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ALGÈBRES À FACTORISATION ET TOPOS SUPÉRIEURS EXPONENTIABLES

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juillet 2016

Je dédie cette thèse à mes parents, Laurent et Catherine.

RÉSUMÉ

Cette thèse est composée de deux parties indépendantes ayant pour point commun l'utilisation intensive de la théorie des ∞-catégories.

Dans la première, on s'intéresse aux liens entre deux approches différentes de la formalisation de la physique des particules : les algèbres vertex et les algèbres à factorisation à la Costello. On montre en particulier que dans le cas des théories dites topologiques, elles sont équivalentes. Plus précisément, on montre que les ∞-catégories de fibrés vectoriels factorisant non-unitaires sur une variété algébrique complexe lisse X est équivalente à l'∞-catégorie des $\mathcal{E}_M$-algèbres non-unitaires et de dimension finie, où M est la variété topologique associée à X.

Dans la seconde, avec Mathieu Anel, nous étudions la caractérisation de l'exponentiabilité dans l'∞-catégorie des ∞-topos. Nous montrons que les ∞-topos exponentiables sont ceux dont l'∞-catégorie de faisceaux est continue. Une conséquence notable est que l'∞-catégorie des faisceaux en spectres sur un ∞-topos exponentiable est un objet dualisable de l'∞-catégorie des ∞-catégories cocomplètes stables munie de son produit tensoriel. Ce chapitre contient aussi une construction des ∞-coends à partir de la théorie du produit tensoriel d'∞-catégories cocomplètes, ainsi qu'une description des ∞-catégories de faisceaux sur un ∞-topos exponentiable en termes de faisceaux de Leray.

ABSTRACT

This thesis is made of two independent parts, both relying heavily on the theory of ∞-categories.

In the first chapter, we approach two different ways to formalise modern particle physics, through the theory of vertex algebras and the theory of factorisation algebras à la Costello. We show in particular that in the case of 'topological field theories', they are equivalent. More precisely, we show that the ∞-category of non-unital factorisation vector bundles on a smooth complex variety X is equivalent to the ∞-category of non-unital finite dimensional $\mathcal{E}_M$-algebras where M is the topological manifold associated to X.

In the second one, with Mathieu Anel, we study a characterisation of exponentiable objects of the ∞-category of ∞-toposes. We show that an ∞-topos is exponentiable if and only if its ∞-category of sheaves of spaces is continuous. An important consequence is the fact that the ∞-category of sheaves of spectra on an exponentiable ∞-topos is a dualisable object of the ∞-category of cocontinuous stable ∞-categories endowed with its usual tensor product. This chapter also includes a

construction of ∞-coends from the theory of tensor products of cocomplete ∞-categories, together with a description of ∞-categories of sheaves on exponentiable ∞-toposes in terms of Leray sheaves.

La plus grave maladie du cerveau,
c'est de réfléchir.

— Proverbe Shadock

REMERCIEMENTS

Faire des mathématiques m'apparait être avant tout une activité mondaine. Les mathématiciens se réunissent régulièrement autour d'un café, s'assoient devant un tableau noir et discutent des heures durant. Lorsqu'ils ne sont pas ensemble à échanger, ils se cultivent seuls, lisant les dernières nouveautés à la mode et réfléchissant à des choses innovantes. Ces dernières leur permettent d'initier et de nourrir de nouvelles discussions, perpétuant ainsi un cycle intellectuel sans fin. Les mathématiques pures n'ayant que rarement des applications lucratives, une équipe de mathématiques s'apparente ainsi à un cercle de poètes.

C'est dans ce monde que je prends plaisir à évoluer depuis quelques années déjà. Cependant, afin de faire partie de ce tout petit groupe et d'y persévérer, il faut d'abord faire ses preuves : c'est l'étape du doctorat. Il est donc l'heure pour moi de remercier tous ceux qui m'ont aidé à faire un jour partie de ce milieu à la fois accueillant et exigeant.

Je remercie en premier lieu mes deux directeurs de thèse, Damien et Grégory. Ils m'ont toujours soutenu durant ces quatre années ; leur disponibilité et leur grande bienveillance ont beaucoup compté.

Je remercie ensuite Carlos Simpson et Dennis Gaitsgory de m'avoir fait l'honneur d'être les rapporteurs de cette thèse. Je remercie également Christian Ausoni, Bruno Klingler, François Loeser et Bertrand Toën d'avoir accepté de faire partie du jury.

Je ne remercierai jamais assez toutes les personnes qui ont pris du temps à répondre à mes innombrables questions et ce, depuis que je sais dire « Mais pourquoi ? ». Par ordre chronologique : mon père Laurent, ma grand-mère Odile, mon oncle Bruno et les nombreux enseignants de toutes disciplines qui se sont succédés durant ma scolarité.

Mathieu mérite un paragraphe séparé. Je lui dois mon éducation mathématique moderne : catégories, homotopie, géométrie algébrique etc. Je n'ai cessé de m'enrichir tout au long de ces nombreuses journées de discussions mathématiques et extra-mathématiques. J'ai aussi eu la chance de pouvoir « faire des mathématiques » à ses côtés. Nos résultats sur les ∞-topos sont probablement ceux qui me rendent le plus heureux.

Je remercie ma femme Agnès et ma famille, pour leur indéfectible soutien et toute l'aide qu'ils m'ont apportés. Enfin je remercie toutes les personnes qui m'ont témoigné leur amitié et avec qui j'aime toujours autant passer du temps, dans le désordre : Alban, Magali, Matthieu, Jérémy, Nicolas, Brice, Matthias, Ombline, Matthieu, Ève, Benoit, Christophe, Sébastien, Nicolas, Benoit, Daniel-Jérôme, Alexandre, Benjamin, Arnaud. Je pense également à mes collègues de travail : Hugo, Marc, Thomas, Ruben et Amiel avec qui j'ai passé de bons moments dans le couloir 15-16.

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Part I

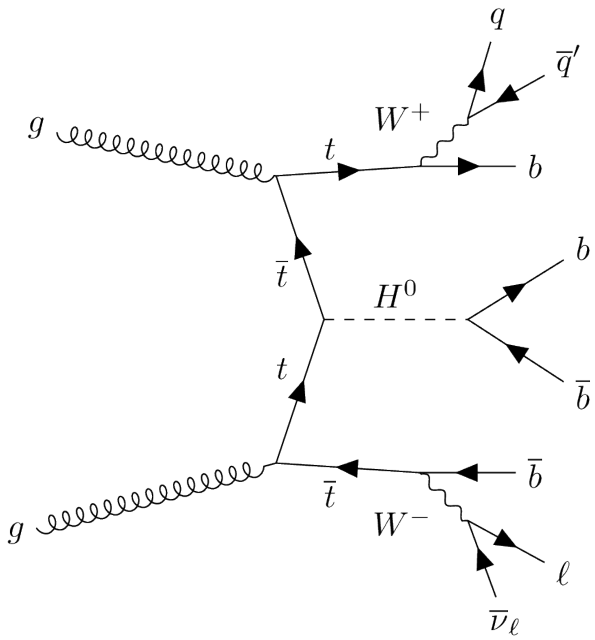
PROPOS LIMINAIRE

INTRODUCTION

Un jour, lors d'une école d'hiver nichée au milieu des Alpes suisses, je discutai avec un ami russe. Appelons-le Vladimir, c'est son prénom. Ayant contemplé une série d'exposés donnés durant la matinée par les étudiants, Vladimir me dit : « tu vois, dans un bon exposé, il doit y avoir des dessins et si possible, dès le début ! » Et si cette phrase est loin de celle que Vladimir prononça, malgré son excellent français, c'est en tous cas l'idée qu'il m'en est restée.

Si les dessins sont de toute première importance lors d'un exposé oral, ils le sont tout autant — si ce n'est davantage — lors de la présentation d'un manuscrit de thèse. Il est toujours bien de soigner la première impression et c'est souvent la seule chose qu'en retiendra le lecteur distrait ou étranger au domaine. Sans aucun doute ces deux catégories de lecteurs formeront l'immense majorité de ceux qui tiendront en leurs mains une copie de ce manuscrit.

Je m'exécute donc. Oyé, voici une image !



1.1 FACTORISATION !

Ce dessin vous dit peut-être quelque chose, il s'agit d'un diagramme de Feynman. Ces diagrammes sont au cœur de la théorie quantique des champs moderne, ils décrivent les interactions élémentaires entre particules. Ce sont eux qui permettent aux physiciens d'établir des prédictions au moyen de calculs, même si personne ne sait encore justifier les fondements mathématiques de la théorie. L'intuition nous donne des recettes de calculs à partir des diagrammes et si une recette est vérifiée expérimentalement, c'est qu'elle est bonne ! L'un des buts de la *physique mathématique* est de rendre conceptuellement rigoureuse la gastronomie des physiciens.

Je ne comprends moi-même pas grand chose au diagramme que je vous ai présenté. Je vais tout de même vous donner un grille de lecture très simple. Il semble être question de gluons, de quarks et de bosons de Higgs ; chaque *arête* du graphe représente en effet une particule élémentaire : g pour *gluon*, q pour *quark*, H^0 pour le *Higgs*, $\overline{\nu}$ est un *lepton* etc. Les sommets *externes* du graphe représentent les particules qui entrent et qui sortent ; les sommets *internes* du graphe représentent les interactions entre particules. À chaque sommet interne correspond une annihilation de particules suivi d'une création d'autres particules. Ces particules fraichement créées iront s'annihiler aux détours du prochain sommet, créant dans la foulée encore d'autres particules et ainsi de suite.

Nous nous arrêterons ici pour ce qui est du graphe. Une des caractéristiques marquantes de la théorie quantique des champs est — comme cela nous saute désormais aux yeux — qu'elle doit manœuvrer à travers un océan de particules qui s'annihilent et se créent, en permanence. Il n'est plus question d'appliquer les anciennes habitudes de la mécanique quantique classique où une expérience consiste à étudier un certain nombre de particules fixées à l'avance et de paramétrer leur évolution à l'aide d'équations différentielles. Non, le nombre total de particules évoluant sous le regard de l'expérimentateur est susceptible de changer à chaque instant. Il est donc nécessaire d'adapter les outils mathématiques à cette réalité.

1.1.1 *L'espace de Ran*

Dans le monde de l'algèbre, un moyen simple de s'accommoder des problèmes de « particules qui disparaissent et apparaissent » existe. Il suffit de bricoler quelque chose à partir de la théorie classique où, si l'on étudie une particule, elle reste là jusqu'à la fin de l'expérience. Supposons que la théorie à une seule particule soit codée par un objet algébrique A. La théorie pour deux particules sera alors codée par $A \otimes A$, celle à trois particules par $A \otimes A \otimes A$ etc. Si l'on veut un objet pour coder un nombre quelconque (mais fini) de particules, on peut tout simplement accoler toutes ces théories en un objet

$$A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \dots$$

Dans cette construction la même particule peut apparaitre plusieurs fois : si une particule p est codée dans A, alors $p \otimes p$ sera dans $A \otimes A$,

$p \otimes p \otimes p$ dans $A \otimes A \otimes A$, etc. On peut décider de tuer toutes ces redondances, on obtient alors un objet noté

$$\Lambda(A).$$

Cependant, il est plus avisé de construire un *espace* qui regrouperait toutes ces particules afin de pouvoir, par la suite, concevoir des objets algébriques plus complexes à partir de celui-ci. C'est à cette fin qu'un jouet pour mathématiciens, l'espace Λ, est utilisé.

Étant donné un espace-temps T — les dimensions dans lesquelles se déplacent les particules — on associe un nouvel espace $\Lambda(T)$ dont les points sont définis de la manière suivante,

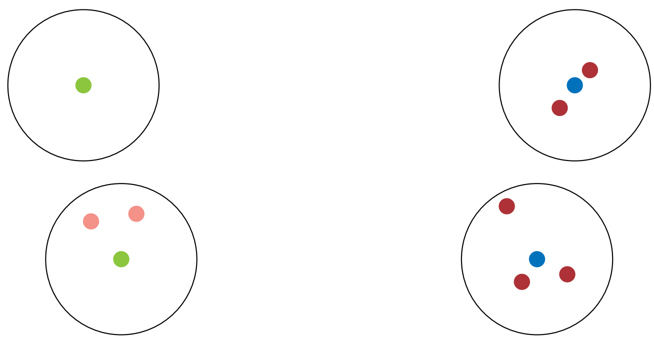
« les points de $\Lambda(T)$ sont les sous-parties finies de T. »

Ainsi, si l'espace temps considéré T est l'ensemble des réels $\mathbb{R}$, l'ensemble $\Lambda(T)$ contient des configurations de points telles que $\{2\}$, $\{\frac{2}{5}, \pi\}$, $\{\sqrt{3}, -9, e, \sin(7)\}$ ou bien encore la configuration vide $\{\}$.

Bien souvent l'espace-temps T est aussi muni d'une *métrique*. Il est en effet heureux de pouvoir mesurer des distances et des durées. Dans ce cas, il aussi possible de mesurer des distances dans $\Lambda(T)$, grâce à la *métrique de Hausdorff* :

« Deux configurations C et C′ de points de T, sont à distance plus petite qu'un réel positif d si, tout point de la configuration C est à une distance inférieure à d d'au moins un point de C′ et si réciproquement, tout point de la configuration C′ est à une distance inférieure à d d'au moins un point de C. »

Exemple :

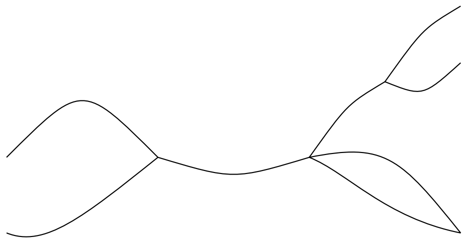


Configurations éloignées   Configurations proches

Notons cet espace métrique $\Lambda_H(T)$. On remarque au passage que toute configuration est à distance nulle de la configuration vide. Pour palier à ce petit inconvénient, on peut extraire le sous-espace de $\Lambda_H(T)$ constitué uniquement des sous-ensembles finis et non-vides. On le notera $\mathcal{R}_H(T)$, il s'agit de *l'espace de Ran*.

Remarque.— Bien que l'espace $\mathcal{R}_H(T)$ porte le nom du mathématicien Ziv Ran, il semble que la notion ait été énoncée pour la première fois par Beilinson et Drinfeld dans [BD].

Maintenant que nous savons mesurer des distances, il est possible de tracer des chemins continus dans l'espace de Ran $\mathcal{R}_H(T)$.



Représentation d'un chemin dans $\mathcal{R}_H(\mathbb{R})$

On retrouve dans ce dessin l'intuition des diagrammes de Feynman. Cet espace de Ran va nous être très utile par la suite.

1.1.2 *Algèbres à factorisation*

Plusieurs tentatives ont été réalisées dans le but d'exprimer rigoureusement la théorie quantique des champs. C'est l'une d'elles dont il va être question ici. Il s'agit des *algèbres à factorisation* telles qu'utilisées par Costello et Gwilliam dans [CG].

La modélisation de la physique par des algèbres à factorisation part du principe que pour décrire la théorie, il faut en décrire les *observables*. Ce sont toutes les mesures qu'un expérimentateur peut faire lorsqu'il étudie un système.

« Une algèbre à factorisation $\mathcal{A}$ sur T est une règle, qui associe à toute boule ouverte B_O de T, un espace vectoriel complexe $\mathcal{A}(B_O)$... »

Cet espace vectoriel complexe $\mathcal{A}(B_O)$ est le gadget qui code tout ce que l'observateur peut voir dans la région spatiotemporelle B_O. La région U, bien qu'inclue dans T, peut aussi être vue comme la région ouverte de $\mathcal{R}_H(T)$ contenant toutes les configurations de points qui tiennent dans B_O. Ainsi $\mathcal{A}(B_O)$ peut être pensé comme toutes les manières d'observer un nombre quelconque de particules présentes dans B_O.

Étant donnée deux boules inclues l'une dans l'autre $B_O \subset B'_O$, on suppose qu'un observateur effectuant des mesures dans le domaine B'_0 sera tout à fait capable d'observer les phénomènes qui apparaissent dans la région plus restreinte B_O.

« Une algèbre à factorisation $\mathcal{A}$ est une règle [...] qui à chaque inclusion de boules ouvertes $B_O \subset B'_O$, associe un morphisme d'espaces vectoriels $\mathcal{A}(B_O) \to \mathcal{A}(B'_O)$... »

Un expérimentateur peut aussi vouloir effectuer plusieurs mesures successives, à la condition que celles-ci ne soient pas réalisées au même endroit et au même moment. C'est l'un des point clés qui distingue les expériences menées dans le monde quantique de celles dont nous avons tous l'habitude dans le monde macroscopique.

Remarque.— Énoncé pour la première fois en 1927 par Werner Heisenberg, le principe d'incertitude stipule qu'il est impossible de connaitre simultanément et avec une précision parfaite à la fois la position d'une particule *et* sa vitesse. [HB]

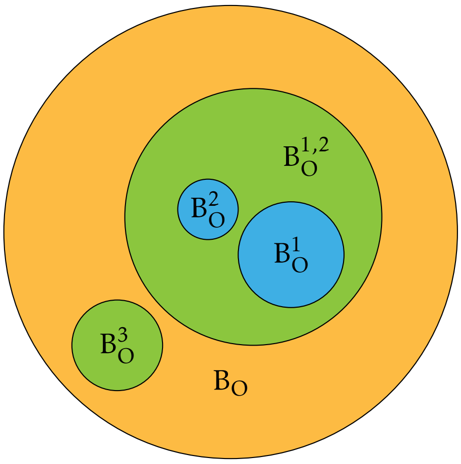
Continuons notre définition,

« Une algèbre à factorisation $\mathcal{A}$ est une règle [...] qui pour toute famille de boules ouvertes *disjointes* $B_O^1, B_O^2, \ldots, B_O^n$ incluses dans une plus grande boule B_O, associe une loi de composition

$$m_{1,\ldots,n} : \mathcal{A}(B_O^1) \otimes \mathcal{A}(B_O^2) \otimes \ldots \otimes \mathcal{A}(B_O^n) \to \mathcal{A}(B_O)$$

associative et commutative... »

L'adjectif commutatif signifie que la règle de composition $m_{1,2}$ est équivalence à la règle de composition $m_{2,1}$ à une permutation des facteurs près. L'adjectif associatif signifie quant à lui qu'étant donné une boule ouverte $B_O^{1,2}$ inclue dans B_O, contenant B_O^1 et B_O^2 et disjointe de B_O^3, alors la règle de composition $m_{1,2,3}$ peut aussi être calculée en utilisant d'abord l'étape intermédiaire $m_{1,2}^{\{1,2\}} : \mathcal{A}(B_O^1) \otimes \mathcal{A}(B_O^2) \to \mathcal{A}(B_O^{1,2})$ puis la composition $m_{\{1,2\},3} : \mathcal{A}(B_O^{1,2}) \otimes \mathcal{A}(B_O^3) \to \mathcal{A}(B_O)$. Cette situation correspond au dessin suivant.



Enfin, on doit être capable de reconstituer les observations faisables sur un très grand domaine, à partir des observations faites sur plein de petits sous-domaines. C'est à dire :

« Une algèbre à factorisation est une règle [...] et telle que, pour toute boule B_O et tout *recouvrement éparpillé* de B_O par une famille de boules $\{B_O^i\}_{i \in I}$, la valeur de $\mathcal{A}(B_O)$ peut être reconstituée à partir des valeurs $\mathcal{A}(B_O^i)$. »

On dit qu'une famille $\{B_O^i\}_{i\in I}$ est un *recouvrement éparpillé* de B_O si pour toute configuration de points C dans B_O, il existe une sous-famille de boules *disjointes* $\{B_O^{i_x}\}_{i_x\in J} \subset \{B_O^i\}_{i\in I}$ telle que pour tout point x de la configuration C, x appartient à $B_O^{i_x}$.

Cela correspond à dire que la famille $\{B_O^i\}_{i\in I}$ contient suffisamment de boules afin de pouvoir construire un voisinage de chaque configuration de B_O dans l'espace de Ran.

Cette définition d'algèbres à factorisation est un peu simplifiée mais tous les concepts fondamentaux sont présents. En langage mathématique moderne, on dira :

Définition.— *Soit* T *un espace topologique quelconque. Notons* $\mathcal{R}_H(T)$ *l'espace topologique construit à partir de* T *dont les points sont les sous-ensembles finis non-vides de* T *et la topologie est définie comme la topologie la moins fine engendrée par les ouverts* $\mathcal{R}(U_1,\dots,U_n)$ *où, pour toute famille finie d'ouverts disjoints* $U_1,\dots,U_n$*, on a*

$$\mathcal{R}(U_1,\dots,U_n) = \{S \in \mathcal{R}_H(T) \mid S \cap U_i \neq \emptyset, \textit{ pour tout } i \leq n\}.$$

Définition.— *Soit* T *un espace topologique quelconque. Notons* $\mathrm{Ouv}_{\amalg}(T)$ *l'opérade colorée dont les couleurs sont les ouverts de* T *et les multimorphismes sont donnés par :*

$$\mathrm{Mult}(U_1,\dots,U_n;V) = \begin{cases} * & \textit{si les } U_i \textit{ sont disjoints et inclus dans } V \\ \emptyset & \textit{sinon} \end{cases}$$

Définition.— *Soit* T *un espace topologique quelconque et soit* $\mathcal{C}$ *une* ∞*-catégorie cocomplète symétrique monoïdale. Une algèbre à factorisation sur* T *à valeurs dans* $\mathcal{C}$*, est une algèbre* $\mathcal{A}$ *sur l'opérade* $\mathrm{Ouv}_{\amalg}(T)$ *à valeurs dans* $\mathcal{C}$*, telle que le précofaisceaux induit sur* $\Lambda_H(T)$ *est un cofaisceau.*

Une algèbre sur $\mathrm{Ouv}_{\amalg}(T)$ *induisant un cofaisceau sur* $\mathcal{R}_H(T)$ *est appelée une algèbre à factorisation non-unitaire.*

Remarque.— Comme on l'a déjà vu avec le tout petit exemple de la métrique sur $\Lambda_H(T)$, la gestion de la configuration vide (respectivement, des unités dans les algèbres) peut poser quelques soucis. C'est pourquoi dans cette thèse, je choisis de travailler uniquement avec l'espace de Ran (respectivement, avec des algèbres sans unités). Ainsi, tout sera le plus souvent non-unitaire et j'oublierai de le préciser.

Un exemple (non-unitaire) tout bête consiste à prendre une $\mathbb{C}$-algèbre commutative A et à déclarer que $\mathcal{A}(B_O) = A$ pour toute boule ouverte B_O non-vide et $\mathcal{A}(\emptyset) = 0$. Les compositions sont données par la multiplication de A et la propriété de reconstruction est assurée automatiquement. Cet exemple fait partie d'une classe d'algèbres à factorisation particulièrement simples à étudier. Ce sont les algèbres à factorisation localement constantes.

Définition.— *Une algèbre à factorisation $\mathcal{A}$ sur une variété topologique* T *est localement constante si pour toute inclusion de boules ouvertes* $B_O \hookrightarrow B'_O$*, le morphisme induit* $\mathcal{A}(B_O) \to \mathcal{A}(B'_O)$ *est une équivalence.*
On appelle aussi les algèbres localement constantes des $\mathcal{E}_T$-algèbres.

Les théories physiques encodées par des algèbres à factorisation localement constantes sont un peu étranges. En effet dans ce cas, la théorie ne dépend plus des propriétés — métriques par exemple — de l'espace-temps T ; elle ne dépend plus que de la topologie de l'espace T. Ces théories n'ont en général aucun sens concret mais elles servent à roder les outils mathématiques. Les théories des champs dont l'algèbre à factorisation est localement constante sont appelées *théories des champs topologiques*. Un exemple célèbre de théorie des champs topologiques est la théorie de Chern-Simons. Bien que ces théories n'aient pas de sens dans le monde réel, elles sont aussi très étudiées en physique théorique et ont des répercussions dans d'autres branches des mathématiques [Wit, Wit2].

Remarque.— Le nom « algèbre à factorisation » provient du fait que la valeur d'une algèbre à factorisation $\mathcal{A}$ sur une union disjointe $U \amalg V$ est équivalente au produit tensoriel de ses valeurs sur U et V :

$$\mathcal{A}(U \amalg V) \simeq \mathcal{A}(U) \otimes \mathcal{A}(V)$$

1.1.3 *Les espaces de Ran*

Je termine par une digression sur l'espace de Ran, ou plutôt *les* espaces de Ran. L'espace de Ran $\mathcal{R}_H(T)$ décrit plus haut est l'espace de Ran utilisé par Lurie dans [HA] ; c'est celui qui nous est le plus familier puisqu'il est métrique dans le cas où T est lui-même métrique. Mais il en existe deux autres.

Le premier est celui qui fut initialement décrit par Beilinson et Drinfeld dans [BD]. Étant donné un entier n, on note $\mathcal{R}_{\leq n}(T)$ le sous-espace topologique de $\mathcal{R}_H(T)$ des configurations d'au plus n points dans T. L'espace $\mathcal{R}_{\leq n}(T)$ hérite de la topologie (métrique si T est métrique) de $\mathcal{R}_H(T)$. Puisque toutes les configurations de T qui sont dans l'un des $\mathcal{R}_{\leq n}(T)$, l'ensemble sous-jacent à l'espace $\mathcal{R}_H(T)$ est égal à l'union des ensembles sous-jacents aux $\mathcal{R}_{\leq n}(T)$. Cependant leurs topologies sont différentes, il y a en général plus d'ouverts dans l'union des $\mathcal{R}_{\leq n}(T)$ que dans $\mathcal{R}_H(T)$. C'est pour cela que l'on pose :

$$\mathcal{R}(T) = \bigcup_{n \in \mathbb{N}} \mathcal{R}_{\leq n}(T).$$

Le second est celui qui est utilisé par Costello et Gwilliam dans [CG]. Il s'agit là encore de changer légèrement la topologie de l'espace de Ran sans en changer les points. Il est clair que les ouverts de Ran doivent être construits à partir des ouverts de l'espace de départ. La manière la plus naïve de procéder est de construire un ouvert $\mathcal{R}(U)$ à partir d'un ouvert U de T où $\mathcal{R}(U)$ est simplement l'ensemble des configurations finies non-vides de U. On notera $\mathcal{R}_W(T)$ l'espace de Ran dont la topologie est engendrée par les $\mathcal{R}(U)$. Les recouvrements ouverts $V \subset \cup_{i\in I} U_i$ dans T tels que $\mathcal{R}(V)$ est recouvert par $\cup_{i\in I}\mathcal{R}(U_i)$ sont appelés *recouvrements de Weiss* dans [CG], d'où la lettre W dans $\mathcal{R}_W(T)$.

Les trois espaces de Ran se comparent à l'aide de bijections continues, on a :

$$\mathcal{R}(T) \to \mathcal{R}_H(T) \to \mathcal{R}_W(T).$$

Chacun donne lieu à une définition *a priori* un peu différente d'algèbre à factorisation. Cependant les notions d'algèbres localement constantes sont les mêmes quelque soit la définition, tant que T est une variété topologique. C'est le corolaire 2.6.3 de [Knud] : les algèbres à factorisation localement constantes sont entièrement déterminées par leurs valeurs sur les boules ouvertes et la propriété de reconstruction est vérifiée automatiquement.

Mystère : pour notre plus grand bonheur, il y aura bientôt un quatrième espace de Ran… Et même un cinquième !

1.2 ALGÈBRES VERTEX

1.2.1 *Un peu d'histoire*

Bien avant que les algèbres à factorisation ne soient inventées, il y eut d'autres tentatives de formalisation de la physique des particules élémentaires. L'une d'elle est celle des axiomes de Wightman. Ils furent publiés en 1964 mais les idées menant à ces axiomes commencèrent à apparaitre dès les années 50 [Wight]. On peut les résumer grossièrement de la manière suivante :

AXIOME 1 Il y a un espace de Hilbert $\mathcal{H}$ sur lequel agit le groupe des spineurs de Poincaré $\widehat{\mathcal{P}}$ (le revêtement universel du groupe de Poincaré) ;

AXIOME 2 Il y a un vecteur préféré dans $\mathcal{H}$ noté $|0\rangle$, représentant l'état sans particules. Il est invariant sous l'action de $\widehat{\mathcal{P}}$;

AXIOME 3 Il y a un certain nombre de champs $\{\phi_i\}_{i\in I}$. Ce sont des distributions tempérées sur l'espace de Minkowski $\mathbb{R}^4$, à valeurs opérateurs densément définis sur $\mathcal{H}$;

AXIOME 4 Les champs ϕ_i sont équivariants sous l'action de $\widehat{\mathcal{P}}$;

AXIOME 5 Les champs ϕ_i commutent ou anti-commutent deux à deux ;

AXIOMES X On ajoute aussi plusieurs conditions de régularité pour que la théorie ait un sens physique.

Les axiomes de Wightman sont l'un des blocs fondamentaux à la base de la théorie quantique des champs moderne. Ces simples suppositions entrainent des théorèmes profonds comme l'invariance CPT ou la corrélation entre le spin d'une particule et son comportement statistique [Green, SW].

Cependant, les axiomes de Wightman souffrent d'un grave problème : les seuls exemples connus de théories des champs satisfaisant rigoureusement ces axiomes sont les théories libres, où les particules se baladent dans l'espace-temps sans jamais interagir avec personne. C'est tellement vrai qu'une partie d'un des problèmes du millénaire est de faire marcher la théorie de Yang-Mills (les recettes de calcul du modèle standard de la physique) avec les axiomes de Wightman — avec au passage un million de dollars américains à la clé [CJW].

Pour le moment, l'idée est de s'attaquer à un problème similaire mais un peu moins complexe. Première chose à faire : diminuer la dimension. On remplace l'espace de Minkowski par $\mathbb{R}^2$ muni de sa métrique euclidienne. Seconde chose à faire : augmenter drastiquement le nombre de symétries. Pour ce faire, on suppose non-seulement l'invariance par le groupe des transformations conformes du plan, mais on suppose aussi l'invariance par toutes les transformations infinitésimales conformes. Et contrairement au cas des dimensions supérieures, en dimension 2, la dimension de l'espace des transformations infinitésimales conformes est infinie !

À l'aide de ces suppositions additionnelles, on se ramène au cas où tous les champs sont en fait holomorphes et on les développe en séries entières. On oublie ensuite tous les problèmes liés aux convergences, ne reste plus que de l'algèbre et de la combinatoire : voici les algèbres vertex.

Les algèbres vertex sont donc les modèles des théories des champs à symétrie conforme en dimension réelle 2. On trouvera les détails des calculs menant aux axiomes des algèbres vertex depuis les axiomes de Wightman dans [4Begginers].

Dans une algèbre vertex, on trouve :

- Un espace vectoriel complexe V ;
- Un opérateur de translation $T : V \to V$;
- Un vecteur invariant $|0\rangle \in V$;
- Une loi de multiplication à valeurs séries de Laurent
$$Y : V \otimes V \to V((z))$$
- Des lois de commutativité et d'associativité faisant intervenir des calculs de séries doublement infinies à deux variables.

1.2.2 *Algèbres à factorisation algébriques*

Malgré — ou plutôt grâce à — son caractère très combinatoire, la théorie des algèbres vertex est riche de nombreux exemples [BZFrenk]. C'est en cherchant à comprendre cette combinatoire de manière géométrique que Beilinson et Drinfeld sont arrivés les premiers à la définition d'une algèbre à factorisation dans leur livre devenu célèbre « Chiral Algebras » [BD].

Qui dit algèbre à factorisation, dit espace de Ran. Or, vu que les algèbres vertex sont définies par des équations algébriques — on a oublié toutes les données analytiques dès le départ — il est naturel de chercher à décrire les algèbres vertex comme des algèbres à factorisation dans le monde de la géométrie algébrique. Cependant, si X est une variété algébrique complexe, même lisse, l'espace $\mathcal{R}_H(X)$ n'a aucune chance de pouvoir être décrit comme un objet de la géométrie algébrique. Il faut donc un remplaçant algébrique à l'espace de Ran : l'espace de Ran champêtre.

L'idée est la suivante : puisque les espaces $\mathcal{R}_{\leq n}(X)$ n'existent pas et sont censés coder les configurations d'au plus n points de X, il faut les remplacer par les variétés algébriques X^n, qui sont elles bien décrites par la géométrie algébrique et qui codent les uplets de point de X, avec des redondances. Et au lieu de tuer les redondances inscrites dans les X^n, nous allons seulement garder les informations indiquant que des configurations sont identiques. Toute surjection $\pi : [\![1, n]\!] \to [\![1, m]\!]$ détermine une injection diagonale

$$\Delta(\pi) : X^m \hookrightarrow X^n,$$

on travaillera donc sur chaque X^n en prenant garde que les objets construits soient compatibles aux identifications définies par les diagonales $\Delta(\pi)$: on travaillera *de manière équivariante.*

Définition.— *Soit X une variété algébrique complexe lisse. On désigne par $\mathrm{Ran}(X)$ l' ∞-préchamp obtenu par colimite*

$$\mathrm{Ran}(X) = \varinjlim_{I} X^I$$

où la colimite est indexée par la catégorie opposée des surjections $\pi : J \to I$ entre ensembles finis non-vides.

On désignera par $\widehat{\mathrm{Ran}}(X)$ le champifié étale de $\mathrm{Ran}(X)$.

Remarque.— On utilise ici les ensembles finis I plutôt que les intervalles $[\![1, n]\!]$ car la construction ne dépend pas de l'ordre choisi sur I.

À titre de comparaison, si nous faisions la même chose avec un espace topologique T et que nous plongions tous les espaces de Ran dans l'∞-catégorie des ∞-préchamps, nous aurions la suite de morphismes de comparaison suivante,

$$\mathrm{Ran}(T) \to \widehat{\mathrm{Ran}}(T) \to \mathcal{R}(T) \to \mathcal{R}_H(T) \to \mathcal{R}_W(T).$$

Dans ce qui suit, on utilisera seulement l'espace de Ran pour paramétrer des données faisceautiques, comme c'est le cas généralement en géométrie algébrique. Puisque les faisceaux (quasi-cohérents) satisfont la descente étale, il n'y aura aucune différence en pratique à travailler avec le préchamp Ran ou son champifié (ils partagent la même réflexion ∞-topossique).

À partir d'une algèbre vertex V et d'une courbe algébrique lisse X, Beilinson et Drinfeld construisent un faisceau quasi-cohérent $\mathcal{A}_X$ sur X — il s'agit de la manière naturelle de parler d'objets paramétrés par une variété en géométrie algébrique — localement équivalent à $V \otimes \mathcal{O}_X$. De plus, puisque V est muni d'un opérateur $T : V \to V$ se comportant comme une dérivation vis à vis de Y, le faisceau $\mathcal{A}_X$ sera muni d'une connexion plate $\nabla : v \otimes f \mapsto Tv \otimes f + v \otimes f'$.

En généralisant un peu, on pourra dire que $\mathcal{A}_X$ est un $\mathcal{D}$-module sur X. La théorie des $\mathcal{D}$-modules a pour but de coder les équations différentielles algébriques intégrables. Un de ses avantages est d'être assez maniable d'un point de vue géométrique : étant donné une application polynômiale $f : X \to Y$ entre deux variétés algébriques lisses, on peut intégrer les équations différentielles le long des fibres de f, c'est l'opération f_* ou « poussé en avant ». De même, il est possible de restreindre les équations le long de f, c'est l'opération $f^!$ ou « tiré en arrière ». Enfin, on peut aussi combiner deux familles d'équations algébriques — mettons à n et p variables — pour en faire une unique famille (à $n+p$ variables), c'est l'opération $\boxtimes$ ou « produit tensoriel extérieur ».

À partir d'une algèbre vertex, Beilinson et Drinfeld construisent aussi d'autres $\mathcal{D}$-modules $\mathcal{A}_{X^I}$ sur tous les X^I. Ceux-ci sont compatibles entre eux vis à vis des immersions diagonales, on pourra donc parler du $\mathcal{D}$-module $\mathcal{A}$ sur $\mathrm{Ran}(X)$. Ces $\mathcal{D}$-modules supplémentaires sont construits à partir de l'opération de multiplication Y et de ses composées. Si $\mathcal{A}_{X^2}$ code l'opération Y, $\mathcal{A}_{X^3}$ code les relations d'associativités etc. Ils permettent aussi de dire que le $\mathcal{D}$-modules $\mathcal{A}$ *se factorise*.

Définition (classique).— *Soit* X *une courbe algébrique lisse. Une algèbre à factorisation algébrique (non-unitaire) sur* X *est la donnée de :*

- *Un $\mathcal{D}$-modules $\mathcal{A}_{X^I}$ sur X^I pour tout* I *fini non-vide ;*
- *Des isomorphismes fonctoriels pour toute surjection* $\pi : J \twoheadrightarrow I$,

$$\Delta(\pi)^! \mathcal{A}_{X^J} \simeq \mathcal{A}_{X^I} ;$$

- *(Factorisation) Des isomorphismes fonctoriels*

$$j(\pi)^! \mathcal{A}_{X^J} \simeq j(\pi)^! \boxtimes_{i \in I} \mathcal{A}_{X^{\pi^{-1}(i)}}$$

  où $j(\pi) : U(\pi) \hookrightarrow X^J$ *est l'immersion ouverte complémentaire de* $\Delta(\pi)$.

Comme les ouverts de Zariski en géométrie algébrique ne sont jamais disjoints, on ne peut pas écrire une propriété de factorisation sur des ouverts. La propriété de factorisation décrite plus haut est alors un bon remplaçant pour le cadre algébrique : on constate que pour deux point *distincts* x et y de X, la valeur de $\mathcal{A}$ au point (x, y) correspond au produit tensoriel des valeurs de $\mathcal{A}$ en x et en y (le produit $\boxtimes$ est fait pour ça).

1.3 DE L'ALGÉBRIQUE VERS LE TOPOLOGIQUE

Nous sommes donc en présence de deux structures mathématiques différentes, ayant toutes les deux pour but de modéliser la théorie quantique des champs et s'exprimant sous la forme d'objet factorisant vivant sur un espace de Ran.

Est-il possible de comparer les deux approches ?

Trois puissants indices nous font tendre vers cette direction. Premièrement les algèbres à factorisation de Costello et Gwilliam sont définies sur des variétés topologiques, tandis que celles de Beilinson et Drinfeld le sont sur des variétés algébriques ; ce sont deux mondes aux propriétés très différentes. Cependant il existe une correspondance Alg $\rightsquigarrow$ Top bien documentée. Elle passe par l'intermédiaire de la théorie des espaces analytiques complexes,

$$\mathrm{Alg} \rightsquigarrow \mathrm{An} \rightsquigarrow \mathrm{Top}$$

La première flèche est *l'analytification*. De manière générale, étant donné une variété algébrique lisse X — soit un objet géométrique décrit par des équations exclusivement polynomiales — on peut lui associer une variété analytique X^{an} — un objet géométrique décrit à l'aide de fonctions analytiques complexes. L'espace X^{an} hérite alors d'une topologie transcendante. Dans le cas qui nous intéresse ici — celui des $\mathcal{D}$-modules — l'analytification consiste à prendre une famille d'équations algébriques et à les considérer comme des équations analytiques.

La seconde consiste à considérer une variété analytique X et à en prendre la variété topologique sous-jacente X^{top} (c'est à dire à en oublier la nature analytique). Quant aux $\mathcal{D}$-modules, elle est donnée par le *foncteur de De Rham*. Cette construction est équivalente au fait de regarder les solutions analytiques de la famille d'équations différentielles.

La composée de deux opérations prends un $\mathcal{D}$-module $\mathcal{A}$ sur une variété algébrique complexe lisse X et produit un faisceau $\mathrm{DR}(\mathcal{A}^{an})$ (de complexes de chaines) d'espaces vectoriels complexes sur la variété topologique X^{top}. On pourra en trouver les détails dans [Hot].

Deuxièmement, si le foncteur de De Rham produit un faisceau sur X^{top}, c'est à dire grosso-modo une « fonction à valeurs dans une catégorie », la définition de Costello et Gwilliam demande un cofaisceau (analogue, très grosso-modo, des distributions à valeurs dans une catégorie). Bien heureusement, un analogue du théorème de Radon-Nikodym existe dans le monde des faisceaux et des cofaisceaux à valeurs dans une ∞-catégorie stable [HA] : dans le cas d'un espace topologique localement compact, il existe un cofaisceau particulier, noté Γ_c qui est tel que tout autre cofaisceau $\mathcal{T}$ s'écrive

$$\mathcal{T} \simeq \mathcal{F} \otimes \Gamma_c$$

où $\mathcal{F}$ est un faisceau. On peut imaginer la composée de cette équivalence avec le foncteur de De Rham comme une alternative à prendre les solutions analytiques d'une famille d'équations : on prends les solutions dans les distributions à la place.

Enfin, même une fois obtenus des cofaisceaux dans le monde topologique, il reste le problème des quatre mousquetaires,

$$\mathrm{Ran}(X^{\mathrm{top}}) \to \mathcal{R}(X^{\mathrm{top}}) \to \mathcal{R}_H(X^{\mathrm{top}}) \to \mathcal{R}_W(X^{\mathrm{top}}).$$

Cet écueil est surmonté grâce au théorème suivant.

☛ Théorème 2.5.21.— *Soit* T *un espace topologique paracompact de Lindelöf. Alors la réflexion ∞-topossique du morphisme préchampêtre*

$$\mathrm{Ran}(T) \to \mathcal{R}(T)$$

est une équivalence.

Et comme les ∞-cofaisceaux ne sont sensibles qu'aux réflexions ∞-topossiques des objets sur lesquels ils sont définis, c'est tout bon !

1.3.1 *Le cas localement constant*

S'il est un cas agréable à traiter, c'est bien celui où tous les objets considérés sont localement constants à fibres de dimension finie (ils sont aussi connus sous le nom de *systèmes locaux*. Dans ce cas le foncteur de De Rham induit une équivalence entre la catégorie dérivée bornée des fibrés vectoriels et celle des systèmes locaux. Du côté topologique, j'ai déjà présenté ce qu'était les algèbres localement constantes. D'après le dictionnaire offert par le foncteur de De Rham, le bon candidat correspondant est celui des algèbres à factorisation algébriques $\mathcal{A}$ sur une variété lisse X telles que $\mathcal{A}_X$ soit dans la catégorie dérivée des fibrés vectoriels.

Dans le cas localement constant, on n'a pas non plus à faire la différence entre les différents espaces de Ran et l'on saute allègrement de Ran vers $\mathcal{R}$ puis $\mathcal{R}_H$. On est alors capable de montrer l'équivalence entre les deux théories (topologiques).

☛ Théorème 2.5.26.— *Soit* X *une variété algébrique lisse et soit* M *sa variété topologique associée. Alors les foncteurs évoqués en amont induisent une équivalence entre les ∞-catégories de fibrés vectoriels factorisant non-unitaires sur* X *et des* $\mathcal{E}_{\mathrm{M}}$*-algèbres non-unitaires de dimension finie.*

1.3.2 *Obstructions au cas général*

Une bonne partie du temps consacré à cette thèse avait pour objectif d'étendre l'équivalence précédente en un foncteur globale envoyant une certaine ∞-catégorie d'algèbres vertex vers des algèbres à factorisation ; en vain. Afin de contempler les gouffres qui surgissent au beau milieu du périple, nous allons devoir entrer plus avant dans les détails de la preuve.

Francis et Gaitsgory ont permis d'étendre la définition de Beilinson et Drinfeld d'algèbre à factorisation algébrique aux variétés de toutes dimensions. Cela fut rendu possible grâce à l'utilisation des ∞-catégories : remplacer les espaces vectoriels par des complexes de chaines offre de nombreux autres degrés de liberté et tandis que la définition classique d'une algèbre à factorisation en dimension plus grande que 2 conduit systématiquement à des algèbres triviales, ce n'est plus le cas dans le monde ∞-catégoriel.

Cette souplesse des ∞-catégories vient avec son lot de désagréments : il faut maintenant décrire un nombre infini de relations d'associativités supérieures qu'il n'est plus question d'écrire à la main ! L'idée de Francis et Gaitsgory dans [FG] et dont l'inspiration provient du chapitre 4 de [BD] est de décrire les algèbres à factorisation comme certaines cogèbres cocommutatives dans l'∞-catégorie des $\mathcal{D}$-modules sur l'espace de Ran munie d'une structure tensorielle adéquate : la structure tensorielle chirale.

