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présentée par

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Contributions to the theory of KZB associators

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Alone this summer morning on the deserted wharf,
I look toward the bar, I look toward the Indefinite,
I look and am glad to see
The tiny black figure of an incoming steamer.
[...]

Fernando Pessoa, *Maritime Ode,*
A Little Larger Than The Entire Universe: Selected Poems.

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À ma mère, à mon père,

Résumé

Dans cette thèse, en suivant les travaux initiés par V. Drinfeld, poursuivis par B. Enriquez, puis par ce dernier, D. Calaque et P. Etingof, nous étudions la connexion KZB elliptique cyclotomique (ellipsitomique en plus court) universelle, associée à l'espace de modules des courbes elliptiques avec n points marqués et une structure de (M, N)-niveau. La platitude de cette connexion nous permet d'étudier des relations de monodromie, ouvrant la voie à une théorie générale des associateurs ellipsitomiques et des groupes de Grothendieck-Teichmüller qui lui correspondent, que l'on dégage via l'utilisation du formalisme des opérades (et certaines de leurs variantes) en nous basant sur les travaux de B. Fresse à ce sujet. D'une part, ce formalisme nous permet par ailleurs d'étudier la structure des associateurs en genre supérieur. D'autre part, l'associateur KZB ellipsitomique nous permet de dégager une théorie des valeurs multizêta elliptiques en des points de torsion, dont on démarque quelques unes de leurs premières propriétés du type associateurs.

On commencera par mettre en place la machinerie opéradique nécessaire pour définir les associateurs ellipsitomiques en partant tour à tour de la situation déjà connue en genre 0, puis de celle en genre 1 et ensuite de leurs variantes cyclotomiques. Enfin, grâce à ce formalisme, nous dégagerons une définition des associateurs en tout genre.

Ensuite, nous entrerons dans le détail de la construction de la connexion KZB ellipsitomique universelle, en premier temps sur l'espace de configurations (M, N)-décorées d'une courbe elliptique puis sur les espaces de modules des courbes à niveau, nous la lieront à sa version réalisée via l'utilisation des algèbres de Hecke doublement affines et des r-matrices classiques dynamiques. Pour finir nous présenterons les applications de cette construction, à savoir : formalité de certains sous-groupes de tresses sur le tore, l'associateur KZB ellipsitomique, valeurs multizêta elliptiques en des points de torsion ainsi qu'une application en représentations d'algèbres de Cherednik cyclotomiques.

Mots-clés

Connexions KZB universelles, associateurs de Drinfeld, groupes de Grothendieck-Teichmüller, valeurs multizêta elliptiques en des points de torsion.

Contributions to the theory of KZB associators

Abstract

In this thesis, following the work initiated by V. Drinfeld and pursued by B. Enriquez, then by the latter together with D. Calaque and P. Etingof, we study the universal twisted elliptic (ellipsitomic in short) KZB connection, associated to the moduli space of elliptic curves with n marked points and a (M, N)-level structure. The flatness of this connection allows us to study monodromy relations satisfied by this connection, opening the way to a general theory of ellipsitomic associators and Grothendieck-Teichmüller groups corresponding to them, which is released via the use of the formalism of operads (and some of their variants) basing ourselves on the work of B. Fresse. On the one hand, this formalism allows us to study the structure of associators in higher genus. On the other hand, the ellipsitomic KZB associator allows us to derive a theory of elliptic multiple zeta values at torsion points, from which some of their first associator-like properties are distinguished.

We will begin by setting up the operadic machinery necessary to define the ellipsitomic associators starting successively with the genus 0 situation, which is well-known, then the genus 1 situation and their cyclotomic variants. Then, in light of this formalism, we will release a definition of genus g associators.

Next, we will go into the details of the construction of the universal ellipsitomic KZB connection, first over the (M, N)-twisted configuration space of an elliptic curve and then over the moduli space of elliptic curves with a level structure. We will associate this connection to its realized version by means of the use of double affine Hecke algebras and of classical dynamical r-matrices. Finally we will present the applications of this construction, namely : the formality of certain subgroups of the braid group on the torus, the ellipsitomic KZB associator, elliptic multiple zeta values at points of torsion as well as an application in representations of cyclotomic Cherednik algebras.

Keywords

Universal KZB connections, Drinfeld associators, Grothendieck-Teichmüller groups, elliptic multiple zeta values at torsion points.

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Chapter 1

Introduction

1.1 Motivation

1.1.1 Associators

The theory of Drinfeld associators was introduced by the ukrainian mathematician Vladimir Drinfeld in his famous article [31]. It is an example of an object that mathematics borrow from physics and whose mathematical significance ends up being independent of its physical importance. In particular the following ideas[1]

- Quantum groups (Drinfeld) : associators produce quantizations of Lie bialgebras.
- Conformal Field theory and Wess-Zumino-Witten models (Witten[2]) : the KZ connection appears naturally in the geometric quantization of 3-dimensional Chern-Simmons theory[3].
- Algebraic topology of varieties and 3-dimensional topological invariants (Witten, Kontsevich[4]) : the universal enveloping algebra of the holonomy Lie $\mathbb{C}$-algebra of the configuration space of the complex plane, which is where the KZ connection is defined, is precisely the algebra of horizontal string diagrams.

served to answer deep problems in

- Number theory (Drinfeld, see [31]) : the KZ Associator is a generating series of all multizeta values, which satisfy associator-like relations.
- Geometric Galois theory (Grothendieck[5]-Drinfeld, see [31] and [61]) : the set of associators is a torsor under the action of a group whose profinite version contains the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

[1] Non-exhaustive list, as well as cited authors with significant contributions in the theory of associators.
[2] see for example his article [105].
[3] The reader can also discover some parts this wide field in the excellent introduction [52] on the subject.
[4] see in particular his article [77].
[5] see in particular his manusscript «Esquisse d'un Program» [61].

- Deformation quantization and formality (Kontsevich, Tamarkin, see [77] and [97]) : each Drinfeld associator provides a universal deformation quantization (i.e. of a universal « star product ») in the space of « observables » of a Poisson variety. Each associator produces a formality morphism of the little disks operad.

Initially, in his seminal work [31] and motivated by the construction of quasi-Hopf algebras, V. Drinfeld was looking to « universalize » the construction associated with the monodromy of a system of differential equations with non-commutative variables coming from high energy physics and showed that, not only associators over $\mathbb{C}$ and over $\mathbb{Q}$ exist, but their existence mobilizes the theory of a mysterious group, the Grothendieck-Teichmüller group (in particular its $\mathbf{k}$-pro-unipotent version), whose existence has been foreseen by Alexander Grothendieck in [61] (see also [27]). This group (and its different completed versions) is very important because it intervenes in several sectors of mathematics (see for example [29] and [23]).

The construction of this connection goes as follows. First observe that the holonomy Lie algebra of the configuration space

$$\mathrm{Conf}(\mathbb{C}, n) := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ if } i \neq j\}$$

of n points on the complex line is isomorphic to the graded Lie $\mathbb{C}$-algebra $\mathfrak{t}_n$ generated by t_{ij}, $1 \leq i \neq j \leq n$, with relations

(S) $t_{ij} = t_{ji}$,

(L) $[t_{ij}, t_{kl}] = 0$ if $\#\{i, j, k, l\} = 4$,

(4T) $[t_{ij}, t_{ik} + t_{jk}] = 0$ if $\#\{i, j, k\} = 3$.

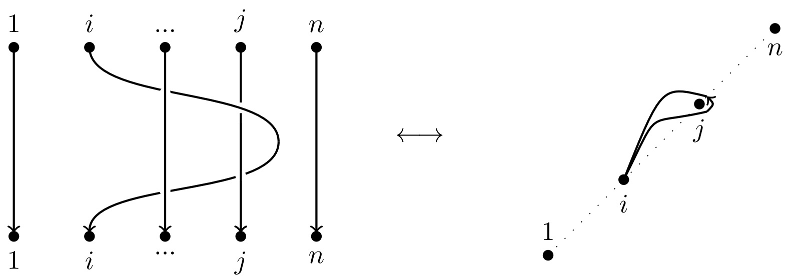
On the one hand, denote by PB_n the fundamental group of $\mathrm{Conf}(\mathbb{C}, n)$, also known as the pure braid group with n strands, and by $\mathfrak{pb}_n$ its Malcev Lie algebra (which is filtered by its lower central series, and complete). Then, one can easily check that PB_n is generated by elementary pure braids P_{ij}, $1 \leq i < j \leq n$, which satisfy (at least) the following relations:

(PB1) $(P_{ij}, P_{kl}) = 1$ if $\{i, j\}$ and $\{k, l\}$ are non crossing,

(PB2) $(P_{kj} P_{ij} P_{kj}^{-1}, P_{kl}) = 1$ if $i < k < j < l$,

(PB3) $(P_{ij}, P_{ik} P_{jk}) = (P_{jk}, P_{ij} P_{ik}) = (P_{ik}, P_{jk} P_{ij}) = 1$ if $i < j < k$.

