n=1$ (compare with [38]). In particular $\omega_{KZB,1}$ is a flat connection form on $\mathcal{E}_\tau^\times$ on the bundle $\mathcal{P}=\overline{\mathcal{P}}^1$ (see (3.8))

$$\omega_{KZB,1}=-\sum_{p\geq 0}g^{(p)}(\xi)d\xi\,\mathrm{Ad}_{X_0}^p(X_1)$$

Theorem 3.3.2. *We fix a $\tau \in \mathbb{H}$ and we consider the functions $\left\{g^{(i)}\right\}_{i\in\mathbb{N}}$. Let $(B, m_\bullet)$ be the 1-model constructed in Lemma 3.1.2 and equipped with the decomposition $(W, \mathcal{M})$ as in (3.5). We consider W equipped with the basis $-\gamma$ and $-d\xi$.*

1. *$(B, m_\bullet) \subset \mathrm{Tot}^\bullet_N \left(A_{DR}\left(\log\left((\mathbb{Z}+\tau\mathbb{Z})_\bullet \mathbb{Z}^2\right)\right)\right)$*
2. *Let C_0 be the associated degree zero geometric connection, we have*

$$r^* C_0 = \omega_{KZB,1}.$$

Proof. Let $(B, m_\bullet)$ be the 1-model constructed in Lemma 3.1.2 and equipped with the decomposition (3.5). We prove 1. The forms $g^{(i)}(\xi)d\xi$ are holomorphic for $i \neq 1$ and $g^{(1)}(\xi)d\xi$ is a form with a logarithmic singularity. By (2.19) we have the statement. The proof of point 2 follows by Theorem 3.1.6. □

For different τ we get different connection forms. We denote the universal KZB-connection on the punctured elliptic curve $\mathcal{E}^\times_\tau$ by $\omega^\tau_{KZB,1}$. As noticed by Hain in [29], $\lim_{\tau\to i\infty} \omega^\tau_{KZB,1}$ is equal to $\omega_{KZ,1}$ modulo a certain endomorphism Q^* of complete Lie algebra. We use the argument of Subsection (2.1.5) to show that Q^* is induced by a strict C_∞-morphism $p_\bullet$.
We denote by z the coordinate on $\mathbb{C}^*$. We define the action of $\mathbb{Z}$ on $\mathbb{C}^*$ via

$$n \cdot z := q^n z. \tag{3.14}$$

There is a morphism $h_\bullet : \mathbb{C}_\bullet \mathbb{Z}^2 \to \mathbb{C}^*_\bullet \mathbb{Z}$ of action groupoids

$$h_0(\xi) = e^{2\pi i\xi}, \quad h_1(\xi, (m, n)) = \left(e^{2\pi i\xi}, n\right)$$

which induces an isomorphism on the quotient. We have $\left\{q^{\mathbb{Z}}\right\} \subset \mathbb{C}^*$ and the maps above give a morphism $\left(\mathbb{C}_\bullet - \left\{\mathbb{Z}^2 + \tau\mathbb{Z}^2\right\}\right)_\bullet \mathbb{Z}^2 \to \left(\mathbb{C}^* - \left\{q^{\mathbb{Z}}\right\}\right)_\bullet \mathbb{Z}$ between action groupoids that induce an isomorphism on the punctured elliptic curve. The quasi-periodicity of F allows us to rewrite the functions as functions on $\left(\mathbb{C}^* - \left\{q^{\mathbb{Z}}\right\}\right)$. Then

$$g^{(0)}(z) = 1,$$
$$g^{(1)}(z) = \pi i + \frac{2\pi i}{z-1} - (2\pi i)\sum_{n=1}^{\infty}\sum_{n|d} d\left(z^{\frac{n}{d}} - z^{\frac{-n}{d}}\right)q^n,$$
$$g^{(l)}(z) = -\frac{(2\pi i)^l}{l!}\left(\sum_{n=1}^{\infty}\left(\sum_{n|d} l d^l \left(z^{\frac{n}{d}} + (-1)^l z^{\frac{-n}{d}}\right)\right)q^n\right) + \frac{(2\pi i)^l B_l}{l!}, \text{ for } l > 1. \tag{3.15}$$

Lemma 3.3.3. *The functions $g^{(l)}$ for $l \geq 0$ can be written as power series on q where the coefficients are rational function on $\mathbb{C}$ with poles on $0, 1$ of the form $\frac{p_1}{p_2}$, where p_i are polynomials over the field $\mathbb{Q}(2\pi i)$ for $i = 1, 2$.*

Given a subfield $\mathbb{Q} \subset \Bbbk \subset \mathbb{C}$, let $D \subset \mathbb{C}^n$ be a normal crossing divisor. We denote by $\mathrm{Rat}^0_\Bbbk(\mathbb{C}^n, D)$ the algebra of rational functions $\frac{p_1}{p_2}$ with poles along D such that p_1, p_2 are polynomials over the field $\Bbbk$. We denote by $\mathrm{Rat}^\bullet_\Bbbk(\mathbb{C}^n, D)$ the differential graded $\Bbbk$-subalgebra of differential forms generated by forms of type $f dx_I$, with $f \in \mathrm{Rat}^0_\Bbbk(\mathbb{C}^n, D)$. In particular $\mathrm{Rat}^\bullet_\Bbbk(\mathbb{C}^n, D) \otimes \mathbb{C} \subset A^*_{DR}(\mathbb{C}^n - D)$.

We consider the differential graded $\mathbb{Q}(2\pi i)$-algebra $\mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})$ and we consider q as a formal variable of degree zero. The functions $g^{(i)}(z)$, $i \geq 0$, as defined in (3.15), are elements of $\mathrm{Rat}^0_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})((q))$ and we denote it by $\underline{g^{(i)}}$. We have a differential graded algebra of formal Laurent series

$$\left(d, \wedge, \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right)\right).$$

We extend the action of $\mathbb{Z}$ defined in (3.14) extend to a map

$$\rho^c : \left(d, \wedge, \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right)\right) \to \coprod_{\mathbb{Z}} \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right).$$

Since $\coprod_{\mathbb{Z}} \mathrm{Rat}^{\bullet}_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left((q)\right) = \mathrm{Map}\left(\mathbb{Z}, \mathrm{Rat}^{\bullet}_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left((q)\right)\right)$ we define ρ^c via

$$\rho^c\left(q\right)\left(n\right) := q, \quad \rho^c\left(dz\right)\left(n\right) := q^n dz, \quad \rho^c\left(\frac{1}{z}\right)\left(n\right) := \frac{q^{-n}}{z}$$

and

$$\rho^c\left(\frac{1}{z-1}\right)\left(n\right) := \begin{cases} \sum_{l=0}^{\infty}\left(q^n z\right)^l & n>0 \\ \sum_{l=0}^{\infty}\frac{q^{-n}}{z}\left(\frac{1}{z^l}\right)^l & n<0. \end{cases}$$

Let

$$i \,:\, \mathrm{Rat}^{\bullet}_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left((q)\right) \hookrightarrow \coprod_{\mathbb{Z}} \mathrm{Rat}^{\bullet}_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left((q)\right)$$

be the map sending $a \mapsto (0,a)$. Hence (ρ^c, i) are the two cofaces map of a 1-truncated cosimplicial unital differential graded algebra. The conerve gives rise to a cosimplicial unital differential commutative graded algebra $A^{\bullet,\bullet}$, where

$$A^{l,m} := \coprod_{\mathbb{Z}^l} \mathrm{Rat}^{\bullet}_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left((q)\right) = \mathrm{Map}\left(\mathbb{Z}^l, \mathrm{Rat}^{m}_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left((q)\right)\right).$$

The differential graded $\mathbb{Q}(2\pi i)$-module $\mathrm{Tot}_N\left(A^{\bullet,\bullet}\right)$ carries a unital C_∞-structure $m_\bullet$ (see Theorem 2.2.2). We set $D := m_1$. Let $\underline{\gamma} \in A^{1,0}$ be the group homomorphism $\gamma : (\mathbb{Z},+) \to (\mathbb{C},+)$ defined by $\underline{\gamma}(n) := 2\pi i n$. We define

$$\underline{\phi}^{(i)} := \underline{g}^{(i)}\frac{dz}{z}$$

for any $i \geq 0$. Hence $\underline{\phi}^{(0)}$, $\underline{\gamma}$ are again closed elements in $\mathrm{Tot}_N\left(A^{\bullet,\bullet}\right)$. Let $\underline{B} \subset \mathrm{Tot}_N\left(A^{\bullet,\bullet}\right)$ be the sub C_∞-algebra generated by

$$1, \underline{\gamma}, \left\{\underline{\phi}^{(i)}\right\}_{i\in\mathbb{N}}.$$

Since all the calculations done for B in the previous section are independent from the choice of $\tau \in \mathbb{H}$, we get that mutatis mutandis some of the results of Lemma 3.1.2 work in formal power series context. In particular there is a strict C_∞-map

$$f^B \,:\, \overline{B} \otimes \mathbb{C} \to B.$$

Proposition 3.3.4. *Let $(m_\bullet, \underline{B}) \subset (m_\bullet, \mathrm{Tot}_N\left(A^{\bullet,\bullet}\right))$ be as above.*

1. $D\left(-\underline{\phi}^{(n)}\right) = \sum_{l=1}^{n} m_{l+1}(\underline{\gamma}, \ldots, \underline{\gamma}, \underline{\phi}^{(p-l)})$*, for any n.*
2. $(m_\bullet, \underline{B}) \subset (m_\bullet, \mathrm{Tot}_N\left(A^{\bullet,\bullet}\right))$ *is a rational sub C_∞-algebra over $\mathbb{Q}(2\pi i)$ where*
$$H^1(\underline{B}, D) = \left(\mathbb{Q}(2\pi i)\right)^2, \text{ and } H^2(\underline{B}, D) = 0.$$
3. *We have*
$$\underline{B}^{0,1} \subset \mathrm{Rat}^1_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left[[q]\right], \quad \underline{B}^{1,1} \subset \mathrm{Map}\left(\mathbb{Z}, \mathrm{Rat}^1_{\mathbb{Q}(2\pi i)}(\mathbb{C},\{0,1\})\left[[q]\right]\right)$$
4. *There exists a $\mathbb{Q}(2\pi i)$-vector space decomposition*

   (3.16) $$\underline{B} = W^\bullet \oplus D\mathcal{M} \oplus \mathcal{M}$$

   which is a Hodge decomposition where
   (a) *W^1 is generated by $-\underline{\phi}^{(0)}$ and $-\underline{\gamma}$, $\mathcal{M}^1$ is generated by $\underline{\phi}^{(i)}$ for $i>0$, and $(D\mathcal{M})^1 = 0$,*
   (b) *and $W^2 = 0$, $\mathcal{M}^2$ is generated by $m_l\left(\underline{\gamma}, \ldots, \underline{\gamma}, \underline{\phi}^{(i)}\right)$ for $i>0, l>2$, and $(D\mathcal{M})^1$ is generated by $D\left(\underline{\phi}^{(i)}\right)$ for $i>1$.*

   We denote the homological pair associated to the decomposition above by $(C_{Ell}, \left(\delta^W\right)^*)$.

5. *We define* $\left(W^1[1]\right)^*$ *as the vector space generated by* Y_0, Y_1*, where* Y_1*, and* Y_0 *resp. denote the dual of* $-s\underline{\phi}^{(0)}$ *and of* $-s\underline{\gamma}$ *respectively. The degree zero connection associated to the decomposition* (3.16) *is*

$$(C_{Ell})_0 = -\underline{\gamma} Y_0 - \sum_{p\geq 0} \underline{\phi}^{(p)} \operatorname{Ad}^p_{Y_0}(Y_1)$$

Let $\underline{F}$ *be denote the Kronecker function considered as a formal power series in* q*. We have*

$$r^* \left(C_{Ell}\right)_0 = -\sum_{p\geq 0} \underline{\phi}^{(p)} \operatorname{Ad}^p_{Y_0}(Y_1) = -\operatorname{ad}_{Y_0} \underline{F}\left(\xi, \operatorname{ad}_{Y_0}\right)(Y_1) d\xi$$

In particular $\left(f^B\right)_* r^* \left(C_{Ell}\right)_0 = \omega_{KZB,1}$.

Proof. Since the functions defined in (3.15) satisfy (3.1), the proof of Lemma 3.1.2 carries over the situation above. This proves 1, 2 and 4. The statement 3 and 5 follow from (3.15) and (3.13) respectively. □

We consider $\operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right)$ as the trivial cosimplicial module. Notice that there is a morphism of cosimplicial differential graded modules

$$i \ : \ A^{\bullet,\bullet} \to \operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right)$$

given as follows: for $f \in A^{l,m}$, $i(f) := f(0,\ldots,0) \in \operatorname{Rat}^m_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right)$. It induces a strict morphism of C_∞-algebras

$$i_\bullet \ : \ \operatorname{Tot}_N\left(A^{\bullet,\bullet}\right) \to \operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\left((q)\right)$$

In particular, $i(B)$ is the commutative differential graded sub algebra of $\operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\,[[q]]$ generated by

$$1, \left\{\underline{\phi}^{(i)}\right\}_{i\in\mathbb{N}}.$$

Note that $i(B^n) = 0$ for $n \geq 2$. Let I_q be the completion of the augmentation ideal of $\operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\,[[q]]$, let $\pi' \ : \ \operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})\,[[q]] \to \operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})$ be the quotient with respect to I_q.

Lemma 3.3.5. *The map*

$$p' := \pi' \circ i \ : \ \underline{B} \to \operatorname{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}, \{0,1\})$$

is a strict C_∞*-algebra morphism such that* $p'(\underline{\gamma}) = 0$ *and*

$$p'(\underline{\phi^{(0)}}) = \frac{dz}{2\pi i z}, \quad p'(\underline{\phi^{(1)}}) = \frac{dz}{2z} + \frac{dz}{z(z-1)}, \quad p'(\underline{\phi^{(l)}}) = \frac{(2\pi i)^{l-1} B_l}{l!}\frac{dz}{z} \text{ for } l > 1.$$

We consider the complex variety $\mathbb{C} - \{0,1\}$. The cohomology is generated by the holomorphic forms

$$\frac{dz}{z}, \quad \frac{dz}{z-1}.$$

Let A be the unital differential graded algebra over $\mathbb{Q}(2\pi i)$ generated by $1, \frac{dz}{z}, \frac{dz}{z-1}$. The inclusion $A \hookrightarrow H^\bullet(A)$ give a quasi-isomorphism, where the latter is considered as a differential graded algebra with vanishing differential. Hence A is a formal differential graded algebra. In particular, there is a canonical Hodge space decomposition for A given by $(A, 0)$, where A^1 is considered to equipped with the basis $\frac{dz}{z}, \frac{dz}{z-1}$. The homological pair $(C_{KZ}, \left(\delta^{W'}\right)^*)$ associated to the decomposition above is

$$C_{KZ} := \frac{dz}{z} Z_0 + \frac{dz}{z-1} Z_0, \quad \delta^{W'} = 0$$

where $Z_0 := \left(s\left(\frac{dz}{z}\right)\right)^*$ and $Z_1 := \left(s\left(\frac{dz}{z-1}\right)\right)^*$. The Maurer-Cartan element C_{KZ} correspond to the strict C_∞-isomorphism $g^{KZ}_\bullet \ : \ A \to W'^\bullet$. In particular there is a differential graded algebra map

$$f^A \ : \ A \otimes \mathbb{C} \to A_{KZ,1}$$

such that $\left(f^A\right)_* C_{KZ} = \omega_{KZ,1}$, where $A_{KZ,1}$ is the complex differential graded algebra defined in Section 1.3.1. We apply the argument of Subsection 2.1.5. We have a diagram of C_∞-algebras

$$\begin{array}{ccc} \underline{B} & \xrightarrow{p'} & A \\ {\scriptstyle g_\bullet}\uparrow & & \downarrow{\scriptstyle f^{KZ}_\bullet} \\ (W^\bullet, m^W_\bullet) & & (W'^\bullet, m^{W'}_\bullet) \end{array}$$

We define $q_\bullet := f^{KZ}_\bullet \circ p' \circ g_\bullet$, where $g_\bullet$ is the C_∞-morphism that corresponds to the Maurer-Cartan element C_{Ell} of Proposition 3.3.4. We get the following diagram $(\mathcal{L}_\infty - \mathrm{ALG})_p$

$$\mathrm{Conv}\left(\left(W, m^W_\bullet\right), \left(B, m^B_\bullet\right)\right) \xrightarrow{p'_*} \mathrm{Conv}\left(\left(W, m^W_\bullet\right), \left(A, m^B_\bullet\right)\right) \xleftarrow[q^*]{} \mathrm{Conv}\left(\left(W', 0\right), \left(A, m^A_\bullet\right)\right).$$

Since A is formal, the discussion done in Subsection 2.1.5 implies that $(p')^*(C_{Ell}) = q_*(C_{KZ})$. Consider the differential graded coalgebra morphsim $Q : T^c(W'[1]) \to T^c(W[1])$ that correspond to $q_\bullet$. Consider the dual Q^* restricted to the degree-zero elements. We are working with C_∞-algebras, then we can apply Corollary 2.1.22. This shows that Q^* is a complete morphism of $\mathbb{Q}(2\pi i)$-Lie algebras

$$Q^* : \widehat{\mathbb{L}}(W'[1]) \to \widehat{\mathbb{L}}\left(W^1[1]\right)$$

which is an inclusion since both of the Lie algebras are free. On the other hand

$$\begin{aligned} (p')^*(C_{Ell}) &= -\frac{dz}{2\pi i z} Y_1 + \left(\frac{dz}{2z} + \frac{dz}{z(z-1)}\right)[Y_0, Y_1] - \sum_{l>1}^{\infty} \frac{(2\pi i)^{l-1} B_l}{l!} \frac{dz}{z} \mathrm{Ad}^l_{Y_0}(Y_1) \\ &= -\frac{dz}{z}\frac{Y_1}{2\pi i} - \left(+\frac{dz}{2z} - \frac{dz}{z-1}\right)\left[2\pi i Y_0, \frac{Y_1}{2\pi i}\right] + \sum_{l>1}^{\infty} \frac{B_l}{l!}\frac{dz}{z} \mathrm{Ad}^l_{2\pi i Y_0}\left(\frac{Y_1}{2\pi i}\right) \\ &= -\frac{dz}{z-1}\left[2\pi i Y_0, \frac{Y_1}{2\pi i}\right] - \sum_{l=0}^{\infty} \frac{B'_l}{l!}\frac{dz}{z} \mathrm{Ad}^l_{2\pi i Y_0}\left(\frac{Y_1}{2\pi i}\right) \\ &= q_*(C_{KZ}) = (Id \otimes Q^*)\, C_{KZ}. \end{aligned}$$

Hence Q^* is given by

$$Z_0 \mapsto -\sum_{l=0}^{\infty} \frac{B'_l}{l!}\frac{dz}{z} \mathrm{Ad}^l_{2\pi i Y_0}\left(\frac{Y_1}{2\pi i}\right), \quad Z_1 \mapsto -\left[2\pi i Y_0, \frac{Y_1}{2\pi i}\right]$$

which is the map found by Hain in [29] (see Section 18).

Proposition 3.3.6. *1. The map p induces a Lie algebra morphism $Q^* : \widehat{\mathbb{L}}(W'[1]) \to \widehat{\mathbb{L}}\left(W^1[1]\right)$ which induces a differential graded Lie algebra map*

$$q_* = (Id \otimes Q^*) : (A') \otimes \widehat{\mathbb{L}}(W'[1]) \to A' \widehat{\otimes} \widehat{\mathbb{L}}\left(W^1[1]\right)$$

for any differential graded algebra A'.

2. Let $\tau \in \mathbb{H}$ we denote the universal KZB-connection with $C_{Ell,\tau}$. We have

$$\lim_{\tau \to i\infty} \omega^\tau_{KZB,1} = (Id \otimes Q)\, \omega_{KZ,1}$$

Proof. Point 1. is proved above. The second part follows from

$$\begin{aligned} \lim_{\tau \to i\infty} \omega^\tau_{KZB,1} &= \lim_{\tau \to i\infty} \left(f^B\right)^* C_{Ell} \\ &= \left(f^B\right)^* p'_* C_{Ell} \\ &= \left(f^B\right)^* q_* C_{KZ} \\ &= \left(f^A\right)^* q_* C_{KZ} \\ &= q_* \omega_{KZ,1}. \end{aligned}$$

where the third equality is a consequence of $\lim_{\tau \to i\infty} q = 0$. □

3.4 A rational Maurer-Cartan element

Let τ be a fixed element of $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$. We have considered the punctured elliptic $\mathcal{E}_\tau^\times$ curve as the quotient $(\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})/\mathbb{Z}^2$. Equivalently it can be written as the solution set $\mathcal{E}_{\tau,\mathrm{alg}}^\times$ of

$$y^2 = x(x-1)(x-\lambda)$$

on $\mathbb{P}^2$ minus the point at infinity. We set

$$e_1 := \wp\left(\frac{1}{2}, \tau\right),\ e_2 := \wp\left(\frac{\tau}{2}, \tau\right),\ \text{and } e_3 := \wp\left(\frac{1+\tau}{2}, \tau\right)$$

The isomorphism between $\mathcal{E}_{\tau,\mathrm{alg}}^\times$ and $\mathcal{E}_\tau^\times$ is given by

$$x = \frac{\wp(\xi,\tau) - e_1}{e_2 - e_1}, \quad y = \frac{\wp'(\xi,\tau)}{(e_2 - e_1)^{\frac{1}{2}}}, \quad \lambda = j(\tau),$$

where j is the elliptic j-function. The two regular algebraic differential forms

$$\frac{dx}{y}, \quad \frac{xdx}{y}$$

are well-defined and they generate the cohomology of $\mathcal{E}_{\tau,\mathrm{alg}}^\times$. Their pullback gives two holomorphic 1-forms

$$2(e_2 - e_1)^{\frac{1}{2}} d\xi, \quad \frac{2\wp(\xi,\tau) - 2e_1}{(e_2 - e_1)^{\frac{1}{2}}} d\xi$$

We denote by $A = \mathbb{C} \oplus \mathbb{C}d\xi \oplus \mathbb{C}\wp(\xi,\tau)d\xi$ the unital differential graded sub algebra generated by $d\xi$ and $\wp(\xi,\tau)d\xi$. We equipped this differential graded algebra with the obvious Hodge decomposition $(W', 0)$, where $W' = A$. We consider W'^1 equipped with the basis $2(e_2 - e_1)^{\frac{1}{2}} d\xi, \frac{2\wp(\xi,\tau)-2e_1}{(e_2-e_1)^{\frac{1}{2}}} d\xi$. Let $\mathbb{L}(W'^*_+[1])$ be the free Lie algebra with degree zero generator T_1 and T_2. The degree zero geometric connection associated to A is given by

$$C_\wp^0 := 2(e_2 - e_1)^{\frac{1}{2}} d\xi T_0 + \frac{2\wp(\xi,\tau) - 2e_1}{(e_2 - e_1)^{\frac{1}{2}}} d\xi T_1 \in A \otimes \mathbb{L}(W'^*_+[1])$$

We denote the Weierstrass zeta function by $\zeta(\xi,\tau)$. It satisfies

$$\begin{aligned}
\frac{\partial}{\partial \xi}\zeta(\xi,\tau) &= -\wp(\xi,\tau), \\
\zeta(\xi+1,\tau) &= \zeta(\xi,\tau) + 2\eta_1, \quad \eta_1 = \zeta\left(\frac{1}{2},\tau\right), \\
\zeta(\xi+\tau,\tau) &= \zeta(\xi,\tau) + 2\eta_2, \quad \eta_2 = \zeta\left(\frac{\tau}{2},\tau\right),
\end{aligned}$$

We have

$$\begin{aligned}
D\left(\zeta(\xi,\tau) - 2\eta_1\xi\right) &= \wp(\xi,\tau)\, d\xi - 2\eta_1 ds + \gamma\frac{(2\eta_1\tau - 2\eta_2)}{2\pi i} \\
&= \wp(\xi,\tau)\, d\xi - 2\eta_1 ds + \gamma
\end{aligned}$$

since $2\eta_1\tau - 2\eta_2 = 2\pi i$. Let $(B, m_\bullet)$ be the 1-model constructed in Lemma 3.1.2 and equipped with the decomposition $(W, \mathcal{M})$ as in (3.5). We consider W equipped with the basis $-\gamma$ and $-d\xi$. Since

$$A, B \subset \mathrm{Tot}_N^\bullet\left(A_{DR}(\mathbb{C} - \{\mathbb{Z} + \tau\mathbb{Z}\})_\bullet \mathbb{Z}^2\right),$$

we have

$$\begin{aligned}
[d\xi] &= \frac{1}{2(e_2 - e_1)^{\frac{1}{2}}}\left[2(e_2 - e_1)^{\frac{1}{2}} d\xi\right] \\
[\gamma] &= \frac{-(e_2 - e_1)^{\frac{1}{2}}}{2}\left[\frac{2\wp(\xi,\tau) - 2e_1}{(e_2 - e_1)^{\frac{1}{2}}}\right] + \frac{2\eta_1 - e_1}{2(e_2 - e_1)^{\frac{1}{2}}}\left[2(e_2 - e_1)^{\frac{1}{2}}\right]
\end{aligned}$$

in $H^1\left(\mathrm{Tot}_N^\bullet\left(A_{DR}\left(\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\}\right)_\bullet\mathbb{Z}^2\right),D\right)$. Consider the Lie algebra morphism $K_1^*:\widehat{\mathbb{L}}\left(({W'}^1_+[1])^*\right)\to\widehat{\mathbb{L}}\left((W^1_+[1])^*\right)$ induced by

$$T_0\mapsto -\frac{1}{2(e_2-e_1)^{\frac{1}{2}}}Y_0+\frac{2\eta_1-e_1}{2(e_2-e_1)^{\frac{1}{2}}}Y_1,\quad T_1\mapsto\frac{(e_2-e_1)^{\frac{1}{2}}}{2}Y_1.$$

Notice that $C^0_\wp$ composed with K^* becomes equal to

$$d\xi Y_0+\left(2\eta_1-\wp\left(\xi,\tau\right)d\xi\right)Y_1\in A^1_{DR}\left(\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\}\right)\widehat{\otimes}\widehat{\mathbb{L}}\left((W^1_+[1])^*\right).$$

By Corollary 2.3.16 we have the following.

