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Quantification par déformation des algébroïdes de Lie, application de la formalité à deux branes.

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CONTENTS

ABSTRACT 7

INTRODUCTION EN FRANÇAIS 9

Contexte 9

Contenu 12

Notations 15

INTRODUCTION 17

Context 17

Content 20

Notations 22

I SOMETHING TO RELY ON 25

I.1 Graded algebraic world 25

I.1.1 Introduction to graded algebra 26

I.1.2 Introduction to higher algebra 35

I.2 Quantization of coisotropic branes 38

I.2.1 The $\mathscr{A}_\infty$-category setting 38

I.2.2 Coisotropic branes into $\mathscr{A}_\infty$-category 43

I.2.3 The $\mathscr{L}_\infty$-quasi-isomorphism 48

II QUANTIZATION OF LIE ALGEBROIDS 53

II.1 From Lie-Rinehart pairs to local Lie algebroids 53

II.1.1 Origins and uses 53

II.1.2 Definitions and links 54

II.2 Local Lie algebroids as coisotropic branes 55

II.2.1 Branes setting 55

II.2.2 The explicit Maurer-Cartan element 55

II.3 Deformation algebras 63

II.3.1 Formal deformation of Lie-Rinehart pairs 63

II.3.2 The Chevalley-Eilenberg algebra 65

II.3.3 The Universal Enveloping algebra 71

II.4 Quantization theorems 86

II.4.1 The dual of a Lie algebra 86

II.4.2 The dual of a local Lie algebroid 87

II.4.3 Toward $\mathscr{L}_\infty$-algebras 88

BIBLIOGRAPHY 93

ABSTRACT

FRANÇAIS

Dans cette thèse, nous montrons un théorème de quantification de l'algèbre des fonctions polynomiales sur le dual d'un algébroïde de Lie local, en appliquant les résultats de la quantification par déformation d'une paire de branes coisotropes obtenus par D. Calaque, G. Felder, A. Ferrario, et C. Rossi. dans [Cal+11]. Dans ce contexte les algèbres déformées obtenues sont l'algèbre différentielle graduée de Chevalley-Eilenberg et l'algèbre associative enveloppante universelle de la déformation formelle triviale de cet algébroïde de Lie local. Ceci généralise un important théorème de quantification du dual d'une algèbre de Lie, obtenu par M. Kontsevich dans [Kon03], au cas des algébroïdes de Lie locaux.

ENGLISH

We prove a quantization theorem of the algebra of polynomial functions on the dual of a local Lie algebroid, by applying the results of the deformation quantization of two coisotropic branes obtained by D. Calaque, G. Felder, A. Ferrario, and C. Rossi. in [Cal+11]. In this context the deformed algebras are the differential graded Chevalley-Eilenberg algebra and the graded Universal Enveloping associative algebra of the trivial formal deformation of this local Lie algebroid. This generalize a famous theorem of quantization of the dual of a Lie algebra obtained by M. Kontsevich in [Kon03] to the local Lie algebroids case.

Introduction en Français

Contexte

Mécanique classique et mécanique quantique

Ce travail dérive de l'étude de l'apparente incompatibilité de deux théories physiques, la mécanique classique et la mécanique quantique.

La mécanique classique est un formalisme mathématique ayant pour objectif de décrire le mouvement d'objets macroscopiques de notre univers. Sa formulation Hamiltonienne consiste en une conceptualisation de l'état physique de l'objet comme un point dans un espace de phase, cet espace étant défini comme une variété de Poisson ou une variété symplectique M avec $\{\bullet,\bullet\}$ pour crochet de Poisson. Une observable est le résultat possible d'une mesure physique du système, celle-ci est représentée, au sein du modèle, par une fonction lisse sur la variété différentielle M. Les lois physiques du système se traduisent au travers d'une fonction spécifique H appelée Hamiltonien du système ou fonction d'énergie, et les prédictions quant à l'évolution du système physique s'obtiennent au travers de l'évolution temporelle d'une observable f. Cette évolution est gouvernée par les équations d'Hamilton:

$$\frac{d}{dt}f = \{f,H\}.$$

À titre d'exemple, si l'on considère un système dynamique à N-degrés de liberté, décrit par une variété lisse X de dimension N, alors l'espace d'état du système est le fibré cotangent de cette variété, $M := T^*X$. Un état du système à un certain temps t est décrit par une position x dans X ayant pour coordonnées locales $(x_i)_{i\leq N}$, et par un moment p dans T^*_xX ayant pour coordonnées locales $(p_i)_{i\leq N}$. Le fibré cotangent T^*X est muni d'une forme symplectique canonique, la 2-forme de Poincaré $\sum_i dx_i \wedge dp_i$, tandis que le Hamiltonien du système est obtenu par transformation de Legendre du Lagrangien classique, on retrouve ainsi les équations canoniques d'Hamilton du mouvement :

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

Ainsi, la loi de conservation de l'énergie s'exprime par la relation $\{H,H\} = 0$.

La mécanique quantique est quant à elle un formalisme mathématique visant à décrire les mouvements d'objets microscopiques de l'univers. Dans sa formulation d'Heisenberg, la variété symplectique de l'espace de phase est remplacée par un espace projectif de Hilbert de dimension infinie, l'espace des états quantiques $\mathcal{H}$. Les observables quantiques et le Hamiltonien H sont alors décrits par des opérateurs autoadjoints sur $\mathcal{H}$. Cependant, comme la composition de tels opérateurs n'est, en toute généralité, pas commutative, il est alors possible de munir $\mathcal{H}$ d'un commutateur non-nul $[\bullet,\bullet]$. L'évolution temporelle du système est alors décrite par les équations d'Heisenberg:

$$\frac{d}{dt}A = \frac{i}{\hbar}[H,A].$$

Si la formulation d'Heisenberg de la mécanique quantique paraît autant similaire à la formulation Hamiltonienne de la mécanique classique c'est parce qu'elles entretiennent le même rôle dans deux contextes différents, de plus comme la formulation d'Heisenberg est équivalente à celle de Schrödinger, qui est plus couramment utilisée en physique, il paraît important de s'intéresser aux liens mathématiques que ces deux théories de la mécanique semblent entretenir. Cependant, la mécanique quantique est également gouvernée par un ensemble de principes externes à cette formulation comme le principe d'incertitude d'Heisenberg, ces principes imposent alors des conditions sur les observables ayant un sens physique. À cause de ces conditions, il se peut que la correspondance entre observables classiques et quantiques ne soit pas toujours possible ou naturelle.

La quantification est alors entendue comme un processus permettant de réaliser, au moins partiellement, cette correspondance en associant à une fonction lisse f un opérateur autoadjoint $Q(f)$ de manière à ce que soit vérifié $Q(1) = Id$ ainsi que l'équation:

$$[Q(f),Q(g)] = i\hbar Q(\{f,g\}).$$

Ce problème amena à de nombreux développements tels que la quantification de Wigner-Weyl [Wey27], la quantification géométrique de Kostant et Souriau [Sou67], la quantification de Berezin [Ber75], la quantification par déformation de Flato, Lichnerowicz et Sternheimer [Bay+77; FLS76], ainsi qu'une abondante diversité de résultats de mathématique pure.

QUANTIFICATION EN MATHÉMATIQUES

Le processus de quantification en mathématiques est à présent un sujet largement étudié à l'aide d'approches et d'interprétations diverses et variées, ici nous allons brièvement revoir l'historique de celui qui nous a mené à ce développement. Une introduction précise sur la quantification des algèbres de Poisson pourra être trouvée dans [ES98], le cadre de travail pour cela est celui des modules libres au dessus de l'algèbre des séries formelles en $\hbar$, notée $\mathbb{K}[\![\hbar]\!]$, pour un corps $\mathbb{K}$ de caractéristique nulle.

Étant donnée une $\mathbb{K}$-algèbre associative et commutative $(A_0, .)$, une déformation de A_0 est une $\mathbb{K}[\![\hbar]\!]$-algèbre associative $(A, *)$ dont le module gradué sous-jacent est un $\mathbb{K}[\![\hbar]\!]$-module libre, et telle que $A_0 = A/\hbar A$ comme $\mathbb{K}$-algèbre. Comme le commutateur de l'algèbre A pourrait être non-nul, A_0 hérite d'une structure d'algèbre de Poisson. Celle-ci est donnée pour une paire d'éléments f_0, g_0 dans A_0, par le commutateur du produit de A évalué sur deux relevés f, g dans A:

$$\{f_0, g_0\} := \frac{1}{\hbar}(f * g - g * f) \ \ mod \ \ \hbar A.$$

L'algèbre de Poisson $(A_0, \{\bullet, \bullet\})$ est alors appelée la limite quasi-classique de A, tandis que A est une quantification (par déformation) de $(A_0, \{\bullet, \bullet\})$. L'algèbre A joue alors le rôle de l'algèbre non commutative des observables quantiques, tandis que l'algèbre de Poisson A_0 joue le rôle de celle des observables classiques. Cependant, bien que la limite quasi-classique soit un objet mathématique unique, la quantification de l'algèbre A_0 n'existe pas en toute généralité, et lorsqu'elle existe, elle n'a pas de raison d'être unique.

En 1997, M. Kontsevitch prouva dans son article [Kon03], que dans le cas où $\mathbb{K} = \mathbb{R}$ et où A_0 est la $\mathbb{R}$-algèbre des fonctions lisses définies sur une variété différentiable réelle M, tout crochet de Poisson $\{\bullet, \bullet\}$ défini sur A_0 induit l'existence d'une quantification de $(A_0, \{\bullet, \bullet\})$, une présentation éclairante du sujet ayant été faite par Bernhard Keller dans [Cat+05]. De plus la $\mathbb{K}[\![\hbar]\!]$-algèbre associative A est ici l'algèbre des séries formelles à coefficients dans A_0, et le processus de quantification s'inspire des diagrammes de Feynmann dans la théorie perturbative des champs topologiques et fut d'ailleurs reliée plus tard au modèle sigma de Poisson dans cette même théorie physique. La preuve de Kontsevitch consiste à démontrer la quantification dans le cas des ouverts de $\mathbb{R}^d$ pour une dimension finie d, puis d'étendre le résultat à l'ensemble de la variété en utilisant des arguments de géométrie formelle et de connexion plate. Le résultat final est souvent exprimé au moyen d'un théorème de formalité pour le complexe décalé des cochaines de Hochschild sur A_0.

En 2005, A. Cattaneo et G. Felder étendirent, dans leur article [CF07], la construction de Kontsevich en considérant une sous-variété $C \subset M$, appelée brane (définissant les conditions de bord d'une théorie des champs quantiques), et en quantifiant l'algèbre de Poisson des sections des puissances extérieures du fibré normal sur C. Le résultat, se réduisant à celui de Kontsevitch pour $C = M$, est une quantification de la structure de Poisson sur M comme structure d'$\mathscr{A}_\infty$-algèbre sur A, et un théorème de formalité relative pour le complexe des cochaines de Hochschild des sections de l'algèbre extérieure sur le fibré normal de C.

En 2010, D. Calaque, G. Felder, A. Ferrario et C. Rossi prolongèrent, dans leur article [Cal+11], le résultat précédent au cas d'une paire de branes linéaires sur un espace vectoriel, X. Le résultat souligna l'importance des $\mathscr{A}_\infty$-structures dans la quantification des structures de Poisson en attribuant une structure d'$\mathscr{A}_\infty$-algèbre à chaque brane déformée, ainsi que celle d'un $\mathscr{A}_\infty$-bimodule sur l'intersection des branes. De plus, en considérant une paire de branes duales de Koszul l'une de l'autre, ils prouvèrent également que la quantification par déformation des bivecteurs quadratiques de Poisson préserve la dualité de Koszul.

CONTENU

ALGÉBROÏDES DE LIE COMME BRANES COÏSOTROPES

Les groupoïdes de Lie ont été grandement utilisés en physique car leur formalisme offre une description unifiée de l'ensemble des actions des groupes de symétrie d'un système physique, remplaçant ainsi les groupes de Lie traditionnels. D'une façon similaire aux algèbres de Lie, les algébroïdes de Lie permettent une description unifiée des actions des groupes de symétries ainsi que des espaces tangents de ces groupes, permettant ainsi une meilleure compréhension des liens entre mécanique classique et mécanique quantique. Ainsi, la question de la quantification des algébroïdes de Lie apparaît comme naturelle.

S'inspirant des travaux de D. Calaque, G. Felder, A. Ferrario et C. Rossi dans [Cal+11], nous présentons une construction d'une paire de branes coisotropes, et nous montrons qu'une structure d'algébroïde de Lie local définit un élément de Maurer-Cartan dans l'algèbre décalée des polyvecteurs de cette configuration.

Théorème II.2.2-1 :

Soient $(M, N, [\bullet, \bullet], \rho)$ un algébroïde de Lie local selon la définition II.1.2-2.
Il existe un élément de Maurer-Cartan d, de l'algèbre de Lie différentielle graduée des polyvecteurs sur $M \oplus N[1]$, uniquement défini à l'aide de $[\bullet, \bullet]$ et ρ :

$$\exists \quad d([\bullet, \bullet], \rho) \in MC(\mathrm{T}^{\bullet}_{\text{poly}}(M \oplus N[1])).$$

Nous montrons ensuite que les $\mathscr{A}_\infty$-algèbres obtenues par l'application de la formalité relative à deux branes, sont l'algèbre différentielle graduée de Chevalley-Eilenberg, et l'algèbre associative enveloppante universelle de l'algébroïde de Lie local en question:

Théorème II.3.2-3 :

Il existe un isomorphisme de $\mathbb{K}[\![\hbar]\!]$-algèbres associatives différentielles graduées :

$$\mathfrak{I}_{A_\hbar} \quad : \quad (C^{\bullet}(L_\hbar, R_\hbar), d_{CE}, \bullet \wedge \bullet) \quad \to \quad (A_\hbar, \mathfrak{L}_{A_\hbar}(d_\hbar), \nabla_{A_\hbar}) \ .$$

Théorème II.3.3-8 :

Il existe un isomorphisme de $\mathbb{K}[\![\hbar]\!]$-algèbres associatives graduées :

$$\mathfrak{I}_{B_\hbar} \quad : \quad (\mathscr{U}(L_\hbar, R_\hbar), \bullet \cdot \bullet) \quad \to \quad (B_\hbar, \mathfrak{L}_{B_\hbar}(d_\hbar) + \nabla_{B_\hbar}) \ .$$

Nous utilisons ensuite ces résultats afin de prouver un théorème de quantification de l'algèbre symétrique sur le dual de l'algébroïde de Lie local, par l'algèbre enveloppante universelle de l'algébroïde de Lie local.

Théorème II.4.2-2 :

Soit $(M,N,[\bullet,\bullet],\rho)$ un algébroïde de Lie local tel que M et N soient concentrés en degré 0 et formant la paire de Lie-Rinehart donnée par:

$$R = \mathrm{S}(M^*) \qquad \text{et} \qquad L = \mathrm{S}(M^*) \otimes N.$$

Si on considère la variété de Poisson formé par le dual R-lineaire L^v, la quantification de l'algèbre de Poisson $\left(O(L^v), \frac{1}{2}\{\bullet,\bullet\}\right)$ des fonctions polynomiales sur L^v, est alors isomorphe à l'algèbre enveloppante universelle de la paire de Lie-Rinehart:

$$(\mathscr{U}(L_\hbar, R_\hbar), \bullet\cdot\bullet).$$

Ce théorème est une généralisation du fameux théorème de quantification de la structure de Poisson de l'algèbre symétrique sur le dual d'une algèbre de Lie, obtenu par M. Kontsevich dans [Kon03, Theorem 8.2].

Organisation

Dans le chapitre I, nous rappelons les résultats et définitions qui sous-tendent cette présentation. La première section est dédiée aux notions classiques de cogèbre et d'algèbre supérieure, nous présentons ici seulement les définitions et propriétés que nous seront amenés à utiliser et nous cherchons à atteindre la définition d'$\mathscr{A}_\infty$-bimodule. Dans la section suivante, nous présentons pas à pas la quantification par déformation d'une paire de branes coisotropes exposée dans l'article [Cal+11], les définitions et propriétés essentielles sont présentées et nous renvoyons le lecteur à des références précises pour les détails techniques.

Le chapitre II commence par une présentation de l'articulation des définitions d'une paire de Lie-Rinehart, d'un algébroïde de Lie ainsi que d'un algébroïde de Lie local. Nous construisons ensuite le cadre coisotrope et montrons l'existence d'un élément de Maurer-Cartan construit à l'aide des morphismes de structure de l'algébroïde de Lie local, le crochet de Lie et l'ancre (Théorème II.2.2-1).

Nous présentons ensuite la déformation formelle triviale d'une paire de Lie-Rinehart et l'explicitons dans le cas d'un algébroïde de Lie local. Puis après avoir calculé les morphismes de structure de l'$\mathscr{A}_\infty$-algèbre obtenue par déformation de la première brane, nous montrons l'existence d'un isomorphisme d'algèbre différentielle graduée entre cette $\mathscr{A}_\infty$-algèbre et l'algèbre de Chevalley-Eilenberg de la déformation formelle triviale de l'algébroïde de Lie local (Théorème II.3.2-3).

Nous changeons de brane, puis nous commençons par définir la notion d'algèbre enveloppante universelle d'une paire de Lie-Rinehart et l'explicitons dans le cas de la déformation formelle triviale d'un algébroïde de Lie local. Nous terminons en montrant l'existence d'un isomorphisme d'algèbres associatives entre l'algèbre obtenue par déformation de la seconde brane et l'algèbre enveloppante universelle de la déformation formelle triviale de l'algébroïde de Lie local (Théorème II.3.3-8).

Nous rappelons ensuite la définition de la structure de Poisson de Kirillov-Kostant-Souriau définie sur le dual d'une algèbre de Lie, et le théorème de quantification associé. Nous terminons cette présentation avec une généralisation de cette structure de Poisson aux algébroïdes de Lie locaux, ainsi que celle du théorème de quantification associé (Théorème II.4.2-2).

NOTATIONS

S'ensuit une courte liste non exhaustive des notations utilisées dans cette présentation.

ENSEMBLES

$\mathbb{N}$	L'ensemble des entiers positifs ou nul.
$\mathbb{Z}$	L'ensemble des entiers relatifs.
$\mathbb{R}$	L'ensemble des nombres réels.
$\mathbb{K}$	Un corps de caractéristique nulle.
$\mathbb{H}$	Le demi-plan supérieure stricte de Poincaré.
$\mathbb{K}[\![\hbar]\!]$	L'ensemble des séries formelles en $\hbar$ à coefficients dans $\mathbb{K}$.
$\mathbb{I}$, $\mathbb{J}$	Un ensemble d'indice.
$\mathrm{Multi}(\mathbb{I})$	L'ensemble des multi-indices sur $\mathbb{I}$
$\mathfrak{S}_n$	L'ensemble des permutations d'un ensemble à n éléments.
$\mathrm{Shuff}_{(p,q)}$	L'ensemble des (p,q)-shuffles.
$\mathrm{UnShuff}_{(p,q)}$	L'ensemble des (p,q)-unshuffles.
$C^+_{p,q}$	L'ensemble des configurations de p points dans $\mathbb{H}$ et q points dans $\mathbb{R}$.
$\mathscr{C}^+_{p,q}$	Le quotient de $C^+_{p,q}$ par les translations réelles et les dilatations positives.
$\mathcal{G}_{p,q}$	Un ensemble de graphes à $p+q$ sommets.
$X_\hbar$	Une abréviation pour $X \otimes \mathbb{K}[\![\hbar]\!]$.
$MC(X)$	L'ensemble des éléments de Maurer-Cartan de X.
$\mathscr{MC}(X)$	L'ensemble des classes d'équivalence de Maurer-Cartan de X.

CATÉGORIES

$\mathbf{Mod}_{\mathbb{K}}$	La catégorie des $\mathbb{K}$-modules.
$\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$	La catégorie des $\mathbb{K}$-modules $\mathbb{Z}$-gradués.
$\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$	La catégorie des objets IxI-gradués dans $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$[1]$	Le foncteur de décalage de $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$*$	Le foncteur de dualité de $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$\otimes$	Le bifoncteur monoïdal de $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$\oplus$	La somme directe de $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.

STRUCTURES ALGÉBRIQUE

$\otimes$	Le produit tensoriel au dessus de $\mathbb{K}$.
$\odot$	Le produit dans l'algèbre symétrique.
$\cdot$	Le produit dans l'algèbre enveloppante universelle.
$\mathscr{A}_\infty$	Une structure A-infini.
$\mathscr{L}_\infty$	Une structure L-infini.

MORPHISMES

$\circ$	La composition usuelle des morphismes.
$f_{\vert X}$	La restriction d'un morphisme f à un sous espace X.
P_X	La projection canonique sur le sous espace X.
$\downarrow$	Le morphisme correspondant à l'identité qui diminue le degré de 1.
$\uparrow$	Le morphisme correspondant à l'identité qui augmente le degré de 1.
∇	Un produit dans une algèbre.
ι	Une unité dans une algèbre.
Δ	Un coproduit dans une cogèbre.
ε	Une counité dans une cogèbre.

INTRODUCTION

CONTEXT

CLASSICAL AND QUANTUM MECHANICS

This work is derived from the study of the apparent incompatibility of two theories of physics, the classical mechanics and the quantum mechanics.

Classical mechanics is a mathematical model which aim to describe the motion of macroscopic objects in our universe, its Hamiltonian formulation consist of a description of the state of the object as a point in a phase space, which is defined as a Poisson or symplectic manifold M (with Poisson bracket $\{\bullet,\bullet\}$). An observable is a possible measurement outcome of the physical system and is defined, in the model, as a smooth function over M. The law of the physical system is then encoded as a specific function H called the Hamiltonian or energy function, and the prediction on the evolution of the physical system is meant to be obtained as the time evolution of an observable f, governed by the Hamiltonian equations:

$$\frac{d}{dt}f = \{f,H\}.$$

For example, if we consider a dynamical system with N-degrees of freedom, that is X is a smooth manifold of dimension N, then the phase space is the cotangent bundle of this manifold, $M := T^*X$. A state of the system at a given time t is described as an element x in X with local coordinates $(x_i)_{i\leq N}$, and momentum p in T^*_xX with local coordinates $(p_i)_{i\leq N}$. The cotangent bundle T^*X is endowed with a canonical symplectic form, the Poincaré two-form $\sum_i dx_i \wedge dp_i$ and the Hamiltonian is the Legendre transformation of the classical Lagrangian, leading to the canonical Hamilton equations of motion:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad , \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

In this context, the law of conservation of energy is expressed by $\{H,H\} = 0$.

Quantum mechanics is also a mathematical model, but it aim to describe the motion of microscopic objects in the universe. Its Heisenberg's formulation replace the symplectic manifold of the classical phase space by an infinite dimensional projective Hilbert space, the space of quantum states $\mathcal{H}$. The observables are now selfadjoint operators on $\mathcal{H}$, with the Hamiltonian H being one of them, and since the composition of such operators is not commutative, this space is endowed with a non-zero commutator $[\bullet,\bullet]$. The time evolution of the system is then described by the Heisenberg equations:

$$\frac{d}{dt}A = \frac{i}{\hbar}[H,A].$$

The Heisenberg's picture of quantum mechanics is very similar to the Hamiltonian formulation of classical mechanics and indeed those have the same role in different context, and since the Heisenberg formulation is equivalent to the Schrödinger formulation, which is more usually used in physics, one should consider the existence of a mathematical link between the two theories.
However, the quantum mechanics comes with its own inviolable principles such as the Heisenberg's uncertainty principle, hence not all of the selfadjoint operators have a physical meaning. Due to those conditions it may not be always possible to make a natural and easy correspondence between the space of classical observables and quantum observables.

Quantization is then understood as a process to realize this correspondence, at least partially, sending a smooth function f to a self adjoint operator $Q(f)$ such that $Q(1) = Id$ and the following equation:

$$[Q(f),Q(g)] = i\hbar Q(\{f,g\}).$$

This problem leads to various developments such as Wigner-Weyl quantization [Wey27], geometric quantization of Kostant and Souriau [Sou67], Berezin's quantization [Ber75], deformation quantization of Flato, Lichnerowicz and Sternheimer [Bay+77] and [FLS76], and an abundant diversity of pure mathematical results in a similar context.

Quantization in mathematics

The quantization process in mathematics is now a wide field of study with various approaches and interpretations, here we briefly review the history that led to our developement. A concise introduction on quantization of Poisson algebras can be found in [ES98], the framework is the one of free modules over the $\mathbb{K}$-algebra of power series in $\hbar$, denoted $\mathbb{K}[\![\hbar]\!]$, for a field, $\mathbb{K}$, of characteristic 0.

Given a commutative associative $\mathbb{K}$-algebra A_0, a deformation of A_0 is an associative $\mathbb{K}[\![\hbar]\!]$-algebra $(A,*)$ whose underlying module is a free $\mathbb{K}[\![\hbar]\!]$-module, and such that $A_0 = A/\hbar A$ as a $\mathbb{K}$-algebra. The algebra A may or may not be commutative, hence A_0 inherits a structure of a Poisson algebra. It is given for a pair of elements f_0, g_0 in

A_0, by the commutator of the product of any two liftings f, g in A:

$$\{f_0, g_0\} := \frac{1}{\hbar}(f * g - g * f) \ \ mod \ \ \hbar A.$$

The Poisson algebra $(A_0, \{\bullet, \bullet\})$ is then called a quasi classical limit of A, and A is called a quantization of $(A_0, \{\bullet, \bullet\})$. The algebra A is then thought as the (non commutative) algebra of quantum observables, whereas the Poisson algebra A_0 is thought as the one of classical observables. Despite the fact that the quasi classical limit is a uniquely defined object, the quantization of the algebra A_0 does not always exist, and if it does, has no reason to be unique.

In 1997, M. Kontsevitch proved in [Kon03] that if $\mathbb{K} = \mathbb{R}$ and A_0 is the $\mathbb{R}$-algebra of smooth functions on a finite dimensional differentiable real manifold M, then any Poisson bracket, $\{\bullet, \bullet\}$, on A_0 induces the existence of a quantization of $(A_0, \{\bullet, \bullet\})$, (see [Cat+05] for an enlightening and encompassing survey of this work). Moreover, the associative $\mathbb{K}[\![\hbar]\!]$-algebra A is the algebra of power series in A_0, and the quantization process is inspired by Feynman diagram perturbation series of topological quantum field theory and was later related to the perturbation series of the Poisson sigma-model in physics. The proof is made by showing an explicit quantization when M is an open subset of $\mathbb{R}^d$ for some finite dimension d, and is then extended using formal geometry and flat connections. The overall result is expressed in terms of a formality theorem for the shifted Hochschild cochain complex over A_0.

In 2005, A. Cattaneo and G. Felder extend Kontsevich's construction in [CF07] by considering a submanifold $C \subset M$ (which may define a boundary condition for the quantum fields), and quantizing the Poisson algebra of sections of the exterior power of the normal bundle of C. The overall results is the quantization of the Poisson structure on M as an $\mathscr{A}_\infty$-algebra structure over A and a relative formality theorem for the Hochschild cochain complex of the sections of the exterior algebra of the normal bundle of C.

In 2010, D. Calaque, G. Felder, A. Ferrario and C. Rossi extend the previous result in [Cal+11] to the case of two branes seen as a pair of linear subspaces of a vector space, X. The result highlight the quantization of Poisson structures as pairs of $\mathscr{A}_\infty$-algebras over the formal deformations of the branes, and $\mathscr{A}_\infty$-bimodule structure over the intersection of the branes. By carefully choosing a specific configuration of branes, Koszul dual to each other, they also prove that this deformation quantization process of quadratic Poisson bivector preserves Koszul duality.

LIE ALGEBROIDS AS COISOTROPIC BRANES

Lie groupoids have been widely used in physics to provide a unified description of the symmetries in a system, they replace the traditional Lie groups and allow the encoding of a number of different symmetries. In a similar way to Lie algebras, Lie algebroids are thought to provide a unified description of both symmetries and tangent spaces in the case of Lie groupoids, leading to a better understanding of the links between classical and quantum mechanics. Because of that, the question of the quantization of Lie algebroid structures appears as a natural step.

Following the work of D. Calaque, G. Felder, A. Ferrario and C. Rossi in [Cal+11], we present the construction of a coisotropic setting with two branes and show that the structure of local Lie algebroids can be encoded as Maurer-Cartan elements of the shifted algebra of polyvector fields on this coisotropic setting.

Theorem II.2.2-1 :

Given a local Lie algebroid, $(M,N,[\bullet,\bullet],\rho)$, w.r.t. definition II.1.2-2.
There exists an element d, which is a Maurer-Cartan element of the differential graded Lie algebra of polyvector fields over $M\oplus N[1]$, uniquely defined by $[\bullet,\bullet]$ and ρ:

$$\exists \quad d([\bullet,\bullet],\rho)\in MC(\mathrm{T}^{\bullet}_{\mathrm{poly}}(M\oplus N[1])).$$

We then show that the deformed $\mathscr{A}_\infty$-algebras obtained by the application of the relative formality for two branes, gives the canonical Chevalley-Eilenberg differential graded algebra, and the canonical universal enveloping associative algebra.

Theorem II.3.2-3 :

There exists an isomorphism of differential graded $\mathbb{K}[\![\hbar]\!]$-algebras:

$$\mathfrak{I}_{A_\hbar} \quad : \quad (C^\bullet(L_\hbar,R_\hbar),d_{CE},\bullet\wedge\bullet) \quad \to \quad (A_\hbar,\mathfrak{L}_{A_\hbar}(d_\hbar),\nabla_{A_\hbar}) \ .$$

Theorem II.3.3-8 :

There exists an isomorphism of associative graded $\mathbb{K}[\![\hbar]\!]$-algebras:

$$\mathfrak{I}_{B_\hbar} \quad : \quad (\mathscr{U}(L_\hbar,R_\hbar),\bullet\cdot\bullet) \quad \to \quad (B_\hbar,\mathfrak{L}_{B_\hbar}(d_\hbar)+\nabla_{B_\hbar}) \ .$$

We then use it to prove a theorem of quantization of the symmetric algebra on the dual of a local Lie algebroid, by the universal enveloping algebra of this local Lie algebroid.

Theorem II.4.2-2 :

Given a local Lie algebroid $(M,N,[\bullet,\bullet],\rho)$ with M and N concentrated in degree 0, and the setting of the underlying Lie-Rinehart pair :

$$R = \mathrm{S}(M^*) \qquad \text{and} \qquad L = \mathrm{S}(M^*) \otimes N.$$

If we consider the R-linear dual Poisson manifold L^v, the quantization of the Poisson algebra $(O(L^v), \frac{1}{2}\{\bullet,\bullet\})$ of polynomial functions over L^v, is then isomorphic to the Universal Enveloping algebra of the Lie-Rinehart pair:

$$(\mathscr{U}(L_\hbar, R_\hbar), \bullet\cdot\bullet).$$

This theorem is a generalization of a well-known quantization theorem of the Poisson structure on the symmetric algebra on the dual of a Lie algebra, due to M. Kontsevich in [Kon03, Theorem 8.2].

Organization

In chapter I, we briefly survey the work on which we rely. The first section is devoted to introduce the readers to the classical notions of coalgebra and higher algebra, where we aim to present only the definitions and properties that we seek, until we reach the definition of $\mathscr{A}_\infty$-modules. The second section presents the quantization of coisotropic branes introduce in [Cal+11], the elementary tools and properties are sketched, and we precisely refer to the original article for the technical details.

In chapter II, we start with the definitions of Lie-Rinehart pairs, Lie algebroids and local Lie algebroids. We then construct the coisotropic setting and show that it encode the local Lie algebroid structure maps, through the existence of a specific Maurer-cartan element made of the Lie-bracket and the anchor map (Theorem II.2.2-1). We then present the trivial formal deformations of Lie-Rinehart pairs, and explicit the Chevalley-Eilenberg algebra of the trivial formal local Lie algebroid.

After computing the structure maps of the $\mathscr{A}_\infty$-algebra obtained as a deformed $\mathscr{A}_\infty$-algebra of the first brane, we show the existence of a DG-isomorphism between this $\mathscr{A}_\infty$-algebra and the Chevalley-Eilenberg algebra of the trivial formal local Lie algebroid (Theorem II.3.2-3).

Switching to the other brane, we define the Universal enveloping algebra of a Lie-Rinehart pair and explicit it in the case of a trivial formal local Lie algebroid. We then show the existence of a morphism of associative algebras between the algebra obtained as a deformed $\mathscr{A}_\infty$-algebra of the second brane and the universal enveloping algebra of the trivial formal local Lie algebroid (Theorem II.3.3-8).

We then recall the definition of the Kirillov-Kostant-Souriau Poisson structure on the dual of a Lie algebra and the quantization theorem associated. We end this presentation with a generalization of the Poisson structure to local Lie algebroid and the quantization theorem associated (Theorem II.4.2-2).