En voici l'idée. L'espace de Ran possède une structure de monoïde commutatif : on peut faire l'union de deux configurations. Notons cette opération par u. On souhaite en fait n'effectuer que l'union des configurations disjointes, on notera à cet effet j l'immersion ouverte correspondante au lieu de Ran × Ran où les configurations sont disjointes. La propriété de factorisation peut alors presque s'écrire

$$j^!(\mathcal{A} \boxtimes \mathcal{A}) \simeq (uj)^! \mathcal{A}.$$

Si le foncteur $(u \circ j)^!$ possède un adjoint à droite $(uj)_*$ — ce qui est toujours le cas — on en déduit donc une flèche

$$\mathcal{A} \to u_* j_* j^! (\mathcal{A} \boxtimes \mathcal{A}) = \mathcal{A} \otimes^{\mathrm{ch}} \mathcal{A}$$

et on voit la structure de cogèbre apparaitre. Bien sûr, si à la place le foncteur $(uj)^!$ avait un adjoint à gauche $u_! j_!$, on pourrait alors parler d'algèbre

$$u_! j_! j^! (\mathcal{A} \boxtimes \mathcal{A}) \to \mathcal{A}$$

mais cela n'est possible qu'avec les $\mathcal{D}$-modules holonômes, il faut donc oublier cette voie.

On commence à toucher là à un premier problème. Pour que la formule $\mathcal{A} \otimes^{\mathrm{ch}} \mathcal{A}$ définisse une structure tensorielle, il faut que les foncteurs u_*, j_*, $j^!$ et $\boxtimes$ vérifient un certain nombre de compatibilités. Cela n'est pas un souci en algébrique où tout se passe bien. Essayez de recopier la même définition dans le cadre topologique et vous tomberez sur un os. Les compatibilités requises ne sont vraies que sur la sous-catégorie des faisceaux constructibles et ceux-ci sont beaucoup plus proches des localement constants qu'autre chose... En fait, c'est la structure tensorielle construite à partir de $u_! j_!$ qui fonctionne très bien dans le cadre des faisceaux !

Remarque.— Tout comme il est possible, étant donné une application $f : X \to Y$, de tirer en arrière et de pousser en avant des $\mathcal{D}$-modules, les mêmes opérations sont définies sur les faisceaux et les cofaisceaux. Dans les bons cas, elles se correspondent à travers les foncteurs de De Rham et Γ_c.

Parallèlement à ce problème, la comparaison Alg $\rightsquigarrow$ Top ne se comporte pas si bien que cela en général. Le foncteur de De Rham ne commute aux opérations élémentaires f_*, $f^!$ que dans des cas bien particuliers. Pire, en dehors des $\mathcal{D}$-modules holonômes il ne commute pas au produit $\boxtimes$. C'est ballot pour faire des algèbres à factorisation... Le foncteur DR est lax symétrique, il est donc capable de transporter les algèbres mais incapable de transporter les cogèbres !

1.4 TOPOS SUPÉRIEURS

1.4.1 *La géométrie tempérée*

C'est en étudiant un chemin de traverse que j'en suis arrivé à étudier les ∞-topos. Dans [FG], Francis et Gaitsgory montrent que les cogèbres factorisantes sur Ran sont en fait équivalentes à des algèbres de Lie dont le support se trouve dans la première strate de l'espace de Ran. On retombe sur la cogèbre à factorisation en prenant le complexe de Chevalley-Eilenberg de l'algèbre de Lie. Une telle algèbre de Lie est appelée *algèbre chirale*.

Si le foncteur de De Rham ne peut pas transporter les cogèbres factorisantes, peut-être peut-il transporter les algèbres chirales à la place. Après tout, il est lax symétrique monoïdale. Pas de chance, on bloque dès la première étape. En cherchant à analytifier l'opération chirale, on tombe sur

$$\begin{array}{ccc} (\mathcal{A} \otimes^{\mathrm{ch}} \mathcal{A})^{\mathrm{an}} & \longrightarrow & \mathcal{A}^{\mathrm{an}} \\ \downarrow & & \\ \mathcal{A}^{\mathrm{an}} \otimes^{\mathrm{ch}} \mathcal{A}^{\mathrm{an}} & & \end{array}$$

ce qui ne permet pas de définir une structure d'algèbre de Lie sur $\mathcal{A}$. Cela vient du fait que le monde algébrique se compare mal au monde analytique ; le monde d'arrivée naturel pour le foncteur analytification est la géométrie tempérée [KS2, Pr]. On peut résumer la situation ainsi

$$\begin{tikzcd}
& \mathrm{An} \arrow[r, rightsquigarrow] \arrow[d, "\mathrm{oubli}"] & \mathrm{Top}_{\mathrm{l.c}} \\
\mathrm{Alg} \arrow[r, "\mathrm{an}"] \arrow[ur, dotted, "\mathrm{an}"] & \mathrm{Temp} \arrow[r, rightsquigarrow] & \mathrm{Top}_{\mathrm{Coh}}
\end{tikzcd}$$

Il est possible de copier la définition du produit tensoriel chiral dans le monde tempéré. Dans ce cas l'analytification d'une algèbre chirale algébrique donnera ce qu'on peut appeler une *algèbre chirale tempérée*. Le passage au monde topologique se fait au moyen d'un *foncteur de De Rham tempéré* [KS3].

Dans ce diagramme $\mathrm{Top}_{\mathrm{Coh}}$ désigne le monde des espaces topologiques localement cohérents et $\mathrm{Top}_{\mathrm{l.c}}$ les espaces localement compacts. Ces deux mondes sont totalement orthogonaux, leur intersection se réduit aux ensembles discrets finis de points.

Dans ce nouveau contexte des espaces localement cohérents, le théorème de Lurie — appelé dualité de Verdier — prônant l'équivalence entre faisceaux et cofaisceaux devient inopérant. D'où la question,

Est-il possible d'étendre la dualité de Verdier à une classe d'espaces topologiques englobant les espaces localement cohérent ?

C'est cette question qui a motivé le présent travail sur les ∞-topos. Le foncteur de De Rham tempéré s'est ensuite révélé décevant, mais trop tard, j'avais déjà trop pris gout aux ∞-topos.

1.4.2 *Exponentiabilité des topos supérieurs*

Le chapitre sur les ∞-topos est une étude systématique de la théorie et des propriétés des ∞-topos exponentiables, elle reprend les idées principales de Johnstone et Joyal [JoJo]. Elle a été menée en collaboration avec Mathieu Anel.

Cette étude a pour but final d'établir une version plus étendue de la dualité de Verdier telle que donnée par Lurie dans [HA]. En ce sens, l'étude de l'exponentiabilité des ∞-topos provient de deux remarques.

Une première remarque est que le résultat de dualité de Verdier ne dépend pas de l'espace topologique mais de son ∞-topos associé. En effet, étant donné une espace topologique T, son ∞-catégorie des faisceaux à valeurs dans une ∞-catégorie bicomplète stable $\mathcal{S}h(T, \mathcal{C})$ ne dépend que de l'∞-topos associé à T, car on a la formule

$$\mathcal{S}h(T, \mathcal{C}) \simeq \mathcal{S}h(T, \mathcal{S}) \otimes \mathcal{C}$$

De même pour l'∞-catégorie des cofaisceaux à valeurs spectrales qui peut s'obtenir comme l'∞-catégorie des foncteurs cocontinus depuis

$\mathcal{S}\mathrm{h}(\mathrm{T}, \mathcal{S}\mathrm{p})$ vers $\mathcal{S}\mathrm{p}$. Si l'on note $\mathcal{S}\mathrm{h}(\mathrm{T}, \mathcal{S}\mathrm{p})^\vee$ cette ∞-catégorie vue comme le dual $\mathcal{S}\mathrm{p}$-linéaire de l'∞-catégorie des faisceaux en spectres, alors la dualité de Verdier s'apparente à un résultat d'autodualité :

$$\mathcal{S}\mathrm{h}(\mathrm{T}, \mathcal{S}\mathrm{p}) \simeq \mathcal{S}\mathrm{h}(\mathrm{T}, \mathcal{S}\mathrm{p})^\vee.$$

La seconde remarque est que le théorème de Lurie fonctionne pour les espaces qui sont localement compacts. Il est malaisé de mimer la définition pour les ∞-topos car il n'y a pas de définition de sous-espace compact pour un ∞-topos. Cependant, il se trouve que dans la catégorie des espaces topologiques, les espaces localement quasi-compacts sont exactement les espaces topologiques exponentiables. On dit qu'un espace topologique T est exponentiable si le foncteur $(-) \times \mathrm{T}$ possède un adjoint à droite (le foncteur $\mathrm{Y} \mapsto \mathrm{Y}^\mathrm{T}$). Cette définition est parfaite pour être implémentée dans les ∞-topos.

Étendant un résultat de Johnstone et Joyal, on trouve la caractérisation suivante des ∞-topos exponentiables :

☛ Théorème 3.4.17.— *Soit $\mathcal{X}$ un ∞-topos. Alors $\mathcal{X}$ est exponentiable si et seulement si son ∞-catégorie de faisceaux d'espaces $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{S})$ est continue, c'est à dire si le foncteur d'évaluation*

$$\varepsilon : \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{S})) \to \mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{S})$$

admet un adjoint à gauche.

Armé de ce théorème, nous en déduisons que bien qu'insuffisante pour démontrer la dualité de Verdier, l'hypothèse d'exponentiabilité est un premier pas décisif dans cette direction.

☛ Théorème 3.6.16.— *Soit $\mathcal{X}$ un ∞-topos exponentiable, alors son ∞-catégorie de faisceaux en spectres $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{S}\mathrm{p})$ est un objet dualisable de l'∞-catégorie des catégories présentables stables munie de son produit tensoriel usuel.*

Des hypothèses supplémentaires seront certainement nécessaires pour passer de dualisable à auto-dual.

1.5 AU FAIT !

Au cours de cette introduction, de nombreux ∞-trucs se sont glissés sans que l'on ne les définisse. Avant de passer à des maths pures et dures, prenons un moment pour parler des affres et délices que nous réserve le monde infini.

1.5.1 ∞-*catégories*

Ce sont les blocs de base sur lesquels sont bâtis les outils du monde infini. Elles partagent essentiellement la même sémantique que la théorie traditionnelle des catégories. Une des principales utilisations des ∞-catégories est de résoudre les problèmes de singularités en mathématiques.

Souvent lorsque l'on se pose un problème, on arrive assez vite à le comprendre génériquement, sur des exemples typiques. Souvent, il se comporte de manière beaucoup plus sauvage sur d'autres exemples, là où tout ne se passe pas comme prévu. Les ∞-catégories permettent de tenir compte des singularités, de manière systématique. L'idée directrice est de conserver plus d'information qu'à l'accoutumée, on ne veut plus simplement voir la singularité, mais retenir la manière dont elle est apparue. Toute la difficulté est ensuite de conserver et transférer toutes ces informations supplémentaires au gré des différentes construction dont nous avons besoin.

Sur un plan élémentaire, les ∞-catégories engendrent une combinatoire bien particulière : la théorie des ensembles nous fournit comme objets de base les collections d'éléments et la relation élémentaire entre deux collection est *l'égalité*. En théorie des ensembles, on passe son temps à tester des égalités entre ensembles. La théorie des catégories va plus loin, deux objets de la théorie peuvent être équivalents, mais de plusieurs manières différentes ! Si je veux savoir si deux villes sont situées sur le même continent, je regarde si on peut tracer un chemin de l'une à l'autre. Oui mais voilà, des chemins, il y en a plein. Le passage aux ∞-catégories se produit lorsqu'on se rend compte que la seule relation disponible pour comparer deux chemins entre eux est de nouveau l'égalité. On va donc chercher à parler de chemin entre des chemins et de chemins entre des chemins de chemins et ainsi de suite, jusqu'à … l'infini.

La théorie des ∞-catégories a été développée par de nombreux mathématiciens au cours des dernières années. Il existe plusieurs manières différentes (mais équivalentes) de les encoder : quasi-catégories, espaces de Segal, espaces de Segal complets, catégories de modèles etc. Sans doute la théorie des quasi-catégories est la plus aboutie grâce à l'œuvre de Jacob Lurie : « Higher Topos Theory », c'est d'ailleurs une référence que je vais abondamment utiliser [HT].

1.5.2 ∞-*topos*

Les ∞-topos sont une généralisation de la théorie des espaces topologiques. Ils sont très utiles, car souvent les constructions fonctorielles associées à des espaces topologiques ne dépendent en fait que de la théorie topossique. L'idée principale est de faire passer les espaces topologiques (qui sont définis à l'aide d'ensembles) dans la théorie des

∞-catégories — on « catégorifie » la notion d'espace topologique . Selon ce point de vue, ce qui compte dans un espace topologique, ce ne sont pas ses points, mais sa topologie, c'est à dire l'ensemble des ses ouverts. En formalisant les axiomes des ouverts, on arrive à des exemples élémentaires de topos. Même dans ces cas très simples, on découvre des espaces qui n'ont pas assez de points ! C'est là probablement le fait le plus marquant de la théorie, il existe des ∞-topos tout à fait non-triviaux, mais sans aucun points ! Pour débuter, on pourra consulter l'excellente bande dessinée d'Alain Prouté [Sha]. Une introduction plus détaillée aux ∞-topos se trouve au chapitre 3.

1.5.3 *∞-algèbres*

Enfin avec la notion d'∞-algèbre, on utilise le relâchement des conditions d'égalité — imposés par la théorie des ensembles — afin de construire de nouvelles structures. C'est plus souple, donc il y a plus d'exemples ! Ainsi, au lieu de demander une relation de commutativité, qui l'on va écrire comme une égalité,

$$ab = ba$$

on demande que ab soit équivalent à ba (on dira aussi homotope). Attention cependant, car il faut maintenant garder une trace des équivalences. De plus, ces équivalences doivent être compatibles entre elles et les compatibilités doivent elles aussi être compatibles etc.

Donnons un aperçu de ce qu'il peut arriver. L'équivalent relâché des algèbres associatives, ce sont les algèbres associatives à homotopie près ; on les appelle aussi des $\mathcal{E}_1$-algèbres. L'équivalent des algèbres commutatives sont les $\mathcal{E}_\infty$-algèbres. Entre les deux, il y a tout un monde pour les $\mathcal{E}_2$-algèbres, les $\mathcal{E}_3$-algèbres etc. À chaque étage, on ajoute un cran de commutativité. Toutes ces étapes intermédiaires sont totalement invisibles en dehors du monde infini.

Part II

MATHEMATICAL CONTENT

2 FROM FACTORISATION VECTOR BUNDLES TO LITTLE DISCS ALGEBRAS

2.1 INTRODUCTION

The aim of this chapter is to link factorisation vector bundles and $\mathcal{E}_M$-algebras (in particular, vertex algebras with $\mathcal{E}_2$-algebras) via the Ran space. This is done using tools to compare $\mathcal{D}$-modules, sheaves and cosheaves. We also need to compare cosheaves on different Ran spaces for the final step. Corner stones for this work are provided papers written by Francis, Gaitsgory [FG] and Knudsen [Knud].

2.1.1 *Notations*

Throughout this chapter, otherwise stated, ∞-sheaves and cosheaves take their values inside the ∞-category of chain complexes over $\mathbb{C}$.

We let $\mathcal{C}\mathrm{at}_{\mathbb{C}}$ be the (very big) ∞-category of (big) stable ∞-categories which are tensored over $\mathcal{V}\mathrm{ect}_{\mathrm{fd}}$ the full subcategory of chain complexes with bounded finite dimensional homology. We shall call them simply 'finite dimensional vector spaces' for convenience.

Functors inside ∞-category $\mathcal{C}\mathrm{at}_{\mathbb{C}}$ are the stable $\mathbb{C}$-linear functors. It is endowed with its natural tensor structure whom $\mathcal{V}\mathrm{ect}_{\mathrm{fd}}$ is the unit and which classifies bilinear maps.These '$\mathbb{C}$-linear stable ∞-categories' are an ∞-analogue of triangulated dg-categories. The analogy is made clear in [Cohn].

2.1.2 *$\mathcal{D}$-modules*

The classical theory of algebraic $\mathcal{D}$-modules — by this we mean the study of the dg-category of chain complexes of quasi-coherent right $\mathcal{D}$-modules — is well understood, standard references include [Hot] and [Kash].

By contrast, the ∞-version of the theory is fairly recent and has been developed by Gaitsgory and Rozenblyum [GR]. The definition of the ∞-category $\mathcal{D}\text{-}\mathrm{Mod}_{\mathrm{qc}}$ is not the biggest issue: for every map between varieties $f : X \to Y$, the theory of $\mathcal{D}$-modules shall supply pushforward and pullback functors f_* and $f^!$ satisfying a list of reasonable assumptions such as a composition law. But in the ∞-world, the equivalence $(f \circ g)_* \simeq f_* \circ g_*$ has to be encoded with all the 'higher information', herein lies the point. We will provide a short summary of the tools we need in this chapter.

The theory of $\mathcal{D}$-modules

The presentation of the theory of algebraic $\mathcal{D}$-modules is borrowed from [FG], we shall only present here the few facts we need in this chapter. Notice also that we are only going to speak of smooth complex varieties and while building the theory requires the use of derived schemes, we do not need it here.

To any smooth complex variety X, one can associate a $\mathbb{C}$-linear stable presentable ∞-category called $\mathcal{D}_{qc}(X)$. Its objects are the chain complexes of quasi-coherent right $\mathcal{D}_X$-modules, where $\mathcal{D}_X$ is the sheaf of differential operators on X. We will refer at an object of $\mathcal{D}_{qc}(X)$ as a $\mathcal{D}$-module, forgetting to say it is a chain complex.

Now that we know what is a $\mathcal{D}$-module, we need to describe the pushforward and pullback operations on $\mathcal{D}$-modules. Let Var be the category of smooth complex varieties, we shall denote by $\mathcal{V}\mathrm{ar}$ the nerve $N(\mathrm{Var})$. The ∞-category $\mathcal{V}\mathrm{ar}$ is endowed with its Cartesian symmetric monoidal structure.

☛ From now on, X is an element of Var, and $f : X \to Y$ is an arrow in Var. Also $j : U \hookrightarrow X$ will always be an open embedding.

Proposition 2.1.1.— *There exists a lax symmetric monoidal functor*

$$\mathcal{D}_* : \mathcal{V}\mathrm{ar} \longrightarrow \mathcal{C}\mathrm{at}_{\mathbb{C}}$$

such that for any $X \in \mathcal{V}\mathrm{ar}$, $\mathcal{D}_*(X) = \mathcal{D}_{qc}(X)$ and for any morphism $f : X \to Y$, the corresponding morphism $f_* : \mathcal{D}(X) \to \mathcal{D}_{qc}(Y)$ is the usual $\mathcal{D}$-module pushforward at the level of homotopy categories.

Proposition 2.1.2.— *There exists a lax symmetric monoidal functor*

$$\mathcal{D}^! : \mathcal{V}\mathrm{ar}^{op} \longrightarrow \mathcal{C}\mathrm{at}_{\mathbb{C}}$$

such that for any $X \in \mathcal{V}\mathrm{ar}$, $\mathcal{D}^!(X) = \mathcal{D}_{qc}(X)$ and for any morphism $f : X \to Y$, the corresponding morphism $f^! : \mathcal{D}_{qc}(Y) \to \mathcal{D}_{qc}(X)$ is the usual $\mathcal{D}$-module pullback at the level of homotopy categories.

In the case where $j : X \to Y$ is an étale map, we shall write j^* for $j^!$ to be consistent with the literature.

Remark 2.1.3.— These theorems encode in particular the 'box product' operation $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \boxtimes \mathcal{N}$ — coming from the lax structure $\mathcal{D}(X) \otimes \mathcal{D}(X) \to \mathcal{D}(X^2)$ — and its compatibility with both f_* and $f^!$: for any two $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{N}$,

$$(f \times f)_*(\mathcal{M} \boxtimes \mathcal{N}) \simeq f_* M \boxtimes f_* \mathcal{N}, \quad (f \times f)^!(\mathcal{M} \boxtimes \mathcal{N}) \simeq f^! \mathcal{M} \boxtimes f^! \mathcal{N}.$$

The two functors $\mathcal{D}_*$ and $\mathcal{D}^!$ can be glued together. For this, we need to enlarge $\mathcal{V}\mathrm{ar}$ and allow correspondences. We shall only need here correspondences obtained from partially defined maps.

Definition 2.1.4.— *Let* $\mathrm{Var}_{\mathrm{pd}}$ *be the category whose objects are smooth complex varieties and whose morphisms are correspondences* $X \rightsquigarrow Y$,

$$\begin{tikzcd} & U \arrow[dl, hook', "j"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where j *is an open embedding. The composition of correspondences is defined by fibre product. We let* $\mathcal{V}\mathrm{ar}_{\mathrm{pd}}$, *the nerve of* $\mathrm{Var}_{\mathrm{pd}}$, *be endowed with its Cartesian symmetric monoidal structure.*

Theorem 2.1.5.— *There exists a lax symmetric monoidal functor*

$$\mathcal{D}_\diamond : \mathcal{V}\mathrm{ar}_{\mathrm{pd}} \longrightarrow \mathcal{C}\mathrm{at}_{\mathbb{C}}$$

such that for any correspondence $(j, f) : X \rightsquigarrow Y$, $\mathcal{D}_\diamond(j, f) = f_* j^*$.

This theorem is an $(\infty, 2)$-consequence of the following flat base change formula.

Proposition 2.1.6.— *Let*

$$\begin{tikzcd} T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

be a commutative square in Var. *If* f *is flat and the square is Cartesian, we have an equivalence* $f^! g_* \simeq g'_* f'^!$

We will also need the fundamental adjunctions:

Proposition 2.1.7.— *Let* X *and* Y *be two objects of* $\mathcal{V}\mathrm{ar}$.

- *If* $i : X \to Y$ *is a closed embedding, then* i_* *is left adjoint to* $i^!$ *and* $i^! i_* \simeq \mathrm{Id}_X$.
- *If* $j : X \to Y$ *is an étale map, then* j^* *is left adjoint to* j_* *and if* j *is an open embedding* $j^* j_* \simeq \mathrm{Id}_X$.

Remark 2.1.8.— The functor $\mathcal{D}_\diamond$ can in fact be extended to the ∞-category of derived schemes of finite type, where the morphisms are not only the partially defined maps but all correspondences,

$$\mathcal{D}_\diamond : \mathcal{S}\mathrm{ch}^{\mathrm{f.t}}_{\mathrm{corr}} \to \mathcal{C}_{\mathbb{C}}$$

see [FG] and chapter V in [GRBook].

2.1.3 *Regular and holonomic $\mathcal{D}$-modules*

The full subcategory of $\mathcal{D}_{qc}(X)$ whose objects are chain complexes of $\mathcal{D}_X$-modules with bounded and regular holonomic homology is stable and $\mathbb{C}$-linear. We shall denote it by $\mathcal{D}_{RH}(X)$ and call its objects simply 'holonomic $\mathcal{D}$-modules'.

Regular holonomic $\mathcal{D}$-modules are stable under the operations f_*, $f^!$ and $\boxtimes$ (6.1.5 in [Hot]), hence we have the following proposition,

Proposition 2.1.9.— *There exists a lax subfunctor of $(\mathcal{D})_\diamond$,*

$$(\mathcal{D}_{RH})_\diamond : \mathcal{V}ar_{pd} \to \mathcal{C}at_{\mathbb{C}}$$

such that for every $X \in \mathcal{V}ar$, $\mathcal{D}_{RH}(X)$ is the ∞-category of regular holonomic $\mathcal{D}$-modules on X.

Remark 2.1.10.— The ∞-category of $\mathcal{D}$-modules that are only holonomic satisfy the same properties, hence we also have a functor $(\mathcal{D}_H)_\diamond$.

2.1.4 *Riemann-Hilbert correspondence*

Let $X = (X, \mathcal{O}_X)$ be a smooth complex variety. From this locally ringed space, we can build two other spaces: its analytification $X^{an} := (X^{an}, \mathcal{O}^{an}_{X^{an}})$ and a purely topological object $X^{top} := (X^{an}, \mathbb{C}_{X^{an}})$. These spaces fit into the following correspondence diagram:

$$\begin{tikzcd}
 & X^{an} \arrow[dl, "\mathrm{oblv}^{an}_{alg}"'] \arrow[dr, "\mathrm{oblv}^{an}_{top}"] & \\
X \arrow[rr, squiggly] & & X^{top}
\end{tikzcd}$$

Heuristically, we could wish to extend the theory $\mathcal{D}_\diamond$ to the spaces X^{an} and X^{top}. The ∞-category $\mathcal{D}_\diamond(X^{an})$ would be the ∞-category of analytic $\mathcal{D}$-modules and $\mathcal{D}_\diamond(X^{top})$ would be the ∞-category $\mathrm{Sh}(X^{top})$ of chain complexes of sheaves of complex vector spaces (here the ring $\mathcal{D}_{X^{top}}$ of differential operation would be $\mathcal{D}_{X^{top}} = \mathbb{C}_{X^{top}}$). Then the functor $\mathcal{D}_\diamond$ applied to the correspondence $X \rightsquigarrow X^{top}$ would give a functor of comparison $\mathrm{DR} : \mathcal{D}(X) \to \mathcal{S}h(X^{top})$.

However in the topological case, although the functors f_* and $f^!$ always exist, they are badly behaved and do not respect the box product $\boxtimes$. So we will need to restrict our attention to the full subcategory of *constructible sheaves* on X^{top}, on which f_* and $f^!$ commute with the box product.

Moreover, the De Rham functor $\mathrm{DR} : \mathcal{D}(X) \to \mathcal{S}h(X^{top})$ does not commute with the box product and the functors f_* in general. When restricted to the full subcategory of regular holonomic $\mathcal{D}$-modules, it does. So we will study $\mathrm{DR} : \mathcal{D}_{RH}(X) \to \mathcal{S}h(X^{top})$.

The following proposition is a consequence of the theory of complex valued sheaves as developed in [HT], [HA] and [SAG], and the theory of correspondences in chapter 5 of [GRBook]. Alternatively, it can be seen as a consequence of the existence of $(\mathcal{D}_{RH})_\diamond$ and the classical de Rham equivalence (see [KS]).

Remark 2.1.11.— For a map $f : X \to Y$, we shall denote also by f the maps $f : X^{an} \to Y^{an}$ and $f : X^{top} \to Y^{top}$. Also in the case of sheaves, we shall denote f^* by f^{-1}.

Proposition 2.1.12.— *There exists a lax symmetric monoidal functor*

$$(\mathcal{S}h_C)_\diamond : \mathcal{V}ar_{pd} \to \mathcal{C}at_{\mathbb{C}}$$

such that $(\mathcal{S}h_C)_\diamond(X)$ *is the* ∞*-category of chain complexes of sheaves of complex vector spaces on the topological manifold* $M = X^{top}$, *with bounded and algebraically constructible homology, for every* $X \in \mathcal{V}ar$. *And for every correspondence* $(j, f) : X \rightsquigarrow Y$, *we have*

$$(\mathcal{S}h_C)_\diamond(j, f) = f_* j^{-1} : \mathcal{S}h_C(X^{top}) \to \mathcal{S}h_C(Y^{top})$$

where for every f, *the functor* f_* *coincide with the usual sheaf pushforward on the homotopy categories and* j^{-1} *coincide with the usual sheaf pullback.*

Together with this 'theory of constructible sheaves', we will make the assumption that the classical results about the de Rham functor between the derived dg-categories hold here.

Definition 2.1.13.— *Let* $\mathcal{D}_{VB}(X)$ *be the full subcategory of* $\mathcal{D}_{RH}(X)$ *whose objects are the* $\mathcal{D}$*-modules* $\mathcal{M}$ *such that for every* n, $H_n(\mathcal{M})$ *is a flat vector bundle on* X.

Let $\mathcal{S}h_{LS}(M)$ *be the full subcategory of* $\mathcal{S}h_{LC}(M)$ *whose objects are the sheaves* $\mathcal{F}$ *such that* $H_n(\mathcal{F})$ *is locally constant with finite dimensional stalks. We shall call such an object, a local system on* M.

We are now able to state the Riemann-Hilbert equivalence.

Theorem 2.1.14 (Riemann-Hilbert)**.—** *The de Rham functor induces an equivalence of lax symmetric monoidal functors*

$$DR : (\mathcal{D}_{RH})_\diamond \Leftrightarrow (\mathcal{S}h_C)_\diamond$$

Moreover, for any $X \in \mathcal{V}ar$ *with associated topological manifold* M, *the equivalence*

$$DR : \mathcal{D}_{RH}(X) \simeq \mathcal{S}h_C(M)$$

restricts to an equivalence of subcategories,

$$DR : \mathcal{D}_{VB}(X) \simeq \mathcal{S}h_{LS}(M).$$

Remark 2.1.15.— Modulo $(\infty, 2)$-considerations, the existence of a morphism

$$DR : (\mathcal{D}_{RH})_\diamond \Rightarrow (\mathcal{S}h_C)_\diamond$$

in Riemann-Hilbert is a consequence of the following properties of the de Rham functor :

- If $\mathcal{M}$ is holonomic then $\mathrm{DR}(\mathcal{M})$ is constructible (4.6.6 in [Hot]);
- For any f, we have $\mathrm{DR} \circ f_* \simeq f_* \circ \mathrm{DR}$ (7.1.1 in [Hot]);
- For any f, we have $f^! \circ \mathrm{DR} \simeq \mathrm{DR} \circ f^!$ (7.1.1 also);
- For any holonomic $\mathcal{M}$ and $\mathcal{N}$, $\mathrm{DR}(\mathcal{M} \boxtimes \mathcal{N}) \simeq \mathrm{DR}(\mathcal{M}) \boxtimes \mathrm{DR}(\mathcal{N})$ (4.7.8 in [Hot]).

Remark 2.1.16.— The ∞-categories of flat vector bundles and local systems are both preserved by the box product and $f^!$ but not in general by f_*. Hence, there are no functors $(\mathcal{D}_{\mathrm{VB}})_\diamond$ or $(\mathcal{S}h_{\mathrm{LC}})_\diamond$.

2.1.5 *Verdier duality*

We shall recall here the main properties of the theory of cosheaves and its link with the theory of sheaves. These results may be found in [HT], [HA], [VD] or [Knud].

Definition 2.1.17.— *Let* M *be a topological space, then its ∞-category of cosheaves (with values in the ∞-category of vector spaces) is defined by*

$$\mathcal{C}\mathrm{osh}(\mathrm{M}) = \mathcal{S}\mathrm{h}(\mathrm{M}, \mathcal{V}\mathrm{ect}^{\mathrm{op}})^{\mathrm{op}}.$$

Remark 2.1.18.— A cosheaf $\mathcal{T}$ on a space M is thus a covariant functor

$$\mathcal{T} : \mathcal{O}(\mathrm{M}) \to \mathcal{V}\mathrm{ect}$$

where $\mathcal{O}(\mathrm{M})$ is the nerve of the category of open subsets of M, sending nerves of open coverings to colimits in $\mathcal{V}\mathrm{ect}$.

The theory of cosheaves does have the following functorialities,

Proposition 2.1.19.— *Let $f : \mathrm{M} \to \mathrm{N}$ be a continuous map between topological spaces. Then there exists a pair of adjoint functors*

$$\mathcal{C}\mathrm{osh}(\mathrm{M}) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathcal{C}\mathrm{osh}(\mathrm{N})$$

such that for every cosheaf $\mathcal{T} \in \mathcal{C}\mathrm{osh}(\mathrm{M})$, the functor

$$f_!\mathcal{T} : \mathcal{O}(\mathrm{N}) \to \mathcal{V}\mathrm{ect}$$

is the composite

$$\mathcal{O}(\mathrm{N}) \xrightarrow{f^{-1}} \mathcal{O}(\mathrm{M}) \xrightarrow{\mathcal{T}} \mathcal{V}\mathrm{ect}$$

DEFINITION 2.1.20.— *The functor $f_!$ shall be called the cosheaf pushforward and $f^!$ the cosheaf pullback. When i is the inclusion of a point $x \in \mathrm{M}$,*

$$i_x : \{x\} \hookrightarrow \mathrm{M}$$

the pullback of a cosheaf $\mathcal{T}$ is called the costalk and it shall be denoted by

$$\mathcal{T}_{x^!} := i_x^! \mathcal{T}$$

REMARK 2.1.21.— Explicitly, the costalk $\mathcal{T}_{x^!}$ is given by

$$\mathcal{T}_{x^!} \simeq \varprojlim_{x \in \mathrm{U}} \mathcal{T}(\mathrm{U}).$$

Since the associated ∞-topos of a topological manifold has enough points, isomorphisms of cosheaves can be checked on costalks.

PROPOSITION 2.1.22.— *Let M be a topological manifold and let $f : \mathcal{T} \to \mathcal{Q}$ be a morphism of cosheaves on M, then f is an equivalence if and only if, for every $x \in \mathrm{M}$,*

$$f_{x^!} : \mathcal{T}_{x^!} \to \mathcal{Q}_{x^!}$$

is an equivalence in $\mathcal{V}$ect.

Proof.— By section 7.2.2 in [HT], the associated ∞-topos of M has finite homotopy dimension because M is paracompact with finite covering dimension. Hence by corollary 7.2.1.17, $(\mathrm{M})_{\mathcal{T}\mathrm{ps}}$ has enough points. The end of the proof is in done in the appendix of [Knud]. $\square$

REMARK 2.1.23.— More generally, if M is a topological space such that the associated ∞-topos $(\mathrm{M})_{\mathcal{T}\mathrm{ps}}$ has enough points then equivalences inside $\mathcal{C}\mathrm{osh}(\mathrm{M})$ can be detected on costalks.

The adjoint pair $f_! \dashv f^!$ satisfy the proper base change,

DEFINITION 2.1.24.— *A topological space* T *is completely regular if it is homeomorphic to a subspace of a compact Hausdorff topological space.*

PROPOSITION 2.1.25 (Proper base change).**—** *Let*

$$\begin{tikzcd} \mathrm{N} \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{M} \arrow[d, "p"] \\ \mathrm{N}' \arrow[r, "g"] & \mathrm{M}' \end{tikzcd}$$

be a Cartesian square in the category Top *of topological spaces, such that* M *is a completely regular topological space and p is a proper map of topological spaces. Then we have a canonical equivalence*

$$g^! \circ p_! \simeq q_! \circ f^!.$$

Proof.— This proposition is a direct consequence of definition of the ∞-category of cosheaves and of section 7.3 in [HT]. Notice however that although corollary 7.8.1.18 requires all the involved spaces to be locally quasi-compact and Hausdorff, the proof uses only the fact that M is completely regular. $\square$

We also have the box product of cosheaves:

Proposition 2.1.26.— *Let* M *and* N *be two topological spaces and suppose that* M *is locally quasi-compact. Let* $\mathcal{T} \in \mathcal{C}osh(M)$ *and* $\mathcal{Q} \in \mathcal{C}osh(N)$ *two cosheaves, then there exists a cosheaf* $\mathcal{T} \boxtimes \mathcal{Q}$ *on* $M \times N$ *defined by*

$$(\mathcal{T} \boxtimes \mathcal{Q})(U \times V) = \mathcal{T}(U) \otimes \mathcal{Q}(V)$$

for every U *open set of* M *and* V *open set of* N*. And if we have inclusions* $U \subset U'$ *and* $V \subset V'$*, the maps* $\mathcal{T}(U) \to \mathcal{T}(U')$ *and* $\mathcal{Q}(V) \to \mathcal{Q}(V')$ *induce the map*

$$(\mathcal{T} \boxtimes \mathcal{Q})(U \times V) \to (\mathcal{T} \boxtimes \mathcal{Q})(U' \times V')$$

Moreover if $f : M \to N$ *is a continuous map and* $\mathcal{T}'$ *is another cosheaf on* M*, then*

$$(f \times f)_!(\mathcal{T} \boxtimes \mathcal{T}') \simeq (f_!\mathcal{T}) \boxtimes (f_!\mathcal{T}')$$

Proof.— From the definition of the ∞-category of sheaves (6.3.5.16 in [HT]), we have

$$\mathcal{C}osh(M) = \mathcal{S}h(M, \mathcal{V}ect^{op})^{op} = [\mathcal{S}h_{\mathcal{S}}(M)^{op}, \mathcal{V}ect^{op}]_c^{op}$$

where $\mathcal{S}h_{\mathcal{S}}(M)$ is the ∞-category of sheaves of spaces on the space M and $[\mathcal{S}h_{\mathcal{S}}(M)^{op}, \mathcal{V}ect^{op}]_c$ is the ∞-category of continuous functors from $\mathcal{S}h_{\mathcal{S}}(M)^{op}$ to $\mathcal{V}ect^{op}$. As a consequence, we have the equivalence,

$$\mathcal{C}osh(M) \simeq [\mathcal{S}h_{\mathcal{S}}(M), \mathcal{V}ect]^{cc}$$

where $[\mathcal{S}h_{\mathcal{S}}(M), \mathcal{V}ect]^{cc}$ is the ∞-category of cocontinuous functors from $\mathcal{S}h_{\mathcal{S}}(M)$ to $\mathcal{V}ect$. Given a cosheaf $\mathcal{T}$, its value on an open set $U \subset M$ is the the value on the associated sheaf $\underline{U}$.

Let $\mathcal{T}$ be a cosheaf on M and $\mathcal{Q}$ be a cosheaf on N, then using the tensor product of vector spaces

$$\mathcal{V}ect \times \mathcal{V}ect \xrightarrow{\otimes} \mathcal{V}ect$$

we can build a functor

$$\begin{tikzcd} \mathcal{S}h_{\mathcal{S}}(M) \times \mathcal{S}h_{\mathcal{S}}(N) \arrow[r, "\mathcal{T}\times\mathcal{Q}"] \arrow[rr, bend right, "\mathcal{T}\otimes\mathcal{Q}"'] & \mathcal{V}ect \times \mathcal{V}ect \arrow[r, "\otimes"] & \mathcal{V}ect. \end{tikzcd}$$

This functor $\mathcal{T} \otimes \mathcal{Q}$ is cocontinuous in each variable because both $\mathcal{T}$ and $\mathcal{Q}$ are cocontinuous and the tensor product of vector spaces is cocontinuous in each variable. Hence, by the universal property of the tensor product of ∞-categories, we can have a functor $\mathcal{T} \boxtimes \mathcal{Q}$,

$$\begin{tikzcd} \mathcal{S}h_{\mathcal{S}}(M) \times \mathcal{S}h_{\mathcal{S}}(N) \arrow[r, "\mathcal{T}\otimes\mathcal{Q}"] \arrow[d] & \mathcal{V}ect \\ \mathcal{S}h_{\mathcal{S}}(M) \otimes \mathcal{S}h_{\mathcal{S}}(N) \arrow[ur, "\mathcal{T}\boxtimes\mathcal{Q}"'] & \end{tikzcd}$$

The restriction of the functor $\mathcal{T} \boxtimes \mathcal{Q}$ on the category $\mathcal{O}(M) \times \mathcal{O}(N)$ is given by $\mathcal{T} \otimes \mathcal{Q}$, as advertised.

Since M is locally quasi-compact, by theorem 1 in [Isb] the locale associated to $M \times N$ is the product of locales $(M)_L \times (N)_L$. By section 6.4.5 in [HT], the inclusion of the ∞-category of locales into the ∞-category of ∞-toposes commutes with limits, so we get the equivalence $(M)_{\mathcal{T}ps} \times (N)_{\mathcal{T}ps} \simeq (M \times N)_{\mathcal{T}ps}$, and thus we have

$$\mathcal{S}h_{\mathcal{S}}(M) \otimes \mathcal{S}h_{\mathcal{S}}(N) \simeq \mathcal{S}h_{\mathcal{S}}(M \times N)$$

which means that $\mathcal{T} \boxtimes \mathcal{Q}$ is a cosheaf on $M \times N$.

If $f : M \to N$ is a continuous map and $\mathcal{T}'$ is another cosheaf on M, then both cosheaves $(f \times f)_!(\mathcal{T} \boxtimes \mathcal{T}')$ and $(f_!\mathcal{T}) \boxtimes (f_!\mathcal{T}')$ are completely determined by their restrictions to the opens $\mathcal{O}(N)\times\mathcal{O}(N) \subset \mathcal{O}(N\times N)$, on which both cosheaves are equivalent by the definition of $\boxtimes$ and $f_!$. □

Remark 2.1.27.— Notice that, although the box product of cosheaves may have a nicer definition than the one for sheaves, it is not compatible with costalks. We usually do not have

$$(\mathcal{T} \boxtimes \mathcal{Q})_{(x,y)^!} \simeq \mathcal{T}_{x^!} \otimes \mathcal{Q}_{y^!}$$

This becomes true however in the case of locally constant cosheaves, or in the case of constructible cosheaves (when M is the underlying topological manifold of a smooth complex variety).

Then we can express the main theorem,

Theorem 2.1.28 (Verdier duality)**.—** *Let* M *be a locally compact Hausdorff topological space. There exits a functor*

$$\Gamma_c : \mathcal{S}h(M) \to \mathcal{C}osh(M)$$

which is an equivalence of ∞-categories and such that, for every continuous map of locally compact Hausdorff topological spaces $f : M \to N$, the functor Γ_c sends the exceptional functors

$$\begin{tikzcd} \mathcal{S}h(M) \arrow[r, shift left, "f_!"] & \mathcal{S}h(N) \arrow[l, shift left, "f^!"] \end{tikzcd}$$

to the pushforward and pullback of cosheaves. Moreover, the functor Γ_C sends the box product of sheaves to the box product of cosheaves: if $\mathcal{F}$ is an object of $\mathcal{S}h(M)$ and $\mathcal{G}$ an object of $\mathcal{S}h(N)$, then

$$\Gamma_c(\mathcal{F} \boxtimes \mathcal{G}) \simeq \Gamma_c(\mathcal{F}) \boxtimes \Gamma_c(\mathcal{G})$$

Remark 2.1.29.— This can be found in [VD] and theorem 5.5.5.1 in [HA]. The last equivalence may be checked on the basis of open sets $U \times V$, with U and V two open subsets of M and N. It is then a direct consequence of the compatibility of Γ_C and the two box products with the functors $f_!$.

Definition 2.1.30.— *For a continuous map between topological manifolds $f : M \to N$, we shall denote by $f^* \dashv f_*$ the adjoint pair between the ∞-categories of cosheaves transferred by Γ_c from the usual adjunction between the ∞-categories of sheaves. We shall call them the exceptional pullback and pushforward of cosheaves.*

Definition 2.1.31.— *Let M be a topological space, a cosheaf $\mathcal{T}$ on M is constant if is can be written as*

$$\mathcal{T} \simeq \pi^! V$$

where $\pi : M \to *$ is the projection and V is an object of $\mathcal{V}\mathrm{ect} \simeq \mathcal{C}\mathrm{osh}(*)$. The cosheaf $\mathcal{T}$ is said to be locally constant if for every $x \in M$, there is an open neighbourhood $x \in U \subset M$ such that $\mathcal{T}_{|U}$ is constant on U. We let $\mathcal{C}\mathrm{osh}_{LC}(M)$ be the full subcategory of $\mathcal{C}\mathrm{osh}(M)$ whose objects are the locally constant cosheaves.

A locally constant cosheaves whose costalks are all finite dimensional shall be called a local system. We shall denote the full subcategory made of local systems by $\mathcal{C}\mathrm{osh}_{LS}(M)$.

Proposition 2.1.32.— *Let M be a topological manifold. The equivalence*

$$\Gamma_c : \mathcal{S}\mathrm{h}(M) \simeq \mathcal{C}\mathrm{osh}(M)$$

induces two equivalences between subcategories:

$$\Gamma_c : \mathcal{S}\mathrm{h}_{LC}(M) \simeq \mathcal{C}\mathrm{osh}_{LC}(M)$$

and

$$\Gamma_c : \mathcal{S}\mathrm{h}_{LS}(M) \simeq \mathcal{C}\mathrm{osh}_{LS}(M)$$

Proof.— Let U be an open disc of M and let $\pi : U \to *$ be the canonical projection, then by proposition 3.3.2 in [KS] we have

$$\pi^*[n] \simeq \pi^!$$

where n is the dimension of M. From this equivalence, we deduce that a sheaf $\mathcal{F}$ is constant on U if and only if $\Gamma_c(\mathcal{F})$ is a constant cosheaf on U. The same argument shows that constant sheaves with finite dimensional stalks on U are equivalent to constant cosheaves with finite dimensional costalks. We then expand the equivalence to locally constant objects using the fact that Γ_c commutes with the restriction to an open subset. Indeed, for an open embedding $j : U \to M$, we have $j^* \simeq j^!$ and thus

$$\Gamma_c \circ j^* \simeq j^! \circ \Gamma_c.$$

□

Definition 2.1.33.— *Let X be a smooth complex variety and let M its associated topological manifold. Then we let*

$$\mathcal{C}\mathrm{osh}_{\mathrm{C}}(M)$$

denote the essential image of

$$\Gamma_c : \mathcal{S}\mathrm{h}_{\mathrm{C}}(M) \to \mathcal{C}\mathrm{osh}(M).$$

We shall refer to an object of $\mathcal{C}\mathrm{osh}_{\mathrm{C}}(M)$ as an (algebraically) constructible cosheaf on M.