Proposition 3.4.1. *There exists an isomorphism of Lie algebras* $K^*=K_1^*+\sum_{i=2}^\infty K_i^*$ *such that* $d-K^*C^0_\wp=d-d\xi Y_0-\left(2\eta_1-\wp\left(\xi,\tau\right)d\xi\right)Y_1-\sum_{i=2}^\infty K_i^*C^0_\wp$ *is gauge equivalent to* $d-\omega_{KZB,1}$.

Chapter 4

The universal KZB connection on the configuration space of points of the punctured elliptic curve

4.1 A connection on the configuration spaces of the punctured torus

For a topological space X we define its configuration spaces as

$$\mathrm{Conf}_n(X) := \{(x_1, \dots, x_n) \in X^n \; : \; x_i \neq x_j \text{ for } i \neq j\}$$

In [9] it is constructed a differential graded algebra A_n which is a model for the differential graded algebra of complex smooth differential forms on $\mathrm{Conf}_n(\mathcal{E}_\tau^\times)$, moreover such a model can be constructed for any $\tau \in \mathbb{H}$ and it gives a family of differential graded algebra parametrized by $\tau \in \mathbb{H}$. In this chapter we construct a 1-extension B_n for A_n equipped with a compatible Hodge type decomposition and we calculate its degree zero geometric connection C_0 (see 2.3.4). We show that $r_* C_0$ corresponds to family of smooth connections (parametrized by $[0,1]$) gauge equivalent to the KZB connection (see Definition 1.3.5). In the last subsection we investigate the relation between the KZB connection and the KZ connection. In particular we construct a Lie algebra morphism between the fibers of the two connections and we give an n-dimensional version of Proposition 3.3.6.

4.1.1 Kronecker function

Let τ be a fixed element of the upper complex plane $\mathbb{H}$. Let $\mathbb{Z} + \tau\mathbb{Z}$ be the lattice spanned by $1, \tau$. Let $(\xi_1, \dots \xi_n)$ be the coordinates on $\mathbb{C}^n$. We define $\xi_0 := 0$ and for $i = 1, \dots, n$ we define r_i, s_i via $\xi_i = s_i + \tau r_i$. We define $\mathcal{D} \subset \mathbb{C}^n$ as

$$\mathcal{D} := \{(\xi_1, \dots \xi_n) \; : \; \xi_i - \xi_j \in \mathbb{Z} + \tau\mathbb{Z} \text{ for some distinct } i, j = 0, \dots n\}$$

We define a $\mathbb{Z}^{2n}$-action on $\mathbb{C}^n$ via translation, i.e.

$$((l_1, m_1), \dots, (l_n, m_n))(\xi_1, \dots \xi_n) := (\xi_1 + l_1 + m_1\tau, \dots, \xi_n + l_n + m_n\tau)$$

Notice that $\mathcal{D}$ is preserved by the action of $\mathbb{Z}^{2n}$. There is a canonical isomorphism

$$(\mathbb{C}^n - \mathcal{D}) / \left(\mathbb{Z}^{2n}\right) \cong \mathrm{Conf}_n\left(\mathcal{E}_\tau^\times\right)$$

since the action is free and properly discontinuous. We denote the action groupoid by

$$(\mathbb{C}^n - \mathcal{D})_\bullet \left(\mathbb{Z}^{2n}\right),$$

it is a simplicial manifold equipped with a simplicial normal crossing divisor (see Section 2.3). Its de Rham complex $A^\bullet_{DR}((\mathbb{C}^n - \mathcal{D})_\bullet(\mathbb{Z}^{2n}))$ is a cosimplicial commutative (non-negatively graded) differential graded algebra. By the simplicial de Rham theorem and the discussion above we have

$$H^\bullet\left(\mathrm{Tot}_N\left(A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right)_\bullet\left(\mathbb{Z}^{2n}\right)\right)\right) \cong H^\bullet\left(\left(\mathbb{C}^n - \mathcal{D}\right)/\left(\mathbb{Z}^{2n}\right), \mathbb{C}\right) \cong H^\bullet\left(\mathrm{Conf}_n\left(\mathcal{E}^\times_\tau\right), \mathbb{C}\right).$$

The normalized total complex of a cosimplicial commutative algebras carries a natural C_∞-structure $m_\bullet$ given by Theorem 2.2.6 .

Proposition 4.1.1. *Let $F(\xi,\eta,\tau)$ be the Kronecher function (see Definition 1.3.4), where η is a formal variable, $F(\xi,\eta,\tau)$ satisfies the* Fay*'s identity, i.e.*

$$\begin{aligned} F(\xi_1,\eta_1,\tau)F(\xi_2,\eta_2,\tau) &= F(\xi_1,\eta_1+\eta_2,\tau)F(\xi_1-\xi_2,\eta_2,\tau) \\ &+ F(\xi_2,\eta_1+\eta_2,\tau)F(\xi_1-\xi_2,\eta_1,\tau). \end{aligned} \tag{4.1}$$

Proof. It follows from standard properties of theta series. ☐

We define the 1-forms $\phi^{(k)}_{i,j}$ for $k \geq 0$ $i,j = 0,1,\ldots,n$ as follows. Let α be a formal variable, then

$$\sum_{k\geq 0}\phi^{(k)}_{i,j}\alpha^k := \alpha F(\xi_i-\xi_j,\alpha,\tau)d(\xi_i-\xi_j).$$

Thanks to the Fourier expansion 3.13

$$\phi^{(0)}_{i,j} = d\xi_i - d\xi_j,$$

$$\phi^{(1)}_{i,j} = \left(\pi i + \frac{2\pi i}{e^{2\pi i(\xi_i-\xi_j)}-1} - (2\pi i)^2\sum_{n=1}^{\infty}\sum_{n|d} d\left(e^{2\pi i\frac{n}{d}(\xi_i-\xi_j)} - e^{-2\pi i\frac{n}{d}(\xi_i-\xi_j)}\right)q^n\right)d(\xi_i-\xi_j),$$

$$\phi^{(l)}_{i,j} = -\left(\frac{(2\pi i)^{l+1}}{l!}\sum_{n=1}^{\infty}\left(\sum_{n|d} d^l\left(e^{2\pi i\frac{n}{d}(\xi_i-\xi_j)} + (-1)^l e^{-2\pi i\frac{n}{d}(\xi_i-\xi_j)}\right)\right)q^n + \frac{B_l}{2\pi i}\right)d(\xi_i-\xi_j),\ \text{for } l>1.$$

i.e. $\phi^{(l)}_{i,j}$ are holomorphic 1-forms for $l \neq 1$ and for $l = 1$ the are meromorphic with a pole of order 1 along the hyperplane $\xi_i = \xi_j$. Let $\Omega(1)$ be the differential graded algebra of polynomials forms on the 1 dimensional simplex with coordinate $0 < u < 1$. We define a parametrized 1-form $\Omega_u(\xi,\alpha) := \exp(2\pi i u r\alpha)F(\xi,\alpha,\tau)d\xi$ on $\mathbb{C} - \{\mathbb{Z}+\mathbb{Z}\tau\}$. For $0 \leq i \leq j \leq n$, we define the $w(u)^{(k)}_{i,j} \in A^1_{DR}(\mathbb{C}^n - \mathcal{D}) \otimes \Omega^0(1)$ as

$$\Omega_u(\xi_i-\xi_j,\alpha) := \sum_{k\geq 0} w(u)^{(k)}_{i,j}\alpha^{k-1}. \tag{4.2}$$

Notice that they are smooth for any $0 \leq u \leq 1$ and they are holomorphic for $u = 0$, in particular $w(0)^{(k)}_{i,j} = \phi^{(k)}_{i,j}$ for any i,j distinct. Thanks to the discussion above we get that they are u-valued smooth forms on $\mathbb{C}^n$ with logarithmic singularities along $\mathcal{D}$. The quasi-periodicity of F implies that the pullback of the action of $\mathbb{Z}^{2n}$ is

$$w(u)^{(k)}_{i,j}(\xi+l_1+m_1\tau,\ldots,\xi_n+l_n+\tau m_n) = \sum_{p=0}^{k} w(u)^{(k-p)}_{i,j}\frac{2\pi i(u-1)(l_i-l_j)^p}{p!}. \tag{4.3}$$

The Fay identity gives the following quadratic relations between the $w(u)^{(k)}_{i,j}$

$$\begin{aligned} &\Omega_u(\xi_i-\xi_l,\alpha)\wedge\Omega_u(\xi_j-\xi_l,\beta) + \Omega_u(\xi_i-\xi_i,\beta)\wedge\Omega_u(\xi_i-\xi_l,\alpha+\beta) \\ &+ \Omega_u(\xi_j-\xi_l,\alpha+\beta)\wedge\Omega_u(\xi_i-\xi_j,\alpha) = 0. \end{aligned} \tag{4.4}$$

For distinct indices we have

$$\begin{aligned} w(u)^{a-1}_{i,l}w(u)^{b}_{j,l} \quad &+ \quad w(u)^{(a)}_{i,j}w(u)^{(b-1)}_{j,l} + \sum_{m=0}^{b}\binom{a+b-1-m}{a-1}w(u)^{(m)}_{j,i}w(u)^{(a+b-1-m)}_{i,l} \\ &+ \quad \sum_{k=0}^{a}\binom{a+b-1-k}{b-1}w(u)^{(a+b-1-k)}_{j,l}w(u)^{(k)}_{i,j} = 0 \end{aligned}$$

The function $F(\xi, \alpha, \tau)$ satisfies $F(\xi, \alpha, \tau) = -F(-\xi, -\alpha, \tau)$. This implies

$$w(u)^{(k)}_{i,j} + (-1)^k w(u)^{(k)}_{j,i} = 0. \tag{4.5}$$

On the other hand we have

$$w(u)^{(0)}_{i,j} = w(u)^{(0)}_{i,0} - w(u)^{(0)}_{j,0}. \tag{4.6}$$

For $i = 0, \ldots n$ we define the group homomorphism

$$\gamma(u)_i \, : \, \mathbb{Z}^{2n} \to \left(A^0_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega^0(1), +\right)$$

via $\gamma(u)_0 := 0$ and for $j \neq 0$ via $\gamma(u)_j\left(\left(l_1, m_1\right), \ldots, \left(l_n, m_n\right)\right) := 2\pi i m_j \otimes (1-u)$. We define $\gamma(u)_{i,j} := \gamma(u)_i - \gamma(u)_j$. For $i = 0, \ldots, n$ we define $\nu(u)_i \in A^1_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega^0(1)$ via $\nu(u)_0 := 0$ and $\nu(u)_i := 2\pi i dr_i \otimes u$ for $i \neq 0$. We set $\nu(u)_{i,j} := \nu(u)_i - \nu(u)_j$. Finally, for $i = 0, \ldots, n$ we define $\beta(u)_i \in A^0_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega^1(1)$ via $\beta(u)_0 := 0$ and $\beta(u)_i := 2\pi i r_i \otimes du$ for $i \neq 0$. We set $\beta(u)_{i,j} := \beta(u)_i - \beta(u)_j$. We denote by d the differential of $A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega^0(1)$.

Lemma 4.1.2. *We have the following relations:* $d\left(\nu(u)_{i,j} + \beta(u)_{i,j}\right) = dw(u)^{(0)}_{i,j} = 0$ *for* $0 \leq i \leq j \leq n$. *For* $k > 0$ *we have*

$$dw(u)^{(k)}_{i,j} = \left(\nu(u)_{i,j} + \beta(u)_{i,j}\right) w(u)^{(k-1)}_{i,j}.$$

Proof. It follows by $d\Omega_u\left(\xi_i - \xi_j, \alpha\right) = \left(\nu(u)_{i,j}\alpha + \beta(u)_{i,j}\alpha\right)\Omega_u\left(\xi_i - \xi_j, \alpha\right)$. □

For a module A over a ring $\mathbb{k}$ and a group G, we denote by $\mathrm{Map}\left(G, A\right)$ the $\mathbb{k}$ module of maps from G to A. An *action of* G *on* A is a morphism of module $\rho^c \, : \, A \to \mathrm{Map}\left(G, A\right)$ such that $\rho^c(a)(e) = a$ for any a where e is the identity, and $\rho^c(a)(gh) = \rho^c\left(\rho^c(a)(g)\right)(h)$, for any $g, h \in G$ and nay a. The tensor product of two actions of G on modules A and B is naturally an action of G on $A \otimes B$. We call the *trivial action* the action given by $\rho^c(a)(g) = a$ for any $g \in G$. The action of $\mathbb{Z}^{2n}$ on $\mathbb{C}^n - \mathcal{D}$ induces an action on $A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right)$ We consider the differential graded algebra $A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right)$. We put the trivial $\mathbb{Z}^{2n}$-action on $\Omega(1)$. We denote the resulting action by $\rho^c \, : \, A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega(1) \to \mathrm{Map}\left(\mathbb{Z}^{2n}, A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega(1)\right)$.

Lemma 4.1.3. *Let* $g = \left(\left(l_1, m_1\right), \ldots, \left(l_n, m_n\right)\right), g' = \left(\left(l'_1, m'_1\right), \ldots, \left(l'_n, m'_n\right)\right) \in \mathbb{Z}^{2n}$. *For* $0 \leq i \leq j \leq n$ *we have*

$$\rho^c(\gamma(u)_{i,j}(g'))(g) = \gamma(u)_{i,j}(g'), \quad \rho^c\left(\nu(u)_{i,j}\right)(g) = \nu(u)_{i,j},$$

$\rho^c\left(\beta(u)_{i,j}\right)(g) = -d\gamma(u)_{i,j}(g) + \beta(u)_{i,j}$ *and*

$$\rho^c\left(w(u)^{(k)}_{i,j}\right)(g) = \sum_{p=0}^{k} w(u)^{(k-p)}_{i,j} \frac{\left(-\gamma(u)_{i,j}(g)\right)^p}{p!}.$$

Moreover the action preserves the differential and the wedge product, i.e. $\rho^c(da)(g) = d\left(\rho^c(a)(g)\right)$, $\rho^c(ab)(g) = \left(\rho^c(a)(g)\right)\left(\rho^c(b)(g)\right)$ *for any* $a, b \in A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \otimes \Omega(1)$ *and* $g \in \mathbb{Z}^{2n}$.

Proof. The first two and the last identities are immediate. The third one follows from the shifting property of F. □

4.1.2 The 1-extension B_n

We define the C_∞-algebras A_n, B_n, A'_n and B'_n. They are rational C_∞-algebras, but we consider them as C_∞-algebras over $\mathbb{C}$. In [9], a model A_n for the configuration space of points of the punctured elliptic curve is constructed. We extend the ideas of [9] for C_∞-algebras.

Definition 4.1.4. Let A_n be the complex unital commutative graded algebra generated by the degree 1 symbols $w(1)^{(k)}_{i,j}$, $\nu(1)_i$, for $k \geq 0$ $i, j = 0, 1, \ldots, n$ modulo the relations (4.4), (4.5), (4.6). We define a differential d via $dw(1)^{(0)}_{i,j} = d\nu(1)_i = 0$ and

$$dw(1)^{(k)}_{i,j} = \left(\nu(1)_i - \nu(1)_j\right) w(1)^{(k-1)}_{i,j}$$

for $k > 0$.

Notice that the elements $w(1)^{(k)}_{i,j}$, $\nu(1)_i$, for $k \geq 0$ $i, j = 0, 1, \ldots, n$ denote elements in $A_{DR}(\mathbb{C}^n - \mathcal{D})$ as well (see previous section). There is an obvious map $\psi^1 : A_n \to A_{DR}(\mathbb{C}^n - \mathcal{D})$ defined by

$$\psi^1\left(w(1)^{(k)}_{i,j}\right) := w(1)^{(k)}_{i,j}, \quad \psi^1(\nu(1)_i) := \nu(1)_i$$

for any i, j, k.

Theorem 4.1.5 ([9]). *The map $\psi^1 : A_n \to A_{DR}(\mathbb{C}^n - \mathcal{D})$ is an inclusion and a quasi-isomorphism.*

We define a parametrized version of A_n.

Definition 4.1.6. Let A'_n be the complex unital commutative graded algebra generates by

1. the degree 0 symbol $\tilde{\gamma}(u)$ and,
2. the degree 1 symbols $w(u)^{(k)}_{i,j}$, $\nu(u)_i$, $\beta(u)_i$ for $k \geq 0$ $i, j = 0, 1, \ldots, n$

modulo the relations (4.4), (4.5), (4.6) and such that

$$\beta(u)_i \beta(u)_j = 0, \quad \nu(u)_0 = \beta(u)_0 = 0$$

We denote $\beta(u)_{i,j} := \beta(u)_i - \beta(u)_j$ and $\nu(u)_{i,j} := \nu(u)_i - \nu(u)_j$. We define a differential d via the relations of Lemma 4.1.2. This makes A'_n a differential graded commutative algebra.

The notation of the generators is justified by the following fact.

Proposition 4.1.7. *The differential graded algebra map $\Psi : A'_n \to A_{DR}(\mathbb{C}^n - \mathcal{D}) \otimes \Omega(1)$ sending $w(u)^{(k)}_{i,j}$,$\nu(u)_{i,j}$ and $\beta(u)_{i,j}$ to the 1-forms represented by these symbols and $\tilde{\gamma}(u)$ to $2\pi i \otimes (1-u)$ is injective.*

Proof. We use some results of Section 4. of [9]. We consider the obvious map of commutative differential graded algebra $\Psi : A'_n \to A_{DR}(\mathbb{C}^n - \mathcal{D}) \otimes \Omega(1)$. In particular [9, Lemma 15] works as well and the same argument of [9, Corollary 16] implies that Ψ is injective. □

Let A'_n as above. For $i = 0, \ldots n$ we define the group homomorphism $\gamma(u)_i : \mathbb{Z}^{2n} \to A_n$ via $\gamma(u)_0 := 0$ and for $j \neq 0$ via $\gamma(u)_j((l_1, m_1), \ldots, (l_n, m_n)) := m_j \tilde{\gamma}(u)$. We define $\gamma(u)_{i,j} := \gamma(u)_i - \gamma(u)_j$.

Lemma 4.1.8. *We define a map $\rho^c : A'_n \to \mathrm{Map}\left(\mathbb{Z}^{2n}, A'_n\right)$ via the relations of Lemma 4.1.3, i.e*

$$\rho^c(\tilde{\gamma}(u))(g) := \tilde{\gamma}(u), \quad \rho^c(\nu(u)_{i,j})(g) := \nu(u)_{i,j}, \quad \rho^c(\beta(u)_{i,j})(g) := -d(\gamma(u)_{i,j}(g)) + \beta(u)_{i,j}$$

and

$$\rho^c\left(w(u)^{(k)}_{i,j}\right)(g) := \sum_{p=0}^{k} w(u)^{(k-p)}_{i,j} \frac{(-\gamma(u)_{i,j}(g))^p}{p!}.$$

1. *ρ^c is a $(\mathbb{Z}^{2n})$-action.*
2. *We have $d\rho^c(a)(g) = \rho^c(da)(g)$ and $\rho^c(ab)(g) = \rho^c(a)(g)\rho^c(b)(g)$ for any $a, b \in A'_n$.*
3. *Ψ respects the $\mathbb{Z}^{2n}$-action.*

Proof. The proof is a direct verification. □

The (co)nerve of the action defines a cosimplicial commutative differential graded algebra, we denote it by $A^{\bullet,\bullet}$. Concretely $A^{p,\bullet}$ is the differential graded algebra $\mathrm{Map}\left(\left(\mathbb{Z}^{2n}\right)^p, A'_n\right)$. The conormalization $N(A)^{\bullet,\bullet}$ is a bidifferential bigraded module where the second differential $\partial_{\mathbb{Z}^{2n}}$ is induced by the action. By Theorem 2.2.6, the differential graded module $\mathrm{Tot}_N(A)$ carries a natural C_∞-structure $m'_\bullet$. We define $m_n := (-1)^n m'_n$. It is a C_∞-structure on $\mathrm{Tot}_N(A)$.

Definition 4.1.9. We denote by $(B'_n, m_\bullet)$ the rational C_∞-subalgebra of $\mathrm{Tot}_N(A)$ generated by

$$w(u)^{(k)}_{i,j}, \quad \alpha(u)_{i,j} := \gamma(u)_{i,j} - \nu(u)_{i,j} - \beta(u)_{i,j},$$

for $k \geq 0$, $i, j = 0, 1, \ldots, n$.

Proposition 4.1.10. *Consider the differential graded algebra A_n. There is a strict C_∞-morphism $p^1 : B'_n \to A_n$ defined via*

$$p^1\left(w(u)^{(k)}_{i,j}\right) := w^{(k)}_{i,j}, \quad p^1\left(\alpha(u)_{i,j}\right) := -\nu_{i,j} \tag{4.7}$$

for any $0 \le i \le j \le n$ and $k \ge 0$.

