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Polarized Quantum Momentum Maps

Master Thesis

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Chapter 1

Introduction

The problem in deformation quantization is to find an associative product, called a $*$-product on $C^\infty(M)[[t]]$, the space of formal power series in functions on a manifold M, which is a deformation of the ordinary product of functions, such that its commutator is a deformation of a given Poisson bracket. For a general Poisson manifold existence is solved by Kontsevich's formality theorem. However, in the symplectic case there are more explicit constructions available due to Fedosov ([6]). Moreover, there might be additional structure that one wishes to quantize, for instance a G-action induced by a Hamiltonian map. One class of such examples arises from considering coadjoint orbits of a Lie algebra, which is what we will be mainly concerned about in this thesis. Also, any $*$-product possesses a canonical trace, that is a cyclic linear functional with values in Laurent series. The Fedosov-Nest-Tsygan index formula relates this trace to characteristic classes of the $*$-product and the underlying symplectic manifold. The goal of this thesis is to find a representation theoretic formula for the terms appearing in the case of a semi-simple coadjoint orbit.

In chapter 2, the basic notions of deformation quantization are recalled, together with some motivating examples. Chapter 3 is a quick summary of Fedosov's construction, together with some generalities about equivalences, and a variation to Fedosov's construction described in [2]. Then in chapter 4, the general theory is applied to the special case of a coadjoint orbit.

At this point, I would like to thank Prof. Damien Calaque for his patience and all the interesting discussions we had.

Chapter 2

Preliminaries

2.1 Deformation Quantization Problem

In the following let (M, ω) denote a symplectic manifold and $\{\cdot, \cdot\}$ the associated Poisson structure.

Definition 2.1 ($*$-Product) *A $*$-product on M is an associative product on $\mathcal{A} := C^\infty(M)[[t]]$ which is t-linear and of the form*

$$f * g = fg + tB_1(f,g) + t^2 B_2(f,g) + \cdots, \quad \textit{where } f,g \in C^\infty(M)$$

where

- *The B_i are bidifferential operators.*
- *The constant function 1 is still the unit in $(\mathcal{A}, *)$ turning it into a unital associative algebra.*

Any $*$-product defines a Poisson structure on the underlying manifold by means of its commutator

$$\{f,g\} := \frac{1}{t}\,[f,g]_*\,|_{t=0} = B_1(f,g) - B_1(g,f), \text{ for } f,g \in C^\infty(M)$$

A $*$-product on a given symlectic manifold (M, ω), such that the Poisson structure induced by the $*$-product and the one induced by the symplectic form coincide, is called a deformation quantization of (M, ω). It follows from the definition of a $*$-product that B_1 is thus always of order 1.

2.2 Polarized Deformation Quantization

A **polarization** of the symplectic manifold (M, ω) is an integrable Lagrangian subbundle P of the complexified tangent bundle $T^{\mathbb{C}}M$. Let $\mathcal{O}$ denote the sheaf of functions locally constant along P, it is in particular a subsheaf of

C^∞, the sheaf of infinitely differentiable functions on M, where the Poisson bracket vanishes. The corresponding quantized structure is roughly speaking a $*$-product together with a deformation of $\mathcal{O}$ that is a commutative subalgebra of $(\mathcal{A}, *)$. More precisely

Definition 2.2 *A quantization of a polarized symplectic manifold* (M, ω, P) *is a pair* $(\mathcal{A}, \mathcal{O})$ *sucht that*

- $\mathcal{A}$ *is a quantization of* (M, ω).
- $\mathcal{O}$ *is a commutative subsheaf of* $\mathcal{A}$ *that comprises of functions vanishing along a deformation* $\mathcal{P}$ *of* P*. That is,* $\mathcal{P}$ *is a locally free* $C^\infty[[t]]$*-module, such that* $\mathcal{P}_0 = P$ *and* $\mathcal{O}$ *the functions* $f \in \mathcal{A}$ *such that* $df|_{\mathcal{P}} = 0$.

2.3 Quantum Momentum Maps

2.3.1 Classical Momentum Maps

Assume that (M, ω) admits a Lie group G of symmetries. Taking differentials this induces a map $\rho : \mathfrak{g} \to \mathcal{X}(M, \omega) = \{\text{vector fields on } M \text{ preserving } \omega\}$ of Lie algebras. Moreover, there is an exact sequence of Lie algebras.

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & (C^\infty(M), \{,\cdot,\cdot\}) & \to & \mathcal{X}(M,\omega) & \to & H^1(M,\mathbb{R}) & \to & 0 \\ & & & & f & \mapsto & \{f, \cdot\} & & & & \end{array}$$

Definition 2.3 *A lift of* ρ *to a Lie algebra map*

$$\mu : \mathfrak{g} \to C^\infty(M)$$

is called a ***momentum map****.*

Equivalently μ can also be defined as a map $M \to \mathfrak{g}^*$ such that for $\mu_\xi \in C^\infty(M)$, $\mu_\xi(x) = \langle \mu(x), \xi \rangle$ the following holds

$$\{\mu_\xi, \mu_\eta\} = \mu_{[\xi,\eta]} \text{ , for } \xi, \eta \in \mathfrak{g}$$

and that the Hamiltonian vector fields X_{μ_ξ}, $\xi \in \mathfrak{g}$ integrate to a G-action.

Remark 2.4 *For connected manifolds such that* $H^1(M) = 0$ *there is a well-defined obstruction class* $\nu_\rho \in H^2(\mathfrak{g}, \mathbb{R})$ *to the existence of such a lift. Equivalence classes of such lifts are controlled by* $H^1(\mathfrak{g}, \mathbb{R})$*. In particular, if we further assume that* $\mathfrak{g}$ *is semi-simple, there is always a unique lift.*

2.3.2 Quantum Momentum Maps

Let now $\mathcal{A} = (C^\infty(M)[[t]], *)$ be a deformation quantization of (M, ω) that is invariant under G, that is

$$(g \cdot f) * (g \cdot h) = g \cdot (f * h) \text{ , where } g \in G \text{ and } f, h \in C^\infty(M)[[t]].$$

In other words, there exists a group homomorphism from G to $Aut(\mathcal{A})$, the group of algebra automorphisms. Again taking derivatives yields a map

$$\rho : \mathfrak{g} \to Der(\mathcal{A}), X \mapsto L_X.$$

Analogously to the classical case there is a map

$$\frac{1}{t}\mathcal{A} \to Der\mathcal{A}, \quad \frac{1}{t} \mapsto \frac{1}{t}ad_* f = \frac{1}{t}[f, \cdot]_*$$

where $Der\mathcal{A}$ denotes the Lie algebra of t-linear derivations on $\mathcal{A}$.

Lemma 2.5 *Under the assumption that $H^1(M, \mathbb{R}) = 0$ and M is connected the following sequence is exact*

$$0 \longrightarrow \mathbb{R}[[t]] \longrightarrow \frac{1}{t}\mathcal{A} \longrightarrow \mathrm{Der}\,\mathcal{A} \longrightarrow 0$$

and a similar remark as in the classical case applies.

Proof Let $\varphi = t^k\varphi_0 + t^{k+1}\varphi_1 + \cdots \in Der\mathcal{A}$ be a derivation on (A). In particular, looking at lowest order terms yields that φ_0 is a derivation on $C^\infty(M)$ and hence a vectorfield. Furthermore, φ is also a derivation on the Lie algebra $(\mathcal{A}, \frac{1}{t}[\cdot, \cdot]_*)$. Looking at lowest order terms one sees, that φ_0 is a derivation of the Poisson-bracket and hence a symplectic vector field. By assumption, any symplectic vector field is Hamiltonian, and hence there is a function $f \in C^\infty(M)$ such that $\varphi_0 = \{f, \cdot\} = \left[\frac{1}{t}f, \cdot\right]|_{t=0}$. Now set $\varphi' = \varphi - \left[\frac{1}{t}t^k f, \cdot\right]$ whose lowest order term is now of order $k+1$. Now the claim follows by induction and taking the limit. □

Definition 2.6 *A* ***quantum momentum map*** *is a lift of ρ to a map*

$$\mu_t : \mathfrak{g} \to \mathcal{A} : \xi \mapsto f_\xi$$

such that the composition $\mathfrak{g} \xrightarrow{\mu_t} \mathcal{A} \xrightarrow{\frac{1}{t}} \frac{1}{t}\mathcal{A}$ is a homomorphism of Lie algebras. In other words, that

$$f_{[\xi,\eta]} = \frac{1}{t}\left[f_\xi, f_\eta\right]_*$$

In order to get a map into the Lie algebra $\mathcal{A}$ we need to change the bracket on $\mathfrak{g}$. In particular, define $\mathfrak{g}_t = \mathfrak{g}[[t]]$ as a $k[[t]]$-module and define the deformed Lie bracket by

$$[X, Y]_t = t\,[X, Y]_{\mathfrak{g}} \text{ for } X, Y \in \mathfrak{g} \subset \mathfrak{g}_\hbar$$

and extend linearly in t.