NOTATIONS

Here is a short and non exhaustive list of notations used throughout this presentation.

SETS

$\mathbb{N}$	The set of positive integers, starting with 0.
$\mathbb{Z}$	The set of all integers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{K}$	A generic field of characteristic 0.
$\mathbb{H}$	The Poincaré upper half-plane.
$\mathbb{K}[\![\hbar]\!]$	The set of power series in $\hbar$ with coefficients in $\mathbb{K}$.
$\mathbb{I}$, $\mathbb{J}$	Sets of indices.
$\mathrm{Multi}(\mathbb{I})$	The set of multi-indices over $\mathbb{I}$
$\mathfrak{S}_n$	The set of permutation of n elements.
$\mathrm{Shuff}_{(p,q)}$	The set of (p,q)-shuffles.
$\mathrm{UnShuff}_{(p,q)}$	The set of (p,q)-unshuffles.
$C^+_{p,q}$	The set of configurations of p points in $\mathbb{H}$ and q points in $\mathbb{R}$.
$\mathscr{C}^+_{p,q}$	The quotient of $C^+_{p,q}$ by real translations and positive dilatations.
$\mathcal{G}_{p,q}$	A set of graphs of $p+q$ vertices.
$X_\hbar$	An abbreviation for $X \otimes \mathbb{K}[\![\hbar]\!]$.
$MC(X)$	The set of Maurer-Cartan elements over X.
$\mathscr{MC}(X)$	The set of Maurer-Cartan equivalence classes over X.

CATEGORIES

$\mathbf{Mod}_{\mathbb{K}}$	The category of $\mathbb{K}$-modules.
$\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$	The category of $\mathbb{Z}$-graded $\mathbb{K}$-modules.
$\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$	The category of IxI-graded objects in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$[1]$	The shift functor on $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$*$	The duality functor on $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$\otimes$	The tensor bifunctor on $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.
$\oplus$	The direct sum on $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$.

ALGEBRAIC STRUCTURES

$\otimes$	The tensor product over $\mathbb{K}$.
$\odot$	The product in the symmetric algebra.
$\cdot$	The product in the Universal enveloping algebra.
$\mathscr{A}_\infty$	An A-infinity related structure.
$\mathscr{L}_\infty$	An L-infinity related structure.

MORPHISMS

$\circ$	The usual composition of morphisms.
$f_{\mid X}$	The restriction of the morphism f to a subspace X.
P_X	The projection onto a subspace X.
$\downarrow$	The morphism corresponding to the identity which lower the degree by 1.
$\uparrow$	The morphism corresponding to the identity which higher the degree by 1.
∇	A product in an algebra.
ι	A unit in an algebra.
Δ	A coproduct in a coalgebra.
ε	A counit in an coalgebra.

I. Something to rely on

I.1 Graded algebraic world

This first section aims at introducing the basics of graded algebra and higher algebra. It can be used as a memorandum by the readers who are already familiar with these notions, and we hope that it will still be useful to them.

Throughout this presentation we consider a field $\mathbb{K}$ of characteristic 0, usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by $\mathbf{Mod}_{\mathbb{K}}$ the category of $\mathbb{K}$-modules, in which the set of $\mathbb{K}$-linear maps between two $\mathbb{K}$-modules V and W is denoted by $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}}(V,W)$. We also admit the axiom of choice, hence the existence of basis for every $\mathbb{K}$-module.

We aim at introducing the reader to higher algebra in the most natural way, therefore we will define a context in which the apparent usual complexity of signs, arities and degrees of classical definitions of $\mathscr{A}_\infty/\mathscr{L}_\infty$-algebras, modules and morphisms is implicit.

I.1.1 Introduction to graded algebra

The categories of graded modules

We start with a recollection of the definitions and properties on the category of $\mathbb{Z}$-graded $\mathbb{K}$-modules. A detailed approach of the subject can be found in [Bou07] or [LV12].

Definition I.1.1-1 : $\mathbb{Z}$-graded $\mathbb{K}$-modules

The category $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded $\mathbb{K}$-modules is as follows:
- Objects of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$ are $\mathbb{K}$-modules endowed with $\mathbb{Z}$-grading:

$$\forall\, k \in \mathbb{Z},\ \exists\, V_k \in \mathbf{Mod}_{\mathbb{K}}, \quad V = \bigoplus_{k\in\mathbb{Z}} V_k \quad \text{in } \mathbf{Mod}_{\mathbb{K}}.$$

An element $v \in V_k$ is called an *homogeneous element of degree* k, we write:

$$deg(v) = |v| = k.$$

We will say that V is of finite type when each of the V_k are finite dimensional.
- Morphisms in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$ are those in $\mathbf{Mod}_{\mathbb{K}}$ that preserve the $\mathbb{Z}$-gradings:

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W) = \left\{ f \in \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}}(V,W) \mid \forall\, k \in \mathbb{Z},\ f \circ P_{V_k} = P_{W_k} \circ f \right\}.$$

Where P_X denotes the projection onto X.
Compositions and identity morphisms are the ones of $\mathbf{Mod}_{\mathbb{K}}$.

This category is equipped with two symmetric monoidal tensor products, $\otimes$ and $\widehat{\otimes}$, coming from the ones on $\mathbf{Mod}_{\mathbb{K}}$. The $\mathbb{Z}$-grading has degree-k component given by:

$$\forall\, k \in \mathbb{Z}, \quad (V \otimes W)_k := \bigoplus_{m+n=k} V_m \otimes W_n \qquad \text{and} \qquad (V \widehat{\otimes} W)_k := \prod_{m+n=k} V_m \otimes W_n.$$

The symmetry isomorphisms, defining the Koszul's sign rule, are as follows.

$$\begin{array}{rcccl} \sigma_{V,W} & : & V \otimes W & \to & W \otimes V \\ & & v \otimes w & \mapsto & (-1)^{|v||w|} w \otimes v \end{array}$$

We also have a shift functor, denoted by $V[1]$, which pushes down the degree of all homogeneous elements such that $(V[1])_k = V_{k+1}$. It then defines internal hom sets, $\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W)$, whose $\mathbb{Z}$-grading is given by:

$$\forall\, k \in \mathbb{Z}, \qquad \underline{\mathrm{Hom}}^k_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W) := \left(\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W) \right)_k := \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V, W[k]),$$

which in the special case of $W = V$, gives rise to suspension and desuspension isomorphisms:

$$\begin{array}{rcccll} \downarrow & : & V & \to & V[1] & \in \underline{\mathrm{Hom}}^{-1}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V, V[1]), \\ \uparrow & : & V[1] & \to & V & \in \underline{\mathrm{Hom}}^{1}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V[1], V). \end{array}$$

Given two elements in two hom sets $f \in \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W)$ and $g \in \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(X,Y)$, we will make use of the symmetry isomorphisms to see their tensor product, $f \otimes g$, as an element of $\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V \otimes X, W \otimes Y)$ by defining:

$$\forall\, v \otimes x \in V \otimes X, \qquad (f \otimes g)\,(v \otimes x) := (-1)^{|v||g|} f(v) \otimes g(x).$$

The décallage isomorphism obtained by tensoring multiple desuspension morphisms is then signed, and we have:

$$\begin{array}{rccl} \downarrow^{n} \circ \uparrow^{\otimes n} \;: & (V[1])^{\otimes n} & \to & V^{\otimes n}[n] \\ & \downarrow v_1 \otimes \cdots \otimes \downarrow v_n & \mapsto & (-1)^{\sum_{i=1}^{n-1}(n-i)(|v_i|-1)} \downarrow \cdots \downarrow v_1 \otimes \cdots \otimes v_n \end{array}.$$

Seeing $\mathbb{K}$ as $\mathbb{Z}$-graded $\mathbb{K}$-module concentrated in degree 0, we define the $\mathbb{Z}$-graded linear dual, V^*, of $\mathbb{Z}$-graded $\mathbb{K}$-module V by:

$$V^* := \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V, \mathbb{K}).$$

Remark I.1.1-2 :

If a $\mathbb{Z}$-graded $\mathbb{K}$-module V has an infinitely non-trivial $\mathbb{Z}$-grading, i.e:

$$\mathrm{Card}\{k \in \mathbb{Z} |\; V_k \neq \{0\}\} = \aleph_0,$$

the dual notion previously introduced does not coincide with the classical notion of dual, as the set of linear forms, since we only have:

$$V^* \;=\; \bigoplus_{k \in \mathbb{Z}} (V_{-k})^* \;\subset\; \prod_{k \in \mathbb{Z}} (V_{-k})^* \;=\; \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}}(V, \mathbb{K}).$$

DIMENSION CONSIDERATIONS

If we consider two $\mathbb{Z}$-graded $\mathbb{K}$-modules of finite type V and W, then since each of their degree-k components are finite dimensional, it is therefore possible to identify graded morphisms with tensors in duals of $\mathbf{Mod}_{\mathbb{K}}$:

$$\forall k \in \mathbb{Z}, \quad \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}}(V_k, W_k) = W_k \otimes (V_k)^* \qquad \text{in } \mathbf{Mod}_{\mathbb{K}}.$$

Hence, since $(V_k)^* = (V^*)_{-k}$, the identification passes to $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$ as follows:

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W) = \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}}(V_k, W_k) = \prod_{k \in \mathbb{Z}} W_k \otimes (V_k)^* = (W \widehat{\otimes} V^*)_0.$$

In this case we will say that our $\mathbb{Z}$-graded $\mathbb{K}$-modules are dualizable, and we will identify the internal hom sets with the completed tensors products of duals:

$$\forall\, k \in \mathbb{Z}, \quad \underline{\mathrm{Hom}}^{k}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W) = (W[k] \widehat{\otimes} V^*)_0 = (W \widehat{\otimes} V^*)_k.$$

For the sake of simplicity we will always work with dualizable $\mathbb{Z}$-graded $\mathbb{K}$-modules, and obviously, in the case of finite dimension we can get rid of all the hats in the previous equalities.

GRADED ALGEBRAS AND MODULES

The context of $\mathbb{Z}$-graded $\mathbb{K}$-modules defines a comprehensive context for defining other algebraic structures such as, chains and cochains complexes, (associative) algebras and coalgebras, and (differential) Lie algebras. We present the definitions in the case of $\mathbb{Z}$-graded $\mathbb{K}$-modules, and it holds with due changes in case of completed ones.

We decide to present each of the properties as equalities between morphisms, by doing so it will always imply the existence of implicit signs that will appear when we will have to explicitly compute those morphisms.

We want to emphasize that an algebra can simply be defined in terms of a monoid object in any monoidal category. Since our study restrict to the categories of $\mathbb{Z}$-graded $\mathbb{K}$-modules, and later on, to differentials $\mathbb{Z}$-graded $\mathbb{K}$-modules we prefer to explicit these realizations as follows:

Definition I.1.1-3 : Graded algebra

A $\mathbb{Z}$-graded $\mathbb{K}$-algebra is a pair (A, ∇_A), where A is an object of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$, and ∇_A is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(A \otimes A, A)$. We will say that (A, ∇_A) is:

- Associative if:
$$\nabla_A \circ (\nabla_A \otimes id_A) = \nabla_A \circ (id_A \otimes \nabla_A),$$
- Commutative iff:
$$\nabla_A = \nabla_A \circ \sigma_{A,A},$$
- Anticommutative if:
$$\nabla_A = -\nabla_A \circ \sigma_{A,A},$$
- Unital if there is a morphism ι_A in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(\mathbb{K}, A)$, such that:
$$\nabla_A \circ (\iota_A \otimes id_A) = id_A = \nabla_A \circ (id_A \otimes \iota_A).$$
In this case ι_A is unique, and we will denote by (A, ∇_A, ι_A) this $\mathbb{Z}$-graded $\mathbb{K}$-algebra.

Given two $\mathbb{Z}$-graded $\mathbb{K}$-algebras, (A, ∇_A) and (B, ∇_B), a morphism of $\mathbb{Z}$-graded $\mathbb{K}$-algebras, $f : (A, \nabla_A) \to (B, \nabla_B)$, is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(A, B)$ such that:
$$\nabla_B \circ (f \otimes f) = f \circ \nabla_A.$$
If both $\mathbb{Z}$-graded $\mathbb{K}$-algebras are unital, then $f \circ \iota_A = \iota_B$.

A special type of algebraic structure that is of a main interest is the one of Lie algebras, in the context of $\mathbb{Z}$-graded $\mathbb{K}$-modules it is defined as follows:

Definition I.1.1-4 : Graded Lie algebra

A $\mathbb{Z}$-graded Lie $\mathbb{K}$-algebra is an anticommutative $\mathbb{Z}$-graded $\mathbb{K}$-algebra, (L, ∇_L), satisfying the (graded) Jacobi identity:
$$\nabla_L \circ (id_L \otimes \nabla_L) = \nabla_L \circ (\nabla_L \otimes id_L) + \nabla_L \circ (id_L \otimes \nabla_L) \circ (\sigma_{L,L} \otimes id_L).$$
The product ∇_L is called a Lie bracket.

As we have a monoidal functor in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$, it is easy to define a graded algebra structure on the space of iterated tensor products of $\mathbb{Z}$-graded $\mathbb{K}$-module V. The free unital graded tensor algebra of V, is the unital associative $\mathbb{Z}$-graded $\mathbb{K}$-algebra $(\mathrm{T}(V), \nabla_{\mathrm{T}(V)}, \iota_{\mathrm{T}(V)})$ defined by:

$$\mathrm{T}(V) := \bigoplus_{n\in\mathbb{N}} V^{\otimes n},$$

where, by convention $V^{\otimes 0} := \mathbb{K}$, and $\iota_{\mathrm{T}(V)}$ is the injection of $\mathbb{K}$ into $V^{\otimes 0}$, while the product $\nabla_{\mathrm{T}(V)}$ is given by concatenation of tensors.
Since the monoidal functor $\otimes$ is symmetric, the graded tensor algebra is endowed with an action of the symmetric groups. It is therefore possible to define another graded algebra: the graded symmetric algebra of V, which is the unital associative $\mathbb{Z}$-graded $\mathbb{K}$-algebra $(\mathrm{S}(V), \nabla_{\mathrm{S}(V)}, \iota_{\mathrm{S}(V)})$ defined by:

$$\mathrm{S}(V) := \bigoplus_{n\in\mathbb{N}} \left(V^{\otimes n} \Big/ \bigoplus_{\sigma\in\mathfrak{S}_n} (id - \sigma).(V^{\otimes n}) \right).$$

The product and unit are the quotient class of the ones of $\mathrm{T}(V)$ and given a tensor of arrity n, $v_1 \otimes \cdots \otimes v_n$ in $V^{\otimes n}$, its quotient class, is denoted by $v_1 \odot \cdots \odot v_n$.

Lastly, as we now have a definition of a graded algebra we recall the definition of a $\mathbb{Z}$-graded A-module over a $\mathbb{Z}$-graded $\mathbb{K}$-algebra, A.

Definition I.1.1-5 : Graded module over a graded algebra

Given a unital associative $\mathbb{Z}$-graded $\mathbb{K}$-algebra (A, ∇_A, ι_A), a $\mathbb{Z}$-graded (left) A-module is a pair (M, ρ_M), where M is a $\mathbb{Z}$-graded $\mathbb{K}$-module and ρ_M is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(A \otimes M, M)$, called the action, such that:

$$\rho_M \circ (id_A \otimes \rho_M) = \rho_M \circ (\nabla_A \otimes id_M),$$

$$\rho_M \circ (\iota_A \otimes id_M) = id_M.$$

Given two A-modules (M, ρ_M) and (N, ρ_N), a morphism of $\mathbb{Z}$-graded A-modules $f : (M, \rho_M) \to (N, \rho_N)$ is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(M, N)$ such that:

$$f \circ \rho_M = \rho_N \circ (id_A \otimes f).$$

Similar definitions hold for right A-modules.

Here also, the monoidal structure provides us with a straightforward example of modules, given two $\mathbb{Z}$-graded $\mathbb{K}$-modules, V and M, the $\mathbb{Z}$-graded free (left) module over M over the free algebra $\mathrm{T}(V)$, is the $\mathbb{Z}$-graded $\mathbb{K}$-module $F_V(M) := \mathrm{T}(V) \otimes M$, with action morphism defined by:

$$\rho_{F_V(M)} : \mathrm{T}(V) \otimes F_V(M) \longrightarrow F_V(M) \qquad , \qquad \rho_{F_V(M)} := \nabla_{\mathrm{T}(V)} \otimes id_M.$$

A morphism of graded free modules $f : F_V(M) \longrightarrow F_V(N)$ is uniquely defined by its restriction on M:

$$f_{|M} : M \longrightarrow F_V(N) \qquad f = id_{\mathrm{T}(V)} \otimes f_{|N}.$$

If we consider a $\mathbb{Z}$-graded unital associative $\mathbb{K}$-algebra (A, ∇_A, ι_A), and a $\mathbb{Z}$-graded A-module (M, ρ_M), we want to consider the set of $\mathbb{Z}$-graded A-linear morphisms from M to A as the A-linear dual. Hence, we denote the A-linear dual of a $\mathbb{Z}$-graded A-module, M, simply by M^v and set:

$$M^v := \mathrm{Hom}_A(M,A) := \left\{ f \in \mathrm{Hom}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(M,A) |\ f \circ \rho_M = \rho_M \circ (id_A \otimes f) \right\}.$$

Definition I.1.1-6 : Tensor product over a $\mathbb{Z}$-graded $\mathbb{K}$-algebra

Given a $\mathbb{Z}$-graded unital associative $\mathbb{K}$-algebra (A, ∇_A, ι_A), and two $\mathbb{Z}$-graded (left and right) A-modules (M, ρ_M) and (N, ρ_N), where the left and right actions coincide, meaning that for any homogeneous elements $a \in A$, $m \in M$ and $n \in N$ we have:

$$\rho_M(a \otimes m) = (-1)^{|a||m|} \rho_M(m \otimes a) \qquad \text{and} \qquad \rho_N(a \otimes n) = (-1)^{|a||n|} \rho_N(n \otimes a).$$

The $\mathbb{Z}$-graded A-module $M \otimes_A N$ is defined by the underlying $\mathbb{Z}$-graded $\mathbb{K}$-module having the following degree-k components:

$$I_k := \{ \rho_M(m \otimes a) \otimes n - m \otimes \rho_N(a \otimes n) |\ m \in M_p,\ n \in N_q,\ a \in A_r,\ p + q + r = k \},$$

$$(M \otimes_A N)_k := {}^{(M \otimes N)_k}\!/\!_{I_k}.$$

The (left) A-module structure over $M \otimes_A N$ is then given by:

$$\rho_{M \otimes_A N} := \rho_M \otimes id_N.$$

Therefore, one can speak about the tensor algebra of a $\mathbb{Z}$-graded A-module M as:

$$\mathrm{T}_A(M) := \bigoplus_{n \in \mathbb{N}} M^{\otimes_A n},$$

where again, $M^{\otimes_A 0} = \mathbb{K}$ and $\iota_{\mathrm{T}_A(M)}$ is the injection of $\mathbb{K}$ into $M^{\otimes_A 0}$, while the product $\nabla_{\mathrm{T}_A(M)}$ is given by concatenation of tensors. We can also define the symmetric algebra of a $\mathbb{Z}$-graded A-module M as:

$$\mathrm{S}_A(M) := \bigoplus_{n \in \mathbb{N}} \left({}^{M^{\otimes_A n}}\!\Big/\!\bigoplus_{\sigma \in \mathfrak{S}_n} (id - \sigma).(M^{\otimes_A n}) \right),$$

where the product and unit are the quotient class of the ones of $\mathrm{T}_A(M)$ and given a tensor of arrity n, $m_1 \otimes_A \cdots \otimes_A m_n$ in $\mathrm{T}^n_A(M)$, its quotient class, is denoted by $m_1 \odot_A \cdots \odot_A m_n$.

Graded derivations

Graded derivations form a subset of the endomorphisms of $\mathbb{Z}$-graded $\mathbb{K}$-modules, of a $\mathbb{Z}$-graded $\mathbb{K}$-algebra, which are characterized by their images on generators of the algebra and recovered using the Leibniz rule.

Definition I.1.1-7 : Graded derivation

Given a $\mathbb{Z}$-graded $\mathbb{K}$-algebra (A, ∇_A), a morphism f in $\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(A,A)$ is called a graded derivation of (A, ∇_A) if:

$$f \circ \nabla_A = \nabla_A \circ (f \otimes id_A) + \nabla_A \circ (id_A \otimes f).$$

This condition is called *the Leibniz rule*.
We denote by $\mathrm{Der}(A)$ the $\mathbb{Z}$-graded $\mathbb{K}$-module of such morphisms. And the graded commutator of derivations endows $\mathrm{Der}(A)$ with a Lie bracket.

$$\forall\, p,q \in \mathbb{Z},\ f \in \mathrm{Der}(A)^p,\ g \in \mathrm{Der}(A)^q, \qquad [f,g]_{\mathrm{Der}(A)} := f \circ g - (-1)^{pq} g \circ f.$$

As an example, a special case of graded derivations that we will need is the one of graded derivations of free algebras:

— Proposition I.1.1-8 : Derivation of free algebra

Given a $\mathbb{Z}$-graded $\mathbb{K}$-module V, a derivation of the free unital graded tensor algebra, f in $\mathrm{Der}(\mathrm{T}(V))$, is uniquely defined by its restrictions on V. It is resumed in the collection of structures maps, also called Taylor components:

$$\forall\, n \in \mathbb{N}, \qquad f^n : V \longrightarrow \mathrm{T}^n(V) \qquad , \qquad f^n := p_{\mathrm{T}^n(V)} \circ f_{|V}.$$

One can recover the whole derivation f using its Taylor components as follows:

$$f_{|\mathrm{T}^n(V)} = \sum_{i \in \mathbb{N}} \sum_{k=0}^{i} id_V^{\otimes k} \otimes f^i \otimes id_V^{\otimes n-1-k}.$$

Chain complexes and DG(L)-algebras

Chains complexes admit a reformulation in terms of $\mathbb{Z}$-graded $\mathbb{K}$-modules, and together with the previously defined structures, they define the so called differential graded (Lie) algebras.

Definition I.1.1-9 : Chain and cochain complexes

A chain (resp. cochain) complex over $\mathbb{K}$, is a pair (A, d_A) where A is an object of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$, and d_A is a morphism in $\underline{\mathrm{Hom}}^{-1}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(A,A)$ (resp. +1), such that:

$$d_A^2 = 0.$$

The morphism d_A is called the differential of A.
Morphisms of (co)chain complexes from (A, d_A) to (B, d_B), are morphisms f in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(A,B)$ such that:

$$d_B \circ f = f \circ d_A.$$

From now on, we choose a cohomological presentation, which means that we decide to consider differentials of degree $+1$, hence in the context of cochain complexes.

Definition I.1.1-10 : Differential graded associative algebra

A differential $\mathbb{Z}$-graded associative $\mathbb{K}$-algebra (DGA-algebra) is a triple (A, d_A, ∇_A), such that:

- (A, d_A) is a cochain complex over $\mathbb{K}$,
- (A, ∇_A) is an associative $\mathbb{Z}$-graded $\mathbb{K}$-algebra,
- d_A is a graded derivation of (A, ∇_A).

Morphisms of DGA-algebras are morphisms of graded algebras which are also morphisms of cochain complexes.

And we also have the Lie algebra definition in the differential $\mathbb{Z}$-graded $\mathbb{K}$-module context.

Definition I.1.1-11 : Differential graded Lie algebra

A differential $\mathbb{Z}$-graded Lie $\mathbb{K}$-algebra (DGL-algebra) is a triple $(L, d_L, [\cdot,\cdot]_L)$, such that:

- (L, d_L) is a cochain complex over $\mathbb{K}$,
- $(L, [\cdot,\cdot]_L)$ is a $\mathbb{Z}$-graded Lie $\mathbb{K}$-algebra,
- d_L is a graded derivation of $(L, [\cdot,\cdot]_L)$.

Graded coalgebras and comodules

Similar to the definition of graded algebras, graded coalgebras are their analogues in the opposite category, these structures appears naturally in various context and often together with an algebra structure, readers may refer to [Swe69] for more details.

Definition I.1.1-12 : Graded coalgebra

A $\mathbb{Z}$-graded $\mathbb{K}$-coalgebra is a pair (C, Δ_C), where C is an object of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$, and Δ_C is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(C, C \otimes C)$. We will say that (C, Δ_C) is:

- Coassociative if:

$$(\Delta_C \otimes id_C) \circ \Delta_C = (id_C \otimes \Delta_C) \circ \Delta_C,$$

- Cocommutative if:

$$\Delta_C = \sigma_{C,C} \circ \Delta_C,$$

- Anticocommutative if:

$$\Delta_C = -\sigma_{C,C} \circ \Delta_C,$$

- Counital if there is a morphism ε_C in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(C, \mathbb{K})$, such that:

$$(\varepsilon_C \otimes id_C) \circ \Delta_C = id_C = (id_C \otimes \varepsilon_C) \circ \Delta_C,$$

  In this case ε_C is unique, and we will denote by $(C, \Delta_C, \varepsilon_C)$ this $\mathbb{Z}$-graded $\mathbb{K}$-coalgebra.
- A coassociative counital coalgebra is said to be conilpotent if:

$$\forall x \in \mathrm{Ker}(\varepsilon),\ \exists n \in \mathbb{N}, \quad \Delta_C^n(x) := \left(\circ_{i=1}^{n-1}(\Delta_C \otimes id_C)\right) \circ \Delta_C(x) = 0$$

Given two $\mathbb{Z}$-graded $\mathbb{K}$-coalgebras (C, Δ_C) and (D, Δ_D), a morphism of $\mathbb{Z}$-graded $\mathbb{K}$-coalgebras $f : (C, \Delta_C) \to (D, \Delta_D)$ is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(C, D)$ such that:

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

If both $\mathbb{Z}$-graded $\mathbb{K}$-coalgebras are counital we also require $\varepsilon_C = \varepsilon_D \circ f$.

Restricting ourself to the case of conilpotent coalgebras, the monoidal structure of the category provides us with a cofree object called the graded tensor coalgebra. Given a $\mathbb{Z}$-graded $\mathbb{K}$-module V, it is defined as the cofree coassociative counital conilpotent graded coalgebra $(\mathrm{T}(V), \Delta_{\mathrm{T}(V)}, \varepsilon_{\mathrm{T}(V)})$, defined by:

$$\mathrm{T}(V) := \bigoplus_{n \in \mathbb{N}} V^{\otimes n},$$

where, by convention, $V^{\otimes 0} := \mathbb{K}$ and $\varepsilon_{\mathrm{T}(V)}$ is the projection onto $\mathbb{K}$.
The coproduct $\Delta_{\mathrm{T}(V)}$ is given by deconcatenation of arity-n tensors:

$$\Delta_{\mathrm{T}(V)} \quad : \quad \begin{array}{rcl} \mathrm{T}(V) & \to & \mathrm{T}(V) \otimes \mathrm{T}(V) \\ v_1 \otimes \cdots \otimes v_n & \mapsto & \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n) \end{array}.$$

With the usual convention that an empty tensor $() = 1 \in \mathbb{K}$.
Similarly, one can define graded comodules.

Definition I.1.1-13 : Graded comodule over a graded coalgebra

Given a coassociative counital $\mathbb{Z}$-graded $\mathbb{K}$-algebra $(C,\Delta_C,\varepsilon_C)$, a $\mathbb{Z}$-graded (left) C-comodule is a pair (M,ρ_M), where M is a $\mathbb{Z}$-graded $\mathbb{K}$-module and ρ_M is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(M,C\otimes M)$, called the coaction, such that:

$$(id_C\otimes\rho_M)\circ\rho_M=(\Delta_C\otimes id_M)\circ\rho_M,$$

$$(\varepsilon_C\otimes id_M)\circ\rho_M=id_M.$$

Given two C-comodules,(M,ρ_M) and (N,ρ_N), a morphism of $\mathbb{Z}$-graded C-comodules, $f:(M,\rho_M)\to(N,\rho_N)$, is a morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(M,N)$ such that:

$$\rho_N\circ f=(id_C\otimes f)\circ\rho_M.$$

Similar definitions hold for right C-comodule.

Again, the monoidal structure provides us with a context of a free comodule. Given two $\mathbb{Z}$-graded $\mathbb{K}$-modules V and M, the $\mathbb{Z}$-graded (left) $\mathrm{T}(V)$-comodule cofreely cogenerated by M over the cofree coalgebra $\mathrm{T}(V)$, is the $\mathbb{Z}$-graded $\mathbb{K}$-module $F_V^c(M):=\mathrm{T}(V)\otimes M$, with coaction morphism defined by:

$$\rho_{F_V^c(M)}:F_V^c(M)\longrightarrow\mathrm{T}(V)\otimes F_V^c(M)\qquad\qquad\rho_{F_V^c(M)}:=\Delta_{\mathrm{T}(V)}\otimes id_M.$$

A morphism of graded cofree comodules $f:F_V(M)\longrightarrow F_V(N)$ is uniquely characterized by its projection onto N, resumed in the collection of structures maps, also called Taylor components:

$$\forall\, n\in\mathbb{N},\qquad f^n:\mathrm{T}^n(V)\otimes M\longrightarrow N\qquad\qquad f^n:=P_{\mathbb{K}\otimes N}\circ f_{|\mathrm{T}^n(V)\otimes M}.$$

One can recover the whole comodule morphism using the following formula:

$$f_{|\mathrm{T}^n(V)\otimes M}=\sum_{i=0}^{n}id_V^{\otimes n-i}\otimes f^i.$$

GRADED CODERIVATIONS

Again, by reversing the direction of the arrows, we get the definitions of coderivations.

Definition I.1.1-14 : Graded coderivation

Given a $\mathbb{Z}$-graded $\mathbb{K}$-coalgebra (C,Δ_C), a morphism f in $\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(C,C)$ is called a graded coderivation of (C,Δ_C) if:

$$\Delta_C\circ f=(f\otimes id_C+id_C\otimes f)\circ\Delta_C.$$

This condition is called *the graded co-Leibniz rule*.
We denote by $\mathrm{Coder}(C)$ the $\mathbb{Z}$-graded $\mathbb{K}$-module of such morphisms.

And the opposite analogue of derivation of free algebras becomes the coderivation of cofree coalgebras, with the following classical property (see [Dol06]).

— Proposition I.1.1-15 : Coderivation of cofree conilpotent coalgebra

Given a $\mathbb{Z}$-graded $\mathbb{K}$-module V, a coderivation of the cofree counital conilpotent graded tensor coalgebra f in $\mathrm{Coder}(\mathrm{T}(V))$ is uniquely characterized by its projection onto V, resumed in the collection of structures maps called Taylor components:

$$\forall\, n \in \mathbb{N}, \qquad f^n : \mathrm{T}^n(V) \longrightarrow V \qquad\qquad f^n := p_V \circ f_{|\mathrm{T}^n(V)}.$$

One can recover the values of the coderivation using the formula:

$$f_{|\mathrm{T}^n(V)} = \sum_{i=0}^{n} \sum_{k=0}^{n-i} id_V^{\otimes k} \otimes f^i \otimes id_V^{\otimes n-i-k}.$$

Again, by choosing a cohomological presentation we combine the previous definitions into the one of a DG-coalgebra:

Definition I.1.1-16 : Codifferential graded coalgebra

A codifferential $\mathbb{Z}$-graded $\mathbb{K}$-coalgebra (DG-coalgebra) is a triple (C, d_C, Δ_C), such that:

- (C, d_C) is a cochain complex over $\mathbb{K}$,
- (C, Δ_C) is $\mathbb{Z}$-graded coassociative $\mathbb{K}$-coalgebra,
- d_C is a graded coderivation of (C, Δ_C).

Morphisms of DG-coalgebras are morphisms of graded coalgebras which are also morphisms of cochain complexes.

I.1.2 Introduction to higher algebra

$\mathscr{A}_\infty/\mathscr{L}_\infty$-structures

The previously defined classical algebraic structures are encompassed in what is now called higher algebra. A survey of $\mathscr{A}_\infty$-structure can be found in [Kel01], here we simply recall the definitions that we will use later on, the ones of $\mathscr{A}_\infty/\mathscr{L}_\infty$-algebras, morphisms and modules.

Definition I.1.2-1 : (Curved) $\mathscr{A}_\infty$-algebra

An $\mathscr{A}_\infty$-algebra is a pair $(A, d_{\mathrm{T}(A[1])})$, such that:

- A is a $\mathbb{Z}$-graded $\mathbb{K}$-module,
- $(\mathrm{T}(A[1]), d_{\mathrm{T}(A[1])}, \Delta_{\mathrm{T}(A[1])}, \varepsilon_{\mathrm{T}(A[1])})$ is a counital DG-coalgebra.