Proof. We construct p^1 in a functorial way. First we consider the map $q : A'_n \to A_n$ defined via

$$q\left(w(u)^{(k)}_{i,j}\right) := w^{(k)}_{i,j}, \quad p^1\left(\nu_{i,j}\right) := \nu_{i,j}, \quad q\left(\beta_{i,j}\right) = 0$$

and $p^1(\tilde{\gamma}(u)) = 0$ for any $0 \le i \le j \le n$ and $k \ge 0$. Then q can be extended to a differential graded algebra map. Now consider A_n equipped with the trivial $\mathbb{Z}^{2n}$ action. The map q is $\mathbb{Z}^{2n}$ equivariant and it can be extended to a map between differential graded cosimplicial algebras $q : \mathrm{Map}\left(\mathbb{Z}^{2n}, A'_n\right) \to \mathrm{Map}\left(\mathbb{Z}^{2n}, A_n\right)$. Then q induces a map of C_∞-algebras

$$\left(\mathrm{Tot}_N\left(\mathbb{Z}^{2n}, A'_n\right), m'_\bullet\right) \to \left(\mathrm{Tot}_N\left(\mathbb{Z}^{2n}, A_n\right), m'_\bullet\right)$$

Moreover this map is strict. The normalized total complex of $\mathrm{Map}\left(\mathbb{Z}^{2n}, A_n\right)$ is A_n and its C_∞-structure corresponds to the ordinary differential graded algebra structure of A_n. In conclusion we have a strict C_∞-map

$$q : \left(\mathrm{Tot}_N\left(\mathbb{Z}^{2n}, A'_n\right), m_\bullet\right) \to (A_n, d, \wedge)$$

Then we set $p^1 := q|_{B'_n}$. □

Proposition 4.1.11. *Let $(B'_n, m_\bullet)$ as above.*

1. *The restriction of m_2 on A'_n coincides with the wedge product.*
2. *Let $x_1, \dots, x_l \in \left\{w(u)^{(k)}_{i,j}, \alpha(u)_{i,j} \mid k \ge 0,\ i,j = 0,1,\dots,n\right\}$.*
   - *(a) If there exists at least one x_s such that $x_s = w(u)^{(k)}_{i,j}$ for some i,j,k, we have*

     $$m_l(x_1, \dots, x_l) = 0$$

     for $l > 2$ and l even.
   - *(b) Let $l > 2$ odd. If there exist more than one x_s such that $x_s = w(u)^{(k)}_{i,j}$ for some i,j,k, then*

     $$m_l(x_1, \dots, x_l) = 0.$$

   - *(c) Let $l > 2$ odd. If there exist exactly one x_s such that $x_s = w(u)^{(k)}_{i,j}$ for some i,j,k, then*

     $$m_l\left(\alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_{l-1},j_{l-1}}, w(u)^{(k)}_{i_l,j_l}\right) = m_l\left(\gamma(u)_{i_1,j_1}, \dots, \gamma(u)_{i_{l-1},j_{l-1}}, w(u)^{(k)}_{i_l,j_l}\right)$$

     and $m_l\left(\alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_{l-1},j_{l-1}}, w(u)^{(k)}_{i_l,j_l}\right)(g) \in A'^1$ is given by

     $$\gamma(u)_{i_1,j_1}(g) \cdots \gamma(u)_{i_{l-1},j_{l-1}}(g) \sum_{p=0}^{k} \frac{w(u)^{(k-p)}_{i_l,j_l}\left(-\gamma(u)_{i,j}(g)\right)^p}{p!}$$

     for $g \in \mathbb{Z}^{2n}$. Moreover

     $$m_l\left(\alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_{l-1},j_{l-1}}, w(u)^{(0)}_{i_l,j_l}\right) = 0.$$

     The $m_{l+1}(w(u)^{(k)}_{i,j}, \alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_l,j_l})$ are invariant under the permutation of the $\alpha(u)$ terms and $m_{l+1}(\alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_r,j_r}, w(u)^{(k)}_{i,j}, \alpha(u)_{i_{r+1},j_{r+1}}, \dots, \alpha(u)_{i_l,j_l})$ is given by

     $$\binom{l}{r} m_{l+1}(w(u)^{(k)}_{i,j}, \alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_l,j_l}).$$

3. *Let $l>2$, then $m_l\left(\alpha(u)_{i_1,j_1},\dots,\alpha(u)_{i_{l-1},j_{l-1}},\alpha(u)_{i_l,j_l}\right)(g)$ can is a form of degree 2 with $(2,0)$-part equal 0, $(1,1)$-part equal to*

$$-\sum_{r=1}^{l}\binom{l}{r}\gamma(u)_{i_1,j_1}(g)\cdots d\gamma(u)_{i_r,j_r}(g)\cdots\gamma(u)_{i_l,j_l}(g)\in {A'}^1$$

for $g\in\mathbb{Z}^{2n}$ and $(2,0)$-part equal to

$$m_l\left(\gamma(u)_{i_1,j_1},\cdots\gamma(u)_{i_l,j_l}\right)(g_1,g_2)$$

for $g_1,g_2\in\mathbb{Z}^{2n}$.

4. $D\left(w(u)^{(n)}_{i,j}\right)=\sum_{l=1}^{n}(-1)^{l+1}m_{l+1}(\alpha(u)_{i,j},\dots,\alpha(u)_{i,j},w(u)^{(n-l)}_{i,j})$*, for any n.*

Proof. The first three points are a consequence of Theorems 2.2.9 and Theorem 2.2.7. The proof of the last point follows from the proof of Lemma 3.1.2. □

By Proposition 4.1.7, for each $k>0$, we have an inclusion

$$\Psi_*\ :\ \mathrm{Map}\left(\left(\mathbb{Z}^{2n}\right)^k,{A'}^{\bullet}_n\right)\to\mathrm{Map}\left(\left(\mathbb{Z}^{2n}\right)^k,\left(A_{DR}\left(\mathbb{C}^n-\mathcal{D}\right)\otimes\Omega(1)\right)^{\bullet}\right).$$

Since Ψ preserves the $\mathbb{Z}^{2n}$-action, the above map is simplicial and we get an inclusion

$$\Psi_*\ :\ \mathrm{Tot}_N\left(A\right)\hookrightarrow\mathrm{Tot}_N\left(A_{DR}\left(\left(\mathbb{C}^n-\mathcal{D}\right)_{\bullet}\left(\mathbb{Z}^{2n}\right)\right)\right)\otimes\Omega(1).$$

We define H via the commutative diagram

$$\begin{array}{ccc}\mathrm{Tot}_N\left(A\right) & \xrightarrow{\Psi_*} & \mathrm{Tot}_N\left(A_{DR}\left(\left(\mathbb{C}^n-\mathcal{D}\right)_{\bullet}\left(\mathbb{Z}^{2n}\right)\right)\right)\otimes\Omega(1)\\ \uparrow & \nearrow_{H} & \\ {B'}_n & & \end{array}$$

In particular H is injective.

Theorem 4.1.12. *Consider $H\ :\ {B'}_n\to\mathrm{Tot}_N\left(A_{DR}\left(\left(\mathbb{C}^n-\mathcal{D}\right)_{\bullet}\left(\mathbb{Z}^{2n}\right)\right)\right)\otimes\Omega(1)$. Let $\tilde{W}\subset{B'}_n$ be the graded vector space generated by*

1. *1 in degree zero,*
2. *$w(u)^{(0)}_{i,0}$, $\alpha(u)_{i,0}$ for $i=1,\dots,n$ in degree 1,*
3.

$$m_2(w(u)^{(0)}_{i,0},\alpha(u)_j),\quad m_2(w(u)^{(0)}_{i,0},w(u)_j),\quad ,m_2(\alpha(u)_i,\alpha(u)_j)$$

and

$$(i<j,j)_1:=m_2(w(u)^{(1)}_{i,j},\alpha(u)_j)-m_2(w(u)^{(1)}_{i,0},\alpha(u)_j)-m_2(w(u)^{(1)}_{j,0},\alpha(u)_i)$$
$$(i<j,j)_2:=m_2(w(u)^{(1)}_{i,j},w(u)^{(0)}_{j,0})-m_2(w(u)^{(1)}_{i,0},w(u)^{(0)}_{j,0})-m_2(w(u)^{(1)}_{j,0},w(u)^{(0)}_{i,0}),$$

for $1\le i<j\le n$ and by

$$\begin{aligned}(i<j<k)_1:=\ &m_2(w(u)^{(1)}_{i,j},\alpha(u)_k)+m_2(w(u)^{(1)}_{i,k},\alpha(u)_j)+m_2(w(u)^{(1)}_{k,j},\alpha(u)_i)\\ &-m_2(w(u)^{(1)}_{i,0},\alpha(u)_k)-m_2(w(u)^{(0)}_{i,k},\alpha(u)_k)-m_2(w(u)^{(1)}_{k,0},\alpha(u)_j)\\ &-m_2(w(u)^{(1)}_{i,0},\alpha(u)_j)-m_2(w(u)^{(0)}_{j,0},\alpha(u)_i)-m_2(w(u)^{(1)}_{k,0},\alpha(u)_i),\end{aligned}$$

and

$$\begin{aligned}(i<j<k)_2:=\ &m_2(w(u)^{(1)}_{i,j},w(u)^{(0)}_{k,0})+m_2(w(u)^{(1)}_{i,k},w(u)^{(0)}_{j,0})+m_2(w(u)^{(1)}_{k,j},w(u)^{(0)}_{i,0})\\ &-m_2(w(u)^{(1)}_{i,0},w(u)^{(0)}_{k,0})-m_2(w(u)^{(0)}_{i,k},w(u)^{(0)}_{k,0})-m_2(w(u)^{(1)}_{k,0},w(u)^{(0)}_{j,0})\\ &-m_2(w(u)^{(1)}_{i,0},w(u)^{(0)}_{j,0})-m_2(w(u)^{(0)}_{j,0},w(u)^{(0)}_{i,0})-m_2(w(u)^{(1)}_{k,0},w(u)^{(0)}_{i,0}),\end{aligned}$$

for $1\le i<j<k\le n$ in degree 2.

Then $H : \bar{W} \to \mathrm{Tot}_N \left(A_{DR}\left((\mathbb{C}^n - \mathcal{D})_\bullet \left(\mathbb{Z}^{2n}\right)\right)\right) \otimes \Omega(1)$ *is an inclusion which is a quasi-isomorphism in degrees* $0, 1, 2$.

Proof. See Subsection 4.2.2. ◻

Definition 4.1.13. We denote by $J \subset B'_n$ the C_∞-ideal generated by all the 2-forms

$$m_l\left(\alpha(u)_{i_1,j_1}, \dots, \alpha(u)_{i_{l-1},j_{l-1}}, \alpha(u)_{i_l,j_l}\right)$$

for $l > 2$. We denote by $B_n := B'_n/J$ the quotient C_∞-algebra.

Remark 4.1.14. By a computer assisted proof we have calculated that

$$m_l\left(\gamma(u)_{i_1,j_1}, \dots, \gamma(u)_{i_l,j_l}\right) = 0$$

for $k = 3, 4$. We conjecture that is true for any k. A consequence is that $J \subset B'_n$ doesn't contain any closed forms and hence that B'_n is a 1-model.

We denote by J_{DR} the image of J via the map H. The map H induces a C_∞-strict morphism

$$H : B_n \to \mathrm{Tot}_N \left(A_{DR}\left((\mathbb{C}^n - \mathcal{D})_\bullet \left(\mathbb{Z}^{2n}\right)\right)\right) \otimes \Omega(1)/J_{DR}.$$

For $0 \le s \le 1$ we denote by

$$ev^s : \mathrm{Tot}_N\, A_{DR}\left((\mathbb{C}^n - \mathcal{D})_\bullet \left(\mathbb{Z}^{2n}\right)\right) \otimes \Omega(1) \to \mathrm{Tot}_N\, A_{DR}\left((\mathbb{C}^n - \mathcal{D})_\bullet \left(\mathbb{Z}^{2n}\right)\right)$$

the evaluation map; it is a strict morphism of C_∞-algebras. Let $J_{s,DR} := ev^s\left(J_{DR}\right)$.

Theorem 4.1.15. *The diagram*

$$\begin{array}{ccc} & A_n & \xrightarrow{\ \psi^1\ } & A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \\ {\scriptstyle p^1}\nearrow & & & \uparrow{\scriptstyle ev^1} \\ B_n & & \xrightarrow{\ H\ } & \left(\mathrm{Tot}_N\, A_{DR}\left((\mathbb{C}^n - \mathcal{D})_\bullet \left(\mathbb{Z}^{2n}\right)\right) \otimes \Omega(1)\right)/J_{DR} \end{array}$$

commutes. Moreover B_n *is a* 1*-extension for* A_n*.*

Proof. See Subsection 4.2.2. ◻

4.1.3 The degree zero geometric connection

Theorem 4.1.16. *There exists a compatible Hodge type decomposition of* B_n

$$B_n = W \oplus \mathcal{M} \oplus d\mathcal{M} \qquad (4.8)$$

such that[1]

1. W^1 *is the vector space generated* $w(u)^{(0)}_{i,0}$, $\alpha(u)_{i,0}$ *for* $i = 1, \dots, n$.

2. $\mathcal{M}^1$ *is the vector space generated* $w(u)^{(k)}_{i,j}$, $i, j = 1, \dots, n$ *and* $k > 0$.

3. W^2 *is the vector space generated for* $1 \le i < j \le n$ *by*

   $$m_2(w(u)^{(0)}_{i,0}, \alpha(u)_j), \quad m_2(w(u)^{(0)}_{i,0}, w(u)_j), \quad m_2(\alpha(u)_i, \alpha(u)_j)$$

   and

   $$(i < j, j)_1, \quad (i < j, j)_2,$$

   and for $1 \le i < j < k \le n$ *by*

   $$(i < j < k)_1, \quad (i < j < k)_2.$$

[1]Note that we make a small abuse of notation here, we consider these elements as elements in B_n.

4. $\mathcal{M}^2$ *is the vector space generated for* $1 \le i < j \le n$

$$
\begin{aligned}
& m_2\left(w(u)^{(k)}_{i,j}, w(u)^{(l)}_{r,s}\right), \quad l,k \geq 1, \\
& m_2\left(w(u)^{(k)}_{i,0}, w(u)^{(0)}_{j,0}\right), \quad k \geq 1, \\
& m_2\left(w(u)^{(1)}_{i,j}, w(u)^{(0)}_{k,0}\right), \quad k < i \text{ or } k < j, \\
& m_2\left(w(u)^{(1)}_{i,j}, w(u)^{(0)}_{i,0}\right), \quad i < j, \\
& m_2(w(u)^{(k)}_{i,j}, \alpha(u)_{r,0}), \quad k > 1, i \neq r \neq j, \\
& m_2(w(u)^{(1)}_{i,j}, \alpha(u)_{r,0}), \quad r < i \text{ or } r < j, \\
& m_2(w(u)^{(k)}_{i,j}, \alpha(u)_{j,0}), \quad k > 1, i < j, \\
& m_2(w(u)^{(1)}_{i,0}, \alpha(u)_{j,0}), \quad k \geq 1, \\
& m_{l+1}(w(u)^{(k)}_{i,j}, \alpha(u)_{i_1,j_1}, \ldots, \alpha(u)_{i_l,j_l}), \quad l > 1, k > 0.
\end{aligned}
$$

Proof. See subsection 4.2.1. □

We calculate the degree zero geometric connection associated to the decomposition (4.8) in the sense of Definition 2.3.24. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be the basis of $\left(W^1_+[1]\right)^*$ dual to

$$
s^{-1}\left(-w(u)^{(0)}_{1,0}\right), \ldots, s^{-1}\left(-w(u)^{(0)}_{n,0}\right), s^{-1}\left(-\alpha(u)_{1,0}\right), \ldots, s^{-1}\left(-\alpha(u)_{n,0}\right) \in W^1_+[1].
$$

On the other hand we denote by

1. $X_{i,j}$ for $i < j$ the element dual to $s^{-1}m_2(-w(u)^{(0)}_{i,0}, -w(u)_{j,0})$;
2. $Y_{i,j}$ for $i < j$ the element dual to $s^{-1}m_2(-\alpha(u)_i, -\alpha(u)_j)$;
3. $U_{i,j}$ for $i < j$ the element dual to $s^{-1}m_2(-w(u)^{(0)}_{i,0}, -\alpha(u)_j)$;
4. $T_{i,j,j}$ for $i < j$ the element dual to $-s^{-1}(i < j, j)_1$,
5. $Z_{i,j,j}$ for $i < j$ the element dual to $-s^{-1}(i < j, j)_2$,
6. $T_{i,j,k}$ for $i < j < k$ the element dual to $-s^{-1}(i < j < k)_1$,
7. $Z_{i,j,k}$ for $i < j < k$ the element dual to $-s^{-1}(i < j < k)_2$.

These elements forms a basis of $\left(W[1]^1\right)^*$. The homotopy transfer theorem (see [35] and [42]) gives a C_∞-structure $m^W_\bullet$ on W and a morphism of C_∞-algebras $g_\bullet : W \to B_n$. In order to calculate the degree zero geometric connection we need to know the maps

$$
m^W_k|_{(W^1)^{\otimes k}}, \quad g^W_k|_{(W^1)^{\otimes k}}
$$

for $k > 0$. Let $g : W \hookrightarrow B_n$ be the inclusion and $f : B_n \to W$ the projection. We define a map $h : B^\bullet_n \to B^{\bullet-1}_n$ as follows: let $b \in B_n$, then the decomposition (4.8) implies that b can be written in a unique way as $b = b_1 + b_2 + Db_3$, then

$$
h(b) := b_3 \in \mathcal{M}.
$$

Following [42] (see Theorem 2.1.26) the maps are given by

$$
m^W_k := f \circ p_k \circ g^{\otimes k}, \quad g_k := h \circ p_k \circ g^{\otimes k}.
$$

where $p_k : B^{\otimes k}_n \to B_n$ is a family of linear maps of degree $2 - k$. In Subsection 4.2.3 we give a formula for

$$
p_k \circ g^{\otimes k}|_{(W^1)^{\otimes k}}
$$

for any $k > 0$. Hence $m_\bullet^W|_{(W^1)^{\otimes \bullet}}$ corresponds to a map $\delta : (T^c(W_+[1]))^1 \to T^c\left(W_+^1[1]\right)$. We are interested to δ^* We use the formulas in Subsection 4.2.3 and we have

$$\begin{aligned} \delta^* X_{i,j} &= [X_i, X_j], \quad \delta^* Y_{i,j} = [Y_i, Y_j], \quad \delta^* X_{i,j} = [X_i, Y_j] - [X_j, Y_i] \\ \delta^* T_{i,j,j} &= -[[X_j, Y_i], Y_i + Y_j], \quad \delta^* Z_{i,j,j} = [X_j + X_i, [Y_i, X_j]], \\ \delta^* T_{i,j,k} &= 2[Y_k, [Y_i, X_j]], \quad \delta^* Z_{i,j,k} = 2[X_k, [Y_i, X_j]], \end{aligned} \tag{4.9}$$

In the same way $g_k^W|_{(W^1)^{\otimes k}}$ corresponds to a map in $\mathrm{Hom}^1\left(T^c\left(W_+^1[1]\right), B_n\right)$ which can be written as

$$\begin{aligned} C_0 &= -\sum_i w(u)_{i,0}^{(0)} X_i - \alpha(u)_i Y_i - \sum_{i,k\geq 1} (-1)^k w(u)_{i,0}^{(k)} [\cdots[[X_i, Y_i], \cdots], Y_i] \\ &- \sum_{j<i,k\geq 1} (-1)^k \left(w(u)_{j,0}^{(k)} - w(u)_{j,i}^{(k)}\right) [\cdots[[X_i, Y_j], \cdots], Y_j] \\ &- \sum_{j<i,k\geq 1} (-1)^k \left(w(u)_{i,0}^{(k)}\right) [\cdots[[X_i, Y_j], \cdots], Y_i]. \end{aligned}$$

This is the degree zero geometric connection associated to the decomposition (4.8) in the sense of Definition 2.3.24. Let $\mathcal{R}_0 \subset \widehat{\mathbb{L}}\left(\left(W_+[1]^0\right)^*\right)$ be the completion of the Lie ideal generated by

$$\delta^* X_{i,j}, \delta^* Y_{i,j}, \delta^* X_{i,j}, \delta^* T_{i,j,j}, \delta^* Z_{i,j,j}, \delta^* T_{i,j,k}, \delta^* Z_{i,j,k}.$$

We denote $\left(\widehat{\mathbb{L}}\left(\left(W_+^1[1]\right)^*\right)/\mathcal{R}_0\right)$ with $\mathfrak{u}$. We denote the L_∞-algebra

$$\mathrm{Conv}_{r,0}\left(\left(B, m_\bullet^B\right), W, \mathcal{M}\right)$$

with $B_n \widehat{\otimes} \mathfrak{u}$ and for any unital non-negatively graded C_∞-algebra $(A, m_\bullet)$ we denote the L_∞-algebra

$$\mathrm{Conv}_{r,0}\left(\left(W_+, m_\bullet^{W_+}\right), (A, m_\bullet)\right)$$

with $A \widehat{\otimes} \mathfrak{u}$.

Theorem 4.1.17. *The degree zero geometric connection associated to the decomposition* (4.8) *is given by*

$$\begin{aligned} C_0 &= -\sum_i w(u)_{i,0}^{(0)} X_i - \alpha(u)_i Y_i - \sum_{i,k\geq 1} w(u)_{i,0}^{(k)} \mathrm{Ad}_{Y_i}^{(k)}(X_i) \\ &- \sum_{j<i,k\geq 1} \left(w(u)_{j,0}^{(k)} + w(u)_{0,i}^{(k)} - w(u)_{j,i}^{(k)}\right) \mathrm{Ad}_{Y_j}^{(k)}(X_i). \end{aligned}$$

Proof. The relation (4.9) implies

$$\mathrm{Ad}_{Y_i}^{(k)}(X_j) = (-1)^{k+1} \mathrm{Ad}_{Y_j}^{(k)}(X_i) \tag{4.10}$$

in $\mathfrak{u}$. □

For $0 \leq s \leq 1$, we set

$$C(s)_0 := (ev_s H)_*(C_0) \in \left(\mathrm{Tot}_N^1 A_{DR}\left((\mathbb{C}^n - \mathcal{D})_\bullet\left(\mathbb{Z}^{2n}\right)\right)/I_{s,DR}\right) \widehat{\otimes} \mathfrak{u}$$

We denote by $w(s)_{i,j}^{(k)}$, $\nu(s)_{i,j}$ and $\gamma_{i,j}(s)$ resp. the elements $ev^s\left(w_{i,j}(u)^{(k)}\right)$, $ev^s\left(\nu(u)_{i,j}^{(k)}\right)$ and $ev^s\left(\gamma(u)_{i,j}\right)$ resp. for some $s \in [0,1]$. Hence $C(s)_0$ is given by

$$\begin{aligned} C(s)_0 &= -\sum_i w(s)_{i,0}^{(0)} X_i - \alpha(s)_i Y_i - \sum_{i,k\geq 1} w(s)_{i,0}^{(k)} \mathrm{Ad}_{Y_i}^{(k)}(X_i) \\ &- \sum_{j<i,k\geq 1} \left(w(s)_{j,0}^{(k)} + w(s)_{0,i}^{(k)} - w(s)_{j,i}^{(k)}\right) \mathrm{Ad}_{Y_j}^{(k)}(X_i) \end{aligned}$$

Notice that $\alpha(s)_i = \gamma(s)_i - \nu(s)_i$.

Theorem 4.1.18. *Consider* $\mathfrak{u}$ *equipped with the action* ad. *For each* $0 \leq s \leq 1$, $r_*C_0(s) \in A_{DR}(\mathbb{C}^n - \mathcal{D}) \widehat{\otimes} \mathfrak{u}$ *is given by*

$$\begin{aligned} r_*C_0(s) \; &= - \sum_i w(s)^{(0)}_{i,0} X_i + \nu(s)_i Y_i - \sum_{i,k\geq 1} w(s)^{(k)}_{i,0} \operatorname{Ad}^{(k)}_{Y_i}(X_i) \\ &- \sum_{j<i,k\geq 1} \left(w(s)^{(k)}_{j,0} + w(s)^{(k)}_{0,i} - w(s)^{(k)}_{j,i} \right) \operatorname{Ad}^{(k)}_{Y_i}(X_j) . \end{aligned}$$

It is a flat connection on $\mathbb{C}^n - \mathcal{D}$ *on the trivial bundle with fiber* $\mathfrak{u}$. *In particular, for* $s = 0$ *the connection is holomorphic. Moreover* r_*C_0 *is a gauge-equivalence between* $r_*C_0(1)$ *and* $r_*C_0(0)$, *where the gauge is given by* $\sum_i 2\pi i r_i Y_i \in A^0_{DR}(\mathbb{C}^n - \mathcal{D}) \widehat{\otimes} \mathfrak{u}$.

Proof. The flatness follows from Proposition 2.3.17. In particular the map $(rev_s H)$ preserves Maurer-Cartan elements. □

4.1.4 The KZB connection

In [11] a meromorphic flat connection $\omega_{KZB,n}$ on the configuration space of the punctured elliptic curve with value in a bundle $\overline{\mathcal{P}}^{n+1}$ is constructed (see Section 1.3.2). We show that $r_*C_0(0)$ corresponds to $\omega_{KZB,n}$. We will use the same notation of [11]. For $n \geq 0$, we define the algebra $\mathfrak{t}_{1,n}$ as the free Lie algebra with generators $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and $t_{i,j}$ for $1 \leq i \neq j \leq n$ modulo

$$\begin{aligned} t_{ij} = t_{ij}, \quad [t_{ij}, t_{ik} + t_{jk}] = 0, \quad [t_{ij}, t_{kl}] = 0 \\ t_{ij} = [X_i, Y_j], \quad [X_i, X_j] = [Y_i, Y_j] = 0, \quad [X_i, Y_i] = - \sum_{j|j\neq i} t_{ij} \\ [X_i, t_{jk}] = [Y_j, t_{ik}] = 0, \quad [X_i + X_j, t_{jk}] = [Y_i + Y_j, t_{ik}] = 0 \end{aligned} \tag{4.11}$$

for i, j, k, l distinct.

Remark 4.1.19. Notice that the the relations $[t_{ij}, t_{ik} + t_{jk}] = 0$, and $[t_{ij}, t_{kl}] = 0$ follow from $[x_i, t_{jk}] = [Y_j, t_{ik}] = 0$,$[x_i + x_j, t_{jk}] = [Y_i + Y_j, t_{ik}] = 0$ and the Jacobi identity.