As a corollary of the things we have said, it is possible to switch between the theory of constructible sheaves and the one of constructible cosheaves.

Proposition 2.1.34.— *There exists a lax symmetric monoidal functor*

$$(\mathcal{C}\mathrm{osh}_{\mathrm{C}})_\diamond : \mathcal{V}\mathrm{ar}_{\mathrm{pd}} \to \mathcal{C}\mathrm{at}_{\mathbb{C}}$$

and an equivalence of lax symmetric monoidal functors

$$\Gamma_c : (\mathcal{S}\mathrm{h}_{\mathrm{C}})_\diamond \Leftrightarrow (\mathcal{C}\mathrm{osh}_{\mathrm{C}})_\diamond$$

such that for every $X \in \mathcal{V}\mathrm{ar}$, $\mathcal{C}\mathrm{osh}_{\mathrm{C}}(X)$ is the ∞-category of constructible cosheaves on the associated topological manifold of X and

$$\Gamma_c : \mathcal{S}\mathrm{h}_{\mathrm{C}}(X) \simeq \mathcal{C}\mathrm{osh}_{\mathrm{C}}(X)$$

is the equivalence of Verdier duality.

2.2 ALGEBRAIC FACTORISATION COALGEBRAS

2.2.1 *The Ran space*

For a smooth complex variety X, the space $\mathrm{Ran}(X)$ is the space of (non-empty) finite subsets of X. Unfortunately, such a space is by no mean a complex variety, even not a scheme. Instead, we will define the Ran space in the most general place where it can be defined: the ∞-category of prestacks $\mathcal{P}(\mathcal{V}\mathrm{ar})$.

Definition 2.2.1.— *Let* Diag *be the opposite category of the category of non-empty finite sets and surjective maps. Let $\mathcal{D}\mathrm{iag}$ denote the nerve of* Diag.

For any variety X and any morphism $\pi : I \twoheadleftarrow J$ in $\mathcal{D}\mathrm{iag}$, we let $\Delta_\pi : X^I \to X^J$ be the associated closed diagonal embedding. Finally, let $j(\pi) : U(\pi) \hookrightarrow X^J$ be the complementary open embedding.

Definition 2.2.2.— *Let X be a smooth complex variety, we define $\mathrm{Ran}(X) \in \mathcal{P}(\mathcal{V}\mathrm{ar})$ to be the colimit*

$$\mathrm{Ran}(X) = \varinjlim_{I \in \mathcal{D}\mathrm{iag}} X^I$$

The Ran space allows parametrisation of objects by families of points with the benefit of giving a notion of 'what it means to vary continuously when two points collide'. The main object we will want to parametrise continuously is the ∞-category of $\mathcal{D}$-modules.

Definition 2.2.3.— *Using Kan extension, we can extend the functor $\mathcal{D}^!$ of proposition 2.1.2 to the ∞-category $\mathcal{P}(\mathcal{V}\mathrm{ar})^{\mathrm{op}}$. As such, the ∞-category of $\mathcal{D}^!$-modules on $\mathrm{Ran}(X)$ is by definition:*

$$\mathcal{D}^!(\mathrm{Ran}(X)) = \varprojlim_{I} \mathcal{D}^!(X^I)$$

Remark 2.2.4.— An element of $\mathcal{D}^!(\mathrm{Ran}(X))$ is a family of equivariant $\mathcal{D}$-modules, $\mathcal{F} \in \mathcal{D}_{qc}(X), \mathcal{F}_2 \in \mathcal{D}_{qc}(X^2), \ldots$ such that for the diagonal embedding $\Delta : X \to X^2$, we have

$$\Delta^! \mathcal{F}_2 \simeq \mathcal{F}$$

with the same data for the other diagonal embeddings, plus the compatible homotopies between these identifications.

Remark 2.2.5.— Using the universal property of $\mathcal{P}(\mathcal{V}\mathrm{ar})$, we can also extend the functor $\mathcal{D}_*$ to the ∞-category of prestacks. A priori the ∞-categories $\mathcal{D}_*(\mathrm{Ran}(X))$ and $\mathcal{D}^!(\mathrm{Ran}(X))$ may not coincide. In our case, the prestack $\mathrm{Ran}(X)$ is obtained by taking a colimit of closed embeddings for which we have the adjunction $\Delta_* \dashv \Delta^!$. Such that in $\mathcal{P}\mathrm{r}^{cc}_{\mathrm{st}}$ we have the equivalence:

$$\varinjlim_{I \in \mathcal{D}\mathrm{iag}} \mathcal{D}_*(X^I) \simeq \varprojlim_{I \in \mathcal{D}\mathrm{iag}^{op}} \mathcal{D}^!(X^I).$$

This is a usual trick when one takes a colimit inside $\mathcal{P}\mathrm{r}^{cc}_{\mathrm{st}}$ the ∞-category of presentable stable ∞-categories with cocontinuous functors (see [Gai]).

For this reason, the theory of $\mathcal{D}$-modules can be nicely extended to certain prestacks.

Definition 2.2.6.— *A pseudo-indvariety is a an object of $\mathcal{P}(\mathcal{V}\mathrm{ar})$ that can be obtained by a colimit of ind-proper maps. We shall denote the ∞-category of pseudo-indvarieties by $\mathrm{pInd}\mathcal{V}\mathrm{ar}$ and let $\mathrm{pInd}\mathcal{V}\mathrm{ar}_{\mathrm{pd}}$ be the ∞-category of pseudo-invarieties with partially defined maps, endowed with its Cartesian symmetric monoidal structure.*

Proposition 2.2.7.— *The lax symmetric monoidal $\mathcal{D}_\diamond$ extends to a lax symmetric monoidal functor along the symmetric monoidal embedding $\mathcal{V}\mathrm{ar}_{\mathrm{pd}} \hookrightarrow \mathrm{pInd}\mathcal{V}\mathrm{ar}_{\mathrm{pd}}$:*

$$\mathcal{D}^!_\diamond : \mathrm{pInd}\mathcal{V}\mathrm{ar}_{\mathrm{pd}} \to \mathcal{C}\mathrm{at}_{\mathbb{C}}.$$

For details on this proposition, see chapter V of [GRBook] or [Gai].

2.2.2 *The chiral tensor product*

The space $\mathrm{Ran}(X)$ comes with a universal property, it is the free idempotent commutative semigroup generated by X. Let us denote by

$$u_X : \mathrm{Ran}(X) \times \mathrm{Ran}(X) \to \mathrm{Ran}(X)$$

the union operation. Because the functor $\mathcal{D}^!_\diamond$ is lax symmetric monoidal, the union induces a (non-unital) tensor structure on the ∞-category $\mathcal{D}^!(\mathrm{Ran}(X))$.

Proposition 2.2.8 ([FG]).— *Let $X \in \mathcal{V}\mathrm{ar}$, the ∞-category $\mathcal{D}^!(\mathrm{Ran}(X))$ may be endowed with a (non-unital) tensor structure, such that for $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{D}^!(\mathrm{Ran}(X))$, we have*

$$\mathcal{F} \otimes^* \mathcal{G} := u_*(\mathcal{F} \boxtimes \mathcal{G})$$

This tensor structure is called the star tensor product.

In order to define the notion of factorisation coalgebra, we need a slight variation on the definition of that tensor product. Instead of using directly the union of two subsets $S \cup T$, we will only allow to take the union of *disjoint* subsets $T \cap S = \emptyset$. Following [Rask], the Ran space becomes a commutative semigroup in $\mathcal{P}(\mathcal{V}\mathrm{ar})$ if we change the morphisms to become correspondences. The multiplication $\amalg_X$ is then defined as

$$\begin{tikzcd}
 & {[\mathrm{Ran}(X) \times \mathrm{Ran}(X)]_{\mathrm{disj}}} \arrow[dl, "j"'] \arrow[dr, "uj"] & \\
\mathrm{Ran}(X) \times \mathrm{Ran}(X) \arrow[rr, squiggly, "\amalg_X"] & & \mathrm{Ran}(X)
\end{tikzcd}$$

Proposition 2.2.9.— *The prestack $\mathrm{Ran}(X)$ endowed with the disjoint multiplication $\amalg_X$ is a commutative semigroup in $\mathcal{P}(\mathcal{V}\mathrm{ar})_{\mathrm{pd}}$.*

As $[\mathrm{Ran}(X) \times \mathrm{Ran}(X)]_{\mathrm{disj}}$ is also a pseudo-indvariety, the theory of $\mathcal{D}$-modules allows us to build a new tensor product.

Proposition 2.2.10 ([FG]).— *Let $X \in \mathcal{V}\mathrm{ar}$, the ∞-category $\mathcal{D}^!(\mathrm{Ran}(X))$ may be endowed with a (non-unital) tensor structure, such that for $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{D}^!(\mathrm{Ran}(X))$, we have*

$$\mathcal{F} \otimes^{\mathrm{ch}} \mathcal{G} := (uj)_* j^*(\mathcal{F} \boxtimes \mathcal{G}).$$

This tensor structure is called the chiral tensor product.

Remark 2.2.11.— The disjoint locus of $\mathrm{Ran}(X) \times \mathrm{Ran}(X)$ is defined as the fibre product

$$\begin{tikzcd}
\mathrm{Ran}(X) \times \mathrm{Ran}(X) \arrow[r, "\cap"] & \mathrm{Ran}(X) \\
{[\mathrm{Ran}(X) \times \mathrm{Ran}(X)]_{\mathrm{disj}}} \arrow[u, "j"] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \emptyset \arrow[u]
\end{tikzcd}$$

Being the pullback of an open embedding, the morphism j is also an open embedding of prestacks. This is why we use the notation j^* in the definition of the chiral tensor product.

For $\mathrm{I} \in \mathcal{D}\mathrm{iag}$, let $(\Delta^{\mathrm{I}})^!$ be the projection $\mathcal{D}^!(\mathrm{Ran}(\mathrm{X})) \to \mathcal{D}(\mathrm{X}^{\mathrm{I}})$. We then have this explicit description of the chiral tensor product.

Proposition 2.2.12 ([FG]).— *Let* $\mathrm{X} \in \mathcal{V}\mathrm{ar}$ *and let* $(\mathcal{F}_j)_{j\in \mathrm{J}}$ *be a finite family of* $\mathcal{D}^!$*-modules on* $\mathrm{Ran}(\mathrm{X})$*. Then the canonical map*

$$\bigoplus_{\pi} j(\pi)_* j(\pi)^* \left(\boxtimes_{j\in\mathrm{J}} (\Delta^{\mathrm{I}_j})^! \mathcal{F}_j \right) \to (\Delta^{\mathrm{I}})^! \left(\otimes^{\mathrm{ch}}_{j\in\mathrm{J}} \mathcal{F}_j \right)$$

is an equivalence, where the direct sum is taken over all surjections $\pi : \mathrm{I} \twoheadrightarrow \mathrm{J}$.

2.2.3 *Factorisation coalgebras*

Definition 2.2.13 ([FG]).— *Let* $\mathcal{F}$ *be a cocommutative, coassossiative coalgebra for the chiral tensor product on* $\mathrm{Ran}(\mathrm{X})$*. We will say that* $\mathcal{F}$ *is a factorisation coalgebra if for every* $\pi : \mathrm{J} \twoheadleftarrow \mathrm{I} \in \mathcal{D}\mathrm{iag}$*, the map*

$$(\Delta^{\mathrm{J}})^!(\mathcal{F}) \to j(\pi)_* j(\pi)^* \left(\boxtimes_{j\in\mathrm{J}} (\Delta^{\mathrm{I}_j})^! \mathcal{F} \right)$$

obtained from proposition 2.2.12, induces by adjunction an equivalence

$$j(\pi)^* (\Delta^{\mathrm{J}})^!(\mathcal{F}) \to j(\pi)^* \left(\boxtimes_{j\in\mathrm{J}} (\Delta^{\mathrm{I}_j})^! \mathcal{F} \right)$$

We shall let $\mathcal{F}\mathcal{D}^{\mathrm{ch}}(\mathrm{X})$ *be the full subcategory of cocommutative, coassociative coalgebras of* $(\mathrm{Ran}(\mathrm{X}), \otimes^{\mathrm{ch}})$ *whose objects are the factorisation coalgebras.*

Remark 2.2.14.— Let $\mathcal{F}$ be a factorisation coalgebra. For every finite set of points $i : x = (x_1, \ldots, x_n) \hookrightarrow \mathrm{X}^n$, let $\mathcal{F}_{x_1,\ldots,x_n} = i^! \mathcal{F}_n$ be the !-stalk at x. Because $\mathcal{F}$ is a $\mathcal{D}^!$-module on $\mathrm{Ran}(\mathrm{X})$, we have $\mathcal{F}_{x,x} \simeq \mathcal{F}_x$ for every $x \in \mathrm{X}$ and because $\mathcal{F}$ has the factorisation property, we also have $\mathcal{F}_{x,y} \simeq \mathcal{F}_x \otimes \mathcal{F}_y$ for $x \neq y$.

Definition 2.2.15.— *We shall call a* $\mathcal{D}^!$*-module* $\mathcal{M}$ *on* $\mathrm{Ran}(\mathrm{X})$ *regular holonomic if for every* $\mathrm{I} \in \mathcal{D}\mathrm{iag}$, $(\Delta^{\mathrm{I}})^! \mathcal{M} \in \mathcal{D}(\mathrm{X}^{\mathrm{I}})$ *is regular holonomic. Thus,*

$$\mathcal{D}^!_{\mathrm{RH}}(\mathrm{Ran}(\mathrm{X})) \simeq \varprojlim_{\mathrm{I}\in\mathcal{D}\mathrm{iag}} \mathcal{D}_{\mathrm{RH}}(\mathrm{X}^{\mathrm{I}})$$

Proposition 2.2.16.— *The* ∞*-category* $\mathcal{D}^!_{\mathrm{RH}}(\mathrm{Ran}(\mathrm{X}))$ *inherits the chiral tensor structure from* $\mathcal{D}^!(\mathrm{Ran}(\mathrm{X}))$.

Proof.— Let $\mathcal{M}$ and $\mathcal{N}$ be two regular holonomic $\mathcal{D}^!$-modules on $\mathrm{Ran}(X)$. Then because of proposition 2.1.9, regular holonomic $\mathcal{D}$-modules are stable under the operations $\boxtimes$, j_* and j^*, so that by proposition 2.2.12, the chiral tensor product

$$\mathcal{M} \otimes^{\mathrm{ch}} \mathcal{N}$$

is again regular holonomic. Then proposition then follows from proposition 2.2.1.1 in [HA]. □

Definition 2.2.17.— *We let $\mathcal{FD}^{\mathrm{ch}}_{\mathrm{RH}}(X)$ be the full subcategory of $\mathcal{FD}^{\mathrm{ch}}(X)$ whose objects are the factorisation coalgebras that are also regular holonomic modules.*

Remark 2.2.18.— The same could be said of the ∞-category of (non necessarily regular) holonomic $\mathcal{D}^!$-modules on $\mathrm{Ran}(X)$. There is an ∞-category $\mathcal{FD}^{\mathrm{ch}}_{\mathrm{H}}(X)$ of holonomic factorisation coalgebras.

2.2.4 *Examples*

In the case where $X = \mathbb{A}^1$, the ordinary category of factorisation coalgebras (which are translation invariant) is equivalent to the category of vertex algebras.

$$\mathrm{VAlg} \simeq \mathrm{FD}^{\mathrm{ch}}_{\mathrm{equi}}(\mathbb{A}^1)$$

The theory of vertex algebras is rich of examples, many can be found in [BZFrenk].

A vertex algebra is in particular a vector space V with an endomorphism $T : V \to V$, so that the sheaf of modules $\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1}$ is a quasi-coherent right $\mathcal{D}$-module. However, this $\mathcal{D}$-module is holonomic if and only if V is finite dimensional and the theory of vertex algebras reduces to the theory of traditional commutative algebras with a derivation, in the case where the underlying vector space is finite dimensional.

For this reason, we shall instead consider dg-vertex algebras. Differential graded vertex algebras are defined straightforwardly from vertex algebras. Given a dg-vertex algebra V, the associated $\mathcal{D}$-module $\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1}$ is holonomic if and only if V has finite dimensional bounded homology. In this case $\mathcal{V}$ is even a flat vector bundle. The factorisation coalgebra is obtained by applying the construction of chapter 19 in [BZFrenk] at the dg-level.

Extending the equivalence of ordinary categories, we obtain a dg-equivalence between the dg-categories of dg-vertex algebras and dg-factorisation coalgebras,

$$\text{dg-VAlg} \simeq \text{dg-FD}^{\mathrm{ch}}_{\mathrm{equi}}(\mathbb{A}^1)$$

and to any dg-factorisation coalgebra it is possible to associate a factorisation coalgebra in the ∞-categorical framework. However, we do not know whether any factorisation coalgebra can be strictified into a dg-factorisation coalgebra.

Proposition 2.2.19.— *There exists a functor,*

$$\text{dg-VAlg}_{\mathrm{f.d}} \to \mathcal{FD}^{\mathrm{ch}}_{\mathrm{VB}}(\mathbb{A}^1)$$

sending finite dimensional dg-vertex algebras to factorisation vector bundles on the affine space $\mathbb{A}^1$.

A trivial class of examples

Thus, we are looking for dg-vertex algebras whose underlying homology is bounded and finite dimensional. Such examples include dg-algebras $(A_\bullet, d, m)$ with a derivation, such that $H_n(A)$ is finite dimensional for every n and trivial for almost every n. These correspond to the so called *commutative* dg-vertex algebras.

A non-commutative example

A non-commutative example of a dg-vertex algebra satisfying our requirements is given by the chiral de Rham complex (see [MSV]). Given a variety X, the authors build a sheaf of dg-vertex algebras whose underlying chain complex is quasi-isomorphic to the de Rham complex of X. Hence, in the case where X is projective, the built dg-vertex algebras have bounded finite dimensional homology.

How to build examples of factorisation $\mathcal{D}_{\mathrm{RH}}$-modules

Both examples above are translation equivariant. A recipe to build regular holonomic factorisation algebras out of a translation equivariant one is to add some modules on points. By the construction of chapter 19.5 in [BZFrenk], given a dg-module M over a dg-vertex algebra V, both having finite dimensional homology, we can construct a regular holonomic factorisation algebra on $\mathbb{A}^1$. Simply choose one point $i : \{x\} \hookrightarrow \mathbb{A}^1$, then the sum $\mathcal{V} \oplus i_* M$ has the structure of a regular holonomic factorisation coalgebra. By induction, given any *finite* number of points $i_k : \{x_k\} \hookrightarrow \mathbb{A}^1$ and any finite number of dg-modules M_k with finite dimensional homology, the sum $\mathcal{V} \oplus (i_1)_* M_1 \oplus \ldots \oplus (i_n)_* M_n$ will also have the structure of a regular holonomic factorisation coalgebra.

2.3 Riemann-Hilbert for factorisation coalgebras

2.3.1 *Constructible factorisation coalgebras*

We have been able to define the ∞-category of $\mathcal{D}^!$-modules on the Ran space by using a right Kan extension of the functor $\mathcal{D}^! : \mathcal{V}\mathrm{ar}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_{\mathbb{C}}$ to $\mathcal{D}^! : \mathcal{P}(\mathcal{V}\mathrm{ar})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_{\mathbb{C}}$. We will use the same definition to define constructible !-sheaves on the Ran space.

Definition 2.3.1.— *Let* M *be the topological manifold associated to a smooth complex variety, then we shall define* $\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M}))$ *as*

$$\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M})) \simeq \varprojlim_{\mathrm{I}\in\mathcal{D}\mathrm{iag}} \mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{M}^{\mathrm{I}})$$

From the existence of $\mathcal{D}^!_{\diamond}$ (proposition 2.2.7) and the equivalence given by theorem 2.1.14, we deduce the following propositions.

Proposition 2.3.2.— *Let* X *be an object of* $\mathcal{V}\mathrm{ar}$ *and let* M *be its associated topological manifold. Then the equivalence*

$$\mathrm{DR} : \mathcal{D}_{\mathrm{RH}}(\mathrm{X}) \simeq \mathcal{S}\mathrm{h}_{\mathrm{C}}(\mathrm{M})$$

extends to an equivalence of ∞*-categories*

$$\mathrm{DR}^! : \mathcal{D}^!_{\mathrm{RH}}(\mathrm{Ran}(\mathrm{X})) \simeq \mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M}))$$

Proof.— For f a morphism of $\mathcal{V}\mathrm{ar}$, we have $\mathrm{DR} \circ f^! \simeq f^! \circ \mathrm{DR}$, hence the result. □

Corollary 2.3.3.— *The* ∞*-category* $\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M}))$ *inherits a (non-unital) symmetric tensor structure from* $\mathcal{D}^!_{\mathrm{RH}}(\mathrm{Ran}(\mathrm{X}))$ *that we shall also call the chiral tensor product.*

Proposition 2.3.4.— *Let* $(\mathcal{F}_j)_{j\in\mathrm{J}}$ *be a finite family of objects of* $\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M}))$*. Then there is a canonical equivalence*

$$\bigoplus_{\pi} j(\pi)_* j(\pi)^{-1}\left(\boxtimes_{j\in\mathrm{J}}(\Delta^{\mathrm{I}_j})^!\mathcal{F}_j\right) \to (\Delta^{\mathrm{I}})^!\left(\bigotimes^{\mathrm{ch}}_{j\in\mathrm{J}}\mathcal{F}_j\right)$$

where the direct sum is taken over all surjections $\pi : \mathrm{I} \twoheadrightarrow \mathrm{J}$.

Proof.— By Riemann-Hilbert, the de Rham functor commutes with all operations j_*, j^* and $\boxtimes$. It also commutes with finite direct sums as it is a stable functor. The proposition then follows from proposition 2.2.12 and the definition of the chiral tensor product on $\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M}))$ given by the equivalence of proposition 2.3.2. □

Definition 2.3.5.— *A cocommutative coalgebra* $\mathcal{F}$ *for the disjoint tensor structure will be called a factorisation coalgebra if for every* $\pi : \mathrm{J} \twoheadleftarrow \mathrm{I}$ *in* $\mathcal{D}\mathrm{iag}$*, the map*

$$j(\pi)_* j(\pi)^{-1}\left(\boxtimes_{j\in\mathrm{J}}(\Delta^{\mathrm{I}_j})^!\mathcal{F}\right) \to (\Delta^{\mathrm{J}})^!\mathcal{F}$$

obtained from proposition 2.3.4, induces by adjunction an equivalence

$$j(\pi)^{-1}\left(\boxtimes_{j\in\mathrm{J}}(\Delta^{\mathrm{I}_j})^!\mathcal{F}\right) \to j(\pi)^{-1}(\Delta^{\mathrm{J}})^!(\mathcal{F})$$

We shall let $\mathcal{F}\mathcal{S}\mathrm{h}_{\mathrm{C}}(\mathrm{M})$ *be the* ∞*-category of factorisation coalgebras in the constructible context.*

Since regular holonomic $\mathcal{D}$-modules and constructible sheaves are equivalence, so are the corresponding factorisation coalgebras.

Proposition 2.3.6.— *The equivalence of symmetric monoidal ∞-categories given by the de Rham functor*

$$\mathrm{DR}^! : (\mathcal{D}^!_{\mathrm{RH}}(\mathrm{Ran}(X)), \otimes^{\mathrm{ch}}) \to (\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(M)), \otimes^{\mathrm{ch}}),$$

induces an equivalence between the ∞-categories of factorisation coalgebras,

$$\mathrm{DR} : \mathcal{F}\mathcal{D}^{\mathrm{ch}}_{\mathrm{RH}}(X) \simeq \mathcal{F}\mathcal{S}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(M)$$

Proof.— Since the extended de Rham functor $\mathrm{DR}^!$ is a symmetric monoidal equivalence, the corresponding ∞-categories of commutative coalgebras are also equivalent. Since by Riemann-Hilbert it also commutes with the operations j^* and $\boxtimes$, it preserves the factorisation property. □

Remark 2.3.7.— In the case of $\mathcal{D}$-modules that are only holonomic, the de Rham functor does not commute with f_* any more. Nonetheless, for every morphism of varieties f, there is always a canonical map

$$\mathrm{DR} \circ f_* \to f_* \circ \mathrm{DR}.$$

and since for any two holonomic $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{N}$, we have the equivalence

$$\mathrm{DR}(M \boxtimes N) \simeq \mathrm{DR}(M) \boxtimes \mathrm{DR}(N),$$

the extended functor

$$\mathrm{DR}^! : (\mathcal{D}^!_{\mathrm{H}}(\mathrm{Ran}(X)), \otimes^{\mathrm{ch}}) \to (\mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(M)), \otimes^{\mathrm{ch}}),$$

is colax symmetric monoidal, which is fine to send coalgebras to coalgebras.

Thus the equivalence $\mathrm{DR} : \mathcal{F}\mathcal{D}^{\mathrm{ch}}_{\mathrm{RH}}(X) \simeq \mathcal{F}\mathcal{S}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(M)$ extends to a functor $\mathrm{DR} : \mathcal{F}\mathcal{D}^{\mathrm{ch}}_{\mathrm{H}}(X) \to \mathcal{F}\mathcal{S}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(M)$.

2.3.2 *Factorisation vector bundles and local systems*

The classical Riemann-Hilbert equivalence is an extension of the equivalence between vector bundles and local systems. We would like to say something about this equivalence at the factorisation level.

Definition 2.3.8.— *Let $\Delta_{\mathrm{main}} : X \to \mathrm{Ran}(X)$ be the main diagonal embedding.*

- *Let $\mathcal{F}\mathcal{D}^{\mathrm{ch}}_{\mathrm{VB}}(X)$ be the full subcategory of $\mathcal{F}\mathcal{D}^{\mathrm{ch}}_{\mathrm{RH}}(X)$ made of factorisation coalgebras $\mathcal{A}$ such that $\Delta^!_{\mathrm{main}}\mathcal{A}$ is a chain complexes of $\mathcal{D}$-modules with vector bundle homology. We shall call them (non-unital) factorisation vector bundles;*

- *Likewise, let* $\mathcal{FS}\mathrm{h}^{\mathrm{ch}}_{\mathrm{LS}}(\mathrm{M})$ *be the full subcategory of* $\mathcal{FS}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(\mathrm{M})$ *made of factorisation algebras* $\mathcal{F}$ *such that* $\Delta^!_{\mathrm{main}}\mathcal{A}$ *is a chain complex of sheaves whose homology consists of local systems i.e locally constant sheaves with finite dimensional stalks. They shall be called (non-unital) factorisation local systems.*

Remark 2.3.9.— Let $\mathcal{A}$ be an element of $\mathcal{FS}\mathrm{h}^{\mathrm{ch}}_{\mathrm{LS}}(\mathrm{M})$, and let $\Delta : \mathrm{M} \hookrightarrow \mathrm{M}^2$ be the diagonal embedding. Then it is almost never true that $\Delta^!\mathcal{A}$ is a local system. Instead it is a constructible sheaf. This is a reminiscence that local systems are usually not preserved by the operations of sheaves. Same can be said about vector bundles.

Proposition 2.3.10.— *The de Rham functor restricts to an equivalence on the subcategories*

$$\mathrm{DR} : \mathcal{FD}^{\mathrm{ch}}_{\mathrm{VB}}(\mathrm{X}) \simeq \mathcal{FS}\mathrm{h}^{\mathrm{ch}}_{\mathrm{LS}}(\mathrm{M})$$

Proof.— The functor $\mathrm{DR}^! : \mathcal{D}^!_{\mathrm{RH}}(\mathrm{Ran}(\mathrm{X})) \to \mathcal{S}\mathrm{h}^!_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M}))$ is defined as a limit functor, hence by definition we have the following commutation relation,

$$\Delta^!_{\mathrm{main}} \circ \mathrm{DR}^! \simeq \Delta^!_{\mathrm{main}}$$

which means that $\mathrm{DR} : \mathcal{FD}^{\mathrm{ch}}_{\mathrm{RH}}(\mathrm{X}) \simeq \mathcal{FS}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(\mathrm{M})$ reduces to

$$\mathrm{DR} : \mathcal{FD}^{\mathrm{ch}}_{\mathrm{VB}}(\mathrm{X}) \simeq \mathcal{FS}\mathrm{h}^{\mathrm{ch}}_{\mathrm{LS}}(\mathrm{M}).$$

by theorem 2.1.14 □

2.4 VERDIER DUALITY FOR FACTORISATION COALGEBRAS

☛ Let M be the associated topological manifold of a smooth complex variety.

Following the exact same path as in the previous section with the functor DR, the Verdier duality functor

$$\Gamma_c : (\mathcal{S}\mathrm{h}_{\mathrm{C}})_\diamond \Leftrightarrow (\mathcal{C}\mathrm{osh}_{\mathrm{C}})_\diamond$$

allows us to define a chiral tensor structure on the ∞-category of constructible cosheaves on Ran(M):

$$\mathcal{C}\mathrm{osh}_{\mathrm{C}}(\mathrm{Ran}(\mathrm{M})) = \varprojlim_{\mathrm{I} \in \mathcal{D}\mathrm{iag}} \mathcal{C}\mathrm{osh}_{\mathrm{C}}(\mathrm{M}^{\mathrm{I}})$$

whose explicit description is given by,

Proposition 2.4.1.— *Let* $(\mathcal{T}_j)_{j\in\mathrm{J}}$ *be a finite family of constructible cosheaves on* Ran(M). *Then, for every* $\mathrm{I} \in \mathcal{D}\mathrm{iag}$, *there is a canonical equivalence*

$$\bigoplus_{\pi} j(\pi)_* j(\pi)^{-1}\left(\boxtimes_{j\in\mathrm{J}} (\Delta^{\mathrm{I}_j})^! \mathcal{T}_j\right) \to (\Delta^{\mathrm{I}})^! \left(\otimes^{\mathrm{ch}}_{j\in\mathrm{J}} \mathcal{T}_j\right)$$

where the direct sum is taken over all surjections $\pi : \mathrm{I} \twoheadrightarrow \mathrm{J}$.

Most importantly, Verdier duality induces an equivalence between the ∞-category of constructible factorisation coalgebras realised as sheaves and as cosheaves.

Definition 2.4.2.— *Let $\Delta_{\mathrm{main}} : M \to \mathrm{Ran}(M)$ be the canonical embedding.*

- *Let $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{ch}}_{\mathrm{C}}(M)$ be the ∞-category of cocommutative coalgebras in $\mathcal{C}\mathrm{osh}_{\mathrm{C}}(\mathrm{Ran}(M))$ for the chiral tensor product and satisfying the factorisation property analogous to definition 2.3.5.*
- *We shall denote by $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{ch}}_{\mathrm{LS}}(M)$, the full subcategory of factorisation coalgebras $\mathcal{T}$ such that $\Delta^!_{\mathrm{main}}\mathcal{T}$ is locally constant. We shall refer to objects of this ∞-category as factorisation local systems.*

Remark 2.4.3.— Let $\Delta_{U^I} : U^I \to \mathrm{Ran}(M)$ be the composition of the open embedding $U^I \to M^I$ (the complement of the diagonals) and the canonical map $M^I \to \mathrm{Ran}(M)$. Let $\mathcal{T}$ be a factorisation local system on M, then because of the factorisation property, the pullback

$$\Delta^!_{U^I}\mathcal{T} \in \mathcal{C}\mathrm{osh}(M^I)$$

lies in fact in $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(M^I)$.

Proposition 2.4.4.— *The Verdier duality functor induces an equivalence between the ∞-categories $\mathcal{F}\mathcal{S}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(M)$ and $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{ch}}_{\mathrm{C}}(M)$,*

$$\Gamma_c : \mathcal{F}\mathcal{S}\mathrm{h}^{\mathrm{ch}}_{\mathrm{C}}(M) \simeq \mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{ch}}_{\mathrm{C}}(M)$$

This equivalence induces an equivalence between the subcategories

$$\Gamma_c : \mathcal{F}\mathcal{S}\mathrm{h}^{\mathrm{ch}}_{\mathrm{LS}}(M) \simeq \mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{ch}}_{\mathrm{LS}}(M)$$

Remark 2.4.5.— Both equivalences are direct consequences of proposition 2.1.34.

2.5 RELATION TO LITTLE DISCS ALGEBRAS

In this section, we prove that the ∞-category of (non-unital) factorisation local systems, as defined in the previous section, is equivalent to the ∞-category of finite dimensional (non-unital) $\mathcal{E}_M$-algebras.

2.5.1 *The disjoint tensor product*

In order to compare factorisation coalgebras and $\mathcal{E}_M$-algebras, we need to translate the definition of $\mathcal{E}_M$-algebras in the language of the Ran space. The beginning of this translation has been made in [HA] section 5.5.4. The complete equivalence has been made by Knudsen in [Knud].

Here, we will use his work extensively and recall the definitions and theorems we are going to need.

The notion of $\mathcal{E}_{\mathrm{M}}$-algebra is fundamentally topological and a key difference between the algebraic world and topological world is that given a topological manifold M, the colimit defining the Ran space can be calculated in Top; there is no need to use prestacks *a priori*.

Definition 2.5.1.— *Let* M *be a topological manifold, then* $\mathcal{R}(\mathrm{M})$ *is the topological Ran space of* M *defined as the colimit inside the category of topological spaces*

$$\mathcal{R}(\mathrm{M}) = \varinjlim_{\mathrm{I}\in\mathrm{Diag}} \mathrm{M}^{\mathrm{I}}$$

☛ Let M be a topological manifold.

Definition 2.5.2.— *An open subset* U *of* M *is a disc if it is homeomorphic to the open unit disc on* $\mathbb{R}^n$ *for some* n*. For a finite family of disjoint discs* $\mathrm{U}_1, \ldots, \mathrm{U}_n$ *of* M*, we let* $\mathcal{R}(\mathrm{U}_1, \ldots, \mathrm{U}_n)$ *be the open subset of* $\mathcal{R}(\mathrm{M})$ *defined by*

$$\mathcal{R}(\mathrm{U}_1, \ldots, \mathrm{U}_n) = \{\mathrm{S} \in \mathcal{R}(\mathrm{M}) | \mathrm{S} \cap \mathrm{U}_i \neq \emptyset, \forall 1 \leq i \leq n\}$$

Remark 2.5.3.— The opens $\mathcal{R}(\mathrm{U}_{\mathrm{I}})$ do not form a basis of the topology of $\mathcal{R}(\mathrm{M})$. The topological spaces induced by the discs can be denoted by $\mathcal{R}_{\mathrm{H}}(\mathrm{M})$ as it is the topology generated from the Hausdorff metric. This is the topological space considered by Lurie in [HA]. This difference between the two topologies disappear when looking exclusively to locally constant cosheaves.

Definition 2.5.4.— *A cosheaf* $\mathcal{T}$ *on* $\mathcal{R}(\mathrm{M})$ *will be called locally constant if the pullback* $r_n^!\mathcal{T}$ *is locally constant, where* $r_n : \mathrm{U}_n \to \mathcal{R}(\mathrm{M})$ *is the composite of the open embedding* $\mathrm{U}_n \subset \mathrm{M}^n$ *corresponding to the complement of the diagonals, and the canonical map* $\mathrm{M}^n \to \mathcal{R}(\mathrm{M})$.

We let $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(\mathrm{M}))$ *denote the* ∞*-category of locally constant cosheaves on* $\mathcal{R}(\mathrm{M})$.

Remark 2.5.5.— Locally constant cosheaves are called 'constructible cosheaves' in [Knud] and [HA], meaning 'constructible with respect to the canonical stratification of the Ran space'.

Knudsen endows the ∞-category $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(\mathrm{M}))$ with a (non-unital) tensor structure that we are going to describe now.

Definition 2.5.6.— *Let* $\mathcal{D}\mathrm{isc}(\mathrm{M})^{\otimes}_{\mathrm{nu}}$ *be the operadic nerve of the (non-unital) coloured operad of discs in* M*. The colours are the discs* $\mathrm{U} \subset \mathrm{M}$*. For a (non-empty) family of discs* U_{I} *and any disc* V*, the set* $\mathrm{Mult}(\mathrm{U}_{\mathrm{I}}, \mathrm{V})$ *is a point if the all the discs* U_i *are disjoint and included in* V*; it is empty otherwise.*

Definition 2.5.7.— *An object U_I of $\mathcal{D}\mathrm{isc}(M)^{\otimes}_{\mathrm{nu}}$ is disjoint if $U_i \cap U_j = \emptyset$ for every $i \neq j \in I$. The full subcategory of disjoint families is $\mathcal{D}_*(M)^{\otimes} \subset \mathcal{D}\mathrm{isc}(M)^{\otimes}_{\mathrm{nu}}$.*

Remark 2.5.8.— $\mathcal{D}_*(M)^{\otimes}$ is not a (non-unital) ∞-operad but it is an approximation of the ∞-operad of discs and we have an inert coCartesian map $\mathcal{D}_*(M)^{\otimes} \to \mathcal{F}\mathrm{in}_*$ (see [Knud] section 2 and Appendix A). Indeed the union of disjoint families of discs is only partially defined.

Definition 2.5.9.— *Let $\mathcal{S}\mathrm{urj}_*$ denote the wide subcategory of $\mathcal{F}\mathrm{in}_*$ containing the surjective functions. The inclusion $\mathcal{S}\mathrm{urj}_* \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad, the non-unital operad.*

Let $\mathcal{D}(M)^{\otimes}$ be the $\mathcal{S}\mathrm{urj}_*$-envelope of $\mathcal{D}_*(M)^{\otimes} \to \mathcal{S}\mathrm{urj}_*$. Its underlying ∞-category is denoted $\mathcal{D}(M)$. Its objects are the non-empty families of disjoint discs and the morphisms $U_I \to V_J$ are given by the surjections $\pi : I \twoheadrightarrow J$ such that $\amalg_{\pi^{-1}(j)} U_i \subset V_j$ for all $j \in J$.

A morphism $U_I \to V_J$ is an isotopy if $\pi : I \to J$ is bijective.

Proposition 2.5.10 ([Knud] 2.6.3).— *The forgetful map*

$$\mathrm{oblv} : \mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(M)) \to [\mathcal{D}(M), \mathcal{V}\mathrm{ect}]$$

is fully faithful and its essential image is made of those functors

$$F : \mathcal{D}(M) \to \mathcal{V}\mathrm{ect}$$

such that for any isotopy $U_I \to V_J$, the induces map

$$F(U_I) \to F(V_J)$$

is an equivalence.

Remark 2.5.11.— In short for the first statement: locally constant cosheaves on $\mathcal{R}(M)$ are completely determined by their restriction along the map $\mathcal{O}(\mathcal{R}_H(M)) \hookrightarrow \mathcal{O}(\mathcal{R}(M))$.

Definition 2.5.12.— *The ∞-category $[\mathcal{D}(M)^{\mathrm{op}}, \mathcal{V}\mathrm{ect}^{\mathrm{op}}]$ is endowed with the (non-unital) Day convolution tensor structure. The corresponding tensor structure on the opposite category is called the disjoint tensor product.*

Proposition 2.5.13 ([Knud] 3.2.1).— *For discs $U_I \in \mathcal{D}(M)$ and functors $F_J \in [\mathcal{D}(M), \mathcal{V}\mathrm{ect}]$, there is a natural $\mathfrak{S}_J$-equivariant equivalence*

$$\left(\bigotimes^{\mathrm{disj}} F_j\right)(U_I) \xrightarrow{\sim} \bigoplus_{\pi : I \twoheadrightarrow J} \bigotimes_{J} F_j(U_{\pi^{-1}(j)})$$

Corollary 2.5.14.— *The full subcategory $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(M))$ inherits a (non-unital) symmetric tensor structure.*

Definition 2.5.15.— *A cocommutative coalgebra $(\mathcal{A}, w)$ for the disjoint tensor product on $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(\mathrm{M}))$ shall be called a locally constant factorisation coalgebra if for every $\mathrm{U}_\mathrm{I} \in \mathcal{D}(\mathrm{M})$, the composite map*

$$\mathcal{A}(\mathrm{U}_\mathrm{I}) \xrightarrow{w^{|\mathrm{I}|}(\mathrm{U}_\mathrm{I})} \mathcal{A}^{\otimes|\mathrm{I}|}(\mathrm{U}_\mathrm{I}) \xrightarrow{\textit{projection}} \bigotimes_{i\in\mathrm{I}} \mathcal{A}(\mathrm{U}_i)$$

is an equivalence.

We shall denote by $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{disj}}_{\mathrm{LC}}(\mathrm{M})$ the full subcategory of the ∞-category of cocommutative coalgebras made of locally constant factorisation coalgebras.

Theorem 2.5.16 ([Knud] 2.2.7 and 3.2.5)**.—** *The ∞-category of non-unital $\mathcal{E}_\mathrm{M}$-algebras (in $\mathcal{V}\mathrm{ect}$) is equivalent to $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{disj}}_{\mathrm{LC}}(\mathrm{M})$.*

We will be only interested in the case of finite dimensional objects. By finite dimensional, we always mean: 'bounded with finite dimensional homology groups'.

Definition 2.5.17.— *A locally constant factorisation coalgebra $\mathcal{A}$ will be called a factorisation local system if $\mathcal{A}(\mathrm{U}_\mathrm{I})$ is finite dimensional for every $\mathrm{U}_\mathrm{I} \in \mathcal{D}(\mathrm{M})$. We shall let $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{disj}}_{\mathrm{LS}}(\mathrm{M})$ be the full subcategory of $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{disj}}_{\mathrm{LC}}(\mathrm{M})$ made of factorisation local systems for the disjoint tensor structure.*

Corollary 2.5.18.— *The ∞-category of (non-unital) finite dimensional $\mathcal{E}_\mathrm{M}$-algebras is equivalent to $\mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{disj}}_{\mathrm{LS}}(\mathrm{M})$.*

Proof.— Thanks to proposition 2.2.7 in [Knud], the ∞-category of (non-unital) finite dimensional $\mathcal{E}_\mathrm{M}$-algebras is equivalent to the ∞-category of locally constant $\mathcal{D}_*(\mathrm{M})$ algebras in $\mathcal{V}\mathrm{ect}_{\mathrm{f.d}}$. And by proposition 3.2.5, those correspond to locally constant coalgebras taking values in $\mathcal{V}\mathrm{ect}_{\mathrm{f.d}}$. □

2.5.2 *Comparison of the Ran spaces*

Having jumped into the topological world, the last mile of the proof consists in a comparison of the Ran spaces $\mathcal{R}(\mathrm{M})$ and $\mathrm{Ran}(\mathrm{M})$. The two compare well when we deal with paracompact Hausdorff Lindelöf spaces. Let's first recall the definition of a paracompact Hausdorff Lindelöf topological space.

Definition 2.5.19.— *A topological space T is said to be paracompact if every open cover T has a locally finite refinement.*

It is said to be Lindelöf if every open covering of T has a countable subcover.

Remark.— Being paracompact Hausdorff is equivalent to having a partition of unity subordonated to every open cover.

In the case where T is paracompact Hausdorff and Lindelöf, we understand well the nature of the ∞-topos associated to $\mathcal{R}(\mathrm{T})$.

Proposition 2.5.20.— *Let* T *be a paracompact Hausdorff Lindelöf space, then we have the following equivalence in the ∞-category of ∞-toposes,*

$$(\mathcal{R}(\mathrm{T}))_{\mathcal{T}\mathrm{ps}} \simeq \varinjlim_{n} (\mathcal{R}_{\leq n}(\mathrm{T}))_{\mathcal{T}\mathrm{ps}}$$

Proof.— The plan for this proof is to check that $\mathcal{R}(\mathrm{T})$ satisfies the hypotheses of 7.1.5.8 in [HT].