We can depict the generator $P_{i,j}$ in the following two equivalent ways:



Therefore one has a surjective morphism of graded Lie algebras $p_n : \mathfrak{t}_n \twoheadrightarrow \mathrm{gr}(\mathfrak{pb}_n)$ sending t_{ij} to $\sigma(\log(P_{ij}))$, $i < j$ where $\sigma : \mathfrak{pb}_n \longrightarrow \mathrm{gr}(\mathfrak{pb}_n)$ is the symbol map.

On the other hand, denote $\exp(\hat{\mathfrak{t}}_n)$ the exponential group associated to the degree completion $\hat{\mathfrak{t}}_n$ of $\mathfrak{t}_n$. The universal KZ connection on the trivial $\exp(\hat{\mathfrak{t}}_n)$-principal bundle over $\mathrm{Conf}(\mathbb{C}, n)$ is then given by the holomorphic 1-form

$$w_n^{\mathrm{KZ}} := \sum_{1 \leqslant i < j \leqslant n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij} \in \Omega^1(\mathrm{Conf}(\mathbb{C}, n), \mathfrak{t}_n),$$

which takes its values in $\mathfrak{t}_n$. It is a fact that the connection associated to this 1-form is flat and descends to a flat connection over the moduli space $\mathcal{M}_{0,n+1} \simeq \mathrm{Conf}(\mathbb{C}, n)/\mathrm{Aff}(\mathbb{C})$ of rational curves with $n + 1$ marked points.

First, the regularized holonomy of this connection along the real straight path from 0 to 1 in $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ gives an element $\Phi_{\mathrm{KZ}} \in \mathbb{C}\langle\langle x_0, x_1\rangle\rangle$ called the KZ associator that is a generating series for values at 0 and 1 of multiple polylogarithms, the latter being precisely multiple zeta values ([80],[49]). Next, using the monodromy representation of the universal KZ connection, one obtains :

1. A morphism of filtered Lie algebras $\mu_n : \mathfrak{pb}_n \longrightarrow \hat{\mathfrak{t}}_n$ such that $\mathrm{gr}(\mu_n) \circ p_n = \mathrm{id}$. Hence one concludes that p_n and μ_n are bijective. This proves that $\mathfrak{pb}_n$ is isomorphic to the degree completion of its associated graded, which is actually $\mathfrak{t}_n$. We will then say that the group PB_n is *formal* (or more accurately *weakly* 1-*formal*, see the comments below).

2. A system of relations (called Pentagon (P) and two Hexagons $(H_\pm)$) satisfied by the KZ associator.

Notice that our definition of formality is weaker than the original one of Dennis Sullivan. If G is a finitely generated group and $\mathfrak{g}$ is the Lie algebra of its prounipotent completion, then G is 1-*formal* in the sense of [28] if $\mathfrak{g}$ is quadratically generated, that is, if there is an isomorphism

$$\mathfrak{g} \simeq \hat{\mathbb{L}}(H_1(G))/I$$

where I is the closed ideal in the free pronilpotent Lie algebra $\hat{\mathbb{L}}(H_1(G))$ generated by the subset

$$S \subset \Lambda^2 H_1(G) = \mathbb{L}_2(H_1(G)$$

of the degree 2 elements of the free Lie algebra $\mathbb{L}(H_1(G)$. In light of the above, we say that G is *weakly* 1-*formal* if

$$\mathfrak{g} \simeq \hat{\mathbb{L}}_2(H_1(G))/I$$

where I is the closure of an homogeneous ideal in $\hat{\mathbb{L}}(H_1(G))$. Then, by Morgan's work [87], the fundamental group of smooth complex algebraic varieties are weakly 1-formal but are not allways 1-formal. For instance, by Bezrukavnikov's results in [11], elliptic braid groups are not 1-formal for $n \geq 2$. Another example is the fundamental group of the C^*-bundle over an elliptic curve E associated to a line bundle of degree 1, as the Lie algebra of its unipotent fundamental group is the Heisenberg Lie algebra

$$\mathfrak{g} \simeq \hat{\mathbb{L}}(X, Y))/(\mathrm{ad}_X^2(Y), \mathrm{ad}_Y^2(X)),$$

which is homogeneous but not quadratically presented. Nevertheless, if $g \geq 2$, all genus g braid groups are 1-formal.

Then, V. Drinfeld showed that the set $\mathrm{Ass}(\mathbf{k})$ is a torsor under the action of an important and somewhat mysterious group : the prounipotent Grothendieck-Teichmüller group, denoted $\mathbf{GT}(\mathbf{k})$. $\mathrm{Ass}(\mathbf{k})$ is also a torsor under the action of its graded version, denoted by $\mathbf{GRT}$. The starting point into the consideration of this group is that it arises in Grothendieck's program of studying the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ through its outer action on the algebraic fundamental group(oid) of the moduli spaces of curves $\mathcal{M}_{g,n}$. The group $\mathbf{GT}(\mathbf{k})$ has at least a profinite and a pro-ℓ version, but it is the easiest of the three to work with. It is then a fact that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ injects into the profinite Grothendieck-Teichmüller group and it has been famously conjectured to be isomorphic to this group. Since then, the KZ equations became popular among mathematicians and they were quickly noticed to have relations to several other mathematical fields such as number theory, quantum group theory and deformation quantization.

Finally, on the "iterated integral" point of view, one is brought to characterise MZVs as being periods of $\mathcal{M}_{0,n}$. In fact, if we denote $\mathrm{MT}(\mathbb{Z})$ for the Tannakian category of mixed Tate motives over $\mathbb{Z}$, then MZVs are periods of $\mathrm{MT}(\mathbb{Z})$ which bring us to consider their motivic versions. Motivic MZVs (mMZVs) proved to be very important as they permit to work with a crucially useful formula due to A. Goncharov ([59]) and F. Brown ([21]) for the coaction of the graded ring of affine functions on the prounipotent part of the Galois group of $\mathrm{MT}(\mathbb{Z})$ over $\mathbb{Q}$. As an application of these tools, F. Brown has shown that all periods of $\mathrm{MT}(\mathbb{Z})$ are $\mathbb{Q}[\frac{1}{2\pi \mathrm{i}}]$-linear combinations of MZVs, that every MZV of weight N is a $\mathbb{Q}$-linear combination of elements of the set $\{\zeta(k_1, ..., k_r),\ \text{where } k_i = 2 \text{ or } 3,\ \text{and } k_1 + \cdots k_r = N\}$ ([21]). Other striking results of the use of mMZVs can be found in perturbative Quantum Field Theory ([22]) and, more recently, in perturbative Superstring Theory ([92]).

1.1.2 Generalisations I : The cyclotomic case

Similarly, one can consider the configuration space

$$\mathrm{Conf}(\mathbb{C}^\times, n) := \{\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n | z_i \neq z_j \text{ if } i \neq j\}$$

of n points on $\mathbb{C}^\times$. Then $\mathrm{Conf}(\mathbb{C}^\times, n) = \mathrm{Conf}(\mathbb{C}, n+1)/\mathbb{C}$ and thus its fundamental group PB_n^1 is isomorphic to PB_{n+1}. More generally, for any $M \in \mathbb{Z} - \{0\}$ one can consider an M-twisted configuration space

$$\mathrm{Conf}(\mathbb{C}^\times, n, M) := \{\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n | z_i^M \neq z_j^M \text{ for some } i \neq j\}.$$

In [33], B. Enriquez used the so-called universal trigonometric KZ connection, to prove that one has an isomorphism $\mathfrak{pb}_n^M \longrightarrow \exp(\hat{\mathfrak{t}}_n^M)$, where $\mathfrak{pb}_n^M$ is the Malcev Lie algebra of the fundamental group $\mathrm{PB}_n^M \subset \mathrm{PB}_n^1$ of $\mathrm{Conf}(\mathbb{C}^\times, n, M)$, and $\mathfrak{t}_n^M$ is the holonomy Lie algebra of $\mathrm{Conf}(\mathbb{C}^\times, n, M)$. The holonomy of this connection along a suitable (non closed) path gives a universal pseudotwist $\Psi_{\mathrm{KZ}}^M \in \exp(\mathfrak{t}_2^M)$ that is a generating series for values of multiple polylogarithms at Mth roots of unity i.e. cyclotomic MZVs (which will be denoted μ_M-MZVs), satisfies relations with Φ_{KZ} and whose monodromy will give us cyclotomic associator relations.

Finally, the set $\mathrm{Ass}(M, \mathbf{k})$ of so-called cyclotomic associators is a torsor under the action of the cyclotomic analog $\widehat{\mathrm{GT}}_M(\mathbf{k})$ of the group $\widehat{\mathrm{GT}}(\mathbf{k})$, which maps to $\widehat{\mathrm{GT}}(\mathbf{k})$ and whose associated Lie algebra is isomorphic to its associated graded $\mathfrak{grt}_M$.