With this notation we can reformulate the above definition to

Definition 2.7 *A* ***quantum momentum map*** *is a map of algebras* $\mu_\hbar : U\mathfrak{g}_\hbar \to \mathcal{A}$ *such that*

$$L_X f = \frac{1}{t} ad\mu_\hbar(X), \ \ X \in \mathfrak{g}, f \in \mathcal{A}.$$

Setting $t = 0$ we see that such a quantum momentum map induces a classical momentum map $\mathfrak{g} \to \frac{1}{t}\mathcal{A} \overset{t=0}{\to} (C^\infty(M), \{\cdot,\cdot\})$.

Definition 2.8 *A quantum momentum map* $\mu_\hbar$ *is called* ***strong*** *if* $\mu_\hbar(X) \in C^\infty(M) \subset \mathcal{A}$*, and thus*

$$[\mu_\hbar(X), \cdot]_* = t\{\mu_\hbar(X), \cdot\}.$$

In other words, that the classical and the quantum momentum map coincide.

2.4 Examples

2.4.1 Moyal-Weyl Product

Let (V, ω) be a symplectic vector space and let $\Pi = \omega^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \in V \otimes V \subset SV[[t]] \otimes SV[[t]]$ denote the associated Poisson bi-vector. Define a product on $C^\infty(V)[[t]]$ as follows

$$f * g = me^{\frac{t}{2}\Pi}(f \otimes g) = \sum_n \frac{1}{n!} \left(\frac{t}{2}\right)^n \omega^{i_1 j_1} \cdots \omega^{i_n j_n} \frac{\partial^n f}{\partial x^{i_1} \cdots \partial x^{i_1}} \frac{\partial^n g}{\partial x^{j_1} \cdots \partial x^{j_n}}$$

where m denotes the multiplication of functions. This product is associative since

$$\begin{aligned}
f * (g * h) &= me^{\frac{t}{2}\Pi}(f \otimes me^{\frac{t}{2}\Pi}(g \otimes h)) \\
&= m(1 \otimes m)((1 \otimes \Delta)e^{\frac{t}{2}\Pi})(1 \otimes e^{\frac{t}{2}\Pi})(f \otimes g \otimes h) \\
&= m(1 \otimes m)e^{\frac{t}{2}\Pi_{1,2} + \frac{t}{2}\Pi_{1,2}} e^{\frac{t}{2}\Pi_{2,3}}(f \otimes g \otimes h) \\
&= m(1 \otimes m)e^{\frac{t}{2}\Pi_{1,3} + \frac{t}{2}\Pi_{2,3}} e^{\frac{t}{2}\Pi_{1,2}}(f \otimes g \otimes h) \\
&= m(1 \otimes m)((\Delta \otimes 1)e^{\frac{t}{2}\Pi})(e^{\frac{t}{2}\Pi} \otimes 1)(f \otimes g \otimes h) \\
&= me^{\frac{t}{2}\Pi} m(e^{\frac{t}{2}\Pi} f \otimes g) \otimes h \\
&= (f * g) * h
\end{aligned}$$

where we used $Dm = m(\Delta D)$ for a differential operator D, and where Δ denotes the usual coproduct on SV, and $\Delta e^x = e^{\Delta x}$.

It follows that this product defines a deformation quantization of (V, ω).

Remark 2.9 *For* f, g *linear functions we have* $f * g = fg + \frac{t}{2}\{f, g\}$*.*

Alternatively, the inclusion $W := V^* \oplus k \hookrightarrow C^\infty(V)$ turns W into a Lie algebra. Namely, the central extension of the trivial Lie algebra V^* by the cocycle

$c(X, Y) = \{X, Y\} \in k$. Multiplying c by t we get a new Lie algebra structure on $W[[t]] = W_t$. Moreover, the canonical map $W_t \to C^\infty(V)[[t]]$ is a map of Lie algebras and further induces a map

$$UW_t \to C^\infty(V)[[t]]$$

that completely determines the $*$-product.

2.4.2 Cotangent bundles

Let $(T^*M, d\lambda)$ be a cotangent bundle and let $\mathrm{Diff}(M)$ denote the algebra of differential operators. Moreover, define

$$\mathrm{Diff}_\hbar(M) = \langle \mathcal{O}_M, TM \rangle \,/ I$$

where the $\mathcal{O}_M$ denotes the algebra of function on M and the ideal I is generated by

$$\begin{aligned} f \otimes g &= fg \\ f \otimes X &= fX \\ X \otimes f - f \otimes X &= tX(f) \\ X \otimes Y - Y \otimes X &= t\,[X, Y] \end{aligned}$$

where $f, g \in \mathcal{O}_M$ and X, Y are vectorfields on M. These relations are actually the same as the ones coming from the deformed Poisson structure following from

Lemma 2.10 *Let $f, g \in \mathcal{O}_M$ be viewed as functions on T^*M constant along the fibers and let $X, Y \in \mathcal{X}(M)$ be vectorfields on M viewed as linear functions on T^*M. Then*

$$\begin{aligned} \{f, g\} &= 0 \\ \{f, X\} &= X(f) \\ \{X, Y\} &= [X, Y]. \end{aligned}$$

Furthermore, the symplectic transformation generated by the Hamiltonian vector fields are given by translation by df in the case of a function, and by the pushforward of covectors via $exp(\varepsilon X)$ in the case of X.

Moreover, $\mathrm{Diff}_\hbar(M)$ is isomorphic to the Rees algebra of $\mathrm{Diff}(M)$ via

$$\begin{aligned} \mathrm{Diff}_\hbar(M) &\to \mathrm{Rees}\,\mathrm{Diff}(M) \\ f &\mapsto f \cdot 1 \\ X &\mapsto X \cdot t \end{aligned}$$

Now pick a torsionfree connection on M and use it to define a bijection from the algebra of functions on T^*M which are polynomial along the fiber to $\mathrm{Diff}_\hbar(M)$

$$\begin{array}{rcl} S(TM)[[t]] & \to & \mathrm{Rees}(\mathrm{Diff}(M)) \\ X_1 \cdots X_n & \mapsto & (f \mapsto t^n \nabla^n_{X_1,\cdots,X_n} f) \end{array}$$

This defines a deformed product on the space of functions that are polynomial when restricted to a fiber. To show that this extends to a $*$-product, namely that the product is given by bidifferential operators, it suffices to show that at every point the product is continuous with respect to the topology induces by the vanishing ideal. However, away from the zero-section, where it follows from the grading that is respected, this is hard to show. One aproach to show that it indeed defines a $*$-product, which is carried out in [10], is to extend the map to pseudo-differential operators, and then show that there is a complete symbol calculus.

Chapter 3

Existence Results

3.1 Fedosov Quantization

In this chapter we describe Fedosov's [6] construction for a quantization of a symplectic manifold. The idea is roughly the following. If our manifold admitted Darboux charts where the transition maps are just affine symplectic transformations, then we could use the standard Weyl algebra as a deformation since it is invariant under affine symplectic transformations. However, in general symplectic manifolds are not of that simple structure and we have to make the charts infinitesimally small and then use a connection to patch them together.

Let now $W = \mathbb{C}[[x_1, \cdots, x_{2n}, t]]$ denote the (completed) Weyl algebra as constructed above where $(x_1, \cdots, x_{2n})$ is a dual basis of the symplectic vector space (V, ω_0). Assign furthermore degrees, such that $deg(t) = 2$ and $deg(x_i) = 1$, and thusly turning W into the completion of a graded algebra. The symplectic group $\mathrm{Sp}(2n) = \mathrm{Sp}(V, \omega_0)$ acts on W. Taking derivatives we get a map of Lie algebras

$$\rho : \mathrm{sp}(2n) \to \mathrm{Der}(W)$$

which admits a lift along

$$\frac{1}{t}W \overset{\mathrm{ad}}{\to} \mathrm{Der}(W)$$

whose image is given by quadratic elements in W. Considering W as functions on V with valus in $\mathbb{C}[[t]]$, the exterior derivative defines a derivation of degree 1, that is a map $W \otimes \Lambda^\bullet \overset{d}{\to} W \otimes \Lambda^\bullet$, also given by $d = \frac{1}{t}\,\mathrm{ad}_*(dx_i \omega_0^{ij} x_j)$, and a corresponding homotopy inverse d^{-1}. Together they define a projection $\sigma := id - \left(dd^{-1} + d^{-1}d\right) : W \to \mathbb{C}[[t]]$. Moreover, all those maps respect the $Sp(2n)$ structure.