For any n, the topology on $\mathcal{R}_{\leq n}(\mathrm{T})$ is generated by the open sets $\mathcal{R}(\mathrm{U}_1, \dots, \mathrm{U}_p)$ for every family of disjoint open subsets $\mathrm{U}_1, \dots, \mathrm{U}_p$ of T with $p \leq n$. Hence,

- For every n the map $i_n : \mathcal{R}_{\leq n}(\mathrm{T}) \hookrightarrow \mathcal{R}_{\leq n+1}(\mathrm{T})$ is a closed embedding. Indeed, let $\mathrm{C} = \{x_1, \dots, x_{n+1}\}$ be a family of $n+1$ distinct points of T. Because T is Hausdorff, there exists disjoint open subsets $\mathrm{U}_1, \dots, \mathrm{U}_{n+1}$ such that $\mathrm{C} \in \mathcal{R}(\mathrm{U}_1, \dots, \mathrm{U}_{n+1})$. By definition any configuration of this open subsets of $\mathcal{R}_{\leq n+1}(\mathrm{T})$ is made of configuration of cardinality $n+1$, so $i_n(\mathcal{R}_{\leq n}(\mathrm{T}))$ is closed in $\mathcal{R}_{\leq n+1}(\mathrm{T})$;

- For every n, the topological space $\mathcal{R}_{\leq n}(\mathrm{T})$ is paracompact Hausdorff and Lindelöf. Let $\{x_1, \dots, x_p\}$ and $\{y_1, \dots, y_q\}$ be two distinct points of $\mathcal{R}_{\leq n}(\mathrm{T})$. Without loss of generality, we may assume that x_1 is not in $\{y_1, \dots, y_q\}$. Because T is Hausdorff, we can find two disjoint open subsets $\mathrm{U}, \mathrm{V} \subset \mathrm{T}$ such that $x_1 \in \mathrm{U}$ and all the other points are in V. Then $\mathcal{R}(\mathrm{U}, \mathrm{V}) \cap \mathcal{R}(\mathrm{V}) = \emptyset$ and $\{x_1, \dots, x_n\} \in \mathcal{R}(\mathrm{U}, \mathrm{V})$ while $\{y_1, \dots, y_p\} \in \mathcal{R}(\mathrm{V})$. This proves that $\mathcal{R}_{\leq n}(\mathrm{T})$ is Hausdorff.

  To prove that it is both paracompact and Lindelöf, we use the canonical quotient map $p_n : \mathrm{T}^n \to \mathcal{R}_{\leq n}(\mathrm{T})$, comming from the definition of $\mathcal{R}_{\leq n}(\mathrm{T})$ as a colimit. Since T is Lindelöf, the space T^n is also Lindelöf, idem for the quotient $\mathcal{R}_{\leq n}(\mathrm{T})$. Moveover, T^n is paracompact and p_n is open with finite fibres, so $\mathcal{R}_{\leq n}(\mathrm{T})$ is also paracompact.

  Indeed, without loss of generality we may assume that T is infinite — otherwise T and everyone else are finite discrete spaces — and let $\mathrm{V}_1 \times \dots \times \mathrm{V}_n$ be an open subset of T. Then because T is Hausdorff and infinite, there exists a point $(x_1, \dots, x_n) \in \mathrm{V}_1 \times \dots \times \mathrm{V}_n$ whose coordinates are pairwise distincts. Using the fact that T is Hausdorff again, we can find disjoint open subsets $\mathrm{U}_1, \dots, \mathrm{U}_n$ such that

  $$x \in \mathrm{U}_1 \times \dots \times \mathrm{U}_n \subset \mathrm{V}_1 \times \dots \times \mathrm{V}_n.$$

  This implies that $\mathcal{R}(\mathrm{U}_1, \dots, \mathrm{U}_n) \subset p_n(\mathrm{V}_1 \times \dots \times \mathrm{V}_n)$ and p_n is open.

Let $\{U_i\}_{i\in I}$ be any open covering of $\mathcal{R}_{\leq n}(T)$, then $\{p_n^{-1}(U_i)\}_{i\in I}$ is an open covering of T^n. Since T^n is paracompact, we let $\{V_j\}_{j\in J}$ be a locally finite refinement. Because p_n is open, $\{p_n(V_j)\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$ and since p_n has finite fibres, it is also locally finite.

Since every paracompact Hausdorff space is normal, every $\mathcal{R}_{\leq n}(T)$ is a normal and Lindelöf topological space. As a consequence, the space $\mathcal{R}(T)$, which is a countable union along closed maps of such spaces, is also normal and Lindelöf.

If $\{U_i\}_{i\in I}$ is a cover of $\mathcal{R}(M)$, then by definition of the colimit topology, for every n, $\{U_i \cap \mathcal{R}_{\leq n}(T)\}_{i\in I}$ is an open cover of $\mathcal{R}_{\leq n}(T)$ which is Lindelöf so there exists a countable subset $I_n \subset I$ such that $\{U_i\}_{i\in I_n}$ covers $\mathcal{R}_{\leq n}(T)$. The union J of all subsets I_n is countable as a countable union of countable sets and $\{U_i\}_{i\in J}$ is a countable subcover of $\mathcal{R}(T)$.

Let F and F′ be two disjoint closed subsets of $\mathcal{R}(T)$. Because for every n, the map i_n is a closed embedding, the subsets $F \cap \mathcal{R}_{\leq n}(T)$ and $F' \cap \mathcal{R}_{\leq n}(T)$ are two closed disjoint subsets of $\mathcal{R}_{\leq n}$ which is a normal topological space. So there exists two disjoint neighbourhoods U_n and U'_n of these two closed subsets of $\mathcal{R}_{\leq n}(T)$. Without loss of generality, we may assume that for every n, the neighbourhoods satisfy $U_n \subset i_n^{-1}(U_{n+1})$ and $U'_n \subset i_n^{-1}(U'_{n+1})$. Let U be the union of all U_n and U′ the union of all U'_n. Then by construction U and U′ are disjoint neighbourhoods of F and F′ (respectively) in $\mathcal{R}(T)$, so $\mathcal{R}(T)$ is a normal topological space.

By a theorem of Morita, every normal Lindelöf space is paracompact [Mor]. The last thing we need to see is that $\mathcal{R}(T)$ is Hausdorff, but the proof is similar to the proof that $\mathcal{R}_{\leq n}(T)$ is Hausdorff.

Hence $\mathcal{R}(T)$ satisfies the hypotheses of 7.1.5.8 in [HT]. □

Theorem 2.5.21.— *Let* T *be a paracompact Hausdorff Lindelöf topological space, then the* ∞*-topos reflection of the canonical map*

$$\pi : \mathrm{Ran}(T) \to \mathcal{R}(T)$$

is an equivalence.

Proof.— For any n, let $\mathrm{Ran}_{\leq n}(T)$ be the prestack defined by

$$\mathrm{Ran}_{\leq n}(T) = \varinjlim_{|I|\leq n} T^I$$

Let $N(\omega)$ be the ∞-category associated to the ordinal ω. There is a functor

$$|\,.\,| : \mathcal{D}\mathrm{iag} \to N(\omega)$$

sending a finite set I to its cardinality $|I|$ and any surjection $\pi : I \twoheadleftarrow J$ to the morphism $|I| \leq |J|$. This implies that for every n, there is a canonical map

$$\Delta_n : \mathrm{Ran}_{\leq n}(T) \hookrightarrow \mathrm{Ran}_{\leq n+1}(T)$$

such that
$$\mathrm{Ran}(M) \simeq \varinjlim_{n \in \mathrm{N}(\omega)} \mathrm{Ran}_{\leq n}(T).$$
Using the unit of the adjunction
$$\mathcal{P}(\mathrm{Top}) \rightleftarrows \mathrm{Top}$$
on $\mathrm{Ran}_{\leq n}(T)$ we get a map $\pi_n : \mathrm{Ran}_{\leq n}(T) \to \mathcal{R}_{\leq n}(T)$. Using the unit of the adjunction again on the map Δ_n, we deduce that the following square is commutative,
$$\begin{tikzcd}
\mathrm{Ran}_{\leq n+1}(T) \arrow[r, "\pi_{n+1}"] & \mathcal{R}_{\leq n+1}(T) \\
\mathrm{Ran}_{\leq n}(T) \arrow[u, "\Delta_n"] \arrow[r, "\pi_n"] & \mathcal{R}_{\leq n}(T) \arrow[u, "i_n"']
\end{tikzcd}$$
Hence,
$$\pi = \varinjlim_n \pi_n.$$
□

Moreover, by definition the ∞-topos $(\mathrm{Ran}(T))_{\mathcal{T}\mathrm{ps}}$ is equivalent to the colimit
$$(\mathrm{Ran}(T))_{\mathcal{T}\mathrm{ps}} \simeq \varinjlim_n (\mathrm{Ran}_{\leq n}(T))_{\mathcal{T}\mathrm{ps}}.$$
In the meantime since T is paracompact Hausdorff and Lindelöf, by proposition 2.5.20 the same is true for $\mathcal{R}(T)$,
$$(\mathcal{R}(T))_{\mathcal{T}\mathrm{ps}} \simeq \varinjlim_n (\mathcal{R}_{\leq n}(T))_{\mathcal{T}\mathrm{ps}}.$$
Hence we have reduced the problem to proving that the reflection of π_n is an equivalence, for every n. We will do this by induction.

The $n = 1$ case is trivial since
$$\mathrm{Ran}_{\leq 1}(T) = T = \mathcal{R}_{\leq 1}(T).$$
Let $n \geq 1$ and suppose that the map
$$\pi_n : \mathrm{Ran}_{\leq n}(T) \to \mathcal{R}_{\leq n}(T)$$
induces an equivalence between the associated ∞-toposes. Then let $\mathcal{R}_{n+1}(T)$ be the topological subspace of $\mathcal{R}_{\leq n+1}(T)$ made of configurations of cardinality n. Likewise, let $\mathrm{Ran}_{n+1}(T)$ be the colimit
$$\mathrm{Ran}_{n+1}(T) = \varinjlim_{|I|=n+1} U^I$$
where $U^I \subset T^I$ is the complement of all the diagonals. Let $s_{n+1} : \mathcal{R}_{n+1}(T) \hookrightarrow \mathcal{R}_{\leq n+1}(T)$ be the associated open embedding, and likewise $k_{n+1} : \mathrm{Ran}_{n+1}(T) \hookrightarrow \mathrm{Ran}_{\leq n+1}(T)$. Let also $\omega_{n+1} : \mathrm{Ran}_{n+1}(T) \to$

$\mathcal{R}_{n+1}(\mathrm{T})$ be the map obtained from the definition of $\mathrm{Ran}_{n+1}(\mathrm{T})$ as a colimit. They fit in the following commutative square

$$\begin{tikzcd} \mathrm{Ran}_{\leq n+1}(\mathrm{T}) \arrow[r, "\pi_{n+1}"] & \mathcal{R}_{\leq n+1}(\mathrm{T}) \\ \mathrm{Ran}_{n+1}(\mathrm{T}) \arrow[r, "\omega_{n+1}"] \arrow[u, "k_{n+1}"] & \mathcal{R}_{n+1}(\mathrm{T}) \arrow[u, "s_{n+1}"'] \end{tikzcd}$$

As the action of the symmetric group $\mathfrak{S}_\mathrm{I}$ on U^I is free for every I, the map ω_{n+1} induces an equivalence between of ∞-toposes :

$$\omega_{n+1} : (\mathrm{Ran}_{n+1}(\mathrm{T}))_{\mathcal{T}\mathrm{ps}} \simeq (\mathcal{R}_{n+1}(\mathrm{T}))_{\mathcal{T}\mathrm{ps}}.$$

The last ingredient to the proof is the fact that for every I of cardinality lesser or equal to $n+1$, the canonical map

$$\mathrm{T}^\mathrm{I} \to \mathcal{R}_{\leq n+1}(\mathrm{T})$$

is proper and T^I is completely regular, hence by 7.3.1.16, the map

$$(\mathrm{T}^\mathrm{I})_{\mathcal{T}\mathrm{ps}} \to (\mathcal{R}_{\leq n+1}(\mathrm{T}))_{\mathcal{T}\mathrm{ps}}$$

is a proper map of ∞-toposes.

Applying the base change formula, we obtain

$$i_n^* \circ (\pi_{n+1})_* \circ \pi_{n+1}^* \simeq (\pi_n)_* \circ \pi_n^*$$

which is the identity by hypothesis. We also get

$$s_{n+1}^* \circ (\pi_{n+1})_* \circ \pi_{n+1}^* \simeq (\omega_{n+1})_* \circ (\omega_{n+1})^*$$

which is also the identity. And as $\mathcal{R}_{\leq n+1}(\mathrm{T})$ is the union of $\mathcal{R}_n(\mathrm{T})$ and $\mathcal{R}_{\leq n}(\mathrm{T})$, we deduce that $(\pi_{n+1})_* \circ \pi_{n+1}^*$ is equivalent to the identity of the ∞-category $\mathcal{S}\mathrm{h}_\mathcal{S}(\mathcal{R}_{\leq n+1}(\mathrm{T}))$.

The same argument shows that $\pi_{n+1}^* \circ (\pi_{n+1})_*$ reduces to $\pi_n^* \circ (\pi_n)_*$ on $\mathrm{Ran}_{\leq n}(\mathrm{T})$ and to $(\omega_{n+1})^* \circ (\omega_{n+1})_*$ on $\mathrm{Ran}_{n+1}(\mathrm{T})$. Hence $\pi_{n+1}^* \circ (\pi_{n+1})_*$ is equivalent to the identity of $\mathcal{S}\mathrm{h}_\mathcal{S}(\mathrm{Ran}_{\leq n+1}(\mathrm{T}))$.

As a consequence π_{n+1} induces an equivalence of ∞-toposes,

$$\pi_{n+1} : (\mathrm{Ran}_{n+1}(\mathrm{T})) \simeq (\mathcal{R}_{n+1}(\mathrm{T}))$$

such that the induction is finished.

2.5.3 *The equivalence*

We are now going to deduce the equivalence between (non-unital) finite dimensional $\mathcal{E}_\mathrm{M}$-algebras and factorisation local systems for the chiral tensor product, from the comparison of the spaces $\mathrm{Ran}(\mathrm{M})$ and $\mathcal{R}(\mathrm{M})$.

Proposition 2.5.22.— *The canonical map inside $\mathcal{P}(\mathrm{Top})$,*

$$\pi : \mathrm{Ran}(M) \to \mathcal{R}(M)$$

induces an adjunction

$$\mathcal{C}\mathrm{osh}(\mathrm{Ran}(M)) \underset{\pi^!}{\overset{\pi_!}{\rightleftarrows}} \mathcal{C}\mathrm{osh}(\mathcal{R}(M))$$

which restricts to locally constant cosheaves with finite dimensional costalks on each side,

$$\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathrm{Ran}(M)) \underset{\pi^!}{\overset{\pi_!}{\rightleftarrows}} \mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(M))$$

The adjoint functors $\pi_!$ and $\pi^!$ are mutually inverse equivalences.

Proof.— Step 1: As the definition of $\mathcal{C}\mathrm{osh}(\mathcal{X})$ for a prestack $\mathcal{X}$ is

$$\mathcal{C}\mathrm{osh}(\mathcal{X}) = \mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{V}\mathrm{ect}^{\mathrm{op}})^{\mathrm{op}},$$

the existence of the adjoint functors $\pi_! \dashv \pi^!$ follows directly from the usual adjoint functors $\pi^* \dashv \pi_*$ between ∞-categories of sheaves on prestacks. We will work out the definitions of these two adjoint functors.

For every $I \in \mathcal{D}\mathrm{iag}$, let

$$\pi_I : M^I \to \mathcal{R}(M)$$

be the canonical map and for every $s : I \twoheadleftarrow J$ in $\mathcal{D}\mathrm{iag}$, let

$$\Delta(s) : M^I \hookrightarrow M^J$$

the associated diagonal embedding. Then thanks to the commutative diagram

$$\begin{tikzcd} M^I \arrow[r, "\Delta(s)"] \arrow[dr, "\pi_I"'] & M^J \arrow[d, "\pi_J"] \\ & \mathcal{R}(M) \end{tikzcd}$$

we have the equivalence of functors

$$\pi_I^! \simeq \Delta(s)^! \circ \pi_J^!.$$

Hence we can describe the functor $\pi^!$ as the following limit,

$$\pi^! \simeq \varprojlim_{I \in \mathcal{D}\mathrm{iag}} \pi_I^!.$$

Since $\pi^!$ is the limit of the $\pi_I^!$, its left adjoint has to be the colimit of the left adjoints $(\pi_I)_!$,

$$\pi_! \simeq \varinjlim_{I \in \mathcal{D}\mathrm{iag}} (\pi_I)_! : \mathcal{C}\mathrm{osh}(\mathrm{Ran}(M)) \to \mathcal{C}\mathrm{osh}(\mathcal{R}(M)).$$

Step 2: We will now show that $\pi^!$ preserves local systems. For every $n \in \mathbb{N}$, let $\mathcal{R}_{\leq n}(M)$ be the subspace of $\mathcal{R}(M)$ whose points are only the finite families with cardinality lesser or equal to n. This is a stratification of $\mathcal{R}(M)$. We let

$$\mathcal{R}_n(M) = \mathcal{R}_{\leq n}(M) - \mathcal{R}_{\leq n-1}(M)$$

be the n-th open stratum of $\mathcal{R}(M)$ and $j_n : \mathcal{R}_n(M) \to \mathcal{R}(M)$ the corresponding open embedding.

Let $\mathcal{T}$ be an object of $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(M))$, then by proposition 2.5.3 in [Knud], the restriction $j_n^! \mathcal{T}$ is a locally constant cosheaf with finite dimensional costalks. Let I be an object of $\mathcal{D}\mathrm{iag}$ and let n be the cardinality of I. We then have the following factorisation

$$\begin{tikzcd} M^I \arrow[r, "\pi_I"] \arrow[dr, "r_I"'] & \mathcal{R}(M) \\ & \mathcal{R}_{\leq n}(M) \arrow[u, hook, "i_n"'] \end{tikzcd}$$

Moreover, the inverse image of the stratum S_n by r_I is the open subset $U^I \subset M^I$ which is the complement to the diagonals in M^I. Let $j_I : U^I \hookrightarrow M^I$ be the corresponding open embedding. The restricted map

$$e_I = r_I \circ j_I : U^I \to S_n$$

is étale since $\mathcal{R}_n(M)$ is homeomorphic to the quotient of U^I by the canonical free action of $\mathfrak{S}_I$. Thus, the cosheaf $e_I^! \circ j_n^! \mathcal{T}$ is a local system and by the commutative diagram

$$\begin{tikzcd} U^I \arrow[r, hook, "j_I"] \arrow[ddrr, "e_I"'] & M^I \arrow[r, "\pi_I"] \arrow[dr, "r_I"'] & \mathcal{R}(M) \\ & & \mathcal{R}_{\leq n}(M) \arrow[u, hook] \\ & & \mathcal{R}_n(M) \arrow[u, hook] \arrow[uu, bend right, "j_n"'] \end{tikzcd}$$

we have

$$e_I^! \circ j_n^! \simeq j_I^! \circ \pi_I^!,$$

which means that $j_I^! \circ \pi_I^! \mathcal{T}$ is a local system so that,

$$\pi^! \mathcal{T} \in \mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathrm{Ran}(M)).$$

Step 3: we show that $\pi_!$ also preserves local systems.

Let $\{\mathcal{T}_I\} \in \mathcal{C}\mathrm{osh}_{\mathrm{LS}}(M)$ be a local system, I be an object of $\mathcal{D}\mathrm{iag}$ and n be its cardinality, then $(\pi_I)_! \mathcal{T}_I$ is a local system on $\mathcal{R}(M)$. Indeed, because $r_I : M^I \to \mathcal{R}_{\leq n}(M)$ is a proper map with finite fibres between two locally compact spaces, by proper base change, every costalk of

$(\pi_I)_!\mathcal{T}_I$ is a finite sum of costalks of $\mathcal{T}_I$ which are finite dimensional by assumption, so the costalks of $(\pi_I)_!\mathcal{T}_I$ are finite dimensional.

Then because π_I is proper and the following square is Cartesian,

$$\begin{tikzcd} M^I \arrow[r, "\pi_I"] & \mathcal{R}(M) \\ U^I \arrow[u, hook, "j_I"] \arrow[r, "e_I"] \arrow[ur, phantom, "\urcorner"] & \mathcal{R}_n(M) \arrow[u, hook, "j_n"'] \end{tikzcd}$$

by proper base change, we have

$$(e_I)_! \circ j_I^! \simeq j_n^! \circ (\pi_I)_!$$

By assumption $j_I^!\mathcal{T}_I$ is locally constant and as e_I is étale, $(e_I)_! \circ j_I^!\mathcal{T}_I$ is also locally constant. We deduce that the restriction of $(\pi_I)_!\mathcal{T}_I$ on the stratum $\mathcal{R}_n(M)$ is locally constant. The same arguments may be used to show that the restrictions on the other strata are also locally constant and thus that $(\pi_I)_!\mathcal{T}_I$ is a local system on $\mathcal{R}(M)$.

We now use the fact that colimits of cosheaves are computed objectwise (see remark 2.1.18), moreover the ∞-categories of locally constant cosheaves in $\mathcal{C}osh(\mathcal{R}_H(M))$ and $\mathcal{C}osh(\mathcal{R}(M))$ are equivalent by proposition 2.5.10, so that we can check whether a cosheaf is locally constant only on the opens $\mathcal{R}(U_I)$ for every $U_I \in \mathcal{D}(M)$. Combining these two results, a colimit of locally constant cosheaves on $\mathcal{R}(M)$ is again locally constant. This means that $\pi_!(\{\mathcal{T}_I\})$ is a locally constant cosheaf.

Finally, let $x = \{x_1, \dots, x_n\}$ be a point in $\mathcal{R}(M)$ and let $\mathcal{T}$ be a locally constant cosheaf on $\mathcal{R}(M)$. Then because it is locally constant its costalk at x is given by

$$\mathcal{T}_{x^!} \simeq \mathcal{T}(\mathcal{R}(U_I))$$

for any $U_I \in \mathcal{D}(M)$ where I has cardinality n and such that $x \in \mathcal{R}(U_I)$. Thus costalks commute with colimits of locally constant cosheaves since colimits of cosheaves are computed objectwise. Computing the costalks we get, for I of cardinality n,

$$(\pi_!(\{\mathcal{T}_I\}))_{x^!} \simeq (\mathcal{T}_I)_{(x_1,\dots,x_n)^!}$$

which is finite dimensional by assumption, so $\pi_!$ preserves local systems.

Step 4: The adjoint functors $\pi_!$ and $\pi^!$ are mutually inverse equivalences. This is a consequence of theorem 2.5.21. It can be proved also directly in this case, by induction. □

Remark 2.5.23.— The idea that cosheaves on both Ran spaces are equivalent takes its source in section 4.2.4 of [BD].

Proposition 2.5.24.— *Let* $X \in \mathcal{V}\mathrm{ar}$ *and let* M *be its associated topological manifold. The* ∞*-category* $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathrm{Ran}(M))$ *inherits the chiral tensor structure from the* ∞*-category* $\mathcal{C}\mathrm{osh}_{\mathrm{C}}(\mathrm{Ran}(M))$. *In the same way, the* ∞*-category* $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(M))$ *inherits the disjoint tensor structure from* $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(M))$. *Moreover, both* $\pi_!$ *and* $\pi^!$ *are symmetric monoidal functors.*

Proof.— Since locally constant cosheaves taking finite dimensional values are stable under finite sums and box product, for $\mathcal{T}, \mathcal{Q}$ two objects of $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathrm{Ran}(M))$, we have

$$\mathcal{T} \otimes^{\mathrm{ch}} \mathcal{Q} \in \mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathrm{Ran}(M))$$

by proposition 2.4.1. Hence by proposition 2.2.1.1 in [HA], the ∞-category of locally constant cosheaves $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathrm{Ran}(M))$ inherits the chiral tensor structure from $\mathcal{C}\mathrm{osh}_{\mathrm{C}}(\mathrm{Ran}(M))$.

Likewise, thanks to proposition 2.5.13, for any two cosheaves $\mathcal{T}$ and $\mathcal{Q}$ in $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(M))$, we have

$$\mathcal{T} \otimes^{\mathrm{disj}} \mathcal{Q} \in \mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(M))$$

and $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(M))$ inherits from $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}(M))$ the disjoint tensor structure.

Thanks to proposition 2.5.22, the canonical morphism $\pi : \mathrm{Ran}(M) \to \mathcal{R}(M)$ gives an equivalence of the associated ∞-category of cosheaves. Let us endow the topological space $\mathcal{R}(M)$ with its structure of partial monoid, defined by the following correspondence in Top :

$$\begin{tikzcd}
 & (\mathcal{R}(M) \times \mathcal{R}(M))_{\mathrm{disj}} \arrow[dl, "j"'] \arrow[dr, "uj"] & \\
\mathcal{R}(M) \times \mathcal{R}(M) \arrow[rr, rightsquigarrow, "\sqcup_M"] & & \mathcal{R}(M)
\end{tikzcd}$$

The canonical morphism π is in fact a morphism of partial monoïds, that becomes an equivalence on the ∞-category of cosheaves. Since $\pi_!$ is also an equivalence on the ∞-category of local systems, we get a symmetric monoidal functor

$$\mathcal{C}\mathrm{osh}_{\mathrm{LS}} : \mathcal{R}(M)^{\otimes}_{*} \to \mathcal{C}\mathrm{at}_{\mathbb{C}}$$

where $\mathcal{R}(M)^{\otimes}_{*}$ is the enveloping symmetric monoidal category of the monoid $\mathcal{R}(M)$ inside the category $\mathrm{Top}_{\mathrm{pd}}$ of topological spaces with partially defined morphisms. The resulting tensor structure on the ∞-category of local systems $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(M))$ is again called the chiral tensor structure.

For convenience, we will use the comparison map $\mathcal{R}(M) \to \mathcal{R}_{H}(M)$, which is also a morphism of partially defined monoids inducing an equivalence on the ∞-category of local systems defined on both sides.

We now need to check that the disjoint tensor structure and the chiral one are the same on $\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}_{H}(M))$. The key idea is the following : all the maps defining the partial multiplication $\sqcup_M$ are *open*. The map j is obviously open but $u \circ j$ is also open. Indeed, the topology on $(\mathcal{R}_H(M) \times \mathcal{R}_H(M))_{disj}$ is generated by the opens $\mathcal{R}(U_1, \dots, U_n) \times \mathcal{R}(V_1, \dots, V_p)$ where all open subsets $U_1, \dots V_p$ of M are disjoint. We have on such open subsets,

$$(u \circ j)(\mathcal{R}(U_1, \dots, U_n) \times \mathcal{R}(V_1, \dots, V_p)) = \mathcal{R}(U_1, \dots, U_n, V_1, \dots, V_p)$$

which proves that $u \circ j$ is open.

The operation which associates to a topological space T its category of open sets $\mathcal{O}(T)$, sends partially defined open maps to partially defined functors. We will say that

$$F : C \rightsquigarrow D$$

is a partially defined functor if there exists a full subcategory $j : C_0 \subset C$ and a functor $f : C_0 \to D$.

Given a partially defined open map

$$\begin{tikzcd} & U \arrow[dl, "j"'] \arrow[dr, "f"] & \\ X \arrow[rr, rightsquigarrow] & & Y \end{tikzcd}$$

the functor $\mathcal{O}$ gives the partially defined functor

$$\begin{tikzcd} & \mathcal{O}(U) \arrow[dl, "j_!"'] \arrow[dr, "f_!"] & \\ \mathcal{O}(X) \arrow[rr, rightsquigarrow] & & \mathcal{O}(Y) \end{tikzcd}$$

where for an open map f, the functor $f_!$ is defined by $f_!(\Omega) = f(\Omega)$ for every open set Ω.

Since for any two topological spaces, we have the canonical map $\mathcal{O}(M) \times \mathcal{O}(N) \to \mathcal{O}(M \times N)$ which is compatible with the functors $f_!$ (that is, for any open map $f : M \to N$ and any pair of open subsets $U, V \subset M$, we have $(f \times f)_!(U \times V) = f_!(U) \times f_!(V)$), the functor $\mathcal{O}$ is lax symmetric monoidal.

Hence the functor $\mathcal{O}$ sends the partial monoid $\mathcal{R}_H(M)$ to the partial monoid $\mathcal{O}(\mathcal{R}_H(M))$ in the category of categories .

Recall that every object U_I of $\mathcal{D}(M)_*$ defines functorially an object of $\mathcal{O}(\mathcal{R}_H(M))$ by the rule

$$U_I \mapsto \mathcal{R}(U_I).$$

This extends to a morphism of partially defined monoids

$$\mathcal{R} : \mathcal{D}(M)_* \to \mathcal{O}(\mathcal{R}_H(M))$$

Indeed, two objects of $\mathcal{D}(M)_*$, U_I and V_J may only be composed if all the opens U_i are disjoint from the opens V_j, for $(i, j) \in I \times J$, in this case

$$\mathcal{R}(U_I, V_J) = (u \circ j)_!(\mathcal{R}(U_I) \times \mathcal{R}(V_J)).$$

Using the universal property of Day convolution, we get a lax symmetric monoidal functor

$$[-, \mathcal{V}\mathrm{ect}] : \mathrm{Cat}_{\mathrm{pd}} \to \mathcal{C}\mathrm{at}_{\mathbb{C}}$$

sending a partially defined functor (j, f) to the composite of Kan extensions $f_* \circ j^{-1}$.

Through this lax symmetric monoidal functor, the morphism of partially defined monoids $\mathcal{R}$ becomes a symmetric monoidal embedding

$$\mathcal{R}_* : ([\mathcal{D}(\mathrm{M}), \mathcal{V}\mathrm{ect}], \otimes^{\mathrm{disj}}) \hookrightarrow ([\mathcal{O}(\mathcal{R}_{\mathrm{H}}(\mathrm{M})), \mathcal{V}\mathrm{ect}], \otimes^{\mathrm{disj}})$$

We are going to compare the lax symmetric monoidal functors $\mathcal{C}\mathrm{osh}_{\mathrm{LC}}$ and $[\mathcal{O}(-), \mathcal{V}\mathrm{ect}]$. There is a fully faithful functor

$$\mathcal{C}\mathrm{osh}(\mathcal{R}_{\mathrm{H}}(\mathrm{M})) \hookrightarrow [\mathcal{O}(\mathcal{R}_{\mathrm{H}}(\mathrm{M})), \mathcal{V}\mathrm{ect}]$$

sending a cosheaf to its underlying precosheaf. Moreover, given any open embedding $j : \mathrm{U} \to \mathrm{X}$ and any cosheaf $\mathcal{T}$ on U, the exceptional pushforward along j is easy to compute, it reads

$$j_*(\mathcal{T})(\mathrm{V}) = \mathcal{T}(\mathrm{V}) \text{ if } \mathrm{V} \subset \mathrm{U}, \text{ and } 0 \text{ otherwise.}$$

this is the right adjoint to j^*. In the same regard, the union map $u : \mathcal{R}_{\mathrm{H}}(\mathrm{M}) \times \mathcal{R}_{\mathrm{H}}(\mathrm{M}) \to \mathcal{R}_{\mathrm{H}}(\mathrm{M})$ is proper, so that $u_! = u_*$, this means that the disjoint tensor structure computed at proposition 2.5.13 may be red as : given any two locally constant cosheaves $\mathcal{T}$ and $\mathcal{Q}$, we have

$$\mathcal{T} \otimes^{\mathrm{disj}} \mathcal{Q} \simeq u_* \circ j_* \circ j^*(\mathcal{T} \boxtimes \mathcal{Q}).$$

An analogue formula can be written for any number of locally constant cosheaves.

Hence, we recognise a morphism of lax symmetric monoidal functors

$$\mathcal{C}\mathrm{osh}_{\mathrm{LC}} \Longrightarrow [\mathcal{O}(-), \mathcal{V}\mathrm{ect}]$$

defined on the category $\mathcal{R}_{\mathrm{H}}(\mathrm{M})^{\otimes}_{*}$. As a consequence, the embedding

$$\mathcal{C}\mathrm{osh}_{\mathrm{LC}}(\mathcal{R}_{\mathrm{H}}(\mathrm{M})) \hookrightarrow [\mathcal{O}(\mathcal{R}_{\mathrm{H}}(\mathrm{T})), \mathcal{V}\mathrm{ect}]$$

intertwine the chiral tensor structure and the disjoint tensor structure, so that π induces a (non-unital) symmetric monoidal equivalence

$$\pi_! : (\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathrm{Ran}(\mathrm{M})), \otimes^{\mathrm{ch}}) \simeq (\mathcal{C}\mathrm{osh}_{\mathrm{LS}}(\mathcal{R}(\mathrm{M})), \otimes^{\mathrm{disj}}) : \pi^!.$$

□

Theorem 2.5.25.— *The functors $\pi_!$ and $\pi^!$ induce an equivalence of ∞-categories,*

$$\pi_! : \mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{ch}}_{\mathrm{LS}}(\mathrm{M}) \simeq \mathcal{F}\mathcal{C}\mathrm{osh}^{\mathrm{disj}}_{\mathrm{LS}}(\mathrm{M}) : \pi^!.$$

Proof.— Because the functors $\pi_! \dashv \pi^!$ are both symmetric monoidal equivalences, they induce inverse equivalences

$$\mathcal{C}\text{om-coalg}^{\text{disj}}(\mathcal{C}\text{osh}_{\text{LS}}(\text{Ran}(M)) \underset{\pi^!}{\overset{\pi_!}{\rightleftarrows}} \mathcal{C}\text{om-coalg}^{\text{ch}}(\mathcal{C}\text{osh}_{\text{LS}}(\mathcal{R}(M)).$$

On both sides, since we work with locally constant cosheaves, the factorisation properties can be rewritten as: a cocommutative coalgebra $\mathcal{A}$ has the factorisation property if for every I, the coalgebra map

$$\mathcal{A} \to \mathcal{A}^{\otimes I}$$

induces by projection, an equivalence

$$\mathcal{A}_{S^!} \simeq \bigotimes_{s \in S} \mathcal{A}_{s^!}$$

for every subset S of cardinality $|I|$.

Hence, since both $\pi^!$ and $\pi_!$ preserve costalks, they also preserve the factorisation property. So they restrict to inverse equivalences between the ∞-categories of factorisation local systems :

$$\pi_! : \mathcal{F}\mathcal{C}\text{osh}^{\text{ch}}_{\text{LS}}(M) \simeq \mathcal{F}\mathcal{C}\text{osh}^{\text{disj}}_{\text{LS}}(M) : \pi^!.$$

□

Theorem 2.5.26.— *Let* X *be a smooth complex variety and let* M *be its associated topological manifold. Then the ∞-category of non-unital factorisation vector bundles on* X *is equivalent to the ∞-category of finite dimensional non-unital $\mathcal{E}_M$-algebras.*

Proof.— We use the chain of equivalences

$$\mathcal{F}\mathcal{D}^{\text{ch}}_{\text{VB}}(X) \xrightarrow{\text{DR}} \mathcal{F}\mathcal{S}\text{h}^{\text{ch}}_{\text{LS}}(M) \xrightarrow{\Gamma_c} \mathcal{F}\mathcal{C}\text{osh}^{\text{ch}}_{\text{LS}}(M) \xrightarrow{\pi_!} \mathcal{F}\mathcal{C}\text{osh}^{\text{disj}}_{\text{LS}}(M)$$

that we have proved in propositions 2.3.10 and 2.4.4 and theorem 2.5.25. The final equivalence is given by corollary 2.5.18,

$$\mathcal{F}\mathcal{C}\text{osh}^{\text{disj}}_{\text{LS}}(M) \simeq \mathcal{E}_M\text{-alg}^{\text{nu}}_{\text{f.d}}$$

□

3 EXPONENTIABLE HIGHER TOPOSES

3.1 INTRODUCTION

In this chapter, with Mathieu Anel, we characterise the class of exponentiable ∞-toposes and we give two applications: if $\mathcal{X}$ is an exponentiable ∞-topos then $\mathcal{S}h(\mathcal{X}, \mathrm{Sp})$ is dualisable and for suitable ∞-categories $\mathcal{C}$, $\mathcal{S}h(\mathcal{X}, \mathcal{C})$ can be described as $\mathcal{C}$-Leray sheaves.

3.1.1 *Motivation: Verdier duality*

Classical Verdier duality — of which standard references are [KS] and of course [SGA5]— says that given a continuous map between two locally compact Hausdorff spaces $f : X \to Y$, we are supplied with a bunch of functors $\mathrm{Hom}, \otimes, f^*, f_*, f_!, f^!$ between the derived categories of bounded chain complexes of sheaves of vector spaces on X and Y, with lots of relations between them. Together they form 'les six opérations de Grothendieck'. Although handy in practice, this theory is not satisfactory in itself as stated.

Derived categories shall be replaced by the stable ∞-categories of chain complexes. The reason is that it is more natural to work with the axioms of stable ∞-categories than with those of triangulated categories and the 'octahedral axiom'. It also allows us to perform infinite derived operations, not only cones and cocones.

Once we work with stable ∞-categories, the theory is still not fully satisfactory. While the geometric construction and interpretation of the first functors is crystal clear, the adjoint pair $f_! \dashv f^!$ seems to be an ad hoc construction. Yet these functors do have strong compatibilities with the others; this cannot be a coincidence. Therefore a natural interrogation comes to the mind of every new apprentice 'what's their geometrical meaning? Where do they come from?'.

The answer is given by a theorem of Lurie in [HA]. To a topological space X one can build the ∞-category of 'functions with (homotopy) linear values' a.k.a the ∞-category of sheaves of spectra $\mathcal{S}h(X, \mathcal{S}p)$. Given a map $f : X \to Y$ it is natural to pullback functions :

$$f^* : \mathcal{S}h(Y, \mathcal{S}p) \to \mathcal{S}h(X, \mathcal{S}p),$$

it is also possible to pushforward functions $f_* : \mathcal{S}h(X, \mathcal{S}p) \to \mathcal{S}h(Y, \mathcal{S}p)$. With the theory of functions comes its dual, the theory of distributions. In this framework, distributions correspond to the ∞-category of cosheaves $\mathcal{C}osh(X, \mathcal{S}p)$. Distributions are naturally pushed forward, so we get a functor $f_! : \mathcal{C}osh(X, \mathcal{S}p) \to \mathcal{C}osh(Y, \mathcal{S}p)$. Since the functor $f_!$ preserves colimits, it has a right adjoint $f^!$.

If X is locally compact, then Lurie's theorem gives us an equivalence of ∞-categories $\mathcal{S}h(X, \mathrm{Sp}) \simeq \mathcal{C}osh(X, \mathrm{Sp})$ which allows the transfer of the adjunction $f_! \dashv f^!$ to a new adjunction on the sheaf side, that we recognise to be the usual functors $f_! \dashv f^!$ of classical Verdier duality.

This explain what are $f_!$ and $f^!$ and where they come from. But we are still not satisfied. The duality theorem between sheaves and cosheaves is proved only for Hausdorff spaces. Should it work for much general topological spaces? Here is a strong hint that it should: in [KS2] Kashiwara and Shapira develop the theory of ind-sheaves, transferred on the subanalytic site by Prelli [Pr]. These are sheaves on a new Grothendieck topology on $\mathbb{R}^n$, the topology is nice enough so that the category of sheaves on this site is equivalent to the category of sheaves on a new topological space $\mathbb{R}^n_{sa}$. This space is highly non Hausdorff but it is locally coherent. In [KS2, Pr] all the Grothendieck operations on the derived categories of sheaves of chain complexes are built up, including $f^!$ and they prove that they satisfy all the expected relations.

3.1.2 *Exponentiability of ∞-toposes*

The duality $\mathcal{S}h(X, \mathrm{Sp}) \simeq \mathcal{C}osh(X, \mathrm{Sp})$ is not really dependant on the topological space X but rather on the ∞-category of sheaves $\mathcal{S}h(X)$. This means that it makes no harm to look for a broader result and to replace topological spaces by ∞-toposes. For an ∞-topos $\mathcal{X}$ both the ∞-category of sheaves and cosheaves can be defined and the sought theorem is the equivalence

$$\mathcal{S}h(\mathcal{X}, \mathrm{Sp}) \simeq \mathcal{C}osh(\mathcal{X}, \mathrm{Sp})$$

The key remark on this equivalence is that it is an autoduality result. The ∞-category of cocomplete and stable ∞-category has a closed symmetric monoidal structure. For any two stable and cocomplete ∞-categories $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ classifies functors with domain $\mathcal{A} \times \mathcal{B}$ and cocontinuous in each variable. For this monoidal structure, the dual of a stable cocomplete ∞-category $\mathcal{C}$ is the ∞-category of cocontinuous functors $[\mathcal{C}, \mathrm{Sp}]^{cc}$, i.e functors $\mathcal{C} \to \mathrm{Sp}$ that commute with all small colimits. But for any ∞-topos $\mathcal{X}$, we have

$$\mathcal{C}osh(\mathcal{X}, \mathrm{Sp}) \simeq [\mathcal{S}h(\mathcal{X}, \mathrm{Sp}), \mathrm{Sp}]^{cc}.$$

Hence we are looking for the ∞-toposes $\mathcal{X}$ whose ∞-category of stable sheaves is *autodual*.

In this chapter we focus on a intermediate step: when is $\mathcal{S}h(\mathcal{X}, \mathrm{Sp})$ dualisable? A sufficient condition is the exponentiability of $\mathcal{X}$.

☛ Definition.— *An ∞-topos $\mathcal{X}$ is said to be exponentiable if the functor*

$$\mathcal{Y} \mapsto \mathcal{X} \times \mathcal{Y}$$

has a right adjoint: $\mathcal{Z} \mapsto \mathcal{Z}^{\mathcal{X}}$.

3.1.3 *Leray sheaves*

When X is a locally quasi-compact topological space, then for any other space Z, the exponential X^Z is the space of continuous maps $f : X \to Z$ and the topology on $C^0(X, Z)$ is the quasi-compact-open topology.

In the same way, given two ∞-toposes $\mathcal{X}$ and $\mathcal{Z}$, the exponential $\mathcal{X}^{\mathcal{Z}}$ should resemble $[\mathcal{X}, \mathcal{Z}]$. It is always possible to look at the ∞-category of sheaves $[\mathcal{S}h(\mathcal{Z}), \mathcal{S}h(\mathcal{X})]^*$ given by the left exact and cocontinuous functors $f^* : \mathcal{S}h(\mathcal{Z}) \to \mathcal{S}h(\mathcal{X})$. But this object usually doesn't have the right universal property. Indeed $[\mathcal{S}h(\mathcal{Z}), \mathcal{S}h(\mathcal{X})]^*$ is an ∞-category of left exact and cocontinuous functors while $\mathcal{S}h(\mathcal{Z})$ and $\mathcal{S}h(\mathcal{X})$ are defined by small limits conditions — in a very explicit way when we deal with sheaves on a site.

It is then natural to ask $\mathcal{S}h(\mathcal{X})$ to be described by finite limits and small colimits conditions instead of small limits conditions. Such a description is what we call *Leray sheaves*.

In fact, the very first definition of sheaves on a topological space X involved abelian groups associated to *compact subsets* of X. A sheaf was then a functor

$$\mathcal{F} : K \mapsto \mathcal{F}(K)$$

commuting to *finite limits* and *filtered colimits*.

☛ Definition.— *Let* X *be a locally compact topological space and* $\mathcal{C}$ *be a cocomplete and finitely complete ∞-category. Let* D_c *be the poset of compact subspaces of* X. *A left exact functor*

$$\mathcal{F} : D_c^{op} \to \mathcal{C}$$

is a $\mathcal{C}$-valued Leray sheaf on X *if it satisfies the additional condition*

$$\mathcal{F}(K) \simeq \varinjlim_{K \ll K'} \mathcal{F}(K')$$

where $K \ll K'$ *means that there exists an open subset* U *such that* $K \subset U \subset K'$.

This alternative definition was quickly abandoned as it was more difficult to work with it than the one on open sets and as it doesn't give the right definition of sheaves if X is not locally compact.

A proof of the equivalence between sheaves on X and Leray sheaves on X is in [HT]. It encompasses both the locally compact and the locally coherent cases. At the end of this chapter, we show an analogous theorem for any exponentiable ∞-topos. In particular, this means that for a new class of quasi-compact topological spaces — the ∞-lqc topological spaces — sheaves can be described as Leray sheaves.

3.1.4 *The use of ∞-categories*

The theory of ∞-categories is a generalization of the theory of categories. Many models for ∞-categories exist; we will be using the model of *quasi-categories* as developed by Lurie in Higher Topos Theory [HT]. They were developed first by Joyal [Joy].

The use of ∞-categories is critical for this work: given a topological space X it is usually not enough to consider $\mathcal{S}h(X, Ab)$ the category of abelian sheaves on X, we need the full chain complex formalism. This is very well treated inside the theory of ∞-categories as *stable* ∞-categories. The axioms are very simple: an ∞-category is stable if and only if it has a zero object and a square is Cartesian if and only if it is coCartesian. The homotopy category of a stable ∞-category is triangulated, it is the usual category people have been working with since they had to 'derive' the functors.

Why do we really need stable ∞-categories and not derived categories? Well, a nice feature of abelian categories is that additive functors commute automatically with both finite sums and finite products; instead in a stable ∞-category exact morphisms commute with all finite limits and colimits. For example a cocontinuous functor $F : \mathcal{C} \to \mathcal{D}$ between two stable ∞-categories automatically commutes with finite limits. This makes the existence of adjoint functors much easier, the pullback of cosheaves for example. What is more, the classical disjunction between coproducts and sifted colimits — that adds up to construct all small colimits — becomes in the stable world the disjunction between finite colimits and filtered colimits. We usually study functors that commute with the first of the pair, it is then easier to look for the commutation with filtered colimits than sifted colimits, so that a lot of construction will be able to take place in the stable world and not in the abelian one.