As iterated integrals, μ_M-MZVs are shown to be periods of $\mathbb{P}^1 - \{0, \mu_M, \infty\}$. In fact, by relying on Deligne's theory of the motivic fundamental group of $\mathbb{G}_m - \mu_M$ and on F. Brown and A. Goncharov's explicit coaction formula, C. Glanois used in [57] motivic μ_M-MZVs to show analog results on generating families for μ_M-MZVs and studied how the periods in $\mathbb{P}^1 - \{0, \mu_M, \infty\}$ relate to each other when taking different choices for M. Now, the main difference with the classical case is that the upper bound for the dimension of μ_M-MZVs of a given weight is reached in the cases[6] $M = 1, 2, 3, 4, 8$ but it is known to be not reached, for instance, if $M = p^s$ for a prime $p \geq 5$. This means that μ_M-MZVs are not enough to describe all periods of $\mathbb{P}^1 - \{0, \mu_M, \infty\}$ in this case.

Now, if we return to consider the set of cyclotomic associators one can show that if M' divides M, then Ψ^M_{KZ} and $\Psi^{M'}_{\mathrm{KZ}}$ satisfy distribution relations, analogously to C. Glanois distribution study. By imposing these relations one obtains a subset of cyclotomic associators which is a torsor under a certain subgroup of GT_M. This subgroup can be seen as an explicit approximation of the motivic fundamental group of $\mathbb{G}_m - \mu_M$.

1.1.3 Generalisations II : The elliptic case

The genus one universal Knizhnik–Zamolodchikov–Bernard (KZB) connection $\nabla^{\mathrm{KZB}}_{1,n}$ was introduced in [24]. This is a flat connection over the moduli space of elliptic curves with n marked points $\mathcal{M}_{1,n}$, which was independently discovered by Levin–Racinet [81] in the specific cases $n = 1, 2$. It restricts to a flat connection over the configuration space

$$\mathrm{Conf}(\mathbb{T}, n) := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i - z_j \notin \Lambda_\tau \text{ if } i \neq j\}/\Lambda_\tau^n$$

of n points on an (uniformized) elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, for $\tau \in \mathfrak{h}$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. More precisely, this connection is defined on a G-principal bundle over $\mathcal{M}_{1,n}$ where the Lie algebra associated to G has as components:

1. the holonomy Lie algebra $\mathfrak{t}_{1,n}$ of $\mathrm{Conf}(\mathbb{T}, n)$ controlling the variations of the marked points: it has generators x_i, y_i, for $i = 1, ..., n$, corresponding to moving z_i along the topological cycles generating $H_1(E_\tau)$;

2. a Lie algebra $\mathfrak{d}$ composed by the Lie algebra $\mathfrak{sl}_2$ with standard generators e, f, h and a Lie algebra $\mathfrak{d}_+ := \mathrm{Lie}(\{\delta_{2m} | m \geq 1\})$ such that each δ_{2m} acts as a highest weight element for $\mathfrak{sl}_2$. The Lie algebra $\mathfrak{d}$ controls the variation of the curve in $\mathcal{M}_{1,n}$.

Now, the connection $\nabla^{\mathrm{KZB}}_{1,n}$ can be locally expressed as $\nabla^{\mathrm{KZB}}_{1,n} := d - \Delta(\mathbf{z}|\tau)d\tau - \sum_i K_i(\mathbf{z}|\tau)dz_i$ where

1. the term $K_i(-|\tau) : \mathbb{C}^n \longrightarrow \hat{\mathfrak{t}}_{1,n}$ is holomorphic on

   $$\mathbb{C}^n - \mathrm{Diag}_{n,\tau} = \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i - z_j \in \Lambda_\tau \text{ if } i \neq j\},$$

   where $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$, with only poles at the diagonal in $\mathbb{C}^n$ and the Λ_τ^n-translates of this diagonal. It is constructed out of a function

   $$k(x, z|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)}, -\frac{1}{x}.$$

   This relates directly the connection $\nabla^{\mathrm{KZB}}_{1,n}$ with Zagier's work [106] on Jacobi forms and to Brown and Levin's work [20].

[6]The case $M = 6$ being also known but treated differently.

2. the term $\Delta(\mathbf{z}|\tau)$ is a meromorphic function $\mathbb{C}^n \times \mathfrak{h} \longrightarrow \mathrm{Lie}(G)$ with only poles at the diagonal in $\mathbb{C}^n \times \mathfrak{h}$ and the Λ_τ^n-translates of this diagonal. In particular, the coefficients of δ_{2m} in $\Delta(\mathbf{z}|\tau)$ are Eisenstein series.

We also refer to Hain's survey [63] and references therein for the Hodge theoretic and motivic aspects of the story. Let us fix $\tau \in \mathfrak{h}$. Recall that the Lie algebra $\bar{\mathfrak{t}}_{1,2}(\mathbb{C})$ is isomorphic to the free Lie algebra $\mathfrak{f}_2(\mathbb{C})$ generated by two elements $x := x_1$ and $y := y_1$. We define the elliptic KZB associators $A(\tau), B(\tau)$ as the regularized holonomies from 0 to 1 and 0 to τ of the differential equation

$$G'(z) = -\frac{\theta_\tau(z + \operatorname{ad} x) \operatorname{ad} x}{\theta_\tau(z)\theta_\tau(\operatorname{ad} x)}(y) \cdot G(z), \tag{1.1}$$

with values in the group $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}(\mathbb{C}))$ More precisely, this equation has a unique solution $G(z)$ defined over $\{a + b\tau,\ \text{for } a, b \in]0,1[\}$ such that $G(z) \simeq (-2\pi\,\mathrm{i}\, z)^{-[x,y]}$ at $z \longrightarrow 0$. In this case,

$$A(\tau) := G(z)^{-1}G(z+1), \quad B(\tau) := G(z)^{-1}e^{2\pi\,\mathrm{i}\,x}G(z+\tau).$$

These are elements of the group $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}(\mathbb{C}))$. Then, one can construct an holomorphic map sending each $\tau \in \mathfrak{h}$ to a couple $e(\tau) := (A(\tau), B(\tau))$ where $A(\tau)$ (resp. $B(\tau)$) is the regularized holonomy of the universal elliptic KZB connection along the the straight paths from 0 to 1 (resp. from 0 to τ) in the once punctured elliptic curve $(\mathbb{C} - \Lambda_\tau)/(\Lambda_\tau) \simeq \mathrm{Conf}(E_\tau, 2)/E_\tau$. Then, B. Enriquez described and studied in [34] the general theory of elliptic $\mathbf{k}$-associators, whose set is denoted $\mathrm{Ell}(\mathbf{k})$ and for which the couple $e(\tau)$ is an example of a $\mathbb{C}$-point. Some of the main features of the so-called elliptic KZB associators $e(\tau)$ are the following:

- They satisfy algebraic and modularity relations.
- They satisfy a differential equation in the variable τ expressed only in terms of iterated integrals of Eisenstein series, which will be called iterated Eisenstein integrals.
- When taking τ to $i\infty$ (which consists on computing the constant term of the q-expansion of the series $A(\tau)$ and $B(\tau)$), they can be expressed in terms of the KZ associator Φ_{KZ}.
- The set $\mathrm{Ell}(\mathbf{k})$ is a torsor under the actions of the elliptic analog $\mathbf{GT}_{e\ell\ell}(\mathbf{k})$ of the (prounipotent) group $\mathbf{GT}(\mathbf{k})$ and of its graded version $\mathbf{GRT}_{e\ell\ell}$.

Next, in [35], B. Enriquez studied the coefficients of the series $A(\tau)$ and $B(\tau)$ and showed they are the elliptic analogs of MZVs. These coefficients were called elliptic multiple zeta values (eMZVs) in analogy to the genus 0 story. They are functions denoted $I(\tau)$ and $J(\tau)$, depending on the elliptic parameter τ, which satisfy the following:

- when taking $\tau \longrightarrow \infty$, eMZVs can be expressed only in terms of MZVs;
- they satisfy a differential equation expressed in terms of iterated Eisenstein integrals which, analogously to the motivic coaction formula in the genus 0 cases, can be used to get results on generating families for eMZVs and their decomposition. In particular, in ([82]) there is a complete description of the algebras of the elliptic multiple zeta values $I(\tau)$ and $J(\tau)$ (modulo 2πi) in terms of multiple zeta values and special linear combinations of iterated Eisenstein integrals.

An important feature of these decompositions is that they are controlled by a special derivation algebra, first studied by H. Tsunogai ([100]) and by A. Pollack ([90]) which is deeply connected with both the Lie algebra of the (graded) elliptic Grothendieck-Teichmüller group and with the bi-graded Lie algebra of the prounipotent radical of $\pi^{geom}(\mathrm{MEM})$, where MEM denotes the Tannakian category of universal mixed elliptic motives constructed by R. Hain and M. Matsumoto in [64].