Let now (M, ω) be a symplectic manifold. Now we want to attach all this structure at every point of our manifold, using that the tangent space is naturally a symplectic vector space. More precisely, let P denote the $\mathrm{Sp}(2n)$-principal bundle of symplectic frames in TM. Define the **Weyl algebra bundle** as the following associated bundle

$$\mathcal{W} = P \times_{\mathrm{Sp}(2n)} W.$$

And the corresponding **Fedosov algebra** as $\mathbf{W} = \mathcal{W} \otimes \Lambda^\bullet$. The operations d, d^{-1}, σ induce pointwise operations on $\mathbf{W}$ which we denote by δ, δ^{-1}, σ satisfying the following properties:

- $\delta^2 = (\delta^{-1})^2 = 0$
- δ defines a derivation on $\mathbf{W}$
- $\sigma = id - \left(\delta\delta^{-1} + \delta^{-1}\delta\right)$

Next, we want to describe connections on $\mathcal{W}$. Originating from an $\mathrm{Sp}(2n)$-pricipal bundle, any principal connection on P induces a connection on $\mathcal{W}$ via extension of structure group along $\mathrm{Sp}(2n) \to \mathrm{Aut}(W)$ followed by taking the associated covariant derivative on the $\mathrm{Aut}(W)$-representation W. Owing to the surjectivity of (3.1) such a connection can be represented locally by section of $\frac{1}{t}\mathcal{W}$. More precisely, let ∂ be a symplectic connection on (M, ω). Locally it is defined by a connection 1-form

$$\alpha = \Gamma^i_{jk} dx^j \otimes \partial_i \otimes dx^k$$

the associated $\mathrm{Aut}(W)$-connection can then be represented by the connection 1-form

$$\tilde{\alpha} = \frac{1}{2}\frac{1}{t}\omega_{li}\Gamma^i_{jk} x^l x^j dx^k.$$

More precisely, we used the homomorphism

$$\rho : \mathrm{sp}(2n) \to \frac{1}{t}\{\text{quadratic functions}\} \in \frac{1}{t}W \tag{3.1}$$

to get a connection 1-form with values in the Lie algebra $\frac{1}{t}W$. Since ρ is an $\mathrm{Sp}(2n)$-invariant homomorphism of Lie algebras one can define connections with values in $\frac{1}{t}W$ via the usual local formulas. In particular, they induce $\mathrm{Aut}(W)$-connections. Now, any two $\frac{1}{t}W$-connections differs by a section $\frac{1}{t}\gamma$ of $\frac{1}{t}\mathcal{W}$.

Remark 3.1 *Since* ad *is not injective, an addition of a section of* $T^*M[[t]]$ *yields the same* $\mathrm{Aut}(W)$*-connection.*

Let now $\tilde{\partial}$ denote the $\frac{1}{t}\mathcal{W}$-connection constructed above and consider

$$\tilde{D} = \tilde{\partial} + \frac{1}{t}\gamma$$

with associated covariant derivative on $\mathcal{W}$

$$Da = \partial a + \frac{1}{t}\left[\gamma, a\right], \quad \text{for } a \text{ a section of } \mathbf{W}$$

The curvature is then given by the following 2-form with values in $\frac{1}{t}\mathcal{W}$

$$\Omega = R + \partial\gamma + \frac{1}{t}\underbrace{\gamma^2}_{\frac{1}{2}[\gamma,\gamma]}$$

and the corresponding curvature of D is then

$$D^2 = \frac{1}{t}\left[\Omega, \cdot\right]$$

Definition 3.2 *A $\frac{1}{t}\mathcal{W}$-connection is called **abelian** if its corresponding covariant derivative is flat, that is if and only if Ω is a central form.*

By the second Bianchi identity $D\Omega = 0$, which reduces to Ω being closed 2-form in the abelian case.

Theorem 3.3 (Fedosov) *Let ∂ be a torsionfree symplectic connection and let $\nu \in Z^2(M)[[t]]$ be a sequence of closed 2-forms. Then there exists an abelian connection of the form*

$$D = \partial - \delta + \frac{1}{t}\left[r, \cdot\right]$$

for some $r \in \mathcal{W} \otimes \Lambda^1$ such that its curvature is given by $-\omega + t\nu$. Furthermore, r can be chosen uniquely such that $\delta^{-1}r = 0$ and such that it only contains terms of degree more than 2.

Remark 3.4 *By looking at the local transformation formula for a connection, one notices that a $\frac{1}{t}W$-connection transforms as a tensor up to a term in the image of ρ. In particular, it makes sense to talk about the linear term. If it conincides with $-\delta$ and the connections is furthermore abelian it is called a **Fedosov connection**.*

Proof (sketch) The curvature of D is given by

$$\Omega = R + \partial r + \frac{1}{t}r^2 - \delta r - \omega.$$

By assumption we have $\delta^{-1}\delta r = r$. Thus our prospective r satisfies

$$r = \delta^{-1}(R - t\nu) + \delta^{-1}(\partial r + \frac{1}{t}r^2), \tag{3.2}$$

which can now be solved by the iteration method. □

Using such an abelian connection we can now construct a $*$-product.

Theorem 3.5 (Fedosov) *Let $\mathcal{W}_D$ denote the flat sections of $\mathcal{W}$. Then*

i) *$\mathcal{W}_D$ is a subalgebra of $\mathcal{W}$.*

ii) *$\sigma = \mathrm{id} - (\delta\delta^{-1} + \delta^{-1}\delta)$ gives a 1-to-1 correspondence between $\mathcal{W}_D$ and $C^\infty(M)[[t]]$.*

iii) *σ induces a $*$-product on (M, ω)*

Proof i) Follows from the fact that D is a derivation on $\mathcal{W}$

ii) Given a function $f \in C^\infty(M)[[t]]$ we want to construct a flat section $a \in \mathcal{W}$ such that $\sigma(a) = f$. From $Da = 0$ it follows that

$$\delta a = \partial a + \frac{1}{t}[r, a].$$

using the definition of σ and that $\delta^{-1}a = 0$ since it has no "form" terms, we get

$$a = \sigma(a) + \delta^{-1}(\partial a + \frac{1}{t}[r, a]),$$

which is again an equation we can solve by the iteration method.

iii) Doing the first two iterations of solving the above equation we get

$$a = f + \partial_i f\, x^i + \frac{1}{2}\partial_i\partial_j f\, x^i x^j + \text{terms of degree} \geq 3.$$

From this is follows that the Poisson structure induced by the $*$-product coincides with the one coming from ω. □

3.2 Obstruction Theory

In this section we introduce the main object controlling differentiable products, (infinitesimal) deformations and equivalences thereof, namely the Gerstenhaber algebra of (local) Hochschild cochains (see [3] for more details). Let A denote the algebra (or sheaf of algebras) of C^∞-functions on a Poisson manifold $(M, \{\cdot,\cdot\})$. Define the vector spaces

$$\begin{aligned} C^k(A) &:= \mathrm{Hom}_{\mathrm{diff}}(A^{\otimes k}, A) \\ &= \{\varphi : \underbrace{A \times \cdots \times A}_{k} \mapsto A \mid \varphi \text{ is a differential operator}\}. \end{aligned}$$

Equivalently, we could ask that the maps φ be maps of sheaves, which implies by Peetre's theorem that they be differential operator. For $f \in C^k(A)$ and $g \in C^l(A)$ we define the brace product

$$\begin{aligned} &f\{g\}(x_1, \cdots, x_{k+l-1}) := \\ &\quad \sum (-1)^{(i-1)(l-1)} f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+l-1}), x_{i+l}, \cdots, x_{i+l-1}) \end{aligned}$$

and finally the **Gerstenhaber bracket**

$$[f, g] := f\{g\} - (-1)^{(k-1)(l-1)} g\{f\}.$$

Any differentiable product on A is given by an element $\mu \in C^2(A)$, which satisfies associativity if and only if

$$2\mu\{\mu\} = [\mu, \mu] = 0.$$

Theorem 3.6 *The Gerstenhaber bracket satisfies the graded Jacobi identity with the grading shifted by* 1.

This implies in particular that any associative product μ induces a differential on $C^\bullet(A)$, turning it into the Hochschild complex for the algebra (A, μ).

Now, assume that μ and μ' are $*$-products on M and we try to show that they are equivalent. For simplicity, assume that they agree up to order $(k-1) \geq 1$. Namely,

$$\mu' = \mu + t^k B + O(t^{k+1}).$$

By associativity we get

$$\begin{aligned} 0 &= [\mu', \mu'] \\ &= t^k 2[\mu, B] + O(t^{2k}) \end{aligned}$$

In particular, looking at the two lowest order terms we get

$$\begin{aligned} [m, B] &= 0 \\ [\Pi, B] &= 0 \end{aligned}$$

where m, Π denote ordinary multiplication and the Poisson tensor, respectively.

We try to find $F \in C^1(A)[[t]]t^l$ such that

$$\begin{aligned} \mu'(f, g) &= \mu^{1+F}(f, g) \\ &:= (1+F)^{-1}\mu((1+F)f, (1+F)g) \\ &= \mu + [\mu, F] + O(t^{2l}). \end{aligned} \tag{3.3}$$

Thus we can easily add (that is by applying an equivalence) to B any coboundary in the complex $(C^\bullet(A), d = [m, \cdot])$, that is the ordinary (differential) Hochschild cohomology of $C^\infty(M)$. Its computation goes back to [11] and is given by

Proposition 3.7 *Let $\varphi \in C^k(A)$ be a Hochschild cocycle (i.e. $[m, \varphi] = 0$), then there exists $\varphi' \in C^{k-1}(A)$ and a polyvector field τ such that*

$$\varphi(f_1, \cdots, f_k) = [m, \varphi'](f_1, \cdots, f_k) + \tau(df_1, \cdots, df_k).$$

This identifies $HH(A, A)$ with $\Gamma(\Lambda^\bullet(TM))$. Moroever, the Poisson tensor Π induces a derivation on $HH(A, A)$.