Using ∞-categories make things simpler. We have already seen that the axioms of stable ∞-categories are simpler that those of a triangulated category. In fact, being stable is a *property* of an ∞-category whereas being triangulated is a *structure*!

It also makes things easier for toposes. In a topos equivalence relations are effective, in an ∞-topos *all* groupoids are effective. Another one: in the ∞-category of sheaves on a topos $\mathcal{X}$ there is a so-called *subobject classifier* $\Omega_{\mathcal{X}}$ such that maps $A \to \Omega_{\mathcal{X}}$ correspond to subobjects of A. In an ∞-topos instead, there is an 'object classifier' i.e we do not need to restrict arbitrarily to subobjects but we can consider all (small) maps $B \to A$.

3.1.5 *What is new in this chapter?*

The exponentiability of toposes was studied and understood in the paper of Johnstone and Joyal [JoJo]. Would the same proof work in the case of ∞-toposes?

The theories of toposes and ∞-toposes share for the most part the same semantic. We list here the main differences between the two theories, each item describes a proposition that is true for toposes.

- The ∞-category of ω-small spaces $\mathcal{S}_{\text{fin}}$ is *not* locally ω-small so it cannot be an ∞-category of sheaves in the universe $\mathcal{U}(\omega)$;
- In the ∞-category world, we do not know whether the ∞-category of spaces $\mathcal{S}$ is *noetherian*, that is if every left exact reflexive localisation of $\mathcal{S}$ is accessible;
- An ∞-topos $\mathcal{X}$ may not be *hypercomplete* i.e Whitehead's theorem may not hold: there can be an arrow f in $\mathcal{S}\text{h}(\mathcal{X})$ such that $\pi_n(f)$ is an equivalence for every n but f may not be an equivalence;
- If X is a topological space, then it is possible that its associated ∞-topos $\mathcal{T}\text{op}(X)$ hasn't enough points, or is even not hypercomplete;
- Not every ∞-category of sheaves may be described as an ∞-category of sheaves over a site, even if the associated ∞-topos is hypercomplete;
- It is not known whether the ∞-category of sheaves of spaces on a topological space $\mathcal{S}\text{h}(X)$ is equivalent to the coherent nerve of a model category with underlying category $\text{Top}_{/X}$. We only know that there exists a Quillen adjunction [HT].

The conclusion being that, most of the time proofs must not rely on those nice properties that we had while working on ordinary toposes.

The proof of Johnstone and Joyal uses sites, 2-categorical constructions and geometric theories [JoWr]. As we just saw, it is not always possible in an $(\infty, 1)$-context to use sites, we see $(\infty,2)$-categorical constructions as a painful formalism that we want to avoid and there is no theory of ∞-topos associated to a geometric theory. This is why, even if the proof structure is similar, we need to use some other tools to prove the exponentiation theorem for ∞-toposes; the most important being the tensor product of ∞-categories. Moreover, the tensor product often allows us to make simpler proofs where we do not have to dive into the *wavy arrows*. And of course, examples are different, as it is harder for the ∞-topos associated to a topological space to be exponentiable in the ∞-category of ∞-toposes than it is for the associated topos, in the category of toposes.

In addition to the characterization of ∞-toposes that is the ∞-version of Johnstone and Joyal theorem, we have further developed the theory of tensor products of ∞-categories, showing in particular that the product of ∞-toposes correspond to the tensor product of their ∞-categories of sheaves.

☛ Theorem 3.4.17.— *An ∞-topos $\mathcal{X}$ is exponentiable if and only if the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous i.e iff the evaluation functor*

$$\varepsilon : \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X})) \to \mathcal{S}\mathrm{h}(\mathcal{X})$$

has a left adjoint.

The part on dualisability is completely new. We characterize the dualisable cocomplete ∞-category and we prove that $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathrm{Sp})$ is dualisable for an exponentiable ∞-topos $\mathcal{X}$. To do so, we build the theory of coends for ∞-categories from scratch, without using dinatural transformations. This is made possible by an intensive use of tensor products of ∞-categories; the constructions and proofs are straightforward and model-independent. Steps in other directions were made in [Cr] and [Gl].

☛ Theorem 3.6.16.— *Let $\mathcal{X}$ be an exponentiable ∞-topos, then the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathrm{Sp})$ of stable sheaves on $\mathcal{X}$ is a dualisable object of $\mathcal{P}\mathrm{r}_{\mathrm{St}}$.*

Finally, we have proved that given an exponentiable ∞-topos $\mathcal{X}$ and a suitable ∞-category $\mathcal{C}$, the ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$ can be expressed as $\mathcal{C}$-valued Leray sheaves. That is as functors satisfying finite limits and small colimits conditions instead of the usual small limits conditions. That kind of result was already known in some cases, see [HT] for the case where X is a locally compact or a locally coherent topological space.

☛ Theorem 3.7.18.— *Let $\mathcal{X}$ be an exponentiable ∞-topos, and let $\mathcal{C}$ be a stable bicomplete ∞-category. Then there exists a finitely cocomplete subcategory $\mathrm{D} \subset \mathcal{S}\mathrm{h}(\mathcal{X})$ and a bimodule $w : \mathrm{D}^{op} \times \mathrm{D} \to \mathcal{S}$ such that the ∞-category of sheaves $\mathcal{S}\mathrm{h}(\mathcal{X})$ is equivalent to the ∞-category of left exact functors $\mathrm{F} : \mathrm{D}^{op} \to \mathcal{C}$ satisfying the coend condition*

$$\mathrm{F}(a) \simeq \int_{b \in \mathrm{D}} w(a, b) \otimes \mathrm{F}(b).$$

for all $a \in \mathrm{D}$.

3.1.6 *The proofs by hands waving*

Let $\mathcal{X}$ be an ∞-topos. To show that $\mathcal{X}$ is exponentiable, the idea is to build by hand the exponential $\mathcal{Y}^{\mathcal{X}}$ for every other ∞-topos $\mathcal{Y}$. Fortunately we won't have to do it for every $\mathcal{Y}$ as the ∞-category of ∞-toposes is sufficiently well behaved; it is generated under limits by the affine ∞-toposes so we would only need to build $(\mathbb{A}^C)^{\mathcal{X}}$ for every small ∞-category C. Ultimately those exponentials can also be build from the unique exponential $(\mathbb{A}^1)^{\mathcal{X}}$.

This ∞-topos is very special. If it exists then the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is equivalent to $\mathrm{pt}((\mathbb{A}^1)^{\mathcal{X}})$ the ∞-category of points of $(\mathbb{A}^1)^{\mathcal{X}}$. This equivalence will give us the necessary and sufficient conditions on $\mathcal{X}$ to be able to build the exponential $(\mathbb{A}^1)^{\mathcal{X}}$. Studying the properties of $\mathbb{A}^1$ we find out that it is an injective ∞-topos and so will be $(\mathbb{A}^1)^{\mathcal{X}}$. Injective ∞-toposes have the very particular property of being determined by their ∞-category of points: that is, knowing $\mathcal{S}\mathrm{h}(\mathcal{X})$ is an ∞-category of points of an injective ∞-topos, we can find which one it is!

As any injective ∞-topos is a retract of an affine ∞-topos, an ∞-category of points of an ∞-topos is a retract by ω-continuous functors of a presheaf ∞-category. Such a characterization is not very useful, we much prefer the equivalent one of being *continuous*. A cocomplete ∞-category $\mathcal{C}$ is continuous if the evaluation functor $\varepsilon : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint. Working a bit more it is possible, if $\mathcal{C} = \mathcal{S}\mathrm{h}(\mathcal{X})$, to replace the very large ∞-category $\mathrm{Ind}(\mathcal{C})$ by a large one $\mathrm{Ind}(D)$, where D is a small ∞-category. We then get a *standard presentation*:

$$\mathrm{Ind}(D) \underset{\alpha}{\overset{\beta}{\leftleftarrows}} \!\!\xrightarrow{\;\varepsilon\;}\! \mathcal{S}\mathrm{h}(\mathcal{X})$$

that is β is a fully faithful left adjoint.

Armed with this presentation, we tensor by Sp and obtain the ∞-category of stable sheaves $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathrm{Sp})$ as a retract of $\mathrm{Ind}(D) \otimes \mathrm{Sp}$ which is dualisable in the ∞-category of cocomplete stable ∞-categories.

Finally, to get a description of $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{C})$ as Leray sheaves for a bicomplete ∞-category $\mathcal{C}$, we tensor the standard presentation by $\mathcal{C}$, so we get another

$$\mathrm{Ind}(D) \otimes \mathcal{C} \underset{\alpha}{\overset{\beta}{\leftleftarrows}} \!\!\xrightarrow{\;\varepsilon\;}\! \mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{C})$$

Then $\mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{C})$ can be alternatively described as left exact functors $D^{op} \to \mathcal{C}$ satisfying limit conditions or colimits conditions whether you consider the fixed points of $\alpha\varepsilon$ or $\beta\varepsilon$. The description of sheaves $\mathcal{X} \to \mathcal{C}$ as functors satisfying finite limits and colimit conditions requires detailed attention.

3.1.7 *Notations and conventions*

- Let $\omega \in \mathbb{U} \in \mathbb{V} \in \mathbb{W}$ be three Grothendieck universes;
- To avoid heavy notations, we establish a dictionary: *small* means $\mathbb{U}$-small, *large* means $\mathbb{V}$-small and *very large* means $\mathbb{W}$-small;
- By a limit or a colimit, we mean a small one;
- By a category or an ∞-category, we mean a large one;
- The large ∞-category of small spaces will be denoted $\mathcal{S}$; its homotopy category is $\mathcal{H}$. The very large ∞-category of large spaces is $\widehat{\mathcal{S}}$, with homotopy category $\widehat{\mathcal{H}}$;
- If $\mathcal{C}$ is an ∞-category and x, y are two objects of $\mathcal{C}$, then $\mathrm{Map}_{\mathcal{C}}(x, y)$ is the space of maps from x to y in $\mathcal{C}$. Most of the time, we only need to think of it as an object of $\widehat{\mathcal{H}}$. When using it inside ends and coends formulas, we shall write $[a, b]$ instead of $\mathrm{Map}(a, b)$;
- For an ∞-category $\mathcal{C}$, we write $\mathrm{Int}(\mathcal{C})$ for the biggest ∞-groupoid inside $\mathcal{C}$;
- If K is a simplicial set, we write $K^{\triangleright}$ for the join $K \star \Delta^0$ and $K^{\triangleleft}$ for $\Delta^0 \star K$;
- The large ∞-category of small ∞-categories is $\mathcal{C}\mathrm{at}_\infty$; the very large one of large ∞-categories will be denoted $\widehat{\mathcal{C}\mathrm{at}}_\infty$;
- Every time we write $X \simeq Y$ we mean 'X is canonically equivalent to Y';
- Every time we draw $X \underset{G}{\overset{F}{\leftrightarrows}} Y$, we mean that F is left adjoint to G i.e the left adjoint is on the top. In the same fashion, for vertical arrows, the left arrow is the left adjoint;
- For $\mathcal{C}$ and $\mathcal{D}$ two ∞-categories, we will denote by $[\mathcal{C}, \mathcal{D}]$ the ∞-category of functors between $\mathcal{C}$ and $\mathcal{D}$, $[\mathcal{C}, \mathcal{D}]_c$ the ∞-category of *continuous* functors. Likewise, $[\mathcal{C}, \mathcal{D}]^{cc}$ will be the ∞-category of *cocontinuous* functors and $[\mathcal{A} \times \mathcal{B}, \mathcal{C}]_{c,c}$ is the ∞-category of functors continuous in each variable;
- We generally use Roman letters like A, B, C, D to designate small ∞-category, while calligraphic letters like $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are for general ∞-categories;
- We abusively do not distinguish notationally between a category and its nerve, as categories can be considered as special kinds of ∞-categories;
- For an ∞-category $\mathcal{C}$ the category of ind-objects in $\mathcal{C}$ is $\mathrm{Ind}(\mathcal{C})$. An object in $\mathrm{Ind}(\mathcal{C})$ is a formal filtered colimit “ $\varinjlim$ ” c_i;

- We will say that a cocomplete functor between two cocomplete ∞-categories $F : \mathcal{A} \to \mathcal{B}$ is a *localisation functor* if there exists a set of arrows S of $\mathcal{A}$ with the following universal property: for every cocomplete ∞-category $\mathcal{C}$, the canonical map induced by F, $[\mathcal{B}, \mathcal{C}]^{cc} \to [\mathcal{A}, \mathcal{C}]^{cc}_S$ is an equivalence. Where $[\mathcal{A}, \mathcal{C}]_S$ is the ∞-category of cocontinuous functors $F : \mathcal{A} \to \mathcal{S}$ such that for any $s \in S$, $F(s)$ is an equivalence in $\mathcal{C}$;

- A localisation functor $\mathcal{A} \to \mathcal{B}$ is said to be *reflexive* if it has a fully faithful right adjoint.

3.1.8 *Unusual notations and terminology*

- The traditional notation for the end of a bimodule $F : C \times C^{op} \to \mathcal{S}$ is $\int_c F(c, c)$ while the traditional notation for coends is $\int^c F(c, c)$. These notations follow the usual tensor calculus conventions but we do not like them. While the integral symbol is quite natural to picture an operation much alike a colimit, the integral symbol is quite misleading when used for ends.

  In our sense, we only fall short of a symbol for *product integration*. Fortunately, J.C Loredo-Osti created the LaTeX package "prodint". So we will write ends like $\prod_c F(c, c)$;

- We call a category *smally presentable* when [HT] calls it presentable and [AdRo] say locally presentable. Indeed, if cocomplete categories are the analogue to commutative monoids, then *finite* presentation monoids correspond to *smally* presentable categories;

- In the same way, we call an ∞-category $\mathcal{C}$ *smally generated* if it has a small and dense full subcategory. Or equivalently, if there exists a reflexive localisation functor $\mathcal{P}(D) \to \mathcal{C}$ with D a small ∞-category;

- A formalism similar to the one of affine schemes will be used for locales and for ∞-toposes: we will use X, Y, Z to denote locales while $\mathcal{O}(X), \mathcal{O}(Y), \mathcal{O}(Z)$ will denote the associated frames. In the same way, we will write $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ for ∞-toposes and $\mathcal{S}h(\mathcal{X})$, $\mathcal{S}h(\mathcal{Y})$, $\mathcal{S}h(\mathcal{Z})$ for their ∞-categories of sheaves.

  This means that $X \mapsto \mathcal{O}(X)$ is the isomorphism $\mathcal{L}oc \to \mathcal{F}rm^{op}$ between locales and frames, while $\mathcal{X} \mapsto \mathcal{S}h(\mathcal{X})$ is the isomorphism $\mathcal{T}op \to \mathcal{S}hv^{op}$ between ∞-toposes and ∞-categories of sheaves.

3.1.9 *Intuition table*

Many objects in ∞-category theory behave very much like usual objects in algebra. We think it is useful to give the intuition behind the use and vocabulary of the different notions in the theory of ∞-category. The following table is a variation on remark 6.1.1.3 in [HT].

∞-Category Theory	Algebra
∞-Category	Set
Cocomplete ∞-category	Commutative monoid
Smally presentable ∞-category	Finite pres. commutative monoid
Cocomplete stable ∞-category	Abelian group
∞-Category of sheaves	Finite presentation semi-ring
∞-Topos	Spectrum of a finite pres. semi-ring

3.2 RECOLLECTION ON LOCALLY QUASI-COMPACT SPACES

In view of the exponentiability theorem for ∞-toposes it is illuminating to recall what happens in the case of topological spaces. It is in fact from there that comes all the intuitions and we always have it in mind when dealing with ∞-toposes. Here are some references: [Elephant, Ma, Sc].

3.2.1 *Usual definitions of compactness*

Definition 3.2.1.— *Let $\mathcal{C}$ be any category and κ any cardinal. An object X of $\mathcal{C}$ is said to be κ-compact if the functor $\mathrm{Hom}(X,-)$ commutes with κ-filtered colimits.*

Definition 3.2.2.— *Let X be a topological space,*

- *X is quasi-compact (or qc for short) if the open set X is an ω-compact object of the category $\mathcal{O}(X)$ of open subsets of X ;*
- *X is compact if it is quasi-compact and Hausdorff.*

Proposition 3.2.3.— *Any retract of a quasi-compact topological space is quasi-compact.*

3.2.2 *The $\ll$ relation*

The usual definition of local quasi-compactness requires the use of *neighbourhoods*. This notion is unfitted with the categorification of topology; all properties must be defined with open sets. To this end, we will replace quasi-compact neighbourhoods with the $\ll$ relation.

Let X be a topological space.

Definition 3.2.4.— *Let* U *and* V *be two open subsets of X. Then* $U \ll V$ *if on only if* $U \subset V$ *and for every cover* $\{V_i\}_{i\in I}$ *of V, there exists a finite subset* $J \subset I$ *such that* $U \subset \cup_{i\in J} V_i$.

Example.— if X is Hausdorff, U is relatively compact and $\overline{U} \subset V$, then $U \ll V$. More generally if there exists a quasi-compact set K such that $U \subset K \subset V$, then $U \ll V$.

Remark 3.2.5.— Notice that $U \ll U$ if and only if U is quasi-compact, which is also equivalent to U being ω-compact in $\mathcal{O}(X)$.

Proposition 3.2.6.— *Suppose* X *is Hausdorff and* $U \ll V$, *then* $\overline{U} \subset V$.

Proof.— Otherwise, let $x \in \overline{U} \cap V^c$. If the filter of neighbourhoods of x has a finite basis, then x is isolated. So we can suppose it is infinite: let $(\mathcal{V}_\alpha)_{\alpha\in\kappa}$ be a basis of closed neighbourhoods of x. Then $\{V \cap \mathcal{V}_\alpha^c, \alpha \in \kappa\}$ is an open cover of V such that no finite subcover covers U. □

Counter-example.— Let $\mathbb{R} \subset \operatorname{Spec} \mathbb{R}[X]$ be endowed with the spectral topology. Then every non-empty open subset is quasi-compact and dense, which means $\overline{U} \subset V$ is often false while $U \ll V$ is always true.

3.2.3 *Quasi-compact and quasi-separated morphisms*

Definition 3.2.7.— *We say that a continuous map* $f : X \to Y$ *is a quasi-compact morphism if*

$$U \ll V \Longrightarrow f^{-1}U \ll f^{-1}V.$$

Remark 3.2.8.— If $f : X \to Y$ is a quasi-compact morphism, then $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ preserves ω-compact objects.

Definition 3.2.9.— *We say that a topological space* X *is quasi-separated if the diagonal map* $X \to X^2$ *is a quasi-compact morphism.*

Proposition 3.2.10.— *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two continuous maps between topological spaces.*

- *If f and g are quasi-compact morphisms, so is $g \circ f$;*
- *If $g \circ f$ is quasi-compact and g is injective, then f is quasi-compact;*

- *Retracts of quasi-compact morphisms are quasi-compact;*
- *Sections of retracts are quasi-compact morphisms.*

Proof.— Let $r : Y \to X$ be a retraction with Y being a locally quasi-compact topological space. Let $s : X \to Y$ be a section of r, i.e $r \circ s = \mathrm{Id}$. Then s is a quasi-compact morphism. Indeed, let $U \ll V$ be open subsets of Y and let $\{O_i\}_{i\in I}$ be an open cover of $s^{-1}(V)$. Then $\{r^{-1}O_i\}_{i\in I}$ in an open cover of $V \subset r^{-1}s^{-1}(V)$. Hence by assumption, there exists a $J \subset I$ finite such that $\{r^{-1}O_j\}_{j\in J}$ is an open cover of U and so, $\{O_j\}_{j\in J}$ is an open cover of $s^{-1}(U)$, which means $s^{-1}(U) \ll s^{-1}(V)$. □

Corollary 3.2.11.— *Any retract of a quasi-separated topological space is quasi-separated.*

Proposition 3.2.12.— *If a topological space* X *has a basis of quasi-compact neighbourhoods, stable by binary intersection, then* X *is quasi-separated.*

Remark 3.2.13.— We never used the fact that X had points. This means that, all of this can be said in the more general framework of *locales*.

3.2.4 *Quick recollection on locales*

The theory of locales is a harmless generalization of the theory of topological spaces.

Definition 3.2.14.— *A frame is a poset $\mathcal{F}$ such that every small family of elements has a supremum (called a join) and every finite family of elements has a infimum element (called a meet) and such that finite meets distribute over joins. A morphism of frames is a morphism of posets that commutes finite meets and all joins. The category of frames is* $\mathcal{F}\mathrm{rm}$.

The category of locales is defined by the equality

$$\mathcal{L}\mathrm{oc} = \mathcal{F}\mathrm{rm}^{op}$$

The isomorphism is denoted $X \mapsto \mathcal{O}(X)$.

Remark 3.2.15.— There is an obvious functor $\mathrm{Top} \to \mathcal{L}\mathrm{oc}$ that associates to a topological space X the locale associated to the frame $\mathcal{O}(X)$ of open subsets of X. A locale is said to be *spacial* if it comes from a topological space.

3.2.5 *Locally quasi-compact spaces*

Definition 3.2.16.— *We say that a topological space (or a locale)* X *is locally quasi-compact (or lqc for short) if for every open subset* V,

$$\bigcup_{U \ll V} U = V.$$

Proposition 3.2.17.— *If a topological space* X *has a basis of quasi-compact neighbourhoods, then it is locally quasi-compact in the above sense.*

Proposition 3.2.18.— *Any retract of a locally quasi-compact topological space is locally quasi-compact.*

Proof.— By prop 3.2.10, if $r : Y \to X$ is a retract with Y an lqc topological space, then the section $s : X \to Y$ is a quasi-compact morphism. Let $\{U_i\}_{i\in I}$ be a basis of open qc subsets of Y, then by remark 3.2.8 $\{s^{-1}(U_i)\}_{i\in I}$ is a family of qc open subsets of X.

To finish, we must show that $\{s^{-1}(U_i)\}_{i\in I}$ is dense in $\mathcal{O}(X)$. But $s^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a localisation functor and $\{U_i\}_{i\in I}$ is dense in $\mathcal{O}(Y)$. □

Theorem 3.2.19.— *If* $U \ll V$ *are two open subsets of a locally quasi-compact space* X*, then there exists a quasi-compact subspace* $K \subset X$ *such that* $U \subset K \subset V$*. In other words, the definition of locally quasi-compact spaces coincides with the usual one.*

Moreover any lqc locale is spacial, so we do not need to distinguish between lqc topological spaces and locales.[Sc]

Theorem 3.2.20.— *The locally quasi-compact locales coincide with the exponentiable objects in the category of locales.[Elephant]*

We are now aiming at a purely categorical description of local quasi-compactness.

Let X be a topological space and consider $\mathcal{O}(X)$ the associated frame of open subsets. The evaluation functor $\varepsilon : \mathrm{Ind}(\mathcal{O}(X)) \to \mathcal{O}(X)$ has a right adjoint $\alpha : \mathcal{O}(X) \to \mathrm{Ind}(\mathcal{O}(X))$ sending an open subset U to the constant diagram $\alpha(U) = U$.

Let us build yet another adjoint functor β.

Definition 3.2.21.— *The functor* $\beta : \mathcal{O}(X) \to \mathrm{Ind}(\mathcal{O}(X))$ *is defined as follows:*

$$\beta(V) = \text{“}\varinjlim_{U \ll V}\text{''}\, U.$$

Theorem 3.2.22.— *The functor* β

- *preserves the final object if and only if* X *is quasi-compact;*
- *is left adjoint to* ε *if and only if* X *is locally quasi-compact;*
- *is left exact if and only if* X *is quasi-compact and quasi-separated.*

In the case where X *is locally quasi-compact, the unit map* $\mathrm{Id} \longrightarrow \varepsilon \circ \beta$ *is an isomorphism.*

Definition 3.2.23.— *Let $\mathcal{F}$ be a frame. We will say, following [Sc] that $\mathcal{F}$ is continuous if the evaluation functor*

$$\mathrm{Ind}(\mathcal{F}) \to \mathcal{F}$$

has a right adjoint.

Corollary 3.2.24.— *In view of theorem 3.2.20, a locale X is exponentiable if and only if $\mathcal{O}(X)$ is continuous.*

3.2.6 *Coherent and locally coherent spaces*

The major source of lqc topological spaces are the retracts of locally coherent spaces.

Definition 3.2.25.— *A topological space (or a locale) X is*

- *locally coherent if it has a basis of quasi-compact open subsets, stable by binary intersections;*
- *coherent if it is locally coherent and quasi-compact.*

Definition 3.2.26.— *A distributive lattice is a poset in which every couple of elements has a meet and a join, and such that non-empty finite meets distribute over non-empty finite joins. A map of distributive lattice is a map of posets that commutes with non-empty finite meets and joins. The category of distributive lattice with those morphisms is $\mathcal{DL}\mathrm{att}$.*

Proposition 3.2.27.— *The forgetful functor* $\mathrm{oblv} : \mathcal{F}\mathrm{rm} \to \mathcal{DL}\mathrm{att}$ *has a left adjoint* $\mathrm{Ind} : \mathcal{DL}\mathrm{att} \to \mathcal{F}\mathrm{rm}$. *Furthermore for every distributive lattice* L, $\mathrm{Ind}(L)$ *is the frame of a locally coherent locale and if* L *has a final object,* $\mathrm{Ind}(L)$ *it is the frame of a coherent one.*

Theorem 3.2.28.— *Let X be a topological space, then X is a retract of a coherent locale if and only if X is quasi-compact, locally quasi-compact and quasi-separated.*

Proof.— Suppose X is qcqs and lqc, then by theorem 3.2.22, the frame $\mathcal{O}(X)$ is a retract of $\mathrm{Ind}(\mathcal{O}(X))$ which is coherent.

The converse is true because coherent locales are qc, lqc and qs; these three properties are all stable under retracts. □

Corollary 3.2.29.— *Let X be a topological space, then $\mathcal{O}(X)$ is a retract of a locally coherent frame if and only if X is locally quasi-compact and quasi-separated.*

Proof.— The "only if" part is the same as in the previous theorem. Let us show the "if" part. Let $\mathcal{O}_c(X)$ be the ordered set of open subsets U of X such that $U \ll X$ and note $L = \mathrm{Ind}(\mathcal{O}_c(X))$. Because $\mathcal{O}_c(X)$ doesn't have a final object, L is only locally coherent.

Then, the map $\beta : \mathcal{O}(X) \to \mathrm{Ind}(\mathcal{O}(X))$ factorizes through $\mathrm{Ind}(\mathcal{O}_c(X))$ because for every $U, V \in \mathcal{O}(X)$, if $U \ll V \subset X$ then $U \ll X$. We will denote this map β'.

Because X is quasi-separated, β commutes with non-empty finite limits and so does β'. By construction $\beta'(X) \simeq$ "$\varinjlim_{U \ll X} U$" is the final object of $\mathrm{Ind}(\mathcal{O}_c(X))$ so that β' is a left exact functor. Finally because X is locally quasi-compact, β is a fully faithful left adjoint to ε. The same is true for β' by restriction. □

3.2.7 *Example: subanalytic topology*

Let X be the analytic variety $\mathbb{R}$ with its usual topology. Because $\mathbb{R}$ is locally compact, it is a retract of a locally coherent space. Let us describe the points of the frame $\mathrm{Ind}(\mathcal{O}_c(X))$: they correspond to ultrafilters in $\mathcal{O}_c(X)$. To the usual point $x \in \mathbb{R}$, we add x^+ the ultrafilter of the open sets $]x, x + \varepsilon[$ and likewise x^-. These are the points of the space X_{sa} associated to this new frame. If one restricts its attention to subanalytic open sets (that is a basis of the topology in $\mathbb{R}$ stable by intersection), we find back the space associated to the *subanalytic site* described in [KS2].

As X_{sa} is only locally coherent, one can coherentise (or compactify if you wish) X_{sa} by adding two new ultrafilters : $+\infty$ and $-\infty$. The corresponding site allows only finite covers of X while the subanalytic site admits locally finite covers.

3.3 ∞-TOPOSES

A standard reference on ∞-toposes is [HT]. You can also have a look at [Rezk, ToVe] for homotopy toposes.

3.3.1 *Definition*

In this paragraph we want to recall the definition of an ∞-topos. We are only going to need the extrinsic definition but we still include the intrinsic one as we will make use of universality of colimits.

Definition 3.3.1.— *Let $\mathcal{C}$ be an ∞-category, we say that $\mathcal{C}$ is an ∞-category of sheaves if $\mathcal{C}$ is smally presentable and if the 'slice functor'*

$$x \mapsto \mathcal{C}_{/x}$$

commutes with limits (see [HT]). In particular colimits in $\mathcal{C}$ are universal, that is: for every $D \in \mathcal{C}$, $C \in \mathcal{C}_{/D}$ and any diagram $f : I \to \mathcal{C}_{/D}$, we have

$$\left(\varinjlim_{i\in I} f(i)\right) \times_D C \simeq \varinjlim_{i\in I} \left(f(i) \times_D C\right)$$

The very large ∞-category of $\mathcal{S}\mathrm{hv}$ of ∞-categories of sheaves is the non-full subcategory of $\widehat{\mathcal{C}\mathrm{at}}^{cc}$ whose objects are the ∞-categories of sheaves and the morphisms are the left exact and cocontinuous functors.

For $\mathcal{C}$ and $\mathcal{D}$ two ∞-categories, the ∞-category of left exact and cocontinuous functors will be denoted $[\mathcal{C}, \mathcal{D}]^*$.

Definition 3.3.2.— *The very large ∞-category of ∞-toposes is defined by*

$$\mathcal{T}\mathrm{op} = \mathcal{S}\mathrm{hv}^{op}$$

The isomorhism sends an ∞-topos $\mathcal{X}$ to its ∞-category of sheaves $\mathcal{S}\mathrm{h}(\mathcal{X})$; a morphism $f : \mathcal{X} \to \mathcal{Y}$ is sent to $f^* : \mathcal{S}\mathrm{h}(\mathcal{Y}) \to \mathcal{S}\mathrm{h}(\mathcal{X})$.

Remark 3.3.3.— Following our intuition table, universality of colimits (and more generally the full 'descent conditions' on colimits) can be though as a distributivity property, making an ∞-topos a sort of 'spectrum of a commutative semi-ring'. For this article, we do not need the full descent axioms. Instead what we need is an external characterisation of ∞-toposes as left exact reflexive localisations of free ∞-categories on sheaves.

Definition 3.3.4.— *Let C be a small category. Let $\overline{C}$ be the free category generated by C by finite limits i.e $(\overline{C})^{op}$ is the smallest subcategory in $\mathcal{P}(C^{op})$ containing C^{op} and closed under finite colimits.*

We call $\mathcal{S}[C] = \mathcal{P}(\overline{C})$ the free ∞-category of sheaves generated by C.

Proposition 3.3.5 (Universal property of free ∞-category of sheaves)**.—** *Let C be a small ∞-category and $\mathcal{D}$ be any ∞-category. Let $i : C \to \mathcal{S}[C]$ be the inclusion functor. Then by construction of $\mathcal{S}[C]$, the restriction functor i^* induces an equivalence between the ∞-category of cocontinuous left exact functors $\mathcal{S}[C] \to \mathcal{D}$ and the ∞-category of functors $C \to \mathcal{D}$.*

Proposition 3.3.6.— *An ∞-category $\mathcal{E}$ is an ∞-category of sheaves if it is a left exact and accessible reflexive localisation of a free ∞-category of sheaves:*

$$\mathcal{S}[C] \underset{\longleftarrow}{\overset{L}{\longrightarrow}} \mathcal{E}$$

that is L is a left exact left adjoint and its right adjoint is fully faithful and accessible.

Proof.— By the results in [HT], an ∞-category of sheaves $\mathcal{E}$ is a left exact and accessible reflexive localisation of a presheaf ∞-category

$$L : \mathcal{P}(C) \to \mathcal{E}$$

with C a small ∞-category. The proposition we want to prove is just a slight variation. Indeed for any small ∞-category C, the Yoneda embedding $C \hookrightarrow \mathcal{P}(C)$ extends to a left exact and cocontinuous functor $T : \mathcal{S}[C] \to \mathcal{P}(C)$. Its right adjoint is the left extension of the inclusion $C \hookrightarrow \mathcal{P}(\overline{C}) = \mathcal{S}[C]$, it is accessible and fully faithful and $LT : \mathcal{S}[C] \to \mathcal{E}$ is the desired reflexive localisation. □

Corollary 3.3.7.— *Any left exact and accessible reflexive localisation of an ∞-category of sheaves is an ∞-category of sheaves.*

Proposition 3.3.8.— *The ∞-category of ∞-toposes has all small limits and colimits.*

Remark 3.3.9.— The final objects of $\mathcal{T}op$ are the punctual ∞-toposes $*$ with ∞-category of sheaves $\mathcal{S}h(*) \simeq \mathcal{S}$ and the initial objects are the empty ∞-toposes $\mathcal{S}h(\emptyset) = 0$; where 0 denotes the ∞-category Δ^0.

3.3.2 *Glossary of maps between ∞-toposes*

We intend to give here all the definitions of maps $\mathcal{X} \to \mathcal{Y}$ between ∞-toposes that we will use in this article (ref [Elephant, HT]).

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism between two ∞-toposes $\mathcal{X}$ and $\mathcal{Y}$. We will say that:

- the ∞-topos $\mathcal{X}$ has trivial $\mathcal{Y}$-shape if $f^* : \mathcal{S}h(\mathcal{Y}) \to \mathcal{S}h(\mathcal{X})$ is fully faithful;
- the morphism f is *essential* if $f^* : \mathcal{S}h(\mathcal{Y}) \to \mathcal{S}h(\mathcal{Y})$ has a left adjoint $f_! : \mathcal{S}h(\mathcal{X}) \to \mathcal{S}h(\mathcal{Y})$;
- the morphism f is *proper* if it satisfies the stable Beck-Chevalley condition (see [HT]);
- the morphism f is *cell-like* if f is proper and $\mathcal{X}$ has trivial $\mathcal{Y}$-shape;
- the morphism f is *étale* if there exists $U \in \mathcal{S}h(\mathcal{Y})$ such that $f^* : \mathcal{S}h(\mathcal{Y}) \to \mathcal{S}h(\mathcal{Y})_{/U}$ is the product by U;
- the ∞-topos $\mathcal{X}$ is a *subtopos* of $\mathcal{Y}$ or that f is an *inclusion* if $f^* : \mathcal{S}h(\mathcal{Y}) \to \mathcal{S}h(\mathcal{X})$ has a fully faithful right adjoint $f_* : \mathcal{S}h(\mathcal{X}) \to \mathcal{S}h(\mathcal{Y})$;

- the ∞-topos $\mathcal{X}$ is an *open subtopos* of $\mathcal{Y}$ if f is an étale inclusion;
- the ∞-topos $\mathcal{X}$ is a *closed subtopos* of $\mathcal{Y}$ if f is a proper inclusion;
- the ∞-topos $\mathcal{X}$ is a *locally closed subtopos* of $\mathcal{Y}$ if f is the intersection of an étale inclusion and a proper inclusion.

3.3.3 *Affine ∞-toposes*

The category of commutative semi-rings is generated under colimits by free semi-rings $\mathbb{N}[x_1, ..., x_n]$, so the category of spectra of commutative semi-rings is generated under limits by the affine spaces $\mathbb{A}^n$. We wish to prove the analogue statement for ∞-toposes.

Definition 3.3.10.— *An affine ∞-topos is an ∞-topos $\mathcal{X}$ such that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is a free ∞-category of sheaves. Let $\mathcal{A}\mathrm{ff}$ be the full subcategory of $\mathcal{T}\mathrm{op}$ whose objects are the affine ∞-toposes.*

We let $\mathbb{A}^{\mathrm{C}}$ be the affine ∞-topos with ∞-category of sheaves $\mathcal{S}[\mathrm{C}]$, for a small ∞-category C. For convenience, we also let $\mathbb{A}$ be the affine ∞-topos $\mathbb{A}^*$; its ∞-category of sheaves will be denoted $\mathcal{S}[\mathrm{X}]$.

Theorem 3.3.11.— *The ∞-category $\mathcal{T}\mathrm{op}$ is generated under pullbacks by affine ∞-toposes .*

Proof.— We are going to prove the dual statement that the ∞-category $\mathcal{S}\mathrm{hv}$ is generated under colimits by the free ∞-categories of sheaves. For any ∞-category of sheaves $\mathcal{E}$, there exists a free ∞-category of sheaves $\mathcal{S}[\mathrm{C}]$ and a left exact and accessible reflexive localisation functor

$$\mathrm{L} : \mathcal{S}[\mathrm{C}] \to \mathcal{S}\mathrm{h}(\mathcal{X}).$$

Let S be the strongly saturated class of morphisms f in $\mathcal{S}[\mathrm{C}]$ such that $\mathrm{L}(f)$ is an equivalence in $\mathcal{E}$. Because both $\mathcal{S}[\mathrm{C}]$ and $\mathcal{E}$ are accessible ∞-categories by proposition 5.5.4.2 in [HT], there exists a small subset $\mathrm{S}_0 \subset \mathrm{S}$ such that S_0 generates S as a strongly saturated class.

We can now identify $\mathcal{E}$ as $\mathrm{S}_0^{-1}\mathcal{S}[\mathrm{C}]$. Let $0 \leftrightarrow 1$ be the ∞-category generated by two objects and one invertible arrow. We then obtain the following pushout in the ∞-category $\mathcal{S}\mathrm{hv}$:

$$\begin{tikzcd}
\mathcal{S}\left[\coprod_{\mathrm{S}_0}\Delta^1\right] \arrow[r] \arrow[d] & \mathcal{S}[\mathrm{C}] \arrow[d] \\
\mathcal{S}\left[\coprod_{\mathrm{S}_0} 0 \leftrightarrow 1\right] \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{E}.
\end{tikzcd}$$

This ends the proof that any ∞-category of sheaves is a poushout of free ∞-categories of sheaves: morphisms $f^* : \mathcal{S}\mathrm{h}(\mathcal{X}) \to \mathcal{S}\mathrm{h}(\mathcal{Y})$ are canonically equivalent to morphisms $g^* : \mathcal{S}[C] \to \mathcal{S}h(\mathcal{Y})$ such that $g^*(s)$ is invertible for any $s \in S_0$. □

Remark 3.3.12.— If we would accept to work with $(\infty, 2)$-categories, then $\mathcal{E}$ becomes the colimit of the 2-arrow diagram $\mathrm{Id} \Rightarrow \mathrm{L}$.

3.3.4 *Tensor product of ∞-categories*

The analogy between the ∞-category $\mathcal{S}hv$ and that of commutative semi-rings is a very powerful source of intuition. Following the analogy, $\mathcal{T}op$ looks like the category of affine schemes and as we are interested in exponentiable objects in $\mathcal{T}op$, it is wise to get a good description of products in $\mathcal{T}op$. Finite products in the category of affine schemes correspond to tensor product of rings. Hence, we gather useful facts from chapter 5.5 of [HT] , 1.4 and 4.8 of [HA] on tensor product of ∞-categories and show that the coproduct of ∞-categories of sheaves is given by the tensor product of the underlying smally presentable ∞-categories.

Definition 3.3.13.— *Let $\widehat{\mathcal{C}at}^{cc}$ be the very large ∞-category of large ∞-categories which are smally cocomplete, with cocontinuous functors. For any two such ∞-categories $\mathcal{C}, \mathcal{D}$, the mapping space is*

$$\mathrm{Map}_{\widehat{\mathcal{C}at}^{cc}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Int}\left([\mathcal{C}, \mathcal{D}]^{cc}\right)$$

Definition 3.3.14.— *The full subcategory of $\widehat{\mathcal{C}at}^{cc}$ made of smally presentable ∞-categories will be denoted $\mathcal{P}r$.*

Theorem 3.3.15.— *The ∞-category $\widehat{\mathcal{C}at}^{cc}$ admits all small limits and colimits and the embedding*

$$\mathcal{P}r \hookrightarrow \widehat{\mathcal{C}at}^{cc}$$

commutes with small limits and colimits, so that $\mathcal{P}r$ also has all small limits and colimits.

Theorem 3.3.16.— *The ∞-category $\widehat{\mathcal{C}at}^{cc}$ has a closed symmetric monoidal structure $\otimes$ such that cocontinuous functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ canonically correspond to functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ cocontinuous in each variable.*

The unit object of $\otimes$ is the cocomplete ∞-category $\mathcal{S}$.

It will be important to understand how $\mathcal{C} \otimes \mathcal{D}$ is build as we need these technical details for future proofs. The basic idea to build $\mathcal{C} \otimes \mathcal{D}$ is to force the commutation $c \otimes \varinjlim d_i \simeq \varinjlim(c \otimes d_i)$ and then to add all the colimits of “pure tensor” $c \otimes d$. Because $\mathcal{C}$ and $\mathcal{D}$ are large, they are $\mathbb{V}$-small so we get a reflexive localisation functor

$$[(\mathcal{C} \times \mathcal{D})^{op}, \widehat{\mathcal{S}}] \to [(\mathcal{C} \times \mathcal{D})^{op}, \widehat{\mathcal{S}}]_{c,c}$$

By composition with the Yoneda embedding, we get a functor

$$\mathcal{C} \times \mathcal{D} \to [(\mathcal{C} \times \mathcal{D})^{op}, \widehat{\mathcal{S}}]_{c,c}$$

it is cocontinuous in each variable. The tensor product $\mathcal{C} \otimes \mathcal{D}$ is then the smallest cocomplete subcategory of $[(\mathcal{C} \times \mathcal{D})^{op}, \widehat{\mathcal{S}}]_{c,c}$ that contains the image of the Yoneda embedding.

Theorem 3.3.17.— *Let $\mathcal{C}$ and $\mathcal{D}$ be two smally presentable ∞-categories, then $\mathcal{C} \otimes \mathcal{D}$ is smally presentable. Moreover $[\mathcal{C}, \mathcal{D}]^{cc}$ is also smally presentable, such that $\mathcal{P}\mathrm{r}$ inherits a closed symmetric monoidal structure from $\widehat{\mathcal{C}\mathrm{at}}^{cc}$.*

For smally presentable ∞-categories the tensor product is particularly nice as we can compute it with the following formula.

Proposition 3.3.18.— *Let $\mathcal{C}$ and $\mathcal{D}$ be two smally presentable ∞-categories, then*

$$\mathcal{C} \otimes \mathcal{D} \simeq [\mathcal{C}^{op}, \mathcal{D}]_c$$

We now describe other properties we will need afterwards,

Proposition 3.3.19.— *Let $\mathcal{A}$ and $\mathcal{B}$ be two cocomplete ∞-categories. Let S be a large set of arrows of $\mathcal{A}$ and T a large set of arrows of $\mathcal{B}$. Let $f : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ be the canonical map and denote by $\mathrm{S} \boxtimes \mathrm{T}$ the set of arrows in $\mathcal{A} \otimes \mathcal{B}$ of the form $f(s \times b)$ with $s \in \mathrm{S}$ or $f(a \times t)$ with $t \in \mathrm{T}$.*

Then

$$(\mathrm{S} \boxtimes \mathrm{T})^{-1} \mathcal{A} \otimes \mathcal{B} \simeq (\mathrm{S}^{-1}\mathcal{A}) \otimes (\mathrm{T}^{-1}\mathcal{B})$$

Moreover, if the localisation maps $\mathcal{A} \to \mathrm{S}^{-1}\mathcal{A}$ and $\mathcal{B} \to \mathrm{T}^{-1}\mathcal{B}$ are reflexive, then $\mathcal{A} \otimes \mathcal{B} \to (\mathrm{S} \boxtimes \mathrm{T})^{-1}\mathcal{A} \otimes \mathcal{B}$ is also reflexive.

Proof.— By composition, it is possible to reduce to the case where T is empty. Denote $\mathrm{S}' = \mathrm{S} \boxtimes \mathrm{T}$, we will show that $\mathrm{S}'^{-1}(\mathcal{A} \otimes \mathcal{B})$ has the same universal property as $(\mathrm{S}^{-1}\mathcal{A}) \otimes \mathcal{B}$. Let $\mathcal{D}$ be a cocomplete ∞-category, then by the universal properties of localisations and the tensor product, both $[\mathrm{S}'^{-1}(\mathcal{A} \otimes \mathcal{B}), \mathcal{D}]^{cc}$ and $[(\mathrm{S}^{-1}\mathcal{A}) \otimes \mathcal{B}, \mathcal{D}]^{cc}$ are equivalent to the ∞-category of functors

$$\mathrm{F} : \mathcal{A} \times \mathcal{B} \to \mathcal{D}$$

cocontinuous in each variable and such that for every $s \in \mathrm{S}$, $b \in \mathcal{B}$, $\mathrm{F}(s \times b)$ is an equivalence in $\mathcal{D}$.