1.2 Contents

The purpose of this thesis is to define a twisted version of the genus one KZB associator introduced in [24] and [34]. The first part concerns foundational grounds which we will use to define ellipsitomic associators. We will redefine by means of our operadic approach elliptic, cyclotomic associators. Then we define ellipsitomic associators. Finally we concentrate in the framed case and give a definition of genus g associators based in our operadic approach.

The second part concerns the proof of the fact that the set of ellipsitomic $\mathbb{C}$-associators is not empty, by providing an ellipsitomic KZB associator. We start by focusing on the universal ellipsitomic KZB connection. This is a flat connection on a principal bundle over the moduli space of elliptic curves with a Γ-structure, where $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, and n marked points. It restricts to a flat connection on the so-called Γ-twisted configuration space of points on an elliptic curve, which can be used for proving the formality of some interesting subgroups of the pure braid group on the torus. Then, we define twisted elliptic associators as renormalized holonomies along certain paths on a once punctured elliptic curve with a Γ-structure. We study the monodromy of this connection and show that it gives rise to a relation between twisted elliptic associators, the KZ associator [31] and the cyclotomic KZ associator [33]. Moreover, twisted elliptic associators can be regarded as a generating series for iterated Eisenstein integrals whose coefficients are elliptic multiple zeta values at torsion points. In the case $M = N$, these coefficients are related to Goncharov's work [58] and also to the recent work [19] of Broedel–Matthes–Richter–Schlotterer. We finally conjecture that the universal KZB connection realizes as the usual KZB connection associated to elliptic dynamical r-matrices with spectral parameter [42, 44].

It is worth mentioning the recent work [98], where Toledano-Laredo and Yang define a similar KZB connection. More precisely, they construct a flat KZB connection on moduli spaces of elliptic curves associated with crystallographic root systems. The type A case coincides with the universal elliptic KZB connection defined in [24], and we suspect that the type B case coincides with the connection of the present paper for $M = N = 2$. It is interesting to point out that a common generalization of their work and ours (for $M = N$) could be obtained by constructing a universal KZB connection associated with arbitrary complex reflection groups, which shall be related to the (genus 0) universal KZ connection associated with finite subroups of $\mathrm{PSL}_2(\mathbb{C})$ ([85]).

The structure of this thesis goes as follows:

- **Chapter 2 :** This chapter sets the basics for the understanding of the rest of the thesis.
  - In Section 2.1 we introduce the formal definition of Drinfeld associators. We set up a lot of terminology involving the exponential group associated to a degree completed Lie algebra.
  - In Section 2.2 we introduce the KZ associator, first by using the universal KZ equations and then by using the universal KZ connection (it is the same construction under two slightly

different languages). By doing so, we elucidate the implicit operadic nature of the associator relations and we explain the word "universal" in a comprehensive manner. Then we use the flatness of the universal KZ connection to reprove the formality of the braid groups and we analyse the anatomy of the KZ associator involving multizeta values.

  – In Section 2.3 we explain how all the genus 0 theory translates to its cyclotomic counterpart.
  – In Section 2.4 we do the same for the elliptic counterpart.
  – In Section 2.5 we give a quick reminder of the general notions of operads, operadic modules, and moperads, in Section 2.1.
  – Finally, in Section 2.6, we associate these structures to the Fulton-MacPherson compactified configuration spaces in genus 0 and to the collections of their fundamental groupoids and of their holonomy Lie algebras. We also recall the operadic definitions of associators and Grothendieck-Teichmüller groups and enhance these notions into a torsor isomorphism between these and their non-operadic (classical) versions.

- **Chapter 3 :** In this chapter we present the main results of this thesis. We then enumerate some perspectives and future directions that can be undertaken after the work done here.
- **Chapter 4 :** This chapter is devoted to the definition of twisted elliptic associators and twisted elliptic Grothendieck-Teichmüller groups by means of operads in groupoids and their variants.
  – Section 4.1 is devoted to the corresponding - and equivalent - operadic definitions in the genus 1 case by using operad modules instead of operads, mainly following [34].
  – Next, in Section 4.2 we turn to the cyclotomic situation and proceed in the same way by using moperads this time.
  – Finally, in Section 4.3, we concentrate on the twisted elliptic (or ellipsitomic) situation and proceed by combining the use of operad modules and the lifting techniques we used in Sections 4.1 and 4.2. In particular we give a definition of ellipsitomic associators in terms of elements satisfying some explicit equations as well as ellipsitomic Grothendieck-Teichmüller groups in their **k**-prounipotent and graded versions.
- **Chapter 5 :** In this chapter we begin the study of genus g associators, for $g > 1$.
  – In Section 5.1 we remind the operadic module structures that are associated to framed Fulton-MacPherson compactified configuration spaces on a genus g oriented surface.
  – In Section 5.2 we concentrate in the genus 0 framed case and we associate operad structures to the collection of the corresponding framed configuration spaces an to the collection of their fundamental groupoids. We also associate an operadic structure to the collection of their holonomy Lie algebras. Then we give definitions of framed associators, show that they do not form an empty set for $\mathbf{k} = \mathbb{C}$ and show that they are the same as non-framed associators.
  – In Section 5.3 we give operadic definitions of genus g associators and Grothendieck-Teichmüller groups, which we relate to their classical point of view in terms of some elements satisfying relations. Then, we conjecture that the set of framed genus g associators

is not empty and we give a start on the study of the framed genus g universal KZB connection over the framed configuration space of points on a genus g surface, with the hope of showing that the the set of genus g associators over the complex numbers is not empty.

- **Chapter 6 :** In this chapter we define and study the universal twisted elliptic KZB connection.
  - In Section 6.1, we introduce Γ-twisted configuration spaces on an elliptic curve and define the universal Γ-KZB connection on them.
  - As in [24] the connection extends from the configuration space to the moduli space $\bar{\mathcal{M}}^{\Gamma}_{1,[n]}$ of elliptic curves with a Γ-level structure and marked points. This is proved in Section 6.3 using some technical definitions introduced in Section 6.2 related to the derivations of the holonomy Lie algebra $\mathfrak{t}^{\Gamma}_{1,n}$ of the twisted configuration space in genus 1. As in the untwisted case, the results of this section also apply to the "unordered marked points" situation.
  - In Section 6.4, we provide a notion of realizations for the Lie algebras previously introduced, and show that the universal KZB connection realizes to a flat connection intimately related to elliptic dynamical r-matrices with spectral parameter.
- **Chapter 7 :** In this chapter we sketch several applications of twisted elliptic associators and the twisted elliptic KZB connection.
  - In Section 7.1, we derive from the monodromy representation the formality of the fundamental group of the twisted configuration space of the torus, which is a subgroup of $\mathrm{PB}_{1,n}$. As in the cyclotomic case, this formality result extends to a relative formality result for the map $\mathrm{B}_{1,n} \longrightarrow \Gamma^n \rtimes \mathfrak{S}_n$.
  - Then, in Section 7.2, we show that this connection gives rise to a monodromy morphism $\gamma_n : \mathrm{B}^{\Gamma}_{1,[n]} \longrightarrow \mathbf{G}^{\Gamma}_n \rtimes \mathfrak{S}_n$. The relations between the generators give rise to twisted elliptic associator relations, providing an example of such an object.
  - In Section 7.3 we study the $A_{s,\gamma}(\tau)$ coefficients that were implicitely used in the definition of the universal twisted elliptic KZB connection by relating them to the so-called Eisenstein-Hurwitz series. We show that these are modular forms for the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defining $\bar{\mathcal{M}}^{\Gamma}_{1,[n]}$ and compute the constant terms in their q_N-expansion, where $q_N = e^{\frac{2\pi\mathrm{i}}{N}\tau}$.
  - Finally, in Section 7.4, we construct a homomorphism from the Lie algebra $\bar{\mathfrak{t}}^{\Gamma}_{1,n} \rtimes \mathfrak{d}^{\Gamma}$ to the twisted Cherednik algebra $H^{\Gamma}_n(k)$. This allows us to consider the twisted elliptic KZB connection with values in representations of the twisted Cherednik algebra.
- **Chapter 8 :** In this chapter we give a quick definition of elliptic multiple zeta values at torsion points in terms of iterated integrals of Eisenstein-Hurwitz series.
  - In Section 8.1 we give a definition of the twisted version of Pollack's Lie algebra of special derivations.
  - In Section 8.2 we use the action of the $\mathbf{k}$-prounipotent ellipsitomic Grothendieck-Teichmüller group on the ellipsitomic KZB associator to establish a differential equation in the variable $\tau \in \mathfrak{h}$ which is satisfied by this associator and which involves exclusively Eisenstein-Hurwitz series.

– In Section 8.3 we use the machinery of iterated integrals developped by B. Enriquez in [35] to give a definition of ellipsitomic multizeta values in terms of iterated integrals of Eisenstein-Hurwitz series strongly related to multiple Hurwitz values.