An $\mathscr{A}_\infty$-morphism, or ∞-morphism of $\mathscr{A}_\infty$-algebras $f : (A, d_{\mathrm{T}(A[1])}) \longrightarrow (B, d_{\mathrm{T}(B[1])})$ is a morphism of DG-coalgebras between $(\mathrm{T}(A[1]), d_{\mathrm{T}(A[1])}, \Delta_{\mathrm{T}(A[1])}, \varepsilon_{\mathrm{T}(A[1])})$ and $(\mathrm{T}(B[1]), d_{\mathrm{T}(B[1])}, \Delta_{\mathrm{T}(B[1])}, \varepsilon_{\mathrm{T}(B[1])})$.

As a coderivation of cofree coalgebra, the structure map $d_{\mathrm{T}(A[1])}$ of an $\mathscr{A}_\infty$-algebra split into its Taylor components $(d^{(n)}_{\mathrm{T}(A[1])})_{n\in\mathbb{N}}$. We say that $(A, d_{\mathrm{T}(A[1])})$ is flat when $d^{(0)}_{\mathrm{T}(A[1])} = 0$. If all Taylor components vanish except for $d^{(1)}_{\mathrm{T}(A[1])}$ then $(A, d_{\mathrm{T}(A[1])})$ is a cochain complex, and if $d^{(2)}_{\mathrm{T}(A[1])}$ also does not vanish then $(A, d_{\mathrm{T}(A[1])})$ is a DG-algebra.

A similar definition with the symmetric cofree conilpotent coalgebra gives rise to $\mathscr{L}_\infty$-algebras:

Definition I.1.2-2 : (Curved) $\mathscr{L}_\infty$-algebra

An $\mathscr{L}_\infty$-algebra is a pair $(L, d_{\mathrm{S}(L[1])})$, such that:

- L is a $\mathbb{Z}$-graded $\mathbb{K}$-module,
- $(\mathrm{S}(L[1]), d_{\mathrm{S}(L[1])}, \Delta_{\mathrm{S}(L[1])}, \varepsilon_{\mathrm{S}(L[1])})$ is a counital cocommutative DG-coalgebra.

An $\mathscr{L}_\infty$-morphism, or ∞-morphism of $\mathscr{L}_\infty$-algebra $f : (L, d_{\mathrm{S}(L[1])}) \longrightarrow (M, d_{\mathrm{S}(M[1])})$ is a morphism of DG-coalgebras between $(\mathrm{S}(L[1]), d_{\mathrm{S}(L[1])}, \Delta_{\mathrm{S}(L[1])}, \varepsilon_{\mathrm{S}(L[1])})$ and $(\mathrm{S}(M[1]), d_{\mathrm{S}(M[1])}, \Delta_{\mathrm{S}(M[1])}, \varepsilon_{\mathrm{S}(M[1])})$.

If all the Taylor components for $n = 0$ or $n > 2$ vanish, then $(L, d^{(1)}_{\mathrm{S}(L[1])})$ is a DGL-algebra. We say that $(L, d_{\mathrm{S}(L[1])})$ is flat when $d^{(0)}_{\mathrm{S}(L[1])} = 0$. In the cases of interest for us, we will always deal with flat $\mathscr{L}_\infty$-algebras.

Definition I.1.2-3 : $\mathscr{L}_\infty$-quasi-isomorphism

Given two flat $\mathscr{L}_\infty$-algebras $(L, d_{\mathrm{S}(L[1])})$ and $(M, d_{\mathrm{S}(M[1])})$, an $\mathscr{L}_\infty$-quasi-isomorphism $f : (L, d_{\mathrm{S}(L[1])}) \longrightarrow (M, d_{\mathrm{S}(M[1])})$ is an $\mathscr{L}_\infty$-morphism such that $f^{(0)} = 0$ and $f^{(1)}$ is a quasi-isomorphism of complexes.
$\mathscr{A}_\infty$-quasi-isomorphisms are defined in a similar way.

We end this short introduction with one of the main object of interest for us, $\mathscr{A}_\infty$-(bi)modules, on which the reader may find a concise introduction in [Kel06].

Definition I.1.2-4 : $\mathscr{A}_\infty$-module

Given an $\mathscr{A}_\infty$-algebra, $(A, d_{\mathrm{T}(A[1])})$, an $\mathscr{A}_\infty$-module is a pair $(M, (d_M^{(n)})_{n\in\mathbb{N}})$, such that:

- M is a $\mathbb{Z}$-graded $\mathbb{K}$-module,
- $d_M^{(n)} \;:\; \mathrm{T}^{n-1}(A[1]) \otimes M[1] \;\to\; M[1]$ is a morphism of degree 1,
- for all $n \in \mathbb{N}^*$, we have:

$$\sum_{i\leq n} d_M^{(n-i+1)} \left(id_{A[1]}^{\otimes n-i} \otimes d_M^{(i)} + \sum_{k\leq n-1-i} id_{A[1]}^{\otimes k} \otimes d_A^{(i)} \otimes id_{A[1]}^{\otimes n-1-i-k} \otimes id_{M[1]} \right) = 0.$$

An $\mathscr{A}_\infty$-module is usually defined without using the shift functor, the reason for this is, as we will see in the next section, that $\mathscr{A}_\infty$-modules and $\mathscr{A}_\infty$-algebras admit an encompassing interpretation in terms of $\mathscr{A}_\infty$-category, which looks like an $\mathscr{A}_\infty$-algebra with many objects.

Deformation of $\mathscr{A}_\infty$-structures

Deformation problem amounts to characterize all the possible algebraic structures of a certain type, which we can define on a space (constructed as a tensor of a predefined algebra with some conditions). For example, it is a well-known fact that the deformations of an associative $\mathbb{K}$-algebra are controlled by the Maurer-Cartan equivalence classes of the shifted Hochschild cochain complex over this algebra. Here we simply recall how these properties appeal together in our context.

We recall that we are working with a field $\mathbb{K}$ of characteristic 0. Given a differential graded Lie algebra $(L, d_L, [\bullet,\bullet]_L)$, the set of Maurer-Cartan elements $MC(L)$ is the set of homogeneous elements v of degree 1 satisfying the Maurer-Cartan equation:

$$d_L(v) + \frac{1}{2}[v,v]_L = 0.$$

When L_0 is a nilpotent Lie algebra and its adjoint action on L_1 is also nilpotent, then L_0 integrates to a right affine action of $exp(L_0)$ on L_1. And it is therefore possible to define the set of Maurer-Cartan equivalence classes as:

$$\mathscr{MC}(L) := MC(L)/exp(L_0).$$

When we are dealing with DGL-algebras we do not expect them to satisfy this nilpotency condition in all cases. To make this property a reality we will tensor our DGL-algebra with an artinian algebra. To do so we consider a graded commutative $\mathbb{K}$-algebra A whose maximal ideal is unique, and finite dimensional, those algebras are sometimes called test algebras. In this condition the DGL-algebra $L \otimes A$ satisfy our nilpotency condition.

A classical example of such an algebra is given by the truncated polynomial algebra $\mathbb{K}[\![\hbar]\!]/(\hbar^n)$ for a given integer n. But we will prefer to work with the algebra of formal power series in $\hbar$ (which is not nilpotent), and we see $L \otimes \mathbb{K}[\![\hbar]\!]$ as the limit over the DGL-algebras $L \otimes \mathbb{K}[\![\hbar]\!]/(\hbar^n)$. The result is called a graded topologically free $\mathbb{K}[\![\hbar]\!]$-module $L_\hbar$, and its set of Maurer-Cartan equivalence classes is:

$$\mathscr{MC}(L_\hbar) := \varprojlim_{n \in \mathbb{N}} \mathscr{MC}(L \otimes \mathbb{K}[\![\hbar]\!]/(\hbar^n)).$$

The study of the Maurer-Cartan equivalence classes is important for deformation theory, but what we are interested in here is how these sets behave with respect to $\mathscr{L}_\infty$-morphisms between their DGL-algebras:

Theorem I.1.2-5 : [Kon03]

Given two DGL-algebras L and M, seen as flat $\mathscr{L}_\infty$-algebras, and an $\mathscr{L}_\infty$-quasi-isomorphism:

$$f : L \to M$$

Then for any test algebra A, the morphism defined by the Taylor components $f^{(n)}$:

$$v \mapsto \sum_{n \in \mathbb{N}^*} \frac{1}{n!} f^{(n)}(v, \cdots, v).$$

Induce a bijection between the sets $\mathscr{MC}(L \otimes A)$ and $\mathscr{MC}(M \otimes A)$.

I.2 Quantization of coisotropic branes

In this section, we review the content of [Cal+11] by gradually unfolding the definitions and the main propositions. Some adjustment have been made on the notations in order to avoid ambiguity and to encompass with previous definitions. Proofs of the claims can be found in the article, and we precisely refer to it when necessary while some technical details are intentionally left aside, as they do not provide a real enlightening of the constructions.

The overall reasoning is as follows: a coisotropic brane setting is used to define two $\mathscr{A}_\infty$-algebras and an $\mathscr{A}_\infty$-bimodule, from which we construct an $\mathscr{A}_\infty$-category. We then explicit an $\mathscr{L}_\infty$-quasi-isomorphism from the Lie algebra of polyvector fields to the shifted Hochschild cochain complex of this $\mathscr{A}_\infty$-category. This $\mathscr{L}_\infty$-quasi-isomorphism induces a correspondence between the sets of equivalence classes of ($k[\![\hbar]\!]$-tensorised) Maurer-Cartan elements, which are Poissons structures on one side and $\mathscr{A}_\infty$-bimodules structures on the other side.

I.2.1 The $\mathscr{A}_\infty$-category setting

Category of I×I-graded objects

Assuming the notations of Section I.1, we define on the category of $\mathbb{Z}$-graded $\mathbb{K}$-modules a new layer of filtering: the category of I×I-graded objects, presented in [Cal+11, §3]. This category is sometimes known as the category of $\mathbb{K}^J$-modules, for $J = I \times I$, and we explicit it in our context:

Definition I.2.1-1 : I×I-graded objects

Given a finite set I, we define the category $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$ as follows:

- Objects of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$ are $\mathbb{Z}$-graded $\mathbb{K}$-modules V endowed with an I×I-grading:

$$\forall\, (i,j) \in \mathrm{I}\times\mathrm{I},\ \exists\, V_{i,j} \in \mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}, \qquad V = \bigoplus_{(i,j)\in\mathrm{I}\times\mathrm{I}} V_{i,j} \quad \text{in } \mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}.$$

- Morphisms in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$ are those in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$ that preserve I×I-gradings:

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V,W) = \left\{ f \in \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V,W) \,|\, \forall\, i,j \in I,\ f(V_{i,j}) \subset W_{i,j} \right\}.$$

Compositions and identity morphisms are the ones of $\mathbf{Mod}_{\mathbb{K}}$.

This category inherits the direct sum and the shift functor of $\mathbb{Z}$-graded $\mathbb{K}$-modules, both defined componentwise on two I×I-graded objects V and W:

$$\forall\, i,j \in I, \quad (V[k])_{i,j} := V_{i,j}[k] \qquad \text{and} \qquad (V \oplus W)_{i,j} := V_{i,j} \oplus W_{i,j}.$$

In fact, the category $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$ is enriched over $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$, with hom-object $\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V,W)$ in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$, whose $\mathbb{Z}$-graduation is defined by:

$$\forall k \in \mathbb{Z}, \quad \underline{\mathrm{Hom}}^k_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V,W) := \bigoplus_{(i,j)\in\mathrm{I}\times\mathrm{I}} \underline{\mathrm{Hom}}^k_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(V_{i,j},W_{i,j}).$$

The composition is made I×I-componentwise:

$$\begin{array}{rcrcl} \circ_{U,V,W} & : & \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V,W)\otimes\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(U,V) & \to & \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(U,W) \\ & & g\otimes f & \mapsto & \sum_{(i,j)\in\mathrm{I}\times\mathrm{I}} g_{|V_{i,j}}\circ f_{|U_{i,j}} \end{array}.$$

And the identity element is simply the direct sum of identity morphisms:

$$\begin{array}{rcrcl} id_V & : & \mathbb{K} & \to & \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V,V) \\ & & 1 & \mapsto & \sum_{(i,j)\in\mathrm{I}\times\mathrm{I}} id_{V_{i,j}} \end{array}.$$

Additionally, $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$ comes with a specific tensor product $\otimes_I$, with unit object $\mathbb{K}^I$, which makes $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$ into a non-symmetric monoidal category:

$$\forall\, i,j\in I,\quad (V\otimes_I W)_{i,j} := \bigoplus_{k\in I} V_{i,k}\otimes W_{k,j} \qquad \text{and} \qquad \left(\mathbb{K}^I\right)_{i,j} := \begin{cases} \{0\} & \text{if } i\neq j \\ \mathbb{K} & \text{otherwise} \end{cases}.$$

In fact, a $\mathbb{K}^{I\times I}$-module is a $\mathbb{K}^I$-bimodule and thus $\otimes_I$ is simply given by the tensor product over $\mathbb{K}^I$. It is worth mentioning that $V\otimes_I W\subset V\otimes W$ (in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$) thus the I×I-grading can be understood as a filtering on the tensor product component in the category of $\mathbb{Z}$-graded $\mathbb{K}$-modules.

Remark I.2.1-2 :

One way to visualize an object V of this category is to provide a total order on I and take a square matrix of size $\#I = n$ with coefficients in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$. Then we place the $\mathbb{Z}$-graded $\mathbb{K}$-module $V_{i,j}$ on the i-th row and j-th column. The modified tensor product then act like a product of matrices, where usual sums and products are replaced by direct sum and tensor products of $\mathbb{K}$-modules.

$$V = \begin{pmatrix} V_{i_1,i_1} & \cdots & V_{i_1,i_n} \\ \vdots & \ddots & \vdots \\ V_{i_n,i_1} & \cdots & V_{i_n,i_n} \end{pmatrix}$$

I×I-GRADED COALGEBRA AND CODERIVATION

The monoidal product $\otimes_I$ gives rise to cofree coalgebras cogenerated by I×I-graded objects.

Definition I.2.1-3 : I×I-graded tensor coalgebra

Given an I×I-graded object V in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$.
The I×I-graded tensor coalgebra $\mathrm{T}_I(V)$ is also defined by iterated tensor products:

$$\mathrm{T}_I(V) := \bigoplus_{n\in\mathbb{N}} V^{\otimes_I n} \subset \mathrm{T}(V).$$

Where $V^{\otimes_I 0}$ stands for the diagonaly I×I-graded unit object $\mathbb{K}^I$. The coproduct $\Delta \in \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(\mathrm{T}_I(V), \mathrm{T}_I(V) \otimes_I \mathrm{T}_I(V))$ is the deconcatenation of tensors:

$$\begin{array}{cccc} \Delta & : & \mathrm{T}_I(V) & \to & \mathrm{T}_I(V) \otimes_I \mathrm{T}_I(V) \\ & & v_1 \otimes \cdots \otimes v_n & \mapsto & \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n) \end{array},$$

where an empty tensor () stands for the unit 1^I in the unit object $\mathbb{K}^I$.
The counit $\varepsilon \in \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(\mathrm{T}_I(V), \mathbb{K}^I)$ is the projection onto $\mathbb{K}^I$.

The I×I-graded tensor coalgebra is a counital conilpotent coassociative cofree coalgebra, but it would be naive to think that this structure is the same as any of the ones on $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}$. To see why let us consider the simplest non-trivial case where $I = \{a, b\}$, and let us consider a generic I×I-graded object V in the category $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$. Following the remark on visualization of I×I-graded objects we get:

$$\mathrm{T}_I^0(V) = \begin{pmatrix} \mathbb{K} & \{0\} \\ \{0\} & \mathbb{K} \end{pmatrix}$$

$$\mathrm{T}_I^1(V) = \begin{pmatrix} V_{a,a} & V_{a,b} \\ V_{b,a} & V_{b,b} \end{pmatrix}$$

$$\mathrm{T}_I^2(V) = \begin{pmatrix} V_{a,a}^{\otimes 2} \oplus V_{a,b} \otimes V_{b,a} & V_{a,a} \otimes V_{a,b} \oplus V_{a,b} \otimes V_{b,b} \\ V_{b,a} \otimes V_{a,a} \oplus V_{b,b} \otimes V_{b,a} & V_{b,b}^{\otimes 2} \oplus V_{b,a} \otimes V_{a,b} \end{pmatrix}$$

$$\vdots$$

Hence, the I×I-graded tensor coalgebra $T^c_I(V)$ contains the counital tensor coalgebras of its diagonal components: $T(V_{i,i})$ for all $i \in I$. But it also contains various different spaces that, as we will see later on, can be used to encode module actions. For the purpose of the presentation we are interested in coderivations of this coalgebra.

Definition I.2.1-4 : I×I-graded coderivations

Given an I×I-graded object V and its I×I-graded tensor coalgebra $(T_I(V), \Delta, \varepsilon)$. An I×I-graded coderivation of $T_I(V)$ is a morphism ψ in $\underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(T_I(V), T_I(V))$ such that:

$$\Delta \circ \psi = (\psi \otimes_I id + id \otimes_I \psi) \circ \Delta.$$

We denote by $\mathrm{Coder}_{\mathrm{I}\times\mathrm{I}}(T_I(V))$ the set of I×I-graded coderivations of $T_I(V)$.

Direct computations show that any I×I-graded coderivation ψ satisfy $\varepsilon \circ \psi = 0$. Moreover, ψ is only defined by its arity-n Taylor components, which in turn decompose with respect to the I×I-grading:

$$\forall\, (i,j) \in \mathrm{I}\times\mathrm{I}, \qquad \psi^{(n)}_{i,j} \;\;:\;\; (V^{\otimes_I n})_{i,j} \;\;\to\;\; V_{i,j}\;\; .$$

One can then recover the map ψ by following an I×I-graded variant of the recovering procedure for coderivations of cofree coalgebra :

$$\psi_{|T^n_I(V)_{i,j}} = \sum_{p+q+r=n} \;\; \sum_{k,l \in I} \left(id_V^{\otimes p}\right)_{i,k} \otimes_I \psi^{(q)}_{k,l} \otimes_I \left(id_V^{\otimes r}\right)_{l,j}.$$

$\mathscr{A}_\infty$-CATEGORY AS I×I-GRADED OBJECT

All of the above arrange in the definition of an $\mathscr{A}_\infty$-category:

Definition I.2.1-5 : $\mathscr{A}_\infty$-category

A (small and finite) $\mathscr{A}_\infty$-category is a triple $\mathscr{E} = (I, E, d_E)$, such that:

- I is a finite set whose elements are called the objects of $\mathscr{E}$,
- $E = \bigoplus_{(i,j)\in \mathrm{I}\times\mathrm{I}} E_{i,j}$ is an object of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$,
- d_E is a codifferential on $T_I(E[1])$, ie a degree 1 I×I-graded coderivation of $T_I(E[1])$ such that $d_E \circ d_E = 0$.

Remark I.2.1-6 :

An $\mathscr{A}_\infty$-category with only one object is an $\mathscr{A}_\infty$-algebra, while one of the form $\mathscr{E} = (\{a,b\}, E, d_E)$ such that $E_{b,a} = \{0\}$, provides us with a new kind of $\mathscr{A}_\infty$-structures: in this case, $(E_{a,a}, (d_E)_{a,a})$ and $(E_{b,b}, (d_E)_{b,b})$ are two $\mathscr{A}_\infty$-algebras, while $(E_{a,b}, (d_E)_{a,b})$ is an $\mathscr{A}_\infty$-$E_{a,a}$-$E_{b,b}$-bimodule.
Conversely, the data of two $\mathscr{A}_\infty$-algebras and an $\mathscr{A}_\infty$-bimodule defines an $\mathscr{A}_\infty$-category having two objects, see [Cal+11] Example 3.3 .

HOCHSCHILD GRADED MODULE OF AN $\mathscr{A}_\infty$-CATEGORY

Definition I.2.1-7 : Hochschild graded module

Given an object $V = \bigoplus_{(i,j)\in \mathrm{I}\times\mathrm{I}} V_{i,j}$ in $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$,
the Hochschild graded module of V is the following object of $\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}$:

$$\mathcal{C}^\bullet(V,V) := \underline{\mathrm{Hom}}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(\mathrm{T}_I(V),V)$$

It is endowed with a total $\mathbb{Z}$-grading coming from a partially shifted $\mathbb{Z}$-bigrading:

$$\forall\,(p,q)\in\mathbb{Z}^2,\qquad \mathcal{C}^{(p,q)}(V,V) := \underline{\mathrm{Hom}}^q_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V^{\otimes_I(p+1)},V),$$

$$\forall\, n\in\mathbb{Z},\qquad \mathcal{C}^n(V,V) = \bigoplus_{p+q=n} \mathcal{C}^{(p,q)}(V,V).$$

An element $f\in\mathcal{C}^{(p,q)}(V,V)$, is said to be an homogeneous cochain of arity $p+1$, degree $|f| = q$, and of total degree $||f|| = p+q$.

The composition of Hochschild elements simply as morphisms is not a well-defined operation due to possibly different arities. But the brace operation on two elements, that we denote by $\psi\{\phi\}$ for two homogeneous morphisms $\psi\in\mathcal{C}^{p_1,q_1}(V,V)$ and $\phi\in\mathcal{C}^{p_2,q_2}(V,V)$, endows $\mathcal{C}^\bullet(V,V)$ with a structure of non-associative algebra.

$$\psi\{\phi\} := \sum_{k=0}^{p_1} \psi\circ\left(id_V^{\otimes_I k}\otimes_I\phi\otimes_I id_V^{\otimes_I p_1-k}\right)\in\mathcal{C}^{(p_1+p_2,q_1+q_2)}(V,V).$$

Nevertheless, its graded commutator is well-known to define a $\mathbb{Z}$-graded Lie bracket over the Hochschild graded module: the Gerstenhaber bracket:

$$[\psi,\phi]_{Gerst} := \psi\{\phi\} - (-1)^{(p_1+q_1)(p_2+q_2)}\phi\{\psi\}.$$

The Hochschild graded module of V can also be identified as a $\mathbb{Z}$-graded $\mathbb{K}$-module (using appropriate suspension) with the space of $\mathrm{I}\times\mathrm{I}$-graded coderivations of $\mathrm{T}_I(V[1])$ since we have:

$$\begin{aligned}\forall\, n\in\mathbb{Z},\qquad \mathrm{Coder}^n_{\mathrm{I}\times\mathrm{I}}(\mathrm{T}_I(V[1])) &= \underline{\mathrm{Hom}}^n_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(\mathrm{T}_I(V[1]),V[1])\\ &= \bigoplus_{p+q=n}\underline{\mathrm{Hom}}^q_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z},\mathrm{I}^2}}(V^{\otimes_I(p+1)},V)\\ &= \mathcal{C}^n(V,V).\end{aligned}$$

Proposition I.2.1-8 :

Under the above identification of $\mathbb{Z}$-graded $\mathbb{K}$-modules the Gerstenhaber bracket is identified with the natural Lie bracket of coderivations.

We now consider an $\mathscr{A}_\infty$-category $\mathscr{E} = (I,E,d_E)$ and we will call the Hochschild graded module of the $\mathscr{A}_\infty$-category the Hochschild graded module of the underlying

I×I-graded object E.
Since the $\mathscr{A}_\infty$-category structure gives a coderivation $d_E \in \mathrm{Coder}_{I\times I}(\mathrm{T}_I(E[1]))$, the proposition I.2.1-8 associates to d_E a Hochschild element γ in $\mathcal{C}^\bullet(E,E)$ of total degree 1 satisfying:

$$[\gamma,\gamma]_{Gerst} = \gamma\{\gamma\} - (-1)^1\gamma\{\gamma\} = 0.$$

Thus one can define the following I×I-graded morphism, which associate to any homogeneous Hochschild element, f, its Gerstenhaber bracket with γ:

$$\begin{array}{rcccl} \partial_\gamma & : & \mathcal{C}^\bullet(E,E) & \to & \mathcal{C}^\bullet(E,E) \\ & & f & \mapsto & [\gamma,f]_{Gerst} \end{array}.$$

It is a straightforward computation, using the graded Jacobi identity, to show that it defines a differential on $\mathcal{C}^\bullet(E,E)$:

$$\begin{aligned} \partial_\gamma \circ \partial_\gamma(f) &= [\gamma,[\gamma,f]_{Gerst}]_{Gerst} \\ &= (-1)^{||f||}[\gamma,[f,\gamma]_{Gerst}]_{Gerst} + (-1)^{1+||f||}[f,[\gamma,\gamma]_{Gerst}]_{Gerst} \\ &= -\partial_\gamma \circ \partial_\gamma(f). \end{aligned}$$

— Proposition I.2.1-9 :

The setting of an $\mathscr{A}_\infty$-category $\mathscr{E} = (I,E,d_E)$ defines the structure of a differential graded Lie algebra over the Hochschild graded module of E:

$$\left(\mathcal{C}^\bullet(E,E), \partial_\gamma, [\bullet,\bullet]_{Gerst}\right),$$

where $[\bullet,\bullet]_{Gerst}$ is the Gerstenhaber bracket, and γ is the MC-element associated to d_E.

I.2.2 Coisotropic branes into $\mathscr{A}_\infty$-category

Coisotropic setting description

We consider a certain configuration of $\mathbb{Z}$-graded $\mathbb{K}$-modules, namely, let X denote a finite d-dimensional $\mathbb{Z}$-graded $\mathbb{K}$-module endowed with two decompositions as a direct sum of pairs of $\mathbb{Z}$-graded $\mathbb{K}$-submodule, $X \cong U \oplus U'$ and $X \cong V \oplus V'$, together satisfying the following equality.

$$X \cong (U\cap V) \oplus (U'\cap V) \oplus (U\cap V') \oplus (U'\cap V'). \tag{I.1}$$

From which we deduce:

$$U \cong (U\cap V) \oplus (U\cap V') \quad , \quad V \cong (U\cap V) \oplus (U'\cap V),$$

$$X^* \cong U^\perp \oplus (U')^\perp \quad , \quad X^* \cong V^\perp \oplus (V')^\perp,$$

where $U^\perp = \{f \in X^* |\ f_{|U} = 0\}$, is the set of linear forms which vanish on U. Our main object of interest is the shifted DGL-algebra of polyvector fields on X:

Definition I.2.2-1 : $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)$

The differential graded Lie algebra of polyvector fields on X is defined by the following underlying $\mathbb{Z}$-graded $\mathbb{K}$-module:

$$\mathrm{T}^{\bullet}_{\mathrm{poly}}(X) := \mathrm{S}_{\mathrm{S}(X^*)}(\mathrm{Der}(\mathrm{S}(X^*))[-1])[1] \cong \mathrm{S}(X^*) \otimes \mathrm{S}(X[-1])[1],$$

where we have used the identification $\mathrm{Der}(\mathrm{S}(X^*)) \cong \mathrm{S}(X^*) \otimes X$. This module is equiped with a zero differential, and the Schouten-Nijenhuis Lie bracket $[\bullet,\bullet]_{\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)}$, which is defined as the extension of the Lie bracket of vector fields on X as a graded biderivation.
Hence if we consider two homogeneous tensors d_1, d_2 in $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)$ of arities n_1, n_2, there exist two finite collections of derivations of $\mathrm{S}(X^*)$, $d_{1,1}, \cdots, d_{1,n_1}$ and $d_{2,1}, \cdots, d_{2,n_2}$ such that:

$$d_k = \downarrow (\uparrow d_{k,1} \odot_{\mathrm{S}(X^*)} \cdots \odot_{\mathrm{S}(X^*)} \uparrow d_{k,n_k}) \in \mathrm{T}^{\bullet}_{\mathrm{poly}}(X), \qquad k \in \{1,2\}.$$

For the sake of simplicity, we omit the notation $\odot_{\mathrm{S}(X^*)}$ of the product in the symmetric algebra. Hence, the Lie bracket of d_1 and d_2 is given by:

$$[d_1,d_2]_{\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)} := \sum_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} sgn_{(d_1,d_2,i,j)}$$
$$\downarrow (\uparrow [d_{1,i},d_{2,j}]_{\mathrm{Der}} \uparrow d_{1,1} \cdots \widehat{\uparrow d_{1,i}} \cdots \uparrow d_{1,n_1} \uparrow d_{2,1} \cdots \widehat{\uparrow d_{2,j}} \cdots \uparrow d_{2,n_1}).$$

Where the sign is given by:

$$sgn_{(d_1,d_2,i,j)} := (-1)^{|d_{2,j}|(|d_1|-|d_{1,i}|-1)+\sum_{k<i}(|d_{1,i}|+1)(|d_{1,k}|+1)+\sum_{k'<j}(|d_{2,j}|+1)(|d_{2,k'}|+1)}.$$

We associate to this setting the following three $\mathbb{Z}$-graded $\mathbb{K}$-modules:

$$\begin{aligned} A &:= \mathrm{S}(U^*) \otimes \mathrm{S}(U'[-1]) \cong \mathrm{S}(U^* \oplus U'[-1]),\\ B &:= \mathrm{S}(V^*) \otimes \mathrm{S}(V'[-1]) \cong \mathrm{S}(V^* \oplus V'[-1]),\\ K &:= \mathrm{S}((U\cap V)^*) \otimes \mathrm{S}((U'\cap V')[-1]). \end{aligned}$$

We see the $\mathbb{Z}$-graded $\mathbb{K}$-modules $A[1]$, $B[1]$ and $K[1]$ as submodules of $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)$. Hence, A and B endowed with zero differential are $\mathbb{Z}$-graded (symmetric) associative commutative $\mathbb{K}$-algebras (i.e, trivial $\mathscr{A}_\infty$-algebras). With these two $\mathscr{A}_\infty$-algebras, we define an $\mathscr{A}_\infty$-A-B-bimodule structure on K, using a Kontsevitch-inspired approach:

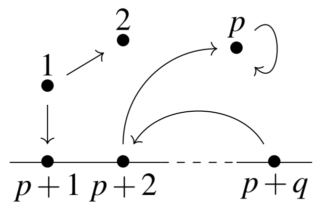
To do so, it is sufficient to define its Taylor components $d_K^{(m,n)}$ and check that they satisfy certain relations. These Taylor components are defined in terms of sums over a finite set of specific graphs, used to set a product of differential forms with coefficients made up of integrals over configuration spaces. The following are the details of this construction, see [Cal+11, §6.2] , and [Kon03, §2].

Graphs and compactified configuration spaces

We will need some preliminary definitions on sets of graphs and configurations spaces which can be found in [Cal+11, §5.1] and [Cat+05] (in French).
Given two non-negative integers $p,q \in \mathbb{N}$ let $\mathcal{G}_{p,q}$ denotes the set of admissible graphs of type (p,q). One such graph Γ is a pair composed of two sets $V(\Gamma)$ and $E(\Gamma)$:

- $V(\Gamma)$: the set of vertices, is a totally ordered set of cardinal $p+q$,
- $E(\Gamma)$: the set of edges, is a finite collection of element of $V(\Gamma) \times V(\Gamma)$.

The first p elements of $V(\Gamma)$ are called the vertices of the first type, while the remnant ones are called the vertices of the second type. A pair (i,j) represent an oriented edge of Γ, going from the i-th vertice to the j-th. Finally, $E(\Gamma)$ can contain multiple copies of the same edge, and loops.



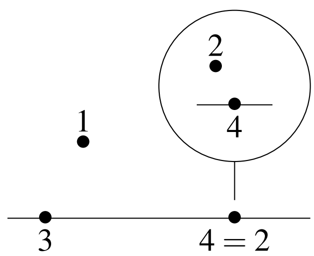
A graph in $\mathcal{G}_{p,q}$

Given two non-negative integers $p,q \in \mathbb{N}$ we denote by $C^+_{p,q}$ the configuration space of p points in the complex strict upper-half plane $\mathbb{H}$ and q points on the real axis $\mathbb{R}$.

$$C^+_{p,q} := \{(v,w) \in \mathbb{H}^p \times \mathbb{R}^q \mid \forall\, k \neq k', v(k) \neq v(k') \text{ and } \forall\, k < k', w(k) < w(k')\}/G_2$$

Where, G_2 denotes the semidirect product $\mathbb{R}^+ \ltimes \mathbb{R}$ which acts diagonally on $\mathbb{H}^p \times \mathbb{R}^q$ by positive dilatation and real translation:

$$\forall\, (\lambda,\mu) \in \mathbb{R}^+ \ltimes \mathbb{R},\ \forall\, v \in \mathbb{H} \cup \mathbb{R}, \qquad (\lambda,\mu)v := \lambda v + \mu$$

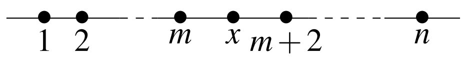


An element of $\mathscr{C}^+_{2,2}$

These configurations spaces admits a compactification à la Fulton-MacPherson, $\mathscr{C}^+_{p,q}$, which can keep track of the relative direction (and speed) of convergence of multiple points collapsing together. See [Kon03], [Cal+11], [Cat+05] or [CR11] for the details. When no ordering is involved among the vertices of second type ($q \leq 1$) we will simply drop the "+" notation.