Using this proposition we can assume that B is in fact a bivector field. Looking again at equation (3.3) an equivalence of the form

$$1 + t^{k-1}F$$

for an F such that $[m, F] = 0$, changes B by $[\Pi, F]$.

Thus the first obstruction to this equivalence problem lies in the second cohomology of the complex

$$(\Gamma(\Lambda^\bullet(TM)), d = [\Pi, \cdot])$$

which is called **Poisson cohomology**. In the symplectic case, polyvectorfields can be identified with differential forms, and the Poisson differential goes over to the ordinary exterior derivative on forms. All in all we get the following

Proposition 3.8 *Assume that $H^2(M;\mathbb{C}) = 0$. Then all $*$-products are equivalent. In particular, all $*$-products are locally equivalent.*

3.3 Classification

Using local equivalence and given one $*$-product $\mathcal{A}$ any other $*$-product can be described by these local equivalences, and its equivalence class is then uniquely defined by the corresponding 1-Čech cohomology class with values in the sheaf $\underline{\mathrm{Aut}}\mathcal{A}$. Here, we take automorphisms to be differential operators equal to the identity (mod t).

In more abstract terms, local equivalence shows that we have a gerbe of $*$-products over our manifold. And by Fedosov's existence result we get a global section $\mathcal{A}$. In this case global sections (up to equivalence) are in one-to-one correspondence with

$$\check{H}^1(M, \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{A})).$$

Since, the automorphisms considered here are equal to the identity (mod t), this automorphism group is pro-unipotent. Hence, any automorhism can be written as $\exp(\varphi)$ for a nilpotent derivation of $\mathcal{A}$. Moreover, Lemma 2.5 shows that locally, any such derivation is inner. Therefore, we get the following short exact sequence of sheaves of groups (and central extension)

$$0 \longrightarrow \underline{\mathbb{R}}[[t]] \longrightarrow \underline{\mathcal{A}} \longrightarrow \underline{\mathrm{Aut}}\mathcal{A} \longrightarrow 1$$

where $\mathcal{A}$ is equipped with the group structure given by the Campbell-Baker-Hausdorff formula. This induces a boundary morphism in non-abelian cohomology

$$\check{H}^1(M, \underline{\mathrm{Aut}}(\mathcal{A})) \to H^2(M, \mathbb{R}[[t]]).$$

In ordinary (abelian) cohomology it would follow that this is an isomorphism since $\underline{\mathcal{A}}$ is soft being isomorphic to $C^\infty[[t]]$ (only as sheaf in **set**) and therefore acyclic. However, one can check (see [4]) that this is still true in general.

Since Fedosov's construction gives us indeed a canonical class of $*$-products, namely the ones with curvature $\Omega = -\omega$, the dependence here being the choice of symplectic connection, we get the following

Proposition 3.9 *Equivalence classes of $*$-products on a symplectic manifold (M, ω) are in 1-to-1 correspondence with*

$$H^2(M, \mathbb{R}[[t]])$$

A similar picture emerges for $*$-products constructed by Fedosov's method. Namely, the pro-unipotent group

$$\exp(F^1(\frac{1}{t}\mathcal{W})),$$

where F^1 denotes elements with terms of positive degree, acts on Fedosov-connections. In [6] it is moreover shown, that the curvature defines a bijection between the resulting coset space and $-\omega + tH^2(M)[[t]]$.

Using this [4] shows that the two classifications coincide.

Theorem 3.10 *The equivalence class $\theta(\mathcal{A})$ of a Fedosov $*$-product with Fedosov connection D is given by the cohomology class of its curvature via*

$$\theta\mathcal{A} = -\frac{1}{t}[\Omega_D].$$

Furthermore, any $*$-product is equivalent to a Fedosov $*$-product.

3.4 Canonical Trace

On W, the Moyal-Weyl quantization of $\mathbb{R}^{2n}$, one can define the following linear functional

$$Tr(a) = \frac{1}{(2\pi t)^n} \int a \frac{\omega^n}{n!}, \quad \text{for } a \in C_c^\infty(\mathbb{R}^{2n})[[t]]. \tag{3.4}$$

It is easily checked to vanish on cummutators, and is unique upto normalization with this property. This functional is hence calle the trace. Since all *-products are locally equivalent to a Moyal-Weyl product, one can extends this definition to global functions with compact support as long as it is invariant under autoequivalences of the Moyal-Weyl product. This follows from the following

Lemma 3.11 *Let (W, BCH) denote the group induced by the pro-nilpotent Lie algebra $(W, [\cdot,\cdot]_*)$ via the Baker-Campbell-Hausdorff formula and $\mathrm{Aut}^+(W)$ the group of autoequivalences of the $*$-product ($= 1$ mod t). Then the following map is onto*

$$\begin{array}{ccc} (W, BCH) & \longrightarrow & \mathrm{Aut}^+(W) \\ f & \mapsto & \exp([f,\cdot]_*) \end{array}$$

Definition 3.12 *The unique linear functional $Tr : \mathcal{A}_c \to \mathbb{C}((t))$ that vanishes on commutators and is locally given by* (3.4) *is called the* ***canonical trace****.*

With this definition we can now cite the Fedosov-Nest-Tsygan theorem (see [6], [8]).

Theorem 3.13 *Let $(\mathcal{A}, *)$ be a deformation quantization of a compact symplectic manifold (M, ω) with characteristic class θ, then*

$$Tr(1) = \int_M e^{\frac{\theta}{2\pi}}\ \hat{A}(TM, \omega), \tag{3.5}$$

where $\hat{A}$ is the multiplicative genus induced by the power series $\left(\frac{z/2}{\sinh(z/2)}\right)^{\frac{1}{2}}$, that is

$$\hat{A}(TM, \omega) = \det{}^{\frac{1}{2}} \frac{R/2}{\sinh(R/2)}$$

for the curvature R of a symplectic connection on TM.

3.5 Polarized Fedosov Quantization

Following [2] there is the following variation of Fedosov's approach if (M, ω) admits a polarization. In the following let $L \subset T^{\mathbb{C}}M$ be a Langrangian subbundle of the complexified tangent bundle. In particular this induces a reduction of structure group of the symplectic frame bundle to

$$\mathrm{Sp}(2n, n) := \{A \in \mathrm{Sp}(2n) |\ A(\mathbb{C}^n \times \{0\}) \subset \mathbb{C}^n \times \{0\}\}.$$

In this case there is a lift of the natural representation on W different from (3.1). More precisely let p_i, q_i be canonical generator of W, then define

$$\begin{array}{rcl} \rho_0 : sp(2n, n) & \to & \frac{1}{t}W \\ A & \mapsto & \frac{1}{t}\sum_{i,j}(\omega(q_i, Ap_j)p_i * q_j + \frac{1}{2}\omega(q_j, Aq_i)p_i * q_j) \\ & & = \rho(A) + \frac{1}{2}\sum_i \omega(Ap_i, q_i) \end{array}$$

Moreover the right ideal generated by the p_i induces a filtration $F^L_\bullet W$ on W, and the image of ρ_0 lies in $F^L_1 W$. In analogue to the non-polarized case we can define an operator d_L^{-1} different from the d^{-1} that preserves this filtration (the d already does this). This also changes the definition of σ_L which now annihilates everything with positive filtration. Again, we consider the Fedosov algebra **W** which has now an additional filtration that is preserved by the natural operations. The notion of a $\frac{1}{t}W_{\rho_0}$-connection depends on ρ_0 and therefore differs. Since the space of such connections is still an affine space over sections of $\frac{1}{t}\mathcal{W} \otimes \Lambda^1$, we can relate those two notions once a torsionfree symplectic connection is fixed. The resulting covariant derivative coincides in either case. Analogously one can define $\frac{1}{t}F^L_1 W_{\rho_0}$-connections.

Definition 3.14 *A **polarized Fedosov connection** $\tilde{D}$ is a $\frac{1}{t}W_{\rho_0}$-connection that is abelian, is Fedosov (that is its linear term is given by $-\delta$), and such that $\tilde{D} + \delta$ is a $\frac{1}{t}F^L_1 W_{\rho_0}$-connection.*

Such a connection again induces a $*$-product. Let $\mathcal{O}_L$ denote the algebra of functions constant along L.

Proposition 3.15 *The $*$-product induced by a polarized Fedosov connection satisfies the semi-separation of variables condition*

$$f * g = fg, \quad \text{for } f \in \mathcal{O}_L, g \in C^\infty[[t]]$$

and is therefore polarized.