The elements $\sum_i X_i$ and $\sum_i Y_i$ are central in $\mathfrak{t}_{1,n}$. We denote by $\bar{\mathfrak{t}}_{1,n}$ the quotient of $\mathfrak{t}_{1,n}$ modulo

$$\sum_i X_i = \sum_i Y_i = 0 \tag{4.12}$$

Proposition 4.1.20. *The lie algebra* $\bar{\mathfrak{t}}_{1,n+1}$ *admits the following presentation: the generators are* $X_1, \ldots, X_n$ *and* $Y_1, \ldots, Y_n$ *and the relations are*

$$\begin{aligned} [X_i, X_j] = [Y_i, Y_j] = 0, \quad [X_i, Y_j] - [X_j, Y_i] = 0, \; i < j \\ [[X_j, Y_i], Y_i + Y_j] = [X_j + X_i, [Y_i, X_j]] = 0, \; i < j \\ [Y_k, [Y_i, X_j]] = [X_k, [Y_i, X_j]] = 0, \; i < j < k \end{aligned} \tag{4.13}$$

Proof. Let $\tilde{L}_n$ be the free Lie algebra on with generators $\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}_1, \ldots, \tilde{Y}_n$ and $\tilde{t}_{i,j}$ for $1 \leq i \neq j \leq n$ modulo relations

$$\begin{aligned} \tilde{t}_{ij} = \tilde{t}_{ij} \\ \tilde{t}_{ij} = \left[\tilde{X}_i, \tilde{Y}_j\right], \quad \left[\tilde{X}_i, \tilde{X}_j\right] = \left[\tilde{Y}_i, \tilde{Y}_j\right] = 0, \\ \left[\tilde{X}_i, \tilde{t}_{jk}\right] = \left[\tilde{Y}_j, \tilde{t}_{ik}\right] = 0, \quad \left[\tilde{X}_i + \tilde{X}_j, \tilde{t}_{jk}\right] = \left[\tilde{Y}_i + \tilde{Y}_j, \tilde{t}_{ik}\right] = 0. \end{aligned} \tag{4.14}$$

for i, j, k, l distinct. We first show that the map $\tilde{h} : \tilde{L}_n \to \bar{\mathfrak{t}}_{1,n+1}$ defined by $\tilde{h}(\tilde{X}_i) = X_i$, $\tilde{h}(\tilde{Y}_i) = Y_i$ is an isomorphism of Lie algebras. The map is clearly well-defined. We define an inverse via $\tilde{h}^{-1}(X_i) := \tilde{X}_i$, $\tilde{h}^{-1}(Y_i) := \tilde{Y}_i$ for $i < n+1$ and with $\tilde{h}^{-1}(X_{n+1}) := -\sum_{i=1}^n \tilde{X}_i$, $\tilde{h}^{-1}(Y_{n+1}) := -\sum_{i=1}^n \tilde{Y}_i$. In order to prove that $\tilde{h}^{-1}$ is well-defined, we have to check that $\tilde{h}^{-1}$ sends the relations (4.11) and (4.12) into (4.14). This is immediate if we consider distinct index i, j, k, l smaller that $n+1$. It is also immediate to show that $\tilde{h}^{-1}(t_{ij}-t_{ij}) = \tilde{h}^{-1}(t_{ij}-t_{ij}) = \tilde{h}^{-1}([X_i,X_j]) = \tilde{h}^{-1}([Y_i,Y_j]) = 0$ if one of the index is equal to $n+1$. On the other hand, consider the cubic relation $[X_{n+1},t_{jk}]=0$. We have

$$\begin{aligned}\tilde{h}^{-1}\left([x_{n+1},t_{jk}]\right) &:= \left[-\sum_{i=1}^{n}\tilde{X}_i,\tilde{t}_{jk}\right]\\ &= \sum_{\substack{i=1\\ i\neq j,k}}^{n}\left[\tilde{X}_i,\tilde{t}_{jk}\right]-\left[\tilde{X}_j+\tilde{X}_k,\tilde{t}_{jk}\right]=0\end{aligned}$$

Similarly, we have $\tilde{h}^{-1}\left(\left[X_i,t_{(n+1)k}\right]\right) = \tilde{h}^{-1}\left([Y_{n+1},t_{jk}]\right) = \tilde{h}^{-1}\left(\left[Y_i,t_{(n+1)k}\right]\right)=0$. We have

$$\begin{aligned}\tilde{h}^{-1}\left(\left[\tilde{X}_{(n+1)}+\tilde{X}_j,t_{(n+1)j}\right]\right) &:= \left[-\sum_{i=1}^{n}\tilde{X}_i+\tilde{X}_j,\left[-\sum_{i=1}^{n}\tilde{X}_i,\tilde{Y}_j\right]\right]\left[-\sum_{i=1}^{n}\tilde{X}_i,\tilde{t}_{jk}\right]\\ &= \left[\sum_{\substack{i=1\\ i\neq j}}^{n}\tilde{X}_i,\sum_{\substack{l=1\\ l\neq j}}^{n}\left[\tilde{X}_l,\tilde{Y}_j\right]+\left[\tilde{X}_j,\tilde{Y}_j\right]\right]\\ &= \sum_{\substack{i,l=1\\ i\neq j\neq l, i\neq l}}^{n}\left[\tilde{X}_i,\left[\tilde{X}_l,\tilde{Y}_j\right]\right]+\sum_{\substack{i=1\\ i\neq j}}^{n}\left[\tilde{X}_i,\left[\tilde{X}_i,\tilde{Y}_j\right]\right]+\left[\tilde{X}_i,\left[\tilde{X}_j,\tilde{Y}_j\right]\right]=0\end{aligned}$$

since the first summand is zero and by (4.14) we have $\left[\tilde{X}_i,\left[\tilde{X}_j,\tilde{Y}_j\right]\right]=\left[\tilde{X}_j,\left[\tilde{X}_i,\tilde{Y}_j\right]\right]$, i.e the second summand is zero as well. The same arguments work for the rest of the cubic relations. This shows that $\tilde{h}$ is an isomorphism of Lie algebras. We define the map $\tilde{h}' : \bar{\mathfrak{t}}_{1,n+1}\to\tilde{L}_n$ via $\tilde{h}'(X_i)=\tilde{X}_i$, $\tilde{h}'(Y_i)=\tilde{Y}_i$ for $i=1,\ldots,n$. The Jacobi identity and the relations $[X_i,X_j]=[Y_i,Y_j]=0$ for $i\neq j$ allow us to extend the relation (4.13) for unordered indices. This shows that $\tilde{h}'$ is an isomorphism and so is $\tilde{h}\tilde{h}'$. □

Let $\tau\in\mathbb{H}$ be as above. We define $\mathcal{D}\subset\mathbb{C}^{n+1}$

$$\mathcal{D}:=\left\{(\xi_1,\ldots\xi_n)\,:\,\xi_i-\xi_j\in\mathbb{Z}+\tau\mathbb{Z}\text{ for some distinct }i,j=1,\ldots,n+1\right\}.$$

We define an action of $(\mathbb{C},+)$ on $\mathbb{C}^{n+1}-\mathcal{D}$ via $z(\xi_1,\ldots,\xi_{n+1}):=(\xi_1-z,\ldots,\xi_{n+1}-z)$. This induces an action of $\mathcal{E}$ on $\mathrm{Conf}_{n+1}(\mathcal{E})$ via $\xi'(\xi_1,\ldots,\xi_{n+1}):=(\xi_1-\xi',\ldots,\xi_{n+1}-\xi')$. We get a projection $\pi_1 : \mathbb{C}^{n+1}-\mathcal{D}\to\left(\mathbb{C}^{n+1}-\mathcal{D}\right)/\mathbb{C}$ defined via $\pi_1(\xi_1,\ldots,\xi_{n+1})=\xi_{n+1}(\xi_1,\ldots,\xi_n)$ which induces $\pi_2 :$ $\mathrm{Conf}_{n+1}(\mathcal{E})\to\mathrm{Conf}_{n+1}(\mathcal{E})/\mathcal{E}$. We fix a section $h_1 : \left(\mathbb{C}^{n+1}-\mathcal{D}\right)/\mathbb{C}\to\left(\mathbb{C}^{n+1}-\mathcal{D}\right)$ which sends $[\xi_1,\ldots,\xi_n]$ to $(\xi_1,\ldots,\xi_n,0)$, this induces also a section $h_2 : \mathrm{Conf}_{n+1}(\mathcal{E})/\mathcal{E}\to\mathrm{Conf}_{n+1}(\mathcal{E})$. There is an isomorphism $\chi_1 : \mathbb{C}^n-\mathcal{D}\to\left(\mathbb{C}^{n+1}-\mathcal{D}\right)/\mathbb{C}$ given by $\chi_1(\xi_1,\ldots,\xi_n)=[\xi_1,\ldots,\xi_n,0]$. Its inverse is $\chi_1^{-1}[\xi_1,\ldots,\xi_n,\xi_{n+1}]=(\xi_1-\xi_{n+1},\ldots,\xi_n-\xi_{n+1})$. In particular such an isomorphism induces another isomorphism $\chi_2 : \mathrm{Conf}_n(\mathcal{E}^\times)\to\mathrm{Conf}_{n+1}(\mathcal{E})/\mathcal{E}$. We define smooth functions on $(\mathbb{C}-\{\mathbb{Z}+\tau\mathbb{Z}\})\times[0,1]$ $f(u)^{(k)}_{i,j}$ via $w(u)^{(k)}_{i,j}=f(u)^{(k)}_{i,j}d(z_i-z_j)$. We fix an integer n. For $0\le i,j\le n+1$, we define

$$k(u)_{ij}:=\sum_k f(u)^{(k)}_{i,j}\,\mathrm{Ad}^{(k)}_{Y_i}(X_j)\in\left(A^0_{DR}(\mathbb{C}^{n+1}-\mathcal{D})\otimes\Omega^0(1)\right)\widehat{\otimes}\hat{\mathfrak{t}}_{1,n+1}.$$

We define

$$K(u)_i:=-X_i+\sum_{\substack{j=1\\ j\neq i}}k(u)_{ij}$$

and

$$\varpi(u):=\sum_{i=1}^{n+1}K(u)_i d\xi_i\in A^1_{DR}(\mathbb{C}^{n+1}-\mathcal{D})\otimes\Omega^0(1)\widehat{\otimes}\hat{\mathfrak{t}}_{1,n+1}.$$

For $0\le s\le 1$ we define the bundle $\mathcal{P}^n_s$ with fiber $\widehat{\mathfrak{t}}_{1,n}$ on $\mathrm{Conf}_n(\mathcal{E})$ via the following equation (see [38], [11]): each section f of $\mathcal{P}^n_s$ satisfies

$$\begin{aligned}&f(\xi_1,\cdots,\xi_j+l,\cdots,\xi_n)=f(\xi_1,\cdots,\xi_n),\\ &f(\xi_1,\cdots,\xi_j+l\tau,\cdots,\xi_n)=\exp(-2\pi il\,(1-s)\,Y_j)\cdot f(\xi_1,\cdots,\xi_n)\end{aligned}$$

for any integer l, where $y_j^k \cdot a := \mathrm{Ad}_{y_j}^k(a)$ for $a \in \widehat{\mathfrak{t}}_{1,n}$. For $0 \leq s \leq 1$ we define the connection form

$$\varpi(s) \in A^1_{DR}(\mathbb{C}^{n+1} - \mathcal{D})\widehat{\otimes}\widehat{\mathfrak{t}}_{1,n+1}$$

as the evaluation of $\varpi(u)$ at s. For $0 \leq s \leq 1$ we define the bundle $\tilde{\mathcal{P}}_s^n$ with fiber $\widehat{\bar{\mathfrak{t}}}_{1,n}$ on $\mathrm{Conf}_n(\mathcal{E})$ as the fiber quotient of $\mathcal{P}_s^n$ via the relation (4.14). We denote by $\overline{\mathcal{P}}_s^n$ the pullback of $\tilde{\mathcal{P}}_s^n$ along $h\chi_2$ and by $\tilde{\varpi}(s) \in A^1_{DR}\left(\mathbb{C}^{n+1} - \mathcal{D}\right)\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$ the image of $\varpi(s)$ via the quotient map $A^1_{DR}(\mathbb{C}^{n+1} - \mathcal{D})\widehat{\otimes}\widehat{\mathfrak{t}}_{1,n+1} \to A^1_{DR}(\mathbb{C}^{n+1} - \mathcal{D})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$. Consider the linear map

$$(h_2\chi_2)^* : A^1_{DR}(\mathbb{C}^{n+1} - \mathcal{D})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1} \to A^1_{DR}(\mathbb{C}^{n} - \mathcal{D})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$$

We define

$$\overline{\varpi}(s) := (h_2\chi_2)^* \tilde{\varpi}(s) + \sum \nu(s)_i Y_i \in A^1_{DR}(\mathbb{C}^{n} - \mathcal{D})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1},$$

in particular, $\overline{\varpi}(0) = \omega_{KZB,n}$.

Theorem 4.1.21. *Let $r_*C_0(s)$ be the connection obtained in Theorem 4.1.18. We have*

$$\overline{\varpi}(s) = r_*C_0(s)$$

in $A^1_{DR}(\mathbb{C}^{n} - \mathcal{D})\widehat{\otimes}\widehat{\bar{\mathfrak{t}}}_{1,n+1}$. In particular, for each $0 \leq s \leq 1$, $r_*C_0(s)$ defines a flat connection form on the bundle $\overline{\mathcal{P}}_s^{n+1}$ and $r_*C_0(0) = \omega_{KZB,n}$ on $\mathrm{Conf}_n(\mathcal{E}^\times)$.

Proof. We first show the equality. Since $X_{n+1} = -X_1 - \cdots - X_n$, we have

$$\begin{aligned}
(h_2\chi_2)^* \tilde{\varpi}(s) &= \sum_{i=1}^n K(s)_i d\xi_i \\
&= \sum_{i=1}^n \left(-X_i + \sum_{j\neq i} k(s)_{ij}\right) d\xi_i \\
&= \sum_{i=1}^n \big(- X_i + \sum_{k\geq 1} f(s)^{(k)}_{i,n+1} \mathrm{Ad}^{(k)}_{Y_i}(-X_i) d\xi_i \\
&+ \sum_{j,\, k\geq 1} f(s)^{(k)}_{i,j} \mathrm{Ad}^{(k)}_{Y_i}(X_j) d\xi_i - f(s)^{(k)}_{i,n+1} \mathrm{Ad}^{(k)}_{Y_i}(X_j) d\xi_i \big) \\
&= \sum_{i=1}^n \left(-X_i - \sum_{k\geq 1} f(s)^{(k)}_{i,n+1} \mathrm{Ad}^{(k)}_{Y_i}(X_i) d\xi_i\right) \\
&+ \sum_{(i>j),\, k\geq 1} \big(f(s)^{(k)}_{i,j} d\xi_i \mathrm{Ad}^{(k)}_{Y_i}(X_j) + f(s)^{(k)}_{j,i} d\xi_j \mathrm{Ad}^{(k)}_{Y_j}(X_i) \\
&- f(s)^{(k)}_{i,n+1} d\xi_i \mathrm{Ad}^{(k)}_{Y_i}(X_j) - f(s)^{(k)}_{j,n+1} d\xi_j \mathrm{Ad}^{(k)}_{Y_j}(X_i) \big).
\end{aligned}$$

We have

$$-f(s)^{(k)}_{i,n+1} d\xi_j \mathrm{Ad}^{(k)}_{Y_i}(X_j) = f(s)^{(k)}_{0,i} \mathrm{Ad}^{(k)}_{Y_j}(X_i) = -w(s)^{(k)}_{0,i} \mathrm{Ad}^{(k)}_{Y_j}(X_i)$$

and

$$f(s)^{(k)}_{i,j} d\xi_i \mathrm{Ad}^{(k)}_{Y_i}(X_j) + f(s)^{(k)}_{j,i} d\xi_j \mathrm{Ad}^{(k)}_{Y_j}(X_i) = \left(-f(s)^{(k)}_{j,i} d\xi_i + f(s)^{(k)}_{j,i} d\xi_j\right) \mathrm{Ad}^{(k)}_{Y_j}(X_i).$$

and we conclude

$$\begin{aligned}
(h_2\chi_2)^* \tilde{\varpi}(s) = &- \sum_i w(s)^{(0)}_{i,0} X_i - \sum_{i,k\geq 1} w(s)^{(k)}_{i,0} \mathrm{Ad}^{(k)}_{Y_i}(X_i) \\
&- \sum_{j<i,k\geq 1} \left(w(s)^{(k)}_{j,0} + w(s)^{(k)}_{0,i} - w(s)^{(k)}_{j,i}\right) \mathrm{Ad}^{(k)}_{Y_i}(X_j).
\end{aligned}$$

The flatness follows from Theorem 4.1.18. □

It follows that the gauge equivalence given in Theorem 4.1.17 has to be understand as a gauge-equivalence between $\left(\overline{\varpi}(s), \overline{\mathcal{P}}_s^{n+1}\right)$ on $\mathrm{Conf}_n(\mathcal{E}^\times)$. Combining this with Proposition 2.4.15, we have that the fundamental group of $\mathrm{Conf}_n(\mathcal{E}^\times)$ is formal. This fact was originally proved in [6]. In [11], this is proved in a more explicit way, by studying the holonomy of the KZB connection (see Section 2 in loc. cit.). By using Proposition 2.4.15, we give a kind of automatic proof by using the Chen's nature of the KZB connection.

4.1.5 KZB and KZ connection

In [29] it is shown that the KZB connection on the punctured elliptic curve can be turned into the KZ connection by taking the restriction of the universal KZB to the first order Tate curve by sending τ to $i\infty$. In the previous chapter (see Proposition 3.3.6), we give an interpretation of that in terms of C_∞-morphism. In this section we prove the same facts for $\omega_{KZB,n}$ as well: $\lim_{\tau\to i\infty}\omega^\tau_{KZB,n}$ is equal to $\omega_{KZ,n}$ modulo a morphism of Lie algebra $Q^* : \widehat{\mathfrak{t}}_n\widehat{\otimes}\mathbb{Q}(2\pi i) \to \widehat{\mathfrak{t}}_{1,n+1}\widehat{\otimes}\mathbb{Q}(2\pi i)$ of complete Lie algebra. Moreover we use the argument of Subsection (2.1.5) to show that Q^* is induced by a strict C_∞-morphism $p_\bullet$.

Let $\tau \in \mathbb{H}$ be fixed as above. We set $q := \exp(2\pi i\tau)$. We define the action of $\mathbb{Z}^n$ on $(\mathbb{C}^*)^n$ via

$$(m_1, \ldots, m_n)\cdot(z_1,\ldots,z_n) := (q^{m_1}z_1,\ldots,q^{m_n}z_n) \tag{4.15}$$

where $(z_1,\ldots,z_n)$ are coordinate on $(\mathbb{C}^*)^n$. We set $z_0 := 1$. We define a map $e : \mathbb{C}^n \to (\mathbb{C}^*)^n$ via $e(\xi_1,\ldots,\xi_n) := (\exp(2\pi i\xi_1),\ldots,\exp(2\pi i\xi_n))$. This map extend ot a simplicial map $e_\bullet : \mathbb{C}^n_\bullet\mathbb{Z}^{2n} \to (\mathbb{C}^*)^n_\bullet\mathbb{Z}^n$ between the two action groupoid

$$\begin{aligned} e_0(\xi_1,\ldots,\xi_n) &:= (\exp(2\pi i\xi_1),\ldots,\exp(2\pi i\xi_n)), \\ e_1((\xi_1,\ldots,\xi_n),((l_1,m_1),\ldots,(l_n,m_n))) &:= (\exp(2\pi i\xi_1),\ldots,\exp(2\pi i\xi_n),(m_1,\ldots,m_n)) \end{aligned}$$

and it induces an isomorphism on the quotient. Let $\mathcal{D}\subset\mathbb{C}^n$ be the divisor defined above. Then $e(\mathcal{D})$ is the normal crossing divisor

$$\left\{(z_1,\ldots,z_n) \mid z_i \neq q^{\mathbb{Z}}z_j \text{ for } 0\le i<j\le n\right\} \subset (\mathbb{C}^*)^n .$$

It is clearly preserved by the action of $\mathbb{Z}^n$ and hence the restriction gives rise to a morphism of simplicial manifolds with simplicial normal crossing divisor

$$e_\bullet : (\mathbb{C}^n - \mathcal{D})_\bullet\mathbb{Z}^{2n} \to ((\mathbb{C}^*)^n - e(\mathcal{D}))_\bullet\mathbb{Z}^{2n}$$

By (3.13) and we have

$$\begin{aligned}
\phi^{(0)}_{i,j} &= \frac{1}{2\pi i}\frac{dz_i}{z_i} - \frac{1}{2\pi i}\frac{dz_i}{z_i}, \\
\phi^{(1)}_{i,j} &= \frac{1}{2\pi i}\left(\pi i + \frac{2\pi i z_j}{z_i - z_j} - (2\pi i)^2\sum_{n=1}^{\infty}\sum_{n|d} d\left(\left(\frac{z_i}{z_j}\right)^{\frac{n}{d}} - \left(\frac{z_i}{z_j}\right)^{\frac{-n}{d}}\right)q^n\right)\left(\frac{dz_i}{z_i} - \frac{dz_i}{z_i}\right), \\
\phi^{(l)}_{i,j} &= -\frac{1}{2\pi i}\left(\frac{(2\pi i)^{l+1}}{l!}\left(\sum_{n=1}^{\infty}\left(\sum_{n|d} d^l\left(\left(\frac{z_i}{z_j}\right)^{\frac{n}{d}} + (-1)^l\left(\frac{z_i}{z_j}\right)^{\frac{-n}{d}}\right)\right)q^n + \frac{B_l}{2\pi i}\right)\right)\left(\frac{dz_i}{z_i} - \frac{dz_i}{z_i}\right) \quad (4.16)
\end{aligned}$$

for $l > 1$, where B_l are the Bernoulli numbers.

Lemma 4.1.22. *Let $D_0, D_1, D_d \subset \mathbb{C}^n$ be defined as follows: D_0 is the set of points $(z_1,\ldots,z_n)$ such that $z_i = 0$ for some i; D_1 is the set of points $(z_1,\ldots,z_n)$ such that $z_i = 1$ for some i; D_d is the set of points $(z_1,\ldots,z_n)$ such that $z_i = z_j$ for some $i\neq j$. The forms $\phi^{(l)}_{i,j}$ for $l\geq 0$ can be written as power series on q where the coefficients are 1-forms of the form $f dz_i$ for some i, where f is rational function on $\mathbb{C}^n$ of the form $\frac{p_1}{p_2}$, where p_j are polynomials over the field $\mathbb{Q}(2\pi i)$ for $j = 1, 2$. Moreover f has only poles of order 1 located in the normal crossing divisor $\underline{\mathcal{D}} := D_0\cup D_1\cup D_d \subset \mathbb{C}^n$.*

Given a subfield $\mathbb{Q} \subset \Bbbk \subset \mathbb{C}$, consider a normal crossing divisor $\mathcal{D}' \subset \mathbb{C}^n$, we denote by $\mathrm{Rat}^0_{\Bbbk}(\mathbb{C}^n, \mathcal{D}')$ the algebra of rational functions $\frac{p_1}{p_2}$ with poles along $\mathcal{D}$ such that p_1, p_2 are polynomials over the field $\Bbbk$. We denote by $\mathrm{Rat}^\bullet_{\Bbbk}(\mathbb{C}^n, \mathcal{D}')$ the differential graded $\Bbbk$- subalgebra of differential forms generated by forms of type $f dx_I$, with $f \in \mathrm{Rat}^0_{\Bbbk}(\mathbb{C}^n, \mathcal{D}')$. In particular $\mathrm{Rat}^\bullet_{\Bbbk}(\mathbb{C}^n, \mathcal{D}') \otimes \mathbb{C} \subset A^*_{DR}(\mathbb{C}^n - \mathcal{D}')$.
We consider the differential graded algebra $\mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \underline{\mathcal{D}})$ and we now assume that q is a formal variable of degree zero. In particular, notice that the function $\phi^{(k)}_{i,j}$, $0 \leq i, j \geq n$, $k \geq 0$, as defined in (4.16), are elements of $\mathrm{Rat}^1_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \mathcal{D}')((q))$. We have a differential graded algebra (over $\mathbb{Q}(2\pi i)$) of formal Laurent series

$$\left(d, \wedge, \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \underline{\mathcal{D}})((q))\right).$$

We extend the action of $\mathbb{Z}^n$ defined in (4.15) extend to an action $\rho^c : \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \underline{\mathcal{D}})((q)) \to \mathrm{Map}\left(\mathbb{Z}^n, \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \underline{\mathcal{D}})((q))\right)$ via

$$\rho^c(q)(m_1, \dots, m_n) := q, \quad \rho^c(dz_i)(m_1, \dots, m_n) := q^{m_i} dz_i, \quad \rho^c\left(\frac{1}{z_i}\right)(m_1, \dots, m_n) := \frac{q^{-m_i}}{z_i}$$

and

$$\rho^c\left(\frac{1}{z_i - 1}\right)(m_1, \dots, m_n) := \begin{cases} \sum_{l=0}^{\infty} (q^n z_i)^l & n > 0 \\ \sum_{l=0}^{\infty} \frac{q^{-n}}{z_i}\left(\frac{1}{z_i^l}\right)^l & n < 0 \end{cases}$$

The nerve gives rise to a cosimplicial unital commutative differential graded algebra $A^{\bullet,\bullet}$, where $A^{p,q} := \mathrm{Map}\left((\mathbb{Z}^n)^p, \mathrm{Rat}^q_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \underline{\mathcal{D}})((q))\right)$. For $0 \leq i, j \leq n$, we denote by $\underline{\gamma}_{i,j} \in A^{1,0}$ the group homomorphism $\gamma(0)_{i,j} : \mathbb{Z}^{2n} \to \mathbb{C}$ defined in the previous section. We denote by $\underline{\phi}_{i,j} \in A^{0,1}$ the 1-forms in (4.16) considered as formal power series in q. By Theorem 2.2.6, $\mathrm{Tot}_N(A)$ carries a C_∞-structure $m_\bullet$. Let $\underline{B'}_n$ be the C_∞-subalgebra of $\mathrm{Tot}_N(A)$ generated by

$$\underline{\phi}^{(k)}_{i,j}, \underline{\gamma}_{i,j} \text{ for } k \geq 0,\ i, j = 0, 1, \dots, n.$$

Let $\underline{J} \subset \underline{B'}$ be the C_∞ ideal generated by all the two forms

$$m_l\left(\underline{\gamma}_{i_1,j_1}, \dots, \underline{\gamma}_{i_l,j_l}\right)$$

for $l > 2$. We denote by $\underline{B}_n := \underline{B'}_n / \underline{J}$ the quotient C_∞-algebra. Notice that a consequence of the conjecture in Remark 4.1.14 were that $\underline{B}_n$ is a 1-model.