4-colored propagators on the I-cube

In order to define the $\mathscr{A}_\infty$-bimodule structure, we only need to understand the graphs and configurations spaces comming from pairs of the form $(0, m+1+n)$ for $m,n \in \mathbb{N}$. In such configurations we have $m+1+n$ ordered points on the real axis, one of which, the $(m+1)$-th acts as a central point that we will call x and that we will put at the origin using the action of G_2.



An element of $\mathscr{C}^+_{0,m+1+n}$

To any edge (j_1, j_2) in a graph $\Gamma \in \mathcal{G}_{0,m+1+n}$, we associate a projection, $\pi_{(j_1,j_2)}$, from $\mathscr{C}^+_{0,m+1+n}$ to $\mathscr{C}^+_{0,3} \subset \mathscr{C}_{2,1}$, which simply extract the triple of points $(j_1, j_2, m+1)$ and send it to the same configuration of three points z, w and x on the real axis line and eventually "collapsed" together (in the sens of the quotient space). The image of the projection $\pi_{(j_1,j_2)}$ lies in a boundary of codimension 2 of the manifold with corners $\mathscr{C}_{2,1}$, also called the I-cube see [Cal+11, §6.2] for the full details.

These projections allow us to pullback some differential forms defined on $\mathbb{H}^2 \times \mathbb{R}$ (which extends smoothly to $\mathscr{C}_{2,1}$) to the more general space $\mathscr{C}^+_{0,m+1+n}$. The very detailed definitions and proofs associated to these 1-forms can be found in [Cal+11, §5.2], here we briefly recall their definitions in the case of the $\mathscr{A}_\infty$-bimodule structure. Given a configuration of three points $(z, w, x) \in \mathscr{C}_{2,1}$ and using the action we can still consider that $x = 0$, hence we set:

$$\omega^{+,-}(z,w) := \frac{1}{2\pi}\mathrm{d}\arg\left(\frac{\sqrt{z}-\sqrt{w}}{\sqrt{z}-\overline{\sqrt{w}}}\frac{\sqrt{z}+\overline{\sqrt{w}}}{\sqrt{z}+\sqrt{w}}\right),$$
$$\omega^{-,+}(z,w) := \frac{1}{2\pi}\mathrm{d}\arg\left(\frac{\sqrt{z}-\sqrt{w}}{\sqrt{z}+\overline{\sqrt{w}}}\frac{\sqrt{z}-\overline{\sqrt{w}}}{\sqrt{z}+\sqrt{w}}\right).$$

Where the complex square root is uniquely determined on the (large) upper half plane by its value having a positive imaginary part (hence in the first quadrant).

Remark I.2.2-2 :

In the most general case, when $z, w \in \mathbb{H}$ and $x \in \mathbb{R}$, we will use the following definition of the 4-colored propagators $\omega^{+,+}$,$\omega^{-,-}$,$\omega^{+,-}$ and $\omega^{-,+}$:

$$\omega^{+,+}(z,w,x) := \frac{1}{2\pi}\mathrm{d}\arg\left(\frac{z-w}{\bar{z}-w}\right),$$
$$\omega^{-,-}(z,w,x) := \frac{1}{2\pi}\mathrm{d}\arg\left(\frac{w-z}{\overline{w}-z}\right),$$
$$\omega^{+,-}(z,w,x) := \frac{1}{2\pi}\mathrm{d}\arg\left(\frac{\sqrt{z-x}-\sqrt{w-x}}{\sqrt{z-x}-\overline{\sqrt{w-x}}}\frac{\sqrt{z-x}+\overline{\sqrt{w-x}}}{\sqrt{z-x}+\sqrt{w-x}}\right),$$
$$\omega^{-,+}(z,w,x) := \frac{1}{2\pi}\mathrm{d}\arg\left(\frac{\sqrt{z-x}-\sqrt{w-x}}{\sqrt{z-x}+\overline{\sqrt{w-x}}}\frac{\sqrt{z-x}-\overline{\sqrt{w-x}}}{\sqrt{z-x}+\sqrt{w-x}}\right).$$

CONSTRUCTION OF THE $\mathscr{A}_\infty$-BIMODULE STRUCTURE

We are now almost ready to define the Taylor components of the $\mathscr{A}_\infty$-A-B-bimodule structure. We consider our coisotropic setting equiped with a set of linear coordinates $\{x_i\}_{i\leq d}$ adapted to our orthogonal decomposition (I.1), by which we mean that there exist two non-disjoint subsets $I_1, I_2 \subset [d] = \{1, \ldots, d\}$ such that:

$$[d] = (I_1 \cap I_2) \sqcup (I_1 \cap I_2^c) \sqcup (I_1^c \cap I_2) \sqcup (I_1^c \cap I_2^c), \tag{I.2}$$

and,

$$Span((x_i)_{i\in I_1\cap I_2}) = (U\cap V)^* \quad , \quad Span((x_i)_{i\in I_1\cap I_2^c}) = (U\cap V')^*,$$
$$Span((x_i)_{i\in I_1^c\cap I_2}) = (U'\cap V)^* \quad , \quad Span((x_i)_{i\in I_1^c\cap I_2^c}) = (U'\cap V')^*.$$

Given any edge (i,j) of a graph $\Gamma \in \mathcal{G}_{0,m+1+n}$ and any subset $J\subset [d]$ we define the following endomorphism of $\left(\mathrm{T}^\bullet_{\mathrm{poly}}(X)\right)^{\otimes m+1+n}$:

$$\tau^J_{(i,j)} := \sum_{k\in J}\left(1^{\otimes i-1}\otimes \iota_{\mathrm{d}x_k}\otimes 1^{\otimes m+n+1-i}\right)\circ\left(1^{\otimes j-1}\otimes \partial_{x_k}\otimes 1^{\otimes m+n+1-j}\right),$$

where $\iota_{\mathrm{d}x_k}$ is the classical interior derivative and ∂_{x_k} the classical partial derivative. We use the previous propagators to construct an $\Omega^1(\mathscr{C}^+_{0,m+1+n})$-valued endomorphism of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)^{\otimes m+1+n}$, by setting for all edges e in $E(\Gamma)$:

$$\omega^K_e := \pi^*_e(\omega^{+,-})\otimes \tau_e^{I_1\cap I_2^c} + \pi^*_e(\omega^{-,+})\otimes \tau_e^{I_1^c\cap I_2}. \tag{I.3}$$

Theorem I.2.2-3 : [Cal+11, Proposition 6.5]

We see the algebras A, B, K as $\mathbb{Z}$-graded associative subalgebras of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)[-1]$. Given two non-negative integers $m,n\in\mathbb{N}$ the morphism $d_K^{(m,n)}$ defined as follows is the (m,n)-Taylor component of an $\mathscr{A}_\infty$-A-B-bimodule structure over K.

$$d_K^{(m,n)} \ : \ A[1]^{\otimes m}\otimes K[1]\otimes B[1]^{\otimes n} \ \rightarrow \ K[1]$$

$$d_K^{(m,n)} := \sum_{\Gamma\in\mathcal{G}_{0,m+1+n}} \mu^{K[1]}_{m+1+n}\circ\left(\int_{\mathscr{C}^+_{0,m+1+n}}\prod_{e\in E(\Gamma)}\omega^K_e\right)$$

Where $\mu^{K[1]}_{m+1+n}$ stands for the $\mathbb{K}$-multilinear map of degree $m+n$ given by iterated products (in the symmetric algebra $\mathrm{T}^\bullet_{\mathrm{poly}}(X)[-1]$) from $\mathrm{T}^\bullet_{\mathrm{poly}}(X)^{\otimes m+1+n}$ to $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$, followed by the projection onto $K[1]$.

Remark I.2.2-4 :

Due to the shift in the definition of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$, the contraction operators τ_e are of degree -1, thus $d_K^{m,n}$ should be a sum of morphisms of degree depending on the number of edges of each graphs Γ. But one should notice that the integral is non-zero only if the integrand is a $dim\left(\mathscr{C}^+_{0,m+1+n}\right)$-form and this dimension is equal to $m+n-1$. Thus $\prod_{e\in E(\Gamma)}\omega^K_e$ is a morphism of degree $1-m-n$ which combines with μ^K_{m+1+n} to make $d_K^{m,n}$ into a morphism of degree 1.

$\mathscr{A}_\infty$-CATEGORY AS $\mathscr{A}_\infty$-BIMODULE

Following the remark in Section I.2.1-6, see [Cal+11, §3.7], given two $\mathscr{A}_\infty$-algebras $(A, d_{\mathrm{T}(A)})$ and $(B, d_{\mathrm{T}(B)})$ the definition of the $\mathscr{A}_\infty$-A-B-bimodule K defines an $\mathscr{A}_\infty$-category (I, E, d_E) with a structure set as follows:

$$I := \{a, b\} \quad , \quad E_{a,a} := A \quad , \quad E_{b,b} := B \quad , \quad E_{a,b} := K \quad , \quad E_{b,a} := \{0\}. \tag{I.4}$$

Thus d_E, as a codifferential of degree 1 on $\mathrm{T}_I(E[1])$, decomposes into three parts due to the $\mathrm{I}\times\mathrm{I}$-grading:

$$\begin{aligned}
&\mathrm{Coder}^1_{\mathrm{I}\times\mathrm{I}}(\mathrm{T}_I(E[1])) \\
&\cong \underline{\mathrm{Hom}}^1_{\mathbf{Mod}^{\mathbb{Z},\mathrm{I}^2}_{\mathbb{K}}}(\mathrm{T}_I(E[1]), E[1]) \\
&\cong \bigoplus_{n\in\mathbb{N}} \underline{\mathrm{Hom}}^1_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A[1]^{\otimes n+1}, A[1]) \oplus \bigoplus_{p,q\in\mathbb{N}} \underline{\mathrm{Hom}}^1_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A[1]^{\otimes p} \otimes K[1] \otimes B[1]^{\otimes q}, K[1]) \\
&\qquad \oplus \bigoplus_{m\in\mathbb{N}} \underline{\mathrm{Hom}}^1_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(B[1]^{\otimes m+1}, B[1]).
\end{aligned}$$

Using the $\mathscr{A}_\infty$-algebra structure on A (resp. B), $d_{\mathrm{T}(A)}$ (resp. $d_{\mathrm{T}(B)}$), and the $\mathscr{A}_\infty$-A-B-bimodule structure previously defined, d_K, we set the Taylor components of d_E to be:

$$\begin{aligned}
&\forall\, n \in \mathbb{N}, \quad (d_E)^{(n)}_{a,a} := d^{(n)}_A \quad , \qquad \forall\, m \in \mathbb{N}, \quad (d_E)^{(m)}_{b,b} := d^{(m)}_B, \\
&\forall\, p,q \in \mathbb{N}, \quad (d_E)^{(p,q)}_{a,b} := d^{(p,q)}_K \quad , \qquad (d_E)_{b,a} = 0.
\end{aligned} \tag{I.5}$$

— Proposition I.2.2-5 :

The triple, (I, E, d_E), defined by (I.4) and (I.5), is an $\mathscr{A}_\infty$-category:

$$\mathrm{Cat}_\infty(A,B,K) := (I, E, d_E)\,.$$

I.2.3 THE $\mathscr{L}_\infty$-QUASI-ISOMORPHISM

In the previous section we have constructed an $\mathscr{A}_\infty$-category $\mathrm{Cat}_\infty(A,B,K)$ out of a coisotropic setting. Using proposition I.2.1-9, we now make $\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K))$ into a differential graded Lie algebra and review the existence of an $\mathscr{L}_\infty$-quasi-isomophism, $\mathfrak{L}$, between the graded Lie algebra $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$, and $\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K))$, see [Cal+11, §7].

$$\mathfrak{L} : \left(\mathrm{T}^\bullet_{\mathrm{poly}}(X), 0, [\bullet,\bullet]\right) \longrightarrow \left(\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K)), [\gamma,\bullet], [\bullet,\bullet]_{Gerst}\right)$$

We recall that such an $\mathscr{L}_\infty$-morphism is nothing more than a morphism of codifferential cocommutative coalgebras between the shifted cocommutative cofree coalgebras $\mathrm{S}(\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1])$ and $\mathrm{S}(\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K))[1])$, endowed with codifferentials $d_{\mathrm{T}^\bullet_{\mathrm{poly}}(X)}$ and $d_{\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K))}$ given by the following non-zero Taylor components (also called Q-manifolds structure) coming from the DGLA structures:

$$\forall\, v_1, v_2 \in \mathrm{T}^\bullet_{\mathrm{poly}}(X), \qquad d^{(2)}_{\mathrm{S}(\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1])}(\downarrow v_1 \otimes \downarrow v_2) = (-1)^{|v_1|} \downarrow [v_1, v_2]$$

$\forall\, f_1, f_2 \in \mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K))$,

$$d^{(1)}_{\mathrm{S}(\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K))[1])}(\downarrow f_1) = \downarrow [\gamma, f_1]_{Gerst}$$
$$d^{(2)}_{\mathrm{S}(\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K))[1])}(\downarrow f_1 \otimes \downarrow f_2) = (-1)^{|f_1|} \downarrow [f_1, f_2]_{Gerst}$$

Therefore, as a morphism of coalgebras, $\mathfrak{L}$ is uniquely defined by its Taylor components:

$$\mathfrak{L}^{(n)} : \mathrm{S}^n(\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1]) \longrightarrow \mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K))[1].$$

THE DECOMPOSITION

We start by a comprehensive description of target object: the DGLA of Hochschild cochains. By definition of the $\mathrm{I}\times\mathrm{I}$-grading on $\mathrm{Cat}_\infty(A,B,K)$, we get the following $\mathbb{K}$-module decomposition in each $\mathbb{Z}$-bidegree (p,q):

$$\begin{aligned} &\mathcal{C}^{(p,q)}(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K)) \\ &\cong \underline{\mathrm{Hom}}^q_{\mathbf{Mod}^{\mathbb{Z},\mathrm{I}^2}_{\mathbb{K}}}(\mathrm{T}^{p+1}_I(\mathrm{Cat}_\infty(A,B,K)), \mathrm{Cat}_\infty(A,B,K)) \\ &\cong \underline{\mathrm{Hom}}^q_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A^{\otimes p+1}, A) \oplus \bigoplus_{0\le i\le p} \underline{\mathrm{Hom}}^q_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A^{\otimes i} \otimes K \otimes B^{\otimes p-i}, K) \oplus \underline{\mathrm{Hom}}^q_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(B^{\otimes p+1}, B). \end{aligned}$$

Hence, by definition of the total degree and appropriate (de)suspension morphisms, we get the total degree n component:

$$\begin{aligned} &\mathcal{C}^n(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K)) \\ &\cong \bigoplus_{p+q=n} \mathcal{C}^{(p,q)}(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K)) \\ &\cong \bigoplus_{i\in\mathbb{N}} \underline{\mathrm{Hom}}^n_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A[1]^{\otimes i+1}, A[1]) \oplus \bigoplus_{i,j\in\mathbb{N}} \underline{\mathrm{Hom}}^n_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A[1]^{\otimes i} \otimes K[1] \otimes B[1]^{\otimes j}, K[1]) \\ &\qquad \oplus \bigoplus_{j\in\mathbb{N}} \underline{\mathrm{Hom}}^n_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(B[1]^{\otimes j+1}, B[1]) \\ &\cong \mathcal{C}^n(A,A) \oplus \mathcal{C}^n(A,K,B) \oplus \mathcal{C}^n(B,B). \end{aligned}$$

Where we implicitly see the $\mathbb{Z}$-graded $\mathbb{K}$-module A (resp. B) as a trivially $\mathrm{I}\times\mathrm{I}$-graded object concentrated in (a,a) (resp. (b,b)) and where the last middle term is just a compact notation for a Hochschild-like space:

$$\mathcal{C}^n(A,K,B) := \bigoplus_{p+q=n} \bigoplus_{0\le i\le p} \underline{\mathrm{Hom}}^q_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(A^{\otimes i} \otimes K \otimes B^{\otimes p-i}, K).$$

This decomposition allows us to split the n-Taylor components $\mathfrak{L}^{(n)}$ in terms of three parts:

$$\begin{aligned}
\mathfrak{L}_A^{(n)} &: \mathrm{S}^n(\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1]) \longrightarrow \mathcal{C}^\bullet(A,A)[1],\\
\mathfrak{L}_B^{(n)} &: \mathrm{S}^n(\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1]) \longrightarrow \mathcal{C}^\bullet(B,B)[1],\\
\mathfrak{L}_K^{(n)} &: \mathrm{S}^n(\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1]) \longrightarrow \mathcal{C}^\bullet(A,K,B)[1].
\end{aligned}$$

Explicit Taylor components

We now define the different parts of the $\mathscr{L}_\infty$-morphism using similar tools as in Section I.2.2. We consider a pair of non-negative integers (n,m) and a general graph $\Gamma \in \mathcal{G}_{n,m}$. To any edge $e \in E(\Gamma)$, we associate two new $\Omega^{|E(\Gamma)|}(\mathscr{C}^+_{n,m})$-valued endomorphisms of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)^{\otimes n+m}$:

$$\begin{aligned}
\omega_e^A &:= \pi_e^*(\omega^+) \otimes \left(\tau_e^{I_1\cap I_2} + \tau_e^{I_1\cap I_2^c}\right) + \pi_e^*(\omega^-) \otimes \left(\tau_e^{I_1^c\cap I_2} + \tau_e^{I_1^c\cap I_2^c}\right),\\
\omega_e^B &:= \pi_e^*(\omega^+) \otimes \left(\tau_e^{I_1\cap I_2} + \tau_e^{I_1^c\cap I_2}\right) + \pi_e^*(\omega^-) \otimes \left(\tau_e^{I_1\cap I_2^c} + \tau_e^{I_1^c\cap I_2^c}\right).
\end{aligned}$$

These definitions are very similar to the ones of ω_e^K, eq. (I.3), but with minor modifications: The form ω^+ (resp. ω^-) is a smooth 1-form on $\mathscr{C}_{2,0}$ taking the same value as $\omega^{+,+}$ (resp. $\omega^{-,-}$). Thus, π_e^* is the pullback of the projection from $\mathscr{C}^+_{m,n}$ to $\mathscr{C}_{2,0}$, see [Cal+11] Section 5.3.1

It is worth mentioning that in this context (as opposed to the one of the $\mathscr{A}_\infty$-bimodule structure definition), the graphs used may have multiple edges and loops. The details on how to handle them can be found in [Cal+11] page 25 and we will not speak about it here as it is not relevant for the whole comprehension.

Theorem I.2.3-1 : [Cal+11, Theorem 7.2]

We see the algebras A, B, K as $\mathbb{Z}$-graded subalgebras of $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)[-1]$.
The morphisms $\mathfrak{L}_A^{(n)}$, $\mathfrak{L}_B^{(n)}$, and $\mathfrak{L}_K^{(n)}$, for all $n \in \mathbb{N}$, defined below on an arity-n tensor $\gamma_1 \otimes \cdots \otimes \gamma_n \in \mathrm{T}^{\bullet}_{\mathrm{poly}}(X)^{\otimes n}$, are the Taylor components of an $\mathscr{L}_\infty$-quasi-isomorphism.

- $\mathfrak{L}_A^{(n)}$ and $\mathfrak{L}_B^{(n)}$ are defined by their arity-m parts:

$$\mathfrak{L}_A^{(n)}(\gamma_1 \otimes \cdots \otimes \gamma_n)_{|A^{\otimes m}} \quad : \quad A^{\otimes m} \quad \to \quad A$$

$$\mathfrak{L}_A^{(n)}(\gamma_1 \otimes \cdots \otimes \gamma_n)_{|A^{\otimes m}} := \sum_{\Gamma \in \mathcal{G}_{n,m}}$$

$$\uparrow \circ \mu_{n+m}^{A[1]} \circ \left(\int_{\mathscr{C}^+_{n,m}} \prod_{e \in E(\Gamma)} \omega_e^A \right) \circ (id_{\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)^{\otimes n}} \otimes \downarrow^{\otimes m})(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes \bullet)$$

$$\mathfrak{L}_B^{(n)}(\gamma_1 \otimes \cdots \otimes \gamma_n)_{|B^{\otimes m}} \quad : \quad B^{\otimes m} \quad \to \quad B$$

$$\mathfrak{L}_B^{n}(\gamma_1 \otimes \cdots \otimes \gamma_n)_{|B^{\otimes m}} := \sum_{\Gamma \in \mathcal{G}_{n,m}}$$

$$\uparrow \circ \mu_{n+m}^{B[1]} \circ \left(\int_{\mathscr{C}^+_{n,m}} \prod_{e \in E(\Gamma)} \omega_e^B \right) \circ (id_{\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)^{\otimes n}} \otimes \downarrow^{\otimes m})(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes \bullet)$$

- $\mathfrak{L}_K^{(n)}$ is defined by its arity-(p,q) parts:

$$\mathfrak{L}_K^{(n)}(\gamma_1 \otimes \cdots \otimes \gamma_n)_{|A^{\otimes p} \otimes K \otimes B^{\otimes q}} \quad : \quad A^{\otimes p} \otimes K \otimes B^{\otimes q} \quad \to \quad K$$

$$\mathfrak{L}_K^{(n)}(\gamma_1 \otimes \cdots \otimes \gamma_n)_{|A^{\otimes p} \otimes K \otimes B^{\otimes q}} := \sum_{\Gamma \in \mathcal{G}_{n,p+1+q}}$$

$$\uparrow \circ \mu_{n+p+1+q}^{K[1]} \circ \left(\int_{\mathscr{C}^+_{n,p+1+q}} \prod_{e \in E(\Gamma)} \omega_e^K \right) \circ (id_{\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)^{\otimes n}} \otimes \downarrow^{\otimes p+1+q})(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes \bullet)$$

Where $\mu_{n+m}^{A[1]}$ (resp. $\mu_{n+m}^{B[1]}$, resp. $\mu_{n+p+1+q}^{K[1]}$) stands for the $\mathbb{K}$-multilinear map of degree $m+n-1$ (resp. $n+p+q$) given by iterated products (in the symmetric algebra $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)[-1]$) from $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)^{\otimes n+m}$ (resp. $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)^{\otimes n+p+1+q}$) to $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)$, followed by the projection onto $A[1]$ (resp. $B[1]$, resp. $K[1]$).

Using appropriate suspension and desuspensions, we consider the suspended versions of those morphisms, which will really defines the Taylor components of the morphism of DG-coalgebras that we seek.

$$
\begin{aligned}
\mathfrak{L}_A^{(n)} &: (\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1])^{\otimes n} \to \mathcal{C}^\bullet(A,A)[1] \ , \\
\mathfrak{L}_B^{(n)} &: (\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1])^{\otimes n} \to \mathcal{C}^\bullet(B,B)[1] \ , \\
\mathfrak{L}_K^{(n)} &: (\mathrm{T}^\bullet_{\mathrm{poly}}(X)[1])^{\otimes n} \to \mathcal{C}^\bullet(A,K,B)[1] \ .
\end{aligned}
$$

Remark I.2.3-2 :

In [Cal+11], the proof of the theorem splits into two parts:
The first part (§7.2), involving graph computations and integrals on configuration spaces by means of Stokes theorem, proves that it is indeed an $\mathscr{L}_\infty$-morphism.
The second part (§7.3), shows that it is in fact a quasi-isomorphism, by using an $\mathscr{A}_\infty$-version of Keller's condition for bimodule (definition in §1.5).

II. QUANTIZATION OF LIE ALGEBROIDS

II.1 FROM LIE-RINEHART PAIRS TO LOCAL LIE ALGEBROIDS

II.1.1 ORIGINS AND USES

Lie algebroids are mathematical objects first introduced by J. Pradines in [Pra70] following works on Lie groupoids by C. Ehresmann and P. Libermann [Ehr52; Lib59]. Like Lie groupoids with respect to Lie groups, Lie algebroids are the "many objects" analogue of Lie algebras. Lie groupoids have been used in physics to encode groups actions coming from internal and external symmetries, and in differential geometry as a tool for desingularization of some quotient spaces. The study of Lie algebroids seems relevant for both fields since they act as an infinitesimal objects for Lie groupoids in the smooth setting, and so, provides us with an unified framework for non-commutative and symplectic geometry which are the mathematical contexts of quantum and classical physics respectively. We recommend [Lan06; Mac87] for an introduction to the subject.

II.1.2 DEFINITIONS AND LINKS

In this section, we will always consider $\mathbb{Z}$-graded $\mathbb{K}$-modules of finite dimension in order to properly use their graded linear duals without having to switch between the categories of completed and not completed $\mathbb{Z}$-graded $\mathbb{K}$-modules. In this context, we present the definition of Lie-Rinehart pairs [Rin63], which are a more algebraic definition of a Lie algebroid in differential geometry and physics.

Definition II.1.2-1 : Lie-Rinehart pair

A Lie-Rinehart pair is a tuple $(R, \nabla_R, \iota_R, L, [\bullet,\bullet]_L, \phi_L, \rho)$, such that:

- (R, ∇_R, ι_R) is a $\mathbb{Z}$-graded associative, commutative, unital $\mathbb{K}$-algebra.
- $(L, [\bullet,\bullet]_L)$ is a $\mathbb{Z}$-graded Lie algebra,
- (L, ϕ_L) is a $\mathbb{Z}$-graded left R-module,
- ρ is a Lie morphism in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(L, \mathrm{Der}(R))$, called the *anchor*, and satisfying a Leibniz-like identity for all v_1, v_2 in L and r in R:

$$[v_1, \phi_L(r \otimes v_2)]_L = \phi_L(\rho(v_1)(r) \otimes v_2) + (-1)^{|v_1||r|}\phi_L(r \otimes [v_1, v_2]_L).$$

To see why the definition of Lie-Rinehart pair encompasses the one of a Lie algebroid as in [Lan06], consider a vector bundle $\pi : E \to X$ over a (differentiable) manifold X.
We set the algebra R to be the associative unital $\mathbb{K}$-algebra $C^\infty(X)$ of smooth functions over X. While the Lie algebra L is set to be the space of global sections $\Gamma(X,E)$ over X equiped with a choosen Lie bracket $[\bullet,\bullet]$ and an action of $C^\infty(X)$ as the product of functions.
We then identify the space of derivations of smooth functions over X with the tangent bundle TX equiped with the natural Lie bracket of vector fields.
The condition for a morphism $\rho \in \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(\Gamma(X,E), TX)$ to be an anchor map for a Lie-Rinehart pair as defined above transposes into the definition of an anchor map as a vector bundle map $\rho : E \to TX$ satisfying the Leibniz rule:

$$\forall\, \sigma_1, \sigma_2 \in \Gamma(X,E),\ \forall\, f \in C^\infty(X), \qquad [\sigma_1, f\sigma_2] = f[\sigma_1, \sigma_2] + (\rho \circ \sigma_1)(f)\sigma_2$$

In the following, we will consider a special case of Lie-Rinehart pairs where the algebra R is a free $\mathbb{K}$-module and the Lie algebra is a free R-module, restricting our study to what we call *local Lie algebroids*:

Definition II.1.2-2 : Local Lie algebroid

A local Lie algebroid is a Lie-Rinehart pair, $(R, \nabla_R, \iota_R, L, [\bullet,\bullet]_L, \phi_L, \rho)$, such that, M and N are two $\mathbb{Z}$-graded $\mathbb{K}$-modules and where:

- $R = \mathrm{S}(M^*)$ is the symmetric graded free algebra over M^*,
- $L = \mathrm{S}(M^*) \otimes N$ and ϕ_L is the concatenation of tensors, in other words, L is freely generated by N as an R-module,
- ρ is a $\mathrm{S}(M^*)$-linear morphism.

Notice that since ρ is $\mathrm{S}(M^*)$-linear it can be understood as an element of the set $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(N, \mathrm{Der}(\mathrm{S}(M^*)))$. For the sake of simplicity we may shorten the notation of a local Lie algebroid from $(R, \nabla_R, \iota_R, L, [\bullet,\bullet]_L, \phi_L, \rho)$ to $(M, N, [\bullet,\bullet], \rho)$. It is straightforward to see that local Lie algebroids generalize the definition of a $\mathbb{Z}$-graded Lie $\mathbb{K}$-algebra.

II.2 LOCAL LIE ALGEBROIDS AS COISOTROPIC BRANES

II.2.1 BRANES SETTING

Simplifying a bit further, we consider local Lie algebroid $(M, N, [\bullet,\bullet], \rho)$ where M and N are $\mathbb{Z}$-graded $\mathbb{K}$-modules of finite dimension and concentrated in degree 0. With it, we construct a coisotropic setting as in Equation (I.1). Following the notations of [Cal+11], we set the four $\mathbb{Z}$-graded $\mathbb{K}$-modules to be:

$$X := U := M \oplus N[1], \quad U' := \{0\} \qquad \text{and} \qquad V := M, \quad V' := N[1]$$

Direct computations then show that the $\mathbb{Z}$-graded $\mathbb{K}$-algebras involved in the quantization of coisotropic branes, as in I.2.2, are:

$$\begin{aligned} \mathrm{T}^{\bullet}_{\mathrm{poly}}(X) &= \mathrm{S}((M \oplus N[1])^*) \otimes \mathrm{S}(M[-1] \oplus N)[1], \\ A &= \mathrm{S}((M \oplus N[1])^*), \\ B &= \mathrm{S}(M^*) \otimes \mathrm{S}(N), \\ K &= \mathrm{S}(M^*). \end{aligned}$$

We will show that the data of a local Lie algebroid gives rise to an explicit Maurer-Cartan element in the DGL-algebra $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)$. And we will later use it to deform the $\mathscr{A}_\infty$-algebras A and B. For now, since $(M, N, [\bullet,\bullet], \rho)$ is a Lie-Rinehart pair in the special case where L is a free R-module, it induces that the morphism $[\bullet,\bullet]$ can be reconstructed from its restriction on $N^{\otimes 2}$, denoted $[\bullet,\bullet]_N$:

$$\forall\, w \in \mathrm{S}(M^*),\ \forall\, v_1, v_2 \in N, \qquad [v_1, w \otimes v_2] = \rho(v_1)(w) \otimes v_2 + w \otimes [v_1, v_2]_N$$

Consequently we shall consider the following two graded morphisms of degree 0:

$$[\bullet,\bullet]_N \in \mathrm{Hom}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(N^{\otimes 2}, \mathrm{S}(M^*) \otimes N) \qquad \text{and} \qquad \rho \in \mathrm{Hom}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(N, \mathrm{Der}(\mathrm{S}(M^*)))$$

II.2.2 THE EXPLICIT MAURER-CARTAN ELEMENT

Given the setting previously introduced, we define two new morphisms associated to ρ and $[\bullet,\bullet]_N$ as follows:

$$\begin{aligned} [\bullet,\bullet]_N^{\triangle} \ : \ N^* &\to \underline{\mathrm{Hom}}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(N[1]^{\otimes 2}, \mathrm{S}(M^*)) \\ \phi &\mapsto (id_{\mathrm{S}(M^*)} \otimes \phi) \circ [\bullet,\bullet]_N \circ \uparrow^{\otimes 2} \end{aligned}$$

$$\begin{aligned} \rho^{\triangle} \ : \ M^* &\to \underline{\mathrm{Hom}}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(N[1], \mathrm{S}(M^*)) \\ \psi &\mapsto ev_\psi \circ \rho \circ \uparrow \end{aligned}$$

where ev_ψ is the evaluation morphism of a derivation of $\mathrm{S}(M^*)$ on ψ. The degree of each of these morphisms is readily computed as being equal to $|[\bullet,\bullet]_N^{\triangle}| = 2$ and $|\rho^{\triangle}| = 1$, and we have the following result:

Theorem II.2.2-1 :

Consider the two morphisms $[\bullet,\bullet]_N^\triangle$ and $\rho^\triangle$ defined above and set d as follow:

$$d := \rho^\triangle - \frac{1}{2}[\bullet,\bullet]_N^\triangle \circ \downarrow \qquad \in \mathrm{S}(M^*)\otimes \mathrm{S}(N^*[-1])\otimes(M\oplus N[1])$$

Then d is a Maurer-Cartan element of the DGL-algebra $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$.