Proof (sketch following [2]) The usual iteration method yields that for $f \in \mathcal{O}_L$

$$\sigma_L^{-1}(f) = f + \hat{f},$$

where $\hat{f} \in F^L_1(\mathcal{W})$. And so for any g

$$\begin{aligned} f * g &= \sigma_L((f + \hat{f})\sigma_L^{-1}(g)) \\ &= \sigma_L(f\sigma_L^{-1}(g)) \\ &= fg. \end{aligned}$$

□

To get such a connection in question, equation 3.2 is used, where the δ^{-1} is replaced by the polarized version, and the R which was to be interpreted as $\rho(R')$ where R' is the curvature 2-form for a torsionfree sysmplecic connection, is now $\rho_0(R')$. A sufficient condition for the resulting r to define a polarized Fedosov connection, that is $r \in F^L_1(\frac{1}{t}\mathcal{W})$, is that $\nu = 0$ in (3.2) and the symplectic connection is polarized. The latter one can be interpreted as a kind of integrability condition. Its existence in particular follows from the following definition, where we use the same terminology as in [2].

Definition 3.16 *A Lagrangian subbundle L is called a **good polarization** if locally there exists functions $a_1, \cdots, a_n$ and $f_1, \cdots, f_n$ such that*

- *The Hamiltonian vector fields X_{a_i} form a local basis in L.*
- *This basis is completed by the Hamiltonian vector fields X_{f_i}.*
- *All the X_{a_i}, X_{f_i} commute pairwise.*

In this local basis the canonical flat connection is a torsionfree $\mathrm{sp}(2n,n)$-connection. One can patch these together to a global connection. Now the above discussion implies

Proposition 3.17 *For a good polarization L there exists a polarized Fedosov connection with curvature $-\omega$.*

To relate the resulting $*$-product to one in the unpolarized construction, we use that for a fixed connection we can assign a $\frac{1}{t}W_\rho$-connection $\bar{D}$ that induces the same covariant derivative and has thusly the same subalgebra of flat sections. Moreover, its curvature differs by $\rho(R') - \rho_0(R')$, whose cohomology class can be shown (see [2] Proposition 2.7) to be $t\frac{1}{t}c_1(L)$, i.e. the first Chern class of L. The last difference is the change of the projection σ_L to σ, but since the underlying algebra is the same, both constructions will yield equivalent $*$-products. In conclusion, we get that the constructed polarized $*$-product has characteristic class

$$\frac{1}{t}\omega + \frac{1}{2}c_1(P).$$

3.6 Characteristic class of a polarized $*$-product

We cite a result from [2] which helps to compute the characteristic class of a polarized $*$-product. Let (M,ω,L) be a symplectic manifold with Lagrangian subbundle. For the result in this section we assume the following integrability condition, which in particular implies that L defines a good polarization.

Definition 3.18 *The complex subbundle L is called integrable if either L is analytic or the following three conditions hold.*

- *L is involutive.*
- *$L \cap \bar{L}$ has constant rank.*
- *$L + \bar{L}$ is involutive.*

Let $(\mathcal{A}, *)$ be a polarized $*$-product. Then define the following objects.

$$\begin{aligned} \mathcal{O} &:= \{f \in C^\infty[[t]] \mid \ df|_L = 0\}, \\ \mathcal{F}(\mathcal{O}) &:= \{f \in \mathcal{A} \mid \ \frac{1}{t}[f,\mathcal{O}]_* \in \mathcal{O}\}, \\ \mathcal{T}_\mathcal{O} &:= \mathrm{Hom}_\mathcal{O}(\Omega^1_{\mathcal{O}/\mathbb{C}}, \mathcal{O}). \end{aligned}$$

Now [2] shows the following.

Proposition 3.19 *The sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{F}(\mathcal{O}) \longrightarrow \mathcal{T}_{\mathcal{O}} \longrightarrow 0$$

is a locally split exact sequence of locally free $\mathcal{O}$-modules and Lie algebras. Locally (in a neighborhood U) there exists functions $a_1, \cdots, a_n \in \mathcal{O} \cap C^\infty$ and pairwise commuting $f_1, \cdots, f_n \in \mathcal{A}$ such that

- *da_i form a basis in $L^\perp$.*
- $\frac{1}{t}[a_i, f_j] = \delta_{ij}$.
- *$1, f_1, \cdots, f_n$ form a local $\mathcal{O}$-module basis of $\mathcal{F}(\mathcal{O})$.*
- *The map*

$$\begin{array}{rcl} \mathcal{F}(\mathcal{O})|_U & \to & \bigoplus_{i=1}^{n} \mathcal{O}|_U \\ g & \mapsto & (\frac{1}{t}[g, a_1], \cdots, \frac{1}{t}[g, a_n]) \end{array}$$

descends to a local isomorphism of $\mathcal{O}$-modules on $\mathcal{T}_{\mathcal{O}}$.

Such a local splitting then forms a Čech-1-cocycle with values in $\Omega^{1,\mathrm{cl}}_{\mathcal{O}}$ and via the boundary homomorphism in cohomology of the short exact sequence

$$0 \longrightarrow \underline{\mathbb{C}}[[t]] \longrightarrow \mathcal{O} \longrightarrow \Omega^{1,\mathrm{cl}}_{\mathcal{O}} \longrightarrow 0$$

it defines a characteristic class

$$cl(\mathcal{A}, \mathcal{O}) \in H^2(M, \mathbb{C}[[t]]).$$

Now we can cite one of the main results (Theorem 4.6) of [2]

Theorem 3.20

$$\theta(\mathcal{A}) = \frac{1}{t} cl(\mathcal{A}, \mathcal{O}) - \frac{1}{2} c_1(L)$$

Chapter 4

Quantization of Coadjoint Orbits

4.1 Classical structure

Let $\mathfrak{g}$ be a Lie algebra and consider its dual $\mathfrak{g}^*$. The dual of the Lie bracket is then a map $\mathfrak{g}^* \to \mathfrak{g}^* \bigwedge \mathfrak{g}^*$, which can be considered as a bivector field Π on $\mathfrak{g}^*$ whose coefficients are linear functions. This indeed induces a Poisson structure called the **Kirillov-Kostant-Souriau** Poisson structure.

Proposition 4.1 *The skew-symmetric pairing $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{g}^*)$ defined by*

$$\{f,g\} = \Pi(df \wedge dg)$$

is indeed a Poisson bracket uniquely defined by the property that on linear functions induced by an element in $\mathfrak{g}$ the Poisson bracket coincides with the Lie bracket on $\mathfrak{g}$.

Proof Let $x,y \in \mathfrak{g}$ induce the linear functions $f_x(\lambda) := \lambda(x)$ for $\lambda \in \mathfrak{g}^*$ and f_y analogously. Their differential one-forms are then given by the constant covector fields given by x and y respectively. Thus

$$\{f_x, f_y\}(\lambda) = \Pi_\lambda(df_x \wedge dg_y) = \lambda([x,y]) = f_{[x,y]}.$$

Now we use that differentials of linear functions generate the cotangent space at every point to conclude unicity and the Jacobi identity since both of those are tensorial on covector fields. □

Since Π is never non-degenerate everywhere (in particular at 0) it does not define a symplectic struture on $\mathfrak{g}^*$. It does however define a symplectic form on an involutive distribution as follows. Let now (M,Π) be any Poisson manifold. Consider Π as a skew-adjoint map $T^*M \to TM$ and denote its image by E, which is not necessarily a vector bundle but still a C^∞-submodule of the tangent sheaf. Over $C^\infty(M)$ it is generated by the image of

$$\begin{array}{rcl} C^\infty(M) & \to & \Gamma(TM) \\ f & \mapsto & X_f := \{f,\cdot\}. \end{array}$$

Since this map is a homomorphism of Lie algebras (where C^∞ is equipped with the Poisson bracket) it follows that E is involutive.

Next we show that E is integrable by symplectic manifolds in the following sense.

Proposition 4.2 *Through any point $x \in M$ there exists a submanifold, whose tangent bundle coincides with E. Furthermore, Π restricts to a non-degenerate Poisson structure, hence a symplectic form.*

Proof We follow roughly [7]. Consider the one-parameter family of diffeomorphisms $exp(tX_f)$ induced by a Hamiltonian vector field X_f, for $f \in C^\infty(M)$. The Jacobi identity translates to $L_{X_f}\Pi = 0$ which shows that $exp(tX_f)$ leaves E invariant, as would any other Poisson diffeomorphism. Let $\dim(E_x) =: r$ and choose r functions $f_1, \cdots, f_r$ such that $X_i := X_{f_i}$ generate E_x. Then define the map

$$(t_1, \cdots, t_r) \mapsto \prod_{i=1}^{i=r} exp(t_i X_i)(x).$$

It follows that this a submanifold whose tangent bundle lies indeed in E. Since the dimension of E does not change under Hamiltonian transformations it is constant along this submanifolds, turning it into an integral submanifold. □

In our case of $M = \mathfrak{g}^*$ we can choose the f_i in the proof to be linear functions. The associated Hamiltonian vector field X_i is then given by

$$\lambda \mapsto (\lambda, \lambda([x_i, \cdot]) \in \mathfrak{g}^* \times \mathfrak{g}^* = T\mathfrak{g}^*.$$

Therefore, the X_i integrate to the coadjoint action of G on $\mathfrak{g}^*$.