Note: A part of the results figuring in this thesis consist on an ongoing collaboration by the author and by Damien Calaque and appear in chapters 4, 6 and 7 in this thesis for sake of convenience.

Consistency

Chapters 4, 5 and 6 are essentially independent. Section 4.1 can be very iluminating for the understanding of chapter 5. Next, Section 4.3 and all sections of chapter 6 are related to each other in chapter 7, Section 7.2, where we use the universal twisted elliptic KZB connection (constructed in chapter 6) to prove that twisted elliptic associators (defined in chapter 3) do exist over the complex numbers. Finally, chapter 8 uses the results in chapter 7, Sections 7.1, 7.2 and 7.3.

$$\begin{tikzcd}
 & \text{Chap. } 2 \arrow[r, dotted] \arrow[dl] \arrow[d] \arrow[dr] & \text{Chap. } 3 \\
\text{Chap. } 4 \arrow[r, dotted] \arrow[dr] & \text{Chap. } 5 & \text{Chap. } 6 \arrow[dl] \\
 & \text{Chap. } 7 \arrow[d] & \\
 & \text{Chap. } 8 &
\end{tikzcd}$$

Chapter 2

Background

In the first part we will make a reminder on the most basic tools in the theory of associative and Lie $\mathbf{k}$-algebras which will be used, taking as an example the Kohno-Drinfeld Lie algebra $\mathfrak{t}_n$ that will be used extensively throughout this thesis. The objective of this first section is to fix all the notations that will be used throughout this thesis in a comprehensive manner, to give a formal definition of Drinfeld $\mathbf{k}$-associators and enunciate the fact that, when $\mathbf{k} = \mathbb{C}$, this set is not empty.

In the second part we will study the KZ equation and we will give a definition of the KZ associator from an analytic viewpoint. Then, we will make a small reminder on the basics of connections on a G-principal bundle. We will then introduce the universal KZ connection defined in a trivial $\exp(\hat{\mathfrak{t}}_n)$-principal bundle over the configuration space of the complex plane. Then we will give a geometrical definition of the KZ associator and we will prove that it provides a Drinfeld $\mathbb{C}$-associator.

In the third and fourth part we sketch the theory of the universal KZ associator in the cyclotomic and elliptic contexts.

In the fifth and sixth sections we give in a clear manner the definitions of Grothendieck-Teichmüller groups and associators by means of operad theory and Fulton-MacPherson compactifications.

Note. The material of this chapter is standard, the author does not claim originality of almost any result that figures in here. Bibliographical references will appear at the end of each section where the reader can extend the work presented in here and of which the author has been inspired to build this introduction.

Notation

- In this thesis $\mathbf{k}$ designates a field of characteristic zero.
- Unless otherwise stated, composition of morphisms are read from left to right.

2.1 Drinfeld associators

2.1.1 Associative and Lie k-algebras

Associative algebras

We recall the definition of an associative $\mathbf{k}$-algebra.

Definition 2.1.1. *An* associative $\mathbf{k}$-algebra *is a pair* $(\mathcal{A}, \cdot)$ *where* $\mathcal{A}$ *is a* $\mathbf{k}$*-vector space along with a bilinear map, called* multiplication

$$\begin{array}{rcl} \cdot : \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ (x, y) & \longmapsto & x \cdot y \end{array}$$

that satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for each* $x, y, z \in \mathcal{A}$*. It is said that the algebra* $\mathcal{A}$ *is* unitary *if there is a neutral element for the multiplication (that is, an element denoted* 1 *that satisfies* $1 \cdot x = 1 = x \cdot 1$ *for all* $x \in \mathcal{A}$*).*

Example 2.1.2. *Let us enumerate some examples of associative algebras.*

1. *The set of square matrices* $n \times n$ *with values in* $\mathbf{k}$ *forms a unitary associative algebra on* $\mathbf{k}$*, which is not commutative in general.*
2. *The set of complex numbers* $\mathbb{C}$ *forms an associative, commutative and unitary* $\mathbb{C}$*-algebra of real dimension 2.*
3. *Polynomials with coefficients in* $\mathbf{k}$ *form an infinite dimensional associative* $\mathbf{k}$*-algebra which is commutative and unitary.*
4. *In particular, the tensor space* TV *can be provided with the structure of an associative* $\mathbf{k}$*-algebra with multiplication*

$$\begin{array}{rcl} TV \times TV & \longrightarrow & TV \\ (x, y) = ((x_0, x_1, \ldots, x_n), (y_0, y_{1,\ldots}, y_m)) & \longmapsto & x \cdot y := (x_0, \ldots, x_n, y_0, \ldots, y_m), \end{array}$$

   where $x_i \in V^{\otimes i}, y_j \in V^{\otimes j}, \forall 1 \leqslant i \leqslant n, \forall 1 \leqslant j \leqslant m$*.*

5. *Let* $\mathbf{k}\langle\langle X_0, X_1\rangle\rangle$ *be the associative* $\mathbf{k}$*-algebra of formal series of powers in two non commutative variables* X_0*,* X_1*. Elements of this* $\mathbf{k}$*-algebra are of the form*

$$f(X_0, X_1) = \sum_{\omega \text{ word in } X_0, X_1} c_\omega \cdot \omega$$

   where X_0 *and* X_1 *are formal symbols that do not commute,* $c_\omega \in \mathbf{k}$*, and where* ω *is a word consisting only on powers of letters* X_0 *and* X_1*,*

$$\omega = X_{j_0}^{n_0} X_{j_1}^{n_1} X_{j_2}^{n_2} \cdots X_{j_p}^{n_p},$$

   where $j_0, \ldots, j_p \in \{0, 1\}$*,* $p, n_0, \ldots, n_p \in \mathbb{N}$*. For example,* $\omega = X_1^3 X_0 X_1^2 X_0^9 X_1$ *is a word.*

Let's move on to the definition of a Lie algebra.

Lie algebras

Definition 2.1.3. *A* Lie algebra over a field $\mathbf{k}$ *is a* $\mathbf{k}$*-vector space* $\mathfrak{g}$ *provided with a* $\mathbf{k}$*-bilinear antisymmetric map called* Lie bracket*:*

$$\begin{array}{rcl} [-,-]:\mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ (X,Y) & \longmapsto & [X,Y] \end{array}$$

that satisfies the Jacobi identity*:*

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0,$$

for each $X,Y,Z\in\mathfrak{g}$*. A map of Lie* $\mathbf{k}$*-algebras is a map between* $\mathbf{k}$*-vector spaces*

$$f:\mathfrak{g}\longrightarrow\mathfrak{h}$$

compatible with the Lie brackets of $\mathfrak{g}$ *and* $\mathfrak{h}$*, that is:*

$$f([x,y]_{\mathfrak{g}})=[f(x),f(y)]_{\mathfrak{h}}$$

for all $x,y\in\mathfrak{g}$*. A* Lie ideal $\mathfrak{i}$ *(resp. a* Lie subalgebra $\mathfrak{h}$*) of* $\mathfrak{g}$ *is a vector subspace of* $\mathfrak{g}$ *such that:*

$$[\mathfrak{g},\mathfrak{i}]\subseteq\mathfrak{i}\,(\textit{resp. } [\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}).$$

Given an ideal $\mathfrak{i}$ *of* $\mathfrak{g}$ *one can form the* Lie quotient $\mathfrak{g}/\mathfrak{i}$*: it is the vector space* $\mathfrak{g}/\mathfrak{i}$ *provided with the bracket*

$$[g+\mathfrak{i},g'+\mathfrak{i}]:=[g,g']+\mathfrak{i}.$$

Remark 2.1.4. *Antisymmetry means* $[x,y]=-[y,x]$*. Bilinearity means*

$$[ax+by,z]=a[x,z]+b[y,z] \text{ and } [z,ax+by]=a[z,x]+b[z,y],$$

for all $a,b\in\mathbf{k}$ *and all* $x,y,z\in\mathfrak{g}$*.*

Example 2.1.5.

1. *Any vector space* E *can be provided with the structure of a Lie algebra by establishing*

$$\forall x,y\in E:[x,y]=0.$$

   Such Lie $\mathbf{k}$*-algebra, where the Lie bracket is zero, is called abelian Lie algebra.*

2. *From an associative algebra* $(\mathcal{A},\cdot)$ *over* $\mathbf{k}$*, one can always build an Lie* $\mathbf{k}$*-algebra <ith underlying set* $\mathcal{A}$ *by setting, for all* $x,y\in\mathcal{A}$*:*

$$[x,y]:=x\cdot y-y\cdot x.$$

   This is called the commutator of the two elements x *and* y*. It is easy to verify that this defines a Lie algebra structure on* $\mathcal{A}$*.*

3. *As a concrete example of the previous situation, consider the space $\mathcal{M}_n(\mathbf{k})$ of matrices $n \times n$ with coefficients in $\mathbf{k}$. This is an associative algebra provided usual matrix product, not abelian in general. We can also give it a structure of an associative $\mathbf{k}$-algebra, with the bracket*

$$[A, B] = AB - BA.$$

We denote $\mathfrak{gl}_n(\mathbf{k})$ this Lie algebra.