Proof :

First, using appropriate identification of hom sets and duals we have:

$$\rho^\triangle \in \mathrm{S}(M^*)\otimes N[1]^*\otimes M \qquad \text{and} \qquad [\bullet,\bullet]_N^\triangle\circ\downarrow \in \mathrm{S}(M^*)\otimes\left(N[1]^{\otimes 2}\right)^*\otimes N[1]$$

The fact that $[\bullet,\bullet]_N$ is a graded anti-symmetric morphism on $N\otimes N$ allow us to pass from $(N[1]^{\otimes 2})^*$ in the middle term to the graded symmetric algebra over $N[1]^*$ since for all homogeneous elements $v_1, v_2 \in N$ we have:

$$\begin{aligned}
[\bullet,\bullet]_N\circ\uparrow^{\otimes 2}(\downarrow v_1\otimes\downarrow v_2) &= (-1)^{|\downarrow v_1||\uparrow|}[v_1,v_2]_N \\
&= -(-1)^{|v_1|+1+|v_1||v_2|}[v_2,v_1]_N \\
&= (-1)^{|v_1||v_2|+|v_1|+|\downarrow v_2||\uparrow|}[\bullet,\bullet]_N\circ\uparrow^{\otimes 2}(\downarrow v_2\otimes\downarrow v_1) \\
&= [\bullet,\bullet]_N\circ\uparrow^{\otimes 2}((-1)^{|\downarrow v_1||\downarrow v_2|}\downarrow v_2\otimes\downarrow v_1) \\
&= [\bullet,\bullet]_N\circ\uparrow^{\otimes 2}\circ\,\sigma_{N[1],N[1]}(\downarrow v_1\otimes\downarrow v_2)
\end{aligned}$$

Thus $d\in \mathrm{S}(M^*)\otimes\mathrm{S}(N^*[-1])\otimes(M\oplus N[1]) = \mathrm{S}((M\oplus N[1])^*)\otimes(M\oplus N[1])$ and we see it as a graded derivation of $\mathrm{S}((M\oplus N[1])^*)$ of degree 1 by extending it using the graded Leibniz rule, which gives for all $l_1,\cdots,l_n$ in $(M\oplus N[1])^*$:

$$d(l_1\odot\cdots\odot l_n) := \sum_{0\le i\le n}(-1)^{\sum_{0\le k<i}|l_k|}\, l_1\odot\cdots\odot d(l_i)\odot\cdots\odot l_n$$

The Schouten-Nijenhuis bracket of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$ is the graded Lie bracket of derivations on this set and the differential is zero, so if we want to show that d is a Maurer-Cartan element only remain to show that:

$$\frac{1}{2}[d,d]_{\mathrm{T}^\bullet_{\mathrm{poly}}(X)} = \frac{1}{2}[d,d]_{\mathrm{Der}} = 0$$

Since $|[\bullet,\bullet]_N^\triangle| = 2$ and $|\rho^\triangle| = 1$ then d is a morphism of degree 1, and so, the graded Lie bracket of derivations of d with itself is equal to the composition $2.d^2$ which is then a derivation, so we only need to show that d^2 vanishes on generators of $\mathrm{S}((M\oplus N[1])^*)$:

$$\forall\,\phi\in M^*,\quad \rho^\triangle(\rho^\triangle(\phi)) - \frac{1}{2}[\bullet,\bullet]_N^\triangle\circ\downarrow(\rho^\triangle(\phi)) = 0 \qquad \text{(II.1)}$$

$$\forall\,\psi\in N^*,\ -\frac{1}{2}\rho^\triangle([\bullet,\bullet]_N^\triangle(\psi)) + \frac{1}{4}[\bullet,\bullet]_N^\triangle\circ\downarrow([\bullet,\bullet]_N^\triangle(\psi)) = 0 \qquad \text{(II.2)}$$

Consider $\{m_i\}_{i\in\mathbb{I}} \in M^{\mathbb{I}}$ (resp. $\{n_j\}_{j\in\mathbb{J}} \in N^{\mathbb{J}}$) to be a basis of M (resp. N), for a finite set $\mathbb{I}$ (resp. $\mathbb{J}$) of cardinal $dim(M)$ (resp. $dim(N)$), and its dual basis $\{m^i\}_{i\in\mathbb{I}}$ (resp. $\{n^j\}_{j\in\mathbb{J}}$). We introduce the following two lemmata, in order to prove (II.1).

— Lemma II.2.2-2 :

The following equality holds true for all $i \in \mathbb{I}$ and all $a,b \in \mathbb{J}$,

$$\left([\bullet,\bullet]_N^{\triangle} \circ \downarrow (\rho^{\triangle}(m^i))\right)(\downarrow n_a \otimes \downarrow n_b) = -\rho \circ [\bullet,\bullet]_N(n_a \otimes n_b)(m^i).$$

— Lemma II.2.2-3 :

The following equality holds true for all $i \in \mathbb{I}$ and all $a,b \in \mathbb{J}$,

$$\rho^{\triangle}\left(\rho^{\triangle}(m^i)\right)(\downarrow n_a \otimes \downarrow n_b) = \frac{1}{2}\rho(n_b)\circ\rho(n_a)(m^i) - \frac{1}{2}\rho(n_a)\circ\rho(n_b)(m^i).$$

Applying these two relations to the left-hand term of Equation (II.1) proves the equality since for all i in $\mathbb{I}$ and all a,b in $\mathbb{J}$ we have:

$$\begin{aligned}
&\left(\rho^{\triangle}(\rho^{\triangle}(m^i)) - \frac{1}{2}[\bullet,\bullet]_N^{\triangle} \circ \downarrow (\rho^{\triangle}(m^i))\right)(\downarrow n_a \otimes \downarrow n_b)\\
&= \frac{1}{2}\rho(n_b)\circ\rho(n_a)(m^i) - \frac{1}{2}\rho(n_a)\circ\rho(n_b)(m^i) + \frac{1}{2}\rho\circ[\bullet,\bullet]_N(n_a\otimes n_b)(m^i)\\
&= \frac{1}{2}[\rho(n_a),\rho(n_b)]_{\mathrm{Der(S}(M^*))}(m^i) + \frac{1}{2}\rho([n_a,n_b]_N)(m^i)\\
&= 0.
\end{aligned}$$

Where in the last step, we use the property that ρ is a Lie morphism.
Following a similar procedure, we introduce two other computational lemmata used to prove Equation (II.2).

— Lemma II.2.2-4 :

The following equality holds true for all $j,a_1,a_2,a_3 \in \mathbb{J}$,

$$\rho^{\triangle}\left([\bullet,\bullet]_N^{\triangle}(n^j)\right)(\downarrow n_{a_1} \otimes \downarrow n_{a_2} \otimes \downarrow n_{a_3}) = \sum_{\sigma\in\mathfrak{S}_3} -\frac{(-1)^{\sigma}}{6}\rho(n_{a_{\sigma(1)}})\left([n_{a_{\sigma(2)}},n_{a_{\sigma(3)}}]_N(n^j)\right).$$

— Lemma II.2.2-5 :

The following equality holds true for all $j,a_1,a_2,a_3 \in \mathbb{J}$,

$$\begin{aligned}
&\left([\bullet,\bullet]_N^{\triangle} \circ \downarrow\right)\left([\bullet,\bullet]_N^{\triangle}(n^j)\right)(\downarrow n_{a_1} \otimes \downarrow n_{a_2} \otimes \downarrow n_{a_3})\\
&\qquad = \sum_{\substack{\sigma\in\mathfrak{S}_3\\ k\in\mathbb{J}}} \frac{(-1)^{\sigma}}{3}[n_{a_{\sigma(2)}},n_{a_{\sigma(3)}}]_N(n^k)\odot[n_{a_{\sigma(1)}},n_k]_N(n^j).
\end{aligned}$$

Since $[\bullet,\bullet]_L$ is a graded Lie bracket on $L = \mathrm{S}(M^*)\otimes N$, it satisfies the graded Jacobi identity. Given three basis elements n_{a_1},n_{a_2},n_{a_3} of N, an element n^j of the dual basis, and by applying the Jacobi identity twice and the property of the anchor, we get the following decomposition of 0:

$$
\begin{aligned}
&= \sum_{\sigma\in\mathfrak{S}_3} (-1)^\sigma [n_{a_{\sigma(1)}}, [n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_L]_L(n^j) \\
&= \sum_{\sigma\in\mathfrak{S}_3} (-1)^\sigma [n_{a_{\sigma(1)}}, \sum_{k\in\mathbb{J}} \phi_L([n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^k)\otimes n_k)]_L(n^j) \\
&= \sum_{\sigma\in\mathfrak{S}_3} (-1)^\sigma \sum_{k\in\mathbb{J}} \Big(\rho(n_{a_{\sigma(1)}})\Big([n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^k)\Big)\odot n_k + [n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^k)\odot[n_{a_{\sigma(1)}}, n_k]_N\Big)(n^j) \\
&= \sum_{\sigma\in\mathfrak{S}_3} (-1)^\sigma \rho(n_{a_{\sigma(1)}})\Big([n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^j)\Big) + \sum_{\substack{k\in\mathbb{J}\\ \sigma\in\mathfrak{S}_3}} (-1)^\sigma [n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^k)\odot[n_{a_{\sigma(1)}}, n_k]_N(n^j).
\end{aligned}
$$

We can now explicit the left-hand term of Equation (II.2) using this relation and lemmata to prove it since for all j, a_1, a_2, a_3 in $\mathbb{J}$:

$$
\begin{aligned}
&\left(-\frac{1}{2}\rho^\triangle([\bullet,\bullet]_N^\triangle(n^j)) + \frac{1}{4}[\bullet,\bullet]_N^\triangle\circ\downarrow([\bullet,\bullet]_N^\triangle(n^j))\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3}) \\
&= \sum_{\sigma\in\mathfrak{S}_3} \frac{(-1)^\sigma}{12}\rho(n_{a_{\sigma(1)}})\Big([n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^j)\Big) \\
&\quad + \sum_{\substack{\sigma\in\mathfrak{S}_3\\ k\in\mathbb{J}}} \frac{(-1)^\sigma}{12}[n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^k)\odot[n_{a_{\sigma(1)}}, n_k]_N(n^j) \\
&= 0.
\end{aligned}
$$

□

The following are the proofs of the four computational lemmata.

Remind that $\{m_i\}_{i\in\mathbb{I}}\in M^{\mathbb{I}}$ (resp. $\{n_j\}_{j\in\mathbb{J}}\in N^{\mathbb{J}}$) stands for a basis of M (resp. N), for a finite set $\mathbb{I}$ (resp. $\mathbb{J}$) and $\{m^i\}_{i\in\mathbb{I}}$ (resp. $\{n^j\}_{j\in\mathbb{J}}$) stands for the dual basis. Since we are working with the tensor algebra, we will denote by $\mathrm{Multi}(\mathbb{I})$ the set of multi-index, that is the set of all finite and totally ordered collections of elements of $\mathbb{I}$. And given a multi-index $I\in\mathrm{Multi}(\mathbb{I})$, we will denote by m_I the tensor $m_{I_1}\otimes\cdots\otimes m_{I_{\#I}}$, with similar notations for N and duals.

As a side note for all the following computations, we will always indistinctively use the notation m^I for $m^{I_1}\otimes\cdots\otimes m^{I_{\#I}}$ and for $m^{I_1}\odot\cdots\odot m^{I_{\#I}}$. We allow ourselves this abuse of notation since it is of no importance in the case of the elements m^I which have all degree 0 and acts, in some sense, simply as coefficients. It is of course not the case for the elements of $N[1]$ and we will take a great care to detail the computations.

Under these notations, the coefficients defining the tensors of the previous morphisms ρ and $[\bullet,\bullet]_N$ are as follows:

$\forall\, I\in\mathrm{Multi}(\mathbb{I}),\ i\in\mathbb{I},\ j\in\mathbb{J},\ \exists\ \mathbf{r}_i^{I,j}\in\mathbb{K}$,

$$
\rho = \sum_{\substack{I\in\mathrm{Multi}(\mathbb{I})\\ j\in\mathbb{J}, i\in\mathbb{I}}} \mathbf{r}_i^{I,j}\, m^I\otimes m_i\otimes n^j,
$$

$\forall I \in \mathrm{Multi}(\mathbb{I}),\ j_1, j_2, j_3 \in \mathbb{J},\ \exists\ \mathbf{b}_{j_3}^{I,j_1,j_2} \in \mathbb{K}$,

$$[\bullet,\bullet]_N = \sum_{\substack{I\in\mathrm{Multi}(\mathbb{I})\\ j_1,j_2,j_3\in\mathbb{J}}} \mathbf{b}_{j_3}^{I,j_1,j_2}\, m^I \otimes n_{j_3} \otimes (n^{j_1} \otimes n^{j_2}).$$

And the coefficients are simply given by the pairing on $\mathrm{S}(M^*)$ and on $\mathrm{S}(M^*)\otimes N$:

$$\mathbf{r}_i^{I,j} :=< m_I, \rho(n_j)(m^i) > \qquad \text{and} \qquad \mathbf{b}_{j_3}^{I,j_1,j_2} :=< m_I \otimes n^{j_3}, [n_{j_1}, n_{j_2}]_N > .$$

For the sake of simplicity we will make use of Einstein notation for tensors by omitting the sum over repeated indices appearing at any position in coefficients or tensors (but not in sign's exponent). For instance, the two previous morphisms are written as:

$$\rho = \mathbf{r}_i^{I,j}\, m^I \otimes m_i \otimes n^j \qquad \text{and} \qquad [\bullet,\bullet]_N = \mathbf{b}_{j_3}^{I,j_1,j_2}\, m^I \otimes n_{j_3} \otimes (n^{j_1} \otimes n^{j_2}).$$

Viewing the two morphisms $\rho^\triangle$ and $[\bullet,\bullet]_N^\triangle$ as derivations of the graded symmetric algebra $\mathrm{S}(M^* \oplus N^*[-1])$, we can compute the expressions of theirs associated tensors as follow:

$$\rho^\triangle = \mathbf{r}_i^{I,j} m^I \odot \uparrow n^j \otimes m_i \qquad \text{and} \qquad [\bullet,\bullet]_N^\triangle = \mathbf{b}_{j_3}^{I,j_1,j_2} m^I \odot (\uparrow n^{j_1} \odot \uparrow n^{j_2}) \otimes n_{j_3}.$$

Viewing back the elements of $\mathrm{S}(M^* \oplus N^*[-1])$ as symmetric multilinear morphisms on $N[1]$ we will not forget to implicitly use the symmetrization isomorphism.
We also use Kronecker delta symbol, δ_α^β, for a given pair of indices (α,β), which takes values 1 or 0 depending if $\alpha = \beta$ or not.

Proof of II.2.2-2:

Given any i in $\mathbb{I}$ and any a, b in $\mathbb{J}$, a direct computation shows that:

$$\begin{aligned}
&\left([\bullet,\bullet]_N^\triangle \circ \downarrow (\rho^\triangle(m^i))\right)(\downarrow n_a \otimes \downarrow n_b)\\
&= \left([\bullet,\bullet]_N^\triangle(\mathbf{r}_i^{I,j} m^I \odot n^j)\right)(\downarrow n_a \otimes \downarrow n_b)\\
&= \mathbf{r}_i^{I,j}\, \mathbf{b}_j^{I',j_1,j_2} m^I \odot m^{I'} \odot (\uparrow n^{j_1} \odot \uparrow n^{j_2})(\downarrow n_a \otimes \downarrow n_b)\\
&= \mathbf{r}_i^{I,j} m^I \odot m^{I'} \otimes \frac{1}{2}\left(\mathbf{b}_j^{I',j_1,j_2} \uparrow n^{j_1} \otimes \uparrow n^{j_2} - \mathbf{b}_j^{I',j_1,j_2} \uparrow n^{j_2} \otimes \uparrow n^{j_1}\right)(\downarrow n_a \otimes \downarrow n_b)\\
&= -\mathbf{r}_i^{I,j}\, \mathbf{b}_j^{I',a,b} m^{I'} \odot m^I,
\end{aligned}$$

where we use the fact that $[\bullet,\bullet]_N$ is a graded anti-symmetric morphism therefore we have the relation $\mathbf{b}_{j_3}^{I,j_1,j_2} = -\mathbf{b}_{j_3}^{I,j_2,j_1}$. On the other side, we have:

$$\begin{aligned}
&\rho \circ [\bullet,\bullet]_N(n_a \otimes n_b)(m^i)\\
&= \rho(\mathbf{b}_{j_3}^{I',a,b} m^{I'} \otimes n_{j_3})(m^i)\\
&= \mathbf{b}_j^{I',a,b}\, \mathbf{r}_i^{I,j} m^{I'} \odot m^I,
\end{aligned}$$

which proves that:

$$\left([\bullet,\bullet]_N^{\triangle}\circ\downarrow(\rho^{\triangle}(m^i))\right)(\downarrow n_a\otimes\downarrow n_b) = -\rho\circ[\bullet,\bullet]_N(n_a\otimes n_b)(m^i).$$

□

Proof of II.2.2-3:

Given any i in $\mathbb{I}$ and any a,b in $\mathbb{J}$, a direct computation shows that:

$$\begin{aligned}
&\rho^{\triangle}\left(\rho^{\triangle}(m^i)\right)(\downarrow n_a\otimes\downarrow n_b)\\
&=\rho^{\triangle}(\mathbf{r}_i^{I,j}m^I\odot\uparrow n^j)(\downarrow n_a\otimes\downarrow n_b)\\
&=\mathbf{r}_i^{I,j}\mathbf{r}_{i'}^{I',j'}\left(m^{I'}\odot\uparrow n^{j'}\odot m_{i'}(m^I)\odot\uparrow n^j\right)(\downarrow n_a\otimes\downarrow n_b)\\
&=m^{I'}\odot m_{i'}(m^I)\otimes\frac{1}{2}\left(\mathbf{r}_i^{I,j}\mathbf{r}_{i'}^{I',j'}\uparrow n^{j'}\otimes\uparrow n^j-\mathbf{r}_i^{I,j}\mathbf{r}_{i'}^{I',j'}\uparrow n^j\otimes\uparrow n^{j'}\right)(\downarrow n_a\otimes\downarrow n_b)\\
&=\frac{1}{2}\left(-\mathbf{r}_i^{I,b}\mathbf{r}_{i'}^{I',a}+\mathbf{r}_i^{I,a}\mathbf{r}_{i'}^{I',b}\right)m^{I'}\odot m_{i'}(m^I),
\end{aligned}$$

and:

$$\begin{aligned}
&\rho(n_b)\circ\rho(n_a)(m^i)\\
&=\rho(n_b)\left(\mathbf{r}_i^{I,a}m^I\right)\\
&=\mathbf{r}_{i'}^{I',b}\mathbf{r}_i^{I,a}m^{I'}\odot m_{i'}(m^I),
\end{aligned}$$

which proves that:

$$\rho^{\triangle}\left(\rho^{\triangle}(m^i)\right)(\downarrow n_a\otimes\downarrow n_b)=\frac{1}{2}\rho(n_b)\circ\rho(n_a)(m^i)-\frac{1}{2}\rho(n_a)\circ\rho(n_b)(m^i).$$

□

Proof of II.2.2-4:

Given any j,a_1,a_2,a_3 in $\mathbb{J}$, the left term reads as follows:

$$\begin{aligned}
&\rho^{\triangle}\left([\bullet,\bullet]_N^{\triangle}(n^j)\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&=\rho^{\triangle}\left(\mathbf{b}_j^{I,j_1,j_2}m^I\odot(\uparrow n^{j_1}\odot\uparrow n^{j_2})\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&=\mathbf{r}_i^{I',j_3}\mathbf{b}_j^{I,j_1,j_2}\left(m^{I'}\odot\uparrow n^{j_3}\odot m_i(m^I)\odot(\uparrow n^{j_1}\odot\uparrow n^{j_2})\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&=m^{I'}\odot m_i(m^I)\otimes\left(\sum_{\sigma\in\mathfrak{S}_3}\frac{(-1)^{\sigma}}{6}\mathbf{r}_i^{I',j_1}\mathbf{b}_j^{I,j_2,j_3}\uparrow n^{j_{\sigma(1)}}\otimes\uparrow n^{j_{\sigma(2)}}\otimes\uparrow n^{j_{\sigma(3)}}\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&=\left(\sum_{j_1,j_2,j_3\in\mathbb{J}}\sum_{\sigma\in\mathfrak{S}_3}-\frac{(-1)^{\sigma}}{6}\mathbf{r}_i^{I',j_1}\mathbf{b}_j^{I,j_2,j_3}\delta_{a_1}^{j_{\sigma(1)}}\delta_{a_2}^{j_{\sigma(2)}}\delta_{a_3}^{j_{\sigma(3)}}\right)m^{I'}\odot m_i(m^I)\\
&=\left(\sum_{\sigma\in\mathfrak{S}_3}-\frac{(-1)^{\sigma}}{6}\mathbf{r}_i^{I',a_{\sigma(1)}}\mathbf{b}_j^{I,a_{\sigma(2)},a_{\sigma(3)}}\right)m^{I'}\odot m_i(m^I).
\end{aligned}$$

While given any $\sigma \in \mathfrak{S}_3$, we have:

$$\begin{aligned}
&\rho(n_{a_{\sigma(1)}})\left([n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N\right)(n^j)\\
&= \rho(n_{a_{\sigma(1)}})\left(\mathbf{b}_{j_3}^{I,a_{\sigma(2)},a_{\sigma(3)}} m^I \odot n_{j_3}\right)(n^j)\\
&= \mathbf{r}_i^{I',a_{\sigma(1)}}\mathbf{b}_j^{I,a_{\sigma(2)},a_{\sigma(3)}} m^{I'} \odot m_i(m^I),
\end{aligned}$$

which together proves that:

$$\rho^{\triangle}\left([\bullet,\bullet]_N^{\triangle}(n^j)\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3}) = \sum_{\sigma\in\mathfrak{S}_3} -\frac{(-1)^{\sigma}}{6}\rho(n_{a_{\sigma(1)}})\left([n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^j)\right). \quad \square$$

Proof of II.2.2-5:

Given any j, a_1, a_2, a_3 in $\mathbb{J}$, the left-hand term reads as follows:

$$\begin{aligned}
&([\bullet,\bullet]_N^{\triangle}\circ\downarrow)\left([\bullet,\bullet]_N^{\triangle}(n^j)\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&= ([\bullet,\bullet]_N^{\triangle}\circ\downarrow)\left(\mathbf{b}_j^{I,j_1,j_2} m^I\odot(\uparrow n^{j_1}\odot\uparrow n^{j_2})\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&= \left(\mathbf{b}_j^{I,j_1,j_2} m^I\odot\left([\bullet,\bullet]_N^{\triangle}(n^{j_1})\odot\uparrow n^{j_2} - \uparrow n^{j_1}\odot[\bullet,\bullet]_N^{\triangle}(n^{j_2})\right)\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&= \left(\mathbf{b}_j^{I,j_1,j_2} m^I\odot\left(\mathbf{b}_{j_1}^{I',j_3,j_4} m^{I'}\odot\uparrow n^{j_3}\odot\uparrow n^{j_4}\odot\uparrow n^{j_2} - \uparrow n^{j_1}\odot\mathbf{b}_{j_2}^{I'',j_5,j_6} m^{I''}\odot\uparrow n^{j_5}\odot\uparrow n^{j_6}\right)\right)\\
&\qquad(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&= \left(\mathbf{b}_j^{I,k,j_3}\mathbf{b}_k^{I',j_1,j_2} - \mathbf{b}_j^{I,j_1,k}\mathbf{b}_k^{I',j_2,j_3}\right) m^I\odot m^{I'}\otimes\left(\sum_{\sigma\in\mathfrak{S}_3}\frac{(-1)^{\sigma}}{6}\uparrow n^{j_{\sigma(1)}}\otimes\uparrow n^{j_{\sigma(2)}}\otimes\uparrow n^{j_{\sigma(3)}}\right)\\
&\qquad(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\
&= \sum_{\sigma\in\mathfrak{S}_3} -\frac{(-1)^{\sigma}}{6}\delta_{a_1}^{j_{\sigma(1)}}\delta_{a_2}^{j_{\sigma(2)}}\delta_{a_3}^{j_{\sigma(3)}}\left(\mathbf{b}_j^{I,k,j_3}\mathbf{b}_k^{I',j_1,j_2} - \mathbf{b}_j^{I,j_1,k}\mathbf{b}_k^{I',j_2,j_3}\right) m^I\odot m^{I'}\\
&= \sum_{\sigma\in\mathfrak{S}_3} -\frac{(-1)^{\sigma}}{6}\left(\mathbf{b}_j^{I,k,a_{\sigma(3)}}\mathbf{b}_k^{I',a_{\sigma(1)},a_{\sigma(2)}} - \mathbf{b}_j^{I,a_{\sigma(1)},k}\mathbf{b}_k^{I',a_{\sigma(2)},a_{\sigma(3)}}\right) m^I\odot m^{I'}\\
&= \sum_{\sigma\in\mathfrak{S}_3}\frac{(-1)^{\sigma}}{6}2.\mathbf{b}_j^{I,a_{\sigma(1)},k}\mathbf{b}_k^{I',a_{\sigma(2)},a_{\sigma(3)}} m^I\odot m^{I'}.
\end{aligned}$$

And given an element σ in $\mathfrak{S}_3$ the the right-hand reads as follows:

$$\begin{aligned}
&\sum_{k\in\mathbb{J}}[n_{a_{\sigma(2)}}, n_{a_{\sigma(3)}}]_N(n^k)\odot[n_{a_{\sigma(1)}}, n_k]_N(n^j)\\
&= \sum_{k\in\mathbb{J}}\mathbf{b}_k^{I,a_{\sigma(2)},a_{\sigma(3)}} m^I\odot\mathbf{b}_j^{I',a_{\sigma(1)},k} m^{I'}\\
&= \sum_{k\in\mathbb{J}}\mathbf{b}_j^{I,a_{\sigma(1)},k}\mathbf{b}_k^{I',a_{\sigma(2)},a_{\sigma(3)}} m^I\odot m^{I'}
\end{aligned}$$

which together proves that:

$$\begin{aligned}([\bullet,\bullet]_N^{\triangle}\circ\downarrow)\left([\bullet,\bullet]_N^{\triangle}(n^j)\right)(\downarrow n_{a_1}\otimes\downarrow n_{a_2}\otimes\downarrow n_{a_3})\\ =\sum_{\substack{\sigma\in\mathfrak{S}_3\\ k\in\mathbb{J}}}\frac{(-1)^{\sigma}}{3}[n_{a_{\sigma(2)}},n_{a_{\sigma(3)}}]_N(n^k)\odot[n_{a_{\sigma(1)}},n_k]_N(n^j).\end{aligned}$$

□

II.3 Deformation algebras

II.3.1 Formal deformation of Lie-Rinehart pairs

We first consider a Lie-Rinehart pair $(R, \nabla_R, \iota_R, L, [\bullet,\bullet]_L, \phi_L, \rho)$ and to properly make use of Theorem I.1.2-5 we consider $\mathbb{K}[\![\hbar]\!]$ the $\mathbb{Z}$-graded $\mathbb{K}$-module concentrated in degree 0. In the context of this theorem, one could in a straightforward way define a $\mathbb{K}[\![\hbar]\!]$-linear extension of all the structure maps and get another Lie-Rinehart pair, we will not do it here and instead we set some of the structure maps to have an $\hbar$-dependent coefficient:

$$R_\hbar := R \otimes \mathbb{K}[\![\hbar]\!] \qquad L_\hbar := L \otimes \mathbb{K}[\![\hbar]\!]$$

$$\nabla_{R_\hbar} := (\nabla_R \otimes \nabla_{\mathbb{K}[\![\hbar]\!]}) \circ (id_R \otimes \sigma_{\mathbb{K}[\![\hbar]\!],R} \otimes id_{\mathbb{K}[\![\hbar]\!]}) \qquad \iota_{R_\hbar} := \iota_R \otimes 1$$

$$[\bullet,\bullet]_{L_\hbar} := ([\bullet,\bullet]_L \otimes \hbar.\nabla_{\mathbb{K}[\![\hbar]\!]}) \circ (id_L \otimes \sigma_{\mathbb{K}[\![\hbar]\!],L} \otimes id_{\mathbb{K}[\![\hbar]\!]})$$

$$\phi_{L_\hbar} := (\phi_L \otimes \nabla_{\mathbb{K}[\![\hbar]\!]}) \circ (id_R \otimes \sigma_{\mathbb{K}[\![\hbar]\!],L} \otimes id_{\mathbb{K}[\![\hbar]\!]})$$

$$\rho_\hbar := \rho \otimes \hbar \in \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(L, Der(R)) \otimes \mathbb{K}[\![\hbar]\!]$$

And we see $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(L, Der(R)) \otimes \mathbb{K}[\![\hbar]\!]$ as the subset of graded $\mathbb{K}[\![\hbar]\!]$-linear morphisms in $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{K}}^{\mathbb{Z}}}(L_\hbar, Der(R_\hbar))$ which takes value in the set of $\mathbb{K}[\![\hbar]\!]$-linear derivations of $R_\hbar$.

— Proposition II.3.1-1 :

$(R_\hbar, \nabla_{R_\hbar}, \iota_{R_\hbar}, L_\hbar, [\bullet,\bullet]_{L_\hbar}, \phi_{L_\hbar}, \rho_\hbar)$, is a Lie-Rinehart pair, called the *formal deformation* of the Lie-Rinehart pair $(R, \nabla_R, \iota_R, L, [\bullet,\bullet]_L, \phi_L, \rho)$.

Proof :

Straightforward since $(R, \nabla_R, \iota_R, L, [\bullet,\bullet]_L, \phi_L, \rho)$ is a Lie-Rinehart pair and the new morphisms are $\mathbb{K}[\![\hbar]\!]$-linear. □

Accordingly, we consider the formal deformation, $(M_\hbar, N_\hbar, [\bullet,\bullet]_{L_\hbar}, \rho_\hbar)$, of the local Lie algebroid defined in Section II.2.1, and we apply to it the $\mathbb{K}[\![\hbar]\!]$-linear extension of the construction of [Cal+11], giving the following setting:

$$\begin{aligned}
\mathrm{T}^\bullet_{\mathrm{poly}}(X)_\hbar &= \mathrm{S}((M \oplus N[1])^*) \otimes \mathrm{S}(M[-1] \oplus N)[1] \otimes \mathbb{K}[\![\hbar]\!],\\
A_\hbar &= \mathrm{S}((M \oplus N[1])^*) \otimes \mathbb{K}[\![\hbar]\!],\\
B_\hbar &= \mathrm{S}(M^*) \otimes \mathrm{S}(N) \otimes \mathbb{K}[\![\hbar]\!],\\
K_\hbar &= \mathrm{S}(M^*) \otimes \mathbb{K}[\![\hbar]\!].
\end{aligned}$$

Hence, using a $\mathbb{K}[\![\hbar]\!]$-linear extension of the $\mathscr{L}_\infty$-quasi-isomorphism of [Cal+11] one gets the map:

$$\begin{array}{cccl}
\mathfrak{L}_\hbar & : & \mathrm{T}^\bullet_{\mathrm{poly}}(X) \otimes \mathbb{K}[\![\hbar]\!] & \to \ \mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K), \mathrm{Cat}_\infty(A,B,K)) \otimes \mathbb{K}[\![\hbar]\!] \\
 & & x \otimes p & \mapsto \ \sum_{n \in \mathbb{N}^*} \frac{1}{n!} \mathfrak{L}^{(n)}(x, \cdots, x) \otimes p^n
\end{array},$$

which, according to Theorem I.1.2-5 together with a limit argument, induces a bijection between the sets of equivalence classes of Maurer-Cartan elements:

$$\mathfrak{L}_\hbar \; : \; \mathscr{MC}(\mathrm{T}^\bullet_{\mathrm{poly}}(X)\otimes\mathbb{K}[\![\hbar]\!]) \;\to\; \mathscr{MC}(\mathcal{C}^\bullet(\mathrm{Cat}_\infty(A,B,K),\mathrm{Cat}_\infty(A,B,K))\otimes\mathbb{K}[\![\hbar]\!])$$

In [Cal+11, §8], the authors studied the deformation quantization of quadratic Koszul algebras, using an $\hbar$-dependant bivector field π in the case of a trivial decomposition: a total brane and a zero brane. Here the situation is more general as we will have a non trivial pair of branes, and we are interested in the deformation quantization of the symmetric algebras A and B under a specific $\hbar$-dependant Maurer-Cartan element, that we present:

— Proposition II.3.1-2 :

Let $d = \rho^\triangle - \frac{1}{2}[\bullet,\bullet]^\triangle_N \circ \downarrow$ be as in Theorem II.2.2-1 and define:

$$d_\hbar := d \otimes \hbar$$

Then $d_\hbar$ is a Maurer-Cartan element of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)_\hbar$.

Proof :

The proof is a direct computation since d is a Maurer-Cartan element of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$ and everything is $\mathbb{K}[\![\hbar]\!]$-linearily extended, it amounts to showing that:

$$[d_\hbar, d_\hbar] = 0$$

□

We will show that the image of $d_\hbar$ under the $\mathscr{L}_\infty$-morphism $\mathfrak{L}_\hbar$ deforms the algebras A and B into DG-algebras which are respectively isomorphic to the Chevalley-Eilenberg algebra and the universal enveloping algebra of the formal deformation $(M_\hbar, N_\hbar, [\bullet,\bullet]_{L_\hbar}, \rho_\hbar)$.