Proposition 4.1.23. *Let $A'_n \subset B'_n$ as in Definition 4.1.6.*

1. *There is a strict morphism of complex differential graded algebras*

$$\varphi : ev_0 H(A'_n) \to \mathrm{Rat}^\bullet_{\mathbb{Q}(2\pi i)}(\mathbb{C}^n, \underline{\mathcal{D}})((q)) \otimes \mathbb{C}$$

   which preserves the action ρ^c,

2. *φ induces a strict morphism of C_∞-algebras*

$$\varphi : ev_0 H(B'_n) \to \underline{B'}_n \otimes \mathbb{C}$$

   such that $\varphi(J_{1,DR}) \subset \underline{J}$, and

3. *φ induces a strict morphism of C_∞-algebras*

$$\varphi : ev_0 H(B'_n)/J_{1,DR} \to \underline{B}_n \otimes \mathbb{C}.$$

Proof. Point 1 follows by (4.16). Point 2 and 3 are straightforward. □

Corollary 4.1.24. *Proposition 4.1.11 holds mutatis mutandis for $\underline{B'}_n$, i.e by replacing $w(u)^{(k)}_{i,j}$ with $\underline{\phi}^{(k)}_{i,j}$, $\gamma(u)_{i,j}$ with $\underline{\gamma}_{i,j}$ and setting $\beta(u)_{i,j} = \nu(u)_{i,j} = 0$.*

Proof. Notice that $\underline{B'}_n$ is a formal version of the image of $ev_0 H\left(B'_n\right)$, in particular the proof of Proposition 4.1.11 is independent by the choice of τ. We get that the statements hold for $\underline{B'}_n$ as well. $\square$

Corollary 4.1.25. *Consider the vector space decomposition of Theorem 4.1.16. The element*

$$\begin{aligned}\underline{C_0} \quad &= - \quad \sum_i \underline{\phi}_{i,0}^{(0)} X_i - \underline{\gamma}_i Y_i - \sum_{i,k\geq 1} \underline{\phi}_{i,0}^{(k)} \operatorname{Ad}_{Y_i}^{(k)}\left(X_i\right) \\ &- \quad \sum_{j<i,k\geq 1} \left(\underline{\phi}_{j,0}^{(k)} + \underline{\phi}_{0,i}^{(k)} - \underline{\phi}_{j,i}^{(k)}\right) \operatorname{Ad}_{Y_j}^{(k)}\left(X_i\right)\end{aligned}$$

is a Maurer-Cartan element in the L_∞-algebra $\operatorname{Conv}_{r,0}\left(\left(W_+, m_\bullet^{W_+}\right), \left(\underline{B}_n^\bullet, m_\bullet\right)\right)$.

Proof. Consider the vector space decomposition of Theorem 4.1.16 and the degree zero geometric connection C_0 of Theorem 4.1.17. The map φ induces a strict morphism of L_∞-algebras

$$\left(\varphi ev_0 H\right)_* \,:\, B_n \widehat{\otimes} \widehat{\mathfrak{t}}_{1,n+1} \to \left(\underline{B}_n \otimes \mathbb{C}\right) \widehat{\otimes} \widehat{\mathfrak{t}}_{1,n+1}$$

that preserves Maurer-Cartan elements. In particular $\underline{C_0} = \left(\varphi ev_0 H\right)_* C_0$. $\square$

The quotient map $\mathbb{Z}^n \to \{e\}$ induce a map between cosimplicial graded module

$$i^{\bullet,\bullet} \,:\, A^{\bullet,\bullet} \to \operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)((q))$$

where the latter carries a trivial cosimplicial structure. This induces a morphism of C_∞-structure

$$i \,:\, \operatorname{Tot}_N^\bullet(A) \to \operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)((q))$$

where the latter is a unital commutative differential graded algebra. In particular we have $i(\underline{B'}_n) \subset \operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)[[q]]$ and $i(\underline{J}) = 0$. Let J_q be the completion of the augmentation ideal of $\operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)[[q]]$. We have a differential graded algebra map $\pi \,:\, \operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)[[q]] \to \operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)$ given by the quotient. Hence we get a strict morphism of C_∞-algebras

$$p := \pi \circ i \,:\, \underline{B}_n \to \operatorname{Rat}_{\mathbb{Q}(2\pi i)}^\bullet\left(\mathbb{C}^n, \underline{\mathcal{D}}\right)$$

such that

$$p(\underline{\gamma}_{i,j}) = 0, \qquad p(\underline{\phi}_{i,j}^{(0)}) = \frac{1}{2\pi i}\frac{dz_i}{z_i} - \frac{1}{2\pi i}\frac{dz_i}{z_i},$$

$$p(\underline{\phi}_{i,j}^{(1)}) = \left(\frac{1}{2} + \frac{z_j}{z_i - z_j}\right)\left(\frac{dz_i}{z_i} - \frac{dz_j}{z_j}\right), \qquad p(\underline{\phi}_{i,j}^{(l)}) = \frac{-\left(2\pi i\right)^{l-1} B_l}{l!}\left(\frac{dz_i}{z_i} - \frac{dz_j}{z_j}\right) \text{ for } l > 1$$

Recall the complex differential graded algebra $A_{KZ,n}$ defined in Section 1.3.1. Let $\underline{A}_{KZ,n}$ be the unital differential commutative graded algebra over $\mathbb{Q}(2\pi i)$ generated by the degree 1 closed forms $\omega_{i,-1}$, $\omega_{i,0}$, and $\omega_{i,j}$ for $1 \leq i < j \leq n$. Consider $\underline{A}_{KZ,n}$ equipped with the Hodge type decomposition $\underline{A}_{KZ,n} = W \oplus d\mathcal{M} \oplus \mathcal{M}$ such that $W = \underline{A}_{KZ,n}$ and $\mathcal{M} = 0$. The degree zero geometric connection associated to that decomposition is given by

$$\underline{\omega}_{KZ,n} := \sum_{1\leq i<j\leq n} \omega_{ij} T_{ij},$$

where $\underline{\omega}_{KZ,n} \in \underline{A}_{KZ,n} \otimes \mathfrak{t}_n$. Moreover there is a differential graded algebra inclusion

$$f^A \,:\, \underline{A}_{KZ,n} \otimes \mathbb{C} \to A_{KZ,n}$$

such that $f_*^A\left(\underline{\omega}_{KZ,n}\right) = \omega_{KZ,n}$. We have a diagram of C_∞-algebras

$$\begin{array}{ccc} \underline{B}_n & \xrightarrow{p} & \underline{A}_{KZ,n} \otimes \mathbb{Q}(2\pi i) \\ {\scriptstyle g_\bullet}\uparrow & & \downarrow{\scriptstyle Id} \\ W_+ & & W'_+ \end{array}$$

where $g_\bullet$ is the C_∞-algebra morphism corresponding to the Maurer-Cartan element $\overline{C(0)_0}$ of Corollary 4.1.25. Hence $q_\bullet := p \circ g_\bullet$ is a morphism of C_∞-algebras. This corresponds to a morphism of differential graded coalgebras

$$Q \,:\, T^c W_+[1] \to T^c W'_+[1] \otimes \mathbb{Q}(2\pi i).$$

The restriction of its dual gives a Lie algebra morphism

$$Q^* \,:\, \widehat{\mathfrak{t}}_n \widehat{\otimes} \mathbb{Q}(2\pi i) \to \widehat{\bar{\mathfrak{t}}}_{1,n+1} \widehat{\otimes} \mathbb{Q}(2\pi i)$$

We calculate Q^* via the method of Subsection 2.1.5. We have $q_*(\underline{\omega}_{KZ,n}) = p^*\left(\underline{C_0}\right)$ since the connection is quadratic, where $q_*(\omega_{KZ,n}) = \sum_{\substack{-1 \leq i \neq j \leq n \\ j>0}} \omega_{i,j} Q^*\left(T_{i,j}\right)$. Moreover

$$\begin{aligned}
p^*\left(\underline{C_0}\right) &= \sum_i p\left(\underline{\phi}_{i,0}^{(0)}\right) X_i - p\left(\underline{\gamma}_i\right) Y_i - \sum_{i,k\geq 1} p^*\left(\underline{\phi}_{i,0}^{(k)}\right) \mathrm{Ad}_{Y_i}^{(k)}(X_i) \\
&- \sum_{j<i,k\geq 1} p\left(\underline{\phi}_{j,0}^{(k)} + \underline{\phi}_{0,i}^{(k)} - \underline{\phi}_{j,i}^{(k)}\right) \mathrm{Ad}_{Y_j}^{(k)}(X_i) \\
&= \sum_i \left(-\frac{dz_i}{z_i} \sum_{i=0}^{\infty} \frac{B_k}{k!} \mathrm{Ad}_{2\pi i Y_i}^{(k)}\left(\frac{X_i}{2\pi i}\right) - \left[2\pi i Y_i, \frac{X_i}{2\pi i}\right] \frac{dz_i}{z_i - 1}\right) \\
&- \sum_{j<i} \frac{dz_j}{z_j - 1} + \frac{dz_i}{z_i - 1} - \frac{dz_j - dz_i}{z_j - z_i} \left[Y_j, X_i\right].
\end{aligned}$$

Hence $Q^* \,:\, \widehat{\mathfrak{t}}_n \otimes \mathbb{Q}(2\pi i) \to \widehat{\bar{\mathfrak{t}}}_{1,n+1} \widehat{\otimes} \mathbb{Q}(2\pi i)$ is given by

$$Q^*\left(T_{-1,i}\right) = -\sum_j \left[Y_j, X_i\right], \quad Q^*\left(T_{0,i}\right) = -\sum_{i=0}^{\infty} \frac{B_k}{k!} \mathrm{Ad}_{2\pi i Y_i}^{(k)}\left(\frac{X_i}{2\pi i}\right), \quad Q^*\left(T_{i,j}\right) = -\left[Y_i, X_j\right].$$

for $1 \leq i, j \leq n$.

Theorem 4.1.26. *Let B_n, $\underline{B_n}$ as above.*

1. *The map p induces a Lie algebra morphism $Q \,:\, \widehat{\mathfrak{t}}_n \otimes \mathbb{Q}(2\pi i) \to \widehat{\bar{\mathfrak{t}}}_{1,n+1} \widehat{\otimes} \mathbb{Q}(2\pi i)$ which induces a differential graded Lie algebra map*

$$q_* = (Id \otimes Q^*) \,:\, A_{KZ,n} \otimes \widehat{\mathfrak{t}}_n \to A_{KZ,n} \widehat{\otimes} \widehat{\bar{\mathfrak{t}}}_{1,n+1}$$

2. *For a fixed $n > 0$ and $\tau \in \mathbb{H}$. We denote the KZB connection with*

$$\omega_{KZB,n}^{\tau} \in A_{DR}\left(\mathbb{C}^n - \mathcal{D}\right) \widehat{\otimes} \widehat{\bar{\mathfrak{t}}}_{1,n+1}.$$

We have

$$\lim_{\tau \to i\infty} \omega_{KZB,n}^{\tau} = q_*\left(\omega_{KZ,n}\right)$$

Proof. The first part is proved above. The second part follows as the proof of Proposition 3.3.6. □

4.2 Proofs and calculations

4.2.1 Proof of Theorem 4.1.16

The commutative differential graded algebra $\mathcal{Y}_n$ is the the free differential commutative graded algebra generated by elements of degree 1

$$w_j, v_j, \text{ for } j = 1, \ldots, n, \quad w_{ij} \text{ for } 0 \leq i, j \leq n.$$

modulo the following relations

$$\begin{aligned}
(4.17) \qquad && w_{ii} &= 0, \\
(4.18) \qquad && w_{ij} - w_{ji} &= 0, \\
(4.19) \qquad && w_i \wedge v_i &= 0, \\
(4.20) \qquad && w_{ij} \wedge w_i - w_{ij} \wedge w_j &= 0, \\
(4.21) \qquad && w_{ij} \wedge v_i - w_{ij} \wedge v_j &= 0, \\
(4.22) \qquad && w_{il} \wedge w_{jl} + w_{jl} \wedge w_{ij} + w_{ji} \wedge w_{il} &= 0.
\end{aligned}$$

The differential is given by

$$dw_{i,j} = w_i \wedge v_j + w_j \wedge v_i, \quad dw_i = dv_j = 0.$$

Lemma 4.2.1. *There exists a vector space decomposition*

$$\mathcal{Y}_n = W'' \oplus \mathcal{M}'' \oplus d\mathcal{M}'' \tag{4.23}$$

as in (4.8) *such that*

1. $W''^1 \subset \mathcal{Y}_n$ *is the vector space generated by* w_i, v_j *for* $j, i = 1, \ldots, n$.
2. $W''^2 \subset \mathcal{Y}_n$ *is the vector space generated by*

   $$w_i \wedge v_j, \quad w_{ij} \wedge v_i, \quad w_i \wedge w_j, \quad w_{ij} \wedge w_i, \quad v_i \wedge v_j,$$

   for $1 \leq i < j \leq n$, *and*

   $$\begin{aligned} &w_{ij} \wedge v_k + w_{ik} \wedge v_j + w_{kj} \wedge v_i \\ &w_{ij} \wedge w_k + w_{ik} \wedge w_j + w_{kj} \wedge w_i \end{aligned}$$

   for $1 \leq i < j < k \leq n$.
3. $\mathcal{M}''^1 \subset \mathcal{Y}_n$ *is the vector space generated by* w_{ij} *for* $j, i = 1, \ldots, n$.
4. $\mathcal{M}''^2 \subset \mathcal{Y}_n$ *is the vector space generated by*

   $$\begin{aligned} &w_{ij} \wedge w_{kl}, \textit{ for any } i < j,\ k < l, \\ &w_{ij} \wedge v_k, \textit{ for any } i < j,\ k < j, \\ &w_{ij} \wedge w_k, \textit{ for any } i < j,\ k < j, \end{aligned}$$

Proof. We define W''^1 and $\mathcal{M}''^1$ as in point 1. and 3. resp. It is immediate to see that all the elements listed at point 2. are closed and not exact and their cohomology classes are linearly independent. It remains to prove that $\mathcal{M}''^2$, as defined above, contains no closed forms except zero. In this proof we will call the relations (4.17) – (4.19) *trivial relations.* For $i = 0$ we set $v_i := 0$, and any letter i, j, k, l is an integer between 0 and n.
We define the vector spaces V_1, V_2 and V_3 as follows.

- V_1 is the vector space generated by $w_{ij} \wedge w_{kl}$, for any $i < j$, $k < l$;
- V_2 is the vector space generated by $w_{ij} \wedge v_k$, for any $i < j$, $k < j$;
- V_3 is the vector space generated by $w_{ij} \wedge w_k$, for any $i < j$, $k < j$;

Note that

$$V_1 \oplus V_2 \oplus V_3 = \mathcal{M}''^2$$

and

$$dV_r \cap dV_s = \{0\} \text{ for } r \neq s.$$

Let a be a closed element of W'' of degree 2. We write

$$a = \underbrace{\sum_{(i<j);(k<l)} \lambda_{(i<j);(k<l)} w_{ij} w_{kl}}_{:=a_1} + \underbrace{\sum_{(i<j);(k<j)} \mu_{(i<j);(k<j)} w_{ij} \wedge v_k}_{:=a_2} + \underbrace{\sum_{(i<j);(k<j)} \alpha_{(i<j);(k<j)} w_{ij} \wedge w_k}_{:=a_3}$$

where $a_i \in V_i$ for any i and we have $da = 0$ if and only if $da_1 = da_2 = da_3 = 0$. We start with a_1. We define four vector subspaces

- $V_1^0 \subset V_1$ is the vector space generated by the $w_{ij} \wedge w_{kl}$, for any $i < j$, $k < l$; such that $|\{i,j\} \cap \{k,l\}| = 0$,
- $V_1^1 \subset V_1$ is the vector space generated by the $w_{ij} \wedge w_{kl}$, for any $i < j$, $k < l$; such that $|\{i,j\} \cap \{k,l\}| = 1$
- ${V'}_1^0$ is the vector space generated by the $w_i \wedge v_j \wedge w_{kl}, v_i \wedge w_j \wedge w_{kl}$ for any $i < j$, $k < l$; such that $|\{i,j\} \cap \{k,l\}| = 0$,
- ${V'}_1^1$ is the vector space generated by the $w_i \wedge v_j \wedge w_{kl}, v_i \wedge w_j \wedge w_{kl}$, for any $i < j$, $k < l$; such that $|\{i,j\} \cap \{k,l\}| = 1$

We have $V_1^0 \cap V_1^1 = \{0\}$ and $dV_1^i \subset {V'}_1^i$ for $i = 0,1$. Notice that there is no relation involving the elements of ${V'}_1^0$. On the other hand the only relations involving elements of ${V'}_1^0$ are (4.20) and (4.21). They are between elements of ${V'}_1^1$. Hence ${V'}_1^0 \cap {V'}_1^1 = \{0\}$ and $dV_1^0 \cap dV_1^1 = \{0\}$. We write $a_1 = a_1^1 + a_1^0$, with $a_1^i \in V_1^i$ for $i = 0,1$. Then $da_1 = 0$ if and only if $da_1^i = 0$ for $i = 0,1$. We have two cases.

1. We can write $a_1^0 = \sum_{i<j,\, k<l,\, j<k} \lambda_{i,j,k,l} w_{i,j} \wedge w_{kl}$. Since there is no relations involving

   $$v_i \wedge w_j \wedge w_{kl}, w_i \wedge w_j \wedge w_{kl}, \text{ for any } i < j,\ k < l;$$

   inside ${V'}_1^0$, we get $da_1^0 = 0$ if each $\lambda_{i,j,k,l} = 0$.

2. For a $k < j < l$ we define $V_1^{k,j,l}$ as the vector space generated by $w_{kj} \wedge w_{kl}, w_{kj} \wedge w_{jl}, w_{kl} \wedge w_{jl}$. We have $V_1^1 = \oplus_{k<j<l} V_1^{k,j,l}$, since the only relation involving elements of V_1^1 is (4.22). We define ${V'}_1^{k,j,l} \subset {V'}_1^1$ as the vector space generated by

   $$w_{s_1} \wedge v_{s_2} \wedge w_{s_3 s_4}, v_{s_1} \wedge w_{s_2} \wedge w_{s_3 s_4},$$

   where $(s_1 < s_2; s_3 < s_4) \in \{(k<j);(k<l), (k<j);(j<l), (k<l);(j<l)\}$. The only non -trivial relations in this subspace are (4.20) and (4.21) and they imply

   $${V'}_1^1 = \oplus_{k<j<l} {V'}_1^{k,j,l}.$$

   We can write a_1 as

   $$a_1^1 = \sum_{k<j<l} \left(\lambda^1_{(k<j);(k<l)} w_{kj} \wedge w_{kl} + \lambda^2_{(k<j);(j<l)} w_{kj} \wedge w_{jl} + \lambda^3_{(k<l);(j<l)} w_{kl} \wedge w_{jl} \right).$$

   In particular since $dV_1^{k,j,l} \subset {V'}_1^{k,j,l}$ we have $da_1^1 = 0$ if and only if

   $$d\left(\lambda^1_{(k<j);(k<l)} w_{kj} \wedge w_{kl} + \lambda^2_{(k<j);(j<l)} w_{kj} \wedge w_{jl} + \lambda^3_{(k<l);(j<l)} w_{kl} \wedge w_{jl} \right) = 0.$$

   The equation above corresponds to $\lambda^1_{(k<j);(k<l)} = \lambda^3_{(k<l);(j<l)}$ and $\lambda^1_{(k<j);(k<l)} = -\lambda^2_{(k<l);(j<l)}$. Hence

   $$a_1^1 = \sum_{k<j<l} \lambda_{(k<j);(k<l)} \left(w_{kj} \wedge w_{kl} - w_{kj} \wedge w_{jl} + w_{kl} \wedge w_{jl} \right)$$

   i.e. $a_1^1 = 0$ by (4.22).

Now assume that $da_2 = 0$. We define $V_2^{i,j,k} \subset V^2$ for any $i < j < k$ as the vector space generated by

$$w_{ik} \wedge v_j, w_{jk} \wedge v_i.$$

Then $V_2 = \oplus_{i<j<k} V_2^{i,j,k}$ since there are no non -trivial relations involving the elements of V_2. We define ${V'}_2$ as the vector space generated by

$$w_i \wedge v_j \wedge v_k, w_j \wedge v_i \wedge v_k, w_k \wedge v_i \wedge v_j.$$

for any $i < j < k$. Hence

$${V'}_2 = \oplus_{i<j<k} {V'}_2^{i,j,k}$$

where ${V'}_2^{i,j,k}$ is the vector space generated by the terms above for a fixed $i < j < k$. We can write

$$a_2 = \sum_{i<j<k} \lambda^1_{i<j<k} w_{ik} \wedge v_j + \lambda^2_{i<j<k} w_{jk} \wedge v_i$$

and since $dV_2^{i,j,k} \subset {V'}_2^{i,j,k}$ we have that $da_2 = 0$ if and only if

$$d\left(\lambda^1_{i<j<k} w_{ik} \wedge v_j + \lambda^2_{i<j<k} w_{jk} \wedge v_i\right) = 0,$$

for any $i < j < k$. This implies $\lambda^2_{i<j<k} = 0 = \lambda^1_{i<j<k}$, i.e $a_2 = 0$.
The proof for a_3 is analogous to the one for a_2. □

There is a quasi-isomorphism $f : A_n \to \mathcal{Y}_n$ defined via $f(\nu_{i,j}) := \nu_i - \nu_j$ for $0 \leq i, j \leq n$ and

$$f(w^{(k)}_{i,j}) := \begin{cases} w_i - w_j & \text{for } 0 \leq i, j \leq n,\ k = 0 \\ w_{i,j} & \text{for } 1 \leq i, j \leq n,\ k = 1 \\ 0 & \text{otherwise .} \end{cases}$$

(see [9]).

Proposition 4.2.2. *There exists a Hodge type vector space decomposition of A_n*

$$A_n = W' \oplus \mathcal{M}' \oplus d\mathcal{M}'$$

such that ${W'}^i = f\left({W''}^i\right)$ and ${\mathcal{M}'}^i = f\left({\mathcal{M}''}^i\right)$ for $i = 0, 1, 2$.

The proof of the proposition above follows by the following lemmas.