Remark 2.1.6. *The* Ado theorem *shows that any Lie $\mathbf{k}$-algebra of finite dimension can be seen as a subalgebra of $\mathfrak{gl}_n(\mathbf{k})$. Unfortunately, the majority of Lie $\mathbf{k}$-algebras which we will work with are infinite dimensional, as in the case of a free associative $\mathbf{k}$-algebra in two generators that we define next.*

Proposition 2.1.7. *Let S be a set. There is a unique (up to unique isomorphism) Lie $\mathbf{k}$-algebra $\mathfrak{f}_S(\mathbf{k})$ provided with a map of sets $\pi : S \longrightarrow \mathfrak{f}_S(\mathbf{k})$ such that, for each Lie algebra $\mathfrak{g}$ and each map of sets $f : S \longrightarrow \mathfrak{g}$, there is a unique morphism of Lie algebras $\widetilde{f} : \mathfrak{f}_S(\mathbf{k}) \longrightarrow \mathfrak{g}$ so that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & \mathfrak{g} \\ \mathfrak{f}_S(\boldsymbol{k}) \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

that is, so that $f = \tilde{f} \circ \pi$. $\mathfrak{f}_S(\mathbf{k})$ is called the free Lie $\mathbf{k}$-algebra over S.

If $S = \{X, Y\}$, we will denote from now on $\mathfrak{f}_S(\mathbf{k}) = \mathfrak{f}(X, Y)$.

Remark 2.1.8. *Let's take a closer look at this definition. A* Lie word *in symbols $X_1, \dots, X_n$ is a formal bracket of these symbols. For example*

$$[[X_1, X_4], [[X_7, [X_9, X_2]], X_1]].$$

The Lie algebra $\mathfrak{f}_S(\mathbf{k})$ must be understood as the $\mathbf{k}$-vector space generated by all (linear combinations of) Lie words *modulo the subspace obtained by applying antisymmetry and the Jacobi identity. Concretely, if we take $S = \{A, B\}$, then an element of $\mathfrak{f}_S(\mathbf{k})$ is a finite sum*

$$f(A, B) = \sum_{\omega Lie\ word\ in A, B} c_w \cdot \omega$$

where $c_w \in \mathbf{k}$.

Remark 2.1.9. *A Lie algebra can be presented by generators and relations: it is simply the quotient Lie $\mathbf{k}$-algebra of the free Lie $\mathbf{k}$-algebra in such generators and the ideal generated by such relations. One has to verify that the vector subspace generated by the relations is indeed an ideal.*

Every Lie algebra $\mathfrak{g}$ is contained in an associative algebra $\mathcal{U}(\mathfrak{g})$ - usually (much) larger than $\mathfrak{g}$ - called the *universal enveloping algebra of* $\mathfrak{g}$ and where $[-,-]_{\mathfrak{g}}$ matches the bracket given by the two-element commutator $[x, y] := x \cdot y - y \cdot x$.

Definition 2.1.10. *The universal enveloping* $\mathbf{k}$*-algebra of* $\mathfrak{g}$*, denoted* $\mathcal{U}(\mathfrak{g})$*, is the unique (up to unique isomorphism) associative* $\mathbf{k}$*-algebra provided with a morphism of* $\mathbf{k}$*-Lie algebras*

$$\pi : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$$

such that for each associative algebra $\mathcal{A}$ *and each map* $f : \mathfrak{g} \longrightarrow \mathcal{A}$ *of vector spaces, there is a unique associative algebra morphism* $\widetilde{f} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{A}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{A} \\ \mathcal{U}(\mathfrak{g}) \arrow[ur, dashed, "\exists!\tilde{f}"'] & \end{tikzcd}$$

that is, so that $f = \tilde{f} \circ \pi$*.*

Remark 2.1.11. *Specifically,* $\mathcal{U}(\mathfrak{g})$ *is the quotient* $T(\mathfrak{g})/\mathcal{I}$ *of tensor algebra modulo the two-sided ideal generated by the relation*

$$x \otimes y - y \otimes x = [x, y].$$

Example 2.1.12. *If* $\mathfrak{g} = \mathfrak{f}_S(\mathbf{k})$ *and* $S = \{x_1, \dots, x_m\}$ *then* $\mathcal{U}(\mathfrak{g}) = \mathbf{k}\langle S\rangle$ *is the free associative algebra in symbols in* S *whose basis is given by the words* $\omega = x_{j_1} \cdots x_{j_n}$ *where* $j_i \in \{1, \dots, m\}$ *for all* $i = 1, \dots, n$*.*

Example: The Kohno-Drinfeld Lie algebra

Definition 2.1.13. *The Kohno-Drinfeld Lie* $\mathbf{k}$*-algebra, denoted* $\mathfrak{t}_n(\mathbf{k})$*, is the Lie algebra freely generated by symbols* t_{ij}*,* $1 \leqslant i \neq j \leqslant n$*, modulo the ideal generated by the following relations:*

$$t_{ij} = t_{ji} \tag{2.1}$$

$$[t_{ij}, t_{kl}] = 0 \tag{2.2}$$

$$[t_{ij}, t_{ik} + t_{jk}] = 0 \tag{2.3}$$

where $\operatorname{card}\{i, j, k, l\} = 4$*. These relations are usually called infinitesimal braids relations. In the next chapter we will justify this denomination. In the case* $\mathbf{k} = \mathbb{C}$*, we will use the notation* $\mathfrak{t}_n(\mathbb{C}) := \mathfrak{t}_n$*.*

As an exercice one can explore the structure of this Lie algebra for low values of n.

Structure of $\mathfrak{t}_n(\mathbf{k})$ for $n \leqslant 3$. The following facts are easy to prove :

1. The element $c_n := \sum_{1 \leqslant i < j \leqslant} t_{ij}$ is central in $\mathfrak{t}_n(\mathbf{k})$ (ie it commutes with every element of $\mathfrak{t}_n(\mathbf{k})$). One deduces that we can define the quotient $\bar{\mathfrak{t}}_n(\mathbf{k}) := \mathfrak{t}_n(\mathbf{k})/\langle c_n\rangle$.
2. The Lie $\mathbf{k}$-algebras $\mathfrak{t}_2(\mathbf{k})$ is the free Lie algebra on one generator and $\bar{\mathfrak{t}}_2(\mathbf{k})$ is the trivial Lie $\mathbf{k}$-algebra.

3. $\bar{\mathfrak{t}}_3(\mathbf{k})$ is nothing but the free associative $\mathbf{k}$-algebra in two generators.
4. The Lie subalgebra of $\mathfrak{t}_n(\mathbf{k})$ generated by t_{ij}, where $i, j \in [1, n]$, identifies with $\mathfrak{t}_{n-1}(\mathbf{k})$.
5. The Lie subalgebra of $\mathfrak{t}_n(\mathbf{k})$ generated by $t_{1n}, t_{2n}..., t_{(n-1)n}$ identifies with the free Lie $\mathbf{k}$-algebra $\mathfrak{f}_n(\mathbf{k})$.
6. There is an isomorphism of Lie $\mathbf{k}$-algebras
$$\mathfrak{t}_n(\mathbf{k}) \simeq \mathfrak{t}_{n-1}(\mathbf{k}) \oplus \mathfrak{f}_n(\mathbf{k}).$$
7. Let $\mathbf{k}c_3$ be the abelian Lie $\mathbf{k}$-algebra generated by $c_3 = t_{12} + t_{13} + t_{23}$. There is an isomorphism of Lie $\mathbf{k}$-algebras
$$\mathfrak{t}_3(\mathbf{k}) \simeq \mathbf{k}c_3 \oplus \mathfrak{f}_2(\mathbf{k})$$
where $\mathfrak{f}_2(\mathbf{k})$ is the free Lie $\mathbf{k}$-algebra generated by t_{13} and t_{23} (or, equivalently, by t_{12} and t_{23}).

2.1.2 The exponential group

Completed filtered associative k-algebras

Definition 2.1.14. *A topological ring is a ring with the structure of a topological space so that the multiplication $A \times A \longrightarrow A$ is a homomorphism of topological spaces. A topological vector space over a topological ring $\mathbf{k}$ is a $\mathbf{k}$-vector space such that the addition and the multiplication by scalars of the vector space are topological homomorphisms.*

In this chapter, we will mainly use the standard and the discrete topologies.

We have notions of a topological associative algebra and a Lie topological algebra that will not be recalled here.