II.3.2 THE CHEVALLEY-EILENBERG ALGEBRA

The Chevalley-Eilenberg algebra of a Lie algebra is a commutative DG-algebra made up of wedge products of duals whose differential encodes the Lie bracket. Similarly, the Chevalley-Eilenberg algebra of a Lie-Rinehart pair (R,L) is the commutative DG-algebra of anti-symmetric R-multilinear maps whose differential encodes both Lie bracket and anchor map.

Definition II.3.2-1 : Chevalley-Eilenberg algebra of a Lie-Rinehart pair

The Chevalley-Eilenberg algebra of a Lie-Rinehart pair, $(R,\nabla_R,\iota_R,L,[\bullet,\bullet]_L,\phi_L,\rho)$, where R and L are concentrated in degree 0 is the DG-algebra, $C^\bullet(L,R)$, of graded anti-symmetric R-multilinear morphisms between L and R:

$$C^\bullet(L,R) := \mathcal{A}lt_R(L,R) = \mathrm{Hom}_R(\wedge_R^\bullet L,R)$$

The grading is set to be the arity of the morphisms and, given an homogeneous morphism f of degree n, the differential $d_{CE}(f)$ on any $n+1$ elements $l_0,\cdots,l_n \in L$ is:

$$\begin{aligned} d_{CE}(f)(l_0,\cdots,l_n) := & \sum_{\sigma\in \mathrm{Shuff}_{(1,n)}} (-1)^\sigma \rho(l_{\sigma(0)})\left(f(l_{\sigma(1)},\cdots,l_{\sigma(n)})\right) \\ & - \sum_{\sigma\in \mathrm{Shuff}_{(2,n-1)}} (-1)^\sigma f([l_{\sigma(0)},l_{\sigma(1)}]_L,l_{\sigma(2)},\cdots,l_{\sigma(n)}) \end{aligned}$$

To any pair of homogeneous morphisms f and g of degree n and m, their product $f\wedge g$ is set on any $n+m$ elements $l_1,\cdots,l_{n+m}\in L$ by:

$$(f\wedge g)(l_1,\cdots,l_{n+m}) := \frac{1}{n!m!}\sum_{\sigma\in\mathfrak{S}_{n+m}} (-1)^\sigma f(l_{\sigma(1)},\cdots,l_{\sigma(n)})g(l_{\sigma(n+1)},\cdots,l_{\sigma(n+m)})$$

We now apply the $\mathfrak{L}_{A_\hbar}$ part of the $\mathbb{K}[\![\hbar]\!]$-linear extension of the $\mathscr{L}_\infty$-morphism from [Cal+11] to the Maurer-Cartan element $d_\hbar$ and show that it is rather simple to compute.

— Proposition II.3.2-2 :

Let d be as in Theorem II.2.2-1, then:

$$\mathfrak{L}_{A_\hbar}(d_\hbar) = d\otimes\hbar$$

Proof :

In regard of Theorems I.1.2-5 and I.2.3-1, we have the following definitions:

$$\mathfrak{L}_{A_\hbar}(d_\hbar) = \sum_{n\in\mathbb{N}^*}\frac{1}{n!}\mathfrak{L}_A^{(n)}(d\otimes\cdots\otimes d)\otimes\hbar^n,$$

where, for all $n \in \mathbb{N}^*$,

$$\mathfrak{L}_A^n(d \otimes \cdots \otimes d)_{|A^{\otimes m}}$$

$$= \sum_{\Gamma \in \mathcal{G}_{n,m}} \uparrow \circ \mu_{n+m}^{A[1]} \circ \left(\int_{\mathscr{C}_{n,m}^+} \prod_{e \in E(\Gamma)} \omega_e^A \right) \circ (id_{\mathrm{T}_{\mathrm{poly}}^\bullet(X)^{\otimes n}} \otimes \downarrow^{\otimes m})(d \otimes \cdots \otimes d \otimes \bullet)$$

In order to compute this morphism, we need to characterize the graphs Γ whose contributions to the sum do not trivially vanish. To do so, we consider $n \in \mathbb{N}^*$, $m \in \mathbb{N}$ and a graph $\Gamma \in \mathcal{G}_{n,m}$ such that, there exist $a_1, \cdots, a_m \in A$ satisfying:

$$\mu_{n+m}^{A[1]} \circ \left(\int_{\mathscr{C}_{n,m}^+} \prod_{e \in E(\Gamma)} \omega_e^A \right) (d \otimes \cdots \otimes d \otimes \downarrow a_1 \otimes \cdots \otimes \downarrow a_m) \neq 0.$$

First, since ω_e^A is a 1-form we know that the integral will vanish on the integrands coming from graphs whose number of edges is different than $dim(\mathscr{C}_{n,m}^+)$, thus, the graph Γ is compelled to satisfy:

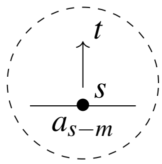
$$\#E(\Gamma) = 2n + m - 2.$$

Then we consider the sets $\mathbb{I}$ and $\mathbb{J}$ as in the proof of Theorem II.2.2-1, that is $\{m_i\}_{i \in \mathbb{I}} \in M^{\mathbb{I}}$ (resp. $\{n_j\}_{j \in \mathbb{J}} \in N^{\mathbb{J}}$) is a basis of M (resp. N) and $\{m^i\}_{i \in \mathbb{I}}$ (resp. $\{n^j\}_{j \in \mathbb{J}}$) is the dual basis of M^* (resp. N^*). To match with the decomposition in Equation (I.2), we consider the set $\mathbb{I} \cup \mathbb{J}$ with a bijection onto $[d]$ and we will thus consider that $I_1 := \mathbb{I} \cup \mathbb{J}$ and $I_2 := \mathbb{I}$, which means that:

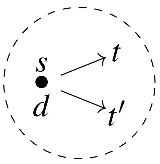
$$\omega_e^A = \pi_e^*(\omega^+) \otimes (\tau_e^{\mathbb{I}} + \tau_e^{\mathbb{J}}).$$

Now recall that the x_k in ∂_{x_k} and $\iota_{\mathrm{d}x_k}$ in the definitions of $\tau_e^{\mathbb{I}}$ and $\tau_e^{\mathbb{J}}$ must be linear coordinates on $X = M \oplus N[1]$, thus we simply have to take $x_k := m^k$ if $k \in \mathbb{I}$ and $\uparrow n^k$ else.

Let $(s,t) \in E(\Gamma)$ be an edge defined by two numbers $1 \leq s,t \leq n+m$. If $s > m$, this edge will involve terms of the form $\iota_{\mathrm{d}x_k}(a_{s-m})$ which vanish since $a_{s-m} \in \mathrm{S}(M^* \oplus N^*[-1])$.



(s,t) with $s > m$



$(s,t),(s,t')$ with $s \leq m$

Similarly, any pair of edges with common starting point $(s,t),(s,t')$ such that $s \leq m$ involve terms of the form $\iota_{\mathrm{d}x_{k_1}} \circ \iota_{\mathrm{d}x_{k_2}}(d)$ which also vanish since $d \in \mathrm{S}(M^* \oplus N^*[-1]) \otimes (M \oplus N[1])$.

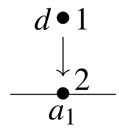
In addition, the whole computation ends with a projection onto A, so each vertex of the first type, which are all colored with a d element, must have at least one edge going out of them in order to have a non zero contribution. Now since the number of edges must be equal to the sum of edges going out the vertices of first type ($= n$) and those going out the vertices of second type ($= 0$), we are left with a second relation that Γ is compelled to satisfy:

$$\#E(\Gamma) = n.$$

Then, we are left with the property that if the graph Γ has a non zero contribution, then $2n+m-2 = n$, thus $n+m = 2$. And we know compute the contributions of the only two graphs satisfying these relations.

- $(n,m) = (1,1)$: There is only one graph, Γ_0, whose contribution is not trivially set to be zero. And we can compute its contribution as follows:

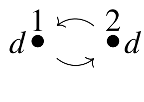
$\Gamma_0 :$



$$\begin{aligned}
&\mu_2^{A[1]} \circ \left(\int_{\mathscr{C}_{1,1}^+} \prod_{e \in E(\Gamma_0)} \omega_e^A \right) (d \otimes \downarrow a_1) \\
&= \left(\int_{\mathscr{C}_{1,1}^+} \pi_{(1,2)}^*(\omega^+) \right) \mu_2^{A[1]} \circ \left(\tau_{(1,2)}^{\mathbb{I}} + \tau_{(1,2)}^{\mathbb{J}} \right) (d \otimes \downarrow a_1) \\
&= \left(\int_{[0;1]} \frac{1}{2\pi} \frac{\partial}{\partial \theta} \arg\left(\frac{e^{i\pi\theta}}{e^{-i\pi\theta}} \right) \mathrm{d}\theta \right) \mu_2^{A[1]} \circ \left(\sum_{k \in \mathbb{I}} \iota_{\mathrm{d}x_k} \otimes \partial_{x_k} + \sum_{k \in \mathbb{J}} \iota_{\mathrm{d}x_k} \otimes \partial_{x_k} \right) (d \otimes \downarrow a_1) \\
&= \mu_2^{A[1]} \circ \left(\sum_{i \in \mathbb{I}} \iota_{\mathrm{d}m^i}(d) \otimes \partial_{m^i}(\downarrow a_1) + \sum_{j \in \mathbb{J}} \iota_{\mathrm{d}\uparrow n^j}(d) \otimes \partial_{\uparrow n^j}(\downarrow a_1) \right) \\
&= \downarrow d(a_1).
\end{aligned}$$

- $(n,m) = (2,0)$: There is also one graph, Γ_1, whose contribution is not trivially computed to be zero. But its coefficient vanishes:

$\Gamma_1 :$



$$\begin{aligned}
&\mu_2^{A[1]} \circ \left(\int_{\mathscr{C}_{2,0}^+} \prod_{e \in E(\Gamma_1)} \omega_e^A \right) (d \otimes d) \\
&= \left(\int_{\mathscr{C}_{2,0}^+} \pi_{(1,2)}^*(\omega^+) \wedge \pi_{(2,1)}^*(\omega^+) \right) \mu_2^{A[1]} \circ \left(\tau_{(1,2)}^{\mathbb{I}} + \tau_{(1,2)}^{\mathbb{J}} \right) \circ \left(\tau_{(2,1)}^{\mathbb{I}} + \tau_{(2,1)}^{\mathbb{J}} \right) (d \otimes d) \\
&= \left(\int_{\mathbb{H} \setminus \{i\}} \omega^+ . \omega^-(i,z) - \omega^+ . \omega^-(z,i) \mathrm{d}z \right) \mu_2^{A[1]} \circ \left(\tau_{(1,2)}^{\mathbb{I}} + \tau_{(1,2)}^{\mathbb{J}} \right) \circ \left(\tau_{(2,1)}^{\mathbb{I}} + \tau_{(2,1)}^{\mathbb{J}} \right) (d \otimes d) \\
&= 0.
\end{aligned}$$

We shall note that, in the context of $\mathfrak{L}_A$, graphs with loops do not appear because their contribution is defined to be zero. Hence, by appealing all the previous computations and using the fact that d is an element of degree 0 in $\mathrm{T}^{\bullet}_{\mathrm{poly}}(X)$, we get:

$$\mathfrak{L}_{A_\hbar}(d_\hbar) := \mathfrak{L}^{(1)}_A(d) \otimes \hbar = d \otimes \hbar.$$

□

We now state one of the two main theorems, which says that the deformed algebra $A_\hbar$ obtained from the deformation quantization of A, is actually the Chevalley-Eilenberg algebra of the formal deformation of the local Lie algebroid.

Recall that $\{m_i\}_{i\in\mathbb{I}} \in M^{\mathbb{I}}$ (resp. $\{n_j\}_{j\in\mathbb{J}} \in N^{\mathbb{J}}$) stands for a basis of M (resp. N), $\{m^i\}_{i\in\mathbb{I}}$ (resp. $\{n^j\}_{j\in\mathbb{J}}$) stands for the dual basis and given a multi-index $I \in \mathrm{Multi}(\mathbb{I})$ we denote by m_I the tensor $m_{I_1} \odot \cdots \odot m_{I_{\#I}}$, with similar notations for N and duals and that we assume Einstein notation, then:

Theorem II.3.2-3 :

Given the formal deformation of a local Lie algebroid, $(M_\hbar, N_\hbar, [\bullet,\bullet]_{L_\hbar}, \rho_\hbar)$, and the setting of Lie algebroids as coisotropic branes of II.3.1. The following is an isomorphism of DG-$\mathbb{K}[\![\hbar]\!]$-algebras:

$$\begin{array}{cccc} \mathfrak{I}_{A_\hbar} & : & (C^\bullet(L_\hbar,R_\hbar), d_{CE}, \bullet\wedge\bullet) & \to & (A_\hbar, \mathfrak{L}_{A_\hbar}(d_\hbar), \nabla_{A_\hbar}) \\ & & C^p(L_\hbar,R_\hbar) \ni f & \mapsto & \frac{1}{p!}\mathbf{f}^{I,(j_1,\cdots,j_p)} m^I \otimes \uparrow n^{j_1} \odot \cdots \odot \uparrow n^{j_p} \end{array}$$

where the coefficient $\mathbf{f}^{I,(j_1,\cdots,j_p)} \in \mathbb{K}[\![\hbar]\!]$ is set by the $\mathbb{K}[\![\hbar]\!]$-linear extension of the tensor pairing on $\mathrm{S}(M^*)$:

$$\mathbf{f}^{I,(j_1,\cdots,j_p)} := < m_I, f(n_{j_1},\cdots,n_{j_p}) >,$$

and where $\nabla_{A_\hbar}$ is the $\mathbb{K}[\![\hbar]\!]$-linear extension of the product of A.

Proof of II.3.2-3:

Since we are dealing with local Lie algebroids, we have $L_\hbar = \mathrm{S}(M^*) \otimes N \otimes \mathbb{K}[\![\hbar]\!]$ and $R_\hbar = \mathrm{S}(M^*) \otimes \mathbb{K}[\![\hbar]\!]$, thus an element of the Chevalley-Eilenberg algebra is completely defined by its values on elements of N, and as a $\mathbb{Z}$-graded $\mathbb{K}$-module we get the following isomorphism.

$$C^\bullet(L_\hbar,R_\hbar) \cong \mathbb{K}[\![\hbar]\!] \otimes \underline{\mathrm{Hom}}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(\mathrm{S}(N[-1]), \mathrm{S}(M^*))$$

Now since $A_\hbar = \mathrm{S}((M \oplus N[1])^*) \otimes \mathbb{K}[\![\hbar]\!] \cong \mathbb{K}[\![\hbar]\!] \otimes \mathrm{S}(M^*) \otimes \mathrm{S}(N^*[-1])$, it is identified as tensor equivalent to the $\mathbb{Z}$-graded $\mathbb{K}$-module $\mathbb{K}[\![\hbar]\!] \otimes \underline{\mathrm{Hom}}_{\mathbf{Mod}^{\mathbb{Z}}_{\mathbb{K}}}(\mathrm{S}(N[1]), \mathrm{S}(M^*))$. Given an element f in $C^p(L_\hbar,R_\hbar)$ it is then decomposed in $A_\hbar$ using coefficients $\mathbf{f}^{I,(j_1,\cdots,j_p)}$ (and by taking care of the suspensions) as:

$$f = (-1)^{p(p-1)/2}\mathbf{f}^{I,(j_1,\cdots,j_p)} m^I \otimes \uparrow n^{j_1} \odot \cdots \odot \uparrow n^{j_p}$$

Thus, the morphism $\mathfrak{I}_{A_\hbar}$ is an isomorphism of $\mathbb{Z}$-graded $\mathbb{K}$-modules, and up to some renormalization factors it is the canonical between hom sets and tensors.

We are then only left with the proof that $\mathfrak{I}_{A_\hbar}$ is a morphism of commutative DG-algebra, and as a preliminary remark, note that for any $f \in C^p(L_\hbar, R_\hbar)$ and any $a_1, \cdots, a_p$ in $\mathbb{J}$ we have:

$$\mathfrak{I}_{A_\hbar}(f)(\downarrow n_{a_1} \otimes \cdots \otimes \downarrow n_{a_p}) = \frac{(-1)^{p(p-1)/2}}{p!} f(n_{a_1}, \cdots, n_{a_p})$$

Firstly, consider a pair of homogeneous morphisms f and g in $C^\bullet(L_\hbar, R_\hbar)$ of degree p and q, and any $a_1, \cdots, a_{p+q} \in \mathbb{J}$, since $f \wedge g$ is also an anti-symmetric morphism we have the following relation:

$$\begin{aligned}
&\mathfrak{I}_{A_\hbar}(f \wedge g)(\downarrow n_{a_1} \otimes \cdots \otimes \downarrow n_{a_{p+q}}) \\
&= \frac{(-1)^{(p+q)(p+q-1)/2}}{(p+q)!} (f \wedge g)(n_{a_1}, \cdots, n_{a_{p+q}}) \\
&= \sum_{\sigma \in \mathfrak{S}_{p+q}} \frac{(-1)^\sigma (-1)^{(p+q)(p+q-1)/2}}{(p+q)!p!q!} f(n_{a_{\sigma(1)}}, \cdots, n_{a_{\sigma(p)}}) g(n_{a_{\sigma(p+1)}}, \cdots, n_{a_{\sigma(p+q)}}) \\
&= \sum_{\sigma \in \mathfrak{S}_{p+q}} \frac{(-1)^\sigma (-1)^{(p+q)(p+q-1)/2}}{(p+q)!p!q!} \mathbf{f}^{I',(a_{\sigma(1)}, \cdots, a_{\sigma(p)})} \mathbf{g}^{I'',(a_{\sigma(p+1)}, \cdots, a_{\sigma(p+q)})} m^{I'} \odot m^{I''},
\end{aligned}$$

and, if we compare it to,

$$\begin{aligned}
&\nabla_{A_\hbar}(\mathfrak{I}_{A_\hbar}(f), \mathfrak{I}_{A_\hbar}(g))(\downarrow n_{a_1} \otimes \cdots \otimes \downarrow n_{a_{p+q}}) \\
&= \frac{1}{p!q!} \mathbf{f}^{I,(j_1, \cdots, j_p)} \mathbf{g}^{I',(j_{p+1}, \cdots, j_{p+q})} m^I \odot m^{I'} \otimes (\uparrow n^{j_1} \odot \cdots \odot \uparrow n^{j_{p+q}})(\downarrow n_{a_1} \otimes \cdots \otimes \downarrow n_{a_{p+q}}) \\
&= \sum_{\sigma \in \mathfrak{S}_{p+q}} \frac{(-1)^\sigma (-1)^{(p+q)(p+q-1)/2}}{(p+q)!p!q!} \mathbf{f}^{I,(a_{\sigma(1)}, \cdots, a_{\sigma(p)})} \mathbf{g}^{I',(a_{\sigma(p+1)}, \cdots, a_{\sigma(p+q)})} m^I \odot m^{I'}.
\end{aligned}$$

We get $\nabla_{A_\hbar} \circ (\mathfrak{I}_{A_\hbar} \otimes \mathfrak{I}_{A_\hbar}) = \mathfrak{I}_{A_\hbar} \circ (\bullet \wedge \bullet)$, hence $\mathfrak{I}_{A_\hbar}$ is a morphism of algebras.

Secondly, consider an homogeneous morphism f in $C^p(L_\hbar, R_\hbar)$ and any $a_0, \cdots, a_p \in \mathbb{J}$, then we can unroll the following equations:

$$\begin{aligned}
&\mathfrak{I}_{A_\hbar}(d_{CE}(f))(\downarrow n_{a_0} \otimes \cdots \otimes \downarrow n_{a_p}) \\
&= \frac{(-1)^{p(p+1)/2}}{(p+1)!} d_{CE}(f)(n_{a_0}, \cdots, n_{a_p}) \\
&= \sum_{0 \le i \le p} \frac{(-1)^{i+p(p+1)/2}}{(p+1)!} \rho_\hbar(n_{a_i}) \left(f(n_{a_0}, \cdots, \hat{n_{a_i}}, \cdots, n_{a_p}) \right) \\
&\quad + \sum_{0 \le i < j \le p} \frac{(-1)^{i+j+p(p+1)/2}}{(p+1)!} f([n_{a_i}, n_{a_j}]_{L_\hbar}, n_{a_0}, \cdots, \hat{n_{a_i}}, \cdots, \hat{n_{a_j}}, \cdots, n_{a_p}) \\
&= \sum_{0 \le i \le p} \frac{(-1)^{i+p(p+1)/2}}{(p+1)!} \hbar \mathbf{r}_{k_1}^{I,a_i} m^I \odot m_{k_1} \left(\mathbf{f}^{I',(a_0,\cdots,\hat{a}_i,\cdots a_p)} m^{I'} \right) \\
&\quad + \sum_{0 \le i < j \le p} \frac{(-1)^{i+j+p(p+1)/2}}{(p+1)!} \hbar f \left(\mathbf{b}_{k_2}^{I'',a_i,a_j} m^{I''} \otimes n_{k_2}, n_{a_0}, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots, a_p \right) \\
&= \sum_{0 \le i \le p} \frac{(-1)^{i+p(p+1)/2}}{(p+1)!} \hbar \mathbf{r}_{k_1}^{I,a_i} \mathbf{f}^{I',(a_0,\cdots,\hat{a}_i,\cdots a_p)} m^I \odot m_{k_1} \left(m^{I'} \right) \\
&\quad + \sum_{0 \le i < j \le p} \frac{(-1)^{i+j+p(p+1)/2}}{(p+1)!} \hbar \mathbf{b}_{k_2}^{I'',a_i,a_j} \mathbf{f}^{I''',(k_2,a_0,\cdots,\hat{n_{a_i}},\cdots,\hat{n_{a_j}},\cdots,n_{a_p})} m^{I''} \odot m^{I'''},
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{L}_{A_\hbar}(d_\hbar)(\mathfrak{I}_{A_\hbar}(f))(\downarrow n_{a_0} \otimes \cdots \otimes \downarrow n_{a_p}) \\
&= \hbar(\rho^\triangle - \frac{1}{2}[\bullet,\bullet]_N^\triangle \circ \downarrow) \left(\frac{1}{p!} \mathbf{f}^{I,(j_1,\cdots,j_p)} m^I \otimes (\uparrow n^{j_1} \odot \cdots \odot \uparrow n^{j_p}) \right) (\downarrow n_{a_0} \otimes \cdots \otimes \downarrow n_{a_p}) \\
&= \Bigg(\hbar \frac{1}{p!} \mathbf{f}^{I,(j_1,\cdots,j_p)} \mathbf{r}_{k_1}^{I',j_0} m^{I'} \odot m_{k_1}(m^I) \otimes (\uparrow n^{j_0} \odot \cdots \odot \uparrow n^{j_p}) \\
&\quad + \sum_{1 \le i \le p} \hbar \frac{(-1)^{1+(i-1)}}{2.p!} \mathbf{f}^{I,(j_1,\cdots,j_p)} \mathbf{b}_{j_i}^{I'',s_1,s_2} m^I \odot m^{I''} \otimes \\
&\qquad\qquad (\uparrow n^{s_1} \odot \uparrow n^{s_2} \odot \uparrow n^{j_1} \odot \cdots \odot \widehat{\uparrow n^{j_i}} \odot \cdots \odot \uparrow n^{j_p}) \Bigg) (\downarrow n_{a_0} \otimes \cdots \otimes \downarrow n_{a_p}) \\
&= \sum_{\sigma \in \mathfrak{S}_{p+1}} \hbar \frac{(-1)^\sigma (-1)^{p(p+1)/2}}{p!(p+1)!} \mathbf{f}^{I,(a_{\sigma(1)},\cdots,a_{\sigma(p)})} \mathbf{r}_{k_1}^{I',a_{\sigma(0)}} m^{I'} \odot m_{k_1}(m^I) \\
&\quad + \sum_{1 \le i \le p} \hbar \frac{-1}{2.p!} \mathbf{f}^{I,(k,j_1,\cdots,\hat{j}_i,\cdots,j_p)} \mathbf{b}_k^{I'',s_1,s_2} m^I \odot m^{I''} \\
&\qquad\qquad (\uparrow n^{s_1} \odot \uparrow n^{s_2} \odot \uparrow n^{j_1} \odot \cdots \odot \widehat{\uparrow n^{j_i}} \odot \cdots \odot \uparrow n^{j_p})(\downarrow n_{a_0} \otimes \cdots \otimes \downarrow n_{a_p})
\end{aligned}$$

$$= \sum_{0\leq i\leq p} \hbar \frac{(-1)^{i+p(p+1)/2}}{p!(p+1)!} \mathbf{f}^{I,(a_0,\cdots,\hat{a}_i,\cdots,a_p)} \mathbf{r}_{k_1}^{I',a_i} m^{I'} \odot m_{k_1}(m^I)$$
$$+ \sum_{\sigma\in\mathfrak{S}_{p+1}} \hbar \frac{p.(-1)^{p(p+1)/2+1}}{2.p!(p+1)!} \mathbf{f}^{I,(k,a_{\sigma(2)},\cdots,a_{\sigma(p)})} \mathbf{b}_k^{I'',a_{\sigma(0)},a_{\sigma(1)}} m^I \odot m^{I''}$$
$$= \sum_{0\leq i\leq p} \hbar \frac{(-1)^{i+p(p+1)/2}}{p!(p+1)!} \mathbf{f}^{I,(a_0,\cdots,\hat{a}_i,\cdots,a_p)} \mathbf{r}_{k_1}^{I',a_i} m^{I'} \odot m_{k_1}(m^I)$$
$$+ \sum_{0\leq i<j\leq p} \hbar \frac{(-1)^{p(p+1)/2+1}}{(p+1)!} \mathbf{f}^{I,(k,a_0,\cdots,\hat{a}_i,\cdots,\hat{a}_j,\cdots,a_p)} \mathbf{b}_k^{I'',a_i,a_j} m^I \odot m^{I''},$$

which proves that $\mathfrak{I}_{A_\hbar} \circ d_{CE} = \mathfrak{L}_{A_\hbar}(d_\hbar) \circ \mathfrak{I}_{A_\hbar}$ meaning that $\mathfrak{I}_{A_\hbar}$ is also a morphism of chain complexes and thus an isomorphism of commutative DG-algebras. □

II.3.3 THE UNIVERSAL ENVELOPING ALGEBRA

As in the Lie algebra case, the universal enveloping algebra of a Lie-Rinehart pair is in some sense the most general associative algebra into which the Lie-Rinehart pair embeds in such a way that the Lie bracket and the anchor map are both transfered only into the commutator of the associative algebra. We now recall the definition and the universal property that it satisfies, following the work of [MM10].

We start with a Lie-Rinehart pair $(R,\nabla_R,\iota_R,L,[\bullet,\bullet]_L,\phi_L,\rho)$ where R and L are concentrated in degree 0. The $\mathbb{Z}$-graded $\mathbb{K}$-module $R\oplus L$ has a natural graded Lie algebra structure defined for any r_1,r_2 in R and any l_1,l_2 in L by:

$$[r_1+l_1,r_2+l_2]_{R\oplus L} := \rho(l_1)(r_2)-\rho(l_2)(r_1)+[l_1,l_2]_L.$$

Consider the classical universal enveloping algebra, $U(R\oplus L)$, and its augmentation ideal $\bar{U}(R\oplus L)$ and denote by $x_1 \cdot x_2$ the quotient class in $U(R\oplus L)$ of a tensor $x_1\otimes x_2$ in $T(R\oplus L)$.

Definition II.3.3-1 : Universal enveloping algebra of a Lie-Rinehart pair

The universal enveloping algebra of the Lie-Rinehart pair, $(R,\nabla_R,\iota_R,L,[\bullet,\bullet]_L,\phi_L,\rho)$, where R and L are concentrated in degree 0, is the quotient algebra:

$$\mathscr{U}(R,L) := \bar{U}(R\oplus L)/(I)$$

Where (I) is the two-sided ideal in $\bar{U}(R\oplus L)$ generated by:

$$I := \{r_2\cdot(r_1+l_1)-r_2r_1-\phi_L(r_2)(l_1) |\ r_1,r_2\in R,\ l_1\in L\}$$

The universal enveloping algebra satisfies the following universal property. Given any unital $\mathbb{K}$-algebra A and any pair of morphisms $\kappa_R : R\to A$ and $\kappa_L : L\to A$, such that, κ_R is a morphism of unital $\mathbb{K}$-algebras and κ_L is a morphism of Lie algebras for the commutator on A satisfying, for any r in R and any l in L,

$$\kappa_R(r)\kappa_L(l) = \kappa_L(\phi_L(r)(l)) \qquad \text{and} \qquad [\kappa_L(l),\kappa_R(r)]_{Com} = \kappa_R(\rho(l)(r)),$$

then there exists a unique morphism of unital $\mathbb{K}$-algebras $\mathcal{K} : \mathcal{U}(R,L) \to A$ such that:

$$\mathcal{K}_{|R} = \kappa_R \qquad \text{and} \qquad \mathcal{K}_{|L} = \kappa_L.$$

As we did with the Chevalley-Eilenberg algebra, we want to apply the $\mathbb{K}[\![\hbar]\!]$-linear extension of the $\mathscr{L}_\infty$-morphism of [Cal+11] but this time we want to compute its $\mathfrak{L}_{B_\hbar}$ part, unfortunately this computation will involve too many diagrams and there is no easy way to deal with all of them. Instead, we will take a different approach by exploiting the derived right action of $B_\hbar$ on $K_\hbar$.

Recall from [Cal+11, §4.1] that the derived right action R_B of B on K, for a given coisotropic setting, is defined as a coalgebra morphism:

$$R_B \;:\; \mathrm{T}(B[1]) \;\to\; \mathrm{T}(\mathrm{End}_{\mathrm{T}(A[1])-\mathrm{comod}}(\mathrm{T}(A[1])\otimes K[1])[1])\;,$$

whose m-th Taylor component, evaluated on a tensor $\downarrow b_1 \otimes \cdots \otimes \downarrow b_m$ in $\mathrm{T}(B[1])$ is an element of $\mathrm{End}_{\mathrm{T}(A[1])-\mathrm{comod}}(\mathrm{T}(A[1])\otimes K[1])$ which again, evaluated on a tensor $\downarrow a_1 \otimes \cdots \otimes \downarrow a_n \otimes \downarrow k$ in $\mathrm{T}(A[1])\otimes K[1]$ is defined by the $\mathscr{A}_\infty$-bimodule structure of Theorem I.2.2-3:

$$R_B^m(\downarrow b_1 \otimes \cdots \otimes \downarrow b_m)^n(\downarrow a_1 \otimes \cdots \otimes \downarrow a_n \otimes \downarrow k)$$
$$:= d_K^{n,m}(\downarrow a_1 \otimes \cdots \otimes \downarrow a_n \otimes \downarrow k \downarrow b_1 \otimes \cdots \otimes \downarrow b_m)$$

It has been shown ([Cal+11, §4.1]) that the derived right action endows the space $\mathrm{End}_{\mathrm{T}(A[1])-\mathrm{comod}}(\mathrm{T}(A[1])\otimes K[1])$ with a structure of an $\mathscr{A}_\infty$-algebra which gives, in the case of a flat $\mathscr{A}_\infty$-algebra, a structure of DG-algebra and that ([Cal+11] Lemma 4.7 and 7.5) the derived actions are $\mathscr{A}_\infty$-quasi-isomorphisms.