Lemma 4.2.3. *Consider ${W'}^i \subset A_n^i$ for $i = 1, 2$ as above. The vector space ${W'}^i \subset A_n^i$ contains only closed not exact elements for $i = 1, 2$.*

Proof. The map f above defines an isomorphism between ${W'}^i \subset A_n^i$ and ${W''}^i \subset A_n^i$ for $i = 1, 2$. Since it is a quasi-isomorphism, then results follow. □

Lemma 4.2.4. *Let V be the vector subspace of closed elements in $\ker(f)$.*

1. *Let ${V'}^1 = 0$ and ${V'}^2 \subset \ker(f)$ be the vector space generated by the elements $w(u)^{(p)}_{i,j} \wedge w(u)^{(q)}_{k,l}$, $0 \leq i, j, k, l \leq n$ such that q or $p > 1$ and elements of the form $w(u)^{(1)}_{i,0} \wedge w(u)^{(q)}_{k,l}$, $0 \leq i, k, l \leq n$. Then we have*
$$\ker(f)^2 = V^2 \oplus {V'}^2.$$
2. *We have $V^1 = 0$ and V^2 is the vector space generated by $((\nu(u)_{i,j} + \beta(u)_{i,j})) \wedge w(u)^{(k)}_{i,j}$ for $0 \leq i, j \leq n$, $k > 1$ and $(\nu(u)_{i,0} + \beta(u)_{i,0}) \wedge w(u)^{(1)}_{i,0}$ for $0 \leq i \leq n$. and $\ker(f)^2 \cap \ker(d) = V$.*

Proof. Point 1 is immediate. Let $a \in \ker(f)^2 \cap \ker(d)$. If $a \in W'$ then this is a contradiction with the fact that f is a quasi-isomorphism. Hence a is exact. Now assume that $a \in V'$. We have $dA_n^1 \cap V' = 0$, then $a = 0$ by the definition of d. □

We can conclude the proof of Proposition 4.2.2. From the map $f : A_n \to \mathcal{Y}_n$ we have

$$\begin{aligned} A_n^2 &\cong \mathcal{Y}_n^2 \oplus \ker f^2 \\ &\cong W''^2 \oplus \left(\mathcal{M}''^2 \oplus V'\right) \oplus \left(d\left(\mathcal{M}''^1\right) \oplus V\right). \end{aligned}$$

The isomorphism above give the desired decomposition.
Consider the strict C_∞-morphism $p^1 : B'_n \to A_n$ defined in (4.7). We have immediately the following lemma.

Lemma 4.2.5. *The map $p^1 : \bar{W} \to W'$ is an isomorphism in degree $0, 1, 2$.*

Recall Proposition 4.1.10. Since $p^1(I) = 0$ we have a well-defined strict morphism of C_∞-algebras $p^1 : B_n \to A_n$. The vector space decomposition (4.8)

$$B_n = W \oplus \mathcal{M} \oplus D\mathcal{M} \tag{4.24}$$

satisfies $W' = p^1(W)$, $\mathcal{M}' = p^1(\mathcal{M})$ in degree $0, 1$ and 2.
We are ready for the proof of Theorem 4.1.16.

Proof. Notice that $p^1 : B_n^i \to A_n^i$ is an isomorphism for $i = 0, 1$. For $i = 2$ we have

$$\begin{aligned} B_n^2 &\cong A_n^2 \oplus \left(\ker p^1\right)^2 \\ &\cong W'^2 \oplus \left(\mathcal{M}'^2 \oplus \left(\ker p^1\right)^2\right) \oplus \left(d\left(\mathcal{M}'^1\right)\right) \end{aligned}$$

By Lemma 4.2.6 below we have that this is a Hodge type decomposition. ☐

4.2.2 Proof of Theorem 4.1.12 and Theorem 4.1.15

We have a commutative diagrams of C_∞-algebras.

$$\begin{array}{ccc} & A_n \xrightarrow{\ \psi^1\ } & \mathrm{Tot}_N\, A_{DR}\left(\left(\mathbb{C}^n - \mathcal{D}\right)_\bullet \left(\mathbb{Z}^{2n}\right)\right) \\ & {\scriptstyle p^1}\nearrow & \uparrow{\scriptstyle ev^1} \\ B'_n & \xrightarrow{\qquad H \qquad} & \left(\mathrm{Tot}_N\, A_{DR}\left(\left(\mathbb{C}^n - \mathcal{D}\right)_\bullet \left(\mathbb{Z}^{2n}\right)\right) \otimes \Omega(1)\right) \end{array}$$

Proof. (of Theorem 4.1.12) Consider the diagram above restricted at $\bar{W}$. The map $p^1 : \bar{W} \to A_n$ is an isomorphism in degree 0, 1 and 2. In [9], it is proved that ψ^1 is a quasi-isomorphism, since ev_u is a quasi-isomorphism as well, we conclude that $H|_{\bar{W}}$ is a quasi-isomorphism in degree 0, 1 and 2. ☐

Lemma 4.2.6. *Consider $p^1 : B_n \to A_n$. The graded vector space $\mathrm{Ker}\left(p^1\right) \subset B_n$ doesn't contain any closed form in dimension 1 and 2. In particular p^1 induces an isomorphism in the cohomology H^i for $i = 0, 1, 2$.*

Proof. Let $a \in \mathrm{Ker}\left(p^1\right)^2$ such that $Da = 0$. We can write a as

$$a = \sum_{l > 2, I \in S} \lambda_I m_l(\phi_{i,j}^{(k)}, \gamma(u)_{i_1,j_1}, \ldots, \gamma(u)_{i_l,j_l}),$$

Note that a is a form of bidegree $(1, 1)$. The element $\partial_{\mathbb{Z}^{2n}} a$ defines a map

$$\partial_{\mathbb{Z}^{2n}} a : \mathbb{Z}^{2n} \to A'_n,$$

in particular if $Da = 0$ then $\partial_{\mathbb{Z}^{2n}} a = 0$. Let $V_{i,j} \subset A'^1_n$ be the vector space generated by

$$\tilde{\gamma}(u)^r w(u)_{i,j}^{(k)}, \tilde{\gamma}(u)^s \left(\nu(u)_{i,j} + \beta(u)_{i,j}\right) w(u)_{i,j}^{(k)}, \quad k, r, s \geq 0. \tag{4.25}$$

By the definition we have $Dc\left(\mathbb{Z}^{2n}\right) \subset \oplus_{i,j} V_{i,j}$ We denote by $(\partial_{\mathbb{Z}^{2n}} a)_{i,j}$ the projection of $\partial_{\mathbb{Z}^{2n}} a$ on $V_{i,j}$. Using the same method of Lemma 3.1.2, $(\partial_{\mathbb{Z}^{2n}} a)_{i,j}$ induces a polynomial P in variables $x_1, \ldots, x_n$ with coefficients in $V_{i,j}$. Moreover since $l > 2$ the polynomial is not linear. Assume $\partial_{\mathbb{Z}^{2n}} a = 0$ then the zero set of P contains $\mathbb{Z}^{2n}$. It follows that all the coefficients of P are 0. This implies that the coefficients vanishes as well, in particular there are non -trivial linear relations between the generators (4.25) which are not contained in (4.1.6), hence a contradiction.
Consider the Hodge type decompositions defined in the subsection above for A'_n and B_n resp. . Notice that $\left(p^1\right)^i : W'^i \to W^i$ is an isomorphism of graded vector space for $i = 1, 2$. Together with the property above, we conclude that it is an isomorphism in the cohomology groups H^i, for $i = 1, 2$. □

Proof. (of Theorem 4.1.15) By the lemma above we have that p^1 is a quasi-isomorphism. On the other hand it is immediate to see that $I_{1,DR} = 0$. The statement follows from Theorem 4.1.5. □

4.2.3 Calculation of the p-kernels

We consider B_n equipped with the C_∞-structure $m_\bullet$ defined in the previous section. The Hodge type decomposition (4.8) induces a homotopy retract diagram between chain complexes

$$f : (B_n^\bullet, D) \overset{\longleftarrow}{\longrightarrow} (W^\bullet, 0) : g \tag{4.26}$$

where f is the projection on W and g is the inclusion. We define a map $h : B_n^\bullet \to B_n^{\bullet-1}$ as follows. Let $a \in B_n^\bullet$, the decomposition (4.8) allows us to write $a = (a_1, a_2, Da_3)$, where $a_1 \in W^\bullet$, $a_2 \in \mathcal{M}^\bullet$ and $a_3 \in \mathcal{M}^{\bullet-1}$, then $h(a_1, a_2, Da_3) = (0, a_3, 0)$. In particular gf is homotopic to Id_{B_n} via the cochain homotopy h. Notice that

$$f \circ g = Id_{B_n} \quad f \circ h = 0 \quad h \circ g = 0 \quad h \circ h = 0 \tag{4.27}$$

We calculate the p-kernel using Proposition 2.1.31. We adopt the following notation: for $v \in V_1 \oplus V_2$ and $w_1 \in V_1$ we say that $v = w_1 \oplus V_2$ if $v = w_1 + w$ for some $w \in V_2$.

Proposition 4.2.7. *For $m = 2$ the p-kernels are as follows.*

1. $$p_2(w(u)_{i,0}^{(0)}, \alpha(u)_{j,0}) = \begin{cases} m_2(w(u)_{i,0}^{(0)}, \alpha(u)_{j,0}), & i < j \\ D\left(w(u)_{i,0}^{(1)}\right), & i = j \\ D\left(w(u)_{i,j}^{(1)} - w(u)_{i,0}^{(1)} - w(u)_{j,0}^{(1)}\right) \oplus \mathcal{M}^2, & i < j \end{cases}$$

2. $$p_2(w(u)_{i,0}^{(0)}, w(u)_{j,0}^{(0)}) = \begin{cases} m_2(w(u)_{i,0}^{(0)}, w(u)_{j,0}^{(0)}), & i < j \\ 0, & i = j \\ -m_2(w(u)_{j,0}^{(0)}, w(u)_{i,0}^{(0)}), & i < j \end{cases}$$

3. $$p_2(w(u)_{i,0}^{(0)}, w(u)_{j,0}^{(0)}) = \begin{cases} m_2(w(u)_{i,0}^{(0)}, w(u)_{j,0}^{(0)}), & i < j \\ 0, & i = j \\ -m_2(w(u)_{j,0}^{(0)}, w(u)_{i,0}^{(0)}), & i < j \end{cases}$$

Proof. It follows from $p_2 = m_2$ and the decomposition(4.8). □

Proposition 4.2.8. *Let i, j, k be distinct. For $m = 3$ the p-kernels are as follows.*

1. $$p_3(w(u)_{i,0}^{(0)}, \alpha(u)_{j,0}, \alpha(u)_{k,0}) = \begin{cases} (j < i < k)_1 \oplus \mathcal{M}^2, & j < i < k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$$

2. $$p_3(w(u)_{i,0}^{(0)}, \alpha(u)_{k,0}, \alpha(u)_{j,0}) = \begin{cases} (k < i < j)_1 \oplus \mathcal{M}^2, & k < i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$$

3. $$p_3(\alpha(u)_{j,0}, w(u)_{i,0}^{(0)}, \alpha(u)_{k,0}) = \begin{cases} -(j < i < k)_1 \oplus \mathcal{M}^2, & j < i < k \\ -(k < i < j)_1 \oplus \mathcal{M}^2, & k < i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$$

4. $p_3(\alpha(u)_{k,0}, w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}) = \begin{cases} -(k<i<j)_1 \oplus \mathcal{M}^2, & k<i<j \\ -(j<i<k)_1 \oplus \mathcal{M}^2, & j<i<k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

5. $p_3(\alpha(u)_{k,0}, \alpha(u)_{j,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (j<i<k)_1 \oplus \mathcal{M}^2, & j<i<k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

6. $p_3(\alpha(u)_{j,0}, \alpha(u)_{k,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (k<i<j)_1 \oplus \mathcal{M}^2, & k<i<j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

7. $p_3(w(u)^{(0)}_{i,0}, (w(u)^{(0)}_{j,0}, \alpha(u)_{k,0}) = \begin{cases} -(k<j<i)_2 \oplus \mathcal{M}^2, & j<i<k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

8. $p_3((w(u)^{(0)}_{j,0}, w(u)^{(0)}_{i,0}, \alpha(u)_{k,0}) = \begin{cases} -(k<i<j)_2 \oplus \mathcal{M}^2, & j<i<k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

9. $p_3(w(u)^{(0)}_{i,0}, \alpha(u)_{k,0}, w(u)^{(0)}_{j,0}) = \begin{cases} (k<i<j)_2 \oplus \mathcal{M}^2, & k<i<j \\ (k<j<i)_2 \oplus \mathcal{M}^2, & k<j<i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

10. $p_3(w(u)^{(0)}_{j,0}, \alpha(u)_{k,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (k<i<j)_2 \oplus \mathcal{M}^2, & k<i<j \\ (k<j<i)_2 \oplus \mathcal{M}^2, & k<j<i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

11. $p_3(\alpha(u)_{k,0}, w(u)^{(0)}_{j,0}, w(u)^{(0)}_{i,0}) = \begin{cases} -(k<j<i)_2 \oplus \mathcal{M}^2, & j<i<k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

12. $p_3(\alpha(u)_{k,0}, w(u)^{(0)}_{i,0}, w(u)^{(0)}_{j,0}) = \begin{cases} -(k<i<j)_2 \oplus \mathcal{M}^2, & j<i<k \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

Proof. Explicitly, the formula for p_3 is given by

$$p_3 = m_2\left((h \otimes Id) \circ (m_2 \otimes Id)\right) - m_2\left((Id \otimes h) \circ (Id \otimes m_2)\right) + m_3$$

and by Proposition 4.1.11 the term m_3 vanishes. The results follows by a direct calculation applying the decomposition (4.8). □

The arguments of the proof above works also for the next two propositions.

Proposition 4.2.9. *Let i, j be distinct. For $m = 3$ the p-kernels are as follows.*

1. $p_3(w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}, \alpha(u)_{j,0}) = \begin{cases} (j<i,i)_1 + D\left(w(u)^{(2)}_{j,0} - w(u)^{(2)}_{j,i}\right) \oplus \mathcal{M}^2, & j<i \\ D\left(w(u)^{(2)}\right) \oplus \mathcal{M}^2, & j=i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

2. $p_3(\alpha(u)_{j,0}, w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}) = \begin{cases} -2\,(j<i,i)_1 - 2D\left(w(u)^{(2)}_{j,0} - w(u)^{(2)}_{j,i}\right) \oplus \mathcal{M}^2, & j<i \\ -2D\left(w(u)^{(2)}_{j,0}\right) \oplus \mathcal{M}^2, & j=i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

3. $p_3(\alpha(u)_{j,0}, \alpha(u)_{j,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (j<i,i)_1 + D\left(w(u)^{(2)}_{j,0} - w(u)^{(2)}_{j,i}\right) \oplus \mathcal{M}^2, & j<i \\ D\left(w(u)^{(2)}_{j,0}\right) \oplus \mathcal{M}^2, & j=i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

Let i, j be distinct, then

1. $p_3(w(u)^{(0)}_{i,0}, \alpha(u)_{i,0}, \alpha(u)_{j,0}) \in \mathcal{M}^2$

2. $p_3(w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}, \alpha(u)_{i,0}) = \begin{cases} (j < i, i)_1 + D\left(w(u)^{(2)}_{i,0}\right) \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

3. $p_3(\alpha(u)_{i,0}, w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}) = \begin{cases} -(j < i, i)_1 - D\left(w(u)^{(2)}_{i,0}\right) \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

4. $p_3(\alpha(u)_{j,0}, w(u)^{(0)}_{i,0}, \alpha(u)_{i,0}) = \begin{cases} -(j < i, i)_1 - D\left(w(u)^{(2)}_{i,0}\right) \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

5. $p_3(\alpha(u)_{i,0}, \alpha(u)_{j,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (j < i, i)_1 + D\left(w(u)^{(2)}_{i,0}\right) \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

6. $p_3(\alpha(u)_{j,0}, \alpha(u)_{i,0}, w(u)^{(0)}_{i,0}) \in \mathcal{M}^2$

Proposition 4.2.10. *For $m = 3$ the p-kernels are as follows.*

1. $p_3(w(u)^{(0)}_{j,0}, w(u)^{(0)}_{j,0}, \alpha(u)_{i,0}) = \begin{cases} -(i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

2. $p_3(w(u)^{(0)}_{j,0}, \alpha(u)_{i,0}, w(u)^{(0)}_{j,0}) = \begin{cases} 2\,(i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

3. $p_3(\alpha(u)_{i,0}, w(u)^{(0)}_{j,0}, w(u)^{(0)}_{j,0}) = \begin{cases} -(i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

4. $p_3(\alpha(u)_{i,0}, \alpha(u)_{j,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

5. $p_3(w(u)^{(0)}_{j,0}, w(u)^{(0)}_{i,0}, \alpha(u)_{i,0}) \in \mathcal{M}^2$

6. $p_3(w(u)^{(0)}_{i,0}, \alpha(u)_{i,0}, w(u)^{(0)}_{j,0}) = \begin{cases} (i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

7. $p_3(w(u)^{(0)}_{j,0}, \alpha(u)_{i,0}, w(u)^{(0)}_{i,0}) = \begin{cases} (i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

8. $p_3(\alpha(u)_{i,0}, w(u)^{(0)}_{i,0}, w(u)^{(0)}_{j,0}) \in \mathcal{M}^2$

9. $p_3(\alpha(u)_{i,0}, w(u)^{(0)}_{j,0}, w(u)^{(0)}_{i,0}) = \begin{cases} -(i < j, j)_2 \oplus \mathcal{M}^2, & i < j \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

Proposition 4.2.11. *Let $k > 2$*

1. $p_{k+1}(w(u)^{(0)}_{i,0}, \underbrace{\alpha(u)_{j,0}, \ldots, \alpha(u)_{j,0}}_{k}) = \begin{cases} (-1)^k \left(w(u)^{(k)}_{j,0} - w(u)^{(k)}_{i,j}\right) \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

2. $p_{k+1}(w(u)^{(0)}_{i,0}, \underbrace{\alpha(u)_{j,0}, \alpha(u)_{i,0}, \ldots, \alpha(u)_{i,0}}_{k}) = \begin{cases} (-1)^k w(u)^{(k)}_{i,0} \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \textit{otherwise} \end{cases}$

3. $p_{k+1}(w(u)^{(0)}_{i,0}, \underbrace{\alpha(u)_{i,0},\dots,\alpha(u)_{i,0}}_{k}) = \begin{cases} (-1)^k w(u)^{(k)}_{i,0} \oplus \mathcal{M}^2, & j < i \\ \mathcal{M}^2, & \text{otherwise} \end{cases}$

4. Let

$$x_1,\dots,x_{k+1} \in \left\{ w(u)^{(0)}_{i,j}, \alpha(u)_{i,j} \text{ for } i,j = 0,1,\dots,n. \right\}$$

such that $(x_1, x_2, \dots, x_{k+1})$ *is not a permutation of either* $(w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}, \dots, \alpha(u)_{j,0})$ *either* $(w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}, \alpha(u)_{i,0}, \dots, \alpha(u)_{i,0})$ *or* $(w(u)^{(0)}_{i,0}, \alpha(u)_{i,0}, \dots, \alpha(u)_{i,0})$*. Then*

$$p_{k+1}(x_1, x_2, \dots, x_{k+1}) \in \mathcal{M}^2.$$

Proof. Since $m_{k+1}\left(\left(B^1_n\right)^{\otimes k+1}\right) \subset \mathcal{M}^2$ for $k > 2$, we conclude that $p_T(w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}, \dots, \alpha(u)_{j,0}) = 0$ if T is not binary. The condition (4.27) implies that the only tree such that

$$p_T(w(u)^{(0)}_{i,0}, \alpha(u)_{j,0}, \dots, \alpha(u)_{j,0}) \neq 0$$

is the one in Figure 4.1. A direct calculation by using the decomposition (4.8) gives the desired result.

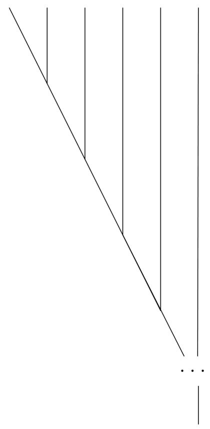


Figure 4.1: The three with non-trivial induced map.

The same argument works in the other cases as well. ☐

Appendix A

Appendix

A.1 Coalgebras

Definition A.1.1. A *coassociative coalgebra* is a graded vector space C over a field $\Bbbk$ equipped with a degree zero linear map $\Delta : C \to C \times C$ such that

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes 1} & C \otimes C \otimes C \end{array}$$

commutes. A *morphism of coalgebras* $f : C \to C'$ is a map that preserves the comultipliciation, i.e $f \otimes f \circ \Delta = \Delta'(f)$.

The *iterated coproduct* $\Delta^n : C^{\otimes n} \to C$ si defined via $\Delta^0 := 1$, $\Delta^1 := \Delta$, and $\Delta^n := (\Delta \otimes 1 \otimes \cdots \otimes 1) \circ \Delta^{n-1}$. The coassociativity implies that

$$\Delta^n = (1 \otimes 1 \otimes \ldots 1 \otimes \Delta \otimes 1 \cdots \otimes 1) \circ \Delta^{n-1}.$$

Let $\Bbbk$ consider as the 1 dimensional vector space. Then $(\Bbbk, \Delta)$ is an associative coalgebra, where $\Delta(1) := 1 \otimes 1$.

Definition A.1.2. A *counital* coassociative coalgebra is a coassociative coalgebra (C, Δ) together with a degree zero linear map $\epsilon : C \to \Bbbk$ (hence a coalgebra map) such that

$$\begin{array}{ccccc} & & C & & \\ & \swarrow{\scriptstyle \cong} & \downarrow{\scriptstyle \Delta} & {\scriptstyle \cong}\searrow & \\ C \otimes \Bbbk & \xleftarrow[1\otimes\epsilon]{} & C \otimes C & \xrightarrow{\epsilon\otimes 1} & \Bbbk \otimes C \end{array}$$

commutes. Observe that this diagram is the dual of the ordinary diagram for the unity. A *morphism of unital coalgebras* $f : C \to C'$ is a map that preserves the comultipliciation, and the counity. We denote the category of counital coassociative coalgebra by $CoAlg_{+}$. A counital coassociative coalgebra is said to be coaugmented if there is a coalgebra morphism $u : \Bbbk \to C$, called *coaugmentation.* An element $x \in C$ is called *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$.

The existence of a coaugmentation implies $\epsilon \circ u = 1$, and then by splitting lemma we have a direct sum of vectorspaces

$$C = \overline{C} \oplus \Bbbk,$$

where $\overline{C} := \ker \epsilon$. This is actually a direct sum of coalgebras. Define $\overline{\Delta} : \overline{C} \otimes \overline{C} \to \overline{C}$ as

$$\overline{\Delta}(x) := \Delta(x) - 1 \otimes x - x \otimes 1,$$

then $\left(\overline{C}, \overline{\Delta}\right)$ is a coassociative coalgebra. We denote the category of counitary coaugmented coalgebra by $CoAlg_{+}^{augm}$. Since the counit has to be unique, we get that $\overline{C}$ is a non-unital coassociative coalgebra. We denote by $CoAlg_{\text{non unital}}$ the category of non unital associative coalgebras.

Proposition A.1.3. *The construction $C \to \overline{C}$ defines a functor from $CoAlg_{+}^{augm} \to CoAlg_{non\ unital}$ which is an equivalence of categories.*

Proof. The inverse of the above functor is builded as follows. Let (C, Δ) be a non unital coassociative coalgebra. Set $C' := C \oplus \mathbb{k}$. Define $\Delta' : C' \to C' \otimes C'$ linearly as

$$\Delta'(x) := \begin{cases} \lambda (1 \otimes 1), & \text{if } x = \lambda \cdot 1, \\ \Delta'(x) := \Delta(x) := \Delta(x) + 1 \otimes x + x \otimes 1. & \end{cases}$$

We define $\epsilon : C \oplus \mathbb{k} \to \mathbb{k}$ as the projection, and $u : \mathbb{k} \to C'$. It si easy show that (C, ϵ, u) is a coassociative counital coalgebra, moreover the above construction is functorial and gives the inverse of the above functor. □

In order to define the universal objects in the category of coassociative coalgebra, we need to add some restrictions.

Definition A.1.4. Let $C = \overline{C} \oplus \mathbb{k}$ be a coaugmented coalgebra. The *coradical filtration* is defined as $F_0 C := \mathbb{k}1$, and for $r \geq 1$

$$F_r C := \mathbb{k}1 \oplus \left\{ x \in \overline{C} \mid \overline{\Delta}^n(x) = 0, \text{ for any } n \geq r \right\}$$

A counital coalgebra C is said to be *conilpotent* (or connected) if it is coaugmented and $C = \bigcup_{r \geq 0} F_r C$.

