Quillen equivalence of topological spaces and simplicial sets

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Contents

1 Introduction

This thesis is a continuation of a talk on model categories I gave in a seminar in Homotopical and Higher Algebra held by prof. D. Calaque in the autumn semester 2012.
Model categories are an important tool in the study of homotopy theory. They were first introduced in 1967 by D. Quillen in his book [6]. The importance of the existence of a Quillen equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$ is the fact that it induces an equivalence of categories between the homotopy categories Ho$\mathcal{C}$ and Ho$\mathcal{D}$, permitting us to study the homotopy theory in $\mathcal{C}$ through the homotopy theory in $\mathcal{D}$, and vice versa. In our particular case, the objective is to prove that we can study the homotopy of the "complicated" category of topological spaces (**Top**) through the homotopy theory of the category **SSet** of simplicial sets, which is of a more combinatorial nature. Namely, the goal of this thesis is to give an exhaustive proof of the existence of a Quillen equivalence between **Top** and **SSet**, starting from scratch and requiring only basic knowledge of category theory and algebraic topology.
In section 1 we introduce the basic tools to treat the subject (lifting properties, the small object argument, model categories, cofibrantly generated model categories, Quillen adjunctions and Quillen equivalences). In sections 2 and 3 we describe the model structure on **Top** and **SSet** respectively. Finally in section 4 we prove the existance of a Quillen equivalence between the two categories.
Throughout this thesis , we will mainly follow the approach of [3], sometimes using some elements from [4] when deemed useful.

2 Cofibrantly generated model categories and Quillen equivalences

2.1 Model categories

2.1.1 Preliminary notions and notation

Definition 2.1. *Let $\mathcal{C}$ be a category. An object X of $\mathcal{C}$ is said to be a* retract *of an object Y if there are arrows such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r] \arrow[dr, equal] & Y \arrow[d] \\ & X \end{tikzcd}$$

An arrow f in $\mathcal{C}$ is the retract of an arrow g if it is the retract of g in the category of arrows of $\mathcal{C}$.

Definition 2.2. *Let a and b be two morphisms in a category $\mathcal{C}$. We say that a has left lifting property with respect to b, and that b has right lifting property with respect to a, if for every commuting square as below, there is a dashed arrow (called a* diagonal filler*) making the following diagram commutes:*

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "a"'] & {} \arrow[d, "b"] \\ {} \arrow[r] \arrow[ur, dashed] & {} \end{tikzcd}$$

We denote this by $a \pitchfork b$. The dashed arrow is often called a diagonal filler of the square.
Let S, T be two subsets of the arrows of $\mathcal{C}$. We say that S has the left lifting property with respect to T, and that T has the right lifting property with respect to S, if for every $a \in S$, $b \in T$ we have $a \pitchfork b$. In this case we write $S \pitchfork T$. If S is any subset of the arrows of $\mathcal{C}$, we define the following two other subsets of the arrows of $\mathcal{C}$:

$$S^{\pitchfork} = \{b | s \pitchfork b, \ \forall s \in S\}$$

$$^{\pitchfork}S = \{a | a \pitchfork s, \ \forall s \in S\}$$

Notice that a map $f \in \mathcal{C}\,(\mathrm{X}, \mathrm{Y})$ with $f \pitchfork f$ is necessarily an isomorphism. Indeed there must be an arrow g such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, "g"] & Y \end{tikzcd}$$

Then g is obviously the inverse of f.

We introduce now some concepts which will be of great importance in the treatment of cofibrantly generated model categories.

Definition 2.3. *Let $\mathcal{C}$ be a category with all small colimits (i.e. colimits indexed by a small category) and let λ be a limit ordinal. A λ-sequence is a colimit-preserving functor $X : \lambda \to \mathcal{C}$. We usually represent the functor as*

$$X_0 \to X_1 \to X_2 \to \ldots \to X_\beta \to \ldots$$

where $X_\beta = X(\beta)$, and the arrows are unique. We refer to the map $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ as the composition of the λ-sequence. If D is a collection of arrows in $\mathcal{C}$ and every map $X_\beta \to X_{\beta+1}$ is in D, then the composition $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ is called a transfinite composition *of arrows in D.*
D is called closed under transfinite compositions *if every transfinite composition of arrows in D is again in D.*

Definition 2.4. *Let γ be a cardinal, α a limit ordinal. We say that α is γ-filtered if from $A \subseteq \alpha$ and $|A| < \gamma$ follows $\sup A < \alpha$.*

Definition 2.5. *Let $\mathcal{C}$ be a category closed under small colimits, D a collection of arrows in $\mathcal{C}$, A an object of $\mathcal{C}$ and κ a cardinal. We say that A is κ-*small relative to D *if for every κ-filtered ordinal λ and every λ-sequence $X : \lambda \to \mathcal{C}$ such that the arrow $X_\beta \to X_{\beta+1}$ is in D every time $\beta < \lambda$, the canonical map of sets*

$$\mathrm{colim}_{\beta<\lambda}\mathcal{C}\left(\mathrm{A}, \mathrm{X}_\beta\right) \to \mathcal{C}\left(\mathrm{A}, \mathrm{colim}_{\beta<\lambda}\mathrm{X}_\beta\right)$$

is an isomorphism. A is said to be small with respect to *D if there is a cardinal κ such that A is κ-small relative to D. A is* small *if it is small with respect to all of $\mathcal{C}$.*

This works as follows: Recall that $\mathrm{colim}_{\beta<\lambda}\mathcal{C}\left(\mathrm{A}, \mathrm{X}_\beta\right) = \coprod_{\beta<\lambda} \mathcal{C}\left(\mathrm{A}, \mathrm{X}_\beta\right)/\sim$, where two maps $f_1 : A \to X_{\beta_1}$ and $f_2 : A \to X_{\beta_2}$ are equivalent if there is a third map $f_\gamma : A \to X_\gamma$ with $\gamma \leq \beta_i$ making the following diagram commute:

$$\begin{tikzcd}
 & X_{\beta_1} \arrow[dr, bend left] & \\
A \arrow[ur, "f_1"] \arrow[rr, "f_\gamma"] \arrow[dr, "f_2"'] & & X_\gamma \\
 & X_{\beta_2} \arrow[ur, bend right] &
\end{tikzcd}$$

Thus every element of the colimit is the equivalence $[f_\beta]$ class of some map $f_\beta \in \mathcal{C}\left(\mathrm{A}, \mathrm{X}_\beta\right)$. The canonical map $\mathrm{colim}_{\beta<\lambda}\mathcal{C}\left(\mathrm{A}, \mathrm{X}_\beta\right) \to \mathcal{C}\left(\mathrm{A}, \mathrm{colim}_{\beta<\lambda}\mathrm{X}_\beta\right)$ sends $[f_\beta]$ to the composite

$$A \xrightarrow{f_\beta} X_\beta \longrightarrow \mathrm{colim}_{\beta<\lambda} X_\beta$$

It can be checked that this map is well defined. Surjectivity of this map means that for any map $f : A \to \mathrm{colim}_{\beta<\lambda} X_\beta$ there is some $\beta < \gamma$ such that f factors through X_β. Injectivity implies that this factorization is unique, in the sense that if f factors both through $f_1 : A \to X_{\beta_1}$ and $f_2 : A \to X_{\beta_2}$, where $\beta_1 < \beta_2$, then f_2 is given by the composition $A \xrightarrow{f_1} X_{\beta_1} \to X_{\beta_2}$.

Example 2.6. *All sets are small in the category* **Sets**. *Indeed let A be a set, λ be an $|A|$-filtered ordinal and $X : \lambda \to$* **Sets** *be a λ-sequence. Take a map $f : A \to \mathrm{colim}_{\beta<\lambda} X_\beta$. We show it factors through some X_α with $\alpha < \lambda$. Indeed for $a \in A$ define $g(a)$ to be an ordinal such that $f(a)$ is in the image of $X_{g(a)}$, and let $S = \{g(a)|a \in A\}$. Then since λ is $|A|$-filtered, $\sup S = \gamma < \lambda$, and f factors through X_γ (since $f(A)$ is completely contained in the image of X_γ).*

Now let $f_1 : A \to X_{\gamma_1}$ and $f_2 : A \to X_{\gamma_2}$ be two different factorizations of f. Then for every $a \in A$ there must be an ordinal $g(a) \geq \max(\gamma_1, \gamma_2)$ such that the images of $f_1(a)$ and $f_2(a)$ are equal in $X_{g(a)}$. Since λ is $|A|$-filtered, we obtain that $\gamma = \sup_{a\in A} g(a) < \lambda$, and that f_1 and f_2 become equal in X_γ.

Example 2.7. *Not every topological space is small.Let λ be any limit ordinal, then we can construct the following topological spaces:*

- *$A = \{0, 1\}$ with the indiscrete topology.*
- *$Y = \lambda \cup \{\lambda\}$ with topology $\tau_Y = \{\{\lambda \geq \beta > \alpha\} : \alpha < \lambda\} \cup \{\emptyset\}$.*
- *For $\alpha < \lambda$, let $X_\alpha = (Y \times \{0, 1\})/ \sim$, where $\{0, 1\}$ is endowed with the discrete topology and $\sim$ identifies $(x, 0)$ and $(x, 1)$ whenever $x < \alpha$.*

Then the obvious maps $X_\alpha \to X_{\alpha+1}$ give us a λ-sequence. The colimit of the sequence is the topological space

$$X = \mathrm{colim}_{\alpha<\lambda} X_\alpha \cong (Y \times \{0\}) \cup \{(\lambda, 1)\}$$

A subset of X is open if, and only if it is of the form

$$U = \{(\beta, 0) : \lambda \geq \beta > \alpha\} \cup \{(\lambda, 1)\}$$

for some $\alpha < \lambda$. Thus we have a continuous map $f : A \to X$ given by $f(i) = (\lambda, i)$ which does not factor continuously *through any of the X_α, where the points $(\lambda, 0)$ and $(\lambda, 1)$ can be separated.*

2.1.2 Weak factorization systems and the small object argument

Definition 2.8. *Let $\mathcal{C}$ be a category. A weak factorization system in $\mathcal{C}$ is a pair of classes of arrows (A, B) satisfying:*

i. Every arrow f of $\mathcal{C}$ can be written as $f = b \circ a$ for some $a \in A$ and some $b \in B$.

ii. $A^\pitchfork = B$ and ${}^\pitchfork B = A$.

A weak factorization system is said to be functorial *if there are two functors γ and δ on the category of arrows of $\mathcal{C}$ such that the factorization is given by $f = \delta(f) \circ \gamma(f)$.*

Definition 2.9. *Let $\mathcal{C}$ be a category containing all its small colimits, and let I be a set of arrows in $\mathcal{C}$. A* relative I-cell complex *is a transfinite composition of pushouts of elements of I. The collection of all relative I-cell complexes is denoted by I-cell.*

Worded more explicitly, $f : A \to B$ is a relative I-cell complex if there is a λ-sequence $X : \lambda \to \mathcal{C}$ such that every map $X_\beta \to X_{\beta+1}$ is the pushout of some map g_β in I, i.e. we have a pushout diagram:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ \cdot \arrow[r] & X_{\beta+1} \end{tikzcd}$$

with $X_0 = A$ and $X_\lambda = B$.

The result we present now allows us to construct a functorial factorization of all arrows in a category in an easy and effective way. Such a factorization will later be used to construct a weak factorization system.

Proposition 2.10. *Let $\mathcal{C}$ be a cocomplete category, I a set of arrows of $\mathcal{C}$ with domains small relative to I-cell. Then there exist two functors γ and δ on the category of arrows of $\mathcal{C}$ such that $\gamma(f) \in I-$cell, $\delta(f) \in I^{\pitchfork}$ and $f = \delta(f) \circ \gamma(f)$ for every arrow f in $\mathcal{C}$.*

In order to prove this result, we will start a map $f_X \to Y$ in $\mathcal{C}$ and decompose it as

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & E_1 \arrow[d, "p_1"] \\ & Y \end{tikzcd}$$

where p_1 is in $I-$cell. We will iterate this decomposition (using pushouts and transfinite induction) in order to obtain a decomposition $f = \bar{p} \circ u$, where u is a transfinite composition of elements of $I-$cell, and thus is still in the set. Finally, we will show that $\bar{p}$ is in $I^{\pitchfork}$. Functoriality will be given by the fact that our argument can be made functorial in every step. The proof in full detail is the following.

Proof. Choose a cardinal κ such that the domain of every arrow in I is κ-small relative to all of $\mathcal{C}$ (notice that such a κ exists because C is a set of arrows) and a κ-filtered ordinal α. Take $f : X \to Y$ any arrow in $\mathcal{C}$. We define $E_0 = X$, $p_0 = f$. Given E_i and p_i, we define E_{i+1} and p_{i+1} as follows.
Let S index the set of commutative squares of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "u"'] & E_i \arrow[d, "p_i"] \\
B \arrow[r] & Y
\end{tikzcd}$$

We construct E_{i+1} as the pushout of the coproduct of the B_s, $s \in S$, and E_i over the coproduct of the A_s, that is:

$$\begin{tikzcd}
\coprod_{s\in S} A \arrow[r] \arrow[d, "\coprod_{s\in S} u"'] & E_i \arrow[d] \\
\coprod_{s\in S} B \arrow[r] & E_{i+1}
\end{tikzcd}$$

Then we take as p_{i+1} the map from E_{i+1} to Y induced by the universal property of the pushout:

$$\begin{tikzcd}
\coprod_{s\in S} A \arrow[r] \arrow[d, "\coprod_{s\in S} u"'] & E_i \arrow[d] \arrow[ddr, bend left, "p_i"] & \\
\coprod_{s\in S} B \arrow[r] \arrow[drr, bend right] & E_{i+1} \arrow[dr, dashed, "p_{i+1}"] & \\
 & & Y
\end{tikzcd}$$

Given j a limit ordinal and E_i, p_i for every $i < j$, we define $E_j = \mathrm{colim}_{i<j} E_i$ and p_j as the map induced by all the p_i using the universal property of the colimit E_j.
Let $\bar{E} = \mathrm{colim}_{i<\alpha} E_i$ and $\bar{p}$ the map induced by the p_i (again by universal property of the colimit). The map $u : X \to \bar{E}$ given as the transfinite composition of the maps $E_i \to E_{i+1}$ is in $I-$cell, since the latter is closed under transfinite compositions.

We are left to show that $\bar{p} \in I^{\pitchfork}$. Let $c : A \to B$ be in I and let the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "c"'] & \bar{E} \arrow[d, "\bar{p}"] \\
B \arrow[r, "b"'] & Y
\end{tikzcd}$$

Since A is κ-small, the map a factorizes as

$$
\begin{tikzcd}
A \arrow[r, "a_i"] \arrow[dd, "c"'] \arrow[dr, "a"] & E_i \arrow[d] \\
& \bar{E} \arrow[d, "\bar{p}"] \\
B \arrow[r, "b"'] & Y
\end{tikzcd}
$$

By construction of E_{i+1}, this commuting square (rectangle) gives us the following commuting diagram.

$$
\begin{tikzcd}
A \arrow[r, "a_i"] \arrow[d, "c"'] & E_i \arrow[d] \\
B \arrow[r, "b_{i+1}"'] & E_{i+1}
\end{tikzcd}
$$

We can use this arrow b_{i+1} to construct the sought diagonal filler as follows:

$$
\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, "c"'] & & \bar{E} \arrow[dd, "\bar{p}"] \\
& E_{i+1} \arrow[ur] & \\
B \arrow[rr, "b"'] \arrow[ur, "b_{i+1}"'] & & Y
\end{tikzcd}
$$

This shows that $\bar{p} \in I^{\pitchfork}$.

Note that the factorization is functorial in every step, since the association of the set S to the map p_i is functorial, and thus all the process can be made functorial. $\square$

As we preannounced, from this we can get a weak factorization system. In the proof we use a result which will be proved later, i.e. the fact that $I-\mathrm{cell} \subseteq {}^{\pitchfork}(I^{\pitchfork})$ (lemma 2.23).

Theorem 2.11. (The small object argument) *Let $\mathcal{C}$ be a cocomplete category, I a set of arrows of $\mathcal{C}$ with domains small relative to I-cell. Then $(I-\mathrm{cof}, I^{\pitchfork})$ is a (functorial) weak factorization system.*

Proof. By proposition 2.10, we have a functorial factorization of every arrow in $\mathcal{C}$ in elements of $I-\mathrm{cell} \subseteq {}^{\pitchfork}(I^{\pitchfork})$ (by lemma 2.23). Then $({}^{\pitchfork}(I^{\pitchfork}), I^{\pitchfork})$ is a weak factorization system, since it satisfies the required lifting properties. $\square$

The full power of this statement will become apparent when we will apply it to cofibrantly generated model categories.

2.1.3 Model structure on a category

We can now give the definition of a model category.

Definition 2.12. *Let $\mathcal{M}$ be a category which is complete and cocomplete. A model structure on $\mathcal{M}$ is a triple of classes of maps (C, F, W) called respectively cofibrations, fibrations and weak equivalences satisfying:*

i. Two-out-of-three property: Let $X, Y, Z \in \mathcal{M}$, f, g, h morphisms in $\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

If two of f, g, h are weak equivalences, then so is the third.

ii. $(C \cap W, F)$ and $(C, F \cap W)$ are weak factorization systems.

The elements of $C \cap W$ and $F \cap W$ are called trivial cofibrations and trivial fibrations respectively, $\mathcal{M}$ together with its model structure is called a model category. When we have a given model category $(\mathcal{M}, C, F, W)$, we will often abuse of notation and call it $\mathcal{M}$.

To simplify the reading of the diagrams, we will often use $\twoheadrightarrow$ for fibrations, $\rightarrowtail$ for cofibrations and put a little $\sim$ on weak equivalences.

The reader familiar with the subject of model categories might have noticed that this definition of model structure seems to differ a bit from the one usually given. In fact our definition is slightly weaker than the one given in [3] (we don't necessarily have a functorial factorization of maps, even though we will see that will be the case for cofibrantly generated model categories), and it is perfectly equivalent to the one given in [1], as can be easily checked using the following three lemmas.