Lemma 4.3 *Let G be connected and $\lambda \in \mathfrak{g}^*$. Then the coadjoint orbit through λ coincides with the maximal symplectic leaf.*

Proof Let $\mathcal{O}_\lambda$ denote the maximal symplectic leaf through λ. From the above we see that we get a map $G \to \mathcal{O}_\lambda$. A point x lies in $\mathcal{O}_\lambda$ if it can be connected to λ by a sequence of paths of the form $t \mapsto exp(tX)(x_i)$ which implies that x lies in the same coadjoint orbit. □

It then follows that $\mathcal{O}_\lambda = G/H$, $H = \{g \in G : g.\lambda = \lambda\}$ is endowed with a G-invariant symplectic structure.

Lemma 4.4 *The symplectic structure is given by*

$$\begin{aligned} \omega : \mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} &\to \mathbb{R} \\ (x, y) &\mapsto \lambda([x, y]) \end{aligned}$$

Proof The map $G \to \mathfrak{g}^*, g \mapsto g.\lambda$ has the differential at 1

$$\begin{array}{rcl} \mathfrak{g} & \to & \mathfrak{g}^* \\ x & \mapsto & \lambda([x,\cdot]). \end{array}$$

And thusly

$$\begin{array}{rcl} \omega(x,y) & = & \omega(\underbrace{\lambda([x,\cdot])}_{\{f_x,\cdot\}}, \lambda([y,\cdot]) \\ & = & \{f_x, f_y\}(\lambda) \\ & = & \lambda([x,y]). \end{array}$$

Since the tangent bundle of $\mathcal{O}_\lambda$ is equivariant with fiber $\mathfrak{g}/\mathfrak{h}$ it can be identified with

$$T\mathcal{O}_\lambda \cong G \times_H \mathfrak{g}/\mathfrak{h}.$$

If follows from [9] (7.3) that for any splitting $\iota : \mathfrak{g}/\mathfrak{h} \hookrightarrow \mathfrak{g}$ there exist local vectorfields X_i whose Lie bracket is given by

$$[X_i, X_j](\lambda) = [\iota(X_i), \iota(X_j)] + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}.$$

In particular, any $\mathfrak{n} \subset \mathfrak{g}/\mathfrak{h}^{\mathbb{C}}$ such that $\mathfrak{n} + \mathfrak{h}^{\mathbb{C}}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ induces an involutive subbundle. Assume that we have such an $\mathfrak{n}$ which is furthermore a Lagrangian subspace of $(\mathfrak{g}/\mathfrak{h}, \lambda([\cdot,\cdot]))$. The resulting involutive subbundle is then integrable by analyticity and thus defines a polarization. To construct such an $\mathfrak{n}$ we need some additional assumption.

For the rest of this section assume that $\mathfrak{g}$ is semi-simple and that λ corresponds to a semi-simple element $X_\lambda \in \mathfrak{g}$ via the Killing form. Thus, we can identify $\mathfrak{h}$ with the centralizer of X_λ, that is the kernel of $\mathrm{ad}(X_\lambda)$. Since $\mathrm{ad}(X_\lambda)$ is H-invariant and semi-simple, its image constitutes an H-invariant complement to $\mathfrak{h}$, showing that $\mathcal{O}_\lambda$ is reductive. Since X_λ is semi-simple it is contained in a Cartan subalgebra. We then get the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ inducing one on $\mathfrak{g}/\mathfrak{h}^{\mathbb{C}}$. Let Δ denote the set of roots α such that $\alpha(X_\lambda) \neq 0$.

$$\mathfrak{g}/\mathfrak{h}^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta} \mathbb{C} e_\alpha.$$

To construct a polarization, we notice that the eigenvalues of $\mathrm{ad}(X_\lambda)$ occur in pairs of opposite signs. Therefore there exists an ordering of all the roots such that

$$\Delta = \Delta_- \cup \Delta_+,$$

$$\{\alpha(X_\lambda) | \, \alpha \in \Delta_-\} \cap \{\alpha(X_\lambda) | \, \alpha \in \Delta_+\} = \emptyset.$$

In particular, $\mathfrak{n} := \bigoplus_{\alpha \in \Delta_-} \mathbb{C} e_\alpha$ is H-invariant and induces a polarization.

Since we eventually want to compute the characteristic class of a $*$-product on $\mathcal{O}_\lambda$, we try to find a description of the cohomology, in particular $H^2(\mathcal{O}_\lambda)$, in terms of $\mathfrak{g}$. Since G and hence $\mathfrak{g}$ acts on $\mathcal{O}_\lambda$ we can consider the subcompex of $\mathfrak{g}$ invariant differential forms. This complex is moreover isomorphic to the relative Lie algebra Chevalley-Eilenberg complex. Namely consider

$$C^k(\mathfrak{g},\mathfrak{h},\mathbb{R}) := \mathrm{Hom}_{\mathfrak{h}}(\Lambda^k\mathfrak{g}/\mathfrak{h},\mathbb{R})$$

with its usual Chevalley-Eilenberg differential and denote the corresponding cohomology by $H^\bullet(\mathfrak{g},\mathfrak{h})$, then we have a map

$$H^k(\mathfrak{g},\mathfrak{h}) \to H^k(\mathcal{O}_\lambda,\mathbb{R}). \tag{4.1}$$

If G is compact, this can be shown to be an isomorphism, by averaging.

Let us now consider the map

$$\begin{array}{rcl} (\mathfrak{g}^*)^{\mathfrak{h}} & \to & C^2(\mathfrak{g},\mathfrak{h},\mathbb{R}) \\ \alpha & \mapsto & -\alpha([\cdot,\cdot]). \end{array}$$

Either by direct computation or by considering the five-term exact sequence of a Hochschild-Serre type spectral sequence for relative cohomology it follows that

$$\frac{(\mathfrak{g}^*)^{\mathfrak{h}}}{((\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}})} \to H^2(\mathfrak{g},\mathfrak{h})$$

is an isomorphism. As was shown above, there exists an H-invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$ and hence an H-invariant (in fact unique since $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g}/\mathfrak{h},\mathfrak{h}) = 0$) projection

$$p : \mathfrak{g} \to \mathfrak{h},$$

and hence we can identify

$$\begin{array}{rcl} \dfrac{\mathfrak{h}}{[\mathfrak{h},\mathfrak{h}]}^{\,*} & \xrightarrow{\cong} & H^2(\mathfrak{g},\mathfrak{h}) \\ \alpha & \mapsto & -\alpha(p([\cdot,\cdot])). \end{array} \tag{4.2}$$

There is another description of this homomorphism. Consider G as a principal H-bundle over $\mathcal{O}_\lambda$. Then to any complex H-representation V, one can assign the associated complex vector bundle $G\times_H V$ over $\mathcal{O}_\lambda$. For a line bundle, the representation is given by $\theta + i\alpha$ for θ,α real characters on $\mathfrak{h}$. Now the first Chern class of the associated bundle only depends on α. Thus we get a mapping from analytically integral characters on h to $H^2(\mathcal{O}_\lambda)$.

Proposition 4.5 *The assignment*

$$\alpha \mapsto c_1(G\times_H \mathbb{C}_{e^{2\pi(\theta+i\alpha)}}) \in H^2(\mathcal{O}_\lambda)$$

for an analytically integral character α, coincides with the composition of (4.1) *and* (4.2).

Proof By considering left-invariant vector fields on G, p represents a connection 1-form. Since p is H-invariant this defines a principal connection. The usual formulas for the curvature Ω show that for X, Y left invariant vectorfields on G

$$\begin{aligned}\Omega(X,Y) &= dp(X,Y) + [p(X), p(Y)] \\ &= -p([X,Y]) + [p(X), p(Y)],\end{aligned}$$

which is a closed G-invariant 2-form on $\mathcal{O}_\lambda$, that is an element of

$$(\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{h})^{\mathfrak{h}}.$$

To get the curvature for an associated bundle we compose with the character. We see that the first Chern class is represented by the invariant 2-form induced by $\alpha - i\theta$. The first Chern class being real valued implies that the imaginary part is null-cohomologous in $H^2(\mathcal{O}_\lambda)$ (even though it is not in $H^2(\mathfrak{g}, \mathfrak{h})$). □