Definition A.1.5. We define the (counital) *quasi-free coalgebra or tensor coalgebra* on a graded vector space V as

$$T^c(V) := \mathbb{k} \oplus V \oplus V^{\otimes 2} \oplus \dots$$

where the coproduct $\Delta : T^c(V) \to T^c(V) \otimes T^c(V)$ is given by the deconcatenation

$$\Delta(v_1 v_2 \cdots v_p) := \sum_{i=0}^{p} v_1 v_2 \cdots v_i \otimes v_{i+1} v_{i+2} \cdots v_p$$

and $\Delta(1) := 1 \otimes 1$. The counit is the projection $T^c(V) \to \mathbb{k}$ and the coaugmentation is the inclusion $\mathbb{k} \to T^c(V)$. The coradical filtration is $F_r T^c(V) := \oplus_{n \leq r} V^{\otimes n}$, thus $T^c(V)$ is conilpotent. The *reduced quasi-free coassociative coalgebra of V or reduced tensor coalgebra on V* is

$$\overline{T}^c(V) := V \oplus V^{\otimes 2} \oplus \dots$$

where the coproduct $\overline{\Delta} : T^c(V) \to T^c(V) \otimes T^c(V)$ is given by the deconcatenation

$$\overline{\Delta}(v_1 v_2 \cdots v_p) := \sum_{i=i}^{p-1} v_1 v_2 \cdots v_i \otimes v_{i+1} v_{i+2} \cdots v_p.$$

Consider two graded vector space V and W. For each degree zero linear map $F : \overline{T}^c(V) \to \overline{T}^c(W)$, we denote by F_n^m, $m, n = 1, 2, 3, ...$ the sets of maps

$$F_n^m : V^{\otimes n} \xhookrightarrow{i} \overline{T}^c(V) \xrightarrow{F} \overline{T}^c(W) \xrightarrow{prow} W^{\otimes m}$$

where the first map is the inclusion, and the last is the projection. The following proposition and the corollary are is extremely useful.

Proposition A.1.6. *A linear map $F : \overline{T}^c(V) \to \overline{T}^c(W)$ is a morphsim of graded tensor coalgebras ($\overline{\Delta} \circ F = (F \otimes F) \circ \overline{\Delta}$) if and only if, for all $m \geq 2$,*

$$F_n^m = \begin{cases} 0, & \text{if } n < m, \\ \oplus_{n_1 + \dots n_m = n} F_{n_1^1} \otimes \dots \otimes F_{n_m}^1 & \text{if } n \geq m. \end{cases}$$

In particular, the set of all (counital coaugmented) coalgebra morphisms $\operatorname{Hom}(T^c(V), T(W))$ *, which corresponds to* $\operatorname{Hom}\left(\overline{T}^c(V), \overline{T}(W)\right)$ *by Proposition A.1.3, can be identified with the vector space of degree zero maps*

$$\operatorname{Hom}^0(\overline{T}^c(V), W) = \prod_{n \geq 1} \operatorname{Hom}^0\left(V^{\otimes n}, W\right).$$

Definition A.1.7. Let C be a coassociative conilpotent coalgebra. A coderivation is a linear map $d : C \to C$ such that

$$\Delta \circ d = (d \otimes 1) \circ \Delta + (1 \otimes d) \circ \Delta$$

If C is counital, then $d(1) = 0$. A differential coassociative graded coalgebra is a coassociative graded coalgebra equipped with a coderivation.

The above proposition depends by the equation $\overline{\Delta} \circ F = (F \otimes F) \circ \overline{\Delta}$. There is a similar story for the coderivations. Fix a graded vector space V. For a linear map $D : \overline{T}^c(V) \to \overline{T}^c(V)$, we denote by D_n^m, $m, n = 1, 2, 3, ...$ the sets of maps

$$D_n^m : V^{\otimes n} \xhookrightarrow{i} \overline{T}^c(V) \xrightarrow{D} \overline{T}^c(W) \xrightarrow{pro_{V^{\otimes m}}} V^{\otimes m}$$

where the first map is the inclusion, and the last is the projection.

Proposition A.1.8. *A linear map $D : \overline{T}^c(V) \to \overline{T}^c(V)$ is a coderivation of graded tensor coalgebras ($\overline{\Delta} \circ D = (D \otimes 1) \circ \overline{\Delta} + (1 \otimes D) \circ \overline{\Delta}$) if and only if, for all $m \geq 2$,*

$$D_n^m = \begin{cases} 0, & \text{if } n < m, \\ \oplus_{i=0}^{m-1} Id^{\otimes i} \otimes D_{n-m+1}^1 \otimes Id^{m-i-1} & \text{if } n \geq m. \end{cases}$$

In particular, there is a one to one correspondence between the graded vector space of coderivations on $T(V)$ and the graded vector space

$$\mathrm{Hom}\left(T^c(V), V\right) \cong \prod_{n \geq 1} \mathrm{Hom}^0\left(V^{\otimes n}, V\right)$$

A.1.1 Bar Construction

We define the bar construction for a differential graded algebra A. Our definition is different from the one of [13]. Consider the graded tensor coalgebra $BA := T^c(A[1])$. Each element of $A[1]^{\otimes}$ can be visualized as a $(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_n)$, where s is cosnidered as a formal variable of degree -1. We define $d_2 : (A[1])^{\otimes n} \to (A[1])^{\otimes n-1}$ of degree $+1$ via

$$d_2(sa) = 0$$

for $n = 1$, and for $n > 1$

$$d_2\left(sa_1 \otimes \cdots \otimes sa_n\right) = \sum_{i=1}^{n-1} (-1)^{i-1+|a_1|+\ldots|a_i|} a_1 \otimes \cdots \otimes \mu\left(a_i, a_{i+1}\right) \otimes \cdots \otimes a_n$$

where $d_2^2 = 0$ follows from the associativity of μ. The differential d_1 is induced by the differentials of A. More precisely $d_1 : (A[1])^{\otimes n} \to (A[1])^{\otimes n}$ is given by

$$d_1\left(sa_1 \otimes \cdots \otimes sa_n\right) = \sum_{i=1}^{n} \left(sa_1 \otimes \cdots \otimes d_{sA}(-) \otimes \cdots \otimes sa_n\right)$$

We have $d_1 \circ d_2 + d_2 \circ d_1 = 0$. The Bar construction of A is the total complex

$$BA, d_1 + d_2, \tag{A.1}$$

it is a conilpotent coalgebras. The above construction is functorial and thus defines a functor. Such a functor has a left adjoint Ω called cobar construction. See [39] section 2.2 for more details.

A.2 Filtered and Complete graded vector spaces

The goal of this subsection is to introduce some basic knowledge about completion of graded vector spaces and to prove Lemma A.2.5, which is fundamental in Section 2.1. We work on a field $\Bbbk$ of charachteristic zero.

Definition A.2.1. A filtered vector space $(V, F^\bullet)$ is a vector space equipped with a filtration

$$V = F^0(V) \supseteq F^1(V) \supseteq F^2(V) \supseteq \dots$$

of graded subspaces $F^i(V) \subseteq V$. A degree l morphism of filtered vector space is a morphism $f : (V, F^\bullet) \to (W, G^\bullet)$ of vector spaces such that $f\left(F^i(V)\right) \subseteq G^i(V)$ and $f(V^i) \subseteq W^{i+l}$. We denote the category of filtered graded vector spaces by $fgVect$.

A filtration makes V a topological graded vector space where $\{F^\bullet(V)\}$ is a local basis of neighborhood at 0. A filtration preserving degree l morphism between filtered graded vector spaces $f : (V, F^\bullet) \to (W, G^\bullet)$ is a continuous map between topological vector spaces such that $f(V^i) \subseteq W^{i+l}$. The topology induced by the filtration is Hausdorff if and only if $\bigcap_{i\geq 0} F^i(V) = 0$.

Given two filtered graded vector spaces $(V, F^\bullet), (W, G^\bullet)$, the tensor product induces a canonical filtration $(F \otimes G)^\bullet$ given by

$$(F \otimes G)^i (V \otimes W) := \oplus_{p+q=i} F^p(V) \otimes G^q(W) \tag{A.2}$$

where $F^p(V) \otimes G^q(W)$ is the ordinary graded tensor product. In particular, the tensor product of graded vector space $\otimes$ together with the induced filtration above defines a symmetric monoidal structure on $fgVect$.

Given a graded vector space W, we define the *trivial filtration* as

$$F^i(W) := \begin{cases} W, & \text{if } i = 0, \\ 0, & \text{otherwise} \end{cases}$$

We consider the ground field $\Bbbk$ equipped with the trivial filtration. It is the unit of the above symmetric monoidal structure.

Lemma A.2.2. *Let $(V, F^\bullet)$ be a filtered graded vector space.*

1. *Let $q_i : V \to V/F^i(V)$ be the projection. There exists a sequence of subsets $A_i \subset V$, $A_i := \left\{v^i_j\right\}_{j\in J_i}$*

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$$

   such that

   (a) $\ker(q_i) \cap A_i = 0$,

   (b) $q_i(A_i)$ *is a basis of* $V/F^i(V)$.

2. *Assuming $\bigcap_{i\geq 0} F^i(V) = 0$, the set $\mathcal{A} = \bigcup_{i\geq 0} A_i$ is a basis of V and for any $j \in \mathbb{N}$*

$$\mathcal{A} \setminus A_j$$

   is a basis of $F^j(V)$.

Proof. 1. The first part is proved by induction. For $i = 0$ we set $A_0 = 0$. Hence we assume that the statements holds for $i - 1$. From the projection $p_{i-1} : V/F^i(V) \to V/F^{i-1}(V)$. we get a decomposition

$$V/F^{i-1}(V) \oplus F^{i-1}(V)/F^i(V) = V/F^i(V).$$

Choose then a basis B_i of $F^{i-1}(V)/F^i(V)$ and set $A_i := A_{i-1} \cup B_i$.

2. Let V' be the vector space generated by $\mathcal{A}$. Then $W := V/V'$ satisfies $V = V' \oplus W$. Let $w \in W$, then the condition

$$V/F^j(V) \oplus F^j(V) = V$$

implies $w \in F^i(V)$ for each i. The rest of the assertion follows directly.

□

Remark A.2.3. A consequence of the above lemma is that we can consider $V/F^i(V)$ as the vector space generated by A_i, and $F^i(V)/F^{i+1}(V)$ as the subspace of V generated by $A_{i+1} \setminus A_i$.

We denote the free algebra on a graded vector space V by $(T(V), \mu)$, where μ is the concatenation product. It is of finite type only in two cases: either if $V^\bullet$ is of finite type and bounded below with $k > 0$ or if $V^\bullet$ is of finite type and and bounded above with $k < 0$, unfortunately we are interested to the case $k = 0$. We fix a finite type bounded below at k graded vector space V. We define two filtrations. Let $\epsilon : T(V) \to \Bbbk$ be the augmentation map, let $I := \ker(\epsilon)$ be the augmentation ideal. The sequence given by the power of I

$$I^0 := T(V) \subset I \subset I^2 \subset \dots$$

is a filtration $I^\bullet$ on $T(V)$. We define the filtration $F^\bullet$ on V as $F^l(V^\bullet) := \oplus_{j \geq k+l} V^j$. Let $\{v_i^l\}_{i \in J(l)}$ be a basis of V^l, hence

$$\bigcup_{i \geq 0} \bigcup_{j \in J(k+i)} v_j^{k+i}$$

is a basis of V. We denote such a basis by $\{v_i\}_{i \in J(l), l \geq 0}$. An element of $T(V)$ can be written in unique way as a non commutative polynomial

$$\sum_{p \geq 0}^{N} \sum_{(i_1, \dots, i_p)} \lambda_{i_1, \dots, i_p} v_{i_1} \cdots v_{i_p},$$

such that only finitely many $\lambda_{i_1, \dots, i_p}$ are different from 0. The elements $f \in I^i(T(V)) \subset T(V)$ can be written in unique way as

$$\sum_{p \geq i}^{N} \sum_{(i_1, \dots, i_p)} \lambda_{i_1, \dots, i_p} v_{i_1} \cdots v_{i_p},$$

such that only finitely many $\lambda_{i_1, \dots, i_p}$ are different from 0. Following the notation of Lemma A.2.2, a basis A_i^I of $T(V)/I^i$ is the set of monomials $v_{i_1} \cdots v_{i_p}$ such that $p < i$. The tensor algebra $T\left(V/F^j(V)\right)$ may be considered as the graded algebra of non commutative polynomial $\Bbbk\left\langle \{v_i\}_{i \in J(l), l < j} \right\rangle$, where $\{v_i\}_{i \in J(l), l < j}$ is a basis of $V^k \oplus \dots V^{k+j-1}$. For each $j \geq 0$, $G^j(T(V)) \subset T(V)$ is the subspace generated by

$$\sum_{p \geq 0}^{N} \sum_{(i_1, \dots, i_p)} \lambda_{i_1, \dots, i_p} v_{i_1} \cdots v_{i_p} \in \Bbbk\left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle,$$

such that for each $v_{i_1} \cdots v_{i_p}$ there exists a $v_{i_n} \in V^s$, $s \geq j + k$. In particular

$$G^j(T(V)) \oplus T\left(V/F^j(V)\right) = T(V).$$

Following the notation of Lemma A.2.2, a basis A_j^G of $T(V)/G^j(T(V)) = T\left(V/F^j(V)\right)$ is given by the monomials $v_{i_1} \cdots v_{i_p}$ such that each $v_{i_j} \in V^k \oplus \dots V^{k+j-1}$. Let

$$H_{i,j} := \left(T(V)/I^i(T(V))\right)/G^j(T(V)),$$

we identify $H_{i,j}$ as the subspace of $T(V)$ generated by $A_{i,j} := A_j^G \setminus A_i^I$. They form a diagram where all the maps are inclusion and two objects $H_{a,b}$, $H_{c,d}$ are connected by a map $H_{a,b} \hookrightarrow H_{c,d}$ if $a \leq b$ and $c \leq d$, in particular $\mathrm{colim}_{i,j} H_{i,j} = T(V)$.
For V bounded above at k, we define the filtrations $(I^\bullet, F^\bullet, G^\bullet)$ in a similar way: $I^\bullet$ is given by the power of the augmentation ideal, the filtration $F^\bullet$ on V is given by $F^l(V^\bullet) := \oplus_{j \leq k-l} V^j$ and the filtration $G^\bullet$ on $T(V)$ is defined such that

$$G^j(T(V)) \oplus T\left(V/F^j(V)\right) = T(V).$$

Again we define $H_{i,j} := \left(T(V)/I^i(T(V))\right)/G^j(T(V))$ and we identify $H_{i,j}$ as the subspace of $T(V)$ generated by $A_{i,j} := A_j^G \setminus A_i^I$.
Let V be a bounded below at k graded vector space endowed with a basis $\{v_i\}_{i \in I(l), l \geq 0}$ as above and W be a graded vector space equipped with the trivial filtration. We consider the tensor product of filtration

$$I^i(W \otimes T(V)) := W \otimes I^i(T(V)), \quad G^j(W \otimes T(V)) := W \otimes G^j(T(V)).$$

Then $W \otimes T(V) \cong W \otimes \Bbbk \left\langle \{v_i\}_{i \in I(l), l \geq 0} \right\rangle$. In particular each element can be written in unique way as

$$\sum_{p \geq 0}^{N} \sum_{q \geq 0}^{M} \sum_{v_{i_1} \cdots v_{i_p} \in A_{p,q}} w_{i_1, \ldots, i_p} \otimes v_{i_1} \cdots v_{i_p} \tag{A.3}$$

for some N, M. Let $W \widehat{\otimes}_I \Bbbk \left\langle \left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle \right\rangle$ be the completion of $W \otimes T(V)$ with respect to $I^\bullet$. It is the graded vector space of formal power series of the form (A.3) with $N = \infty$. On the other hand, let $W \widehat{\otimes}_G \Bbbk \left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle$ be the completion of $W \otimes T(V)$ with respect to $G^\bullet$, it is the formal graded vector space of formal power series (A.3) with $M = \infty$. We denote by $\mathcal{I}_G^\bullet$ the filtration obtained by the completion of $I^\bullet$ with respect to $G^\bullet$ on $W \widehat{\otimes}_G \Bbbk \left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle$. We denote by $\mathcal{G}_I^\bullet$ the filtration obtained by the completion of $G^\bullet$ with respect to $I^\bullet$ on $W \widehat{\otimes}_I \Bbbk \left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle$. The completion of $W \widehat{\otimes}_G \Bbbk \left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle$ with respect to $\mathcal{I}_G^\bullet$ coincides with the completion of $W \widehat{\otimes}_I \Bbbk \left\langle \{v_i\}_{i \in J(l), l \geq 0} \right\rangle$ with respect to $\mathcal{G}_I^\bullet$. We denote the obtained vector space by $W \widehat{\otimes} \widehat{T}(V)$, it is the vector space of formal power series

$$\sum_{p \geq l}^{\infty} \sum_{q \geq 0}^{\infty} \sum_{v_{i_1} \cdots v_{i_p} \in A_{p,q}} w_{i_1, \ldots, i_p} \otimes v_{i_1} \cdots v_{i_p}$$

We denote the induced filtrations on the completion by $\mathcal{I}$ and resp. $\mathcal{G}$. The same arguments are true for bounded above graded vector spaces as well.

Let Σ_n be the group of permutations on $\{1, 2, \ldots, n\}$. Consider two finite strings of natural numbers $1 \leq i_1 < \cdots < i_p$, $1 \leq j_1 < \cdots < j_q$ for $p, q \geq 0$. We associate a permutation $\sigma \in \Sigma_{p+q}$ via

$$\sigma(l) := \begin{cases} i_l, & \text{if } l \leq p \\ j_l & \text{otherwise} \end{cases}$$

A permutation obtained in this way is called (p,q)-*shuffle*. We denote the set of (p,q)-shuffles by $Sh(p,q)$. We define the sfuffles coproduct $\Delta' : T^c(V) \to T^c(V) \otimes T^c(V)$ via

$$\Delta'(v_1 \cdots v_n) := \sum_{p+q=n, \sigma \in Sh(p,q)} \pm \operatorname{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(p+q)} \tag{A.4}$$

where the signs $\pm$ are given by the signs rule. This makes $(T^c(V), \mu, \Delta')$ a bialgebra. The dual statement is true as well. The product $\mu' : T^c(V) \otimes T^c(V) \to T^c(V)$ given via

$$\mu'(v_1 \cdots v_p \otimes v_{p+1} \cdots v_{q+p}) := \sum_{\sigma \in Sh(p,q)} \pm \operatorname{sign}(\sigma) g v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(p)} \cdots v_{\sigma^{-1}(q+p)} \tag{A.5}$$

makes $(T^c(V), \mu', \Delta)$ a bialgebra. Let $NTS(V) \subset T^c(V)$ be the graded subspace of non trivial shuffles, it is generated by $\mu'(a,b)$ where $a, b \in T^c(V)$, $a, b \notin \Bbbk$. We get a (non exact) sequence

$$T^c(V) \otimes T^c(V) \xrightarrow{\overline{\mu}'} T^c(V) \xrightarrow{p} T^c(V)/NTS(V^\bullet), \tag{A.6}$$

where p is the projection. In particular, $T^c(V)/NTS(V^\bullet)$ is the universal enveloping coalgebra of the free co Liealgebra $\mathbb{L}^c(V)$. We denote $T^c(V)/NTS(V^\bullet)$ by $\mathbb{U}^c(\mathbb{L}^c(V))$.

The *graded dual* of a graded vector space $V^\bullet$ is given by $(V^*)^\bullet := \operatorname{Hom}^\bullet(V, \Bbbk)$. Given a filtered vector space (V, F), we denote by F_* the dual filtration where $F_*^i \operatorname{Hom}(V, \Bbbk)$ is the set of graded morphism such that $f|_{F^i(V)} = 0$. If U is finite dimensional we have

$$U^* \otimes V \cong \operatorname{Hom}^\bullet(U, V). \tag{A.7}$$

Let V be a graded vector space bounded below at k equipped with a basis $\{v_i\}_{i \in I(l), l \geq 0}$ as above and W be a graded vector space equipped with the trivial filtration. We define two filtrations $\mathcal{I}$ and $\mathcal{G}$ on

$Hom^{\bullet}(T(V),W)$ as follows: $\mathcal{I}^i\left(Hom^{\bullet}(T(V),W)\right) \subseteq Hom^{\bullet}(T(V),W)$ (resp. $\mathcal{G}^i\left(Hom^{\bullet}(T(V),W)\right) \subseteq Hom^{\bullet}(T(V),W)$) is the subspace of map $f : T(V) \to W$ such that

$$f|_{T(V)/I^i} \equiv 0, \quad (\text{resp. } f|_{T(V)/G^i} \equiv 0)$$

We consider V^* equipped with the filtration $F^\bullet$ and $T(V^*)$ equipped with the filtration $I^\bullet$ and $G^\bullet$. We define $H^*_{i,j} := \left(T(V^*)/G^j(T(V))\right)/I^i$. If the $H_{i,j}$ are finite, the vector space $H^*_{i,j} \subset T(V^*)$ is actually the dual (as a finite graded vector space) of $H_{i,j}$. The dual of the inclusion $H_{a,b} \hookrightarrow H_{c,d}$ is precisely the projection map

$$H^*_{c,d} = \left(T(V^*)/G^d(T(V^*))\right)/I^c \twoheadrightarrow H^*_{a,b} = \left(T(V^*)/G^b(T(V^*))\right)/I^a.$$

Assume that V is of finite type. For a graded vector space W, let $\Psi_{i,j} : H^*_{i,j} \otimes W \to \operatorname{Hom}^\bullet(H_{i,j}, W)$ be the obvious isomorphism. Since $\operatorname{Hom}(-,W)$ commutes with the colimit, the limit of the $\Psi_{i,j}$ give an isomorphism

$$\Psi : \operatorname{Hom}\left(T^c(V[1]), W\right) \to \widehat{T}\left((V[1])^*\right)\widehat{\otimes}W. \tag{A.8}$$

such that $\operatorname{Hom}^\bullet(T(V),W)$ is isomorphic as a bifiltered vector space to the completion of $T(V^*)\otimes W$ with respect to $I^\bullet$ and $G^\bullet$. In a similar way there is an isomorphism of bifiltered completed graded vector space

$$\Phi : \operatorname{Hom}^\bullet(T^c(V)\otimes T^c(V), \Bbbk) \to \widehat{T}(V^*)\widehat{\otimes}\widehat{T}(V^*). \tag{A.9}$$

Moreover, since the maps μ, Δ, μ' and Δ' are all degree zero filtration preserving morphisms with respect to I and G, we have the following result.

Lemma A.2.4. *Let V be a bounded below or bounded above graded vector space. Then*

$$p^*\left(\mathbb{U}^c\left(\mathbb{L}^c(V)\right)\right)^* \subset \operatorname{Hom}^\bullet(T^c(V), \Bbbk)$$

is the graded vector space of morphisms in $\mathcal{I}^1$ that vanish on non-trivial shuffles. Assume that V is of finite type, there is a commutative diagram where the vertical arrows are isomorphisms.

$$\begin{array}{ccccc}
p^*\left(\mathbb{U}^c\left(\mathbb{L}^c(V)\right)\right)^* & \xrightarrow{i} & \operatorname{Hom}^\bullet(T^c(V),\Bbbk) & \xrightarrow{(\mu')^*} & \operatorname{Hom}^\bullet(T^c(V)\otimes T^c(V),\Bbbk) \\
\downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Phi} \\
\widehat{\mathbb{P}}\left(\widehat{T}(V^*)\right) & \xrightarrow{i} & \widehat{T}(V^*) & \xrightarrow{\widehat{\Delta'}} & \widehat{T}(V^*)\widehat{\otimes}\widehat{T}(V^*)
\end{array}$$

where $\widehat{\mathbb{P}}\left(\widehat{T}(V')\right) := \left\{x \mid \widehat{\Delta'}(x) = \operatorname{Id}^*\widehat{\otimes}x + x\widehat{\otimes}\operatorname{Id}^*\right\}$.

Proof. The commutativity of the first square is straightforward and follows by the construction of Ψ and Φ. For the first square we need only to show that Ψ is well-defined. Let $f \in p^*\left(\mathbb{U}^c\left(\mathbb{L}^c(V)\right)\right)^*$. Then for $v, w \notin \Bbbk$ we have $(\mu')^*\Psi(f)(v\otimes w) = \Psi(f)(\mu'(v\otimes w)) = 0$. Analogously $(\mu')^*\Psi(f)(1\otimes w) = \Psi(f)(w)$ and $(\mu')^*\Psi(f)(w\otimes 1) = \Psi(f)(w)$, thus $\Delta'\Psi(f) = (\mu')^*\Psi(f) = \operatorname{Id}^*\widehat{\otimes}\Psi(f) + \Psi(f)\widehat{\otimes}\operatorname{Id}^*$. $\square$

We denote $p^*\left(\mathbb{U}^c\left(\mathbb{L}^c(V)\right)\right)^*$ by L_{V^*}.