Lemma 2.13. *C, F, $C \cap W$ and $F \cap W$ are closed under retracts.*

Proof. Let $c \in C$, a be a retraction of c, i.e. there are arrows such that the following diagram commutes.

$$
\begin{tikzcd}
{} \arrow[r] \arrow[d, "a"'] \arrow[rr, bend left, equal] & {} \arrow[r] \arrow[d, tail, "c"'] & {} \arrow[d, "a"] \\
{} \arrow[r] \arrow[rr, bend right, equal] & {} \arrow[r] & {}
\end{tikzcd}
$$

We prove that a is a cofibration by showing that it has the left lifting property with respect to all trivial fibrations. Indeed, let $f \in F \cap W$, and assume there are two arrows such that the following diagram commutes:

$$
\begin{tikzcd}
{} \arrow[r] \arrow[d, "a"'] & {} \arrow[d, "f"] \\
{} \arrow[r] & {}
\end{tikzcd}
$$

Then since a is a retraction of c and c has left lifting property with respect to every trivial fibration, we obtain the following commutative diagram:

$$
\begin{tikzcd}
{} \arrow[r] \arrow[d, "a"'] \arrow[rr, bend left, equal] & {} \arrow[r] \arrow[d, tail, "c"'] & {} \arrow[r] \arrow[d, "a"] & {} \arrow[d, "f"] \\
{} \arrow[r, "x"'] \arrow[rr, bend right, equal] & {} \arrow[r] \arrow[urr, dashed, bend left] & {} \arrow[r] & {}
\end{tikzcd}
$$

The composition of x with the dashed arrow is then the desired lifting for a. The proof for fibrations is dual to this one, and for trivial cofibrations and fibrations we can proceed in a similar way. $\square$

Remark 2.14. In fact, the proof works exactly the same way for the following statement:
Let $\mathcal{C}$ be a category, A and B two classes of maps in $\mathcal{C}$. Assume $A \pitchfork B$. Then both A and B are closed under retracts.

Lemma 2.15. *The class W of weak equivalences of a model category is closed under retracts.*

Proof. Let $w \in W$ and r be a retract of w, i.e. we have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "r"'] \arrow[rr, bend left, equal] & X \arrow[r] \arrow[d, tail, "w"', "\wr"] & A \arrow[d, "r"] \\
B \arrow[r] \arrow[rr, bend right, equal] & Y \arrow[r, "b"'] & B
\end{tikzcd}$$

We factorize $r = f \circ c$, where $c \in C \cap W$ and $f \in F$. We get the following diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "c"', "\wr"] \arrow[rr, bend left, equal] & X \arrow[r] \arrow[dd, "w"', "\wr"] & A \arrow[d, tail, "c"', "\wr"] \\
E \arrow[d, two heads, "f"'] & & E \arrow[d, two heads, "f"'] \\
B \arrow[r] \arrow[rr, bend right, equal] & Y \arrow[r, "b"'] & B
\end{tikzcd}$$

Taking the pushout of E and X (whose existence is guaranteed by the cocompleteness of $\mathcal{M}$) we obtain the element $E \cup_A X$. The universal property of the pushout gives us the two dashed arrows in the following diagrams.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "c"', "\wr"] & X \arrow[d, "i"] \arrow[dd, bend left=60, "\wr\, w"] \\
E \arrow[r, "a"] \arrow[d, two heads, "f"'] & E \cup_A X \arrow[d, dashed, "p"] \\
B \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "c"', "\wr"] \arrow[rr, bend left, equal] & X \arrow[r] \arrow[d, "i"] & A \arrow[d, tail, "\wr\, c"] \\
E \arrow[r, "a"'] \arrow[rr, bend right, equal] & E \cup_A X \arrow[r, dashed] & E
\end{tikzcd}$$

with $w = p \circ i$. We see that i is in fact a retraction of c, and thus a trivial cofibration (by lemma 2.13). By the 2-out-of-3 property, $p \in W$. We are left to show that f is a weak equivalence, then we can conclude that $r = f \circ c$ is in W using the 2-out-of-3 property one more time.
We start with the lower part of the last diagram:

$$
\begin{tikzcd}
E \arrow[r, "a"] \arrow[d, two heads, "f"'] \arrow[rr, bend left, equal] & E \cup_A X \arrow[r] \arrow[d, "p"', "\wr"] & E \arrow[d, two heads, "f"] \\
B \arrow[r] \arrow[rr, bend right, equal] & Y \arrow[r, "b"'] & B
\end{tikzcd}
$$

We factorize $p = q \circ j$, with $j : E \cup_A X \to Z$ a trivial cofibration and $q : Z \to Y$ a fibration. This permits us to write the following commutative diagram:

$$
\begin{tikzcd}
E \cup_A X \arrow[rr] \arrow[d, tail, "j"', "\wr"] & & E \arrow[d, two heads, "f"] \\
Z \arrow[r, two heads, "\sim", "q"'] \arrow[rru, dashed, "d"] & Y \arrow[r, "b"'] & B
\end{tikzcd}
$$

where the diagonal filler is obtained by the fact that $(C \cap W, F)$ is a weak factorization system. We use it to write down this diagram, showing that f is a retract of q:

$$
\begin{tikzcd}
E \arrow[r, "a"] \arrow[d, two heads, "f"'] \arrow[rrr, bend left, equal] & E \cup_A X \arrow[r, tail, "\sim", "j"'] & Z \arrow[r, "d"] \arrow[d, two heads, "q"', "\wr"] & E \arrow[d, two heads, "f"] \\
B \arrow[rr] \arrow[rrr, bend right, equal] & & Y \arrow[r, "b"'] & B
\end{tikzcd}
$$

By lemma 2.13, $f \in (F \cap W)$ and in particular, $f \in W$. This concludes the proof. $\square$

Lemma 2.16. *Let $\mathcal{C}$ be any category, A and B two classes of maps in $\mathcal{C}$. Suppose that every map f in $\mathcal{C}$ can be factored as $f = b \circ a$ with $a \in A$ and $b \in B$, that A and B are closed under retracts and that $A \pitchfork B$. Then $A^{\pitchfork} = B$ and ${}^{\pitchfork}B = A$. In particular (A, B) is a weak factorization system in $\mathcal{C}$.*

Proof. We already have $B \subset A^{\pitchfork}$ and $A \subset {}^{\pitchfork}B$ since $A \pitchfork B$. So let $f \in {}^{\pitchfork}B$. We will show that $f \in A$. By assumption, we can factorize $f = b \circ a$ with $a \in A$ and

$b \in B$. The lifting property then gives us the diagonal arrow c in the following diagram.

$$\begin{tikzcd} {} \arrow[r, "a"] \arrow[d, "f"'] & {} \arrow[d, "b"] \\ {} \arrow[r, equal] \arrow[ur, "c"] & {} \end{tikzcd}$$

This permits us to write down a diagram showing that f is in fact a retraction of a.

$$\begin{tikzcd} {} \arrow[r, equal] \arrow[d, "a"'] & {} \arrow[r, equal] \arrow[d, "f"] & {} \arrow[d, "a"] \\ {} \arrow[r, "b"'] \arrow[rr, bend right, equal] & {} \arrow[r, "c"'] & {} \end{tikzcd}$$

Since A is closed under retracts, $f \in A$. A similar argument shows that $B = A^{\pitchfork}$. $\square$

2.2 Cofibrantly generated model categories

We introduce now a particular type of model category which has some useful advantages, for example an increased simplicity to prove the axioms for a model structure and to check if a functor is Quillen. The categories in which we are interested - topological spaces and simplicial sets - are in fact categories of this type.

Let $\mathcal{C}$ be any category, I a class of maps in $\mathcal{C}$. In the literature (e.g. [3, p. 30]), we often find the notation I-inj for the class $I^{\pitchfork}$ and I-proj for ${}^{\pitchfork}I$. We will not adopt such convention for those classes of arrows, but we will use I-cof for ${}^{\pitchfork}(I^{\pitchfork})$ as is usually done, since we find this denomination helpful in the reading.

Motivation for fibrantly and cofibrantly generated model categories comes from the fact that in some examples fibrations are defined as $F = J^{\pitchfork}$ for some set of arrows J (for example Serre fibrations and Kan fibrations are defined this way, as we will see). This gives the hope of being able to reconstruct the whole model structure starting from the class of weak equivalences and some *sets* of arrows to retrieve the whole classes of fibrations and cofibrations using lifting properties.

Before going on with a formal definition of cofibrantly generated model categories, we give a couple of results which will be useful later.

Lemma 2.17. (The Retract Argument) *Let $\mathcal{C}$ be a category, f an arrow in $\mathcal{C}$. Assume we have a factorization $f = p \circ i$ and that f has the left lifting property relative to p. Then f is a retract of i.*

Remark 2.18. The dual statement is: if f has the right lifting property relative to i, then f is a retract of p.

Proof. We have the following commutative square, with the lift r induced by the left lifting property of f relative to p.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "p"] \\ C \arrow[r, equal] \arrow[ur, dashed, "r"] & C \end{tikzcd}$$

Then the retraction diagram for f is the following:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "r"'] \arrow[rr, equal, bend right] & B \arrow[r, "p"'] & C \end{tikzcd}$$

□

Lemma 2.19. *Let I be a set of maps in a cocomplete category $\mathcal{C}$. such that the domains of the maps in I are small relative to I-cell. Then every map in I-cof is the retract of some map in I-cell with the same domain.*

Proof. Let $f \in I-\text{cof}$. Then by theorem 2.11, we have a factorization $f = p \circ i$, where $i \in I-\text{cell}$ and $p \in I^{\pitchfork}$. But $f \in I-\text{cof}$, so $f \pitchfork I^{\pitchfork}$. Then lemma 2.17 tells us that f is a retract of i, concluding the proof. □

Definition 2.20. *Let $(\mathcal{M}, C, F, W)$ be a model category. We call $\mathcal{M}$* cofibrantly generated *if there are two sets of maps $I \subseteq C$ and $J \subseteq C \cap W$ such that:*

i. The domains of the maps in I, J are small with respect to I-cell and J-cell respectively.

ii. The class of fibrations is $F = J^{\pitchfork}$ and the class of trivial fibrations is $F \cap W = I^{\pitchfork}$.

Then I is called the set of generating cofibrations*, J the set of* generating trivial cofibrations*.*

Remark 2.21. If a category is cofibrantly generated, then it follows directly from the small object argument (theorem 2.11) that the weak factorization systems $(C, F \cap W)$ and $(C \cap W, F)$ are functorial.

Remark 2.22. There is an analogous (dual, in fact) notion of *fibrantly generated model category.*

Given the two generating sets, we can reconstruct the sets of cofibrations and trivial cofibrations. Those are given by I-cof and J-cof respectively. Indeed, $C = {}^{\pitchfork}(F \cap W) = {}^{\pitchfork}(I^{\pitchfork}) = I{-}\mathrm{cof}$, and similarly for $C \cap W$.

We present now a result which help us to identify cofibrantly generated model structures on categories.

Lemma 2.23. *Let $\mathcal{C}$ be a cocomplete category, I a class of arrows in $\mathcal{C}$. Then $I{-}\mathrm{cell} \subseteq I{-}\mathrm{cof}$.*

Proof. It is enough to show that $I{-}\mathrm{cof}$ is closed under pushouts and transfinite compositions.
Pushouts: Let $i \in I{-}\mathrm{cof}$, x a pushout of i, that is: we have a pushout diagram as follows.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "x"] \\
B \arrow[r] & Y
\end{tikzcd}$$

We show that x is in $I{-}\mathrm{cof}$ by proving it has the left lifting property with respect to every map in $I^{\pitchfork}$. Let $j \in I^{\pitchfork}$ be such that we have a commuting square as follows.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "x"'] & C \arrow[d, "j"] \\
Y \arrow[r] & D
\end{tikzcd}$$

We want to construct a diagonal filler. Left lifting property of i with respect to j gives us the diagonal filler d in the next diagram (given by the dashed arrow).

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[r, "a"] \arrow[d, "x"] & C \arrow[d, "j"] \\
B \arrow[r] \arrow[urr, dashed] & Y \arrow[r] & D
\end{tikzcd}$$

We use d and the universal property of the pushout to construct the required diagonal filler for the previous diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "x"] \arrow[ddr, bend left, "a"] & \\
B \arrow[r] \arrow[drr, bend right, "d"'] & Y \arrow[dr, dashed] & \\
 & & C
\end{tikzcd}$$

<u>Transfinite compositions:</u> Let c be a transfinite composition of a λ-sequence $X : \lambda \to \mathcal{C}$ in I-cof. We prove that c is again an element of I-cof by transfinite induction. It is obvious that the composition of two elements of I-cof is again in I-cof. Let $\alpha \leq \lambda$ be a limit ordinal and assume that the composition of arrows up to α (excluded) is always in I-cof. Then, if we have a commutative square as follows:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "c_0"'] \arrow[dddd, bend right, "c"'] & A \arrow[dddd, "\in I^{\pitchfork}"] \\
X_1 \arrow[d, "c_1"'] \arrow[ur, dashed] & \\
X_2 \arrow[d, "c_2"'] \arrow[uur, dashed] & \\
\vdots & \\
X_\alpha \arrow[r] & B
\end{tikzcd}$$

where the dashed arrows exist by induction hypothesis. Then since X_α is a colimit, a diagonal filler is induced by universal property. $\square$

Theorem 2.24. *Let $\mathcal{C}$ be a category which is complete and cocomplete. Let W be a subcategory of $\mathcal{C}$, I and J two sets of maps in $\mathcal{C}$. The following are equivalent:*

i. There is a cofibrantly generated model structure on $\mathcal{C}$ with I as generating cofibrations, J as generating trivial cofibrations and W as weak equivalences.

ii. The following conditions are satisfied:

a) W satisfies the 2-out-of-3 property and is closed under retracts *b) the domains of the maps in I, J are small relative to I-cell and J-cell respectively*

c) $J{-}\mathrm{cell} \subseteq (I{-}\mathrm{cof} \cap W)$

d) $I^{\pitchfork} \subseteq (J^{\pitchfork} \cap W)$

e) either $(I{-}\mathrm{cof} \cap W) \subseteq J{-}\mathrm{cof}$ *or* $(J^{\pitchfork} \cap W) \subseteq I^{\pitchfork}$

Proof. Assume (i.) is true. Then by definition of model structure, W satisfies the 2-out-of-3 property. By lemma 2.15, W is also closed under retracts. The domains of the maps in I and J are small by definition of cofibrantly generated model category. By lemma 2.23, $J{-}\mathrm{cell} \subseteq J{-}\mathrm{cof} = I{-}\mathrm{cof} \cap W$. Again by definition of cofibrantly generated model category, $I^{\pitchfork} = J^{\pitchfork} \cap W$.

Now assume the conditions of (ii.) hold. We define the fibrations by $F = J^{\pitchfork}$ and the cofibrations by $C = I{-}\mathrm{cof}$. Then a proof very similar to the one for lemma 2.13 gives us closeness under retracts for F and C. Closeness under retracts of W implies then that $C \cap W$ and $F \cap W$ are also closed under retracts. By hypothesis, every map in $I^{\pitchfork}$ is a trivial fibration. Since every map in J-cell is a trivial cofibration, and every map in J-cof is a retract of a map in J-cell (by lemma 2.19), every map in J-cell is a trivial cofibration. By the small object argument (theorem 2.11), $(I{-}\mathrm{cof}, I^{\pitchfork})$ and $(J{-}\mathrm{cof}, J^{\pitchfork})$ are (functorial) weak factorization systems. By definition of C and F, and by lemma 2.23, $(C, F \cap W)$ and $(C \cap W, F)$ factorize every arrow in $\mathcal{C}$. We show they form weak factorization systems. In order to do this we only have to prove the lifting properties.

We notice that the two last conditions are equivalent (this is why we require only one).
<u>Claim:</u> $(I{-}\mathrm{cof} \cap W) \subseteq J{-}\mathrm{cof} \Leftrightarrow (J^{\pitchfork} \cap W) \subseteq I^{\pitchfork}$
<u>Proof:</u> Assume $(J^{\pitchfork} \cap W) \subseteq I^{\pitchfork}$, let $i \in I{-}\mathrm{cof} \cap W$. By what we have proven earlier, we can factorize $i = j \circ p$ with $p \in J{-}\mathrm{cell} \subseteq (I{-}\mathrm{cof} \cap W)$ and $j \in J^{\pitchfork}$ (here we used the first version of the small object argument, proposition 2.10). By 2-out-of-3 property of W, $j \in J^{\pitchfork} \cap W \subseteq I^{\pitchfork}$. Thus we get the diagonal filler in the following diagram.

$$\begin{tikzcd} {} \arrow[r, "p"] \arrow[d, "i"'] & {} \arrow[d, "j"] \\ {} \arrow[r, equal] \arrow[ur, dashed, "d"] & {} \end{tikzcd}$$

Then we see that i is a retract of j (the retraction diagram is given by the obvious arrows), and thus a map in $J{-}\mathrm{cof}$ (the proof of this last fact is exactly the same as for lemma 2.13).

An almost equal proof applies for the other direction. Assume $(I{-}\mathrm{cof} \cap W) \subseteq J{-}\mathrm{cof}$ and let $j \in J^{\pitchfork} \cap W$, then we can factorize $j = i \circ p$, where $i \in I^{\pitchfork}$ and $p \in I{-}\mathrm{cof} \cap W \subseteq J{-}\mathrm{cof}$. We have the diagram:

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "p"'] & \cdot \arrow[d, "j"] \\ \cdot \arrow[r, "i"'] \arrow[ur, dashed, "d"] & \cdot \end{tikzcd}$$

And again we can conclude by a retraction argument. ■

Now assume $(I-\text{cof}\cap W) \subseteq J-\text{cof}$. This means that every trivial cofibration is in $J-\text{cof} = {}^{\pitchfork}(J^{\pitchfork})$ and has the left lifting property relative to the class of fibrations $J^{\pitchfork}$. Let f be a fibration, then we can factorize $f = i \circ p$, where $i \in I-\text{cell} \subseteq I-\text{cof}$ (by lemma 2.23) and $p \in I^{\pitchfork} \subseteq (I-\text{cof}\cap W)$ (by assumption). By the 2-out-of-3 property, i is also a weak equivalence, and thus a trivial cofibration. By the first part of the proof of the lifting property, $i \pitchfork F$. By lemma 2.17, f is a retract of p, and so $f \in I^{\pitchfork}$ (by remark 2.14). Notice that this implies that f has the right lifting property relative to all maps in $I-\text{cof} = C$. This proves that $(C \cap W, F)$ is a weak factorization system.