Definition 2.1.15. *An associative $\mathbf{k}$-algebra $\mathcal{A}$ is filtered if it is equipped with a descending sequence of ideals*
$$\mathcal{A} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_2 \cdots$$

Remark 2.1.16. *A $\mathbf{k}$-filtered associative algebra $(\mathcal{A}, \{\mathfrak{m}_i\}_{i\in I})$ induces a direct system of quotient rings*
$$\cdots \longrightarrow \mathcal{A}/\mathfrak{m}_{i+1} \longrightarrow \mathcal{A}/\mathfrak{m}_i \longrightarrow \cdots \longrightarrow \mathcal{A}/\mathfrak{m}_2 \longrightarrow \mathcal{A}/\mathfrak{m}_1 \longrightarrow 0.$$

Definition 2.1.17. *The filtered completion of the filtered associative algebra $(\mathcal{A}, \{\mathfrak{m}_i\}_{i\in I})$ is the $\mathbf{k}$-filtered associative algebra $(\hat{\mathcal{A}}, \{\hat{\mathfrak{m}}_i\}_{i\in I})$ where*
$$\begin{aligned} \hat{\mathcal{A}} &:= \lim_{\leftarrow i} \mathcal{A}/\mathfrak{m}_i \\ &= \left\{ a = (a_0, a_1, \dots) \in \prod_{i=1}^{\infty} \mathcal{A}/\mathfrak{m}_i \middle| a_j \equiv a_i [\operatorname{mod} \mathfrak{m}_i], \forall j > i \right\} \end{aligned}$$
and where, for all $i \in I$:
$$\hat{\mathfrak{m}}_i := \{a = (a_0, a_1, \dots) \in \hat{\mathcal{A}} | a_j = 0, \forall j \leqslant i\}.$$

Remark 2.1.18. *One can identify the quotient* $\mathbf{k}$*-algebras* $\mathcal{A}/\mathfrak{m}_i$ *and* $\hat{\mathcal{A}}/\hat{\mathfrak{m}}_i$.

Proposition 2.1.19. *If* $(\mathcal{A}, \{\mathfrak{m}_i\}_{i\in I})$ *is a filtered associative* $\mathbf{k}$*-algebra, then we can endow it with a topology, called Krull topology, defined, for each point* $a \in \mathcal{A}$*, by the basis of neighborhoods*

$$\{a + \mathfrak{m}_i\}_{i\in\mathbb{N}}.$$

Remark 2.1.20. *In case the ideals* $\mathfrak{m}_i$ *are equal to powers* $\mathfrak{m}_i := \mathcal{I}^i$ *for the same ideal* $\mathcal{I}$ *of* $\mathcal{A}$*, the associated completion* $\hat{\mathcal{A}}$ *of* $\mathcal{A}$ *is usually called* $\mathcal{I}$*-adic completion of* $\mathcal{A}$ *and its associated Krull topology is called* $\mathcal{I}$*-adic topology.*

Proposition 2.1.21. *Viewed as a filtered topological Lie* $\mathbf{k}$*-algebra with respect to the Krull topology, the completion* $(\hat{\mathcal{A}}, \{\hat{\mathfrak{m}}_i\}_{i\in I})$ *of an associative filtered* $\mathbf{k}$*-algebra* $(\mathcal{A}, \{\mathfrak{m}_i\}_{i\in I})$ *is precisely its topological completion.*

Proof. Let $\{a_i\}_{i\geqslant 1}$ be a Cauchy sequence in $\mathcal{A}$: for each open set U of $\mathcal{A}$, there is an integer N_U such that, for all $i, j > N_U$, we have $a_i - a_j \in U$. This is verified if, and only if, for every integer n, there exists an integer N_n such that, for all $i, j > N_n$, we have

$$a_i - a_j \in \mathfrak{m}_i.$$

Now, such a sequence always converges in $\hat{\mathcal{A}}$ towards point $a = (a_0, a_1, \dots) \in \prod_{n\geqslant 1} \mathcal{A}/\mathfrak{m}_n$, where, for all n, we have $a_n \equiv a_{N_n} [\mathrm{mod}\, \mathfrak{m}_n]$.

Conversely, every point of $\hat{\mathcal{A}}$ defines a Cauchy sequence in $\mathcal{A}$. □

Example 2.1.22. *If* $\mathcal{A} = \mathbf{k}[X_1, \dots, X_n]$ *is the polynomial* $\mathbf{k}$*-algebra on* $X_1, \dots, X_n$ *and* $\mathcal{I}$ *is its maximal ideal, then the* $\mathcal{I}$*-adic completion of* $\mathcal{A}$ *is the* $\mathbf{k}$*-algebra*

$$\hat{\mathcal{A}} = \mathbf{k}[[X_1, \dots, X_n]]$$

of formal series over $\mathbf{k}$ *in* n *commutative variables.*

Degree completion

The Baker-Cambell-Hausdorff (BCH) formula is essentially useful to associate a group to any completed Lie $\mathbf{k}$-algebra (where the exponential application is not necessarily a group morphism).

Definition 2.1.23. *A graded Lie* $\mathbf{k}$*-algebra is a Lie algebra* $\mathfrak{g}$ *provided with a graduation of vector spaces:*

$$\mathfrak{g} = \bigoplus_{n=-\infty}^{+\infty} \mathfrak{g}_n$$

so that the Lie bracket is compatible with the graduation, that is to say:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

Remark 2.1.24. *If $\mathfrak{g}$ is graded, then $\mathfrak{g}$ induces the same graduation at the level of its associated universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.*

Let $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}_n$ be a positively graded Lie **k**-algebra so that each $\mathfrak{g}_n$ is of finite dimension. We can equip it with a *decreasing filtering of Lie ideals* $\mathfrak{m}_n := \underset{n \leqslant i}{\oplus} \mathfrak{g}_i$ so we get a decreasing sequence:

$$\mathfrak{g} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \cdots$$

Proposition 2.1.25. *The degree completion of $\mathfrak{g}$ is the completion of $\mathfrak{g}$ with respect to filtering $\{\mathfrak{m}_i\}_{i \geqslant 1}$, and is identified with the following product:*

$$\hat{\mathfrak{g}} := \prod_{n=1}^{\infty} \mathfrak{g}_n.$$

Remark 2.1.26. *The difference between $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ lies in that the elements in $\hat{\mathfrak{g}}$ can be written as eventually infinite sums, unlike the elements of $\mathfrak{g}$.*

Example 2.1.27. *Let $\mathfrak{f}_S(\mathbf{k})_n \subset \mathfrak{f}_S(\mathbf{k})$ be the vector subspace spanned by Lie words with $(n-1)$ brackets. For example $\mathfrak{f}(X,Y)_1 = \mathbf{k}\langle X, Y\rangle$, $\mathfrak{f}(X,Y)_2 = \mathbf{k}\langle [X,Y]\rangle$ y $\mathfrak{f}(X,Y)_3 = \mathbf{k}\langle [X,[X,Y]],[Y,[Y,X]]\rangle$. We can notice that*

$$[\mathfrak{f}_S(\mathbf{k})_n, \mathfrak{f}_S(\mathbf{k})_m] \subset \mathfrak{f}_S(\mathbf{k})_{n+m},$$

so we can build a grading $\mathfrak{f}_S(\mathbf{k}) = \bigoplus_{n=1}^{\infty} \mathfrak{f}_S(\mathbf{k})_n$. Then, the degree completion of $\mathfrak{f}_S(\mathbf{k})$ is $\hat{\mathfrak{f}}_S(\mathbf{k}) = \prod_{n=1}^{\infty} \mathfrak{f}_S(\mathbf{k})_n$. It is easy to prove that $\hat{\mathfrak{f}}_S(\mathbf{k}) \subset \mathbf{k}\langle\langle S\rangle\rangle$. If $S = \{X, Y\}$, we will denote from now on $\hat{\mathfrak{f}}_S(\mathbf{k}) = \hat{\mathfrak{f}}(X,Y)$.

Example 2.1.28. *The Kohno-Drinfeld Lie **k**-algebra $\mathfrak{t}_n(\mathbf{k})$ has a positive grading by setting $\deg(t_{ij}) := 1$ and we have*

$$\mathfrak{t}_n(\mathbf{k}) = \bigoplus_{m=1}^{\infty} \mathfrak{t}_n(\mathbf{k})_m,$$

where, for example, $\mathfrak{t}_n(\mathbf{k})_1 = \bigoplus_{i<j} \mathbf{k} t_{ij}$ and $\mathfrak{t}_n(\mathbf{k})_2 = \bigoplus_{i<j<k} \mathbf{k}[t_{ij}, t_{ik}]$. This allows us to define its degree completion $\hat{\mathfrak{t}}_n(\mathbf{k})$.