Now since $d_\hbar$ is a Maurer-Cartan element, general deformation theory induces the existence of deformed $\mathscr{A}_\infty$-algebra $(A_\hbar, \mathfrak{L}_{A_\hbar}(d_\hbar)+\nabla_{A_\hbar})$, $(B_\hbar, \mathfrak{L}_{B_\hbar}(d_\hbar)+\nabla_{B_\hbar})$ where $\nabla_{A_\hbar}$,$\nabla_{B_\hbar}$ are just the $\hbar$-linear extensions of the products in A and B. This in turn gives us new derived actions as an $\mathscr{A}_\infty$-morphism $R_{B_\hbar}$ to the deformed $\mathscr{A}_\infty$-algebra $\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1])\otimes K_\hbar[1])$, and since it restricts to R_B for $\hbar = 0$ a general perturbation theory argument induces that $R_{B_\hbar}$ is also an $\mathscr{A}_\infty$-quasi-isomorphism:

— Proposition II.3.3-2 :

Given a local Lie algebroid viewed as a coisotropic setting as in II.2.1 and the $\mathscr{A}_\infty$-quasi-isomorphism of right derived action of [Cal+11]:

$$R_{B_\hbar} \;:\; (B_\hbar, \mathfrak{L}_{B_\hbar}(d_\hbar)+\nabla_{B_\hbar}) \;\to\; (\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1])\otimes K_\hbar[1]), Q)$$

Where the $\mathscr{A}_\infty$-algebra structure is defined by its non-zero Taylor components:

$$Q^{(0)}(1) = R_{B_\hbar}(\mathfrak{L}_{B_\hbar}(d_\hbar)^0(1)), \quad Q^{(1)}(\psi) = [d_{A_\hbar,K_\hbar}, \psi]_{Com}, \quad Q^{(2)}(\psi_1 \otimes \psi_2) = \psi_1 \circ \psi_2$$

And where $d_{A_\hbar,K_\hbar} := P_{\mathrm{T}(A_\hbar[1])\otimes K_\hbar[1]} \circ d_{K_\hbar}{}_{|\mathrm{T}(A_\hbar[1])\otimes K_\hbar[1]}$ and $d_{K_\hbar}^{m,n} := d_K^{m,n} + \mathfrak{L}_{K_\hbar}(d_\hbar)^{m,n}$.
Then $R_{B_\hbar}$ induces in homology an isomorphism of associative algebras:

$$H(R_{B_\hbar}) \;:\; (B_\hbar, \mathfrak{L}_{B_\hbar}(d_\hbar)+\nabla_{B_\hbar}) \;\to\; H^0(\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1])\otimes K_\hbar[1]))\;.$$

Proof :

We claim that both $\mathscr{A}_\infty$-algebras are associative algebras and since $R_{B_\hbar}$ is an $\mathscr{A}_\infty$-morphism and $H(R_{B_\hbar})$ is an isomorphism of vector spaces it becomes an isomorphism of quotient algebras.
Note that $B_\hbar = \mathrm{S}(M^* \oplus N) \otimes \mathbb{K}[\![\hbar]\!]$ is concentrated in degree 0, so we know that $\mathfrak{L}_{B_\hbar}(d_\hbar) + \nabla_{B_\hbar}$ have only arity-2 (or equivalently, degree 0) non zero component, thus $H^0(B_\hbar) = B_\hbar$ is an associative algebra. On the other side, the arity-0 of the $\mathscr{A}_\infty$-structure on $B_\hbar$ is only made up of $\mathfrak{L}_{B_\hbar}(d_\hbar)^{(0)}$ and we claim that it vanishes. Indeed, we have:

$$\mathfrak{L}_{B_\hbar}(d_\hbar)^{(0)}(1) = \sum_{n\in\mathbb{N}^*} \frac{1}{n!} \sum_{\Gamma\in\mathcal{G}_{n,0}} \uparrow \circ \mu_n^{B[1]} \circ \left(\int_{\mathscr{C}^+_{n,0}} \prod_{e\in E(\Gamma)} \omega_e^B \right) (d^{\otimes n}) \otimes \hbar^n$$

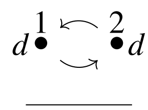
As before, due to the dimension of the integrated space and the vanishing property of two edges going out of a vertex of the first type which are colored with a copy of d, the graphs having non-zero contribution to the sum must satisfy:

$$dim(\mathscr{C}^+_{n,0}) = \#E(\Gamma) \qquad \text{and} \qquad \#E(\Gamma) \leq n.$$

Hence $n \leq 2$ since $dim(\mathscr{C}^+_{n,0}) = 2n-2$.
Recall that $|d| = 1$ and $|\omega_e^B| = -1$ and, as a submodule of $\mathrm{T}^\bullet_{\mathrm{poly}}(X)$, $B[1]$ is concentrated in degree -1, therefore each vertex of the graph must have two edges joining them and thus only $n = 2$ is possible. Consequently, we are left with the following graph, whose contribution also vanishes:

The only remaining graph Γ_0, whose contribution is not trivially computed to be zero, must have two edges and no pair of edges sharing the same initial point. The whole computation also vanish thanks to:

Γ_0 :



$$\begin{aligned}
&\mu_2^{B[1]} \circ \left(\int_{\mathscr{C}^+_{2,0}} \prod_{e\in E(\Gamma_0)} \omega_e^B \right) (d \otimes d) \\
&= \mu_2^{B[1]} \circ \left(\int_{\mathscr{C}^+_{2,0}} \pi^*_{(1,2)}(\omega^+) \wedge \pi^*_{(2,1)}(\omega^+) \right) (\tau^{\mathbb{I}}_{(1,2)} \circ \tau^{\mathbb{I}}_{(2,1)})(d \otimes d) \\
&+ \mu_2^{B[1]} \circ \left(\int_{\mathscr{C}^+_{2,0}} \pi^*_{(1,2)}(\omega^+) \wedge \pi^*_{(2,1)}(\omega^-) \right) (\tau^{\mathbb{I}}_{(1,2)} \circ \tau^{\mathbb{J}}_{(2,1)})(d \otimes d) \\
&+ \mu_2^{B[1]} \circ \left(\int_{\mathscr{C}^+_{2,0}} \pi^*_{(1,2)}(\omega^-) \wedge \pi^*_{(2,1)}(\omega^+) \right) (\tau^{\mathbb{J}}_{(1,2)} \circ \tau^{\mathbb{I}}_{(2,1)})(d \otimes d) \\
&+ \mu_2^{B[1]} \circ \left(\int_{\mathscr{C}^+_{2,0}} \pi^*_{(1,2)}(\omega^-) \wedge \pi^*_{(2,1)}(\omega^-) \right) (\tau^{\mathbb{J}}_{(1,2)} \circ \tau^{\mathbb{J}}_{(2,1)})(d \otimes d).
\end{aligned}$$

And the coefficient of the first and fourth term vanish for the same reason as in the proof of Proposition II.3.2-2, while the two middle terms simply do not end up in $B[1]$ both

due to the elements $\partial_{m^i} \circ \iota_{d\uparrow n^j}(d)$ which lies in $\mathrm{S}(M^*) \otimes \mathrm{S}^2(N^*[-1])$.
So the two $\mathscr{A}_\infty$-algebras are actual DG-algebras and $R_{B_\hbar}$ is a DG-quasi-isomorphism and thus gives an isomorphism of associative algebras in homology. □

We now want to explicitly describe the differential of $\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])$ in order to prove that it is indeed isomorphic to the universal enveloping algebra of our Lie-Rinehart pair.

— Proposition II.3.3-3 :

Given a local Lie algebroid viewed as a coisotropic setting as in II.2.1.
The morphism $d_{A_\hbar,K_\hbar}$ is defined by its only non-zero Taylor component:

$$\left(d_{A_\hbar,K_\hbar}\right)^1 = \nabla_{K_\hbar[1]} \circ (P_{K_\hbar[1]} \otimes id_{K_\hbar[1]}).$$

Proof :

Since $d_{A_\hbar,K_\hbar} = P_{\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1]} \circ d_{K_\hbar}{}_{|\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1]}$ the study restricts to Taylor components $d_{K_\hbar}^{m,0}$, for $m \in \mathbb{N}$, taking value into $A_\hbar[1] \otimes K_\hbar[1]$. As $d_{K_\hbar}^{m,0} = d_K^{m,0} + \mathfrak{L}_{K_\hbar}(d_\hbar)^{m,0}$, we first study the Taylor component $d_K^{m,0}$, which will then be extended $\mathbb{K}[\![\hbar]\!]$-linearily. Consider then an integer m together with elements $a_1, \cdots, a_m$ in A and k in K and a graph Γ in $\mathcal{G}_{m,0}$ such that:

$$\mu_{m+1}^{K[1]} \circ \left(\int_{\mathscr{C}_{0,m+1}^+} \prod_{e \in E(\Gamma)} \omega_e^K \right) (\downarrow a_1 \otimes \cdots \otimes \downarrow a_m \otimes \downarrow k) \neq 0.$$

Again, due to the 1-form part of the operator ω_e^K the graph is compelled to satisfy $dim(\mathscr{C}_{0,m+1}^+) = \#E(\Gamma)$ and since $A = \mathrm{S}(M^* \oplus N^*[-1])$ and $K = \mathrm{S}(M^*)$ if any edge goes out of any point it will involve an element of the form ι_{dx^k} whish all vanish on M^* and N^*, thus $\#E(\Gamma) = 0$ and $m = 1$. And the only graph with no edges reduces to:

$$d_K^{(1,0)}(\downarrow a_1 \otimes \downarrow k) = \mu_2^{K[1]}(\downarrow a_1 \otimes \downarrow k) = \nabla_{K[1]} \circ (P_{K[1]} \otimes id_{K[1]})(\downarrow a_1 \otimes \downarrow k),$$

and since it is $\mathbb{K}[\![\hbar]\!]$-linearily extended, the advised reader will know that it only remains to prove that $\mathfrak{L}_{K_\hbar}(d_\hbar)^{m,0} = 0$ for all $m \in \mathbb{N}$, and indeed since we have:

$$\mathfrak{L}_{K_\hbar}(d_\hbar) = \sum_{n \in \mathbb{N}^*} \frac{1}{n!} \mathfrak{L}_K^{(n)}(d \otimes \cdots \otimes d) \otimes \hbar^n.$$

We can consider a pair of non-negative integers n, m together with elements $a_1, \cdots, a_m$ in A and k in K such that:

$$\mathfrak{L}_K^{(n)}(d \otimes \cdots \otimes d)(a_1 \otimes \cdots \otimes a_m \otimes k) \neq 0.$$

By unfolding the definition of Theorem I.2.3-1, it implies that there exists a graph Γ in $\mathcal{G}_{n,m+1}$ such that:

$$\uparrow \circ \mu_{n+m+1}^{K[1]} \circ \left(\int_{\mathscr{C}_{n,m+1}^+} \prod_{e \in E(\Gamma)} \omega_e^K \right) (d^{\otimes n} \otimes \downarrow a_1 \otimes \cdots \downarrow a_m \otimes \downarrow k) \neq 0$$

As always, since vertices of the first type are colored with a copy of d, there can not be any pair of edges sharing a same vertex of the first type as a starting point, and as before there cannot be any edge having a vertex of the second type for a starting point. Thus, $\#E(\Gamma) \leq n$, and for degree reason, since $|d| = 1$ with $|\omega_e^K| = -1$ and $K[1]$ is concentrated in degree-1, there is no solution for $n \neq 0$. Thus, $\mathfrak{L}_{K_\hbar}(d_\hbar)^{m,0} = 0$ and the claim follows. □

With this new insight we will show that there exists an isomorphism of associative algebras from the universal enveloping algebra of the formal deformation of a local Lie algebroid to the associative algebra $H^0(\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1]))$. To do so we will use the universal property of the universal enveloping algebra on the following setting of morphisms.

— Proposition II.3.3-4 :

Given the formal deformation of a local Lie algebroid, $(M_\hbar, N_\hbar, [\bullet,\bullet]_{L_\hbar}, \rho_\hbar)$, there exist two $\mathbb{K}[\![\hbar]\!]$-linear morphisms:

$$\kappa_{R_\hbar} \;:\; \mathrm{S}(M^*) \otimes \mathbb{K}[\![\hbar]\!] \;\to\; \mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1]) \;,$$

$$\kappa_{L_\hbar} \;:\; \mathrm{S}(M^*) \otimes N \otimes \mathbb{K}[\![\hbar]\!] \;\to\; \mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1]) \;,$$

whose images are defined by the following only non-zero and $\mathbb{K}[\![\hbar]\!]$-linear Taylor components, sets on basis elements $m^I \in \mathrm{S}(M^*)$ and $n_j \in N$ and evaluated on any $a \in A = \mathrm{S}(M^* \oplus N^*[-1])$ and $k \in K = \mathrm{S}(M^*)$:

$$\begin{aligned}
\kappa_{R_\hbar}(m^I)^{(0)}(\downarrow k) &:= \downarrow m^I \odot k, \\
\kappa_{L_\hbar}(m^I \otimes n_j)^{(0)}(\downarrow k) &:= \downarrow m^I \odot \rho_\hbar(n_j)(k), \\
\kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}(\downarrow a \otimes \downarrow k) &:= \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^I \odot k,
\end{aligned}$$

inducing, by the universal property of the universal enveloping algebra $U(L_\hbar, R_\hbar)$, the existence of a morphism of associative unital $\mathbb{K}$-algebras:

$$\mathcal{K} \;:\; \mathscr{U}(L_\hbar, R_\hbar) \;\to\; H^0(\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])) \;.$$

Proof :

We first need to show that each of the Taylor components takes values into the 0-cocycles of $\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])$. Then we will prove that their composition with the quotient map of the homology H satisfies the conditions of the universal property of the universal enveloping algebra $\mathscr{U}(R_\hbar, L_\hbar)$.
For this reason, the proof split into the following lemmata:

— Lemma II.3.3-5 :

The maps $\kappa_{R_\hbar}$ and $\kappa_{L_\hbar}$ take value into the morphisms of degree 0 which vanish under $Q^{(1)}$.

— Lemma II.3.3-6 :

The map $H \circ \kappa_{R_\hbar}$ is a morphism of associative unital $\mathbb{K}$-algebras.
The map $H \circ \kappa_{L_\hbar}$ is a morphism of Lie algebras.

— Lemma II.3.3-7 :

For all $r \in R$ and $l \in L$, we have:

$$\kappa_{R_\hbar}(r)\kappa_{L_\hbar}(l) = \kappa_{L_\hbar}(\phi_{L_\hbar}(r)(l)),$$

$$[\kappa_{L_\hbar}(l), \kappa_{R_\hbar}(r)]_{Com} = \kappa_{R_\hbar}(\rho_\hbar(l)(r)).$$

Note that the conditions of the last lemma could have hold up to 0-coboundaries, but actually hold strictly. In consequence, the two morphisms $H \circ \kappa_{R_\hbar}$ and $H \circ \kappa_{L_\hbar}$ satisfy the universal property of $\mathscr{U}(R_\hbar, L_\hbar)$ and there exists a unique morphism of associative unital $\mathbb{K}$-algebras $\mathcal{K}$ such that:

$$\mathcal{K} \ : \ \mathscr{U}(R_\hbar, L_\hbar) \ \to \ H^0(\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])) \ ,$$

$$\mathcal{K}_{|R_\hbar} = H \circ \kappa_{R_\hbar} \qquad \text{and} \qquad \mathcal{K}_{|L_\hbar} = H \circ \kappa_{L_\hbar}. \qquad \square$$

We now expose the proofs of the previous lemmata.

Proof of II.3.3-5:

The degree of $\kappa_{R_\hbar}(m^I)^{(0)}$ and $\kappa_{L_\hbar}(m^I \otimes n_j)^{(0)}$ is readily computed as being equal to 0, while for $\kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}$ one should notice the projection onto N^*, which is non-zero only if $a \in N^*[-1]$, also leading to a morphism of degree 0, and we now prove that the comodule endomorphisms defined by these Taylor components vanishes under the differential $Q^{(1)}$. But before going further we should recall that as an $\mathscr{A}_\infty$-bimodule structure morphism $d_{K_\hbar}$ is reconstructed from its Taylor components using the ones of the $\mathscr{A}_\infty$-algebra structure of $A_\hbar$ and $B_\hbar$ which are in this case DG-algebras:

$$d_{A_\hbar, K_\hbar | \mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} = id_{A_\hbar[1]}^{\otimes n-1} \otimes d_{K_\hbar}^{(1,0)} + \sum_{\substack{1 \le i \le 2 \\ p \le n-i}} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(i)} \otimes id_{A_\hbar[1]}^{\otimes n-p-i} \otimes id_{K_\hbar[1]}$$

We first study the restriction of the morphisms to expose their Taylor components:

$$\begin{aligned}
& Q^{(1)}(\kappa_{R_\hbar}(m^I))_{|\mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} \\
&= [d_{A_\hbar, K_\hbar}, \kappa_{R_\hbar}(m^I)]_{Com | \mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} \\
&= d_{A_\hbar, K_\hbar} \circ \kappa_{R_\hbar}(m^I)_{|\mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} - \kappa_{R_\hbar}(m^I) \circ d_{A_\hbar, K_\hbar | \mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} \\
&= id_{A_\hbar[1]}^{\otimes n-1} \otimes \left(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{R_\hbar}(m^I)^{(0)}) \right) + \sum_{\substack{1 \le i \le 2 \\ p \le n-i}} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(i)} \otimes id_{A_\hbar[1]}^{\otimes n-p-i} \otimes \kappa_{R_\hbar}(m^I)^{(0)} \\
&\quad - id_{A_\hbar[1]}^{\otimes n-1} \otimes \left(\kappa_{R_\hbar}(m^I)^{(0)} \circ d_{K_\hbar}^{(1,0)} \right) \\
&\quad - \sum_{\substack{1 \le i \le 2 \\ p \le n-i}} \left(id_{A_\hbar[1]}^{\otimes n-i+1} \otimes \kappa_{R_\hbar}(m^I)^{(0)} \right) \circ \left(id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(i)} \otimes id_{A_\hbar[1]}^{\otimes n-p-i} \otimes id_{K_\hbar[1]} \right) \\
&= id_{A_\hbar[1]}^{\otimes n-1} \otimes \left(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{R_\hbar}(m^I)^{(0)}) - \kappa_{R_\hbar}(m^I)^{(0)} \circ d_{K_\hbar}^{(1,0)} \right)
\end{aligned}$$

where the two sums cancels each other due to the degree-0 and arity-0 of the only Taylor component of $\kappa_{R_\hbar}(m^I)$. And it is straightforward to see that both morphism in the last parenthesis cancel each other on elements of the form $\downarrow a \otimes \downarrow k$ since they both take the value $-\downarrow m^I \odot P_{K_\hbar}(a) \odot k$.
Similarily if we consider an element $v \in \mathrm{S}(M^*) \otimes N$ we have:

$$
\begin{aligned}
&Q^{(1)}(\kappa_{L_\hbar}(v))_{|\mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} \\
&= [d_{A_\hbar,K_\hbar}, \kappa_{L_\hbar}(v)]_{Com_{|\mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]}} \\
&= d_{A_\hbar,K_\hbar} \circ \kappa_{L_\hbar}(v)_{|\mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} - \kappa_{L_\hbar}(v) \circ d_{A_\hbar,K_\hbar|\mathrm{T}^n(A_\hbar[1]) \otimes K_\hbar[1]} \\
&= \sum_{0 \le j \le 1} \left(id_{A_\hbar[1]}^{\otimes n-1-j} \otimes d_{K_\hbar}^{(1,0)} + \sum_{\substack{1 \le i \le 2 \\ p \le n-i-j}} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(i)} \otimes id_{A_\hbar[1]}^{\otimes n-p-i-j} \otimes id_{K_\hbar[1]} \right) \\
&\qquad\qquad \circ \left(id_{A_\hbar[1]}^{\otimes n-j} \otimes \kappa_{L_\hbar}(v)^{(j)} \right) \\
&\quad - \sum_{0 \le j \le 1} \left(id_{A_\hbar[1]}^{\otimes n-1-j} \otimes \kappa_{L_\hbar}(v)^{(j)} \right) \\
&\qquad\qquad \circ \left(id_{A_\hbar[1]}^{\otimes n-1} \otimes d_{K_\hbar}^{(1,0)} + \sum_{p \le n-2} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(2)} \otimes id_{A_\hbar[1]}^{\otimes n-p-2} \otimes id_{K_\hbar[1]} \right) \\
&\quad - \sum_{0 \le j \le 1} \left(id_{A_\hbar[1]}^{\otimes n-j} \otimes \kappa_{L_\hbar}(v)^{(j)} \right) \circ \left(\sum_{p \le n-1} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(1)} \otimes id_{A_\hbar[1]}^{\otimes n-p-1} \otimes id_{K_\hbar[1]} \right) \\
&= id_{A_\hbar[1]}^{\otimes n-1} \otimes \left(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(v)^{(0)}) \right) + id_{A_\hbar[1]}^{\otimes n-2} \otimes \left(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(v)^{(1)}) \right) \\
&\quad + \sum_{0 \le j \le 1} \sum_{\substack{1 \le i \le 2 \\ p \le n-i-j}} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(i)} \otimes id_{A_\hbar[1]}^{\otimes n-p-i-j} \otimes \kappa_{L_\hbar}(v)^{(j)} \\
&\quad - \sum_{0 \le j \le 1} id_{A_\hbar[1]}^{\otimes n-j-1} \otimes \left(\kappa_{L_\hbar}(v)^{(j)} \circ (id_{A_\hbar[1]}^{\otimes j} \otimes d_{K_\hbar}^{(1,0)}) \right) \\
&\quad - \sum_{p \le n-2} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(2)} \otimes id_{A_\hbar[1]}^{\otimes n-p-2} \otimes \kappa_{L_\hbar}(v)^{(0)} \\
&\quad - \sum_{p \le n-3} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(2)} \otimes id_{A_\hbar[1]}^{\otimes n-p-3} \otimes \kappa_{L_\hbar}(v)^{(1)} \\
&\quad - id_{A_\hbar[1]}^{\otimes n-2} \otimes \left(\kappa_{L_\hbar}(v)^{(1)} \circ (d_{A_\hbar}^{(2)} \otimes id_{K_\hbar[1]}) \right) \\
&\quad - \sum_{p \le n-1} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(1)} \otimes id_{A_\hbar[1]}^{\otimes n-p-1} \otimes \kappa_{L_\hbar}(v)^{(0)} \\
&\quad - \sum_{p \le n-2} id_{A_\hbar[1]}^{\otimes p} \otimes d_{A_\hbar}^{(1)} \otimes id_{A_\hbar[1]}^{\otimes n-p-2} \otimes \kappa_{L_\hbar}(v)^{(1)} \\
&\quad - id_{A_\hbar[1]}^{\otimes n-1} \otimes \left(\kappa_{L_\hbar}(v)^{(1)} \circ (d_{A_\hbar}^{(1)} \otimes id_{K_\hbar[1]}) \right)
\end{aligned}
$$

$$= id_{A_\hbar[1]}^{\otimes n-1} \otimes \Big(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(v)^{(0)}) - \kappa_{L_\hbar}(v)^{(1)} \circ (d_{A_\hbar}^{(1)} \otimes id_{K_\hbar[1]}) - \kappa_{L_\hbar}(v)^{(0)} \circ d_{K_\hbar}^{(1,0)} \Big)$$
$$+ id_{A_\hbar[1]}^{\otimes n-2} \otimes \Big(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(v)^{(1)}) - \kappa_{L_\hbar}(v)^{(1)} \circ (d_{A_\hbar}^{(2)} \otimes id_{K_\hbar[1]}) \Big)$$
$$+ id_{A_\hbar[1]}^{\otimes n-2} \otimes \Big(-\kappa_{L_\hbar}(v)^{(1)} \circ (id_{A_\hbar[1]} \otimes d_{K_\hbar}^{(1,0)}) \Big)$$

where, again, some of the equalities holds due to the degree 0 of the morphism $\kappa_{L_\hbar}(v)$. The last line exposes the two apparently non-zero Taylor components of $Q^{(1)}(\kappa_{L_\hbar}(v))$, namely the arity-1 and arity-2. But a direct evaluation show that they also both vanish, since for any basis elements $m^I \in \mathrm{S}(M^*)$, $n_j \in N$, and any $a \in A$ and $k \in K$, we have:

$$\begin{aligned}
&\kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}(\downarrow a \otimes \downarrow k) \\
&= d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(m^I \otimes n_j)^{(0)})(\downarrow a \otimes \downarrow k) \\
&\quad - \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)} \circ (d_{A_\hbar}^{(1)} \otimes id_{K_\hbar[1]})(\downarrow a \otimes \downarrow k) \\
&\quad - \kappa_{L_\hbar}(m^I \otimes n_j)^{(0)} \circ d_{K_\hbar}^{(1,0)}(\downarrow a \otimes \downarrow k) \\
&= d_{K_\hbar}^{(1,0)}(\downarrow a \otimes \downarrow m^I \odot \rho_\hbar(n_j)(k)) \; - \; \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}(\downarrow d_\hbar(a) \otimes \downarrow k) \\
&\quad + \kappa_{L_\hbar}(m^I \otimes n_j)^{(0)}(\downarrow P_{K_\hbar}(a) \odot k) \\
&= - \downarrow P_{K_\hbar}(a) \odot m^I \odot \rho_\hbar(n_j)(k) \; - \; \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow d_\hbar(a)), n_j > \odot m^I \odot k \\
&\quad + \downarrow m^I \odot \rho_\hbar(n_j)(P_{K_\hbar}(a)) \odot k \; + \; \downarrow m^I \odot P_{K_\hbar}(a) \odot \rho_\hbar(n_j)(k) \\
&= - \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow d_\hbar(a)), n_j > \odot m^I \odot k \; + \; \downarrow m^I \odot \rho_\hbar(n_j)(P_{K_\hbar}(a)) \odot k \\
&= - \downarrow \rho^{\triangle}(P_{K_\hbar}(a))(n_j) \odot m^I \odot k \otimes \hbar \; + \; \downarrow m^I \odot \rho(n_j)(P_{K_\hbar}(a)) \odot k \otimes \hbar \\
&= 0.
\end{aligned}$$

And for every pair $a_1, a_2 \in A$ and $k \in K$ we have:

$$\begin{aligned}
&\kappa_{L_\hbar}(m^I \otimes n_j)^{(2)}(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
&= \Big(d_{K_\hbar}^{(1,0)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}) - \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)} \circ (d_{A_\hbar}^{(2)} \otimes id_{K_\hbar[1]}) \Big)(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
&\quad - \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)} \circ (id_{A_\hbar[1]} \otimes d_{K_\hbar}^{(1,0)})(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
&= d_{K_\hbar}^{(1,0)}(\downarrow a_1 \otimes \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_2), n_j > \odot m^I \odot k) \\
&\quad + (-1)^{|a_1|} \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}(\downarrow a_1 \odot a_2 \otimes \downarrow k) \\
&\quad + (-1)^{|\downarrow a_1|} \kappa_{L_\hbar}(m^I \otimes n_j)^{1}(\downarrow a_1 \otimes \downarrow P_{K_\hbar}(a_2) \odot k) \\
&= - \downarrow P_{K_\hbar}(a_1) \odot < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_2), n_j > \odot m^I \odot k \\
&\quad + (-1)^{|a_1|} \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_1 \odot a_2), n_j > \odot m^I \odot k \\
&\quad - (-1)^{|a_1|} \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_1), n_j > \odot m^I \odot P_{K_\hbar}(a_2) \odot k \\
&= 0.
\end{aligned}$$

Consequently, the images of the morphisms $\kappa_{R_\hbar}$ and $\kappa_{L_\hbar}$ are subsets of the set of 0-cocycles of the DG-algebra $\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])$. □

Proof of II.3.3-6:

Since $\kappa_{R_\hbar}$ and $\kappa_{L_\hbar}$ take value into the 0-cocycles, we can compose these morphisms with the quotient map coming from the homology functor that we denote by H. Showing that $H \circ \kappa_{L_\hbar}$ is a morphism of Lie algebra and $H \circ \kappa_{R_\hbar}$ is a morphism of associatives algebras is therefore equivalent to showing that the properties holds up to a 0-coboundary. For the later one, it is rather simple as the condition holds strictly since $\kappa_{R_\hbar}$ have only one non trivial Taylor component, hence, given any basis elements $m^I, m^{I'} \in \mathrm{S}(M^*)$ and any $k \in K$, we have:

$$\begin{aligned}
&\kappa_{R_\hbar}(m^I) \circ \kappa_{R_\hbar}(m^{I'})(\downarrow k)\\
&= \kappa_{R_\hbar}(m^I)(\downarrow m^{I'} \odot k)\\
&= \downarrow m^{I'} \odot m^I \odot k\\
&= \kappa_{R_\hbar}(m^I \odot m^{I'})(\downarrow k).
\end{aligned}$$

To show that $\kappa_{L_\hbar}$ is a Lie morphism, we first start with a description of $\kappa_{L_\hbar} \circ [\bullet,\bullet]_{L_\hbar}$. Consider basis elements $m^I, m^{I'} \in \mathrm{S}(M^*)$, $n_j, n_{j'} \in N$, then we have:

$$\begin{aligned}
&\kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar})\\
&= \kappa_{L_\hbar}([m^I \otimes n_j, \phi_L(m^{I'})(1 \otimes n_{j'})]_L) \otimes \hbar\\
&= \kappa_{L_\hbar}(\rho(m^I \otimes n_j)(m^{I'}) \otimes n_{j'} + \phi_L(m^{I'})([m^I \otimes n_j, 1 \otimes n_{j'}]_L)) \otimes \hbar\\
&= \kappa_{L_\hbar}(m^I \odot \rho(n_j)(m^{I'}) \otimes n_{j'}) \otimes \hbar - \kappa_{L_\hbar}(\phi_L(m^{I'})(\rho(n_{j'})(m^I) \otimes n_j)) \otimes \hbar\\
&\quad - \kappa_{L_\hbar}(\phi_L(m^{I'})(\phi_L(m^I)([n_{j'}, n_j]_N))) \otimes \hbar\\
&= \kappa_{L_\hbar}(m^I \odot \rho(n_j)(m^{I'}) \otimes n_{j'}) \otimes \hbar - \kappa_{L_\hbar}(m^{I'} \odot \rho(n_{j'})(m^I) \otimes n_j) \otimes \hbar\\
&\quad + \kappa_{L_\hbar}(\phi_L(m^{I'} \odot m^I)([n_j, n_{j'}]_N)) \otimes \hbar.
\end{aligned}$$

We can now study the two Taylor components of $\kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar})$, starting with the arity-0 one and an element $k \in K$:

$$\begin{aligned}
&\kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar})^{(0)}(\downarrow k)\\
&= \downarrow m^I \odot \rho(n_j)(m^{I'}) \odot \rho(n_{j'})(k) \otimes \hbar^2 - \downarrow m^{I'} \odot \rho(n_{j'})(m^I) \odot \rho(n_j)(k) \otimes \hbar^2\\
&\quad + \downarrow m^{I'} \odot m^I \odot \rho([n_j, n_{j'}]_N)(k) \otimes \hbar^2\\
&= \downarrow m^I \odot \rho(n_j)(m^{I'}) \odot \rho(n_{j'})(k) \otimes \hbar^2 + \downarrow m^{I'} \odot m^I \odot \rho(n_j)(\rho(n_{j'})(k)) \otimes \hbar^2\\
&\quad - \downarrow m^{I'} \odot \rho(n_{j'})(m^I) \odot \rho(n_j)(k) \otimes \hbar^2 - \downarrow m^{I'} \odot m^I \odot \rho(n_{j'})(\rho(n_j)(k)) \otimes \hbar^2\\
&= \downarrow m^I \odot \rho(n_j) \left(m^{I'} \odot \rho(n_{j'})(k)\right) \otimes \hbar^2 - \downarrow m^{I'} \odot \rho(n_{j'}) \left(m^I \odot \rho(n_j)(k)\right) \otimes \hbar^2\\
&= \kappa_{L_\hbar}(m^I \otimes n_j)^{(0)} \circ \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(0)}(\downarrow k) - \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(0)} \circ \kappa_{L_\hbar}(m^I \otimes n_j)^{(0)}(\downarrow k)\\
&= [\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]^{(0)}_{Com}(\downarrow k)
\end{aligned}$$

Here again, the relation holds strictly. But it is not the case anymore for the arity-1

Taylor component since for any $a \in A$ and $k \in K$, we have:

$$
\begin{aligned}
&\kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar})^{(1)}(\downarrow a \otimes \downarrow k) \\
=&\downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_{j'} > \odot m^I \odot \rho(n_j)(m^{I'}) \odot k \otimes \hbar \\
&\quad - \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^{I'} \odot \rho(n_{j'})(m^I) \odot k \otimes \hbar \\
&\quad + \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), [n_j, n_{j'}]_N > \odot m^I \odot m^{I'} \odot k \otimes \hbar,
\end{aligned}
$$

and

$$
\begin{aligned}
&[\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]^{(1)}_{Com}(\downarrow a \otimes \downarrow k) \\
=&\Big(\kappa_{L_\hbar}(m^I \otimes n_j)^{(0)} \circ \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(1)} + \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)} \circ \left(id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(0)} \right) \\
&\quad - \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(0)} \circ \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)} \\
&\quad - \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(1)} \circ \left(id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(m^I \otimes n_j)^{(0)} \right) \Big)(\downarrow a \otimes \downarrow k) \\
=&\downarrow m^I \odot \rho(n_j) \left(< P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_{j'} > \odot m^{I'} \odot k \right) \otimes \hbar \\
&\quad + \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^I \odot m^{I'} \odot \rho(n_{j'})(k) \otimes \hbar \\
&\quad - \downarrow m^{I'} \odot \rho(n_{j'}) \left(< P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^I \odot k \right) \otimes \hbar \\
&\quad - \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_{j'} > \odot m^{I'} \odot m^I \odot \rho(n_j)(k) \otimes \hbar \\
=&\downarrow m^I \odot \rho(n_j) \left(< P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_{j'} > \odot m^{I'} \right) \odot k \otimes \hbar \\
&\quad - \downarrow m^{I'} \odot \rho(n_{j'}) \left(< P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^I \right) \odot k \otimes \hbar.
\end{aligned}
$$

As a result, the non-trivial difference is:

$$
\begin{aligned}
&\kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar})^{(1)}(\downarrow a \otimes \downarrow k) - [\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]^{(1)}_{Com}(\downarrow a \otimes \downarrow k) \\
=&\downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), [n_j, n_{j'}]_N > \odot m^I \odot m^{I'} \odot k \otimes \hbar \\
&\quad - \downarrow m^I \odot m^{I'} \odot \rho(n_j) \left(< P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_{j'} > \right) \odot k \otimes \hbar \\
&\quad + \downarrow m^{I'} \odot m^I \odot \rho(n_{j'}) \left(< P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \right) \odot k \otimes \hbar
\end{aligned}
$$

And we claim that it lies in the image of $Q^{(1)}$, but before proving that, one should notice that the arity-2 Taylor component of $[\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]_{Com}$ is also non-trivial since we have for any pair $a_1, a_2 \in A$ and $k \in K$:

$$\begin{aligned}
&[\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]^{(2)}_{Com}(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
&= \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(1)})(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
&\quad - \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(1)} \circ (id_{A_\hbar[1]} \otimes \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)})(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
&= \kappa_{L_\hbar}(m^I \otimes n_j)^{(1)}(\downarrow a_1 \otimes \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_2), n_{j'} > \odot m^{I'} \odot k) \\
&\quad - \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})^{(1)}(\downarrow a_1 \otimes \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_2), n_j > \odot m^I \odot k) \\
&= \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_1), n_j > \odot m^I \odot < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_2), n_{j'} > \odot m^{I'} \odot k \\
&\quad - \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_1), n_{j'} > \odot m^{I'} \odot < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a_2), n_j > \odot m^I \odot k
\end{aligned}$$

To prove that both remnant Taylor components are the ones of a 0-coboundary we define an element, θ, in $\mathrm{End}^{-1}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])$ and show that $Q^{(1)}(\theta)^{(1)}$ and $Q^{(1)}(\theta)^{(2)}$ coincide with the previous expressions.