The proof for $(C, F \cap W)$ is done in a similar way assuming $(J^{\pitchfork} \cap W) \subseteq I^{\pitchfork}$. □

2.3 The homotopy category $\text{Ho}\mathcal{C}$

As already mentioned, the reason behind model categories is the study of homotopy categories, that is, we take a category $\mathcal{C}$ with some set of arrows W, and we want to study $\mathcal{C}$ up to identification given by arrows in W. A good example is the study of homotopy theory in algebraic topology (the set of arrows W will be given by the weak homotopy equivalences $W_{\mathbf{Top}}$, which we will define in the section dedicated to topological spaces).

Definition 2.25. *Let $\mathcal{C}$ be a category, W a subcategory of $\mathcal{C}$. The* homotopy category $\mathcal{C}[W^{-1}] = \text{Ho}\mathcal{C}$ *is the localization of $\mathcal{C}$ in W. It is determined by the following universal property: let $\text{Ho} : \mathcal{C} \to \text{Ho}\mathcal{C}$ be the localization functor, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor in a category $\mathcal{D}$ such that the image of every arrow in W is invertible. Then there is a functor (indicated by the dashed arrow) making the following diagram commute:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\text{Ho}"] \arrow[dr, "F"'] & \text{Ho}\mathcal{C} \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd}$$

Said in other words, we take the category $\mathcal{C}$ and add (formal) inverses to every arrow in W and complete the so obtained object with all possible compositions, in order to have a (meta)category again.

For general categories, $\text{Ho}\mathcal{C}\,(\text{X}, \text{Y})$ is not necessarily a set for $X, Y \in \mathcal{C}$, thus in general $\text{Ho}\mathcal{C}$ is *not* a category. However, this is true in case $\mathcal{C}$ is a model category, whence their importance.

Definition 2.26. *Let $\mathcal{C}$ be a model category, X an object in $\mathcal{C}$. Denote by $\mathbf{0}$ the initial object of $\mathcal{C}$, by $\mathbf{1}$ its final object. X is said to be* fibrant *if the unique map $X \to \mathbf{1}$ is a fibration. It is said to be* cofibrant *if the map $\mathbf{0} \to X$ is a cofibration.*
A fibrant replacement *is a functor $P : \mathcal{C} \to \mathcal{C}$ which assigns to every object $X \in \mathcal{C}$ a fibrant objet $P(X)$, together with a natural transformation $\mathbf{id} \Rightarrow P$ that is a weak equivalence at every object.*
In the case of a cofibrantly generated model category, $P(X)$ can be found by factoring the unique morphism $X \to \mathbf{1}$ into a trivial cofibration and a fibration:

$$\begin{tikzcd} X \arrow[r, tail, "\sim"] \arrow[dr] & P(X) \arrow[d, two heads] \\ & \mathbf{1} \end{tikzcd}$$

Similarly, a cofibrant replacement *$Q : \mathcal{C} \to \mathcal{C}$ is a functor assigning (through an analogue process) to every $X \in \mathcal{C}$ a cofibrant object $Q(X)$ together with a natural transformation $Q \Rightarrow \mathbf{id}$ that is a weak equivalence at every object.*

Remark 2.27. In a cofibrantly generated model category, the fibrant and cofibrant replacements are easily seen to be functors since the factorization systems $(C, F \cap W)$ and $(C \cap W, F)$ are functorial.

The importance of the fibrant and cofibrant replacements is due to the fact that they are used to prove that the class $\mathrm{Ho}\mathcal{M}\,(\mathrm{X}, \mathrm{Y})$ is actually a set. More precisely we have a bijection:

$$\mathrm{Ho}\mathcal{M}\,(\mathrm{X}, \mathrm{Y}) \cong \mathcal{M}\,(\mathrm{Q}(\mathrm{X}), \mathrm{P}(\mathrm{Y}))/\sim$$

where the equivalence relation $\sim$ is the homotopy equivalence relation defined from the axioms of the model category (see [3, p. 13] for details). This is true for any choice of fibrant and cofibrant replacements. In particular, this shows that the homotopy category of a model category is really a category with small hom-sets.

In [3, p. 13], we find the following result, which we state as a lemma (without giving a proof here) in order to refer to it later:

Lemma 2.28. *Let $\mathcal{M}$ be a model category. Then a map f is a weak equivalence in $\mathcal{M}$ if, and only if it is an isomorphism in the homotopy category* $\mathrm{Ho}\mathcal{M}$*.*

2.4 Quillen adjunctions

Definition 2.29. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ (from $\mathcal{C}$ to $\mathcal{D}$) is a triple (L, R, φ) where $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are functors and φ assigns to every couple of objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$ a bijection $\varphi_{X,Y} : \mathcal{D}\,(\mathrm{L}(\mathrm{X}), \mathrm{Y}) \xrightarrow{\sim} \mathcal{C}\,(\mathrm{X}, \mathrm{R}(\mathrm{Y}))$ which is natural in both X and Y. L is called a left adjoint, R a right adjoint.*

The following general result about adjunction will be useful in the treatment of simplicial sets.

Lemma 2.30. *Let (L, R, φ) be an adjunction from $\mathcal{C}$ to $\mathcal{D}$, A a class of maps in $\mathcal{C}$ and B a class of maps in $\mathcal{D}$. Then:*

i. $R(L(A)^{\pitchfork}) \subseteq A^{\pitchfork}$

ii. $L(A-\mathrm{cof}) \subseteq L(A)-\mathrm{cof}$

iii. $L({}^{\pitchfork}R(B)) \subseteq {}^{\pitchfork}B$

iv. $R(({}^{\pitchfork}B)^{\pitchfork}) \subseteq {}^{\pitchfork}(R(B))^{\pitchfork}$

Proof. For (i.), let $g \in L(A)^{\pitchfork}$ and $f \in A$. Then g has the right lifting property relative to $L(f)$. By adjointness, $R(g)$ has the right lifting property relative to f, and so $R(g) \in A^{\pitchfork}$.

For (ii.), let $f \in A-\mathrm{cof}$ and $g \in L(A)^{\pitchfork}$. Then by (i.), $R(g) \in A^{\pitchfork}$, and thus f has the left lifting property relative to $R(g)$. By adjointness, $L(f)$ has the left lifting property relative to g, and thus $L(f) \in L(A)-\mathrm{cof}$.

(iii.) and (iv.) are dual statements to (i.) and (ii.). □

Definition 2.31. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. A functor $L : \mathcal{M} \to \mathcal{N}$ is a* left Quillen functor *if it is a left adjoint and it preserves cofibrations and trivial cofibrations.*
A functor $R : \mathcal{N} \to \mathcal{M}$ is a right Quillen functor *if it is a right adjoint ad it preserves fibrations and trivial fibrations.*

Definition 2.32. *A* Quillen adjunction *between two model categories $\mathcal{M}$ and $\mathcal{N}$ (from $\mathcal{M}$ to $\mathcal{N}$) is an adjunction (L, R, φ) from $\mathcal{M}$ to $\mathcal{N}$ with L a left Quillen functor and R a right Quillen functor.*

Given a Quillen adjunction (L, R, φ), we will usually denote the unit $X \to RL(X)$ by η and the counit $LR(X) \to X$ by ε. The unit is obtained as the right adjoint of the identity map $\mathbf{id}_{L(X)} : L(X) \to L(X)$, similarly the counit is the left adjoint of the identity map of $R(X)$.

In fact, it is enough that one of the two functors be Quillen to have a Quillen adjunction, as the following lemma shows.

Lemma 2.33. *Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories, (L, R, φ) an adjunction between them. The following are equivalent:*

i. R is a right Quillen functor.

ii. L is a left Quillen functor.

iii. (L, R, φ) is a Quillen adjunction.

Proof. We show that (i.) implies (ii.). Assume (L, R, φ) is an adjunction and R is a right Quillen functor. Then L is a left adjoint and we only have to prove that it preserves cofibrations and trivial cofibrations. Remember that $C = {}^{\pitchfork}(F \cap W)$ and $C \cap W = {}^{\pitchfork}F$.
Let f be a cofibration in $\mathcal{M}$, p a trivial fibration in $\mathcal{N}$. Then since R is Quillen, for every commuting square there is a diagonal arrow a such that:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, tail, "f"'] & R(Z) \arrow[d, two heads, "\wr R(p)"] \\ Y \arrow[r, "l"'] \arrow[ur, dashed, "a"] & R(W) \end{tikzcd}$$

By naturality of φ we have than the following commuting diagram:

$$\begin{tikzcd} L(X) \arrow[r, "\varphi^{-1}(k)"] \arrow[d, tail, "L(f)"'] & Z \arrow[d, two heads, "\wr p"] \\ L(Y) \arrow[r, "\varphi^{-1}(l)"'] \arrow[ur, dashed, "\varphi^{-1}(a)"] & W \end{tikzcd}$$

Thus $L(f)$ has the left lifting property with respect to every trivial cofibration, and thus it is a fibration in $\mathcal{N}$. The proof of the fact the L preserves trivial cofibrations is done similarly.
Dually we obtain that (ii.) implies (i.). Thus if we have that one of the adjoints is Quillen, so is the other and thus the adjunction is Quillen (by definition). Conversely, if the adjunction is Quillen, both of the functors are Quillen. $\square$

Thank to this result, it is possible to express any Quillen adjunction by its left Quillen functor alone.

As we have mentioned before, if we have a cofibrantly generated model category it is much easier to show that an adjunction is Quillen. The next lemma shows how.

Proposition 2.34. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, assume that $\mathcal{M}$ is cofibrantly generated by a set of generating fibrations I and a set of generating trivial cofibrations J. Let $(L, R, \varphi) : \mathcal{M} \to \mathcal{N}$ be an adjunction. The following are equivalent:*

i. *(L, R, φ) is a Quillen adjunction.*

ii. *For every $i \in I$ and for every $j \in J$, $L(i)$ is a cofibration and $L(j)$ is a trivial cofibration.*

Proof. (ii.) follows from (i.) by definition of Quillen functors.

Now assume (ii.) holds. We show that L is a left Quillen functor. This will be enough by what we proved in the last lemma. Let $I_{\mathcal{C}}$, $I_{\mathcal{D}}$ denote the sets generating cofibrations in $\mathcal{C}$ and $\mathcal{D}$ respectively. By lemma 2.30, $L(I_{\mathcal{C}}-\text{cof}) \subseteq L(I_{\mathcal{C}})-\text{cof}$. But by assumption we know that $L(I_{\mathcal{C}}) \subseteq I_{\mathcal{D}}-\text{cof}$, and thus that $L(I_{\mathcal{C}})-\text{cof} \subseteq I_{\mathcal{D}}-\text{cof}$. So $L(I_{\mathcal{C}}-\text{cof}) \subseteq L(I_{\mathcal{C}})-\text{cof} \subseteq I_{\mathcal{D}}-\text{cof}$, or, said in words, L preserves cofibrations. A similar argument shows that L preserves trivial cofibrations. □

Quillen adjunctions between model categories induce adjunctions in the homotopy categories. More precisely, if we have an adjunction between two model categories $\mathcal{M}$ and $\mathcal{N}$:

$$\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N}$$

then it induces an adjunction (called the *derived adjunction*) between the homotopy categories as follows:

$$\mathrm{Ho}\mathcal{M} \underset{RP}{\overset{LQ}{\rightleftarrows}} \mathrm{Ho}\mathcal{N}$$

A proof that this is really an adjunction can be found in [3, p. 18].

Remark 2.35. Let $X \in \mathrm{Ho}\mathcal{C}$ be a cofibrant object, then the unit at X is given by the composition

$$X \xrightarrow{\eta} RL(X) \xrightarrow{R(i)} RPL(X)$$

where η is the unit of the original adjunction, and i is the fibrant replacement of $L(X)$. Notice that i is a weak equivalence. For general $X \in \mathrm{Ho}\mathcal{C}$, the unit is given by

$$X \xleftarrow{\sim} Q(X) \longrightarrow RPLQ(X)$$

where the leftmost arrow is the cofibrant replacement of X. Note that since it is a weak equivalence it is invertible in the homotopy category. The counit is constructed in a similar way.

The functors in the new adjunction are called the *derived functors* of L and R (see for example [3, p. 16] for more on derived functors). We will see that if the Quillen adjunction is a Quillen equivalence, then the adjunction induced on the homotopy categories is in fact an equivalence of categories.

2.5 Quillen equivalences

Definition 2.36. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories A* Quillen equivalence *between $\mathcal{M}$ and $\mathcal{N}$ is a Quillen adjunction (L, R, φ) from $\mathcal{M}$ to $\mathcal{N}$ such that for every cofibrant $X \in \mathcal{M}$ and every fibrant $Y \in \mathcal{N}$, a map $f : L(X) \to Y$ is a weak equivalence in $\mathcal{M}$ if and only if $\varphi(f) : X \to R(Y)$ is a weak equivalence in $\mathcal{N}$.*

Definition 2.37. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An* equivalence of categories *between $\mathcal{C}$ and $\mathcal{D}$ is composed by two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\varepsilon : FG \Rightarrow \mathbf{id}_{\mathcal{D}}$ and $\eta : \mathbf{id}_C \Rightarrow GF$.*

Having an equivalence between two categories means that they are "essentially the same". The importance of Quillen equivalences comes from the fact that even if it isn't an equivalence of categories on its own, the derived adjunction is an equivalence of categories. More precisely:

Proposition 2.38. *Let $(L, R, \varphi) : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction. Then the following are equivalent:*

i. *(L, R, φ) is a Quillen equivalence.*

ii. *The composite map*

$$X \xrightarrow{\eta} RL(X) \xrightarrow{R(i)} RPL(X)$$

is a weak equivalence for every cofibrant X, and the composite

$$LQR(X) \xrightarrow{L(p)} LR(X) \xrightarrow{\varepsilon} X$$

is a weak equivalence for every fibrant X.

Those are the maps contructed in remark 2.35 and represent unit and counit in the homotopy category.

iii. *$(LQ, RP, \tilde{\varphi}) : \mathrm{Ho}\mathcal{M} \to \mathrm{Ho}\mathcal{N}$ is an equivalence of categories (for some choice of $\tilde{\varphi}$).*

Proof. We will prove that $i.$ is equivalent to $ii.$, and that $ii.$ is equivalent to $iii.$.
<u>$i. \Rightarrow ii.$:</u> Assume (L, R, φ) is a Quillen equivalence. Recall that the map $L(X) \to PL(X)$ is a weak equivalence, $L(X)$ is cofibrant (since L is a left Quillen functor) and $PL(X)$ is fibrant. Thus the adjoint map $X \to RPL(X)$ is also a weak equivalence, because the adjunction is a Quillen equivalence. The derivation for the second map is dual.
<u>$ii. \Rightarrow i.$:</u> Assume $ii.$ holds, let $X \in \mathcal{M}$ be cofibrant and $Y \in \mathcal{N}$ be fibrant, and $f : L(X) \to Y$ be a weak equivalence in $\mathcal{N}$. Then $\varphi(f)$ is given by the composition

$$X \xrightarrow{\eta} RL(X) \xrightarrow{R(f)} R(Y)$$

We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, equal] & RL(X) \arrow[r, "R(f)"] \arrow[d, "R(p_{L(X)})"] & R(Y) \arrow[d, "R(p_Y)"] \\
X \arrow[r, "\sim"] & RPL(X) \arrow[r, "RP(f)"'] & RP(Y)
\end{tikzcd}$$

where p_Y denotes the arrow $Y \to P(Y)$ obtained when applying the fibrant replacement, and $p_{L(X)}$ is obtained in the same way. Those two maps are trivial cofibrations by construction. The map $X \to RPL(X)$ is a weak equivalence by assumption. Since f is a weak equivalence, so is $P(f)$, as the following diagram indicates:

$$\begin{tikzcd}
L(X) \arrow[r, "f"] \arrow[d, tail, "\wr", "C \cap W \ni"'] & Y \arrow[d, tail, "\wr"', "\in C \cap W"] \\
PL(X) \arrow[r, "P(f)"'] \arrow[d, two heads, "F \ni"'] & P(Y) \arrow[d, two heads, "\in F"] \\
\mathbf{1} \arrow[r, equal] & \mathbf{1}
\end{tikzcd}$$

Notice that R, being a right Quillen functor, preserves weak equivalences between fibrant objects, and thus $RP(f)$ is a weak equivalence. Now the 2-out-of-3 property assures that $\varphi(f)$ is a weak equivalence. A similar proof shows that if $\varphi(f) : X \to R(Y)$ is a weak equivalence, then so is f.
<u>*ii.* $\Rightarrow$ *iii.*:</u> Assume *ii.* holds. Notice that by construction, the map $X \to Q(X)$ is a weak equivalence. Now consider the following sequence of arrows:

$$X \xleftarrow{\sim} Q(X) \xrightarrow{\sim} RPLQ(X)$$

where the rightmost map is a weak equivalence by assumption. Recall that when localizing, weak equivalences become isomorphisms, and thus that gives us an isomorphism $\eta_X : X \to RPLQ(X)$ which is in fact a natural transformation $\eta : \mathbf{id}_{\mathrm{Ho}\mathcal{M}} \Rightarrow (RP)(LQ)$. In the same way we obtain a natural isomorphism $\epsilon : (LQ)(RP) \Rightarrow \mathbf{id}_{\mathrm{Ho}\mathcal{N}}$. Those are in fact the unit and counit of the derived adjunction, as we have stated in remark 2.35. Thus we have proved that the adjunction induced on the homotopy categories is in fact an equivalence of categories.
<u>*iii.* $\Rightarrow$ *ii.*:</u> We can proceed as in the proof of *ii.* $\Rightarrow$ *iii.*, and use lemma 2.28 to deduce that the arrow $Q(X) \to RPLQ(X)$ is a weak equivalence. If X is taken to be cofibrant, then $X \cong Q(X)$, and we are done. The proof for the second arrow is similar. $\square$

To conclude this section, we present a criterion which permits us to check if a Quillen adjunction is a Quillen equivalence.