Suppose now that $\mathrm{L} : \mathcal{A} \to \mathrm{S}^{-1}\mathcal{A}$ is reflexive. By an analogue of proposition 5.5.4.20 of [HT], precomposition by L induces a fully faithful functor

$$[(\mathrm{S}^{-1}\mathcal{A} \times \mathcal{B})^{op}, \widehat{\mathcal{S}}]_{c,c} \to [(\mathcal{A} \times \mathcal{B})^{op}, \widehat{\mathcal{S}}]_{c,c}$$

Its left adjoint is a reflexive localisation. We deduce that $\mathcal{A} \otimes \mathcal{B} \to \mathrm{S}^{-1}\mathcal{A} \otimes \mathcal{B}$ is also reflexive. □

Corollary 3.3.20.— *The following square is a pushout in* $\widehat{\mathcal{C}\mathrm{at}}^{cc}$,

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{B} \arrow[r] \arrow[d] & \mathrm{S}^{-1}\mathcal{A}\otimes\mathcal{B} \arrow[d] \\
\mathcal{A}\otimes\mathrm{T}^{-1}\mathcal{B} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & (\mathrm{S}^{-1}\mathcal{A})\otimes(\mathrm{T}^{-1}\mathcal{B})
\end{tikzcd}$$

Proof.— Let $\mathcal{D}$ be a cocomplete ∞-category, then $[(\mathrm{S}^{-1}\mathcal{A})\otimes(\mathrm{T}^{-1}\mathcal{B}),\mathcal{D}]^{cc}$ is equivalent to $[\mathcal{A}\otimes\mathcal{B},\mathcal{D}]^{cc}_{\mathrm{S}\boxtimes\mathrm{T}}$, the ∞-category of cocontinuous functor $f:\mathcal{A}\otimes\mathcal{B}\to\mathcal{D}$ sending arrows in $\mathrm{S}\boxtimes\mathrm{T}$ to equivalences. Let $\mathrm{S}'=\mathrm{S}\boxtimes\emptyset$ and $\mathrm{T}'=\emptyset\boxtimes\mathrm{T}$, then the following pullback square ends the proof,

$$\begin{tikzcd}
{[\mathcal{A}\otimes\mathcal{B},\mathcal{D}]^{cc}_{\mathrm{S}\boxtimes\mathrm{T}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{A}\otimes\mathcal{B},\mathcal{D}]^{cc}_{\mathrm{T}'}} \arrow[d] \\
{[\mathcal{A}\otimes\mathcal{B},\mathcal{D}]^{cc}_{\mathrm{S}'}} \arrow[r] & {[\mathcal{A}\otimes\mathcal{B},\mathcal{D}]^{cc}}.
\end{tikzcd}$$

□

We finish with what we were here for: a description of coproducts inside $\mathcal{S}\mathrm{hv}$. Notice that this theorem is stated in [HA] but the proof is left to the reader as it has already been proved in [HT] for the case where one of the two ∞-toposes is localic.

Theorem 3.3.21.— *If $\mathcal{X}$ and $\mathcal{Y}$ are two ∞-toposes, then $\mathcal{S}\mathrm{h}(\mathcal{X})\otimes\mathcal{S}\mathrm{h}(\mathcal{Y})$ is a coproduct of $\mathcal{S}\mathrm{h}(\mathcal{X})$ and $\mathcal{S}\mathrm{h}(\mathcal{Y})$ in $\mathcal{S}\mathrm{hv}$.*

Proof.— Let C and D be two small ∞-categories, we first remark that

$$\mathcal{S}[\mathrm{C}]\otimes\mathcal{S}[\mathrm{D}]\simeq\mathcal{S}[\mathrm{C}\amalg\mathrm{D}].$$

To see this, we remark that

$$\mathcal{S}[\mathrm{C}]\otimes\mathcal{S}[\mathrm{D}]=\mathcal{P}(\overline{\mathrm{C}})\otimes\mathcal{P}(\overline{\mathrm{D}})\simeq\mathcal{P}(\overline{\mathrm{C}}\times\overline{\mathrm{D}}).$$

Now look at the finite completion functor $\mathrm{C}\mapsto\overline{\mathrm{C}}$. It starts from $\mathcal{C}\mathrm{at}$ and goes to $\mathcal{C}\mathrm{art}$, the large ∞-category of finitely complete small ∞-categories with left exact functors. This functor is left adjoint to the forgetful functor. Hence it sends coproducts to coproducts. But in $\mathcal{C}\mathrm{art}$ products and coproducts coincide, and because the forgetful functor preserves limits, we have:

$$\overline{\mathrm{C}\amalg\mathrm{D}}\simeq\overline{\mathrm{C}}\times\overline{\mathrm{D}}\Longrightarrow\mathcal{S}[\mathrm{C}]\otimes\mathcal{S}[\mathrm{D}]\simeq\mathcal{P}(\overline{\mathrm{C}}\times\overline{\mathrm{D}})\simeq\mathcal{P}(\overline{\mathrm{C}\amalg\mathrm{D}})=\mathcal{S}[\mathrm{C}\amalg\mathrm{D}].$$

Remark.— The category of commutative monoids and $\mathcal{C}\mathrm{art}$ are very similar; products and coproducts coincide in both, for the same reason: they are enriched over $\mathbb{E}_\infty$-algebras of spaces. Such ∞-categories are called 0-semiadditive and an ∞-category is 0-semiadditive if and only if its homotopy category is semiadditive. See 4.4.9 and 4.4.11 in [HoLu].

Remark.— For a short proof that $\mathcal{C}\mathrm{art}$ is 0-semiadditive, see lemma 7.3.3.4 in [HT].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two ∞-toposes, we will now show that $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ is an ∞-category of sheaves. There exist two small ∞-categories C and D together with two left exact reflexive localisation functors $\mathrm{L} : \mathcal{S}[\mathrm{C}] \to \mathcal{S}\mathrm{h}(\mathcal{X})$ and $\mathrm{G} : \mathcal{S}[\mathrm{D}] \to \mathcal{S}\mathrm{h}(\mathcal{Y})$. Then both $\mathrm{L}^{\overline{\mathrm{D}}^{op}} : \mathcal{S}[\mathrm{C}]^{\overline{\mathrm{D}}^{op}} \to \mathcal{S}\mathrm{h}(\mathcal{X})^{\overline{\mathrm{D}}^{op}}$ and $\mathrm{G}^{\overline{\mathrm{C}}^{op}} : \mathcal{S}[\mathrm{D}]^{\overline{\mathrm{C}}^{op}} \to \mathcal{Y}^{\overline{\mathrm{C}}^{op}}$ are left exact and accessible reflexive localisation functors. By lemma 7.3.2.3 in [HT] and corollary 3.3.20, we deduce that the pushout morphism

$$\mathcal{S}[\mathrm{C}] \otimes \mathcal{S}[\mathrm{D}] \to \mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$$

is also a left exact and accessible reflexive localisation. And as we have just shown above, $\mathcal{S}[\mathrm{C}] \otimes \mathcal{S}[\mathrm{D}]$ is equivalent to a free ∞-category of sheaves, so that $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ is indeed an ∞-category of sheaves.

Let $p^* : \mathcal{S} \to \mathcal{S}\mathrm{h}(\mathcal{X})$ be a morphism of ∞-category of sheaves (unique up to contractible choice) and let $q^* : \mathcal{S} \to \mathcal{S}\mathrm{h}(\mathcal{Y})$ be another. We claim that the maps $p^* \otimes \mathrm{Id}_{\mathcal{Y}} : \mathcal{S}\mathrm{h}(\mathcal{X}) \to \mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ and $\mathrm{Id}_{\mathcal{S}\mathrm{h}(\mathcal{X})} \otimes q^* : \mathcal{S}\mathrm{h}(\mathcal{Y}) \to \mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ exhibit $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ as a pushout of $\mathcal{S}\mathrm{h}(\mathcal{X})$ and $\mathcal{S}\mathrm{h}(\mathcal{Y})$ in $\mathcal{S}\mathrm{hv}$.

Indeed for any ∞-category of sheaves $\mathcal{E}$, those two maps induce a commutative square

$$\begin{tikzcd}
{[\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y}), \mathcal{E}]^*} \arrow[r] \arrow[d, hook] & {[\mathcal{S}\mathrm{h}(\mathcal{X}), \mathcal{E}]^* \times [\mathcal{S}\mathrm{h}(\mathcal{Y}), \mathcal{E}]^*} \arrow[d, hook] \\
{[\mathcal{S}[\mathrm{C} \amalg \mathrm{D}], \mathcal{E}]^*} \arrow[r] & {[\mathcal{S}[\mathrm{C}], \mathcal{E}]^* \times [\mathcal{S}[\mathrm{D}], \mathcal{E}]^*}
\end{tikzcd}$$

In the above diagram, the vertical arrows are inclusions and the bottom one is an equivalence as $\mathcal{S}[\mathrm{C} \amalg \mathrm{D}]$ is the coproduct $\mathcal{S}[\mathrm{C}] \amalg \mathcal{S}[\mathrm{D}]$.

So we only need to show that if $(\varphi, \psi) \in [\mathcal{S}[\mathrm{C}], \mathcal{E}]^* \times [\mathcal{S}[\mathrm{D}], \mathcal{E}]^*$ factorises through $\mathcal{S}\mathrm{h}(\mathcal{X})$ and $\mathcal{S}\mathrm{h}(\mathcal{Y})$ then the associated map $\varphi \amalg \psi$ factorises through $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$. Let S be a set of arrows of $\mathcal{S}[\mathrm{C}]$ such that $\mathcal{S}\mathrm{h}(\mathcal{X}) \simeq \mathrm{S}^{-1}\mathcal{S}[\mathrm{C}]$ and let be T such that $\mathcal{S}\mathrm{h}(\mathcal{Y}) \simeq \mathrm{T}^{-1}\mathcal{S}[\mathrm{D}]$. If φ and ψ factorise, it means that φ sends arrows in S to equivalences and ψ sends arrows in T to equivalences. Let $\mathrm{S} \boxtimes \mathrm{T}$ be the set of arrows of the form $s \otimes x$ for $s \in \mathrm{S}, x \in \mathcal{S}[\mathrm{D}]$ or $y \otimes t$ with $t \in \mathrm{T}, y \in \mathcal{S}[\mathrm{C}]$, in $\mathcal{S}[\mathrm{C}] \otimes \mathcal{S}[\mathrm{D}]$. By the proof that $\mathcal{S}[\mathrm{C}] \otimes \mathcal{S}[\mathrm{D}] \simeq \mathcal{S}[\mathrm{C} \amalg \mathrm{D}]$ above, we have that the map from $\mathcal{S}[\mathrm{C} \amalg \mathrm{D}]$ to $\mathcal{E}$ associated to (φ, ψ) is equivalent to the map $\varphi \otimes \psi : \mathcal{S}[\mathrm{C}] \otimes \mathcal{S}[\mathrm{D}] \to \mathcal{E}$. But $\varphi \otimes \psi$ sends arrows in $\mathrm{S} \boxtimes \mathrm{T}$ to equivalences so it factorises through $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y}) \simeq (\mathrm{S} \boxtimes \mathrm{T})^{-1}\mathcal{S}[\mathrm{C}] \otimes \mathcal{S}[\mathrm{D}]$. □

3.3.5 *The ∞-category of points of an ∞-topos*

Definition 3.3.22.— *Let $\mathcal{X}$ be a ∞-topos, its ∞-category of points* $\mathrm{pt}(\mathcal{X})$ *is defined by*

$$\mathrm{pt}(\mathcal{X}) = [\mathcal{S}\mathrm{h}(\mathcal{X}), \mathcal{S}]^*$$

An object of $\mathrm{pt}(\mathcal{X})$ *is a morphism* $* \to \mathcal{X}$.

Remark 3.3.23.— The construction $\mathcal{X} \mapsto \mathrm{pt}(\mathcal{X})$ is functorial. For any morphism $f : \mathcal{X} \to \mathcal{Y}$, $\mathrm{pt}(f) : \mathrm{pt}(\mathcal{X}) \to \mathrm{pt}(\mathcal{Y})$ is the precomposition by $f^* : \mathcal{S}\mathrm{h}(\mathcal{Y}) \to \mathcal{S}\mathrm{h}(\mathcal{X})$.

Proposition 3.3.24.— *Let* $\mathcal{X}$ *be an* ∞*-topos, then* $\mathrm{pt}(\mathcal{X})$ *admits filtered colimits. Furthermore, if* $f : \mathcal{X} \to \mathcal{Y}$ *is a morphism between two* ∞*-toposes, then the associated functor* $\mathrm{pt}(f) : \mathrm{pt}(\mathcal{X}) \to \mathrm{pt}(\mathcal{Y})$ *is* ω*-accessible.*

Remark.— The ∞-category $\mathrm{pt}(\mathcal{X})$ is not necessarily ω-accessible, or even smally generated under filtered colimits.

Proof.— Limits and colimits are computed pointwise in $[\mathcal{S}\mathrm{h}(\mathcal{X}), \mathcal{S}]$ and filtered colimits commute with finite limits in $\mathcal{S}$, so $\mathrm{pt}(\mathcal{X})$ has small filtered colimits and $\mathrm{pt}(f)$ preserves filtered colimits. □

Definition 3.3.25.— *We will say that an* ∞*-topos* $\mathcal{X}$ *has enough points if for every arrow* f *of* $\mathcal{S}\mathrm{h}(\mathcal{X})$, f *is an equivalence when* $p(f)$ *is an equivalence in* $\mathcal{S}$ *for every point* $p \in \mathrm{pt}(\mathcal{X})$.

3.3.6 ∞-*Topos associated to a locale*

The main supplier of ∞-toposes is the adjunction between locales and ∞-toposes. We will describe it here. Details are in [HT].

Proposition 3.3.26.— *Let* $\mathcal{X}$ *be an* ∞*-topos, then the full subcategory of* $\mathcal{S}\mathrm{h}(\mathcal{X})$ *of subobjects of terminal objects is a frame. This gives rise to a functor*

$$\tau_{\leq -1} : \mathcal{S}\mathrm{hv} \to \mathcal{F}\mathrm{rm}$$

We let $\mathcal{L}\mathrm{oc}$ *be defined as* $\tau^{op}_{\leq -1}$, *the functor between opposite* ∞*-categories.*

Proposition 3.3.27.— *The functor* $\mathcal{L}\mathrm{oc}$ *is a reflexive localisation functor, with right adjoint* $\mathcal{T}\mathrm{op}$

$$\mathcal{L}\mathrm{oc} \underset{(--)_{\mathcal{T}\mathrm{ps}}}{\overset{(--)_{\mathrm{L}}}{\rightleftarrows}} \mathcal{T}\mathrm{op}$$

In this case, for a locale X, *the* ∞*-category* $\mathcal{S}\mathrm{h}(\mathcal{T}\mathrm{op}(\mathrm{X}))$ *is denoted* $\mathcal{S}\mathrm{h}(\mathrm{X})$.

If X *is a locale, then* $\mathcal{O}(\mathrm{X})$ *can be endowed canonically with the structure of a site by declaring any set of maps* $\{\mathrm{U}_i \to \mathrm{U}\}$ *covering when* $\cup_i \mathrm{U}_i = \mathrm{U}$. *Then* $\mathcal{S}\mathrm{h}(\mathrm{X})$ *is the* ∞*-category of sheaves of spaces on the site* $\mathcal{O}(\mathrm{X})$.

Definition 3.3.28.— *An* ∞*-topos* $\mathcal{X}$ *is said to be localic if* $\mathcal{X} \simeq ((\mathcal{X})_{\mathcal{L}})_{\mathcal{T}\mathrm{ps}}$.

3.4 EXPONENTIABLE ∞-TOPOSES

This section is the heart of the article, we give a precise characterization of exponentiable ∞-toposes. The main theorem states that exponentiable ∞-toposes $\mathcal{X}$ are those whose ∞-category of sheaves $\mathcal{S}\mathrm{h}(\mathcal{X})$

is *continuous*. This result is an ∞-version of the theorem of Johnstone and Joyal [JoJo].

Let us begin here with a heuristic argument. If we are given such an exponentiable ∞-topos then, the exponential $(\mathbb{A}^1)^{\mathcal{X}}$ exists. And we immediately see that

$$\mathcal{S}\mathrm{h}(\mathcal{X}) \simeq \mathrm{pt}((\mathbb{A}^1)^{\mathcal{X}}).$$

This imposes strong constrains on $\mathcal{X}$ because the exponential $(\mathbb{A}^1)^{\mathcal{X}}$ has itself the strong property of being an *injective* ∞-topos: namely, the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ has to be *continuous*.

Definition 3.4.1.— *Let $\mathcal{X}$ be an ∞-topos, we will say that $\mathcal{X}$ is exponentiable if the functor $\mathcal{Y} \mapsto \mathcal{Y} \times \mathcal{X}$ has a right adjoint.*

For an ∞-topos $\mathcal{Y}$ we will say that the particular exponential $\mathcal{Y}^{\mathcal{X}}$ exists if there is an ∞-topos $\mathcal{Y}^{\mathcal{X}}$ and a map $\mathcal{X} \times \mathcal{Y}^{\mathcal{X}} \to \mathcal{Y}$ such that the induced map $\mathrm{Map}(\mathcal{Z}, \mathcal{Y}^{\mathcal{X}}) \to \mathrm{Map}(\mathcal{Z} \times \mathcal{X}, \mathcal{Y})$ *is an equivalence in $\widehat{\mathcal{H}}$ for every $\mathcal{Z} \in$* $\mathcal{T}\mathrm{op}$.

Remark 3.4.2.— By proposition 5.2.2.12 in [HT], an ∞-topos $\mathcal{X}$ is exponentiable if and only if for any $\mathcal{Y} \in \mathcal{T}\mathrm{op}$, the particular exponential $\mathcal{Y}^{\mathcal{X}}$ exists.

3.4.1 *Injective ∞-toposes and their ∞-categories of points*

Definition 3.4.3.— *An ∞-topos $\mathcal{X}$ is injective if for every inclusion $m : \mathcal{Y} \to \mathcal{Z}$, the composition morphism* $\mathrm{Map}(\mathcal{Z}, \mathcal{X}) \to \mathrm{Map}(\mathcal{Y}, \mathcal{X})$ *has a section.*

Remark.— Injective ∞-topos in the ∞-category $\mathcal{T}\mathrm{op}$ should correspond dually to projective objects in $\mathcal{S}\mathrm{hv}$. In this ∞-category , ∞-toposes can be considered as 'commutative semi-algebras' over the 'commutative semi-ring' $\mathcal{S}$. Hence, we guess that injective ∞-toposes are retracts of affine ∞-toposes .

Remark 3.4.4.— This notion of injective ∞-topos would correspond to the notion of *weakly injective* topos defined in [Elephant]. We do not investigate the notions of *complete injective* and *strongly injective* ∞-toposes.

Theorem 3.4.5.— *An ∞-topos is injective if and only if it is a retract in* $\mathcal{T}\mathrm{op}$ *of an affine ∞-topos.*

Proof.— Let $\mathcal{X}$ be an injective ∞-topos, then by definition, there exists an inclusion $\mathcal{X} \to \mathbb{A}^C$ with C a small ∞-category. Because $\mathcal{X}$ is injective, this morphism must split.

On the contrary we will prove that any affine ∞-topos is injective: let $\mathcal{F} = \mathcal{S}\mathrm{h}(\mathcal{Y})$ and $\mathcal{G} = \mathcal{S}\mathrm{h}(\mathcal{Z})$ be two ∞-toposes and $f : \mathcal{Y} \to \mathcal{Z}$ be an inclusion of ∞-toposes. Thanks to the universal property of affine ∞-toposes , we have the following equivalences $\mathrm{Map}(\mathcal{Y}, \mathbb{A}^C) \simeq \mathrm{Int}([C, \mathcal{F}])$

and $\mathrm{Map}(\mathcal{Z}, \mathbb{A}^{\mathrm{C}}) \simeq \mathrm{Int}([\mathrm{C}, \mathcal{G}])$. Then the reflexive localisation f^* gives the desired reflexive localisation $(f^*)^{\mathrm{C}}$.

Finally, let's prove that a retract of an injective ∞-topos is still injective: let $r : \mathcal{X} \to \mathcal{X}'$ be a retraction in $\mathcal{T}\mathrm{op}$ with $\mathcal{X}$ injective and $s : \mathcal{X}' \to \mathcal{X}$ a section. Let $i : \mathcal{Y} \hookrightarrow \mathcal{Z}$ be an inclusion and $f : \mathcal{Y} \to \mathcal{X}'$ be any map. Then $sf : \mathcal{Y} \to \mathcal{X}$ can be extended in $g : \mathcal{Z} \to \mathcal{X}$ because $\mathcal{X}$ is injective. Then $rg : \mathcal{Z} \to \mathcal{X}$ extends f. □

Injective ∞-toposes have the very particular property of being characterized by their ∞-categories of points. That is, knowing $\mathrm{pt}(\mathcal{X})$, we can recover $\mathcal{X}$ in the case where $\mathcal{X}$ is injective. In particular, injective ∞-toposes have enough points.

Definition 3.4.6.— *Let* $\mathrm{pt}(\mathcal{I}\mathrm{nj})$ *be the non-full subcategory of* $\widehat{\mathcal{C}\mathrm{at}}$ *whose objects are presheaves ∞-categories* $\mathcal{P}(\mathrm{C})$ *with* C *a small ∞-category and their retracts by ω-continuous functors. The morphisms of this category are the ω-continuous functors.*

Let $\mathcal{I}\mathrm{nj}$ *be the full subcategory of* $\mathcal{T}\mathrm{op}$ *made of injective ∞-toposes .*

Theorem 3.4.7.— *The functor of points* $\mathrm{pt} : \mathcal{I}\mathrm{nj} \to \mathrm{pt}(\mathcal{I}\mathrm{nj})$ *is an equivalence of ∞-categories.*

Proof.— Let C be a small ∞-category, then $\mathrm{pt}(\mathbb{A}^{\mathrm{C}}) \simeq \mathcal{P}(\mathrm{C}^{op})$ so, by proposition 3.3.24 and theorem 3.4.5, pt is a well defined functor from $\mathcal{I}\mathrm{nj}$ to $\mathrm{pt}(\mathcal{I}\mathrm{nj})$.

We build a new functor $\psi : \mathcal{A} \mapsto [\mathcal{A}, \mathcal{S}]^{\omega}$, where $[\mathcal{A}, \mathcal{S}]^{\omega}$ is the ∞-category of ω-continuous functors between $\mathcal{A}$ and $\mathcal{S}$.

We claim that ψ is a functor from $\mathrm{pt}(\mathcal{I}\mathrm{nj})$ to $\mathcal{I}\mathrm{nj}^{op}$. For this, let $m : \mathcal{A} \to \mathcal{B}$ be an ω-continuous functor, then it induces a functor $m^* : [\mathcal{B}, \mathcal{S}]^{\omega} \to [\mathcal{A}, \mathcal{S}]^{\omega}$. Because filtered colimits are left exact in $\mathcal{S}$, we see that finite limits and all colimits in $\psi(\mathcal{A})$ and $\psi(\mathcal{B})$ are computed pointwise, so m^* is cocontinous and left exact. Finally, $\psi(\mathcal{P}(\mathrm{C}^{op})) \simeq \mathcal{S}[\mathrm{C}] = \mathcal{S}\mathrm{h}(\mathbb{A}^{\mathrm{C}})$, so that by theorem 3.4.5 ψ is well defined.

By the above computation, the functor of points pt induces an equivalence on the subcategory of affine ∞-toposes to the subcategory of $\mathrm{pt}(\mathcal{I}\mathrm{nj})$ made of presheaves ∞-categories . This equivalence extends to their Cauchy-completion. □

3.4.2 *Continuous ∞-categories*

As we have said, we now know that if $\mathcal{X}$ is exponentiable, then $\mathcal{S}\mathrm{h}(\mathcal{X})$ is equivalent to the ∞-category of points of an injective ∞-topos. Which by theorem 3.4.7 means that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is a retract by ω-continuous functors of a presheaf ∞-category. This characterization seems far away from the one we had in the case of locales.

Here we will work out what it means to be an 'ω-retract' of an ∞-category of presheaves and we will see that we can find an adequate definition. This involves the definition of a *continuous* ∞-category . The definition in the 1-categorical context was first given in [JoJo]. We shall prove here the propositions in the ∞-setting directly related to ∞-toposes.

Definition 3.4.8.— *Let $\mathcal{C}$ be an ∞-category with filtered colimits. We will say that $\mathcal{C}$ is continuous if the evaluation functor $\varepsilon : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $\beta : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.*

Remark 3.4.9.— In such a case, we end up with a triple adjunction

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{C}) \arrow[r, "\varepsilon" description] & \mathcal{C} \arrow[l, bend right, "\beta"'] \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

Remark that because α is fully faithful, so is β.

There is a particular difference between continuous ∞-categories and continuous frames. Indeed, the ∞-category $\mathrm{Ind}(\mathcal{C})$ is very large. We will now focus on continuous categories with smallness properties. Namely, we wish to replace $\mathrm{Ind}(\mathcal{C})$ by a locally small ∞-category $\mathrm{Ind}(\mathrm{D})$.

Definition 3.4.10.— *Let $\mathcal{C}$ be a continuous ∞-category, then if there exists a small full subcategory $\mathrm{D} \subset \mathcal{C}$, such that D is stable in $\mathcal{C}$ under finite limits and colimits and such that the evaluation functor $\mathrm{Ind}(\mathrm{D}) \to \mathcal{C}$ has a fully faithful left adjoint, then we call the triple adjunction*

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{D}) \arrow[r, "\varepsilon" description] & \mathcal{C} \arrow[l, bend right, "\beta"'] \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

a standard presentation

Proposition 3.4.11.— *Let $\mathcal{C}$ be a smally presentable and continuous ∞-category. Then $\mathcal{C}$ has a standard presentation.*

Proof.— Because $\mathcal{C}$ is smally presentable, there exists a small and dense full subcategory $\mathrm{D} \subset \mathcal{C}$. We can then take D' the smallest full subcategory of $\mathcal{C}$ containing D and closed in $\mathcal{C}$ under finite limits and colimits. As such, D' is dense in $\mathcal{C}$ so that the evaluation functor $\varepsilon : \mathrm{Ind}(\mathrm{D}') \to \mathcal{C}$ has a fully faithful right adjoint $\alpha : \mathcal{C} \to \mathrm{Ind}(\mathrm{D}')$.

As $\varepsilon : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is continuous and $\mathrm{Ind}(\mathrm{D}') \subset \mathrm{Ind}(\mathcal{C})$ commutes with limits, we deduce that $\varepsilon : \mathrm{Ind}(\mathrm{D}') \to \mathcal{C}$ is continuous and then has a left adjoint β because $\mathrm{Ind}(\mathrm{D}')$ and $\mathcal{C}$ are smally presentable. $\square$

Remark.— We do not really need $\mathcal{C}$ to be smally presentable, but only smally generated. The proof is a bit longer.

Proposition 3.4.12.— *Let $\mathcal{D}$ be an ∞-category. Then $\mathrm{Ind}(\mathcal{D})$ is continuous.*

Proof.— As the ∞-category $\mathrm{Ind}(\mathrm{Ind}(\mathcal{D}))$ is a bit messy, we denote a generic object of $\mathrm{Ind}(\mathrm{Ind}(\mathcal{D}))$ as "$\varinjlim_{\mathrm{I}}$"'$\varinjlim_{\mathrm{J}_i}$' d_{ij}.

Then, the functor $\alpha : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathrm{Ind}(\mathcal{D}))$ right adjoint to ε is given by "$\varinjlim$" $d_i \mapsto$ '$\varinjlim$' d_i.

We claim that the left adjoint β is given by sending "$\varinjlim$" d_i in $\mathrm{Ind}(\mathcal{D}))$ on "$\varinjlim$" d_i in $\mathrm{Ind}(\mathrm{Ind}(\mathcal{D}))$, that is $\beta = \mathrm{Ind}(\alpha)$. We have a *unit transformation* of (β, ε): $\mathrm{Id} \simeq \varepsilon\beta$. So we can check the adjunction on Map-spaces. Any $d \in \mathcal{D}$ is an ω-compact object of $\mathrm{Ind}(\mathcal{D})$, so that for any $d =$ "$\varinjlim_i d_i$", $\beta(d)$ is a formal colimit of ω-compact objects of $\mathrm{Ind}(\mathrm{D})$. Let $a =$ "$\varinjlim_{j\in \mathrm{J}}$"'$\varinjlim_{k\in \mathrm{K}_j}$' d_{jk}. Then we have

$$\mathrm{Map}(\beta(d), a) \simeq \varinjlim_{j\in\mathrm{J}} \varinjlim_{k\in\mathrm{K}_j} \varprojlim_{i\in\mathrm{I}} \mathrm{Map}(d_i, d_{jk}) \simeq \mathrm{Map}(d, \varepsilon(a)).$$

□

Proposition 3.4.13.— *Any retract by ω-continuous functors of a continuous ∞-category is continuous.*

Proof.— Let $r : \mathcal{C} \to \mathcal{D}$ be a retraction by ω-continuous functors and suppose $\mathcal{C}$ is continuous. Let s be an ω-continuous section of r. Because both commute with filtered colimits, we have $\varepsilon_{\mathcal{D}} \circ \mathrm{Ind}(r) \simeq r \circ \varepsilon_{\mathcal{C}}$ and $s \circ \varepsilon_{\mathcal{D}} \simeq \varepsilon_{\mathcal{C}} \circ \mathrm{Ind}(s)$. This means we get the following retract diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{D}) \arrow[r, "\mathrm{Ind}(s)"] \arrow[d, "\varepsilon_{\mathcal{D}}"] & \mathrm{Ind}(\mathcal{C}) \arrow[r, "\mathrm{Ind}(r)"] \arrow[d, "\varepsilon_{\mathcal{C}}"] & \mathrm{Ind}(\mathcal{D}) \arrow[d, "\varepsilon_{\mathcal{D}}"] \\
\mathcal{D} \arrow[r, "s"] & \mathcal{C} \arrow[r, "r"] & \mathcal{D}
\end{tikzcd}$$

Let $\theta = \mathrm{Ind}(r) \circ \beta_{\mathcal{C}} \circ s$. The functor θ is a good candidate to be the left adjoint to $\varepsilon_{\mathcal{D}}$. Indeed, from the unit transformation $\mathrm{Id} \simeq \varepsilon_{\mathcal{C}} \circ \beta_{\mathcal{C}}$ we get $u : \mathrm{Id} \simeq \varepsilon_{\mathcal{D}} \circ \theta$. From the counit transformation $\beta_{\mathcal{C}} \circ \varepsilon_{\mathcal{C}} \to \mathrm{Id}$ we also get a counit transformation $k : \theta \circ \varepsilon_{\mathcal{D}} \to \mathrm{Id}$. Finally $k\theta \circ \theta u : \theta \to \theta$ is homotopic to the identity transformation. Unfortunately $\varepsilon_{\mathcal{D}} k \circ u \varepsilon_{\mathcal{D}} : \varepsilon_{\mathcal{D}} \to \varepsilon_{\mathcal{D}}$ is *not* homotopic to the identity transformation (in this case, one would call θ a weak adjoint). Instead $\varepsilon_{\mathcal{D}} k$ is idempotent.

Fortunately, $[\mathcal{D}, \mathrm{Ind}(\mathcal{D})]$ has all filtered colimits and so, idempotents split (see corollary 4.4.5.16 in [HT]). Let $\theta \xrightarrow{\tau} \beta \xrightarrow{\sigma} \theta$ be such a splitting. We get a new counit map $k' = k \circ (\sigma \varepsilon_{\mathcal{D}}) : \beta \varepsilon_{\mathcal{C}} \to \mathrm{Id}$ and a new unit map $u' = (\varepsilon_{\mathcal{D}} \tau) \circ u : \mathrm{Id} \simeq \varepsilon_{\mathcal{D}} \beta$. This time $\varepsilon_{\mathcal{D}} k' \circ u' \varepsilon_{\mathcal{D}}$ is homotopic to the unit transformation, as well as $k'\beta \circ \beta u'$.

So β is a left adjoint to $\varepsilon_{\mathcal{D}}$, hence $\mathcal{D}$ is a continuous ∞-category. □

Theorem 3.4.14.— *Let $\mathcal{X}$ be an ∞-topos, then the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is equivalent to the ∞-category of points of an injective ∞-topos if and only if $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous.*

Proof.— Suppose $\mathcal{S}\mathrm{h}(\mathcal{X}) \in \mathrm{pt}(\mathcal{I}\mathrm{nj})$, then by theorem 3.4.7, we know that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is a retract by ω-continuous functors of an ∞-category of presheaves $\mathcal{P}(\mathrm{C})$. But $\mathcal{P}(\mathrm{C})$ is finitely presentable, so it is continuous by proposition 3.4.12, and $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous by proposition 3.4.13.

Conversely, suppose that the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous. Because $\mathcal{S}\mathrm{h}(\mathcal{X})$ is smally presentable, by proposition 3.4.11, we get a standard presentation $\mathrm{Ind}(\mathrm{D}) \to \mathcal{S}\mathrm{h}(\mathcal{X})$. In particular, $\mathcal{S}\mathrm{h}(\mathcal{X})$ is a retract by ω-continuous functors of $\mathrm{Ind}(\mathrm{D})$, and $\mathrm{Ind}(\mathrm{D})$ is itself such a retract of $\mathcal{P}(\mathrm{D})$, so that $\mathcal{S}\mathrm{h}(\mathcal{X}) \in \mathrm{pt}(\mathcal{I}\mathrm{nj})$. □

Remark 3.4.15.— More generally an ∞-category $\mathcal{C}$ is in $\mathrm{pt}(\mathcal{I}\mathrm{nj})$ if and only if it is smally generated and continuous.

Corollary 3.4.16.— *If $\mathcal{X}$ is an exponentiable ∞-topos, then its ∞-category of sheaves $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous.*

Proof.— We know that $\mathbb{A}$ is injective and the functor $(-)\times\mathcal{X}$ preserves inclusions, so $\mathbb{A}^{\mathcal{X}}$ is also injective. Now, by definition of $\mathbb{A}$, we have the equivalence of ∞-categories

$$\mathrm{pt}(\mathbb{A}^{\mathcal{X}}) \simeq \mathcal{S}\mathrm{h}(\mathcal{X})$$

which implies the result. □

3.4.3 *Exponentiability theorem*

In corollary 3.4.16, we have seen that the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous for an exponentiable ∞-topos $\mathcal{X}$. In theorem 3.4.17, we show the reciprocal statement.

Theorem 3.4.17.— *An ∞-topos $\mathcal{X}$ is exponentiable if and only if the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous.*

Proof.— The first part of the theorem is done by corollary 3.4.16. For the other part, use lemmas 3.4.19, 3.4.20 and proposition 3.4.18, in this order. □

Proposition 3.4.18.— *An ∞-topos $\mathcal{X}$ is exponentiable if and only if, the particular exponentials $\mathcal{A}^{\mathcal{X}}$ exists for every $\mathcal{A} \in \mathcal{A}\mathrm{ff}$.*

Proof.— By remark 3.4.2, we only need to show that the particular exponentials $\mathcal{Y}^{\mathcal{X}}$ exist for every $\mathcal{Y} \in \mathcal{T}\mathrm{op}$. But by theorem 3.3.11, any $\mathcal{Y} \in \mathcal{T}\mathrm{op}$ is a limit of affine ∞-toposes i.e $\mathcal{Y} \simeq \varprojlim_{i\in\mathrm{I}} \mathcal{A}_i$ with $\mathcal{A}_i \in \mathcal{A}\mathrm{ff}$ for every $i \in \mathrm{I}$. As every exponential $\mathcal{A}_i^{\mathcal{X}}$ exists, we get a map

$$\mathcal{X} \times \varprojlim_{i\in\mathrm{I}} \mathcal{A}_i^{\mathcal{X}} \to \varprojlim_{i\in\mathrm{I}} \mathcal{A}_i$$

that exhibits $\varprojlim_{i\in\mathrm{I}} \mathcal{A}_i^{\mathcal{X}}$ as the exponential $\mathcal{Y}^{\mathcal{X}}$. □

Lemma 3.4.19.— *Let $\mathcal{X}$ be an ∞-topos such that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is a continuous ∞-category, then the exponential $\mathbb{A}^{\mathcal{X}}$ exists in* $\mathcal{T}\mathrm{op}$.

Proof.— In this proof we will be using ∞-versions of ends and coends. These are developed in section 3.5 and behave like ends and coends in category theory.

Let $\mathcal{X}$ be an ∞-topos such that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous. To show that $\mathbb{A}^{\mathcal{X}}$ exists, we have to find an injective ∞-topos $\mathcal{I}$ and functorial isomorphisms

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}}(\mathcal{S}\mathrm{h}(\mathcal{I}), \mathcal{S}\mathrm{h}(\mathcal{Y})) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}}(\mathcal{S}[X], \mathcal{S}\mathrm{h}(\mathcal{Y}) \otimes \mathcal{S}\mathrm{h}(\mathcal{X}))$$

in $\widehat{\mathcal{H}}$.

First we build $\mathcal{I}$. For this take a standard presentation of $\mathcal{S}\mathrm{h}(\mathcal{X})$,

$$\mathrm{Ind}(D) \underset{\alpha}{\overset{\beta}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\ \varepsilon\ }}}}} \mathcal{S}\mathrm{h}(\mathcal{X})$$

and let $W = \beta\varepsilon$. Now because β and ε are adjoint and that β is fully faithful, we have that W is an idempotent cocontinuous comonad on $\mathrm{Ind}(D)$ and β induces an equivalence between $\mathcal{S}\mathrm{h}(\mathcal{X})$ and the fixed points of W.

Let $w : D^{op} \times D \to \mathcal{S}$ be the corresponding bimodule. Notice that because W has its values in ind-objects, the bimodule w is left exact in the first variable. Let $w' : D \times D^{op} \to \mathcal{S}$ be the bimodule defined by $w'(a, b) = w(b, a)$. Because w' is left exact in the second variable its left Kan extension $w'_! : \mathcal{P}(D^{op}) \to \mathcal{P}(D^{op})$ is left exact and cocontinuous.

This new functor is also idempotent. Indeed, the fact that W is idempotent can be translated in terms of coends by the formula

$$w(a, b) \simeq \int_c w(a, c)w(c, b)$$

while the construction of $w'_!$ by left Kan extension is

$$w'_!(F)(a) = \int_c F(c)w(c, a)$$

so we conclude that $w'_!$ is idempotent by Fubini. This implies that the ∞-category of fixed points of $w'_!$ is a retract by cocontinuous and left exact functors of $\mathcal{P}(D^{op})$, hence it is an ∞-category of sheaves whose associated ∞-topos is injective. This is our $\mathcal{I}$.

Let $\mathcal{Y}$ be any ∞-topos. We will show that the ∞-categories $[\mathcal{S}\mathrm{h}(\mathcal{I}), \mathcal{S}\mathrm{h}(\mathcal{Y})]^*$ and $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ are equivalent by describing each separately.

The ∞-category $[\mathcal{S}\mathrm{h}(\mathcal{I}), \mathcal{S}\mathrm{h}(\mathcal{Y})]^*$ is equivalent, by definition of $\mathcal{I}$, to the ∞-category of cocontinuous and left exact functors $\mathcal{F} : \mathcal{P}(D^{op}) \to$

$\mathcal{S}\mathrm{h}(\mathcal{Y})$ such that $\mathcal{F} \circ w'_! \simeq \mathcal{F}$. This ∞-category is also equivalent to the ∞-category of left exact functors $\mathcal{F} : \mathrm{D}^{op} \to \mathcal{S}\mathrm{h}(\mathcal{Y})$ such that

$$\forall d \in \mathrm{D}^{op}, \quad \mathcal{F}(d) \simeq (\mathcal{F} \otimes w)(d) := \int^{c} \mathcal{F}(c) \otimes w(d,c)$$

Remark.— It is possible to write this formula because $\mathcal{S}\mathrm{h}(\mathcal{Y})$ is cocomplete and so, tensored over $\mathcal{S}$.

At the opposite, the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{S}\mathrm{h}(\mathcal{Y})$ is equivalent by proposition 3.3.18 to $[\mathcal{S}\mathrm{h}(\mathcal{X})^{op}, \mathcal{S}\mathrm{h}(\mathcal{Y})]_c$ and as $\mathcal{S}\mathrm{h}(\mathcal{X})$ is equivalent to the fixed points of W which is cocontinuous, this ∞-category is equivalent to continuous functors $\mathcal{F} : \mathrm{Ind}(\mathrm{D})^{op} \to \mathcal{S}\mathrm{h}(\mathcal{Y})$ such that $\mathrm{F} \circ \mathrm{W}^{op} \simeq \mathrm{F}$. Using ends, this is equivalent to left exact functors $\mathcal{F} : \mathrm{D}^{op} \to \mathcal{S}\mathrm{h}(\mathcal{Y})$ such that

$$\forall d \in \mathrm{D}^{op}, \quad \mathcal{F}(d) \simeq [w, \mathcal{F}](d) := \prod_{c} \mathcal{F}(c)^{w(c,d)}$$

Remark.— It is possible to write this formula because $\mathcal{S}\mathrm{h}(\mathcal{Y})$ is complete and so, cotensored over $\mathcal{S}$.

The constructions $- \otimes w$ and $[w, -]$ are functorial and they are adjoint (proposition 3.5.19).

$$[\mathrm{D}^{op}, \mathcal{S}\mathrm{h}(\mathcal{Y})]_{\mathrm{lex}} \underset{[w,-]}{\overset{-\otimes w}{\rightleftarrows}} [\mathrm{D}^{op}, \mathcal{S}\mathrm{h}(\mathcal{Y})]_{\mathrm{lex}}$$

Furthermore, because W is an idempotent comonad, the functor $- \otimes w$ is an idempotent comonad and $[w, -]$ is an idempotent monad. This implies that they restrict to equivalences between their ∞-category of fixed points.

Remark.— This fact is classic. Given an idempotent monad $\mathrm{M} : \mathcal{C} \to \mathcal{C}$, the image of M is contained inside $\mathrm{Fix}(\mathrm{M})$ so M can be decomposed into $\mathrm{M} : \mathcal{C} \xrightarrow{\varepsilon} \mathrm{Fix}(\mathrm{M}) \xrightarrow{\alpha} \mathcal{C}$. Then we have $\varepsilon \dashv \alpha$ and ε is a reflexive localisation. If M has an adjoint comonad W, set $\beta = \mathrm{W}\alpha$, then we get the triple adjunction $\beta \dashv \varepsilon \dashv \alpha$ and β is an equivalence between $\mathrm{Fix}(\mathrm{M})$ and $\mathrm{Fix}(\mathrm{W})$.

As this equivalence is functorial in $\mathcal{Y}$, we have proved the existence of $\mathbb{A}^{\mathcal{X}}$. □

Lemma 3.4.20.— *Let $\mathcal{X}$ be an ∞-topos for which the exponential $\mathbb{A}^{\mathcal{X}}$ exists, then all exponentials $(\mathbb{A}^{\mathrm{C}})^{\mathcal{X}}$ exist for every affine ∞-topos $\mathbb{A}^{\mathrm{C}}$.*

Proof.— The first part of the proof consist in showing that the ∞-topos $\mathcal{T}^{\mathrm{C}}$ defined by $\mathcal{P}(\mathrm{C}) = \mathcal{S}\mathrm{h}(\mathcal{T}^{\mathrm{C}})$ is exponentiable.

For this, we will show that $\mathcal{P}(\mathrm{C})$ is coexponentiable in $\mathcal{S}\mathrm{hv}$. The map $\mathcal{S}[\mathrm{D}] \to \mathcal{S}[\mathrm{D} \times \mathrm{C}^{op}]$ gives the unit map $\mathcal{S}[\mathrm{D}] \to \mathcal{S}[\mathrm{D} \times \mathrm{C}^{op}] \otimes \mathcal{P}(\mathrm{C})$. For every $\mathcal{S}\mathrm{h}(\mathcal{X}) \in \mathcal{S}\mathrm{hv}$, we then have a map $\mathrm{Map}_{\mathcal{S}\mathrm{hv}}(\mathcal{S}[\mathrm{D} \times \mathrm{C}^{op}], \mathcal{S}\mathrm{h}(\mathcal{X})) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}}(\mathcal{S}[\mathrm{D}], \mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{P}(\mathrm{C}))$ which is an isomorphism in $\widehat{\mathcal{H}}$. So by proposition 3.4.18, $\mathcal{T}^{\mathrm{C}}$ is exponentiable and by the calculation we have just done

$$(\mathbb{A}^{\mathrm{D}})^{\mathcal{T}^{\mathrm{C}}} \simeq \mathbb{A}^{\mathrm{D} \times \mathrm{C}^{op}}.$$

A bit of abstract non-sense about exponentials ends the proof as the particular exponential $(\mathbb{A}^{C})^{\mathcal{X}}$ can be defined as $((\mathbb{A})^{\mathcal{X}})^{\mathcal{T}^{C^{op}}}$ for any small ∞-category C.