4.2 $*$-product

Let $\mathfrak{g}$ be semi-simple and $\lambda \in \mathfrak{g}^*$ a semi-simple element. Following [1] we construct a $*$-product on $\mathcal{O}_\lambda$. Let $\mathfrak{h} = \{x \in \mathfrak{g}^{\mathbb{C}} | \, x.\lambda = 0\}$ be the subalgebra of the complexified $\mathfrak{g}$ fixing λ. In case λ is regular, that is $\dim \mathrm{Fix}_\lambda$ is minimal among all $\lambda \in \mathfrak{g}^*$, it follows that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Otherwise, there exists a Cartan subalgebra in $\mathfrak{h}$. Choose an ordering of the roots as above. In any case we get a root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = n_- \oplus \mathfrak{h} \oplus n_+ = \bigoplus_{\alpha \in \Delta^+} \mathbb{C}e_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathbb{C}e_\alpha$$

where Δ^+ is a subset of the positive roots (or all positive roots in case λ regular). Furthermore, we can normalize the root vector e_α such that

$$\omega(e_{-\alpha}, e_\beta) = \lambda([e_{-\alpha}, e_\beta]) = \delta_{\alpha,\beta}$$

Set $\mathfrak{p}_\pm = \mathfrak{h} \oplus n_\pm$ and define the generalized Verma modules

$$M^\pm = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\pm)} \mathbb{C}_{\pm\lambda}$$

Now, $M^+ \otimes_{U\mathfrak{g}} M^-$ can be uniquely identified with $\mathbb{C}$, therefore there exists a unique $\mathfrak{g}$-invariant bilinear pairing $(\cdot, \cdot)$ on $M^+ \times M^-$ called the **Shapovalov pairing**. More concretely, using the PBW decomposition $U\mathfrak{g} \cong Un_- \otimes U\mathfrak{h} \otimes Un+$ define the projection $\varphi : U\mathfrak{g} \to U\mathfrak{h}$. Thus we have for any $x \in U\mathfrak{g}$

$$x = \underbrace{\varphi(x)}_{\in U\mathfrak{h}} + \underbrace{x_-}_{\in n_- U(\mathfrak{g})} + \underbrace{x_+}_{\in U(\mathfrak{g})n_+}$$

Let v_λ, $v_{-\lambda}$ denote the generators of M^+ and M^-, respectively, and let $x \in Un_-$, $y \in Un_+$,

$$\begin{aligned} xv_\lambda \otimes yv_{-\lambda} &= S(y)xv_\lambda \otimes v_{-\lambda} \\ &= (\varphi(S(y)x) + (S(y)x)_- + (S(y)x)_+)v_\lambda \otimes v_{-\lambda} \\ &= \varphi(S(y)x)v_\lambda \otimes v_{-\lambda} + v_\lambda \otimes S((S(y)x)_+)v_{-\lambda} \\ &= \lambda(\varphi(S(y)x))v_\lambda \otimes v_{-\lambda} \end{aligned}$$

where we S denotes the antipode on $U\mathfrak{g}$ which appears since we are pairing two left-modules. Now it follows that

$$(xv_\lambda, yv_{-\lambda}) = \lambda(\varphi(S(y)x)).$$

By identifying $Un_- \cong M^+$ and $Un_+ \cong M^-$ we get a family of pairings $(\cdot,\cdot)_\lambda \to \mathbb{C}$ parametrized by characters on $\mathfrak{h}$. As is shown in [1] Proposition 3.1. , this pairing is non-degenerate for almost all λ even restricted to any line $\mathbb{C}\lambda$. Moreover, it respects the grading in the root lattice, implying the the inverse pairing is given by an element $F_\lambda \in Un_- \check{\otimes} Un_+$ in the completed tensor product. Furthermore [1] show that F_λ is holomorphic at infinity enabling us to define

$$B := F_{\frac{1}{t}\lambda} \in Un_- \check{\otimes} Un_+[[t]]$$

This B now defines a $*$-product on $\mathcal{O}_\lambda$ as follows. Let $\pi : G \to \mathcal{O}_\lambda$ denote the cannonical projection and let $\mathfrak{g}$ act on functions on G by right-invariant vector fields, that is for $f \in C^\infty(G)$ and $X \in \mathfrak{g}$

$$Xf(g) = \frac{d}{dt}|_{t=0} f(ge^{tX}).$$

Now for $f, g \in C^\infty(\mathcal{O}_\lambda)$ we define

$$f * g = mB(f \otimes g) \in C^\infty(\mathcal{O}_\lambda)[[t]],$$

where m denotes the multiplication of functions.

Now we are ready to cite

Theorem 4.6 (Alekseev-Lachowska) *This indeed defines a $*$-product on $\mathcal{O}_\lambda$. In particular,*

- *$f * g$ is H invariant and thus a function on $\mathcal{O}_\lambda$.*
- *B satisfies the associativity condition*
- *$B = 1 + t\sum_{\alpha\in\Delta^+} e_{-\alpha} \otimes e_\alpha + O(t^2)$ implying that $[\cdot,\cdot]_* = t\{\cdot,\cdot\} + O(t^2)$.*

This $*$-product has the following additional properties

Proposition 4.7 **i)** *It is polarized with the commutative subalgebra*

$$\mathcal{O} = \{f \in \mathbb{C}^\infty[[t]] | \, e_\alpha f = 0 \ \ \forall \alpha \in \Delta^+\}$$

ii) *It posesses a strong quantum momentum map.*

The first part is obvious since $B \in Un_- \check{\otimes} Un_+[[t]]$. And for the second part we need the following

Lemma 4.8 ([1] Propostion 4.1) *Let $\mathcal{V}$ be any $U\mathfrak{g}$-module and $w \in \mathcal{V}$. Let furthermore λ be such that the Shapovalov pairing is non-degenerate. Assume that there exists an element $z \in V\check{\otimes}M^-$ that is Un_--invariant and of the form $z \in w \otimes v_{-\lambda} + \mathcal{V}\check{\otimes}(Un_+)_{>0}$. Then such a z is unique.*

Proof Let x_s be a homogeneous basis of $(Un_+)_{>0}$. Since $(\cdot, \cdot)_\lambda$ is assumed to be non-degenerate and homogenous, we can find a basis $y_l \in Un_-$ such that $(S(y_l), x_s) = \delta_{ls}$. Now let z be of the form

$$z = w \otimes v_{-\lambda} + \sum_s w_s \otimes x_s v_{-\lambda}.$$

By Un_--invariance we have for $y_l \in Un_-$

$$0 = y_l w \otimes v_{-\lambda} + \sum_s y_l w_s \otimes x_s v_{-\lambda} + \sum_s w_s \otimes y_l x_s v_{-\lambda}.$$

□

Write $y_l x_s$ as $y_l x_s \in S(\varphi(S(y_l x_s))) + U\mathfrak{g}n_- + n_+ U\mathfrak{g}$ to get

$$\begin{aligned} -y_l w &= \sum_s (-\lambda)(S(\varphi(S(y_l x_s))))w_s \\ &= \sum_s \lambda(\varphi(S(x_s)S(y_l)))w_s \\ &= \sum_s (S(y_l), x_s)_\lambda w_s \\ &= w_s. \end{aligned}$$

Let now $f_H : \mathcal{O}_\lambda \to \mathbb{C}, x \mapsto x(H)$, for an $H \in \mathfrak{g}$ be a classical Hamiltonian function. Pull it back to a function on G and consider the action of $X \in \mathfrak{g}$,

$$\begin{aligned} X f_H(g) &= \frac{d}{dt}|_{t=0}((g e^{tX}).\lambda)(H) \\ &= \lambda(-[X, g^{-1}H]) \\ &= (X.\lambda)(g^{-1}H). \end{aligned}$$

From this is follows that the $\mathfrak{g}$-module generated by f_H is isomorphic to the $\mathfrak{g}$-submodule of $\mathfrak{g}^*$ generated by λ. Under this isomorhpism we can view the differential operator $g \mapsto f_H * g$ as an element in $g^* \otimes Un_+[[t]]$ which is given by the Taylor expansion in t of

$$F_{\frac{\lambda}{t}}(\lambda \otimes v_{-\lambda}) \in \mathfrak{g}^* \otimes M_- \cong_{Un_+} \mathfrak{g}^* \otimes Un_+.$$

Since this is the image of the $\mathfrak{g}$-invariant $F_{\frac{\lambda}{t}}$ under the Un_--homomorphism

$$M^+ \check{\otimes} M^- \cong Un_- \check{\otimes} M^- \to \mathfrak{g}^* \otimes M^-$$

it is clearly Un_--invariant.