Definition 2.39. *A functor $F : \mathcal{C} \to \mathcal{D}$ is said to* reflect *some property if for any arrow $f \in \mathcal{C}$ such that $F(f)$ has said property, f also has the property.*

Proposition 2.40. *Let $(L, R, \varphi) : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction between two model categories $\mathcal{M}$ and $\mathcal{N}$. The following are equivalent:*

i. *(L, R, φ) is a Quillen equivalence.*

ii. L reflects weak equivalences between cofibrant objects and for every fibrant $Y \in \mathcal{N}$ *the map* $LQR(Y) \to Y$ *is a weak equivalence in* $\mathcal{N}$*.*

iii. R reflects weak equivalences between fibrant objects and for every cofibrant $X \in \mathcal{M}$ *the map* $X \to RPL(X)$ *is a weak equivalence in* $\mathcal{M}$*.*

The map $LQR(Y) \to Y$ *is found by applying the bijection* $\mathcal{N}(\mathrm{LQR(Y), X}) \cong \mathcal{M}(\mathrm{QR(Y), R(X)})$ *to the cofibrant replacement* $QR(Y) \to R(Y)$*. The map* $X \to RPL(X)$ *is obtained similarly.*

Proof. We will prove that $i.$ is equivalent to $ii.$, the proof that $i.$ is equivalent to $iii.$ is dual.
$\underline{i. \Rightarrow ii.}$: Assume (L, R, φ) is a Quillen equivalence. As we have seen in proposition 2.38 the map $LQR(Y) \to Y$ is a weak equivalence for every Y which is fibrant.

Now let $X, Y \in \mathcal{M}$ be cofibrant objects, $f : X \to Y$ be such that $L(f)$ is a weak equivalence. Since L preserves weak equivalences between cofibrant objects, $LQ(f)$ is a weak equivalence. But this means that $LQ(f)$ is an isomorphism in the homotopy category $\mathrm{Ho}\mathcal{N}$. Since the adjunction $(LQ, RP, \tilde{\varphi})$ is an equivalence of categories (again by proposition 2.38), f is an isomorphism in $\mathrm{Ho}\mathcal{M}$. This is true if, and only if, f is a weak equivalence (by lemma 2.28). Thus L reflects weak equivalences between cofibrant objects.
$\underline{ii. \Rightarrow i.}$: We show that $(LQ, RP, \tilde{\varphi})$ is an equivalence of categories, then proposition 2.38 concludes the claim. We have to show that unit and counit maps for this adjunction are isomorphisms in the homotopy categories. As stated in remark 2.35, the counit map makes the following diagram commute:

$$\begin{tikzcd} LQRP(X) \arrow[r, "\sim"] \arrow[rr, bend right] & P(X) & X \arrow[l, hook', "\sim"'] \end{tikzcd}$$

where the left arrow is a weak equivalence by assumption, and the right one by construction. Thus the counit map is an isomorphism at every level in the homotopy category.

Notice that the composition of the unit map $X \to RPLQ(X)$ with LQ gives us the map $LQ(X) \to LQRPLQ(X)$ which is inverse to the counit, and thus an isomorphism (in the homotopy category). L reflects weak equivalences between cofibrant objects, thus $Q(X) \to QRPLQ(X)$ is an isomorphism. Q reflects all weak equivalences, thus the unit map is an isomorphism at every level, and $(LQ, RP, \varphi Q)$ is an equivalence of categories. $\square$

3 The model structure on topological spaces

The category of topological spaces, denoted by **Top**, is the category with all topological spaces as objects, and the continuous functions between topological spaces as arrows.

3.1 Completeness and cocompleteness of Top

The category **Top** is both complete and cocomplete. In fact there is an easy way to construct limits and colimits in it. Let $U : \mathbf{Top} \to \mathbf{Sets}$ be the forgetful functor associating to every topological space its underlying set. Let $\mathcal{I}$ be a small category and $F : \mathcal{I} \to \mathbf{Top}$ be a functor of which we would like to find the limit. In order to do that, we construct first the limit of the functor $G = U \circ F : \mathcal{I} \to \mathbf{Sets}$ (which we know to exist, since **Sets** is complete), then we topologize it with the initial topology from the maps $\lim G \to F(i)$, i.e. by defining a subset of $\lim G$ to be open if, and only if it is the preimage of an open set in some of the $F(i)$ (an alternative description of this topology is the subspace topology of $\prod_{i \in \mathcal{I}} F(i)$ in the product topology). Similarly, if we want to find the colimit of F, we first find the colimit of G and put on it the final topology from the maps $F(i) \to \mathrm{colim} G$, i.e. by defining a subset of $\mathrm{colim} G$ to be open if, and only if its preimage under every one of said maps is open (alternative definition: the quotient topology from $\coprod_{i \in \mathcal{I}} F(i)$).

We give as example the explicit construction of pushouts.

Example 3.1. *We want to construct the colimit of the following diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X & \end{tikzcd}$$

The colimit of the diagram in the category **Sets** *of sets is given by*

$$X \cup_A Y = (X \sqcup Y)/\{f(a) \sim g(a) | a \in A\}$$

We put on this set the final topology with respect to the inclusions of X and Y, given by $i_X(x) = [x]$ and $i_y(y) = [y]$ respectively. That means that a subset V of $X \cup_A Y$ is open if, and only if, $i_X^{-1}(V)$ is open in X and $i_Y^{-1}(V)$ is open in Y.

3.2 The model structure on Top

Here and in the rest of this section, we denote the closed unit interval by $I = [0, 1]$. Every map is an arrow in **Top** , and thus a continuous function. We also denote by S^{n-1} the unit sphere in $\mathbb{R}^n$, and by D^n the closed unit ball in $\mathbb{R}^n$ (with the standard topology).

We define the following sets of maps:

- $W_{\mathbf{Top}}$ is the set of maps map $f : X \to Y$ such that $X \neq \emptyset$ and

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a group isomorphism for every $n \geq 0$ and every $x \in X$ plus the identity map of the empty set; we call those arrows *weak (homotopy) equivalences*

- $I_{\mathbf{Top}} = \{S^{n-1} \hookrightarrow D^n | n \geq 0\}$
- $J_{\mathbf{Top}} = \{D^n \to D^n \times I,\ x \mapsto (x, 0) | n \geq 0\}$
- $F_{\mathbf{Top}} = J_{\mathbf{Top}}^{\pitchfork}$, we call those arrows *fibrations*
- $C_{\mathbf{Top}} = I_{\mathbf{Top}}-\text{cof} = {}^{\pitchfork}(I_{\mathbf{Top}}^{\pitchfork})$, we call those arrow *cofibrations*

We claim those sets define a model structure on **Top**.

Theorem 3.2. $(\mathbf{Top}, C_{\mathbf{Top}}, F_{\mathbf{Top}}, W_{\mathbf{Top}})$ *is a model category which is cofibrantly generated with* $I_{\mathbf{Top}}$ *as set of generating cofibrations and* $J_{\mathbf{Top}}$ *as set of generating trivial cofibrations.*

Remark 3.3. Maps in $I_{\mathbf{Top}}$-cell are usually called *relative cell complexes.* A special case are the *relative CW-complexes.* In particular, every map in $J_{\mathbf{Top}}$ can be described as a relative CW-complex, thus we have the relation $J_{\mathbf{Top}} \subseteq I_{\mathbf{Top}}-\text{cell} \subseteq I_{\mathbf{Top}}-\text{cof}$, and thus $J_{\mathbf{Top}}-\text{cof} \subseteq I_{\mathbf{Top}}-\text{cof}$. The fibrations are often called Serre fibrations in the literature.

Proof. We prove this theorem by verifying the assumptions of theorem 2.24. We do this in a number of smaller lemmas and theorems, of which we give an overview below:

a) $W_{\mathbf{Top}}$ satisfies the 2-out-of-3 property and is closed under retracts (**lemma 3.4**)

b) the domains of the maps in $I_{\mathbf{Top}}$, $J_{\mathbf{Top}}$ are small relative to $I_{\mathbf{Top}}$-cell and $J_{\mathbf{Top}}$-cell respectively (**proposition 3.9**)

c) $J_{\mathbf{Top}}-\text{cell} \subseteq (I_{\mathbf{Top}}-\text{cof} \cap W_{\mathbf{Top}})$ (**proposition 3.11**)

d) $I_{\mathbf{Top}}^{\pitchfork} \subseteq (J_{\mathbf{Top}}^{\pitchfork} \cap W_{\mathbf{Top}})$ (**lemma 3.12**)

e) $(J_{\mathbf{Top}}^{\pitchfork} \cap W_{\mathbf{Top}}) \subseteq I_{\mathbf{Top}}^{\pitchfork}$ (**proposition 3.13**)

We will not prove point (e). As it would have required us to introduce more technical results of algebraic topology. $\square$

We begin by proving that the weak equivalences satisfy the required properties. After we have done this, we will turn to prove the properties needed on $I_{\mathbf{Top}}$ and $J_{\mathbf{Top}}$.

Lemma 3.4. $W_{\mathbf{Top}}$ *satisfies the 2-out-of-3 property and is closed under retracts.*

Proof. Let $f : X \to Y$, $g : Y \to Z$, $h : X \to X$ be maps such that $g \circ f = h$. If f and g are in $W_{\mathbf{Top}}$, then obviously so is h. The same is true for g and h in $W_{\mathbf{Top}}$. The problem that could arise if we have f and h in $W_{\mathbf{Top}}$ and we want to show that g is in $W_{\mathbf{Top}}$ is that we could have a point $y \in Y$ which is not in the image of f (and thus we would have, a priori, no means to check if $\pi_n(g, y)$ is a bijection). This potential problem is solved by noticing that if $y_1, y_2 \in Y$ are in the same path connected component, then $\pi_n(Y, y_1) \cong \pi_n(Y, y_2)$, and that with $\pi_0(f)$ the problem is not present.

If we have a retract f of some $w \in W_{\mathbf{Top}}$, we can take the image of the retraction diagram under π_n. Then it is easy to directly obtain an inverse for $\pi_n(w, \cdot)$ using the fact that $\pi_n(f, \cdot)$ is a bijection. $\square$

We attack now the problem of showing that the domains of the maps of $I_{\mathbf{Top}}$ and $J_{\mathbf{Top}}$ are small relative to $I_{\mathbf{Top}}$-cell and $J_{\mathbf{Top}}$-cell respectively.

Definition 3.5. *A map $f : X \to Y$ is a* closed T_1 inclusion *if it is an inclusion of X in Y, it is a closed map (equivalently: $f(X)$ is closed in Y) and every point in $Y \backslash f(X)$ is closed in Y.*

Lemma 3.6. *Closed T_1 inclusions are closed under pushouts and transfinite compositions.*

In particular, every map in $I_{\mathbf{Top}}$-cell is a closed T_1 inclusion.

Proof. Let $i \in I_{\mathbf{Top}}$, j a pushout of i. This means we have a pushout diagram:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "b"'] & X \cup_A Y \end{tikzcd}$$

Then j is injective. We show j is a closed inclusion. By the construction of the topology on the pushout $X \cup_A Y$, $j(Y)$ is closed if and only if $b^{-1}(j(Y))$ is closed in X. Since i is injective we have $a^{-1}(j(Y)) = i(b^{-1}(Y))$. Since i is a closed map, this set is closed in X, and thus $j(Y)$ is closed in the pushout. Hence, j is a closed inclusion. Now let $p \in X \cup_A Y \backslash j(Y)$. Then, by construction of the pushout, $b^{-1}(p)$ is a single point $x \in X \backslash i(A)$. Since i is a closed T_1 inclusion, $\{x\}$ is closed in X. It follows (again by how pushouts are contructed in **Top**) that $\{p\}$ is closed in $X \cup_A Y$. Thus j is a closed T_1 inclusion.

Let $X : \lambda \to$ **Top** be a λ-sequence of closed T_1 inclusions. This means that every map $X_\alpha \to X_{\alpha+1}$ is a closed T_1 inclusion. We use transfinite induction to show that the composition of the sequence is again a closed T_1 inclusion. The case of a finite composition is trivial, so let α be a limit ordinal and assume $X_0 \to X_\beta$ is a closed T_1 inclusion for every $\beta < \alpha$. Denote by i_β the composition $X_0 \to X_\beta$, and by i^β the composition $X_\beta \to X_\alpha$. We have to show that f is injective, a homeomorphism with its image, a closed map and that every point

not in $f(X_0)$ is closed.
i_α is injective: It is easy to prove that in fact any transfinite composition of injective arrows is again injective.
i_α is an inclusion: We show that i_α is a homeomorphism with its image by proving it is a closed map. Let $U \subseteq X_0$ be closed, then by induction hypothesis, $i_\beta(U)$ is closed in X_β. Since $(i^\beta)^{-1}(i_\alpha(U)) = i_\beta(U)$ (by injectivity of all the maps), we have that $i_\alpha(U)$ is closed in X_α (recall that the colimit X_α has the final topology of the maps i^β).
i_α is closed: By induction hypothesis, $i_\beta(X_0)$ is closed in X_β for every $\beta < \alpha$. Notice that since every i^β is injective, we have $(i^\beta)^{-1}(i_\alpha(X_0)) = i_\beta(X_0)$. Again by construction of the colimit, this means that $i_\alpha(X_0)$ is closed in X_α.
Every $x \in X_\alpha \backslash i_\alpha(X_0)$ is closed: Let $x \in X_\alpha \backslash i_\alpha(X_0)$. By injectivity of all the i^β, we have that $(i^\beta)^{-1}(\{x\})$ is always either empty of a point in $X_\beta \backslash i_\beta(X_0)$, and thus closed. This implies that x is closed in X_α.

Notice that every map in $I_{\mathbf{Top}}$ is a closed T_1 inclusion. Thus every map in $I_{\mathbf{Top}}-$cell is a closed T_1 inclusion by definition. □

Lemma 3.7. *Compact topological spaces are finite (i.e. ω-small) relative to closed T_1 inclusions.*

Proof. Let λ be a limit ordinal (any limit ordinal is good, since every ordinal is ω-filtered), $X : \lambda \to \mathbf{Top}$ a λ-sequence of closed T_1 inclusions. Then for every $\alpha < \lambda$, lemma 3.6 shows that the map $X_\alpha \to \mathrm{colim}_{\beta<\lambda} X_\beta =: X_\lambda$ is again a closed T_1 inclusion. Because of this fact, we will abuse of notation and denote the image of X_α in X_λ again by X_α.

Let A be a compact topological space and $f : A \to \mathrm{colim}_{\beta<\lambda} X_\beta$. We have to prove that f factors through some X_α, i.e. that there exist some $\alpha < \lambda$ such that $f(A)$ is completely contained in the image of X_α in $\mathrm{colim}_{\beta<\lambda} X_\beta$.

Assume it is not true. Then there exist a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of ordinals smaller than λ together with a sequence $S = \{x_n\}_{n\in\mathbb{N}} \subseteq f(A)$ such that $x_n \in X_{\alpha_n} \backslash X_{\alpha_{n-1}}$, where we take $\alpha_0 = 0$. We will show that S is closed and discrete as a subspace of X_λ. This implies that it has no limit points, which is in contradiction with the fact that $f(A)$ is compact.

Let $\mu = \sup_n \alpha_n \leq \lambda$. Then μ is a limit ordinal and $X_\mu = \mathrm{colim}_{\beta<\mu} X_\beta = \mathrm{colim}_n X_{\alpha_n}$. By construction, the intersection of any subset $S' \subseteq S$ with any of the X_{α_n} is finite and don't intersect $X_0 = X_{\alpha_0}$, and thus closed. Recall that X_μ is endowed with the final topology induced by the inclusions of the X_{α_n}, thus S' is closed in the subspace topology of X_μ. Then, since the map $X_\mu \to X_\lambda$ is a closed T_1 inclusion, and in particular a closed map, S' is closed as a subspace of X_λ. In particular we have that S is closed (take $S' = S$) and discrete (every subset of S is closed implies that every subset of S is also open).

Uniqueness of the factorization is a trivial consequence of the fact that any (transfinite) composition of closed T_1 inclusions is a closed T_1 inclusion (lemma 3.6). □

This, together with lemma 3.6, shows that the domains of the maps of $I_{\mathbf{Top}}$ (which are compact) are small relative to $I_{\mathbf{Top}}$-cell. For $J_{\mathbf{Top}}$ we have:

Lemma 3.8. *Every map in $I_{\mathbf{Top}}$-cof is a closed T_1 inclusion.*

Proof. Since we know that the domain of every map in $I_{\mathbf{Top}}$ is small relative to $I_{\mathbf{Top}}$-cell, we can apply lemma 2.19, which tells us that every map in $I_{\mathbf{Top}}$-cof is the retract of some map in $I_{\mathbf{Top}}$-cell. Thus it is enough to show that closed T_1 inclusions are closed under retracts.

Suppose f is the retract of some closed T_1 inclusion g with the following retraction diagram:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] \arrow[rr, bend left, equal] & X \arrow[r, "r"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ B \arrow[r, "i"'] \arrow[rr, bend right, equal] & Y \arrow[r, "b"'] & B \end{tikzcd}$$

Then it is easy to see that f must be injective. To show that f is a closed inclusion, it is enough to show that for every $C \subseteq A$ closed, $f(C)$ is closed in B. But $f(C) = (i^{-1} \circ g \circ r^{-1})(C)$, and thus it is closed.

Now let $y \in B \backslash f(A)$. Then we have that $i(y) \notin g(X)$. Indeed, if $i(y) = g(x)$ for some $x \in X$, then we would have $y = b(g(x)) = f(r(x))$, which would be a contradiction. Thus $i(y)$ is closed. i is injective, so we have that $y = i^{-1}(i(y))$ is closed. It follows that f is a closed T_1 inclusion. □

Thanks to those results, we can complete the proof of point (b):

Proposition 3.9. *The domains of the maps in $I_{\mathbf{Top}}$ are small relative to $I_{\mathbf{Top}}-$cell, and the domains of the maps in $J_{\mathbf{Top}}$ are small relative to $J_{\mathbf{Top}}-$cell.*

Proof. As stated before, lemma 3.7 and lemma 3.6 imply that the domains of the maps in $I_{\mathbf{Top}}$ are small relative to $I_{\mathbf{Top}}-$cell.

For the second part of the theorem, we have that $J_{\mathbf{Top}}-\text{cell} \subseteq J_{\mathbf{Top}}-\text{cof} \subseteq I_{\mathbf{Top}}-\text{cof}$, thus every map in $J_{\mathbf{Top}}-$cell is a closed T_1 inclusion. Again by lemma 3.7, we conclude that the domains of the maps in $J_{\mathbf{Top}}$ are small relative to $J_{\mathbf{Top}}-$cell. □

In order to go on, we need a couple of results.