The evaluation map $\mathcal{X} \times \mathbb{A}^{\mathcal{X}} \to \mathbb{A}$ gives $\mathcal{X}^{\mathcal{T}^{C^{op}}} \times (\mathbb{A}^{\mathcal{X}})^{\mathcal{T}^{C^{op}}} \to \mathbb{A}^{C}$ and using the map $\mathcal{X} \to \mathcal{X}^{\mathcal{T}^{C^{op}}}$ (from exponential of the first projection $\mathcal{X} \times \mathcal{T}^{C^{op}} \to \mathcal{X}$), we end up with the evaluation map $\mathcal{X} \times ((\mathbb{A})^{\mathcal{X}})^{\mathcal{T}^{C^{op}}} \to \mathbb{A}^{C}$. Finally for every ∞-topos $\mathcal{Y}$, we get

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}}\left(\mathcal{Y}, (\mathbb{A}^{\mathcal{X}})^{\mathcal{T}^{C^{op}}}\right) \simeq \mathrm{Map}_{\mathcal{T}\mathrm{op}}(\mathcal{Y} \times \mathcal{T}^{C^{op}} \times \mathcal{X}, \mathbb{A}) \simeq \mathrm{Map}_{\mathcal{T}\mathrm{op}}(\mathcal{Y} \times \mathcal{X}, \mathbb{A}^{C})$$

Using proposition 3.4.18 again, $\mathcal{X}$ is exponentiable. □

3.4.4 *Exponentiability of n-toposes*

Since the beginning of this article we have never used that groupoids are effective in an ∞-category of sheaves, in order to prove the exponentiability theorem. What have we used? Essentially that an ∞-category of sheaves $\mathcal{C}$ is smally presentable and colimits are universal in $\mathcal{C}$. This means that the same proof will work in the case of n-toposes. For details about n-toposes, see [HT].

Definition 3.4.21.— *Let $-1 \leq n \leq \infty$. An ∞-category $\mathcal{C}$ is an n-categories of sheaves if there exists a small ∞-category* D *and an accessible reflexive left exact localisation*

$$\mathcal{P}_{\leq n-1}(\mathrm{D}) \to \mathcal{C}$$

where $\mathcal{P}_{\leq n-1}(\mathrm{D})$ is the full subcategory of $\mathcal{P}(\mathrm{D})$ spanned by the $(n-1)$-truncated objects.

A morphism of n-categories of sheaves is a cocontinuous and left exact functor. The ∞-category of n-categories of sheaves $n\mathcal{S}\mathrm{hv}$, is the full subcategory of $\widehat{\mathcal{C}\mathrm{at}}$ whose objects are n-categories of sheaves and whose morphisms are the morphisms of n-categories of sheaves.

The ∞-category of n-toposes is

$$n\mathcal{T}\mathrm{op} = n\mathcal{S}\mathrm{hv}^{op}$$

and the isomorphism is denoted by $\mathcal{X} \mapsto \mathcal{S}\mathrm{h}_{\leq n-1}(\mathcal{X})$.

Remark 3.4.22.— If $\mathcal{C}$ is an n-category of sheaves, then $\mathcal{C}$ is smally presentable and colimits is $\mathcal{C}$ are universal.

From the definition, we have the following correspondence table

$$\begin{array}{ccc} -1\text{-topos} & \leftrightarrow & \emptyset \\ 0\text{-topos} & \leftrightarrow & \text{locale} \\ 1\text{-topos} & \leftrightarrow & \text{topos} \\ \dots & & \\ n\text{-topos} & & \\ \dots & & \\ \infty\text{-topos} & & \end{array}$$

Let us recall the notion of an n-category. Fix $-1 \le n \le \infty$.

Definition 3.4.23.— *A ∞-category $\mathcal{C}$ is an n-category if for every pair of objects $x, y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is (n-1)-truncated. We let $n\widehat{\mathcal{C}\mathrm{at}}$ be the full subcategory of $\widehat{\mathcal{C}\mathrm{at}}$ spanned by n-categories.*

Remark 3.4.24.— Definition 3.4.21 can be improved: $\mathcal{C}$ is an n-category of sheaves if there exists a small n-category D with a left exact reflexive localisation $\mathcal{P}_{\le n-1}(\mathrm{D}) \to \mathcal{C}$.

In order to be sure that the proof in the ∞-case transposes well to the n-case, we need to check some lemmas.

Lemma 3.4.25.— *The very large ∞-category $n\widehat{\mathcal{C}\mathrm{at}}^{cc}$ inherits a symmetric monoidal structure from $\widehat{\mathcal{C}\mathrm{at}}^{cc}$, and the reflexive localisation functor $\tau_{\le n-1} : \widehat{\mathcal{C}\mathrm{at}}^{cc} \to n\widehat{\mathcal{C}\mathrm{at}}^{cc}$ associating to a cocomplete ∞-category $\mathcal{C}$ its cocomplete n-category of $(n-1)$-truncated objects, is symmetric monoidal.*

Proof.— By remark 4.8.2.17 of [HA], for a cocomplete ∞-category $\mathcal{C}$, we have

$$\tau_{\le n-1}\mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S}_{\le n-1}$$

this means that the tensor product of cocomplete n-categories is again an n-category, so that $n\widehat{\mathcal{C}\mathrm{at}}^{cc}$ inherits a symmetric monoidal structure from $\widehat{\mathcal{C}\mathrm{at}}^{cc}$. Furthermore, $\tau_{\le n-1} : \mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{S}_{\le n-1}$ is a symmetric monoidal reflexive localisation functor (see corollary 5.5.6.22 in [HT]). □

Lemma 3.4.26.— *The functor*

$$\mathcal{P}_{\le n-1} : (n\widehat{\mathcal{C}\mathrm{at}}, \times) \to (n\widehat{\mathcal{C}\mathrm{at}}^{cc}, \otimes)$$

is a symmetric monoidal left adjoint to the forgetful functor.

Proof.— The fact that $\mathcal{P}_{\le n-1}$ is symmetric monoidal is a direct consequence of its universal property. Which is itself a immediate consequence of corollary 5.5.6.22 of [HT]. □

Corollary 3.4.27.— *The tensor product of n-categories of sheaves coincide with the coproduct.*

From now on, using the same proofs as for the ∞-case, we deduce all propositions and corollaries we want. Up to:

Theorem 3.4.28.— *An n-topos $\mathcal{X}$ is exponentiable in $n\mathcal{T}\mathrm{op}$ if and only if the n-category $\mathcal{S}\mathrm{h}_{\leq n-1}(\mathcal{X})$ is continuous.*

Remark 3.4.29.— This theorem includes the case of locales, toposes and ∞-toposes.

3.4.5 *Examples of exponentiable ∞-toposes*

We will begin by giving very basic examples of exponentiable ∞-toposes and then make a list of all operations that can build exponentiable ∞-toposes. The specific examples of exponentiable localic ∞-toposes and coherent ∞-toposes will be discussed in the next subsections.

Remark 3.4.30.— We state all the propositions for ∞-toposes but ∞ shall be replaced by $n < \infty$ with no problem.

Let us start with the easiest criterion that will already give us a lot of examples.

Proposition 3.4.31.— *Let $\mathcal{X}$ be an ∞-topos and suppose that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is ω-presentable. Then $\mathcal{X}$ is exponentiable.*

Proof.— If $\mathcal{S}\mathrm{h}(\mathcal{X})$ is ω-presentable, then there exists a small ∞-category D such that

$$\mathcal{S}\mathrm{h}(\mathcal{X}) \simeq \mathrm{Ind}(\mathrm{D})$$

and by proposition 3.4.12, the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{X})$ is continuous. □

This tells us that all the affine ∞-toposes $\mathbb{A}^{\mathrm{C}}$ are exponentiable. That includes also all ∞-toposes $\mathcal{X}$ such that $\mathcal{S}\mathrm{h}(\mathcal{X})$ is a presheaf ∞-category. In particular if G is a discrete group then $\mathcal{B}\mathrm{G}$ is an exponentiable ∞-topos.

Yet another class of examples: locally coherent n-toposes.

Definition 3.4.32.— *Let C be an n-category which admits finite limits. We will say that a Grothendieck topology on C is finitary if for every object $c \in \mathrm{C}$ and every covering sieve $\mathrm{C}^{(0)}_{/c} \subset \mathrm{C}_{/c}$ there exists a finite collection of morphisms $\{c_i \to c\}_{i\in \mathrm{I}}$ in $\mathrm{C}^{(0)}_{/c}$ which generates the sieve $\mathrm{C}^{(0)}_{/c}$.*

Definition 3.4.33.— *Let $n < \infty$, an n-topos is locally coherent if it is an n-topos associated to a finitary n-site.*

Proposition 3.4.34.— *Let $n < \infty$ and $\mathcal{X}$ be a locally coherent n-topos, then $\mathcal{X}$ is exponentiable.*

Proof.— If C is a finitary n-site, then $\mathcal{S}h(C)$ is ω-presentable. Indeed the sheaf condition for $\mathcal{F} \in \mathcal{S}h(C)$ boils down to *finite limit* conditions. All sieves are generated by finite collections $\{c_i \to c\}_{i \in I}$ so the sheaf condition

$$\prod_i \mathcal{F}(c_i) \rightleftarrows \prod_{i \to j} \mathcal{F}(c_j) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; \dots$$

involves only finite products at each level and there is only a finite number of levels because in $\mathcal{S}_{\leq n-1}$ limits of cosimplicial objects can be computed after being truncated at level n.

The consequence is that the inclusion

$$\mathcal{S}h(C) \hookrightarrow \mathcal{P}(C)$$

commutes with filtered colimits, which means that the reflexive localisation $\mathcal{P}(C) \to \mathcal{S}h(C)$ is ω-accessible, so $\mathcal{S}h(C)$ is ω-presentable and $\mathcal{X}$ is exponentiable by proposition 3.4.31. □

We go on with recipes to build more examples. The following two lemmas are not specific to the case of toposes and apply more generally to any kind of exponentiable objects but we like to use local proves.

Lemma 3.4.35.— *Let $\mathcal{X}$ and $\mathcal{Y}$ be two exponentiable ∞-toposes, then $\mathcal{X} \times \mathcal{Y}$ is exponentiable.*

This lemma is itself a corollary of the following proposition,

Proposition 3.4.36.— *Let $\mathcal{C}$ and $\mathcal{D}$ be two cocomplete and continuous ∞-categories. Then $\mathcal{C} \otimes \mathcal{D}$ is also continuous.*

Proof.— Let $\otimes_{\mathcal{S}_{\text{fin}}}$ be the monoidal structure of the large ∞-category of finitely cocomplete ∞-categories with right exacts functors as morphisms (its construction is the same as for $\otimes$, see [HA]).

Because $\mathcal{C}$ and $\mathcal{D}$ are cocomplete, $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Ind}(\mathcal{D})$ are also cocomplete. Now let

$$\varepsilon = \varepsilon_{\mathcal{C}} \otimes \varepsilon_{\mathcal{D}}; \qquad \beta = \beta_{\mathcal{C}} \otimes \beta_{\mathcal{D}}$$

We have $\varepsilon\beta \simeq \mathrm{Id}_{\mathcal{C}\otimes\mathcal{D}}$ so that $\mathcal{C} \otimes \mathcal{D}$ is a retract by cocontinuous functor of $\mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{D})$. But

$$\mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{D}) \simeq \mathrm{Ind}(\mathcal{C} \otimes_{\mathcal{S}_{\text{fin}}} \mathcal{D})$$

so $\mathcal{C}\otimes\mathcal{D}$ is a retract of a continuous ∞-category (by proposition 3.4.12) so it is continuous by proposition 3.4.13. □

Lemma 3.4.37.— *Let $\mathcal{X}$ be an exponentiable ∞-topos and $r : \mathcal{X} \to \mathcal{Y}$ a retraction. Then $\mathcal{Y}$ is also exponentiable.*

Proof.— By proposition 3.4.13, a retract of a continuous ∞-category by cocontinuous functors is continuous, so any retract of an exponentiable ∞-topos is exponentiable. $\square$

Lemma 3.4.38.— *Let $\mathcal{X} \to \mathcal{Y}$ be an étale morphism. If $\mathcal{Y}$ is exponentiable, so is $\mathcal{X}$. In particular open subtoposes of $\mathcal{Y}$ are exponentiable.*

Proof.— The ∞-category $\mathcal{S}\mathrm{h}(\mathcal{Y})_{/U}$ is continuous because colimits in the slice ∞-topos can be computed using the projection $\pi_! : \mathcal{Y}_{/U} \to \mathcal{Y}$. $\square$

Corollary 3.4.39.— *Let X be a locally quasi-compact and quasi-separated topological space, then $\mathcal{T}\mathrm{op}(X)$ is an exponentiable ∞-topos.*

Proof.— If X is qcqs, then the frame $\mathcal{O}(X)$ is a retract of $\mathrm{Ind}(\mathcal{O}(X))_{/X}$. Passing to the associated toposes and using the previous two lemmas proves the corollary. $\square$

All the previous lemmas are particular cases of the following proposition.

Proposition 3.4.40.— *Let $f : I \to \mathcal{T}\mathrm{op}$ be a small diagram of exponentiable ∞-toposes. Suppose also that for any arrow $i \to j$ in I, the following square commutes*

$$\begin{array}{ccc} \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X}_j)) & \xrightarrow{\mathrm{Ind}(f_{ij}^*)} & \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X}_i)) \\ {\scriptstyle \beta_j}\uparrow & & {\scriptstyle \beta_i}\uparrow \\ \mathcal{S}\mathrm{h}(\mathcal{X}_j) & \xrightarrow{f_{ij}^*} & \mathcal{S}\mathrm{h}(\mathcal{X}_i) \end{array}$$

Then, if the colimit of f is exponentiable and if I is cofiltered, the limit of f is exponentiable.

Proof.— By sections 6.3.2 and 6.3.3 in [HT], limits and filtered colimits of ∞-categories of sheaves can be computed in $\widehat{\mathcal{C}\mathrm{at}}$. By direct computation,

$$\varprojlim \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X}_i)) \simeq \mathrm{Ind}(\varprojlim \mathcal{S}\mathrm{h}(\mathcal{X}_i))$$

and thanks to the commuting squares we requested, we get a functor

$$\beta : \varprojlim \mathcal{S}\mathrm{h}(\mathcal{X}_i) \to \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X}_i))$$

left adjoint to the evaluation functor, so that $\varprojlim \mathcal{S}\mathrm{h}(\mathcal{X}_i)$ is continuous.

In the same way, if I is cofiltered, then

$$\varinjlim \mathrm{Ind}(\mathcal{S}\mathrm{h}(\mathcal{X}_i)) \simeq \mathrm{Ind}(\varinjlim \mathcal{S}\mathrm{h}(\mathcal{X}_i))$$

and we end up with the same conclusion. $\square$

Remark 3.4.41.— In particular, a colimit of a diagram of exponentiable ∞-toposes with étale maps is exponentiable and the filtered limit of a diagram of proper maps between ∞-toposes with ω-presentable ∞-category of sheaves is exponentiable.

For example, let $G = \varprojlim G_i$ be a profinite group and let $\mathcal{B}G$ be the 1-topos defined by the following limit in the category of 1-toposes,

$$\mathcal{B}G = \varprojlim_{i \in I} \mathcal{B}G_i$$

Then each map $\mathcal{B}G_i \to \mathcal{B}G_{i+1}$ is a proper map of toposes and by proposition 3.4.40, $\mathcal{B}G$ is an exponentiable topos. In particular, for any field k, the étale spectrum of k is an exponentiable topos.

The next question is: when is a subtopos of an exponentiable ∞-topos exponentiable?

Proposition 3.4.42.— *Let $\mathcal{X}$ be an exponentiable ∞-topos and $i : \mathcal{Y} \subset \mathcal{X}$ be a subtopos. If the reflexive localisation $i^* : \mathcal{S}h(\mathcal{X}) \to \mathcal{S}h(\mathcal{Y})$ is ω-accessible, then $\mathcal{Y}$ is exponentiable.*

Proof.— If the right adjoint to i^* is ω-accessible, then $\mathcal{S}h(\mathcal{Y})$ becomes a retract by ω-continuous functors and we conclude with proposition 3.4.13. □

Corollary 3.4.43.— *Let $\mathcal{X} \hookrightarrow \mathcal{Y}$ be a closed subtopos of $\mathcal{Y}$. Suppose $\mathcal{Y}$ is exponentiable, then $\mathcal{X}$ is also exponentiable.*

Finally combining the results we have on open and closed subtoposes, we get the following proposition:

Proposition 3.4.44.— *A locally closed subtopos of an exponentiable ∞-topos is exponentiable.*

Remark 3.4.45.— In the case of topological spaces, we already know that if X is a locally closed subspace of an lqc space, then X is also lqc.

Other constructions:

Proposition 3.4.46.— *Let $f : \mathcal{X} \to \mathcal{Y}$ be a map between two ∞-toposes. Suppose moreover that f is cell-like or that f is essential with $\mathcal{X}$ having trivial $\mathcal{Y}$-shape. In such circumstances, if $\mathcal{X}$ is exponentiable, then $\mathcal{Y}$ is also exponentiable.*

Proof.— In both cases, f^* is fully faithful with a (left or right) adjoint that commutes with filtered colimits. Then apply proposition 3.4.13. □

Remark 3.4.47.— An analogue proposition for topological spaces would be: let $f : X \to Y$ be a map with X lqc. If f is perfect or surjective open, then Y is lqc.

We shall end this list with a specific discussion on localic exponentiable ∞-toposes and locally coherent ∞-toposes.

3.4.6 *Exponentiable localic ∞-toposes*

A locale is exponentiable in the category of locales if and only if it is locally quasi-compact; a localic topos is exponentiable in the category of toposes if and only if the underlying locale is *metastably locally quasi-compact* [Elephant]. What about localic ∞-toposes?

We have already seen in the examples that the ∞-topos associated to an lqc-qs locale is exponentiable. Here, we begin with a necessary condition on the underlying locale of an exponentiable ∞-topos.

Proposition 3.4.48.— *Let $\mathcal{X}$ be an exponentiable ∞-topos, then its underlying locale is locally quasi-compact.*

Proof.— Let $\mathcal{X}$ an exponentiable ∞-topos and let $\mathcal{O}(\mathcal{X})$ be the underlying frame of $\mathcal{S}h(\mathcal{X})$. The embedding $i : \mathcal{O}(\mathcal{X}) \to \mathcal{S}h(\mathcal{X})$ has a left adjoint given by the *support functor* $\sigma : \mathcal{S}h(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$. For every $U \in \mathcal{S}h(\mathcal{X})$, $\sigma(U)$ is the biggest subobject $\sigma(U) \rightarrowtail U$ that is also a subobject $\sigma(U) \rightarrowtail *$ of the terminal object.

We deduce that i commutes with filtered colimits. And we will show that σ also commutes with filtered colimits. For this, we just need remark that for $\{U_I \rightarrowtail *\}_{i\in I}$ a family of subobjects of $*$ and I filtered, $\varinjlim_{i\in I} U_i$ is also a subobject of $*$.

Indeed, the condition $U_i \rightarrowtail *$ is equivalent to $U_i \simeq U_i \times U_i$. Now because colimits are universal in $\mathcal{S}h(\mathcal{X})$ and $I \to I \times I$ is cofinal, we get

$$\varinjlim_{i\in I} U_i \times \varinjlim_{i\in I} U_i \simeq \varinjlim_{i\in I} U_i.$$

The full subcategory $\mathcal{O}(\mathcal{X})$ becomes a retract by ω-continuous functor of a continuous ∞-category, so it is itself a continuous ∞-category by proposition 3.4.13 which means by corollary 3.2.24 that the underlying locale of $\mathcal{X}$ is locally quasi-compact. □

Remark 3.4.49.— The same kind of proof shows that for any $\mathcal{F} \in \mathcal{S}h(\mathcal{X})$, the frame of subobjects of $\mathcal{F}$ is locally quasi-compact.

The necessary conditions given by proposition 3.4.48 and corollary 3.4.39 are quite good; they are the most useful in practice but we can do better. Here is the ∞-versions of the theorem of Johnstone and Joyal [JoJo].

Definition 3.4.50.— *Let* X *be a locale and* U, V *two objects of* $\mathcal{O}(X)$. *Let* $\underline{U}$ *be the constant sheaf associated to* U. *We will note* $U \ll_\infty V$ *if for any filtered diagram* $\{A_i\}_{i\in I}$ *in* $\mathrm{Sh}(X)$ *such that* $\mathrm{colim}_{i\in I} A_i \simeq \underline{V}$, *there exists* $i_0 \in I$ *such that* A_{i_0} *has a section on* U.

Remark.— Note that $U \ll_\infty V \Longrightarrow U \ll V$.

Theorem 3.4.51.— *A localic ∞-topos is exponentiable if and only if its underlying locale X is locally ∞-quasi-compact (or ∞-lqc) i.e if for any $U \in \mathcal{O}(X)$,*

$$\bigcup\{V, V \ll_\infty U\} = U.$$

Lemma 3.4.52.— *Let X be a locale such that $\mathcal{T}op(X)$ is exponentiable, then X is ∞-lqc.*

Proof.— The first thing to remark is that in this case, because the ∞-category $\mathcal{S}h(X)$ is continuous, $U \ll_\infty V$ for $U, V \in \mathcal{O}(X)$ if and only if there exists an arrow

$$\alpha(\underline{U}) \to \beta(\underline{V})$$

in $\mathrm{Ind}(\mathcal{S}h(X))$ (consider $\beta(\underline{V}) \simeq \text{“}\varinjlim_{i\in I}\text{”}A_i$). So we may only test the relation $U \ll_\infty V$ on the colimit $\beta(\underline{V})$.

Let $U \in \mathcal{O}(X)$, the object $\beta(\underline{U})$ of $\mathrm{Ind}(\mathcal{S}h(X))$ can be written as a filtered colimit

$$\beta(\underline{U}) \simeq \text{“}\varinjlim_{i\in I}\text{”}A_i$$

From the proof of proposition 3.4.48, we deduce that

$$U = \bigcup_{i\in I} \sigma(A_i).$$

By the counit of the adjunction $(\underline{\ }) \dashv \sigma$, for every $i \in I$ we have a map $\underline{\sigma(A_i)} \to A_i$ i.e A_i has a section on $\sigma(A_i)$. This means that $\sigma(A_i) \ll_\infty U$ and X is ∞-lqc. □

We are now going to prove that ∞-lqc locales give rise to exponentiable ∞-toposes. For this, we will adapt the proof of [JoJo] to the ∞-setting. The proof is based on building a left adjoint to $\varepsilon : \mathrm{Ind}(\mathcal{S}h(X)) \to \mathcal{S}h(X)$ by hand. Indeed, thanks to proposition 5.2.2.12 in [HT], to build a fully faithful left adjoint $\beta : \mathcal{S}h(X) \to \mathrm{Ind}(\mathcal{S}h(X))$ we only need to find for every $\mathcal{F} \in \mathcal{S}h(X)$ an ind-object $\beta(\mathcal{F})$ with colimit $\mathcal{F}$ having the right property. As ε is nice enough, this is not so complicated.

Let X be an ∞-lqc locale.

Lemma 3.4.53.— *The evaluation map $h\varepsilon : h(\mathrm{Ind}(\mathcal{S}h(X))) \to h(\mathcal{S}h(X))$ is a Cartesian fibration of $\widehat{\mathcal{H}}$-enriched categories.*

Proof.— Let $\text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{F}_i$ be an object of $h(\mathrm{Ind}(\mathcal{S}h(X)))$ and $f : \mathcal{G} \to \mathcal{F} = \varinjlim_{i\in I} \mathcal{F}_i$ be an arrow in $h(\mathcal{S}h(X))$. The first step is to form the fibre products $\mathcal{G}_i = \mathcal{G} \times_{\mathcal{F}} \mathcal{F}_i$. Then because colimits in $\mathcal{S}h(X)$ are universal, $\varinjlim_{i\in I} \mathcal{G}_i \simeq \mathcal{G}$ in $\mathcal{S}h(X)$. Hence we get a map $\varphi : \text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{G}_i \to \text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{F}_i$ such that $\varepsilon(\varphi) \simeq f$.

The last thing we must show is that φ is Cartesian. Let

$$\psi : \text{“}\varinjlim_{j\in J}\text{”}\,\mathcal{R}_i \to \text{“}\varinjlim_{i\in I}\text{”}\,\mathcal{F}_i$$

be any arrow in $h(\mathrm{Ind}(\mathcal{S}h(X)))$ and let $g : \mathcal{R} \to \mathcal{G}$ be such that $h\varepsilon(\varphi)g \simeq h\varepsilon(\psi)$. Thanks to ψ, for every $j \in J$, there is $i_j \in I$ such that $fg : \mathcal{R}_j \to \mathcal{F}$ factorizes through $\mathcal{F}_{i_j}$. But as $\mathcal{G}_{i_j} \simeq \mathcal{G} \times_{\mathcal{F}} \mathcal{F}_{i_j}$ we get a factorization $\mathcal{R}_j \to \mathcal{G}_{i_j} \to \mathcal{F}_{i_j}$ inducing a map $\theta : \text{“}\varinjlim_{j\in J}\text{”}\mathcal{R}_j \to \text{“}\varinjlim_{i\in I}\text{”}G_i$ with $h\varepsilon(\theta) \simeq g$ and $\varphi\theta \simeq \psi$.

If we have another map $\theta' : \text{“}\varinjlim\text{”}\,\mathcal{R}_j \to \text{“}\varinjlim\text{”}\,G_i$ then the above calculation shows that θ and θ' must agree once composed with $\mathcal{R}_j \to \text{“}\varinjlim\text{”}\,\mathcal{R}_j$ for every $j \in J$. So $\theta \simeq \theta'$ and φ is Cartesian. □

As a consequence, to construct β we only need to check that the ∞-category of ind-objects with colimit $\mathcal{F}$ has an initial object for every $\mathcal{F} \in \mathcal{S}h(X)$. In fact, because the sheaves $\underline{U}$ for $U \in \mathcal{O}(X)$ generate $\mathcal{S}h(X)$ under colimits, we only need to build $\beta(\underline{U})$ for all $U \in \mathcal{O}(X)$.

Lemma 3.4.54.— *Let $U \in \mathcal{O}(X)$, then the ind-object*

$$\beta(\underline{U}) = \text{“}\varinjlim_{V \ll_\infty U}\text{”}\,\underline{V}$$

is initial in the ∞-category of ind-objects with colimit U.

Proof.— The ind-object $\beta(\underline{U})$ has colimit U by assumption that X is ∞-lqc. Let $\text{“}\varinjlim\text{”}_{i\in I}\,\mathcal{F}_i$ be another ind-object with colimit U. For every $V \ll_\infty U$ we get a map $\underline{V} \to \mathcal{F}_{i_V}$ so that we have a map $\beta(\underline{U}) \to \text{“}\varinjlim\text{”}_{i\in I}\,\mathcal{F}_i$. Which means $\beta(\underline{U})$ is initial. □

Summing all this we get the last part of the proof of theorem 3.4.51:

Proposition 3.4.55.— *Let X be an ∞-lqc locale, then $\mathcal{T}op(X)$ is exponentiable.*

3.4.7 *Exponentiability vs coherence and having enough points*

A remarkable result about locally compact locales is that they always have enough points. So every lqc locale is spatial. On the other side, the locally coherent locales are all exponentiable. In fact the same is true for ordinary toposes, all coherent toposes are exponentiable. We will discuss these cases in the ∞-world.

The following results on coherent ∞-toposes are taken from [DAG7].

Definition 3.4.56.— *Let $\mathcal{X}$ be an ∞-topos. We say that $\mathcal{X}$ is locally coherent if there exists a finitary site $\mathcal{C}$ together with a morphism $f : \mathcal{X} \to \mathcal{T}op(\mathcal{C})$ such that:*

- *the morphism f induces an equivalence between the hypercomplete ∞-toposes $f^\wedge : \mathcal{X}^\wedge \simeq \mathcal{T}op(\mathcal{C})^\wedge$;*
- *for every* $U \in \mathcal{S}h(\mathcal{X})$, *there exists* $D \in \mathcal{S}h(\mathcal{C})$ *and an effective epimorphism* $f^*U \to D$.

Remark 3.4.57.— The hypercompletion of a locally coherent ∞-topos is locally coherent.

Now if $\mathcal{C}$ is a finitary site, then $\mathcal{S}h(\mathcal{C})$ is a finitely presentable ∞-category so that $\mathcal{T}op(\mathcal{C})$ is locally coherent and exponentiable. But the ∞-category of sheaves of the hypercompletion of a locally coherent ∞-topos may have no ω-compact object except the initial object as showed in counter-example 6.5.4.5 in [HT].

The conclusion is that, at the opposite of what happens in the case of locally coherent toposes, which are all finitely presentable, some locally coherent ∞-topos are not, so we expect that not every locally coherent ∞-topos is exponentiable.

In the other way, the ∞-topos associated to $\mathbb{R}$, for example, is exponentiable but not locally coherent.

The same counter-example from [HT] gives us an exponentiable ∞-topos which is *not hypercomplete*. From [Sc] we know that every exponentiable locale has enough points, so the situation here is quite different. The reader that would like an example of a non-exponentiable ∞-topos with enough points can check that the ∞-topos associated to $\mathbb{R}^\infty$ does the job.

3.5 COENDS FOR ∞-CATEGORIES

As a prerequisite for the study of dualisable objects in $\widehat{\mathcal{C}at}^{cc}$ and the ∞-category of Leray sheaves, we must develop the theory of coends in the ∞-setting. Traditional references on ends and coends for categories include [Ke, MacL]. We also like [Up] for a short introduction. The beginning of the theory of coends for quasi-categories has been developed in [Cr, Gl]. Cranch develops the definition of dinatural transformations between bifunctors and proves it extends the usual definition for categories, while Glasman proves that the space of natural transformations can be written as an end.

The price to pay when developing the theory of coends in full abstractness is that we have to work with dinatural transformations. Those transformations are not very handy and they do not often appear in mathematics.

Instead, the tensor product of ∞-category of presheaves allow us to develop the theory of coends in a straightforward way. It also doesn't depend on a particular model for ∞-categories. We will be able to prove all the fundamental lemmas used in everyday life, like the co-Yoneda lemma.

Warning! In this section we use the following fact: for a small ∞-category C and a cocomplete ∞-category $\mathcal{D}$,

$$[C^{op}, \mathcal{D}] \simeq \mathcal{P}(C) \otimes \mathcal{D}$$

This is easily proven when $\mathcal{D}$ is smally presentable by proposition 3.3.18. In the general case, we need to know that $\mathcal{P}(C)$ is dualisable. This will be shown in the next section in theorem 3.6.12. To avoid a loop in the proofs, here is the safe path: all coend formulas are true when the target ∞-category is $\mathcal{S}$. This allows us to prove theorem 3.6.12, which in return allows us to prove all coend formulas for any cocomplete target ∞-category $\mathcal{D}$.

3.5.1 *Definition and first properties*

Definition 3.5.1.— *Let C be a small ∞-category, the extension of the map functor*

$$\mathrm{Map}_C : C^{op} \times C \to \mathcal{S}$$

to $\mathcal{P}(C^{op} \times C)$ is called the coend functor and is denoted

$$\int_C : \mathcal{P}(C^{op} \times C) \to \mathcal{S}$$

Remark.— By definition $\int_C$ is cocontinuous.

Proposition 3.5.2.— *The functor $\mathcal{P}(C^{op}) \times \mathcal{P}(C) \to \mathcal{S}$ defined by*

$$(F, G) \mapsto \int_{c \in C} F(c)G(c)$$

is cocontinuous in each variable.

Proof.— This functor is the composition of the cocontinuous functor $\int_C$ with the canonical map $\mathcal{P}(C^{op}) \times \mathcal{P}(C) \to \mathcal{P}(C^{op}) \otimes \mathcal{P}(C) \simeq \mathcal{P}(C^{op} \times C)$ which is cocontinuous in each variable. □

Thanks to the tensor product it is possible to extend the definition of the coend to bimodules with values in any cocomplete ∞-category $\mathcal{D}$.

Proposition-Definition 3.5.3.— *Let C be a small ∞-category and $\mathcal{D}$ be a cocomplete ∞-category. Then the coend functor induces a cocontinuous functor*

$$\int_C : [C \times C^{op}, \mathcal{D}] \longrightarrow \mathcal{D}$$

still called the coend functor.

Proof.— The map is obtained by tensoring with $\mathrm{Id}_{\mathcal{D}}$. We then have a cocontinuous functor

$$\int_{\mathrm{C}} \otimes \mathrm{Id}_{\mathcal{D}} : \mathcal{P}(\mathrm{C}^{op} \times \mathrm{C}) \otimes \mathcal{D} \to \mathcal{D}.$$

But the tensor product $\mathcal{P}(\mathrm{C}^{op} \times \mathrm{C}) \otimes \mathcal{D}$ is canonically equivalent to the ∞-category $[\mathrm{C} \times \mathrm{C}^{op}, \mathcal{D}]$. □

Remark 3.5.4.— The functor $\mathcal{P}(\mathrm{C}^{op}) \times [\mathrm{C}^{op}, \mathcal{D}] \to \mathcal{D}$ defined by

$$(\mathrm{F}, \mathrm{G}) \mapsto \int_{c \in \mathrm{C}} \mathrm{F}(c) \otimes \mathrm{G}(c)$$

where $\otimes$ denotes the canonical tensoring of $\mathcal{D}$ over $\mathcal{S}$, is cocontinuous in each variable.

3.5.2 *Fubini formula*

An immediate consequence of proposition 3.5.3 is the *parametrized* version of the coend. Namely, let D be a small ∞-category and let $\mathcal{E}$ be a cocomplete ∞-category, then we have a the coend functor

$$\int_{\mathrm{C}} : [\mathrm{C}^{op} \times \mathrm{C}, [\mathrm{D}, \mathcal{E}]] \to [\mathrm{D}, \mathcal{E}]$$

Because $[\mathrm{C}^{op} \times \mathrm{C}, [\mathrm{D}, \mathcal{E}]]$ is equivalent to $[\mathrm{C}^{op} \times \mathrm{C} \times \mathrm{D}, \mathcal{E}]$, for any functor $\mathrm{F} : \mathrm{C}^{op} \times \mathrm{C} \times \mathrm{D} \to \mathcal{E}$, we have the coend

$$\int_{c \in \mathrm{C}} \mathrm{F}(c, c, -)$$

which is functorial in the last parameter.

This allows us to state the Fubini formula for coends.

Proposition 3.5.5 (Fubini).— *Let* C *and* D *be two small ∞-categories and $\mathcal{E}$ be a cocomplete ∞-category. For any functor $\mathrm{C}^{op} \times \mathrm{C} \times \mathrm{D}^{op} \times \mathrm{D} \to \mathcal{E}$ we have*

$$\int_{c \in \mathrm{C}} \int_{d \in \mathrm{D}} \mathrm{F}(c, c, d, d) \simeq \int_{d \in \mathrm{D}} \int_{c \in \mathrm{C}} \mathrm{F}(c, c, d, d)$$

Proof.— Thanks to the equivalences

$$[\mathrm{C}^{op} \times \mathrm{C} \times \mathrm{D}^{op} \times \mathrm{D}, \mathcal{E}] \simeq \mathcal{P}(\mathrm{C} \times \mathrm{C}^{op}) \otimes [\mathrm{D}^{op} \times \mathrm{D}, \mathcal{E}] \simeq \mathcal{P}(\mathrm{D} \times \mathrm{D}^{op}) \otimes [\mathrm{C}^{op} \times \mathrm{C}, \mathcal{E}]$$

the following coend square commutes

$$\begin{tikzcd}
{[\mathrm{C}^{op} \times \mathrm{C} \times \mathrm{D}^{op} \times \mathrm{D}, \mathcal{E}]} \arrow[r, "\int_{\mathrm{D}}"] \arrow[d, "\int_{\mathrm{C}}"] & \mathcal{P}(\mathrm{C} \times \mathrm{C}^{op}) \otimes \mathcal{E} \arrow[d, "\int_{c}"] \\
\mathcal{P}(\mathrm{D} \times \mathrm{D}^{op}) \otimes \mathcal{E} \arrow[r, "\int_{\mathrm{D}}"] & \mathcal{E}
\end{tikzcd}$$

□

3.5.3 *Co-Yoneda lemma*

The Yoneda lemma states that for any small ∞-category C and any functor $F : C \to \mathcal{S}$, we have

$$\mathrm{Nat}([-, c]_C, F) \simeq F(c)$$

for $c \in C$. This can be rewritten with the help of ends — that we do not develop here — as

$$F(c) \simeq \prod_{d \in C} [[d, c]_C, F(d)]]_{\mathcal{S}}$$

The co-Yoneda lemma is the dual formula using coends.

Proposition 3.5.6 (co-Yoneda).— *Let* C *be a small* ∞*-category,* $\mathcal{D}$ *be a cocomplete* ∞*-category and* $F : C \to \mathcal{D}$ *be any functor. Note* $\otimes$ *the canonical tensoring of* $\mathcal{D}$ *over* $\mathcal{S}$*. Then for any* $c \in C$,

$$F(c) \simeq \int_{d \in C} [d, c] \otimes F(d)$$

To prove the proposition, we will need a new definition.

Definition 3.5.7.— *For a functor* $F : C^{op} \to \mathcal{S}$*, its* ∞*-category of elements is the subcategory* $C_{/F} \subset \mathcal{P}(C)_{/F}$ *given by the Yoneda embedding* $C \subset \mathcal{P}(C)$. *It is denoted* $\mathrm{el}(F)$.

Remark 3.5.8.— For any functor $F : C^{op} \to \mathcal{S}$, the nice property of $\mathrm{el}(F)$ is that in $\mathcal{P}(C)$,

$$\varinjlim_{c \in \mathrm{el}(F)} c \simeq F$$

Proof of the co-Yoneda lemma.— Let's prove the case where $\mathcal{D} = \mathcal{S}$. Let $F : C \to \mathcal{S}$ be any functor and let $y : C^{op} \to \mathcal{P}(C^{op})$ be the Yoneda embedding.

By cocontinuity of the coend functor, we have for $c \in C$,

$$\int_{d \in C} [d, c] F(d) \simeq \varinjlim_{x \in \mathrm{el}(F)} \int_{d \in C} [d, c] x(d)$$

But by definition of the coend functor

$$\int_{d \in C} [d, c] x(d) \simeq x(c)$$

And the formula is proved

$$\int_{d \in C} [d, c] F(d) \simeq \varinjlim_{x \in \mathrm{el}(F)} x(c) \simeq F(c)$$

Hence the functor $F \mapsto \int_{d\in C}[d,-]F(d)$ is homotopic to the identity. Tensoring it with the identity of $\mathcal{D}$, we obtain an endofunctor of $[C,\mathcal{D}]$

$$F \mapsto \int_{d\in C} [d,-] \otimes F(d)$$

homotopic to the identity, which proves the formula. □

3.5.4 *Left Kan extensions as coends*

When a bimodule $C^{op} \times C \to \mathcal{D}$ is given by the tensor product of two functors, the coend is easily expressible in terms of colimits. In return, we are able to calculate left Kan extension along the Yoneda embedding with the coend functor.

Theorem 3.5.9.— *Let* C *be a small* ∞*-category ,* $\mathcal{D}$ *a cocomplete* ∞*-category ,* G *an object of* $\mathcal{P}(C)$ *and* $F : C \to \mathcal{D}$ *any functor. Then*

$$\int_{c\in C} W(c) \otimes F(c) \simeq \varinjlim_{d\in \mathrm{el}(G)} F(d).$$

Proof.— The functor $\mathcal{P}(C) \times [C,\mathcal{D}] \to [C^{op} \times C,\mathcal{D}]$ sending (W,F) to $W \otimes F$ is cocontinuous in the first variable and the coend functor is cocontinuous. We then have

$$G \simeq \varinjlim_{d\in \mathrm{el}(G)} [-,d] \quad \text{implies}$$

$$\begin{aligned}
\int_{c\in C} G(c) \otimes F(c) &\simeq \int_{c\in C} \left(\varinjlim_{d\in \mathrm{el}(W)} [c,d] \right) \otimes F(c) \\
&\simeq \int_{c\in C} \varinjlim_{d\in \mathrm{el}(G)} [c,d] \otimes F(c) \\
&\simeq \varinjlim_{d\in \mathrm{el}(G)} \int_{c\in C} [c,d] \otimes F(c) \\
&\simeq \varinjlim_{d\in \mathrm{el}(G)} F(d) \qquad \text{(coYoneda)}
\end{aligned}$$

□

Corollary 3.5.10.— *Let* C *be a small* ∞*-category and* $\mathcal{D}$ *be a cocomplete* ∞*-category ,* $F : C \to \mathcal{D}$ *be any functor. Then the left Kan extension of* F *along the inclusion* $i : C \to \mathcal{P}(C)$ *is given by*

$$\mathrm{Lan}_i F : G \mapsto \int_{c\in C} G(c) \otimes F(c)$$

Proof.— For any functor $G \in \mathcal{P}(C)$, write $\mathrm{el}(G)$ for its ∞-category of elements. Then

$$G \simeq \varinjlim_{c \in \mathrm{el}(G)} c$$

in $\mathcal{P}(C)$, so the left Kan extension is given by

$$(\mathrm{Lan}_i F)(G) \simeq \varinjlim_{c \in \mathrm{el}(G)} F(c).$$

Then apply theorem 3.5.9. □

3.5.5 *Ends*

As the theory of limits is dual to that of colimits, the theory of ends is dual to that of coends.

Definition 3.5.11.— *Let* D *be a small* ∞*-category and* $\mathcal{C}$ *be a complete* ∞*-category. The opposite of*

$$\int^{D^{op}} : [D \times D^{op}, \mathcal{C}^{op}] \to \mathcal{C}^{op}$$

is called the end and is denoted by

$$\prod_{D} : [D^{op} \times D, \mathcal{C}] \to \mathcal{C}$$

Remark.— We see right away that $\prod_D$ is a continuous functor.

Remark 3.5.12.— Let $\mathcal{C}$ be a complete ∞-category and let $(K, c) \in \mathcal{S} \times \mathcal{C}$. Then the cotensor c^K defined by

$$c^K = \varprojlim_{K} c$$

corresponds to the tensor $K \otimes c$ in $\mathcal{C}^{op}$. This will allow us to dualise all the properties of the coend.

Notice that the functor

$$\begin{aligned} \mathcal{S}^{op} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (K, c) &\longmapsto c^K \end{aligned}$$

is continuous in each variable.

We list here all the properties of ends obtained by dualising the properties of coends.

Proposition 3.5.13 (Parametrised end).— *Let* C *and* D *be two small* ∞*-categories and let* $\mathcal{E}$ *be a complete* ∞*-category. Then for any functor* $F : C^{op} \times C \times D \to \mathcal{E}$*, the coend*

$$\prod_{c \in C} F(c, c, -)$$

is functorial in the last parameter.

Proposition 3.5.14 (Fubini).— *Let* C *and* D *be two small* ∞*-category and* $\mathcal{E}$ *be a complete* ∞*-category. For any functor* $F : C^{op} \times C \times D^{op} \times D \to \mathcal{E}$, *we have*

$$\prod_{c \in C} \prod_{d \in D} F(c, c, d, d) \simeq \prod_{d \in D} \prod_{c \in C} F(c, c, d, d)$$

Proposition 3.5.15 (Yoneda lemma).— *Let* C *be a small* ∞*-category,* $\mathcal{D}$ *be a complete* ∞*-category and* $F : C \to \mathcal{D}$ *be any functor. Then for any* $c \in C$,

$$F(c) \simeq \prod_{d \in C} F(d)^{[c,d]}$$

Proposition 3.5.16 (Right Kan extensions).— *Let* C *be a small* ∞*-category and* $\mathcal{D}$ *be a complete* ∞*-category. Then for any* $F : C \to \mathcal{D}$, *the right Kan extension of* F *along* $i : C \hookrightarrow \mathcal{P}(C^{op})^{op}$ *is given by*

$$\mathrm{Ran}_i F(G) \simeq \prod_{c \in C} F(c)^{G(c)}$$

We finish with the famous formula between natural transformations and ends.