Lemma 4.9

$$F_{\frac{\lambda}{t}}(\lambda \otimes v_{\frac{-\lambda}{t}}) = \lambda \otimes v_{\frac{-\lambda}{t}} + t \sum_\alpha e_{-\alpha}\lambda \otimes e_\alpha v_{\frac{-\lambda}{t}}$$

Proof By Lemma 4.8 it is enough to show that the righthand side is n_- invariant. Let Φ denote the righthand side and compute $e_{-\beta}\Phi$.

$$\begin{aligned} e_{-\beta}\Phi &= e_{-\beta}\lambda \otimes v_{\frac{-\lambda}{t}} + t\sum_\alpha e_{-\beta}e_{-\alpha}\lambda \otimes e_\alpha v_{\frac{-\lambda}{t}} + t\sum_\alpha e_{-\alpha}\lambda \otimes e_{-\beta}e_\alpha v_{\frac{-\lambda}{t}} \\ &= (e_{-\beta}\lambda + t \sum_{\alpha \text{ s.t. } [e_{-\beta},e_\alpha]\in\mathfrak{h}} (\frac{-\lambda}{t})([e_{-\beta}, e_\alpha])e_{-\alpha}\lambda) \otimes v_{\frac{-\lambda}{t}} \\ &\quad + t\sum_\alpha (e_{-\beta}e_{-\alpha}\lambda + \sum_{\gamma \text{ s.t. } [e_{-\beta},e_\gamma]=c^\alpha_{\beta\gamma}e_\alpha} c^\alpha_{\beta\gamma}e_{-\alpha}\lambda) \otimes e_\gamma v_{\frac{-\lambda}{t}}. \end{aligned}$$

Now the first term vanishes by normalization of the root vectors. And using that $c^\alpha_{\beta\gamma} = \lambda([e_{-\alpha}, [e_{-\beta}, e_{\alpha+\beta}]])$ the coefficients in the second term reduce to

$$e_{-\beta}e_{-\alpha}\lambda + \lambda([e_{-\alpha}, [e_{-\beta}, e_{\alpha+\beta}]])e_{-(\alpha+\beta)}\lambda = 0$$

□

Proof (of Proposition 4.7) By Lemma 4.8 and the analogue for f_H in the second argument we get

$$f_H * g - g * f_H = t\sum_\alpha (e_{-\alpha} \wedge e_\alpha)(f_H \otimes g)$$

proving the condition for a strong quantum momentum map. □

This $*$-product satisfies something stronger than just having a polarization, namely it admits **separation of variables**.

Definition 4.10 *Let (M, ω) admit two transverse integrable Lagrangian P, Q subbundles of $T^{\mathbb{C}}M$. More precisely, assume that the map*

$$\mathcal{O} \otimes \tilde{\mathcal{O}} \to C^\infty(M)$$

$$\begin{aligned} \mathcal{O}(U) &:= \{f \in C^\infty_U \,|\, df|_P = 0\} \\ \tilde{\mathcal{O}}(U) &:= \{f \in C^\infty_U \,|\, df|_Q = 0\} \end{aligned}$$

is onto on infinity jets at every point. A $*$-product is the said to have separation of variables if

$$\begin{aligned} f * g = fg, &\text{ for } f \in \mathcal{O} \text{ and } g \in C^\infty \\ f * g = fg, &\text{ for } f \in C^\infty \text{ and } g \in \tilde{\mathcal{O}} \end{aligned}$$

The $*$-product constructed above obviously satisfies this condition. Moreover, we can deduce the following uniqueness statement

Proposition 4.11 *There is a unique G-invariant $*$-product on $\mathcal{O}_\lambda$ with separation of variables for given G-invariant polarization that furthermore has a strong quantum momentum map.*

Proof First of all, since the polarization is G-invariant, it is also $\mathfrak{h}$-invariant, so is splits into one-dimensional root spaces. Thus, after a suitable choice of positivity, the above construction applies. For uniqueness, since

$$\mathrm{Diff}^2_G(G/H) = ((U\mathfrak{g}/U\mathfrak{g}\mathfrak{h})^{\otimes 2})^H$$

and $*$ admits separation of variables, we can write

$$f * g = B(f \otimes g) \text{ for } B \in Un_- \check{\otimes} Un_+[[t]].$$

From this form it follows that the $*$-commutator determines the product itself. In particular, left multiplication by a Hamiltonian momentum function is determined and concretely given by Lemma 4.9. Now uniqueness follows from the fact that Hamiltonian momentum functions together with 1 generate a subalgebra in $(C^\infty[[t]], *)$ that is dense in a formal neighborhood, that is for any $f \in C^\infty[[t]]$, $x \in M$ and $k \in \mathbb{N}$, there exists an element $g \in U_\hbar\mathfrak{g}$ such that

$$f(x,t) - \mu(g)(x,t) = t^k h(x,t) + h'(x,t),$$

where h' vanishes at x up to order k. □

4.3 Characteristic class

In this section we compute the characteristic class of the unique G-invariant $*$-product with separation of variable and strong quantum momentum map using theorem 3.20.

Since the $*$-commutator with a Hamiltonian function is given by the Lie derivative, and the polarization is assumed to be G-invariant, it follows that the quantum momentum map has image in $\mathcal{F}(\mathcal{O})$.

Proposition 4.12 *The map of $\mathcal{O}$-modules*

$$\begin{array}{rcl} \mathcal{O} \otimes \mathfrak{g} & \to & \mathcal{F}(\mathcal{O}) \\ f \otimes X & \mapsto & f * \mu(X) = f\mu(X), \end{array}$$

where μ is the classical (and quantum) momentum map, is onto. Furthermore, there is a local commuting basis of $\mathcal{F}(\mathcal{O})$ in C^∞ (independent of t). Moreover, the Lie bracket on $\mathcal{F}(\mathcal{O})$ coincides with the Poisson bracket.

Proof We apply proposition 3.19. That is, we consider the map

$$\begin{array}{rcl} \mathfrak{g} & \to & \bigoplus_{i=1}^{n} \mathcal{O}|_U \\ X & \mapsto & (\frac{1}{t}[\nu(X), a_1], \cdots, \frac{1}{t}[\nu(X), a_n]). \end{array}$$

By the strong quantum momentum map property, we have

$$\frac{1}{t}[\nu(X), a_1] = L_X a_i = da_i(X).$$

Now, after tensoring with $C^\infty[[t]]$ which sends a vectorfield spanned by some Hamiltonians to its evaluation on da_i. Since the da_i are independent and the Hamiltonians span the entire tangent space, this map is onto. Now we have a map of free finitely generated $\mathcal{O}|_U$ modules that is onto after tensoring with $C^\infty[[t]]|_U$. It follows then that the original map is onto as well (by using that a map of free modules is onto iff the appropriate minor is a unit, and $\mathcal{O}^\times = \mathcal{O} \cap C^\infty[[t]]^\times$). To get the statement about the Lie bracket we compute for $f, g \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$

$$\begin{array}{rcl} [f\mu(X), g\mu(Y)] & = & [f * \mu(X), g * \mu(Y)] \\ & = & [f, g] * \mu(X) * \mu(Y) + f * [\mu(X), g] * \mu(Y) + \\ & & g * [f, \mu(Y)] * \mu(X) + g * f * [\mu(X), \mu(Y)] \\ & = & tf\{\mu(X), g\}\mu(Y) + gt\{f, \mu(Y)\}\mu(X) + gft\{\mu(X), \mu(Y)\} \\ & = & t\{f\mu(X), g\mu(Y)\}. \end{array}$$

Finally, by separation of variables, the $\mathcal{O}$-module structure on $\mathcal{F}(\mathcal{O})$ is trivial, and since the a_i are independent of t, we can just take the $t = 0$ part of the f_i in proposition 3.19. □

Proposition 4.13 *$cl(\mathcal{A}, \mathcal{O})$ does not depend on t.*

Proof The last proposition show that there exists a local splitting independent of t. This automatically induces a characteristic class independent of t. □

Proposition 4.14

$$\theta(\mathcal{A}) = \frac{1}{t}[\omega] - \frac{1}{2}c_1(L)$$

Proof Follows with theorem 3.20 and the fact that the lowest order term is given by the symplectic form. □

Let us view this under the isomorphism (4.2) in the case of a semi-simple coadjoint orbit. For this, we use the same decomposition of $\mathfrak{g}^{\mathbb{C}}$ as in the last section. In particular,

$$\mathfrak{g}/\mathfrak{h}^{\mathbb{C}} = n_- \oplus n_+$$

and thus

$$L = G \times_H n_-.$$

As above we construct a G-invariant connection with curvature 2-form Ω whose trace is given by

$$c_1(L) = [tr(\Omega)] = \sum_{\alpha\in\Delta^+} \alpha([\cdot,\cdot]).$$

And thus under the homomorphism (4.2) the characteristic class corresponds to

$$\theta(\mathcal{A}) = -\frac{1}{t}\lambda + \rho.$$

4.4 Tr(1)

The only thing missing to compute $Tr(1)$ using the index formula (3.5) is the $\hat{A}$ genus. Note that all the roots $\alpha \in \mathfrak{h}^*$ square to zero under (4.2), and thus

$$\begin{aligned}
\hat{A}(TM,\omega) &= \prod_{\alpha\in\Delta}\left(\frac{\alpha/2}{\sinh(\alpha/2)}\right)^{\frac{1}{2}} \\
&= \prod_{\alpha\in\Delta^+}\frac{\alpha/2}{\sinh(\alpha/2)} \\
&= 1,
\end{aligned}$$

using that $\frac{z/2}{\sinh(z/2)}$ has no linear term.

Assuming that $\mathcal{O}_\lambda$ is compact we get the following

Proposition 4.15

$$Tr(1) = \frac{1}{(2\pi)^n}\frac{1}{n!}\int_{\mathcal{O}_\lambda}\left(\frac{1}{t}\omega + \tilde{\rho}\right)^n,$$

where $\tilde{\rho}$ is the image of ρ under (4.2).

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