Lemma 3.10. *Let $X : \lambda \to \mathbf{Top}$ be a λ-sequence of maps which are both weak equivalences and closed T_1 inclusions. Then the map $X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha$ is both a weak equivalences and a closed T_1 inclusions.*

Proof. Since we have shown that closed T_1 inclusions are closed under transfinite compositions (lemma 3.6), it will be enough to show that for every $\alpha \leq \lambda$, the map $i_\alpha : X_0 \to X_\alpha$ is a weak equivalence. We will do so using using transfinite induction. This is trivial for $\alpha = 0$, and if we assume we have proved the fact up to α, then the case $\alpha+1$ is also true, since weak equivalences are stable under composition. So let α be a limit ordinal, $n>0$ and fix a base point $x_0 \in X_0$.

We show $\pi_n(i_\alpha, x_0)$ is surjective. Let $[f] \in \pi_n(X_\alpha, i_\alpha(x_0))$ be the equivalence class of some map $f : (S^n, *) \to (X_\alpha, x_0)$. Then by lemma 3.7, f factors through some map $g : (S^n, *) \to X_\beta$ for some $\beta < \alpha$. Thus $[f]$ is in the image of $\pi_n(X_\beta, i_\beta(x_0))$. Since i_β is a weak equivalence, $[f]$ is also in the image of i_α.

We show $\pi_n(i_\alpha, x_0)$ is injective. Let $[f], [g] \in \pi_n(X_0, x_0)$ be two different elements of $\pi_n(X_0, x_0)$. Assume by contradiction that $i_\alpha(f)$ and $i_\alpha(g)$ are homotopic, with a basepoint-preserving homotopy $H : S^n \wedge I_+ \to X_\alpha$, where $S^n \wedge I_+ = (S^n \times I)/(* \times I)$ (i.e. the "path" of the basepoint in $S^n \times I$ is identified to a single point). Again by lemma 3.7, H factors through a map $H_\beta : S^n \wedge I_+ \to X_\beta$ for some $\beta < \alpha$. Then H_β is a basepoint-preserving homotopy between $i_\beta(f)$ and $i_\beta(g)$. Since i_β is a weak equivalence, f and g are homotopic in X_0, contradicting the assumption.

We are left to prove the case $n = 0$. Again, we use transfinite induction. If it is true for every ordinal up to α (included), then it is also true for $\alpha+1$, since the map $X_\alpha \to X_{\alpha+1}$ induces a bijection $\pi_0(X_\alpha) \cong \pi_0(X_{\alpha+1})$. So let α be a limit ordinal.

<u>Surjectivity:</u> We take a point $x \in X_\alpha$. Since X_α is the colimit of the λ-sequence restricted to all ordinals less than α, x must be in the image of X_β for some $\beta < \alpha$. This implies that the path-connected component $[x] \in \pi_0(X_\alpha)$ is in the image of $\pi_0(X_\beta)$, and thus the map induced on the π_0 by the composition of the sequence is surjective on path components.

<u>Injectivity:</u> Let $x, y \in X_\alpha$ be in the same path-connected component, and let $\gamma : I \to X_\alpha$ be a path from x to y. Then, by lemma 3.7 and the fact that α is ω-filtered, γ factors through X_β for some $\beta < \alpha$, and thus the preimages of x and y are in the same path-connected component of X_β. This implies that the map $\pi_0(X_0) \to \pi_0(X_\alpha)$ induced by the (partial) transfinite composition is injective. Indeed, if $[x_0], [y_0] \in \pi_0(X_0)$ are two path-connected components with the same image in $\pi_0(X_\alpha)$, by the reasoning above we get some $\beta < \alpha$ such that the image of $[x_0]$ and $[y_0]$ are the same. Thus, by induction hypothesis, $[x_0] = [y_0]$. $\square$

We prove $J_{\mathbf{Top}}-\mathrm{cell} \subseteq (I_{\mathbf{Top}}-\mathrm{cof} \cap W_{\mathbf{Top}})$. The following lemma does the job, together with lemma 2.23.

Proposition 3.11. *Every map in $J_{\mathbf{Top}}$-cof is a trivial cofibration.*

Let $X \subseteq Y$, $i : X \hookrightarrow Y$ the inclusion map. Recall that a *deformation retraction* is a continuous map $H : Y \times I \to Y$ such that $H(i(x), t) = i(x)$ for every $x \in X$ and $t \in I$, $H(y, 0) = y$ for every $y \in Y$ and $H(y, 1) \in i(X)$ for every $y \in Y$. Mote that $H(-, 1)$ is a retraction. If a deformation retraction exists, then i is called the *inclusion of the deformation retract*. An inclusion of a deformation retract is a homotopy equivalence, and thus a weak equivalence.

Proof. As remarked before, we have that $J_{\mathbf{Top}}-\mathrm{cof} \subseteq I_{\mathbf{Top}}-\mathrm{cof}$, thus every map in $J_{\mathbf{Top}}$-cof is a cofibration. We are left to show that every such map is a weak equivalence.

Notice that every map in $J_{\mathbf{Top}}$ is the inclusion of a deformation retract. We have to show that the class of inclusions of deformation retracts is closed under pushouts. Once we have done so, knowing that pushouts of maps in $J_{\mathbf{Top}}$ are also closed T_1 inclusions (since closed T_1 inclusions are closed under pushouts, as we have seen in the proof of lemma 3.6), we also have that transfinite compositions of pushouts of maps in $J_{\mathbf{Top}}$ are weak equivalences (by lemma 3.10), i.e. that $J_{\mathbf{Top}}-\text{cell} \subseteq W_{\mathbf{Top}}$. Thus the fact that weak equivalences are closed under retracts gives us that every map in $J_{\mathbf{Top}}$-cof is also a weak equivalence, by lemma 2.19.

So, let i be the inclusion of a deformation retract and suppose we have a pushout diagram as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "a"'] & X \cup_A Y \end{tikzcd}$$

Then (because the functor $- \times I$ commutes with colimits, since I is locally compact) we also have the following diagram, which is again a pushout diagram:

$$\begin{tikzcd} A \times I \arrow[r] \arrow[d] & Y \times I \arrow[d] \\ X \times I \arrow[r] & (X \cup_A Y) \times I \end{tikzcd}$$

where the maps are the same as in the previous diagram in product with the identity map of I.

Let $K : X \times I \to X$ be the homotopy that makes i into an inclusion of a deformetion retract. Then we induce the map H in the following diagram by universal property of the pushout.

$$\begin{tikzcd} A \times I \arrow[r] \arrow[d] & Y \times I \arrow[d] \arrow[ddr, bend left, "(y,t) \mapsto j(y)"] & \\ X \times I \arrow[r] \arrow[d, "K"'] & (X \cup_A Y) \times I \arrow[dr, dashed, "H"] & \\ X \arrow[rr, "a"'] & & X \cup_A Y \end{tikzcd}$$

Then H is a homotopy with j as inclusion. Indeed, by construction we have $H(j(y), t) = j(y)$. We also have $H(p, 0) = p$. Indeed, if p is in the image of j, then what just said does the job. Else $p = a(x)$ for some $x \in X$, and thus by construction $H(p, 0) = H(a(x), 0) = a(K(x, 0)) = a(x) = p$. $\square$

We show $I_{\mathbf{Top}}^{\pitchfork} \subseteq (J_{\mathbf{Top}}^{\pitchfork} \cap W_{\mathbf{Top}})$.

Lemma 3.12. *Every map in $I_{\mathbf{Top}}^{\pitchfork}$ is a trivial fibration.*

Proof. From $J_{\mathbf{Top}}-\mathrm{cof} \subseteq I_{\mathbf{Top}}-\mathrm{cof}$ follows $I_{\mathbf{Top}}^{\pitchfork} \subseteq J_{\mathbf{Top}}^{\pitchfork}$, and thus every element of $I_{\mathbf{Top}}^{\pitchfork}$ is a fibration. We are left to show that they are weak equivalences.

Let $p : X \to Y$ be in $I_{\mathbf{Top}}^{\pitchfork}$, $x_0 \in X$. We show p is a weak equivalence by proving that $\pi_n(p, x_0) : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is bijective for every $n > 0$, and that $\pi_0(p) : \pi_0(X) \to \pi_0(Y)$ is also bijective. Begin by taking $n > 0$.

$\underline{\pi_n(p, x_0) \text{ is surjective:}}$ Note that the map $* \to S^n$ is the pushout of the inclusion map $S^{n-1} \hookrightarrow D^n$:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, hook] & * \arrow[d] \\
D^n \arrow[r] & S^n
\end{tikzcd}$$

As we have explained previously, the pushout is obtained by taking the quotient of $* \sqcup D^n$ on the image of S^{n-1}, that is by identifying all of the boundary of D^n with the point $*$. The resulting space is obviously the n-sphere S^n. Thus the map $* \to S^n$ is in $I_{\mathbf{Top}}$-cof. Indeed, let $i \in I_{\mathbf{Top}}^{\pitchfork}$, and assume we have a square diagram as follows.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & A \arrow[d, "i"] \\
S^n \arrow[r] & B
\end{tikzcd}$$

Then the fact that the inclusion $S^{n-1} \hookrightarrow D^n$ is in $I_{\mathbf{Top}}$ induces the diagonal filler denoted by d in the next diagram. The filler for the lifting property (denoted by a dashed arrow) is the induced by universal property of the pushout.

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, hook] & * \arrow[r] \arrow[d] & A \arrow[d, "i"] \\
D^n \arrow[r] \arrow[urr, "d"] & S^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

Now let $f : (S^n, *) \to (Y, p(x_0))$. We show that there is an arrow $g : (S^n, *) \to (Y, p(x_0))$ such that $f = p \circ g$. We can write our arrows as follows:

$$
\begin{tikzcd}
(S^n, *) \arrow[r, "f"] & (Y, p(x_0)) & (X, x_0) \arrow[l, "p"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
* \arrow[r, equal] \arrow[d] & * \arrow[r, equal] \arrow[d, "* \mapsto p(x_0)"] & * \arrow[d, "* \mapsto x_0"] \\
S^n \arrow[r, "f"'] & Y & X \arrow[l, "p"]
\end{tikzcd}
$$

The required map $g : S^n \to X$ is then induced by lifting property through the following diagram.

$$
\begin{tikzcd}
* \arrow[r] \arrow[d, "{}^{\pitchfork}(I_{\mathbf{Top}}^{\pitchfork}) \ni"'] & X \arrow[d, "p \in I_{\mathbf{Top}}^{\pitchfork}"] \\
S^n \arrow[r, "f"'] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}
$$

<u>$\pi_n(p, x_0)$ is injective:</u> Let $f, g : (S^n, *) \to (X, x_0)$ be such that $[p \circ f] = [p \circ g]$ in $\pi_n(Y, p(x_0))$. Then we have a basepoint preserving homotopy $H : S^n \times I \to Y$ between the two maps $p \circ f$ and $p \circ g$. We can see this homotopy as a map $\bar{H} : S^n \wedge I_+ \to Y$ (recall that $S^n \wedge I_+ = (S^n \times I)/(* \times I)$). The maps f and g induce a map $(f, g) : S^n \vee S^n \to X$, where $S^n \vee S^n = (S^n \times \{0, 1\})/(* \times \{0, 1\})$. Since the inclusion map $S^n \vee S^n \hookrightarrow S^n \wedge I_+$ is a relative CW-complex, and thus in $I_{\mathbf{Top}}$-cof, the desired homotopy between f and g is obtained by lifting property in the diagram:

$$
\begin{tikzcd}
S^n \vee S^n \arrow[r, "{(f, g)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
S^n \wedge I_+ \arrow[r, "\bar{H}"'] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

Now for the case $n = 0$.

<u>$\pi_0(p)$ is surjective:</u> Notice that the map $\emptyset \to *$ is in $I_{\mathbf{Top}}$. The following commuting diagram (induced by lifting property) shows that in fact p is surjective (which is stronger than what we were set to prove):

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
* \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

<u>$\pi_0(p)$ is injective:</u> Notice that the inclusion $\{0,1\} \hookrightarrow I$ is in $I_{\mathbf{Top}}$. Let $x, y \in X$ be two points such that $p(x)$ and $p(y)$ are in the same path connected component, and let γ be a path in Y from $p(x)$ and $p(y)$ then the lifting property shows that x and y are in the same path connected component of X. In a diagram:

$$\begin{tikzcd}
\{0,1\} \arrow[r, "{(x,y)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
I \arrow[r, "\gamma"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

$\square$

Finally, we have the following result which, by theorem 2.24, concludes the proof of the fact that the structure we have defined is effectively a model structure on **Top**:

Proposition 3.13. *Every trivial fibration is in* $I_{\mathbf{Top}}^{\pitchfork}$.

The interested reader can refer to [3, p. 54] for a proof.

We have the following nice result about fibrant and cofibrant objects in **Top**, which will be useful later.

Lemma 3.14. *Every topological space is fibrant, and CW-complexes are cofibrant.*

Proof. Let X be any topological space. Then the (unique) map $X \to *$ to the terminal object $*$ (the one-point-set) is a fibration. Indeed, assume we have a commutative square as follows:

$$\begin{tikzcd}
D^n \arrow[r, "a"] \arrow[d, hook] & X \arrow[d] \\
D^n \times I \arrow[r] & *
\end{tikzcd}$$

Then we always have a diagonal filler given by $h : D^n \times I \to X$, $h(x,t) = a(x)$. Thus every element of **Top** is fibrant.

Now let X be a CW-complex. By definition of CW-complexes, the map $\emptyset \to X$ from the initial object (the empty set) to X is in $I-$cell $\subseteq I-$cof. Thus all CW-complexes are cofibrant. $\square$

We will now turn to the study of the model structure on the category of simplicial sets.

4 The model structure on simplicial sets

4.1 The category SSet

Definition 4.1. *The* simplicial category Δ *is the category with objects*

$$[n] = \{0, 1, \ldots, n\}$$

for $n \in \mathbb{N}_0$ and arrows the weakly order preserving maps. Said otherwise, we have $f \in \Delta([\mathrm{n}], [\mathrm{k}])$ if and only if $x \leq y \Rightarrow f(x) \leq f(y)$ for every $x, y \in [n]$.

We can interpret geometrically the objects of Δ as follows. The simplex $[n]$ is the convex closure of the points $e_0, e_1, \ldots, e_n$ in $\mathbb{R}^{n+1}$, where e_i denote the elements of the standard basis. Otherwise said, it is the subset of $\mathbb{R}^{n+1}$ given by $\{(x_0, \ldots, x_n) : \sum_{i=0}^{n} = 1,\ x_i \geq 0\ \forall i\}$.

Δ has the two subcategories Δ_+ and Δ_- of injective and surjective order-preserving maps. Every map in Δ can be factored in one element in Δ_- followed by one in Δ_+. Δ is in fact generated by the morphisms $d^i : [n-1] \to [n] \in \Delta_+$ ($n \geq 1$, $0 \leq i \leq n$) whose image doesn't include i, and the morphisms $s^i : [n] \to [n-1] \in \Delta_-$ ($n \geq 1$, $0 \leq i \leq n-1$) identifying i and $i+1$.

The maps d^i can be seen as the inclusion of a face of dimension $n-1$ in the n-simplex $[n]$, and the maps s^i as "collapsing one dimension" of the n-simplex. This extends to a nice geometrical interpretation of all the arrows in Δ, which can be formalized in a functor $r : \Delta \to \mathbf{Top}$ as follows:

- $r[n] = \{(x_0, \ldots, x_n) : \sum_{i=0}^{n} = 1,\ x_i \geq 0\ \forall i\} \subset \mathbb{R}^{n+1}$
- $r[d^i] : r[n-1] \to r[n]$ is given by the inclusion of coordinates

$$r[d^i](x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

- $r[s^i] : r[n] \to r[n-1]$ is obtained crushing two dimensions together as follows

$$r[s^i](x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_{n-1})$$

Given any category $\mathcal{C}$ we define the category of *simplicial objects in* $\mathcal{C}$ as the category of functors from Δ^{op} to $\mathcal{C}$, usually denoted by $\mathcal{C}^{\Delta^{op}}$. The case we are interested in is when $\mathcal{C}$ is the category **Sets** of sets.

Definition 4.2. *The category* **SSet** *of simplicial sets is defined as the category of presheaves over Δ. In other words, it is the category with functors from Δ^{op} to* **Sets** *as objects and natural transformations between such functors as morphisms.*

Let $K \in \mathbf{SSet}$, i.e. a functor $K : \Delta^{op} \to \mathbf{Sets}$. Then we denote $K([n])$ by K_n.

There are two important types of maps in **SSet**, dual to the maps d^i and s^i respectively. Let $K \in \mathbf{SSet}$. For $n \geq 1$, $0 \leq i \leq n$ we have the *face maps*

$d_i : K_n \to K_{n-1}$, which associate to an n-simplice (i.e. an element of K_n) its faces (which are $(n-1)$-simplices). For $n \geq 1$, $0 \leq i \leq n-1$ we have the *degeneracy maps* $s_i : K_{n-1} \to K_n$. A simplex is called *non-degenerate* if it is not in the image of a degeneracy map. A *face* of a simplex x of K is the image of x under any iteration of face maps. Every map of simplicial sets $f : K \to L$ is equivalent to a collection of maps $f_n : K_n \to L_n$ all commuting with the face and degeneracy maps.

We can see a simplicial set X geometrically by gluing all of its (non degenerate) simplices together along their common faces (this concept will then be formalized to a functor in section 4.2).

Examples

There are some important simplicial sets that we will need later when we discuss the model structure on **SSet** .

- We define a functor $\Delta[-] : \Delta \to \mathbf{SSet}$ by $\Delta[n] : \Delta^{op} \to \mathbf{Sets}$ as the Yoneda embedding of Δ into $\mathbf{SSet} = \mathbf{Sets}^{\Delta^{op}}$. It sends $[k]$ to the set of order preserving maps from $[k]$ to $[n]$. Its geometrical interpretation is the topological space $r[n]$, i.e. the closed convex hull of $0, e_1, \ldots, e_n$.

- The boundary of the simplicial set $\Delta[n]$, denoted by $\partial\Delta[n]$ is obtained as the coequalizer gluing all $(n-1)$-simplices together along their boundaries according to the face maps:

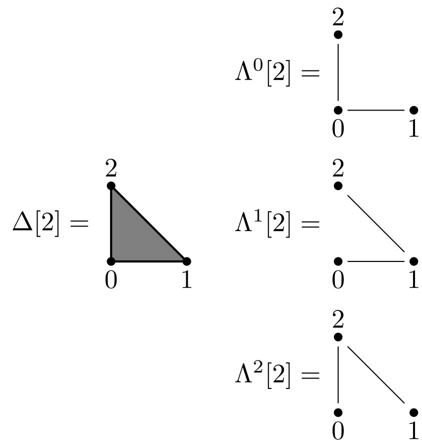
$$\coprod_{[n-2]\to[n] \text{ inj.}} \Delta[n-2] \rightrightarrows \coprod_{[n-1]\to[n] \text{ inj.}} \Delta[n-1] \to \partial\Delta[n]$$

  It is given by all the injective, order preserving maps $[k] \to [n]$ for $0 \leq k < n$. It can be seen as topological boundary of the geometrical interpretation of $\Delta[n]$.