As a comodule endomorphism, θ is defined by its only non-zero Taylor component $\theta^{(1)}$, set as the $\mathbb{K}[\![\hbar]\!]$-linear extension of the following morphism:

$$\begin{array}{rccl}
\theta^{(1)} & : & A_\hbar[1] \otimes K_\hbar[1] & \to & K_\hbar[1] \\
& & \downarrow a \otimes \downarrow k & \mapsto & 2. \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*)\otimes \mathrm{S}^2(N^*[-1])}(a), \downarrow n_j \otimes \downarrow n_{j'} > \odot k
\end{array}$$

The degree of $\theta^{(1)}$ is readily computed to be -1 and it is not hard to see that $Q^{(1)}(\theta)$ has only two Taylor components, $Q^{(1)}(\theta)^{(1)}$ and $Q^{(1)}(\theta)^{(2)}$, which we evaluate:

$$\begin{aligned}
&Q^{(1)}(\theta)^{(1)}(\downarrow a \otimes \downarrow k) \\
&= [d_{A_\hbar, K_\hbar}, \theta]^{(1)}_{Com}(\downarrow a \otimes \downarrow k) \\
&= -\theta^{(1)} \circ (d^{(1)}_{A_\hbar} \otimes id_{K_\hbar[1]})(\downarrow a \otimes \downarrow k) \\
&= -\theta^{(1)} \left(\downarrow \left(\rho^\triangle(a) - \frac{1}{2}[\bullet, \bullet]^\triangle_N \circ \downarrow (a) \right) \otimes \downarrow k \right) \otimes \hbar \\
&= -2. \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*)\otimes \mathrm{S}^2(N^*[-1])}(\rho^\triangle(a)), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \otimes \hbar \\
&\quad + \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*)\otimes \mathrm{S}^2(N^*[-1])}([\bullet, \bullet]^\triangle_N \circ \downarrow (a)), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \otimes \hbar \\
&= -2. \downarrow m^I \odot m^{I'} \odot < \rho^\triangle(P_{\mathrm{S}(M^*)\otimes N^*[-1]}(a)), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \otimes \hbar \\
&\quad + \downarrow m^I \odot m^{I'} \odot < [\bullet, \bullet]^\triangle_N \circ \downarrow (P_{\mathrm{S}(M^*)\otimes N^*[-1]}(a)), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \otimes \hbar,
\end{aligned}$$

where in the last equality we use the fact that both $\rho^\triangle$ and $[\bullet, \bullet]^\triangle_N \circ \downarrow$ are derivations of A of degree 1, hence the projection vanishes if $a \notin \mathrm{S}(M^*) \otimes N^*[-1]$. We can therefore reduce the problem to the case $a = m^T \otimes \uparrow n^t$ with $T \in \mathrm{Multi}(\mathbb{I})$ and $t \in \mathbb{J}$:

$$
\begin{aligned}
& Q^{(1)}(\theta)^{(1)}(\downarrow (m^T \otimes \uparrow n^t) \otimes \downarrow k) \otimes \hbar \\
= & -2. \downarrow m^I \odot m^{I'} \odot < \rho^{\triangle}(m^T) \odot \uparrow n^t, \downarrow n_j \otimes \downarrow n_{j'} > \odot k \otimes \hbar \\
& + \downarrow m^I \odot m^{I'} \odot m^T \odot < [\bullet, \bullet]_N^{\triangle}(n^t), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \otimes \hbar \\
= & - \downarrow m^I \odot m^{I'} \odot <\downarrow \rho^{\triangle}(m^T) \otimes n^t, n_j \otimes n_{j'} > \odot k \otimes \hbar \\
& + \downarrow m^I \odot m^{I'} \odot < n^t \otimes \downarrow \rho^{\triangle}(m^T), n_j \otimes n_{j'} > \odot k \otimes \hbar \\
& + \downarrow m^I \odot m^{I'} \odot m^T \odot n^t (< [\bullet, \bullet]_N, n_j \otimes n_{j'} >) \odot k \otimes \hbar \\
= & - \downarrow m^I \odot m^{I'} \odot \rho(n_j)(m^T) \odot n^t(n_{j'}) \odot k \otimes \hbar \\
& + \downarrow m^I \odot m^{I'} \odot n^t(n_j) \odot \rho(n_{j'})(m^T) \odot k \otimes \hbar \\
& + \downarrow m^I \odot m^{I'} \odot m^T \odot n^t([n_j, n_{j'}]_N) \odot k \otimes \hbar \\
= & - \downarrow m^I \odot m^{I'} \odot \rho(n_j) \left(< m^T \otimes n^t, n_{j'} >\right) \odot k \otimes \hbar \\
& + \downarrow m^{I'} \odot m^I \odot \rho(n_{j'}) \left(< m^T \otimes n^t, n_j >\right) \odot k \otimes \hbar \\
& + \downarrow < m^T \otimes n^t, [n_j, n_{j'}]_N > \odot m^I \odot m^{I'} \odot k \otimes \hbar \\
= & \kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar})^{(1)}(\downarrow (m^T \otimes \uparrow n^t) \otimes \downarrow k) \\
& - [\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]_{Com}^{(1)}(\downarrow (m^T \otimes \uparrow n^t) \otimes \downarrow k)
\end{aligned}
$$

So we obtain the desired relation for arity-1 Taylor components, and it only remains to show it for $Q^{(1)}(\theta)^{(2)}$. Indeed, for any pair $a_1, a_2 \in A$ and $k \in K$ we have:

$$
\begin{aligned}
& Q^{(1)}(\theta)^{(2)}(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
= & [d_{A_\hbar, K_\hbar}, \theta]_{Com}^{(2)}(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
= & \left(d_{K_\hbar}^{(1,0)} \circ \left(id_{A_\hbar[1]} \otimes \theta^{(1)} \right) + \theta^{(1)} \circ \left(d_{A_\hbar}^{(2)} \otimes id_{K_\hbar[1]} + id_{A_\hbar[1]} \otimes d_{K_\hbar}^{(1,0)} \right) \right) (\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k) \\
= & -2. d_{K_\hbar}^{(1,0)}(\downarrow a_1 \otimes \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*) \otimes \mathrm{S}^2(N^*[-1])}(a_2), \downarrow n_j \otimes \downarrow n_{j'} > \odot k) \\
& + \theta^{(1)}((-1)^{1+|a_1|} \downarrow (a_1 \odot a_2) \otimes \downarrow k) + \theta^{(1)}(\downarrow a_1 \otimes \downarrow P_{K_\hbar}(a_2) \odot k) \\
= & 2. \downarrow P_{K_\hbar}(a_1) \odot m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*) \otimes \mathrm{S}^2(N^*[-1])}(a_2), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \\
& - (-1)^{|a_1|} 2. \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*) \otimes \mathrm{S}^2(N^*[-1])}(a_1 \odot a_2), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \\
& + 2. \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*) \otimes \mathrm{S}^2(N^*[-1])}(a_1), \downarrow n_j \otimes \downarrow n_{j'} > \odot P_{K_\hbar}(a_2) \odot k \\
= & 2. \downarrow m^I \odot m^{I'} \odot < P_{\mathrm{S}(M^*) \otimes N^*[-1]}(a_1) \odot P_{\mathrm{S}(M^*) \otimes N^*[-1]}(a_2), \downarrow n_j \otimes \downarrow n_{j'} > \odot k \\
= & + \downarrow < P_{\mathrm{S}(M^*) \otimes N^*}(\downarrow a_1), n_j > \odot m^I \odot < P_{\mathrm{S}(M^*) \otimes N^*}(\downarrow a_2), n_{j'} > \odot m^{I'} \odot k \\
& - \downarrow < P_{\mathrm{S}(M^*) \otimes N^*}(\downarrow a_1), n_{j'} > \odot m^{I'} \odot < P_{\mathrm{S}(M^*) \otimes N^*}(\downarrow a_2), n_j > \odot m^I \odot k \\
= & -[\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]_{Com}^{(2)}(\downarrow a_1 \otimes \downarrow a_2 \otimes \downarrow k).
\end{aligned}
$$

So we have proved that $\kappa_{L_\hbar}([m^I \otimes n_j, m^{I'} \otimes n_{j'}]_{L_\hbar}) - [\kappa_{L_\hbar}(m^I \otimes n_j), \kappa_{L_\hbar}(m^{I'} \otimes n_{j'})]_{Com}$ is a 0-coboundary and so $H \circ \kappa_{L_\hbar}$ is a Lie morphism. $\square$

Proof of II.3.3-7:

We now prove that $H\circ\kappa_{L_\hbar}$ and $H\circ\kappa_{R_\hbar}$ satisfy the conditions of the universal property of $\mathscr{U}(R_\hbar, L_\hbar)$. As before, we prove that the conditions applied to $\kappa_{L_\hbar}$ and $\kappa_{R_\hbar}$ hold up to 0-coboundaries, and in this case it holds stricly (the 0-coboundary is 0):
We consider basis elements $m^{I'}\in R=\mathrm{S}(M^*)$ and $m^I\otimes n_j\in L=\mathrm{S}(M^*)\otimes N$, then we have the following arity-0 Taylor component applied to any $k\in K$,

$$\begin{aligned}
&(\kappa_{R_\hbar}(m^{I'})\circ\kappa_{L_\hbar}(m^I\otimes n_j))^{(0)}(\downarrow k)\\
&=\kappa_{R_\hbar}(m^{I'})^{(0)}\circ\kappa_{L_\hbar}(m^I\otimes n_j)^{(0)}(\downarrow k)\\
&=\kappa_{R_\hbar}(m^{I'})^{(0)}(\downarrow m^I\odot\rho_\hbar(n_j)(k))\\
&=\downarrow m^{I'}\odot m^I\odot\rho_\hbar(n_j)(k)\\
&=\kappa_{L_\hbar}(m^{I'}\odot m^I\otimes n_j)^{(0)}(\downarrow k)\\
&=\kappa_{L_\hbar}(\phi_{L_\hbar}(m^{I'})(m^I\otimes n_j))^{(0)}(\downarrow k).
\end{aligned}$$

Similarily, given elements $a\in A$ and $k\in K$ we have the following arity-1 Taylor component:

$$\begin{aligned}
&(\kappa_{R_\hbar}(m^{I'})\circ\kappa_{L_\hbar}(m^I\otimes n_j))^{(1)}(\downarrow a\otimes\downarrow k)\\
&=\kappa_{R_\hbar}(m^{I'})^{(0)}\circ\kappa_{L_\hbar}(m^I\otimes n_j)^{(1)}(\downarrow a\otimes\downarrow k)\\
&=\downarrow m^{I'}\odot<P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a),n_j>\odot m^I\odot k\\
&=\kappa_{L_\hbar}(m^{I'}\odot m^I\otimes n_j)^{(1)}(\downarrow a\otimes\downarrow k)\\
&=\kappa_{L_\hbar}(\phi_{L_\hbar}(m^{I'})(m^I\otimes n_j))^{(1)}(\downarrow a\otimes\downarrow k),
\end{aligned}$$

and since all other Taylor components trivially vanish, it proves that for all $r\in R$ and $l\in L$, we have:

$$\kappa_{R_\hbar}(r)\kappa_{L_\hbar}(l)=\kappa_{L_\hbar}(\phi_{L_\hbar}(r)(l)).$$

It remains the second relation, here again we consider basis elements $m^{I'}\in R=\mathrm{S}(M^*)$ and $m^I\otimes n_j\in L=\mathrm{S}(M^*)\otimes N$, then we have the following arity-0 Taylor component applied to any $k\in K$,

$$\begin{aligned}
&[\kappa_{L_\hbar}(m^I\otimes n_j),\kappa_{R_\hbar}(m^{I'})]^{(0)}_{Com}(\downarrow k)\\
&=\kappa_{L_\hbar}(m^I\otimes n_j)^{(0)}\circ\kappa_{R_\hbar}(m^{I'})^{(0)}(\downarrow k)-\kappa_{R_\hbar}(m^{I'})^{(0)}\circ\kappa_{L_\hbar}(m^I\otimes n_j)^{(0)}(\downarrow k)\\
&=\kappa_{L_\hbar}(m^I\otimes n_j)^{(0)}(\downarrow m^{I'}\odot k)-\kappa_{R_\hbar}(m^{I'})^{(0)}(\downarrow m^I\odot\rho_\hbar(n_j)(k))\\
&=\downarrow m^I\odot\rho_\hbar(n_j)(m^{I'}\odot k)-\downarrow m^{I'}\odot m^I\odot\rho_\hbar(n_j)(k)\\
&=\downarrow m^I\odot\rho_\hbar(n_j)(m^{I'})\odot k\\
&=\kappa_{R_\hbar}(\rho_\hbar(m^I\otimes n_j)(m^{I'}))^{(0)}(\downarrow k).
\end{aligned}$$

Similarily, given elements $a\in A$ and $k\in K$, we have the following arity-1 Taylor component:

$$\begin{aligned}
&[\kappa_{L_\hbar}(m^I\otimes n_j),\kappa_{R_\hbar}(m^{I'})]^{(1)}_{Com}(\downarrow a\otimes\downarrow k)\\
&=\kappa_{L_\hbar}(m^I\otimes n_j)^{(1)}\circ(id_{A_\hbar[1]}\otimes\kappa_{R_\hbar}(m^{I'})^{(0)})(\downarrow a\otimes\downarrow k)-\kappa_{R_\hbar}(m^{I'})^{(0)}\circ\kappa_{L_\hbar}(m^I\otimes n_j)^{(1)}(\downarrow a\otimes\downarrow k)\\
&=\kappa_{L_\hbar}(m^I\otimes n_j)^{(1)}(\downarrow a\otimes\downarrow m^{I'}\odot k)-\kappa_{R_\hbar}(m^{I'})^{(0)}(\downarrow <P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a),n_j>\odot m^I\odot k)\\
&=\downarrow <P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a),n_j>\odot m^I\odot m^{I'}\odot k-\downarrow m^{I'}\odot <P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a),n_j>\odot m^I\odot k\\
&=0,
\end{aligned}$$

which is what we wanted since $\kappa_{R_\hbar}$ have a trivial arity-1 Taylor component. Again since all other Taylor components trivially vanish, it proves that for all $r\in R$ and $l\in L$ we have:

$$[\kappa_{L_\hbar}(l),\kappa_{R_\hbar}(r)]_{Com}=\kappa_{R_\hbar}(\rho_\hbar(l)(r)).$$

□

We now state the second main theorem, which says that the algebra $B_\hbar$ obtained from the deformation quantization of B, is actually the universal enveloping algebra of the formal deformation of the local Lie algebroid.

Theorem II.3.3-8 :

Given the formal deformation of a local Lie algebroid, $(M_\hbar,N_\hbar,[\bullet,\bullet]_{L_\hbar},\rho_\hbar)$, the following is an isomorphism of associative $\mathbb{Z}$-graded $\mathbb{K}[\![\hbar]\!]$-algebras:

$$\mathfrak{I}_{B_\hbar}\quad:\quad(\mathscr{U}(L_\hbar,R_\hbar),\bullet\cdot\bullet)\quad\to\quad(B_\hbar,\mathfrak{L}_{B_\hbar}(d_\hbar)+\nabla_{B_\hbar})\ ,$$

which is defined using Proposition II.3.3-4 and Proposition II.3.3-2 by:

$$\mathfrak{I}_{B_\hbar}:=H(R_{B_\hbar})^{-1}\circ\mathcal{K}.$$

Proof :

Since $H(R_{B_\hbar})$ is an isomorphism of $\mathbb{Z}$-graded associative $\mathbb{K}$-algebras and since $\mathcal{K}$ is a morphism of $\mathbb{Z}$-graded associative $\mathbb{K}$-algebras, then $\mathfrak{I}_{B_\hbar}$ is an isomorphism if and only if $\mathcal{K}$ is also one. To prove it, we use a formal deformation theory argument, by showing that the evaluation $\hbar=0$ also gives an isomorphism. As an associative algebra, $\mathscr{U}(L_0,R_0)$ is generated by $\mathrm{S}(M^*)\otimes N$, thus consider a basis element $m^I\otimes n_j$, since $\hbar=0$ and $m^I\otimes n_j\in L$ we know that:

$$\mathcal{K}(m^I\otimes n_j)=\kappa_{L_0}(m^I\otimes n_j)^{(1)}.$$

From [Cal+11] Proposition 7.5, we also know that R_{B_0} is an isomorphism, and we want to make $R_{B_0}(m^I\otimes n_j)$ more explicit. By the very definition of the right action, given m elements $a_1,\cdots,a_m\in A$ and an element $k\in K$ we have:

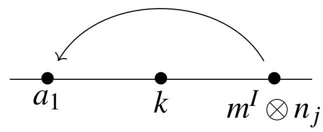
$$\begin{aligned}
&R_{B_0}(m^I\otimes n_j)^{(m)}(\downarrow a_1\otimes\cdots\otimes\downarrow a_m\otimes\downarrow k)\\
&=d_K^{(m,1)}(\downarrow a_1\otimes\cdots\otimes\downarrow a_m\otimes\downarrow k\otimes\downarrow(m^I\otimes n_j))\\
&=\sum_{\Gamma\in\mathcal{G}_{0,m+2}}\mu^{K[1]}_{m+2}\circ\left(\int_{\mathscr{C}^+_{0,m+2}}\prod_{e\in E(\Gamma)}\omega^K_e\right)(\downarrow a_1\otimes\cdots\otimes\downarrow a_m\otimes\downarrow k\otimes\downarrow(m^I\otimes n_j))
\end{aligned}$$

As usual, if we consider a graph, $\Gamma \in \mathcal{G}_{0,m+2}$, whose contribution in the previous sum is non-zero, then Γ is compelled to satisfy:

$$m = dim(\mathscr{C}^{+}_{0,m+2}) = \#E(\Gamma).$$

And, as before, any edge going out of one of the first $m+1$-th vertices induce terms of the forms ι_{dx_k} which trivially vanish on A or K.

It therefore only remains edges going out of the $m+2$-th point colored by $m^I \otimes n_j$ which vanish if more than one edge share this point as a starting point. In consequence, $m = 1$ and Γ has only one edge as in the graph on the right.



We can compare the two elements of $\mathrm{End}_{\mathrm{T}(A_\hbar[1])-\mathrm{comod}}(\mathrm{T}(A_\hbar[1]) \otimes K_\hbar[1])$ by evaluating them. Consider an element $a \in A$ and $k \in K$, then:

$$\kappa_{L_0}(m^I \otimes n_j)^{(1)}(\downarrow a \otimes \downarrow k) = \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^I \odot k$$

While:

$$\begin{aligned}
&R_{B_0}(m^I \otimes n_j)^1(\downarrow a \otimes \downarrow k) \\
&= \mu_3^{K[1]}\left(\int_{\mathscr{C}^+_{0,3}} \pi^*_{(3,1)}(\omega^{+,-}) \otimes \tau^{\mathbb{J}}_{(3,1)}(\downarrow a \otimes \downarrow k \otimes \downarrow (m^I \otimes n_j))\right) \\
&= \left(\int_{\mathscr{C}^+_{0,3}} \pi^*_{(3,1)}(\omega^{+,-})\right) \mu_3^{K[1]}(\partial_{\uparrow n^j}(\downarrow a) \otimes \downarrow k \otimes \downarrow m^I) \\
&= \left(\frac{1}{\pi}\int_0^{+\infty} \frac{\partial}{\partial z}\arg\left(\frac{\sqrt{z}-i}{\sqrt{z}+i}\right)\mathrm{d}z\right) \downarrow ((-1) < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot k \odot m^I) \\
&= \downarrow < P_{\mathrm{S}(M^*)\otimes N^*}(\downarrow a), n_j > \odot m^I \odot k.
\end{aligned}$$

So the two morphisms agree, which means that the image of R_{B_0} and the image of $\mathcal{K}$ coincides, and in consequence $\mathcal{K}$ is an isomorphism of $\mathbb{Z}$-graded $\mathbb{K}$-modules for $\hbar = 0$ which lead to the final claim. □

II.4 Quantization theorems

II.4.1 The dual of a Lie algebra

In his paper [Kon03], Kontsevich gave a direct application of his formality theorem, by quantizing the linear Poisson structure of the dual of a Lie algebra. We review this application and generalize the result to Lie algebroids.

Consider a finite d-dimensional Lie $\mathbb{R}$-algebra $\mathfrak{g}$ and its $\mathbb{R}$-linear dual $\mathfrak{g}^*$ It is a well-known fact that $\mathfrak{g}^*$ is naturally endowed with a Poisson structure called the Kirillov-Kostant-Souriau Poisson bracket. Given a point p in $\mathfrak{g}^*$, the differential at p of a smooth function f in $C^\infty(\mathfrak{g}^*)$ is an element df_p in $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$. Hence, given two smooth functions f,h in $C^\infty(\mathfrak{g}^*)$ one can define a new smooth function:

$$\{f,h\}(p) :=< p, [df_p, dh_p] > .$$

This bilinear bracket is known to obey the Leibniz rule and the Jacobi identity, and thus defines a structure of a Poisson algebra on $C^\infty(\mathfrak{g}^*)$, which can be expressed in local coordinates as follows. The identification $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ associates to each element e in $\mathfrak{g}$, an $\mathbb{R}$-linear map e^* defined by,

$$\begin{array}{rcccc} e^* & : & \mathfrak{g}^* & \to & \mathbb{R} \\ & & \psi & \mapsto & \psi(e) \end{array}$$

By considering a basis $(e_1, \cdots, e_d)$ of $\mathfrak{g}$, this identification gives a set of linear coordinates $(x_1, \cdots, x_d)$ on $\mathfrak{g}^*$ by setting $x_i := e_i^*$. The Kirillov-Kostant-Souriau Poisson bracket is then locally expressed as:

$$\{f,h\} = \sum_{1\leq i,j\leq d} [e_i, e_j]^* \frac{\partial f}{\partial x_i} \frac{\partial h}{\partial x_j}$$

Stricly speaking, Kontsevich's work does not apply to all smooth functions but rather only on polynomial functions, therefore we consider the symmetric algebra $\mathrm{S}(\mathfrak{g})$ as being identified with the subalgebra of polynomial functions over $\mathfrak{g}^*$.

On the other side, the formal $\mathbb{R}$-algebra $\mathfrak{g}[\![\hbar]\!]$, endowed with the bracket $\hbar[\bullet,\bullet]$ is naturally a Lie algebra, and we denote by $U_\hbar(\mathfrak{g})$ the Universal Enveloping algebra of $(\mathfrak{g}[\![\hbar]\!], \hbar[\bullet,\bullet])$.

Theorem II.4.1-1 : [Kon03, Theorem 8.2]

Consider the Poisson manifold $\mathfrak{g}^*$. Kontsevitch's quantization of the Poisson algebra $(\mathrm{S}(\mathfrak{g}), \{\bullet,\bullet\})$ of polynomial functions over $\mathfrak{g}^*$, is isomorphic to $U_\hbar(\mathfrak{g})$.

This result has been reproduced in [CFR11] using a different approach and in the framework of [Cal+11]. Since Kontsevich's proof of this theorem relies on the deformation of the Lie algebra of functions over $\mathfrak{g}^*$, it is not clear if and how to extend this result to Lie algebroids. In the next section, we show that it is indeed a direct application of the previously shown theorems.

II.4.2 THE DUAL OF A LOCAL LIE ALGEBROID

The definition of the dual of a Lie algebroid in the sens of differential geometry can be found in [CW99, §16.5]. Here we consider the setting of a local Lie algebroid $(M,N,[\bullet,\bullet],\rho)$ as in Definition II.1.2-2, where M and N are finite dimensional $\mathbb{Z}$-graded $\mathbb{R}$-modules concentrated in degree 0 and the local Lie algebroid $L = \mathrm{S}(M^*)\otimes N$ is a free $\mathrm{S}(M^*)$-module.

The $\mathrm{S}(M^*)$-linear dual of L is then defined by $L^v := \mathrm{S}(M^*)\otimes N^*$, and is also a free $\mathrm{S}(M^*)$-module, hence $(L^v)^v \cong L$. We then consider the algebra $O(L^v)$ of $\mathrm{S}(M^*)$-linear polynomial functions over L^v:

$$O(L^v) := \mathrm{S}_{\mathrm{S}(M^*)}((L^v)^v) \cong \mathrm{S}(M^*)\otimes \mathrm{S}(N) \cong \mathrm{S}(M^*\oplus N)$$

As for classical Lie algebra, one can define a Poisson bracket, $\{\bullet,\bullet\}$, on $O(L^v)$ by setting it on generators of the symmetric algebra, for all x_1,x_2 in N and all f_1,f_2 in M^*, and then extending it as a biderivation:

$$\begin{aligned}
\{x_1,x_2\} &:= [x_1,x_2],\\
\{f_1,f_2\} &:= 0,\\
\{x_1,f_1\} &:= \rho(x_1)(f_1),\\
\{f_1,x_1\} &:= -\rho(x_1)(f_1).
\end{aligned}$$

— Proposition II.4.2-1 :

The pair $(O(L^v),\{\bullet,\bullet\})$ is a Poisson algebra.

Proof :

The previously defined bracket $\{\bullet,\bullet\}$ is an anti-symmetric bilinear map subject to the Leibniz rule in each argument. Hence, we only need to check the Jacobi identity:

$$\begin{aligned}
\forall x_1,x_2,x_3\in N,\qquad & \{x_1,\{x_2,x_3\}\}+\{x_3,\{x_1,x_2\}\}+\{x_2,\{x_3,x_1\}\}\\
&= [x_1,[x_2,x_3]]+[x_3,[x_1,x_2]]+[x_2,[x_3,x_1]]\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\forall x_1,x_2\in N,\ f_3\in M^*\qquad & \{x_1,\{x_2,f_3\}\}+\{f_3,\{x_1,x_2\}\}+\{x_2,\{f_3,x_1\}\}\\
&= \rho(x_1)\circ\rho(x_2)(f_3)-\rho([x_1,x_2])(f_3)-\rho(x_2)\circ\rho(x_1)(f_3)\\
&= 0,
\end{aligned}$$

where the last equality holds because ρ is a morphism of Lie algebras and other identities involving more than one term of M^* trivially vanishes. □

We recall that $\{m_i\}_{i\in\mathbb{I}}\in M^{\mathbb{I}}$ (resp. $\{n_j\}_{j\in\mathbb{J}}\in N^{\mathbb{J}}$) stands for a basis of M (resp. N), $\{m^i\}_{i\in\mathbb{I}}$ (resp. $\{n^j\}_{j\in\mathbb{J}}$) stands for the dual basis and given a multi-index $I\in\mathrm{Multi}(\mathbb{I})$ we denote by m_I the tensor $m_{I_1}\odot\cdots\odot m_{I_{\#I}}$, with similar notations for N and duals and that we assume Einstein notation. The tensor decomposition of the structure morphisms of the local Lie algebroid are:

$$\rho = \mathbf{r}_i^{I,j}\, m^I \otimes m_i \otimes n^j \qquad \text{and} \qquad [\bullet,\bullet]_N = \mathbf{b}_{j_3}^{I,j_1,j_2}\, m^I \otimes n_{j_3} \otimes (n^{j_1} \otimes n^{j_2}).$$

Hence, as an anti-symmetric biderivation, the Poisson bracket, $\{\bullet,\bullet\}$, has a tensor representation in $\mathrm{S}(M^*) \otimes \mathrm{S}(N) \otimes \mathrm{S}^2((M \oplus N^*)[1])$ given by:

$$\{\bullet,\bullet\} = -\mathbf{b}_{j_3}^{I,j_1,j_2}\, m^I \otimes n_{j_3} \otimes (\downarrow n^{j_1} \odot \downarrow n^{j_2}) + 2.\mathbf{r}_i^{I,j}\, m^I \otimes (\downarrow n^j \odot \downarrow m_i).$$

It is therefore quite obvious that one half of the Poisson bracket $\frac{1}{2}\{\bullet,\bullet\}$ is identified under appropriate suspensions with the Maurer-Cartan element d of Theorem II.2.2-1, and so we get the following theorem.

Theorem II.4.2-2 :

Given a local Lie algebroid, $(M, N, [\bullet,\bullet], \rho)$, with M and N concentrated in degree 0, and the setting of the underlying Lie-Rinehart pair where,

$$R = \mathrm{S}(M^*) \qquad \text{and} \qquad L = \mathrm{S}(M^*) \otimes N.$$

Consider the Poisson manifold L^v. The quantization of the Poisson algebra, $(O(L^v), \frac{1}{2}\{\bullet,\bullet\})$, of polynomial functions over L^v, is isomorphic to the Universal Enveloping algebra of the Lie-Rinehart pair, $(\mathscr{U}(L_\hbar, R_\hbar), \bullet\cdot\bullet)$.

Proof :

It is a direct consequence of Theorem II.3.3-8, since $\frac{1}{2}\{\bullet,\bullet\} = d$ and $O(L^v) = B$. □

Remark II.4.2-3 :

In the special case of a trivial local Lie algebroid where $M = \{0\}$ and $\rho = 0$, $(L, [\bullet,\bullet])$ is a Lie algebra, and we recover Theorem II.4.1-1.

II.4.3 Toward $\mathscr{L}_\infty$-algebras

Going back to the begining, the reader may have noticed that instead of working with Lie algebroids, we could have worked with $\mathscr{L}_\infty$-algebras. Indeed, as we already mentionned, the Lie bracket $[d,d]_{\mathrm{T}^\bullet_{\mathrm{poly}}(X)}$ is here equal to $d \circ d$ and thus, d can be seen as a graded derivation of the algebra $\mathrm{S}((M \oplus N[1])^*)$ squaring to 0. This leads to the following proposition:

Proposition II.4.3-1 :

Given a local Lie algebroid $(M, N, [\bullet,\bullet], \rho)$ where M and N are finite dimensional $\mathbb{Z}$-graded $\mathbb{K}$-modules concentrated in degree zero.
The Maurer-Cartan element d constructed in Theorem II.2.2-1 defines the structure of an $\mathscr{L}_\infty$-algebra over $M[-1] \oplus N$.

Proof :

Since M and N are concentrated in degree zero. The anchor ρ is a graded morphism of degree zero, thus d as a graded derivation of $\mathrm{S}((M[-1] \oplus N)[1]^{*})$ is a graded morphism of degree 1 squaring to zero. □

Therefore, since $X = M \oplus N[1]$, we may hope to apply the same procedure for a general $\mathscr{L}_\infty$-algebra structure over $X[-1]$, which is then called a local $\mathscr{L}_\infty$-algebroid, and to recover an $\mathscr{A}_\infty$-algebraic analogue of the Chevalley-Eilenberg algebra and the Universal enveloping algebra of the $\mathscr{L}_\infty$-algebroid as seen in [Bar08].

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