Corollary 3.5.17.— *Let* C *be a small* ∞*-category and* $F, G \in \mathcal{P}(C^{op})$. *Then*

$$\mathrm{Nat}(F, G) \simeq \prod_{c \in C} \mathrm{Map}_{\mathcal{S}}(F(c), G(c))$$

Remark 3.5.18.— The analogue formula for $F, G \in [C, \mathcal{D}]$ with $\mathcal{D}$ a complete ∞-category, can be obtained by tensoring with $\mathcal{D}^{op}$, using the fact that $[C, \mathcal{D}] \simeq (\mathcal{P}(C) \otimes \mathcal{D}^{op})^{op}$.

3.5.6 *Adjunction between ends and coends*

Let C be a small ∞-category and $w : C^{op} \times C \to \mathcal{S}$ a bimodule. Let $\mathcal{D}$ be a bicomplete ∞-category and for $F \in [C^{op}, \mathcal{D}]$ define

$$[w, F](c) = \prod_{c \in C^{op}} F(d)^{w(d,c)}$$

Thanks to proposition 3.5.13, the construction $[w, -]$ is an endofunctor of $[C^{op}, \mathcal{D}]$.

In the same way, we get a functor $w \otimes -$ defined by

$$(w \otimes F)(c) = \int^{d \in C^{op}} w(c, d) F(d)$$

Proposition 3.5.19.— *The functor* $w \otimes -$ *is left adjoint to* $[w, -]$.

Proof.— Let $F, G \in [C^{op}, \mathcal{D}]$, then using the characterisation of left/right Kan extensions, Fubini and the adjunction between tensors and cotensors, we get in $\widehat{\mathcal{H}}$,

$$\begin{aligned}
\mathrm{Nat}(w \otimes F, G) &\simeq \prod_{c \in C^{op}} \mathrm{Map}_{\mathcal{D}}((w \otimes F)(c), G(c)) \\
&\simeq \prod_{c \in C^{op}} \mathrm{Map}_{\mathcal{D}}\left(\int^{d \in C^{op}} w(c,d) \otimes F(d), G(c)\right) \\
&\simeq \prod_{c \in C^{op}} \prod_{d \in C^{op}} \mathrm{Map}_{\mathcal{D}}(F(d), G(c)^{w(c,d)}) \\
&\simeq \prod_{d \in C^{op}} \mathrm{Map}_{\mathcal{D}}(F(d), [w, G](d)) \\
&\simeq \mathrm{Nat}(F, [w, G])
\end{aligned}$$

□

3.6 DUALISABILITY OF THE ∞-CATEGORY OF STABLE SHEAVES

The theory of smally presentable stable ∞-categories is very similar to the one of abelian groups or vector spaces. Finite limits/colimits behave like sums in a vector space and the small generation condition is similar to a "finite presentation" condition.

We can also think of smally presentable stable ∞-categories as an 'abelianised version' of ∞-toposes . See proposition 3.6.6.

In this section, we will see that dualisability is the 'abelian conterpart' to exponentiability. Namely, given any exponentiable ∞-topos $\mathcal{X}$, its stabilization $\mathcal{S}h(\mathcal{X}) \otimes \mathrm{Sp} \simeq \mathcal{S}h(\mathcal{X}, \mathrm{Sp})$ is dualisable.

3.6.1 *Stabilisation for cocomplete ∞-categories*

We shall recall the definition of stable ∞-categories and the stabilization functor and their very first properties. Please refer to [HA].

Definition 3.6.1.— *An ∞-category $\mathcal{C}$ is stable if*

- *$\mathcal{C}$ has finite limits and colimits;*
- *$\mathcal{C}$ has a zero object;*
- *Cartesian diagrams in $\mathcal{C}$ are also cocartesian and vice-versa.*

Remark 3.6.2.— The axioms imply that the loop functor $\Sigma : \mathcal{C} \to \mathcal{C}$ defined by the pushout diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X.
\end{tikzcd}$$

is invertible. Its inverse is the loop functor Ω, defined by

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X. \end{tikzcd}$$

In general those two are only adjoint and it can be shown that requiring Ω and Σ to be inverse implies the stability of the ∞-category.

Proposition 3.6.3.— *Let $\mathcal{C}$ be an ∞-category with finite colimits and a zero object. Then $\mathcal{C}$ is stable if and only if the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is invertible.*

Definition 3.6.4.— *The ∞-category of spectra* Sp *is the inverse limit of the tower*

$$\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \dots$$

where $\mathcal{S}_*$ is the ∞-category of pointed spaces and Ω is the loop functor.

Remark.— The ∞-category of spectra can alternatively be defined as the colimit inside $\mathcal{P}\mathrm{r}$ of

$$\dots \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \dots$$

Proposition 3.6.5.— *The ∞-category of spectra is stable and finitely presentable.*

Proposition 3.6.6.— *An ∞-category $\mathcal{C}$ is smally presentable and stable if and only if there exists a small ∞-category* I *and an exact and accessible reflexive localisation*

$$[\mathrm{I}, \mathrm{Sp}] \longrightarrow \mathcal{C}$$

Definition 3.6.7.— *The very large ∞-category $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\mathrm{st}}$ of cocomplete stable large ∞-categories is the full subcategory of $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\infty}$ whose objects are the cocomplete stable large ∞-categories .*

Remark.— While functors between cocomplete ∞-categories shall be cocontinuous functors, functors between stable ∞-categories should preserve finite limits and colimits. Fortunately, any cocontinuous functor between stable ∞-categories preserves finite limits.

Theorem 3.6.8.— *The ∞-category $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\mathrm{st}}$ inherits a closed symmetric monoidal structure from the one of $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\infty}$. Furthermore, the inclusion functor* $\mathrm{oblv} : \widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\mathrm{st}} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\infty}$ *has a left adjoint, the stabilization functor:*

$$\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C}) = \mathcal{C} \otimes \mathrm{Sp}$$

making $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\mathrm{st}}$ a symmetric monoidal reflexive localisation of $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\infty}$.

Remark.— The ∞-category Sp is the unit of $\widehat{\mathcal{C}\mathrm{at}}{}^{cc}_{\mathrm{st}}$ and the inclusion oblv commutes with (non-empty) tensor products.

3.6.2 *Dualisability in* $\widehat{\mathcal{C}\mathrm{at}}^{cc}$

We start by recalling the notion of dualisable object in a symmetric monoidal category $(\mathcal{C}, \otimes)$, see chapter 4.6.1 of [HA].

Definition 3.6.9.— *An object* X *of* $\mathcal{C}$ *is dualisable if there exists another object* $X^\vee \in C$ *with two maps*

$$\eta : 1_\mathcal{C} \to X \otimes X^\vee; \ \varepsilon : X^\vee \otimes X \to 1_\mathcal{C}.$$

where $1_\mathcal{C}$ *is the unit of* $\mathcal{C}$*, such that the composite maps*

$$X \xrightarrow{\eta\otimes \mathrm{Id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{Id}\otimes\varepsilon} X$$

$$X^\vee \xrightarrow{\mathrm{Id}\otimes\eta} X^\vee \otimes X \otimes X^\vee \xrightarrow{\varepsilon\otimes\mathrm{Id}} X^\vee$$

are homotopic to the identities on X *and* $X^\vee$ *respectively.*

Proposition 3.6.10.— *In the case where* $\mathcal{C}$ *is a closed symmetric monoidal category, a dualisable object* X *has its dual given by* $X^\vee = [X, 1_\mathcal{C}]$ *where* $[-,-]$ *is the internal hom associated to the monoidal structure.*

Proposition 3.6.11.— *In a closed symmetric monoidal category, any retract of a dualisable object is dualisable.*

Proof.— Let $r : X \to Y$ be a retraction with X a dualisable object and let $s : Y \to X$ be a section. Set $Y^\vee = [Y, 1_\mathcal{C}]$ an let's show that $Y^\vee$ has the right property. Because $r : X \to Y$ is a retraction, the same is true for $s^\vee : X^\vee \to Y^\vee$.

We are then supplied with maps

$$\eta_Y : 1_\mathcal{C} \xrightarrow{\eta_X} X \otimes X^\vee \xrightarrow{r\otimes s^\vee} Y \otimes Y^\vee$$

$$\varepsilon_Y : Y^\vee \otimes Y \xrightarrow{r^\vee\otimes s} X^\vee \otimes X \xrightarrow{\varepsilon_X} 1_\mathcal{C}$$

The composition $(\mathrm{Id}_Y \otimes \varepsilon_Y) \circ (\eta_Y \otimes \mathrm{Id}_X) : Y \to Y$ is then a retract of Id_X, hence homotopic to the identity itself. The same is true for the other composition. □

It will be very useful to understand which objects in $\widehat{\mathcal{C}\mathrm{at}}^{cc}_\infty$ are dualisable, to get some intuition on why $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathrm{Sp}$ should be dualisable and to actually prove it.

Proposition 3.6.12.— *The presheaves* ∞*-categories* $\mathcal{P}(D)$ *with* D *a small* ∞*-category and their retracts are dualisable objects of* $\widehat{\mathcal{C}\mathrm{at}}^{cc}_\infty$*.*

Proof.— Let D be a small ∞-category , then if $\mathcal{P}(\mathrm{D})$ has a dual, it has to be $\mathcal{P}(\mathrm{D}^{op})$, so let's introduce $\mathcal{P}(\mathrm{D}^{op} \times \mathrm{D})$ the ∞-category of bimodules on D; we have $\mathcal{P}(\mathrm{D}^{op}) \otimes \mathcal{P}(\mathrm{D}) \simeq \mathcal{P}(\mathrm{D}^{op} \times \mathrm{D})$.

Then let η be the cocontinuous functor

$$\eta : \mathcal{S} \to \mathcal{P}(\mathrm{D}^{op} \times \mathrm{D})$$

sending the point $* \in \mathcal{S}$ to the map-bimodule $[-,-]_{\mathrm{D}}$. And finally, let

$$\varepsilon : \mathcal{P}(\mathrm{D}^{op} \times \mathrm{D}) \to \mathcal{S}$$

be the coend functor.

The composition $(\mathrm{Id}\otimes\varepsilon)(\eta\otimes\mathrm{Id})$ corresponds to the co-Yoneda lemma for presheaves:

$$\int_{b\in \mathrm{D}} \mathrm{F}(b)[a,b] = \mathrm{F}(a)$$

for a functor $\mathrm{F} \in \mathcal{P}(\mathrm{D})$.

The sister formula comes from

$$\int_{a\in \mathrm{D}} [a,b]\mathrm{F}(a) = \mathrm{F}(b)$$

for a functor $\mathrm{F} \in \mathcal{P}(\mathrm{D}^{op})$.

Because a retract of a dualisable object is dualisable by proposition 3.6.11, we are done for the first half.

□

Remark 3.6.13.— There is a usefull analogy between cocomplete ∞-categories and commutative monoids. The dual of a free commutative monoid $\mathbb{N}^n$ is $\mathbb{N}^n$. We then have $\mathbb{N}^n \otimes \mathbb{N}^n \simeq \mathrm{M}_n(\mathbb{N})$ and $\varepsilon : \mathrm{M}_n(\mathbb{N}) \to \mathbb{N}$ is the trace map. In parallel, the ∞-category of bimodules $\mathcal{P}(\mathrm{D}^{op} \times \mathrm{D})$ can be thought as the 'matrices with coefficient in $\mathcal{S}$ and indexed by D' and the coend is as a trace map; the bimodule $[-,-]$ is nothing but the 'identity matrix'.

We have the following theorem: 'dualisable commutative monoids are the retracts of free commutative monoids of finite type', hence we conjecture that the 'dualisable cocomplete ∞-categories are precisely the retracts of the free cocomplete ∞-categories with a small ∞-category of generators'.

It would make sense to coin the name *free cocomplete ∞-category* for ∞-categories of the form $\mathcal{P}(\mathcal{D})$ for any ∞-category $\mathcal{D}$. Only when $\mathcal{D}$ is a small ∞-category does $\mathcal{P}(\mathcal{D})$ coincide with a presheaf ∞-category. It would also make sense to call *projective* the retracts of free cocomplete ∞-categories. The conjecture would then become: 'the dualisable objects of $\widehat{\mathcal{C}\mathrm{at}}^{cc}$ are the smally generated and projective ∞-categories'.

3.6.3 *Dualisability of stable sheaves*

The ∞-category of sheaves of an exponentiable ∞-topos is not dualisable in general in $\widehat{\mathcal{C}\mathrm{at}}^{cc}_{\infty}$. This comes from the fact that in general, an ∞-category of ind-objects is not dualisable either. This changes in the ∞-category of presentable stable ∞-categories .

Proposition 3.6.14.— *Let* D *be a small ∞-category with finite colimits. Then* $\mathrm{Ind}(\mathrm{D}) \otimes \mathrm{Sp}$ *is a dualisable object of* $\widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{st}}$.

Proof.— Let $\mathrm{Stab} : \mathcal{P}\mathrm{r} \to \widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{st}}$ be the stabilization functor. It is symmetric monoidal and hence sends dualisable objects to dualisable objects. This means that for any small ∞-category D, $\mathcal{P}(\mathrm{D}) \otimes \mathrm{Sp}$ is a dualisable object of $\widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{st}}$.

If moreover D has small colimits, then by proposition 3.3.18 the ∞-category $\mathrm{Ind}(\mathrm{D}) \otimes \mathrm{Sp}$ is equivalent to the ∞-category of left exact functors $[\mathrm{D}^{op}, \mathrm{Sp}]_{lex}$ and $\mathcal{P}(\mathrm{D}) \otimes \mathrm{Sp}$ is equivalent to $[\mathrm{D}^{op}, \mathrm{Sp}]$.

Because Sp is stable and colimits in functor ∞-categories are computed pointwise, the embedding

$$[\mathrm{D}^{op}, \mathrm{Sp}]_{lex} \hookrightarrow [\mathrm{D}^{op}, \mathrm{Sp}]$$

commutes with all limits and colimits. It then has a left adjoint such that $\mathrm{Ind}(\mathrm{D}) \otimes \mathrm{Sp}$ is a retract in $\widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{st}}$ of $\mathcal{P}(\mathrm{D}) \otimes \mathrm{Sp}$. And by proposition 3.6.11, any retract of a dualisable object is dualisable. □

Definition 3.6.15.— *Let* $\mathcal{P}\mathrm{r}_{\mathrm{St}}$ *be the full subcategory of* $\widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{St}}$ *made of smally presentable stable ∞-categories .*

Remark.— The inclusion $\mathcal{P}\mathrm{r}_{\mathrm{St}} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{St}}$ is symmetric monoidal and $\mathcal{P}\mathrm{r}_{\mathrm{St}}$ is closed symmetric monoidal.

Theorem 3.6.16.— *Let* $\mathcal{X}$ *be an exponentiable ∞-topos, then* $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathrm{Sp}$, *the ∞-category of stable sheaves on* $\mathcal{X}$, *is a dualisable object of* $\mathcal{P}\mathrm{r}_{\mathrm{St}}$.

Proof.— Take a standard presentation of $\mathcal{S}\mathrm{h}(\mathcal{X})$:

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{D}) \arrow[r, "\varepsilon" description] & \mathcal{S}\mathrm{h}(\mathcal{X}) \arrow[l, shift right=2, "\beta"'] \arrow[l, shift left=2, "\alpha"] \end{tikzcd}$$

After tensoring by Sp we get a retraction in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$:

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{D}) \otimes \mathrm{Sp} \arrow[r, shift right, "\varepsilon'"'] & \mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathrm{Sp} \arrow[l, shift right, "\beta'"'] \end{tikzcd}$$

Thanks to proposition 3.6.14, we know that $\mathrm{Ind}(\mathrm{D}) \otimes \mathrm{Sp}$ is in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$ and is dualisable in $\widehat{\mathcal{C}\mathrm{at}}^{cc}_{\mathrm{St}}$ so it is dualisable in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$ and $\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathrm{Sp}$ as a retract of a dualisable object is also dualisable by proposition 3.6.11.

□

3.7 LERAY SHEAVES

When the topological space X is locally quasi-compact and Hausdorff, the target category $\mathcal{C}$ is bicomplete and filtered colimits in $\mathcal{C}$ are left exact, the category of $\mathcal{C}$-valued sheaves on X has two equivalent descriptions:

- as continuous functors from $(\mathcal{O}(X))^{op}$ to $\mathcal{C}$;
- or as functors from the opposite category of compact subsets of X to $\mathcal{C}$ preserving finite limits and some filtered colimits, that we call Leray sheaves.

We wish to prove that, more generally the category of $\mathcal{C}$-valued sheaves on an exponentiable ∞-topos can be described with small colimits and finite limits condition instead of small limits conditions.

3.7.1 *Leray sheaves of spaces*

Given an exponentiable ∞-topos $\mathcal{X}$, as $\mathcal{S}h(\mathcal{X})$ is a continuous ∞-category we have a standard presentation:

$$\begin{tikzcd} \mathrm{Ind}(D) \arrow[r, "\varepsilon" description] & \mathcal{S}h(\mathcal{X}) \arrow[l, bend right, "\beta"'] \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

We then obtain a continuous monad $M = \varepsilon\alpha$ and a cocontinuous comonad $W = \beta\varepsilon$ on $\mathrm{Ind}(D)$. Here is the relations they satisfy:

$$W \dashv M,\ M^2 \simeq M,\ W^2 \simeq W,\ MW \simeq W,\ WM \simeq M.$$

Moreover, by the fully faithfulness of α and β, the ∞-category of fixed points of M and W are equivalent to $\mathcal{S}h(\mathcal{X})$.

All these information can be summarised in the following diagram of equivalences of ∞-category :

$$\begin{tikzcd} & \mathcal{S}h(\mathcal{X}) \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ \mathrm{Fix}(W) \arrow[rr, shift right, "M"'] & & \mathrm{Fix}(M) \arrow[ll, shift right, "W"'] \end{tikzcd}$$

where $\mathrm{Fix}(W)$ is the ∞-category of objects $\mathcal{F} \in \mathrm{Ind}(D)$ such that the canonical morphism $W(\mathcal{F}) \to \mathcal{F}$ is an equivalence and $\mathrm{Fix}(M)$ is the ∞-category of objects such that $\mathcal{F} \to M(\mathcal{F})$ is an equivalence.

We now wish to give a different description of $\mathcal{S}h(\mathcal{X})$ by mean of $\mathrm{Fix}(W)$. But first we need to pin down some definitions.

Definition 3.7.1.— *The idempotent comonad* $W : \mathrm{Ind}(D) \to \mathrm{Ind}(D)$ *is cocontinuous, we write*

$$w : D \nrightarrow D$$

for the corresponding bimodule. That is $w(a, b) = \mathrm{Map}(a, \beta\varepsilon b)$. An object of $w(a, b)$ will be called a wavy arrow and will be denoted

$$a \rightsquigarrow b.$$

Remark.— Wavy arrows were initially defined in [JoJo]. Notice that our definition is slightly different as wavy arrows are not directly defined on $\mathcal{S}\mathrm{h}(\mathcal{X})$. The two definitions differ only by a composition by α.

Remark 3.7.2.— Because W is an idempotent comonad, the comultiplication $\mathrm{W} \simeq \mathrm{W}^2$ allows us to compose wavy arrows: from two wavy arrows $a \rightsquigarrow b$ and $b \rightsquigarrow c$ we get a new one $a \rightsquigarrow c$ given by composition $a \to \mathrm{W}b \to \mathrm{W}^2 c \simeq \mathrm{W}c$.

Let $\mathrm{W} \to \mathrm{Id}$ be the counit of the comonad W, then every wavy arrow $a \rightsquigarrow b$ has an underlying 'straight' arrow $a \to b$ given by composition $a \to \mathrm{W}b \to b$. This operation is compatible with composition.

Theorem 3.7.3.— *Let $\mathcal{E} = \mathcal{S}\mathrm{h}(\mathcal{X})$ be an exponentiable ∞-topos together with a standard presentation. Then $\mathcal{S}\mathrm{h}(\mathcal{X})$ is canonically equivalent to the ∞-category of left exact functors $\mathcal{F} : \mathrm{D}^{op} \to \mathcal{S}$ satisfying the condition*

$$\mathcal{F}(a) \simeq \int_{b \in \mathrm{D}} w(a, b)\mathcal{F}(b)$$

for all $a \in \mathrm{D}$.

Remark.— The coend formula can be rewritten as

$$\mathcal{F}(a) \simeq \varinjlim_{a \rightsquigarrow b} \mathcal{F}(b)$$

where the index ∞-category is the ∞-category of elements of $w(a, -)$.

Proof.— Let $i : \mathrm{Ind}(\mathrm{D}) \to \mathcal{P}(\mathrm{D})$ be the canonical embedding and write $w_!$ for the left Kan extension of $w : \mathrm{D} \to \mathcal{P}(\mathcal{D})$ along $\mathrm{D} \to \mathcal{P}(\mathrm{D})$. That is for $\mathcal{F} : \mathrm{D}^{op} \to \mathcal{S}$ we have

$$w_!\mathcal{F} = \int_{b \in \mathrm{D}} w(-, b)\mathcal{F}(b).$$

Now suppose $\mathcal{F}$ is a left exact functor, then we claim that $w_! i\mathcal{F} \simeq i\mathrm{W}\mathcal{F}$. Indeed, the comonad W is cocontinuous, hence it coincides with the left Kan extension of its own restriction to D. Furthermore, the embedding i commutes with filtered colimits and D generates $\mathrm{Ind}(\mathrm{D})$ under filtered colimits, hence $i\mathrm{W}$ is also a left Kan extension of its restriction to D.

The next step is to show that the two functors $w_!$ and $i\mathrm{W}$ coincide on D. This is true by definition as $w(-, b) = i\mathrm{W}b$. The conclusion is that W is a restriction to $\mathrm{Ind}(\mathrm{D})$ of the functor $w_!$. Because of this, we can deduce that

$$i(\mathrm{Fix}(\mathrm{W})) \simeq \mathrm{Fix}(w_!) \cap i(\mathrm{Ind}(\mathrm{D}))$$

which proves the theorem: the functor $\beta i : \mathcal{S}h(\mathcal{X}) \to \mathrm{Fix}(w_!) \cap i(\mathrm{Ind}(\mathrm{D}))$ is an equivalence of ∞-categories . □

Remark 3.7.4.— A similar theorem can be written for the ∞-category of fixed points of M. An interesting fact is that the category of functors $\mathcal{F} : \mathrm{D}^{op} \to \mathcal{S}$ satisfying the dual condition

$$\mathcal{F}(a) \simeq \prod_{b \in \mathrm{D}} \mathcal{F}(b)^{w(b,a)}$$

is precisely the ∞-category of sheaves on the injective ∞-topos $\mathcal{I}$ such that $\mathcal{X} = \mathrm{pt}(\mathcal{I})$. See [JoJo].

Definition 3.7.5.— *Let $w_!$ be the left Kan extension of $w : \mathrm{D} \to \mathcal{P}(\mathrm{D})$ along the Yoneda embedding $y : \mathrm{D} \to \mathcal{P}(\mathrm{D})$. We call an object of* $\mathrm{Fix}(w_!) \cap i(\mathrm{Ind}(\mathrm{D}))$ *a Leray sheaf (of spaces). In other words, a Leray sheaf of spaces is a left exact functor $\mathrm{D}^{op} \to \mathcal{S}$ such that*

$$\mathcal{F}(a) \simeq \int_{b \in \mathrm{D}} w(a,b)\mathcal{F}(b)$$

for all $a \in \mathrm{D}$.

Remark 3.7.6.— Sheaves and Leray sheaves are equivalent in a very non trivial way. It basically takes the monad M to pass from a Leray sheaf to a usual sheaf and the comonad W to go the other way. In particular, the ∞-category $\mathrm{Fix}(\mathrm{M})$ and $\mathrm{Fix}(\mathrm{W})$ are equivalent and both live inside the same ∞-category $\mathrm{Ind}(\mathrm{D})$ but they do not have the same objects in $\mathrm{Ind}(\mathrm{D})$.

3.7.2 *$\mathcal{C}$-valued sheaves*

Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ be any ∞-category. The usual definition of $\mathcal{C}$-valued sheaves on $\mathcal{X}$ is the following:

$$\mathrm{Sh}(\mathcal{X}, \mathcal{C}) = [\mathcal{S}h(\mathcal{X})^{op}, \mathcal{C}]_c$$

However, in the case where $\mathcal{C}$ is a bicomplete ∞-category, we wish to show there is a much simpler expression to work with:

$$\mathrm{Sh}(\mathcal{X}, \mathcal{C}) \simeq \mathcal{S}h(\mathcal{X}) \otimes \mathcal{C}.$$

This result is a slightly different version of proposition 3.3.18 where the assumptions on the two ∞-categories are weakened; essentially by replacing the *small presentation* condition by a *small generation* one.

We begin with the most simple case.

Lemma 3.7.7.— *Let* D *be a small ∞-category and $\mathcal{C}$ be a bicomplete ∞-category, then*

$$[\mathcal{P}(\mathrm{D})^{op}, \mathcal{C}]_c \simeq \mathcal{P}(\mathrm{D}) \otimes \mathcal{C}.$$

Proof.— By theorem 3.6.12 $\mathcal{P}(\mathrm{D})$ is a dualisable object of $\widehat{\mathcal{C}\mathrm{at}}^{cc}$; its dual is $\mathcal{P}(\mathrm{D}^{op})$. Because $\mathcal{C}$ is supposed to be bicomplete, we now have the equivalences

$$\mathcal{P}(\mathrm{D}) \otimes \mathcal{C} \simeq [\mathcal{P}(\mathrm{D}^{op}), \mathcal{C}]^{cc} \simeq [\mathrm{D}^{op}, \mathcal{C}] \simeq [\mathcal{P}(\mathrm{D})^{op}, \mathcal{C}]_c.$$

□

Theorem 3.7.8.— *Let $\mathcal{A}$ and $\mathcal{B}$ be two ∞-categories. Suppose that $\mathcal{A}$ is cocomplete and smally generated and $\mathcal{B}$ is bicomplete, then*

$$\mathcal{A} \otimes \mathcal{B} \simeq [\mathcal{A}^{op}, \mathcal{B}]_c$$

Proof.— Without loss of generality, we can suppose that there exists a small ∞-category D and a large set of arrows S of $\mathcal{P}(\mathrm{D})$ such that $\mathcal{A}$ is the subcategory of S-local objects of $\mathcal{P}(\mathrm{D})$. Let $f : \mathcal{P}(\mathrm{D}) \times \mathrm{B} \to \mathcal{P}(\mathrm{D}) \otimes \mathcal{B}$ be the canonical map and let T be the large set of all morphisms in $\mathcal{P}(\mathrm{D}) \otimes \mathcal{B}$ having the form $f(s \times \mathrm{Id}_b)$ for every $s \in \mathrm{S}$ and $b \in \mathcal{B}$. Then by proposition 3.3.19, we have

$$\mathrm{T}^{-1}(\mathcal{P}(\mathrm{D}) \otimes \mathcal{B}) \simeq \mathcal{A} \otimes \mathcal{B}.$$

By the previous lemma we have $\mathcal{P}(\mathrm{D}) \otimes \mathcal{B} \simeq [\mathcal{P}(\mathrm{D})^{op}, \mathcal{B}]_c$ so that $\mathcal{A} \otimes \mathcal{B}$ correspond to the ∞-category of T′-local objects of $[\mathcal{P}(\mathrm{D})^{op}, \mathcal{B}]_c$, where T′ is the large set of all morphisms of the form $f'(s \times \mathrm{Id}_b)$ with $f' : \mathcal{P}(\mathrm{D}) \times \mathcal{B} \to [\mathcal{P}(\mathrm{D})^{op}, \mathcal{B}]_c$ the corresponding canonical map.

We only need to check that $[\mathcal{A}^{op}, \mathcal{B}]_c$ is the subcategory of $[\mathcal{P}(\mathrm{D})^{op}, \mathcal{B}]_c$ made of T′-local objects. For this, we draw the following commutative diagram

$$\begin{tikzcd}
{[\mathcal{P}(\mathrm{D})^{op}, \mathcal{B}]_c} \arrow[r, "\varphi"] & {[\mathcal{P}(\mathrm{D})^{op} \times \mathcal{B}^{op}, \mathcal{S}]_{c,c}} \\
{[\mathcal{A}^{op}, \mathcal{B}]_c} \arrow[r, "\psi"] \arrow[u] & {[\mathcal{A}^{op} \times \mathcal{B}^{op}, \mathcal{S}]_{c,c}} \arrow[u]
\end{tikzcd}$$

where φ and ψ are embeddings into ∞-categories of functors continuous in each variable and the upward arrows are fully faithful. Let T″ be the large set of objects of $\varphi(\mathrm{T}')$ and let F be an object of $[\mathcal{P}(\mathrm{D})^{op}, \mathcal{B}]_c$. The morphism F is T′-local if and only if $\varphi(\mathrm{F})$ is T″-local and T″-local objects of $[\mathcal{P}(\mathrm{D})^{op} \times \mathcal{B}^{op}, \mathcal{S}]_{c,c}$ are precisely the objects of $[\mathcal{A}^{op} \times \mathcal{B}^{op}, \mathcal{S}]_{c,c}$ by direct computation (use Yoneda lemma and the proof of proposition 5.5.4.20 of [HT]). Hence, $\varphi(\mathrm{F})$ is T″-local if and only if it lies in the image of ψ. We have proved the desired equivalence. □

Remark 3.7.9.— As any cocomplete smally generated ∞-category is also complete, this formula shows that the very large ∞-category of cocomplete smally generated ∞-categories is closed symmetric monoidal.

In order to get a simple proof of this fact, we can adapt the proof in [HA] for presentable ∞-categories to smally generated categories. Indeed, if $\mathcal{A}$ and $\mathcal{B}$ are cocomplete and smally generated, so is $[\mathcal{A}^{op}, \mathcal{B}]_c$. Moreover, every cocontinuous functor $L : \mathcal{A} \to \mathcal{B}$ has a right adjoint (proposition 5.5.2.2 and corollary 5.5.2.9 from [HT] only depends on the ∞-category to be smally generated), hence we get the embedding

$$[\mathcal{A}, \mathcal{B}]^{cc} \hookrightarrow [\mathcal{B}, \mathcal{A}]_c^{op}$$

which is the key element of the proof.

Corollary 3.7.10.— *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ be a bicomplete ∞-category, then*

$$\mathrm{Sh}(\mathcal{X}, \mathcal{C}) \simeq \mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{C}.$$

Corollary 3.7.11.— *Let $\mathcal{A}$ and $\mathcal{B}$ be two cocomplete and smally generated ∞-categories and let $\mathcal{C}$ be a bicomplete ∞-category. Then for every cocontinuous functor $f : \mathcal{A} \to \mathcal{B}$, the cocontinuous functor $f' = f \otimes \mathrm{Id}_{\mathcal{C}}$ has a right adjoint $f^* : [\mathcal{A}^{op}, \mathcal{C}]_c \to [\mathcal{B}^{op}, \mathcal{C}]_c$ given by precomposition by f^{op}.*

Remark.— Notice that this is not an instantiation of what we said in remark 3.7.9, for $\mathcal{A} \otimes \mathcal{C}$ may not be smally generated.

Proof.— For this proof, we need to understand concretely how f' is built. We draw the diagram:

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{C} \arrow[r] \arrow[d, shift right, "f'"'] & {[\mathcal{A}^{op} \times \mathcal{C}^{op}, \widehat{\mathcal{S}}]_{c,c}} \arrow[r, shift right, hook] \arrow[d, shift right, "Lf_!"'] & {[\mathcal{A}^{op} \times \mathcal{C}^{op}, \widehat{\mathcal{S}}]} \arrow[l, shift right] \arrow[d, shift right, "f_!"'] \\
\mathcal{B} \otimes \mathcal{C} \arrow[r] \arrow[u, shift right, "f^*"'] & {[\mathcal{B}^{op} \times \mathcal{C}^{op}, \widehat{\mathcal{S}}]_{c,c}} \arrow[r, shift right, hook] \arrow[u, shift right, "f^*"'] & {[\mathcal{B}^{op} \times \mathcal{C}^{op}, \widehat{\mathcal{S}}]} \arrow[l, shift right] \arrow[u, shift right, "f^*"']
\end{tikzcd}$$

With the horizontal arrows going to the right being fully faithful.

So how is f' built? By left Kan extension, we get the functor $f_!$; localizing it we have $Lf_!$. Then by construction of the tensor product, $Lf_!$ sends the subcategory $\mathcal{A} \otimes \mathcal{C}$ to $\mathcal{B} \otimes \mathcal{C}$, the restriction of $Lf_!$ to $\mathcal{A} \otimes \mathcal{B}$ is the desired f'.

Meanwhile, f^* is well defined on the right and restrict to the central column. The key point is that it can also be restricted to the first column thanks to theorem 3.7.8.

By proposition 4.3.3.7 in [HT], $f_!$ is left adjoint to f^*. This implies that $Lf_!$ is left adjoint to f^* and because the restriction of an adjunction is still an adjunction, we deduce that f' is left adjoint to f^*. □

Remark 3.7.12.— Corollary 3.7.10 and 3.7.11 imply in particular that for every topological space X, there always exists a sheafification functor adjoint to the natural inclusion

$$\mathrm{PSh}(X)\otimes\mathcal{C} \rightleftarrows \mathrm{Sh}(X)\otimes\mathcal{C}.$$

Even if $\mathcal{C}$ has no nice properties and even if $\mathrm{Sh}(X)$ hasn't enough points! Notice however, that if no assumption is made on $\mathcal{C}$, this sheafification functor is usually not left exact.

3.7.3 *$\mathcal{C}$-valued Leray sheaves*

Going back to an exponentiable ∞-topos $\mathcal{X}$ and a standard presentation:

$$\begin{tikzcd} \mathrm{Ind}(D) \arrow[r, "\varepsilon" description] & \mathcal{S}\mathrm{h}(\mathcal{X}) \arrow[l, shift right=2, "\beta"'] \arrow[l, shift left=2, "\alpha"] \end{tikzcd}$$

we let $\mathcal{C}$ be a bicomplete ∞-category. By tensoring the standard presentation with $\mathcal{C}$ be get another triple adjunction:

$$\begin{tikzcd} \mathrm{Ind}(D)\otimes\mathcal{C} \arrow[r, "\varepsilon'" description] & \mathcal{S}\mathrm{h}(\mathcal{X})\otimes\mathcal{C} \arrow[l, shift right=2, "\beta'"'] \arrow[l, shift left=2, "\alpha'"] \end{tikzcd}$$

By the theorems we obtained when discussing $\mathcal{C}$-valued sheaves, we have: the ∞-category $\mathrm{Ind}(D)\otimes\mathcal{C}$ is canonically equivalent to $[D^{op},\mathcal{C}]_{lex}$, the ∞-category of left exact functors $D^{op}\to\mathcal{C}$, in the same way $\mathcal{S}\mathrm{h}(\mathcal{X})\otimes\mathcal{C}$ is identified with $\mathcal{S}\mathrm{h}(\mathcal{X},\mathcal{C})$. The functors β' and ε' are given by $\beta'=\beta\otimes\mathrm{Id}_{\mathcal{C}}$, $\varepsilon'=\varepsilon\otimes\mathrm{Id}_{\mathcal{C}}$. And we also identify ε' with β^* and α' with ε^*.

Exactly as in the case of Leray sheaves of spaces, we obtain both a continuous monad M' and a cocontinuous comonad W' on $\mathrm{Ind}(D)\otimes\mathcal{C}$:

$$M'=\alpha'\varepsilon',\ W=\beta'\varepsilon'.$$

From the properties of the triple adjunction, we can deduce the same formulas as before:

$$W'\dashv M',\ M'^2\simeq M',\ W'^2\simeq W',\ M'W'\simeq W',\ W'M'\simeq M'.$$

Moreover, by the fully faithfulness of α' and β', the ∞-categories of fixed points of M' and W' are equivalent to $\mathcal{S}\mathrm{h}(\mathcal{X},\mathcal{C})$.

All these pieces of information can be summarised in the following diagram of equivalences of ∞-categories:

$$\begin{tikzcd} & \mathcal{S}\mathrm{h}(\mathcal{X},\mathcal{C}) \arrow[dl, "\beta'"'] \arrow[dr, "\alpha'"] & \\ \mathrm{Fix}(W') \arrow[rr, shift right, "M'"'] & & \mathrm{Fix}(M') \arrow[ll, shift right, "W'"'] \end{tikzcd}$$

At this point, the situation is clearly identical to the one we discussed earlier. However, if we want to be able to describe the fixed points of W′ as we did for W, we are poised to make some new assumptions.

Why?

Note $L : \mathcal{P}(D) \to \mathrm{Ind}(D)$ the localisation functor adjoint to $i : \mathrm{Ind}(D) \to \mathcal{P}(D)$, and $L' = L \otimes \mathrm{Id}_{\mathcal{C}} : \mathcal{P}(D) \otimes \mathcal{C} \to \mathrm{Ind}(D) \otimes \mathcal{C}$. Note i' the right adjoint to L'.

By construction, in the proof of theorem 3.7.3, the left Kan extension $w_! : \mathcal{P}(D) \to \mathcal{P}(D)$ satisfies

$$Lw_! \simeq WL.$$

and as $w_!$ is cocontinuous, also by contruction, if $w'_!$ is defined as $w_! \otimes \mathrm{Id}_{\mathcal{C}}$, we get

$$L'w'_! \simeq W'L'.$$

This guaranties the following inclusion

$$\mathrm{Fix}(w'_!) \cap i'([D^{op}, \mathcal{C}]_{lex}) \subset i'(\mathrm{Fix}(W')).$$

We define $\mathcal{C}$-valued Leray sheaves as:

Definition 3.7.13.— *Let $\mathcal{X}$ be an exponentiable ∞-topos together with a standard presentation with generators* D *and let $\mathcal{C}$ be a cocomplete and finitely complete ∞-category . We define the ∞-category $\mathrm{Ler}_D(\mathcal{X}, \mathcal{C})$ of $\mathcal{C}$-valued Leray sheaves on $\mathcal{X}$ as the ∞-category of left exact functors $\mathcal{F} : D^{op} \to \mathcal{C}$ such that*

$$\mathcal{F}(a) = \int_b w(a,b) \otimes \mathcal{F}(b) \quad \Big(\text{or } \mathcal{F}(a) \simeq \varinjlim_{a \rightsquigarrow b} \mathcal{F}(b)\Big)$$

for all $a \in D$. Where $\otimes$ denotes the canonical tensoring of the cocomplete ∞-category $\mathcal{C}$ over $\mathcal{S}$.

We are able to state this intermediate proposition.

Proposition 3.7.14.— *Let $\mathcal{X}$ be an exponentiable ∞-topos , D an ∞-category of generators of a standard presentation and $\mathcal{C}$ a bicomplete ∞-category , then $\varepsilon' L' : \mathrm{Ler}_D(\mathcal{X}, \mathcal{C}) \to \mathcal{S}h(\mathcal{X}, \mathcal{C})$ is fully faithful.*

The remaining key proposition is to show that $w'_!$ sends left exact functors to left exact functors. The proof we used in theorem 3.7.3 was based on the fact that D generates Ind(D) under filtered colimits and $i : \mathrm{Ind}(D) \to \mathcal{P}(D)$ commutes with such colimits.

After tensoring with $\mathcal{C}$ none of them is still true: generally $\mathrm{Ind}(D) \otimes \mathcal{C}$ is not generated under filtered colimits by the image of $D \times \mathcal{C} \to \mathrm{Ind}(D) \otimes \mathcal{C}$ and the inclusion $[D^{op}, \mathcal{C}]_{lex} \to [D^{op}, \mathcal{C}]$ doesn't commute with filtered colimits.

That answers the question 'why?'.

We have identified three classes of assumptions so that the equivalence between sheaves and Leray sheaves is still true:

1) The ∞-category $\mathcal{C}$ is a presheaf ∞-category $\mathcal{P}(\mathrm{I})$ with I a small ∞-category;

2) The ∞-category $\mathcal{C}$ is stable;

3) The functor β commutes with non-empty finite limits and filtered colimits in $\mathcal{C}$ commute with finite limits.

Remark.— Notice that in each of these cases, filtered colimits in $\mathcal{C}$ commute with finite limits.

The first case is a simple lemma.

Lemma 3.7.15.— *Let $\mathcal{X}$ be an exponentiable ∞-topos and let* D *the generators of a standard presentation. Let $\mathcal{C} = \mathcal{P}(\mathrm{I})$ be a presheaf ∞-category with* I *a small ∞-category . Then*

$$\mathrm{Ler}_{\mathrm{D}}(\mathcal{X}, \mathcal{C}) \to \mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{C})$$

is an equivalence.

Proof.— By the universal property of the tensor product,

$$\mathcal{S}\mathrm{h}(\mathcal{X}) \otimes \mathcal{P}(\mathrm{I}) \simeq \mathcal{S}\mathrm{h}(\mathcal{X})^{\mathrm{I}^{op}}; \quad \mathrm{Ind}(\mathrm{D}) \otimes \mathcal{P}(\mathrm{I}) \simeq \mathrm{Ind}(\mathrm{D})^{\mathrm{I}^{op}}.$$

In this case W' is just $\mathrm{W}^{\mathrm{I}^{op}}$ and $w'_!$ is $w_!^{\mathrm{I}^{op}}$. As such, $w'_!$ sends left exact functors to left exact functors, which guarantees that $\mathrm{Ler}_{\mathrm{D}}(\mathcal{X}, \mathcal{C}) \simeq \mathrm{Fix}(\mathrm{W}') \simeq \mathcal{S}\mathrm{h}(\mathcal{X}, \mathcal{C})$. □

The second case is crystal clear. We will only need two lemmas about stable ∞-categories .

Lemma 3.7.16.— *Let* D *be a small ∞-category and $\mathcal{C}$ be a stable bicomplete ∞-category . Then the right adjoint to $\mathrm{L} \otimes \mathrm{Id}_{\mathcal{C}}$,*

$$\mathrm{Ind}(\mathrm{D}) \otimes \mathcal{C} \to \mathcal{P}(\mathrm{D}) \otimes \mathcal{C}$$

is cocontinuous.

Proof.— Indeed this functor is the embedding $[\mathrm{D}^{op}, \mathcal{C}]_{lex} \to [\mathrm{D}^{op}, \mathcal{C}]$. What is more, in any stable ∞-category any colimit is left exact. This means that all colimits in $[\mathrm{D}^{op}, \mathcal{C}]_{lex}$ can be computed pointwise as in $[\mathrm{D}^{op}, \mathcal{C}]$. □

Lemma 3.7.17.— *Let $\mathcal{C}$ be a cocontinuous stable ∞-category . Then the canonical tensoring over $\mathcal{S}$:*

$$\begin{aligned} \mathcal{S} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (K, c) &\longmapsto K \otimes c = \operatorname*{colim}_{K} c \end{aligned}$$

is exact in the first variable.

Proof.— The tensoring $(K, c) \mapsto K \otimes c$ is cocontinuous in the first variable and $\mathcal{C}$ is stable, hence it is also left exact in the first variable. □

Theorem 3.7.18.— *Let $\mathcal{X}$ be an exponentiable ∞-topos together with D an ∞-category of generators of a standard presentation for $\mathcal{S}h(\mathcal{X})$. Let $\mathcal{C}$ be a stable bicomplete ∞-category . Then*

$$\mathrm{Ler}_D(\mathcal{X}, \mathcal{C}) \to \mathcal{S}h(\mathcal{X}, \mathcal{C})$$

is an equivalence.

Proof.— Let $\mathcal{F} : D^{op} \to \mathcal{C}$ be any functor. Then for any $b \in D$, the functor $w(-, b) \otimes \mathcal{F}(b)$ is left exact, because $w(-, b)$ is left exact and thanks to lemma 3.7.17. From lemma 3.7.16, we deduce that the coend

$$\int_{b \in D} w(-, b) \otimes \mathcal{F}(b)$$

calculated inside $\mathcal{P}(D) \otimes \mathcal{C}$ still belongs to $\mathrm{Ind}(D) \otimes \mathcal{C}$. This means that for any functor $\mathcal{F} : D^{op} \to \mathcal{C}$, $w'_!(\mathcal{F})$ is left exact. From this we get that $\mathrm{Ler}_D(\mathcal{X}, \mathcal{C}) \simeq \mathrm{Fix}(W') \simeq \mathcal{S}h(\mathcal{X}, \mathcal{C})$. □

The proof of the third case is quite pedestrian and uninteresting; we shall not include it in this thesis, for the reader's sake. It is completely analogous to the proof of theorem 7.3.4.9 in [HT].

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colophon

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