- For $0 \leq r \leq n$ we define the r-horn of $\Delta[n]$, denoted by $\Lambda^r[n]$ and given by the coequalizer gluing toghether all $(n-1)$-simplices with r as a vertex along their boundaries according to the face maps:

$$\coprod_{\substack{[n-2]\to[n] \text{ inj.} \\ r \text{ in the image}}} \Delta[n-2] \rightrightarrows \coprod_{\substack{[n-1]\to[n] \text{ inj.} \\ r \text{ in the image}}} \Delta[n-1] \to \partial\Delta[n]$$

  Geometrically, $\Lambda^r[n]$ is obtained by taking away the interior and the face opposite to the vertex r from the simplicial set $\Delta[n]$. The following drawing illustrates the concept for $n = 2$:



4.2 Geometric realization

We will now construct an adjunction $(|\cdot|, \mathrm{Sing}, \varphi) : \mathbf{SSet} \to \mathbf{Top}$. The left adjoint $|\cdot| : \mathbf{SSet} \to \mathbf{Top}$ is called *geometric realization* and the right adjoint $\mathrm{Sing} : \mathbf{Top} \to \mathbf{SSet}$ is called *singular functor*. The geometric realization will make the following diagram commute:

$$\begin{tikzcd} \Delta \arrow[r, "{\Delta[-]}"] \arrow[dr, "r"'] & \mathbf{SSet} \arrow[d, "|\cdot|"] \\ & \mathbf{Top} \end{tikzcd}$$

The geometric realization of a simplicial set $K \in \mathbf{SSet}$ is defined as the coend:

$$|K| = \int^{[n]\in\Delta} K_n \cdot r[n]$$

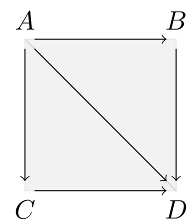
where $K_n \cdot r([n])$ is the coproduct $\bigsqcup_{K_n} r([n])$. It can also be seen as a coequalizer (i.e. a generalized quotient):

$$\bigsqcup_{[n]\to[m]\in\Delta} K_m \cdot r[n] \rightrightarrows \bigsqcup_n K_n \cdot r[n] \longrightarrow |K|$$

where the two parallel arrows are given by the maps $K_m \times r[n] \to K_n \times r[n]$ and $K_m \times r[n] \to K_m \times r[m]$ induced by the associated function $[n] \to [m]$.

This can be seen as taking the geometric interpretation of every simplex in K and gluing them together along their common faces. We try to make this clearer with an example.

Example 4.3. *Take the simplicial set K given by $K_0 = \{A, B, C, D\}$, $K_1 = \{AB, AC, AD, BC, CD\}$ and $K_2 = \{ABD, ACD\}$ (where the order of the letters gives us the orientation). It can be represented as follows:*



Then its geometric realization is (quite obviously) the square $|K| = I \times I$.

Now we construct the singular functor. Given $X \in \mathbf{Top}$, we define $\mathrm{Sing}(X)$ to be the simplicial set having as n-simplices the set $\mathbf{Top}\left(\mathrm{r[n], X}\right)$. This can be seen as trying to fit as many geometrical simplices as possible into the topological space X.

Example 4.4. *The singular functor is very sensible of the topology, as the following three examples on $X = \{0, 1\}$ show:*

- *Let $X = \{0, 1\}$ with the discrete topology. Then $\mathrm{Sing}(X)$ has only two non-degenerate simplex, both of dimension 0, corresponding to the maps $r[0] \mapsto 0$ and $r[1] \mapsto 1$.*
- *Let $X = \{0, 1\}$ with the indiscrete topology. Then every map $r[n] \to X$ is continuous, and thus $\mathrm{Sing}(X)_n \cong \{0, 1\}^{r[n]}$.*
- *Let $X = \{0, 1\}$ with the Sierpinski topology (i.e. the open sets are $\emptyset$, $\{0\}$ and X). Then $\mathrm{Sing}(X)_n$ is in bijective correspondence with the open subsets of $r[n]$, corresponding to the possible inverse images of 0.*

Finally, we show that they form in fact an adjunction. Let $K \in \mathbf{SSet}$ and $X \in \mathbf{Top}$. The bijection $\varphi_{K,X} : \mathbf{Top}\left(\mathrm{|K|, X}\right) \to \mathbf{SSet}\left(\mathrm{K, Sing(X)}\right)$ is given by the following composition:

$$\begin{aligned}
\mathbf{SSet}\left(\mathrm{K, Sing(X)}\right) &\cong \int_{[n]\in\Delta} \mathbf{Sets}\left(\mathrm{K_n, Sing(X)_n}\right) \\
&= \int_{[n]\in\Delta} \mathbf{Sets}\left(\mathrm{K_n}, \mathbf{Top}\left(\mathrm{r[n], X}\right)\right) \\
&\cong \int_{[n]\in\Delta} \mathbf{Top}\left(\mathrm{K_n \cdot r[n], X}\right) \\
&\cong \mathbf{Top}\left(\int^{\mathrm{[n]}\in\Delta} \mathrm{K_n \cdot r[n], X}\right) \\
&= \mathbf{Top}\left(\mathrm{|K|, X}\right)
\end{aligned}$$

The first isomorphism is given by the fact that if $F, G : \mathcal{C} \to \mathcal{D}$ are two functors, then the set of natural transformations from F to G equals $\int_{c\in\mathcal{C}} \mathcal{C}\left(\mathrm{F(c), G(c)}\right)$ (see [5, p. 223] for details). The second one (third line) is given by associating to every $f \in \mathbf{Sets}(\mathrm{K_n}, \mathbf{Top}(\mathrm{r[n]}, \mathrm{X}))$ the element $g \in \mathbf{Top}(\mathrm{K_n} \cdot \mathrm{r[n]}, \mathrm{X})$ given by $g(k, x) = f(k)(x)$. The third isomorphism comes from the fact that a coend is a colimit.

4.3 Completeness and cocompleteness

Since **SSet** is the category of presheaves on Δ, completeness and cocompleteness are given. Limits and colimits are computed termwise. This means that if $\mathcal{I}$ is an index category and we want to compute the limit of $F : \mathcal{I} \to \mathbf{SSet}$, then we can define $L = \lim_{i \in \mathcal{I}} F(i)$ by $L_n = \lim_{i \in \mathcal{I}} F(i)_n$ in **Sets**.

4.4 The model structure on SSet

We define the following sets of maps in **SSet**:

- $W_{\mathbf{SSet}}$ is the set of arrows in **SSet** such that $|f| \in W_{\mathbf{Top}}$; we call those maps *weak equivalences*
- $I_{\mathbf{SSet}} = \{\partial\Delta[n] \hookrightarrow \Delta[n] | n \geq 0\}$
- $J_{\mathbf{SSet}} = \{\Lambda^r[n] \hookrightarrow \Delta[n] | n > 0, 0 \leq r \leq n\}$
- $F_{\mathbf{SSet}} = J_{\mathbf{SSet}}^{\pitchfork}$, we call those maps *fibrations*
- $C_{\mathbf{SSet}} = I_{\mathbf{SSet}}-\mathrm{cof} = {}^{\pitchfork}(I_{\mathbf{Top}}^{\pitchfork})$, we call those maps *cofibrations*

We claim those sets define a model structure on **SSet**.

Theorem 4.5. $(\mathbf{SSet}, C_{\mathbf{SSet}}, F_{\mathbf{SSet}}, W_{\mathbf{SSet}})$ *is a model category which is cofibrantly generated with* $I_{\mathbf{SSet}}$ *as set of generating cofibrations and* $J_{\mathbf{SSet}}$ *as set of generating trivial cofibrations.*

Remark 4.6. In the literature, the elements of $J_{\mathbf{SSet}}^{\pitchfork}$ (the fibration) are often called *Kan fibrations*, and the elements of $J_{\mathbf{SSet}}$-cof *anodyne extensions*.

Proof. We prove this theorem by verifying the assumptions of theorem 2.24. We do this in a number of smaller lemmas and theorems, of which we give an overview below:

a) $W_{\mathbf{SSet}}$ has the 2-out-of-3 property and is closed under retracts (**lemma 4.7**)

b) the domains of the maps in $I_{\mathbf{SSet}}$, $J_{\mathbf{SSet}}$ are small relative to $I_{\mathbf{SSet}}$-cell and $J_{\mathbf{SSet}}$-cell respectively (by **lemma 4.8**)

c) $J_{\mathbf{SSet}}-\mathrm{cell} \subseteq (I_{\mathbf{SSet}}-\mathrm{cof} \cap W_{\mathbf{SSet}})$ (**proposition 4.11**)

d) $I_{\mathbf{SSet}}^{\pitchfork} \subseteq (J_{\mathbf{SSet}}^{\pitchfork} \cap W_{\mathbf{SSet}})$ (**proposition 4.17**)

e) $(J_{\mathbf{SSet}}^{\pitchfork} \cap W_{\mathbf{SSet}}) \subseteq I_{\mathbf{SSet}}^{\pitchfork}$ (**proposition 4.18**)

Again as for **Top**, point (e) is the hardest to prove, and we will not give a proof of it. □

Point (a) is really easy to prove.

Lemma 4.7. *$W_{\mathbf{SSet}}$ satisfies the 2-out-of-3 property and is closed under retracts.*

Proof. This is a trivial consequence of the definition of $W_{\mathbf{SSet}}$, the fact that $|\cdot|$ is a functor and the fact that $W_{\mathbf{Top}}$ satisfies the 2-out-of-3 property and is closed under retracts (which was proven in lemma 3.4). □

We will now prove the desired smallness properties.

Lemma 4.8. *Every simplicial set is small.*

Proof. Let $K \in \mathbf{SSet}$ be a simplicial set. Let $\kappa = \max\{2^\omega, |\bigsqcup_{n\in\mathbb{N}} K_n|\}$. Notice that κ is strictly greater than both ω and than the cardinality of every set of simplices of K. We will show that K is κ-small. Let λ be a κ-filtered ordinal, $X : \lambda \to \mathbf{SSet}$ a λ-sequence. In order to show smallness of K, we have to show that the canonical map of sets:

$$\mathrm{colim}_{\beta<\lambda}\mathbf{SSet}\,(\mathrm{K}, \mathrm{X}_\beta) \to \mathbf{SSet}\,(\mathrm{K}, \mathrm{colim}_{\beta<\lambda}\mathrm{X}_\beta)$$

is an isomorphism.
Surjectivity: Let $f : K \to \mathrm{colim}_{\beta<\lambda} X_\beta =: X_\lambda$. Then for every $n \in \mathbb{N}$ there is some $\beta_n < \lambda$ such that f_n factors through X_{β_n}, since K_n is $|K_n|$-small in **Sets** (example 2.6). We have the commutative diagram:

$$\begin{tikzcd} & (X_{\beta_n})_n \arrow[d] \\ K_n \arrow[ur] \arrow[r, "f_n"'] & (X_\lambda)_n \end{tikzcd}$$

Since λ is κ-filtered, and $\kappa > \omega$), we have $\alpha := \sup_n \beta_n < \lambda$. Thus f_n factors through X_α for every $n \in \mathbb{N}$. The map thus obtained is only a map of sets, it is not necessarily a map of simplicial sets. In other words, let $u : [n] \to [m]$ be any map in Δ, then the following square does *not* necessarily commute:

$$\begin{tikzcd} K_m \arrow[r] \arrow[d, "K(u)"'] & (X_\alpha)_m \arrow[d, "X_\alpha(u)"] \\ K_n \arrow[r] & (X_\alpha)_n \end{tikzcd}$$

But it commutes after composition with $(X_\alpha)_n \to (X_\lambda)_n$. By smallness of K_m we deduce the existence of an ordinal $\alpha < \alpha_u < \lambda$ such that the two composite arrows $K_m \rightrightarrows (X_\alpha)_n$ coequalize in X_{α_u}. In a diagram:

$$\begin{tikzcd}
 & (X_{\alpha_u})_n \arrow[dr, bend left] & \\
K_m \arrow[r, shift left] \arrow[r, shift right] & (X_\alpha)_n \arrow[u, bend left] \arrow[r] & (X_\lambda)_n
\end{tikzcd}$$

Now notice that the number of arrows in Δ is ω, and thus $\theta := \sup_u \alpha_u < \lambda$. Thus f splits through X_θ in two maps of simplicial sets.
Injectivity: Let $f_1, f_2 \in \mathrm{colim}_{\beta<\lambda}\mathbf{SSet}\,(\mathrm{K}, \mathrm{X}_\beta)$ be two maps having the same image $g \in \mathbf{SSet}\,(\mathrm{K}, \mathrm{colim}_{\beta<\lambda}\mathrm{X}_\beta)$. The restriction of f_1, f_2 to n-simplices gives us the maps of sets $(f_1)_n, (f_2)_n \in \mathrm{colim}_{\beta<\lambda}\mathbf{Sets}\,(\mathrm{K_n}, (\mathrm{X}_\beta)_\mathrm{n})$. By assumption, both have the same image $g_n \in \mathbf{Sets}\,(\mathrm{K_n}, (\mathrm{colim}_{\beta<\lambda}\mathrm{X}_\beta)_\mathrm{n})$. Since all sets are small (as shown in example 2.6), we get $(f_1)_n = (f_2)_n$. This is true for every $n \in \mathbb{N}$, so $f_1 = f_2$. $\square$

In particular, the domains of the maps in $I_{\mathbf{SSet}}$ and $J_{\mathbf{SSet}}$ are small relative to $I_{\mathbf{SSet}}$ -cell and $J_{\mathbf{SSet}}$ -cell respectively.

We show that $J_{\mathbf{SSet}}{-}\mathrm{cell} \subseteq I_{\mathbf{SSet}}{-}\mathrm{cof} \cap W_{\mathbf{SSet}}$. In order to do this, we have first to characterize all cofibrations in **SSet** , which is surprisingly simple.

Lemma 4.9. *$f \in I_{\mathbf{SSet}}{-}\mathrm{cof} \Leftrightarrow f$ is injective.*

Proof. Every map in $I_{\mathbf{SSet}}$ is obviously injective, and injections are closed under pushouts, transfinite compositions and retracts, thus every map in $I_{\mathbf{SSet}}$-cell is injective. Then lemma 2.19 (a corollary of the retract argument) implies that every map in $I_{\mathbf{SSet}}$-cof is also injective.

Suppose now that $f : K \to L$ is injective. We show that $f \in I_{\mathbf{SSet}}$-cell by writing it as a (countable) transfinite composition of pushouts of coproducts of maps in $I_{\mathbf{SSet}}$. We start by setting $X_0 = K$, $f_0 = f : X_0 \to L$. Let S_0 indicate the set of 0-simplices of L (i.e. elements of $L_0 = L[0]$) which are not in the image of f_0. Every element $s \in S_0$ corresponds to a map $\Delta[0] \to L$. We get X_1 and f_1 by pushout as follows (the first horizontal map is obtained as the coproduct of the restriction of the inclusions $s \to L$ to the boundaries):

$$\begin{tikzcd}
\emptyset = \coprod_{s \in S_0} \partial\Delta[0] \arrow[r] \arrow[d, hook] & X_0 \arrow[d] \arrow[ddr, bend left, "f_0"] & \\
\coprod_{s \in S_0} \Delta[0] \arrow[r] \arrow[drr, bend right, "a"'] & X_1 \arrow[dr, dashed, "f_1"'] & \\
 & & L
\end{tikzcd}$$

Notice that by construction, the map f_1 is injective (since both a and f_0 are, and their images are disjoint) and surjective on the 0-simplices.

Assume we have constructed simplices X_k and maps $f_k : X_k \to L$ for k up to n such that every f_k is injective and surjective on the simplices of dimension less than k. Then we can construct a similar map f_{n+1} in a fashion similar to the one used before for f_1: define S_n to be the set of all n-simplices in L which are not in the image of f_n. Notice that every element $s \in S_n$ is non-degenerate (degenerate simplexes are already taken care of in the lower dimensional cases) and corresponds to a map $\Delta[n] \to L$. Again (by restricting those maps to the boundary) we obtain X_{n+1} and f_{n+1} by pushout:

$$\begin{tikzcd}
\coprod_{s \in S_n} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X_n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\
\coprod_{s \in S_n} \Delta[n] \arrow[r] \arrow[drr, bend right, "b"'] & X_{n+1} \arrow[dr, dashed, "f_{n+1}"'] & \\
 & & L
\end{tikzcd}$$

By construction, the map f_{n+1} is injective (again, since b and f_n are, and have disjoint images) and surjective on the n-simplices (and thus on all simplices of dimension less or equal to n).

Then the map f is a composition of the sequence X_n, and thus an element in $I_{\mathbf{SSet}}$-cell. $\square$

Remark 4.10. In fact, the proof shows more than what is stated: it also implies that every cofibration is in $I_{\mathbf{SSet}}$-cell, so $I_{\mathbf{SSet}}-\text{cell} = I_{\mathbf{SSet}}-\text{cof}$.

Proposition 4.11. *Every anodyne extension is a trivial cofibration.*

Proof. Every map in $J_{\mathbf{SSet}}$ is injective, thus by lemma 4.9 $J_{\mathbf{SSet}} \subseteq I_{\mathbf{SSet}}-\text{cof}$. It follows that $J_{\mathbf{SSet}}-\text{cof} \subseteq I_{\mathbf{SSet}}-\text{cof}$.