Lemma A.2.5. *Let V as above.*

1. *$\left(\operatorname{Hom}^\bullet(T^c(V),\Bbbk), (\Delta)^*\right)$ is an associative graded algebra. The filtration $\mathcal{I}$ is a filtration of ideals with respect to $(\Delta)^*$.*

2. *Let $[f,g] := \Delta^*(f,g) - (-1)(|f||g|)\Delta^*(g,f)$. Then $(\operatorname{Hom}^\bullet(T^c(V),\Bbbk), [-,-])$ is a Lie algebra and $(L_{V^*}, [-,-])$ is a sub Lie algebra. We denote by $\mathcal{I}$ and $\mathcal{G}$ the restrictions of the two filtrations on $(L_{V^*}, [-,-])$. Then $\mathcal{I}$ is a filtration of Lie ideals with respect to $[-,-]$.*

3. *Assume that V is of finite type. Then L_{V^*} is isomorphic to the completion of the free Lie algebra $\widehat{\mathbb{L}(V)}$ on V with respect to the filtration I and G. In particular each element of L_{V^*} can be written as as a formal power series of Lie monomials*

$$\sum_{p\geq 1}^{\infty}\sum_{q\geq 0}^{\infty}\sum_{v_{i_1}\cdots v_{i_p}\in A_{p,q}}\lambda_{i_1,\ldots,i_p}\left[v_{i_1^*},\left[v_{i_2^*},\ldots,\left[v_{i_{p-1}^*},v_{i_p^*}\right],\ldots\right]\ldots\right],$$

Proof. 1. The map

$$\mathrm{Hom}^\bullet(T^c(V),\Bbbk)\otimes\mathrm{Hom}^\bullet(T^c(V),\Bbbk)\xrightarrow{i}\mathrm{Hom}^\bullet(T^c(V)\otimes T^c(V),\Bbbk)\xrightarrow{(\Delta)^*}\mathrm{Hom}^\bullet(T^c(V),\Bbbk)$$

is an associative product.

2. Clearly $[f,g]:=\Delta^*(f,g)-(-1)^{(|f||g|)}\Delta^*(g,f)$ satisfies the Jacobi identity. Since $(T^c(V),\Delta,\mu')$ is a bialgebra

$$\begin{aligned}&\left(\Delta^*(f,g)-(-1)^{(|f||g|)}\Delta^*(g,f)\right)\mu'(a\otimes b)\\=&\left(f\otimes g-(-1)^{(|f||g|)}g\otimes f\right)\Delta\circ\mu'(a\otimes b)\\=&\left(f\otimes g-(-1)^{(|f||g|)}g\otimes f\right)(\mu'\otimes\mu')(Id\otimes\tau\otimes Id)\,\Delta\otimes\Delta\,(a\otimes b)\\=&\left(f\otimes g-(-1)^{(|f||g|)}g\otimes f\right)\left(1\otimes\mu'(a\otimes b)+(-1)^{|a||b|}b\otimes a+a\otimes b+\mu'(a\otimes b)\otimes 1\right)\\=&0\end{aligned}$$

and hence $[p^*\mathrm{Hom}^\bullet(T^c(V),\Bbbk),p^*\mathrm{Hom}^\bullet(T^c(V),\Bbbk)]\subset p^*\mathrm{Hom}^\bullet(T^c(V),\Bbbk)$. The proof that $\mathcal{I}$ is a filtration of Lie ideals follows immediately.

3. See [46] for a proof that $\mathbb{P}(T(V))\cong\mathbb{L}(V)$. The above statement follows by showing $\widehat{\mathbb{L}(V^*)}\cong\lim_{i,j}\mathbb{L}(H^*_{i,j})$ and by the previous lemma.

□

A.2.1 Proof of Lemma 2.1.15 and corollary (2.1.18)

We start with the proof of Lemma 2.1.15.

Proof. Clearly $\partial^2=0$ and $M_1^2=0$. The relations (2.1) for $n>1$ are equivalent to

$$\sum_{\substack{p+q+r=n\\p+1+r>1,q>1}}(-1)^{p+qr}m_{p+1+r}\circ\left(Id^{\otimes p}\otimes m_q\otimes Id^{\otimes r}\right)=D\circ m_n,\tag{A.10}$$

where $D\circ m_n:=m_1\circ m_n+(-1)^{n+1}m_n\circ\left(\sum_{i=1}^{n-1}Id^{\otimes i}\otimes m_1\otimes Id^{\otimes n-1-i}\right)$. Chose homegeneous elements $f_1,\ldots,f_n$. The expression (A.10) in our case is

$$\begin{aligned}&\sum_{\substack{p+q+r=n\\p+1+r>1,q>1}}(-1)^{p+qr}M_{p+1+r}\circ\left(Id^{\otimes p}\otimes M_q\otimes Id^{\otimes r}\right)(f_1\otimes\cdots\otimes f_n)=\\&\sum_{\substack{p+q+r=n\\p+1+r>1,q>1}}(-1)^{p+qr}\tilde{m}^A_{p+1+r}\left(\left(Id^{\otimes p}\otimes M_q\otimes Id^{\otimes r}\right)\circ(f_1\otimes\cdots\otimes f_n)\right)\circ\Delta^{p+r}=\\\pm&\sum_{\substack{p+q+r=n\\p+1+r>1,q>1}}(-1)^{p+qr}\tilde{m}^A_{p+1+r}\left(f_1\otimes\cdots\otimes f_p\otimes\left(\tilde{m}^A_q(f_{p+1},\ldots,f_{p+q})\circ\Delta^{q-1}\right)\otimes f_{p+q+1}\otimes\cdots\otimes f_n\right)\circ\Delta^{p+r}=\\\pm&\sum_{\substack{p+q+r=n\\p+1+r>1,q>1}}(-1)^{p+qr}\tilde{m}^A_{p+1+r}\left(f_1\otimes\cdots\otimes f_p\otimes\tilde{m}^A_q(f_{p+1},\ldots,f_{p+q})\otimes f_{p+q+1}\otimes\cdots\otimes f_n\right)\circ\Delta^{n-1}=\\&\sum_{\substack{p+q+r=n\\p+1+r>1,q>1}}(-1)^{p+qr}\tilde{m}^A_{p+1+r}\left(Id^{\otimes p}\otimes\tilde{m}^A_q\otimes Id^{\otimes r}\right)\circ(f_1\otimes\cdots\otimes f_n)\circ\Delta^{n-1}.\end{aligned}$$

Set $m_1 = \partial$, we have

$$\left(\partial \circ M_n + (-1)^{n+1} M_n \circ \left(\sum_{i=1}^{n-1} Id^{\otimes i} \otimes \partial \otimes Id^{\otimes n-1-i}\right)\right)(f_1 \otimes \cdots \otimes f_n).$$

The first summand is

$$\partial \circ M_n(f_1 \otimes \cdots \otimes f_n)$$
$$= \tilde{m}_1^A \tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} + (-1)^{|M_n(f_1 \otimes \cdots \otimes f_n)|+1} \tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} \circ \delta,$$

the second is

$$(-1)^{n+1} M_n \circ \left(\sum_{i=1}^{n-1} Id^{\otimes i} \otimes \partial \otimes Id^{\otimes n-1-i}\right)(f_1 \otimes \cdots \otimes f_n) =$$

$$(-1)^{n+1+|f_1|+\cdots+|f_p|} \sum_{i=1}^{n-1} M_n \left(f_1 \otimes \cdots \otimes f_p \otimes \left(\tilde{m}_1^A f_{p+1} + (-1)^{|f_{p+1}|+1} f_{p+1} \circ \delta\right) \otimes f_{p+2} \otimes \cdots \otimes f_n\right) =$$

$$(-1)^{n+1} \sum_{i=1}^{n-1} \tilde{m}_n^A \left(Id^{\otimes p} \otimes \tilde{m}_1^A \otimes Id^{\otimes n-p-1}\right)(f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} +$$

$$(-1)^{n+|f_1|+\cdots+|f_n|} \tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \circ \left(\sum_{i=1}^{n-1} Id^{\otimes p} \otimes \delta \otimes Id^{\otimes r}\right) \circ \Delta^{n-1} =$$

$$(-1)^{n+1} \sum_{i=1}^{n-1} \tilde{m}_n^A \left(Id^{\otimes p} \otimes \tilde{m}_1^A \otimes Id^{\otimes n-p-1}\right)(f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} +$$

$$(-1)^{n+|f_1|+\cdots+|f_n|} \tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} \circ \delta.$$

Since $\tilde{m}_\bullet^A$ is an A_∞-structure

$$\sum_{\substack{p+q+r=n \\ p+1+r>1, q>1}} (-1)^{p+qr} \tilde{m}_{p+1+r}^A \left(Id^{\otimes p} \otimes \tilde{m}_q^A \otimes Id^{\otimes r}\right) \circ (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} =$$

$$\tilde{m}_1^A \tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} + (-1)^{n+1} \sum_{i=1}^{n-1} \tilde{m}_n^A \left(Id^{\otimes p} \otimes \tilde{m}_1^A \otimes Id^{\otimes n-p-1}\right)(f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1}.$$

This shows that $(\partial, M_2, M_3, \dots)$ is an A_∞-structure. The second summand is canceled by the second summand of the first summand and this shows that $(M_\bullet)$ is an A_∞-structure. We prove the third statement. As proved in Lemma A.2.5, For $n = 1, 2$, we have that l_n and l'_n are well-defined on $p^* \left(\mathrm{Hom}^\bullet \left(T^c(V[1])/NTS(V), A\right)\right)$. Let $n > 2$, and $f_1, \dots, f_n \in L_{V[1]^*}(A)$ be homogeneous elements. Let $\mu'(a, b)$ be a non trivial shuffle in $T^c(V[1]) \otimes T^c(V[1])$. For any n we have

$$\tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{n-1} \circ \mu'(a, b) =$$
$$= \tilde{m}_n^A (f_1 \otimes \cdots \otimes f_n) \Big(\sum_{i \neq j}^{n} 1 \otimes \cdots 1 \otimes a \otimes 1 \cdots 1 \otimes b \otimes 1 \cdots \otimes 1$$
$$+ (-1)^{|a||b|} \sum_{i \neq j}^{n} 1 \otimes \cdots 1 \otimes b \otimes 1 \cdots 1 \otimes a \otimes 1 \cdots \otimes 1 \Big)$$
$$= 0$$

since $f_i(1) = 0$ for each i. □

Proof of corollary (2.1.18)

Proof. The first claim is immediate. We prove the second statement. Let $f_1, \dots, f_n \in \mathrm{Hom}^\bullet\left(T^c(V[1]^0), A\right)$ and assume that there is a g with $\delta^* g = f_i$ for some i. Then

$$\begin{aligned}
&\tilde{m}_n^A\left(f_1 \cdots \otimes \delta^* g \otimes \cdots \otimes f_n\right) \circ \Delta^{n-1} = \\
&= \tilde{m}_n^A\left(f_1 \cdots \otimes g \otimes \cdots \otimes f_n\right) \circ \left(Id \otimes \cdots \otimes \delta \otimes \cdots \otimes Id\right) \circ \Delta^{n-1} \\
&= \tilde{m}_n^A\left(f_1 \cdots \otimes g \otimes \cdots \otimes f_n\right) \circ \left(\sum_{i=1}^{n} Id \otimes \cdots \otimes \delta \otimes \cdots \otimes Id\right) \circ \Delta^{n-1}
\end{aligned}$$

since $\delta^* f_i = 0$ for any i. Then

$$\begin{aligned}
&\tilde{m}_n^A\left(f_1 \cdots \otimes g \otimes \cdots \otimes f_n\right) \circ \left(\sum_{i=1}^{n} Id \otimes \cdots \otimes \delta \otimes \cdots \otimes Id\right) \circ \Delta^{n-1} = \\
&= \tilde{m}_n^A\left(f_1 \cdots \otimes g \otimes \cdots \otimes f_n\right) \circ \Delta^{n-1} \circ \delta \\
&= \delta^* \circ \tilde{m}_n^A\left(f_1 \cdots \otimes g \otimes \cdots \otimes f_n\right) \circ \Delta^{n-1}.
\end{aligned}$$

The third assertion is straightforward. □

A.3 Conormalized graded module

The goal of this subsection is to construct an isomorphism $\psi : \mathrm{Tot}_N(A) \to \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ between differential modules. Fix a field $\Bbbk$ of charactersitic zero. Let $B^\bullet$ be a cosimplicial module. The *conormalized graded module* $N(B)^\bullet$ is a (cochain) graded modules defined as follows,

$$N(B)^p := \begin{cases} B^0, & \text{if } p = 0, \\ \bigcap_{i=0}^{p} \mathrm{Ker}\left(s_{p-1}^i : B^p \to B^{p-1}\right), & \text{otherwise.} \end{cases}$$

where s_i^p for $i = 0, \dots p-1$ are the codegenerancy maps. The differential $\partial : N(B)^p \to N(B)^{p+1}$ is given by the alternating sum of the coface maps

$$\partial = \sum_{i=0}^{p} (-1)^i d^i.$$

In particular $N^\bullet$ is a functor from the category of cosimplicial modules *cMod* toward the category of (cochain) differential graded modules *dgMod*.
A cosimplicial differential graded module is a cosimplicial objects in $A^{\bullet,\bullet} \in dgMod$ where the first slot denotes the cosimplicial degree and the second slot denotes the differential degree. It can be visualized as a sequence of cosimplicial modules

$$A^{\bullet,0} \xrightarrow{d} A^{\bullet,1} \xrightarrow{d} A^{\bullet,2} \xrightarrow{d} \dots$$

If we apply the functor N we turn the cosimplicial structure of each terms into a differential graded structures

$$N\left(A^{\bullet,0}\right)^\bullet \xrightarrow{d} N\left(A^{\bullet,1}\right)^\bullet \xrightarrow{d} N\left(A^{\bullet,2}\right)^\bullet \xrightarrow{d} \dots$$

moreover since each cosimplicial maps commutes with the differentials, we get a bicomplex $(N(A), d, \partial)$, where $N(A)^{p,q} := N(A^{p,\bullet})^q$. We define $\mathrm{Tot}_N(A) \in dgMod$ as the total complex associated to the bicomplex above. Explicitly an element $a \in \mathrm{Tot}\left(N(A)\right)^k$ is a collection

$$(a_0, \dots, a_k) \in A^{k,0} \oplus A^{1,k} \oplus \dots A^{0,k}$$

such that each a_i is contained in the kernel of some codegenrancy map. The following is well-known.

Lemma A.3.1. *Let $A^{\bullet,\bullet}$ be a cosimplicial differential graded module.*

1. *Let v be an elements of bidegree (p,q). Then each $v_n \in NC_n^p \otimes A^{n,q}$ is equal to zero for $p > n$.*

2. *NC_p^p is a one dimensional module.*

3. *An element v with bidegree (p,q) is completely determined by*

$$v_p \in NC_p^p \otimes A^{p,q}.$$

4. *There is an isomorphism between differential modules $\psi : \mathrm{Tot}_N(A) \to \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ such that for v with bidegree (p,q) we have*

$$\psi(b)_p = b \in NC_p^p \otimes A^{p,q}$$

Proof. The first two points are immediate. Fix a n and a $p \leq n$. Notice that each inclusion $[p] \hookrightarrow [n]$ is equivalent to an ordered string $0 \leq i_0 < i_i < \cdots < i_p \leq n$ contained in $\{0,1,\ldots,n\}$. For each string $0 \leq i_0 < i_i < \cdots < i_p \leq n$ we denote the associated inclusion by $\sigma_{i_0,\ldots i_p} : [p] \hookrightarrow [n]$, and we define the maps $\lambda_{i_0,\ldots,i_p} : \Delta[n]_p^+ \to \mathbb{k}$, via

$$\lambda_{i_0,\ldots,i_p}(\phi) := \begin{cases} 1 \text{ if } \sigma_{i_0,\ldots i_p} = \phi, \\ 0, \text{ otherwhise} \end{cases}$$

Clearly $\left\{\lambda_{i_0,\ldots i_p}\right\}_{0\leq i_0<i_i<\cdots<i_p\leq n}$ is a basis of NC_n^p. It turns out that v_n can be written as

$$v_n = \sum_{\theta\in\Delta[n]_p^+} \lambda_{i_0,\ldots,i_p} \otimes b^{i_0,\ldots,i_p}$$

for some $b^{i_0,\ldots i_p} \in B^{n,q}$. Let $v_p \in NC_p^p \otimes A^{p,q}$. Since NC_p^p is one dimensional we write $v_p = \lambda_{0,\ldots,p} \otimes b$, for some $b \in A^{p,q}$. We shows that $b^{i_0,\ldots i_p}$ is completely determined by b. Fix a $0 \leq i_0 < i_i < \cdots < i_p \leq n$. Then the above relations read as follows

$$\left(1 \otimes \sigma^*_{i_0,\ldots i_p}\right) v_p = \left(\sigma_{i_0,\ldots i_p *} \otimes 1\right) v_n.$$

In particular the map $\sigma_{i_0,\ldots i_p *} : NC_n^p \to NC_p^p$ is the linear map that sends $\lambda_{i_0,\ldots i_p}$ to $\lambda_{0,\ldots,p}$ and $\lambda_{j_0,\ldots j_p}$ to 0 for $(j_0,\ldots j_p) \neq (i_0,\ldots i_p)$. Then

$$\lambda_{0,\ldots,p} \otimes \sigma^*_{i_0,\ldots i_p}(b) = \left(1 \otimes \sigma^*_{i_0,\ldots i_p}\right) v_p = \left(\sigma_{i_0,\ldots i_p *} \otimes 1\right) v_n = \lambda_{0,\ldots,p} \otimes b^{i_0,\ldots i_p}.$$

Since each degree k elements can be written uniquely as a sum of elements of bidegree (p,q) with $p+q = k$ we get a natural isomorphism $\psi : \mathrm{Tot}_N(A) \to \int^{[n]\in\mathbf{\Delta}} NC_n \otimes A^{n,\bullet}$ of differential graded modules. □

A.4 The C_∞-structure on the 2 dimensional simplex

We first write down the formulas for (2.14). This diagram originally defined in [20] is intensively studied in [25]. We define two maps between simplicial differential graded module (see [20])

$$E_\bullet : NC_\bullet^\bullet \rightleftarrows \Omega^\bullet(\bullet) : \int_\bullet .$$

1. Fix a $[n] \in \mathbf{\Delta}$. For each $p \leq n$ we define $\int_n : \Omega^p(n) \to NC_n^p$ via

$$\int_n (w)(\sigma_{i_0,\ldots,i_p}) := \int_{\Delta_{geo}[p]} \sigma^*_{i_0,\ldots,i_p}(w)$$

i.e, we pulled back w along the smooth inclusion $\sigma_{i_0,\ldots,i_p} : \Delta[p]_{geo} \to \Delta[n]_{geo}$ and we integrate along the geometric standard p-simplex. $\int : \Omega^\bullet(n) \to NC_n$ is indeed a map between graded modules. The Stokes'theorem implies

$$\int_n (dw)(\sigma_{i_0,\ldots,i_p}) = \sum_{j=0}^{p} (-1)^j \int_n (w)(\sigma_{i_0,\ldots,\hat{i_j},\ldots i_p}),$$

i.e $\int_n : \Omega^\bullet(n) \to NC_n$ is a map between differential graded modules. Moreover, the above construction is compatible with the simplicial structure, i.e $\int_\bullet : \Omega^\bullet(\bullet) \to NC^*_\bullet$ is a map between simplicial differential graded modules.

2. We define the quasi inverse of $\int_\bullet$. We fix a $[n] \in \mathbf{\Delta}$. For each string $0 \leq< i_0 < i_1 < \cdots < i_p \leq n$ we define the *Whitney elementary form* $\omega_{i_0,\ldots,i_p} \in \Omega^p(n)$ via

$$\omega_{i_0,\ldots,i_p} := k! \sum_{j=0}^{p} (-1)^j t_{i_j} dt_{i_0} \wedge \cdots \wedge \hat{dt_{i_j}} \wedge \cdots \wedge dt_{i_p}$$

We define $E_n : NC^p_n \hookrightarrow \Omega^p(n)$ via

$$E_n(\lambda) := \sum_{0\leq i_0<\cdots<i_p\leq n} \lambda(\sigma_{i_0,\ldots,i_p})\omega_{i_0,\ldots,i_p}$$

The above map defines a map between differential graded modules $E_n : NC_n \hookrightarrow \Omega^\bullet(n)$. Since the construction is compatible with the simplicial maps we get a map between simplicial differential graded modules $E_\bullet : NC^\bullet_\bullet \hookrightarrow \Omega^\bullet(\bullet)$. Moreover since

$$\int_{\Delta[n]_{geo}} t_1^{a_1} \cdots t_n^{a_n} dt_1 \wedge dt_n := \frac{a_1! \cdots a_n!}{(a_1 + \ldots a_n + n)!}$$

an easy computation demonstrates that

$$\left(\int_\bullet\right) \circ E_\bullet = Id_{NC^\bullet_\bullet}.$$

It remains to construct an explicit simplicial homotopy between $E_\bullet \circ \left(\int_\bullet\right)$ and $\mathrm{Id}_{\Omega^\bullet(\bullet)}$. We recall the construction of [20] (see also [25]). Fix a n, for $0 \leq i \leq n$ we define the map $\phi_i : [0,1] \times \Delta_{geo}[n] \to \Delta_{geo}[n]$ via

$$\phi_i(u, t_0, \ldots, t_n) := (t_0 + (1-u)\delta_{i0}, \ldots, t_0 + (1-u)\delta_{in})$$

Let $\pi : [0,1] \times \Delta_{geo}[n] \to \Delta_{geo}[n]$ be the projection at the second coordinate and let $\pi_* : \Omega^\bullet(n) \to \Omega^{\bullet-1}(n)$ be the integration along the fiber. We define $h^i_n : \Omega^\bullet(n) \to \Omega^{\bullet-1}(n)$ via

$$h^i_n(w) := \pi_* \circ \phi^*_i(w)$$

We define $s_n : \Omega^p(n) \to \Omega^{p-1}(n)$ as follows:

$$s_n w := \sum_{j=0}^{p-1} \sum_{0\leq i_0<\cdots<i_j\leq n} \omega_{i_0\ldots i_j} \wedge h^{i_j}_n \ldots h^{i_0}_n(w)$$

We start with the proof of Proposition 2.2.5.

Proof. Recall that $\Omega^\bullet(2)$ is the free differential graded commutative algebra generated by the degree zero variables t_0, t_1 and t_2 modulo the relations

$$t_0 + t_1 + t_2 = 1, \quad dt_0 + dt_1 + dt_2 = 0.$$

On the other hand NC^0_2 is the vector space generated by λ_0, λ_1 and λ_2, NC^1_2 is generated by $\lambda_{01}, \lambda_{02}$ and λ_{12}, and NC^2_2 is the one dimensional vector space generated by λ_{0012}. We have

1. $E_2(\lambda_0) = t_0$, $E_2(\lambda_1) = t_1$, and $E_2(\lambda_2) = t_2$.
2. $E_2(\lambda_{01}) = t_0 dt_1 - t_1 dt_0$, $E_2(\lambda_{02}) = t_0 dt_2 - t_2 dt_0$, and $E_2(\lambda_{12}) = t_1 dt_2 - t_2 dt_1$.
3. $E_2(\lambda_{012}) = 2t_0 dt_1 dt_2 - 2t_1 dt_0 dt_2 + 2t_2 dt_0 dt_1$.

We prove a). For degree reason we have $m_2(\lambda_{01}, \lambda_{02}) = \mu_{01/02}\lambda_{012}$. By the homotopy transfer theorem we have

$$\begin{aligned}\mu_{01/02} &= \left(\int_{\Delta[2]} E_2(\lambda_{01}) E_2(\lambda_{02})\right) \\ &= \int_{\Delta[2]} t_0(t_0 dt_1 dt_2 - t_1 dt_0 dt_2 + t_2 dt_0 dt1) \\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}\mu_{01/12} &= \left(\int_{\Delta[2]} E_2(\lambda_{01}) E_2(\lambda_{12})\right) \\ &= \int_{\Delta[2]} t_1(t_2 dt_0 dt_1 - t_1 dt_0 dt_2 + t_0 dt_1 dt_2) \\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}\mu_{02/12} &= \left(\int_{\Delta[2]} E_2(\lambda_{01}) E_2(\lambda_{12})\right) \\ &= \int_{\Delta[2]} t_2(t_2 dt_0 dt_1 - t_1 dt_0 dt_2 + t_0 dt_1 dt_2) \\ &= \frac{1}{6}\end{aligned}$$

□

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