It is left to show that every map in $J_{\mathbf{SSet}}-\text{cof}$ is a weak equivalence, i.e. that its geometric realization is a weak equivalence in **Top**. Let $f \in J_{\mathbf{SSet}}$, then f has the form of an inclusion $\Lambda^r[n] \hookrightarrow \Delta[n]$. Notice that $|\Delta[n]| \cong D^n \cong D^{n-1} \times I$, and we can choose those homeomorphisms in a way such that $|\Lambda^r[n]|$ is taken to $D^{n-1} \times \{0\}$. Thus $|f| \in J_{\mathbf{Top}}-\text{cof}$, as the following diagram suggests:

$$\begin{tikzcd}
{|\Lambda^r[n]|} \arrow[r, equal, "\simeq"] \arrow[d, hook, "|f|"'] & D^{n-1} \arrow[r] \arrow[d, hook] & {} \arrow[d, "\in J_{\mathbf{Top}}^{\pitchfork}"] \\
{|\Delta[n]|} \arrow[r, equal, "\simeq"] & D^{n-1} \times I \arrow[r] \arrow[ur, dashed] & {}
\end{tikzcd}$$

This means that $|J_{\mathbf{SSet}}| \subseteq J_{\mathbf{Top}}-\text{cof}$, and thus we have

$$|J_{\mathbf{SSet}}-\text{cof}| \subseteq |J_{\mathbf{SSet}}|-\text{cof} \subseteq J_{\mathbf{Top}}-\text{cof}$$

where we used lemma 2.30 for the first inequality. Thus the geometric realization of any map in $J_{\mathbf{SSet}}$-cof is a weak equivalence in **Top**, meaning that every map in $J_{\mathbf{SSet}}$-cof is a weak equivalence in **SSet**. $\square$

We show $I^{\pitchfork}_{\mathbf{SSet}} \subseteq (J_{\mathbf{SSet}} \cap W)$. In order to do that, we need to prove that the geometric realization preserves finite limits.

Lemma 4.12. *Let $i : L \to K$ be an injective arrow in* $\mathbf{SSet}$. *Then $|i| \in I_{\mathbf{Top}}-$cell. In particular it is a closed T_1 inclusion.*

Proof. By lemma 4.9, $i \in I_{\mathbf{SSet}}-$cell. That means that i is a transfinite composition of pushouts of maps in $I_{\mathbf{SSet}}$. Since the geometric realization is a left adjoint it preserves colimits, and thus $|i|$ is a transfinite composition of maps in $\{|j| : j \in I_{\mathbf{SSet}}\}$. Let j_n denote the inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$, then $|j_n|$ can be seen as the inclusion $S^{n-1} \hookrightarrow D^n$. Thus $|i| \in I_{\mathbf{Top}}-$cell. □

Remark 4.13. Note that in particular if we take $L = \emptyset$, the lemma 4.12 above shows that $|K|$ is a CW-complex. Thus we can take the geometric realization as a functor from $\mathbf{SSet}$ to one of many convenient subcategories of $\mathbf{Top}$, for example the subcategory $\mathbf{CGHaus}$ of compactly generated Hausdorff spaces, or the subcategory $\mathbf{K}$ of k-spaces (sometimes called compactly generated spaces).

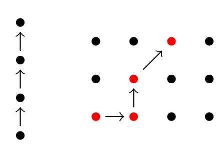
Proposition 4.14. *The geometric realization preserves finite products.*

Proof. Since the product preserves colimits in both $\mathbf{SSet}$ and $\mathbf{K}$, it will be enough to prove that the continuous map

$$|\Delta[n] \times \Delta[m]| \to |\Delta[n]| \times |\Delta[m]|$$

(obtained by universal property) is a homeomorphism. We will shortly prove that both the domain and range of that map are compact Hausdorff. Thus it will suffice to prove that the map is bijective.

The first step is to characterize all non-degenerate simplices of $\Delta[n] \times \Delta[m]$. A (non necessarily non-degenerate) p-simplex of $\Delta[n] \times \Delta[m]$ is basically equivalent to an order preserving map $[p] \to [n] \times [m]$, where in $[n] \times [m]$ we say that $(a_1, b_1) \leq (a_2, b_2)$ if, and only if, $a_1 \leq a_2$ and $b_1 \leq b_2$. This can be better visualized by drawing $[n] \times [m]$ as the integer lattice between $(0, 0)$ and (n, m). Then the more on the right and up elements are, the bigger they are. A *non-degenerate* p-simplex is an injective order preserving map $[p] \to [n] \times [m]$, also called a chain in $[n] \times [m]$. We give a graphical example for the case $p, n = 3$, $m = 2$.



$[3]$ $\qquad$ $[3] \times [2]$

Every such p-chain can be maximally extended to an $(n + m)$-chain (which is not necessarily unique), thus every non-degenerate simplex of $\Delta[n] \times \Delta[m]$ is a face of some $(n + m)$-simplex. Now notice that, since a step in a chain can only be either a step up or a step to the right, an $(n + m)$-chain is completely determined by the vertical coordinates at the end of the horizontal segments, thus there is a bijective correspondence between such chains and m-subsets of $\{1, 2, \ldots, n+m\}$. Thus we have $\binom{n+m}{m}$ maximally extended chains.

Now let $c(i)$, $1 \leq i \leq \binom{n+m}{m}$ be an enumeration of all maximally extended chains, and for any chain c denote by n_c the number of edges of c (e.g. in our example above, $n_c = 3$). Then $\Delta[n] \times \Delta[m]$ is given as the following coequalizer (in **SSet**):

$$\coprod_{1 \leq i < j \leq \binom{n+m}{m}} \Delta[n_{c(i) \cap c(j)}] \rightrightarrows \coprod_{1 \leq i \leq \binom{n+m}{m}} \Delta[n_{c(i)}]$$

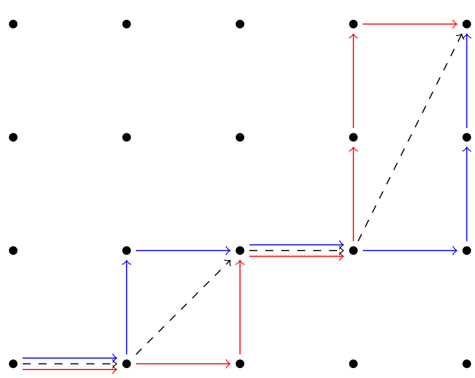
where the two horizontal arrows are induced by the inclusion of $c(i) \cap c(j)$ into $c(i)$ and $c(j)$ respectively. An example of this fact which is easy to visualize is the case $n = m = 1$. Indeed $\Delta[1] \times \Delta[1]$ is given by two copies of $\Delta[2]$, corresponding to the chains $\{(0,0),(0,1),(1,1)\}$ and $\{(0,0),(1,0),(1,1)\}$, glued together along the 1-simplex given by $\{(0,0),(1,1)\}$, which is the intersection of the two chains.

Claim: $|\Delta[n] \times \Delta[m]|$ is compact Hausdorff.

Proof: The geometric realization is a left adjoint, thus it preserves coequalizers. It is then easy to see that $|\Delta[n] \times \Delta[m]|$ is compact. It is Hausodrff by remark 4.13. ■

We will now prove that the map $|\Delta[n] \times \Delta[m]| \to |\Delta[n]| \times |\Delta[m]|$ is bijective.

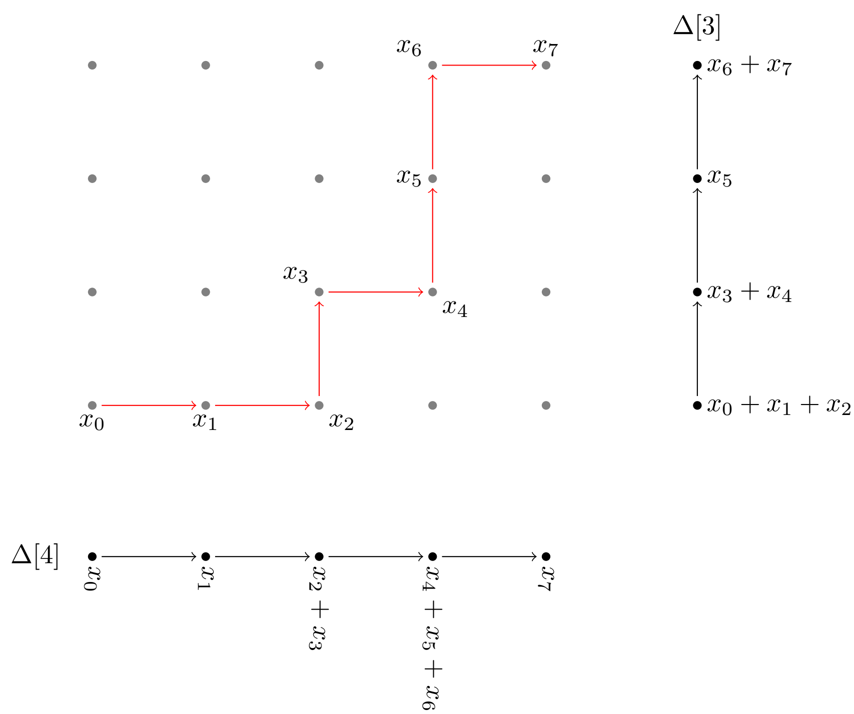
Consider a simplex $\Delta[k]$. Then the coordinates on $|\Delta[k]|$ are an assignment of weights of total sum 1 to the $k+1$ vertices of $\Delta[k]$. Given such an assignment $(x_0, \ldots, x_k)$, the position on $|\Delta[k]| = r[k]$ is the baricenter of the vertices with the respective weights, i.e. $\sum_{i=0}^{k} x_i e_i$ (if you prefer, it is a convex combination of the basis vectors $e_0, \ldots, e_k$ corresponding to the vertices). Such assignemnt is called *barycentric coordinates.* Now every point of $|\Delta[n] \times \Delta[m]|$ corresponds to an assignment of coordinates $(c, x_0, \ldots, x_{m+n})$, where c is a maximally extended chain and $(x_0, \ldots, x_{m+n})$ barycentric coordinates on $\Delta[m+n]$. The following drawing illustrates the concept:



$[4] \times [3]$

Here the red path, which we will denote c_1, and the blue path, c_2, are both maximally extended, and represent two $(5+4)$-simplices of $\Delta[n] \times \Delta[m]$. The dashed path $c_1 \cap c_2$ is a 5-simplex along which the two other simplices fit together. Two assignments of coordinates $(c_1, x_1, \ldots, x_9)$ and $(c_2, y_1, \ldots, y_9)$ represent the same point of $|\Delta[n] \times \Delta[m]|$ if, and only if they have the same weights on $c_1 \cap c_2$ and weight 0 everywhere else.

The map $|\Delta[n] \times \Delta[m]| \to |\Delta[n]| \times |\Delta[m]|$ can be described as follows: Start with a coordinate $(c, x_0, \ldots, x_{m+n}) \in |\Delta[n] \times \Delta[m]|$, write the dots in the grid representing $\Delta[n] \times \Delta[m]$. Then the weights on $|\Delta[n]|$ (respectively $|\Delta[m]|$) are obtained by giving weight 0 to the vertices not in c and summing over the coloumns (respectively the rows). Again, we illustrate the concept with a drawing.



It is then an easy exercise to show surjectivity (by finding an algorithm associating to every coordinate of $|\Delta[n]| \times |\Delta[m]|$ a maximal chain and coordinates on $|\Delta[n] \times \Delta[m]|$) and injectivity of the map (beware that some points of $|\Delta[n] \times \Delta[m]|$ can be on more than one $(m+n)$-simplex).

Now given this, let $K, L \in \mathbf{SSet}$. Then the co-Yoneda lemma tells us that K and L can be expressed as a coend:

$$K \cong \int^{[m] \in \Delta} K_m \cdot \Delta[m] \qquad L \cong \int^{[n] \in \Delta} L_n \cdot \Delta[n]$$

The following formal argument concludes the proof:

$$
\begin{aligned}
|K \times L| &\cong \left| \left(\int^{[m]\in\Delta} K_m \cdot \Delta[m] \right) \times \left(\int^{[n]\in\Delta} L_n \cdot \Delta[n] \right) \right| \\
&\cong \left| \int^{[m]\in\Delta} \int^{[n]\in\Delta} (K_m \times L_n) \cdot (\Delta[m] \times \Delta[n]) \right| \\
&\cong \int^{[m]\in\Delta} \int^{[n]\in\Delta} (K_m \times L_n) \cdot |\Delta[m] \times \Delta[n]| \\
&\cong \int^{[m]\in\Delta} \int^{[n]\in\Delta} (K_m \times L_n) \cdot |\Delta[m]| \times |\Delta[n]| \\
&\cong \left(\int^{[m]\in\Delta} K_m \cdot r[m] \right) \times \left(\int^{[n]\in\Delta} L_n \cdot r[n] \right) \\
&= |K| \times |L|
\end{aligned}
$$

where the first isomorphism is given by the expressions for K and L given by the co-Yoneda lemma, the second and fifth by the fact that the product commutes with colimits in **Sets** and in the nice category **K** (here we overlook some subtility in the treatement of the fifth isomorphism, in a fully formal derivation we should treat products in **K** and their relationship with products in **Top** more carefully). The third isomorphism derives from the fact that the geometric realization is a left adjoint, and thus commutes with colimits. Finally the fourth isomorphism is what we have proven above. $\square$

Proposition 4.15. *The geometric realization preserves finite limits. In particular, it preserves pullbacks.*

Proof. We have just proved that it preserves finite products. By a classical result of category theory (see for example [5, p. 113] for a reference), it is enough to show that it preserves equalizers.

Let $f, g : L \to M$ be two arrows in **SSet** with equalizer $i : K \to L$, and let $j : Z \to |L|$ be the equalizer of $|f|, |g| : |L| \to |M|$ in **Top**. Then i and j are injective. By lemma 4.12, $|i|$ is a closed T_1 inclusion. By universal property, we have the following dashed arrow:

$$
\begin{tikzcd}
Z \arrow[r, "j"] & {|L|} \arrow[r, shift left, "|f|"] \arrow[r, shift right, "|g|"'] & {|M|} \\
{|K|} \arrow[u, dashed, "u"] \arrow[ur, "|i|"'] & &
\end{tikzcd}
$$

Since both $|i|$ and j are injective, so is u. In particular, since $|i|$ is an inclusion, $|K|$ is homeomorphic to a subspace of Z. It is thus enough to prove that u is a surjection to conclude that $Z \cong |K|$. We will prove the equivalent assertion that the images of $|i|$ and j are equal. Let $z \in Z$, then $j(z) \in |x|$ for some non-degenerate simplex $x \in L$. By definition of the geometric realization, $(|f| \circ j)(z) = (|g| \circ j)(z)$ if, and only if $f(x) = g(x)$. Thus x must be a (non-degenerate) simplex of the image of K in L, and thus $j(z)$ is in the image of $|K|$ in $|L|$. $\square$

Lemma 4.16. *If $f \in I_{\mathbf{SSet}}^{\pitchfork}$, then $|f| \in J_{\mathbf{Top}}^{\pitchfork}$.*

Proof. Since $f \in I_{\mathbf{SSet}}^{\pitchfork}$, by lemma 4.9 it has the right lifting property relative to every injective map of simplicial sets, in particular to inclusions. We can thus find a lift h in the following diagram.

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, "(\mathbf{id}_K, f)"'] & K \arrow[d, "f"] \\ K \times L \arrow[r, "\pi_L"'] \arrow[ur, dashed, "h"] & L \end{tikzcd}$$

where π_L is the projection onto L. Thus f is a retract of π_L, with the obvious diagram. This implies that $|f|$ is a retraction of $|\pi_L|$. Notice that $|\pi_L|$ is a fibration in **Top**. Indeed, since the geometric realization preserves finite products (proposition 4.14), $|\pi_L| = \pi_{|L|} : |K| \times |L| \to |L|$ is again the projection on the second factor. If we have a diagram as follows:

$$\begin{tikzcd} D^n \arrow[r, "a"] \arrow[d, hook] & {|K| \times |L|} \arrow[d, "\pi_{|L|}"] \\ D^n \times I \arrow[r, "b"'] & {|L|} \end{tikzcd}$$

then a diagonal filler $c : D^n \times I \to |K| \times |L|$ is given by $c(x, t) = (\pi_{|K|} \circ a)(x) \times b(x, t)$.

Thus by lemma 2.13, $|f|$ is a fibration in **Top**, i.e. an element of $J_{\mathbf{Top}}^{\pitchfork}$. $\square$

Proposition 4.17. *Every map in $I_{\mathbf{SSet}}^{\pitchfork}$ is a trivial fibration.*

Proof. As we already noted before, $J_{\mathbf{SSet}} \subseteq I_{\mathbf{SSet}}$-cof, thus $I_{\mathbf{SSet}}^{\pitchfork} \subseteq J_{\mathbf{SSet}}^{\pitchfork}$ (i.e. every map in $I_{\mathbf{SSet}}^{\pitchfork}$ is a fibration).

Let $f : K \to L$ be any map in $I_{\mathbf{SSet}}^{\pitchfork}$. In order to show that every map in $I_{\mathbf{SSet}}^{\pitchfork}$ is in fact a *trivial* fibration, we must show that $|f| \in W_{\mathbf{Top}}$, i.e. that $|f|$ induces isomorphisms in every homotopy group.

Let v be a vertex (i.e. a 0-simplex) of L, let $F = f^{-1}(v)$ be the fiber of f over v. Then we have the following pullback diagram.

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & K \arrow[d, "f"] \\ \Delta[0] \arrow[r, "v"'] & L \end{tikzcd}$$

By proposition 4.15, we have the pullback diagram:

$$\begin{tikzcd} {|F|} \arrow[r, hook] \arrow[d] & {|K|} \arrow[d, "{|f|}"] \\ * \arrow[r, "{|v|}"'] & {|L|} \end{tikzcd}$$

Thus $|f|$ has fibre $|F|$ above the point $|v|$.

Note that the map $F \to \Delta[0]$ is in $I^{\pitchfork}_{\mathbf{SSet}}$, as the following diagram shows:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & F = f^{-1}(v) \arrow[r, hook] \arrow[d] & K \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[urr, dashed] & \Delta[0] \arrow[r, "v"'] & L \end{tikzcd}$$

The dashed arrow is induced by lifting property of f and shows that the image of $\Delta[n]$ in K is completely contained in F. Thus we can find a lift $\Delta[n] \to F$. This implies, by lemma 4.9, that it has the right lifting property relative to every injective map, in particular to every inclusion, and also that F contains a 0-simplex w. We denote the map collapsing all of F to w by c_w. It is given by the composition $F \to \Delta[0] \xrightarrow{w} F$. Then we have a lift $H : F \times \Delta[1] \to F$ in the following diagram (induced by lifting property).

$$\begin{tikzcd} F \times \partial\Delta[1] \arrow[r, "(\mathbf{id}_F, c_w)"] \arrow[d, hook] & F \arrow[d] \\ F \times \Delta[1] \arrow[r] \arrow[ur, dashed, "H"] & \Delta[0] \end{tikzcd}$$

By proposition 4.14, the geometric realization preserves products. Thus our diagram in **Top** reads (forgetting the one-point-set in the lower right corner):

$$\begin{tikzcd} {|F| \times \{0,1\}} \arrow[r, "(\mathbf{id}_{|F|}, |w|)"] \arrow[d, hook] & {|F|} \\ {|F| \times I} \arrow[ur, "{|H|}"'] & \end{tikzcd}$$

So $|H| : |F| \times I \to |F|$ is a homotopy from the identity map $\mathbf{id} : |F| \to |F|$ to the constant map $|w|$, and thus $|F|$ is contractible.

By lemma 4.16, $|f|$ is a fibration. The maps $|F| \hookrightarrow |K| \xrightarrow{|f|} |L|$ induce a long exact sequence in homotopy (for further details on the topic, refer for example to [2, p. 344]):

$$
\begin{tikzcd}
 & & (\dots) \arrow[dll] \\
\pi_2(|F|,|w|) \arrow[r] & \pi_2(|K|,|w|) \arrow[r, "\pi_2(|f|,|w|)"] & \pi_2(|L|,|v|) \arrow[dll] \\
\pi_1(|F|,|w|) \arrow[r] & \pi_1(|K|,|w|) \arrow[r, "\pi_1(|f|,|w|)"] & \pi_1(|L|,|v|) \arrow[dll] \\
\pi_0(|F|) \arrow[r, "d"'] & \pi_0(|K|) \arrow[r, "\pi_0(|f|)"] & \pi_0(|L|)
\end{tikzcd}
$$

Exactness at the level of π_0 means that $d(\pi_0(|F|)) = \pi_0(|f|)^{-1}([|v|])$.

Since $|F|$ is contractible, $\pi_n(|F|,|w|)$ is trivial for every $n \geq 0$, and thus $\pi_n(|f|,|w|)$ is an isomorphism for every $n \geq 1$. We are left to show that $\pi_0(|f|) : \pi_0(|K|) \to \pi_0(|L|)$ is also an isomorphism. It is certainly injective, since the long sequence is exact for every $|v|$. In order to see that $|f|$ induces a surjective map on path components, notice that every point in $|L|$ is in the same component of the realization of some vertex $v \in L_0$. Since f has the right lifting property relative to all inclusions, it is surjective on vertices. Indeed we have the following diagram:

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & K \arrow[d, "f"] \\
\Delta[0] \arrow[r, "v"'] \arrow[ur, dashed] & L
\end{tikzcd}
$$

So there is a vertex $w \in K_0$ which is mapped to v, giving us surjectivity on path components. $\square$

The last thing left to show is that $(J_{\mathbf{SSet}} \cap W)^{\pitchfork} \subseteq I^{\pitchfork}_{\mathbf{SSet}}$. Then theorem 2.24 will ensure that what we defined is truly a model structure on **SSet**. To prove this fact would require to introduce a great deal of technical notation and auxiliary objects. Since this would exceed the goal of this thesis , we give the result as a theorem and refer to [3, p. 98] for a complete treatment.

Proposition 4.18. *Every trivial fibration is an element of* $I^{\pitchfork}_{\mathbf{SSet}}$.

We will now move on to the proof of the Quillen equivalence between **Top** and **SSet**.

5 Proof of the Quillen equivalence

We conclude this thesis with the proof that the adjunction we constructed in the previous section is in fact a Quillen equivalence. In order to do so, we have to develop a bit of theory about the homotopy theory of simplicial sets.

5.1 Homotopy groups

We define the notion of homotopy groups for simplicial sets. We begin by defining what it means for two 0-simplices to be homotopic in a fibrant simplicial set.

Definition 5.1. *Let K be a fibrant simplicial set, and let $x, y \in K_0$ be 0-simplices. We say that x and y are homotopic (denoted by $x \sim y$) if there is some 1-simplex $z \in K_1$ such that $d_0(z) = x$ and $d_1(z) = y$ (i.e. x and y are the two faces of z).*

Lemma 5.2. *If K is a fibrant simplicial set, then the homotopy forms an equivalence relation on K_0.*

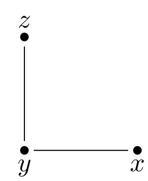
Proof. We have to prove that the relation $\sim$ is reflexive, symmetric and transitive.
Reflexivity: Let $x \in K_0$. Then if we consider $s_0(x) \in K_1$ we have $d_0(s_0(x)) = d_1(s_0(x)) = x$, and thus that $x \sim x$.
Symmetry: Let $x, y \in K_0$ such that $x \sim y$, and let $z \in K_1$ be the 1-simplex such that $d_0(z) = x$ and $d_1(z) = y$. Then we have a map $f : \Lambda^0[2] \to K$ given by the inclusion of the 2-horn drawn below in K.

$$\begin{tikzcd} y \arrow[d, dash, "z"'] & \\ x \arrow[r, dash, "s_0(x)"'] & x \end{tikzcd}$$

Since K is fibrant, f induces an arrow $g : \Delta[2] \to K$ (a 2-simplex) extending f by the diagram:

$$\begin{tikzcd} \Lambda^0[2] \arrow[r, "f"] \arrow[d, hook] & K \arrow[d] \\ \Delta[2] \arrow[r] \arrow[ur, dashed, "g"] & \Delta[0] \end{tikzcd}$$

Then $d_0(g)$ is the required homotopy to get $y \sim x$.
Transitivity: Let $x, y, z \in K_0$ be such that $x \sim y$ and $y \sim z$. Then we have a map $f : \Lambda^0[2] \to K$ given by the inclusion of the following 2-horn in K:



Again as before, since K is fibrant this induces a 2-simplex $g : \Delta[2] \to K$. Then $d_0(g)$ is the desired homotopy giving $x \sim z$. □

We can now define the notion of 0-th homotopy set for fibrant simplicial sets.

Definition 5.3. *Let K be a fibrant simplicial set. Then we denote by $\pi_0(K)$ the set $K_0/\sim$.*

Lemma 5.4. *Let K be a fibrant simplicial set. Then there is an isomorphism $\pi_0(K) \cong \pi_0(|K|)$.*

Proof. Let $a : \pi_0(K) \to \pi_0(|K|)$ be the map taking $[v] \in \pi_0(K)$ to the path connected component of $|X|$ containing $|v|$.

Since $|\Delta[n]|$ is path connected for $n \geq 0$, by construction of the geometric realization every point in $|K|$ is in the same path connected component of the geometric realization of some vertex in K. Thus a is surjective.

To show injectivity, consider for every $[v] \in \pi_0(K)$ the set $K_{[v]}$ of simplices of K having all vertices in $[v]$. Notice that every such $K_{[v]}$ is a sub-simplicial set of K. We can consider them as the path connected components of K. We have that K is the coproduct $\bigsqcup_{[v]\in\pi_0(K)} K_{[v]}$. Since the geometric realization preserves colimits (since it is a left adjoint), we have:

$$|K| = \bigsqcup_{[v]\in\pi_0(K)} |K_{[v]}|$$

where each of the $|K_{[v]}|$ is obviously path connected. This expresses a bijection $\pi_0(K) \cong \pi_0(|K|)$. □

This lemma tells us that to call this set $\pi_0(K)$ makes sense. We will call elements of $\pi_0(K)$ *path components* of K, and if $v \in K_0$ we will also denote by $\pi_0(K, v)$ the pointed set $\pi_0(K)$ with basepoint v.

We want now to extend our definition to higher homotopy groups.

Definition 5.5. *Let $K, L \in$ **SSet** be two simplicial sets. Then we define the* internal hom *of K and L as the simplicial set $[K, L] \in$ **SSet** with sets of simplices $[K, L]_n = \mathbf{SSet}\,(\mathrm{K} \times \Delta[\mathrm{n}], \mathrm{L})$. The arrows between the various sets of simplices are induced by the maps between the $\Delta[k]$. Notice that $[K, L]_0$ is the set of maps of simplicial sets between K and L, and $[K, L]_1$ is the set of homotopies between such maps.*

Definition 5.6. *Let K be a fibrant simplicial set, $v \in K_0$ a vertex of K. If L is any simplicial set, we denote again by v the constant map $L \to \Delta[0] \xrightarrow{v} K$. Let F be the fibre over $v : \partial\Delta[n] \to K$ of the fibration $[\Delta[n], K] \to [\partial\Delta[n], K]$ induced by the inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$. Then we define the n-th homotopy group $\pi_n(K, v)$ to be the pointed set $\pi_0(F, v)$.*

The map $[\Delta[n], K] \to [\partial\Delta[n], K]$ is a fibration by [3, p. 81].

Remark 5.7. An equivalent definition is to say that $\pi_n(K, v)$ is the set of equivalence classes $[x]$ of n-simplices whose entire boundary is degenerate in the vertex v, where two n-simplices x and y are equivalent if there is a homotopy $H : \Delta[n] \times \Delta[1] \to K$ such that $H(\cdot, 0)$ is the inclusion of x in K, $H(\cdot, 1)$ is the inclusion of y and the restriction of H to $\partial\Delta[n] \times \Delta[1]$ is the constant map v.

It is not clear yet that those homotopy groups are in fact groups. The next proposition implies it.

Lemma 5.8. *Let $K, L \in$ **SSet** be simplicial sets and assume that there exist a trivial fibration $f : K \to L$. Then K and L have the same homotopy groups.*

Proof. We will show that the maps $\pi_0(f)$ and $\pi_n(f, v)$ induced by f are all bijections.
Injectivity: Let $[p], [q] \in \pi_n(K)$ (where we omit the basepoint) with the same image in $\pi_n(L)$, i.e. $[f \circ p] = [f \circ q]$. We show that there exist a homotopy between p and q (representants of the classes $[p]$ and $[q]$ respectively). Consider the following diagram:

$$\begin{tikzcd}
\partial\Delta[n] \times \partial\Delta[1] \arrow[r, "{(p,q)}"] \arrow[d, hook] & K \arrow[d, "f"] \\
\partial\Delta[n] \times \Delta[1] \arrow[r, "H"'] \arrow[ur, dashed] & L
\end{tikzcd}$$

where H is the homotopy between $f \circ p$ and $f \circ q$. Notice that the left vertical map is injective, and thus an element of $I_{\mathbf{SSet}}-$cof by lemma 4.9. Thus the dashed filler is induced by lifting property, providing the desired homotopy.
Surjectivity: Let $[p] \in \pi_n(L)$, then the following diagram proves surjectivity:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & K \arrow[d, "f"] \\
\partial\Delta[n] \arrow[r, "p"'] \arrow[ur, dashed] & L
\end{tikzcd}$$

where again we have used the fact that the left vertical map is injective, and thus a cofibration. $\square$

Proposition 5.9. *Let K be a fibrant simplicial set. Then for every $n \geq 0$ and every vertex v of K there is an isomorphism $\pi_n(K, v) \cong \pi_n(|K|, |v|)$.*

Proof. The proof goes by induction on n. Notice that the case $n = 0$ is given by lemma 5.4.

We define the simplicial set P_vK as the following pullback:

$$\begin{tikzcd} P_vK \arrow[r] \arrow[d] & {[\Delta[1], K]} \arrow[d, "{(s,t)}"] \\ K \arrow[r, "v \times \mathbf{id}_K"'] & K \times K \cong [\partial\Delta[1], K] \end{tikzcd}$$

where s and t denote the source and target maps induced by the two maps $\partial\Delta[1] \to \Delta[1]$. We define a second simplicial set ΩK, again as a pullback:

$$\begin{tikzcd} \Omega K \arrow[r] \arrow[d] & P_vK \arrow[d] \\ \Delta[0] \arrow[r, "v"'] & K \end{tikzcd}$$

Using those two commutative squares, we obtain the following diagram:

$$\begin{tikzcd} \Omega K \arrow[r] \arrow[d] & P_vK \arrow[r] \arrow[d] & {[\Delta[1], K]} \arrow[d, "{(s,t)}"] \\ \Delta[0] \arrow[r, "v"'] & K \arrow[r, "v \times \mathbf{id}_K"'] \arrow[d] & K \times K \arrow[d, "p_1"] \\ & \Delta[0] \arrow[r, "v"'] & K \end{tikzcd}$$

where $p_1 : K \times K \to K$ denotes the projection on the first factor. Notice that all cells in the diagram are pullback squares. it can be proven that the vertical composite $p_1 \circ (s, t)$ is in $I^{\pitchfork}_{\mathbf{SSet}}$, i.e. it is a trivial fibration (we will not fill in the details fo the proof of this fact, since in order to do so we would need much of the theory we skipped in the last section; the interested reader can refer to [3, p. 97]). Then the universal properties of the pullbacks give us that the composite map $f : P_vK \to K \to \Delta[0]$ is a trivial fibration. In particular, it follows by lemma 5.8 that all the homotopy groups of P_vK are trivial. Notice that ΩK is the fibre of the map $P_vK \to K$ over v, thus we have a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(K) \longrightarrow \pi_n(\Omega K) \longrightarrow \pi_n(P_vK) \longrightarrow \pi_n(K) \longrightarrow \cdots$$

where we omitted the basepoints. Since the $\pi_n(P_vK)$ are trivial, we obtain that $\pi_{n+1}(K) \cong \pi_n(\Omega K)$.

Now we will show that a similar thing is true in **Top**. Indeed we can take the geometric realization of the diagram above. By proposition 4.15, pullbacks are preserved. Also by definition of geometric realization, the map $|f|$ is a weak equivalence. Then the long homotopy sequence tells us that $\pi_{n+1}(|K|) \cong \pi_n(|\Omega K|)$. We can now conclude our induction. Assume that for every simplicial set L we have $\pi_k(L) \cong \pi_k(|L|)$ for every $k \leq n$. Then:

$$\pi_{n+1}(K) \cong \pi_n(\Omega K) \cong \pi_n(|\Omega K|) \cong \pi_{n+1}(|K|)$$

$\square$

5.2 Proof of the Quillen equivalence

Theorem 5.10. *The adjunction* $(|\cdot|, \mathrm{Sing}, \varphi) : \mathbf{SSet} \to \mathbf{Top}$ *is a Quillen equivalence.*

Proof. We will first prove that it is a Quillen adjunction, and then that it is a Quillen equivalence.

Quillen adjunction: To prove this, we will apply proposition 2.34, that states that for an adjunction it is equivalent to be Quillen or to satisfy $i \in I_{\mathbf{SSet}} \Rightarrow |i| \in I_{\mathbf{Top}}-$cof and $j \in J_{\mathbf{SSet}} \Rightarrow |j| \in J_{\mathbf{Top}}-$cof.

So let $i : \partial\Delta[n] \hookrightarrow \Delta[n]$ be an element of $I_{\mathbf{SSet}}$. Then we have an arrow such that the following diagram commutes.

$$\begin{tikzcd}
|\partial\Delta[n]| \arrow[r, equal, "\sim"] \arrow[d, hook, "|i|"'] & S^{n-1} \arrow[d, hook, "\in I_{\mathbf{Top}}"] \\
|\Delta[n]| \arrow[r, equal, "\sim"] & D^n
\end{tikzcd}$$

This shows that $|i| \in I_{\mathbf{Top}}$-cof. Similarly for $j \in J_{\mathbf{SSet}}$ we have that $|j| \in J_{\mathbf{Top}}$-cof. Thus, by proposition 2.34, $(|\cdot|, \mathrm{Sing}\varphi)$ is a Quillen adjunction.

Quillen equivalence: We want now to use proposition 2.40 to show that this Quillen adjunction is a Quillen equivalence. It states that a Quillen adjunction is a Quillen equivalence if, and only if it satisfies the following two conditions:

a) $|\cdot|$ reflects weak equivalences between cofibrant objects.

b) For every $X \in \mathbf{Top}$ which is fibrant, the map $|Q \circ \mathrm{Sing}(X)| \to X$ is a weak equivalence.

By definition of $W_{\mathbf{SSet}}$, the geometric realization reflects weak equivalences, so condition *a*) is trivially satisfied. We are left to show that the map

$$|\mathbf{Sing}(X)| \to X$$

is a weak equivalence for every X which is fibrant, i.e for every topological space X, as shown in lemma 3.14 (notice that since **Sing** is a right Quillen functor,

Sing(X) is fibrant). So, we have to show that for every $n \geq 0$ and every point $v \in X$ the map

$$\pi_n(|\mathbf{Sing}(X)|, v) \to \pi_n(X, v)$$

is an isomorphism (where we have denoted again by $v \in |\mathbf{Sing}(X)|$ the image of $v \in X$). Indeed, the set of points of X is in bijective correspondence with the vertices of **Sing**(X) and every point of $|\mathbf{Sing}(X)|$ is in the same path component of some vertex of **Sing**(X). Now, since **Sing**(X) is fibrant, by proposition 5.9 we have an isomorphism

$$\pi_n(\mathbf{Sing}(X), v) \cong \pi_n(|\mathbf{Sing}(X)|, v)$$

By composing the map induced by the adjunction with this isomorphism, we get a map $\pi_n(\mathbf{Sing}(X), v) \to \pi_n(X, v)$.

In order to conclude the proof, we will elaborate a bit on remark 5.7. Consider the simplicial set obtained through the following pushout:

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r, hook] \arrow[d] & \Delta[n] \arrow[d] \\
\Delta[0] \arrow[r] & S[n]
\end{tikzcd}$$

Then every element of $\pi_n(\mathbf{Sing}(X), v)$ can be represented by some map $p : S[n] \to \mathbf{Sing}(X)$ sending the unique 0-simplex of $S[n]$ to v. Now, applying the adjunction we obtain:

$$\mathbf{SSet}\left(\mathrm{S[n]}, \mathbf{Sing}(\mathrm{X})\right) \cong \mathbf{Top}\left(\mathrm{S^n}, \mathrm{X}\right)$$

(notice that $|S[n]| \cong S^n$ since the geometric realization preserves pushouts). It is enough to show that the adjunction sends homotopies into homotopies, but since the geometric realization preserves finite products we have:

$$\mathbf{SSet}\left(\mathrm{S[n]} \times \Delta[1], \mathbf{Sing}(\mathrm{X})\right) \cong \mathbf{Top}\left(\mathrm{S^n} \times \mathrm{I}, \mathrm{X}\right)$$

and we're done. $\square$

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