The Baker-Cambell-Hausdorff formula

Let X, Y two elements of an associative **k**-algebra $\mathcal{A}$. Recall the expressions of the exponential and the logarithm in terms of series

$$e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!} \text{ and } \log(1+X) := \sum_{n=1}^{\infty} \frac{(-1)^n X^n}{n}.$$

These are well defined if $\mathcal{A}$ is a completed associative **k**-algebra. In particular, in the algebra $\mathbf{k}[[X,Y]]$ of formal series in commutative variables, we have the relation

$$e^X e^Y = e^{X+Y}.$$

However, in the algebra $\mathbf{k}\langle\langle X,Y\rangle\rangle$ this relation is not true in general. The goal of the Baker-Cambell-Hausdorff formula is to fix this problem.

Definition 2.1.29. *The* Baker-Cambell-Hausdorff element *is the formal series* BCH *of* $\mathbf{k}\langle\langle X,Y\rangle\rangle$ *defined, for every* $X,Y\in\mathbf{k}\langle\langle X,Y\rangle\rangle$, *by*

$$\begin{aligned}\mathrm{BCH}(X,Y):=\log(e^Xe^Y) &= -\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\sum_{k,l=0}^{\infty}\frac{X^kY^l}{k!l!}\right)^n\\ &= X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+\cdots\end{aligned}$$

One can prove that $\mathrm{BCH}(X,Y)\in\hat{\mathfrak{f}}(X,Y)$.

Proposition 2.1.30. *Let* $\mathfrak{g}$ *be a completed Lie* $\mathbf{k}$*-algebra. The* exponential group $\exp(\mathfrak{g})$ associated to $\mathfrak{g}$ *is the group whose underlying set is the set of formal elements of the form* $\{e^X, X\in\mathfrak{g}\}$ *(which is isomorphic to the underlying set of* $\mathfrak{g}$*) provided with the multiplication law defined by the* Baker-Cambell-Hausdorff *formula:*

$$\begin{aligned}\exp(\mathfrak{g})\times\exp(\mathfrak{g}) &\longrightarrow \exp(\mathfrak{g})\\ (e^X,e^Y) &\longmapsto e^{\mathrm{BCH}(X,Y)}.\end{aligned}$$

We have two morphisms, inverse from each other

$$\begin{aligned}e:\mathfrak{g} &\longleftrightarrow \exp(\mathfrak{g}):\log\\ X &\longleftrightarrow e^X\end{aligned}$$

Proof. We need to show that $\mathrm{BCH}(X,Y)$ converges, which is satisfied autotically because $\hat{\mathfrak{g}}=\varprojlim_n(\mathfrak{g}/\mathfrak{m}_n)$. Exercise: Set the following equations:

$$\begin{aligned}\mathrm{BCH}(X,0)=\mathrm{BCH}(0,X) &= 0\\ \mathrm{BCH}(X,-X) &= 1\\ \mathrm{BCH}(\mathrm{BCH}(X,Y),Z) &= \mathrm{BCH}(X,\mathrm{BCH}(Y,Z))=\log(e^Xe^Ye^Z),\end{aligned}$$

the last equation taking place in $\hat{\mathfrak{f}}(X,Y,Z)$. □

Remark 2.1.31. *The definition of* $\exp(\mathfrak{g})$ *makes sense only when the characteristic of* $\mathbf{k}$ *is zero and when* $\mathfrak{g}$ *is complete, otherwise the* $\mathrm{BCH}(X,Y)$ *element does not make sense.*

Example 2.1.32. *The injection of Lie algebras*

$$\begin{aligned}\mathfrak{f}(X,Y) &\hookrightarrow \mathfrak{t}_3(\mathbf{k})\\ X &\longmapsto t_{12}\\ y &\longmapsto t_{23}\end{aligned}$$

induces an injection of groups $\exp(\hat{\mathfrak{f}}_{\mathbf{k}}(X,Y))\hookrightarrow\exp\left(\hat{\mathfrak{t}}_3(\mathbf{k})\right)$.

Finally, if $\mathfrak{g}$ is a pronilpotent Lie $\mathbf{k}$-algebra, we denote $\mathrm{gr}(\mathfrak{g})$ its associated graded Lie algebra. We are ready to define

2.1.3 Drinfeld associators

The first goal of this chapter will be to give a geometrical understanding of the following definition that was introduced by Drinfeld in [31].

Definition 2.1.33. *A Drinfeld* $\mathbf{k}$*-associator is a pair* (λ, Φ) *where* $\lambda \in \mathbf{k}^{\times}$ *and*

$$\Phi(X,Y) := e^{\phi(X,Y)} \in \exp(\hat{\mathfrak{f}}(X,Y)) \subset \mathbf{k}\langle\langle X,Y\rangle\rangle$$

which satisfies the following equations:

$$\Phi(X,Y) = \Phi^{-1}(Y,X) \text{ in } \exp(\hat{\mathfrak{f}}(X,Y)) \tag{2.4}$$

$$e^{\frac{\pm\lambda}{2}t_{12}}\Phi(t_{13},t_{12})e^{\frac{\pm\lambda}{2}t_{13}}\Phi(t_{23},t_{13})e^{\frac{\pm\lambda}{2}t_{23}}\Phi(t_{12},t_{23}) = 1 \text{ in } \exp\left(\hat{\mathfrak{t}}_3\left(\mathbf{k}\right)\right) \tag{2.5}$$

and

$$\Phi(t_{13}+t_{23},t_{34})\Phi(t_{12},t_{23}+t_{24}) = \Phi(t_{12},t_{23})\Phi(t_{12}+t_{13},t_{24}+t_{34})\Phi(t_{23},t_{34}) \text{ in } \exp\left(\hat{\mathfrak{t}}_4\left(\mathbf{k}\right)\right) \tag{2.6}$$

The set of Drinfeld $\mathbf{k}$*-associators will be denoted* $\mathrm{Ass}(\mathbf{k})$*.*

Remark 2.1.34. *The equation (2.4) is called* antisymmetry *relation. The two relations (2.5) are called* two hexagons *relation and the relation (2.6) is called* pentagon *relation.*

While we have taken the time to define each mathematical object involved in this definition, we ignore - for the moment - the particular interest of this mathematical concept, which the reason of being of those equations - at first sight arbitrary - and, above all, wether such a pair does indeed exist.

The second objective of the following section will be then to prove the following theorem, due to Drinfeld:

Theorem A. *The set of* $\mathbb{C}$*-associators is not empty.*

In particular, the proof lies in the existence of a particular $\mathbb{C}$-associator coming from the regularized holonomy of a differential equation in two noncommutative variables called the Knizhnik-Zamolodchikov equation, well-known in physics. The connection associated to these equations will induce an isomorphism between the pure braid group, that is the fundamental group of the configuration space of the complex plane, and the Kohno-Drinfeld algebra, which is the holonomy Lie algebra of these spaces. These concepts will be introduced in the next section.

2.2 The KZ associator

In the previous section we took some time to present a formal definition of the Drinfeld associators by means of the Kohno-Drinfeld Lie algebra and the exponential group of its associated degree completion. At the moment we do not know what is the reason to be of these equations but we will dedicate some time into proving that such a set is not empty when taking $\mathbf{k} = \mathbb{C}$.

In particular, we are going to explicit an example of such an associator through the resolution of a certain system of differential equations in two non-commutative variables, whose geometric version will allow us to understand the the architecture of the definition of Drinfeld associators. We will mainly follow [31].

2.2.1 Solutions of the universal Knizhnik-Zamolodchikov equation

In this section, we will introduce the Knizhnik[1]-Zamolodchikov[2] (KZ) equations in its universal version. Initially, these equations, which form a system of partial differential equations in the complex plane with regular singular points, were born in quantum field theory (especially in condensed matter and high-energy physics) as equations that satisfy a set of additional restrictions for the correlation functions in the Wess-Zumino-Witten model in two dimensional field theory and which are associated to an associative $\mathbf{k}$-algebra of a fixed level. The reader interested in learning about the KZ equations in the context of quantum field theory may consult the introduction [52] on the subject.

The universal KZ equation

The universal version of these equations was established by Drinfeld in [31] and are defined for any type of associative $\mathbf{k}$-algebra that satisfies the infinitesimal braid relations - that is, defined in the Kohno-Drinfeld Lie algebra. Remember that the configuration space of n points on the complex plane is the following open subspace of $\mathbb{C}^n$:

$$\mathrm{Conf}(\mathbb{C}, n) := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j, \text{ if } i \neq j\}.$$

Definition 2.2.1. *For each $n \geqslant 2$, the Knizhnik-Zamolodchikov differential system over (any open subset within) the configuration space* $\mathrm{Conf}(\mathbb{C}, n)$ *is*

$$(\mathrm{KZ})_n : dW \quad = \quad \frac{1}{2i\pi} \sum_{1 \leqslant i < j \leqslant n} \frac{t_{ij}}{z_i - z_j}(dz_i - dz_j)W,$$

that is, for $i = 1, \dots, n$:

$$(\mathrm{KZ})_n : \frac{\partial W}{\partial z_i} = \frac{1}{2i\pi} \sum_{1 \leqslant i < j \leqslant n} \frac{t_{ij}}{z_i - z_j} W,$$

where W is a function defined in any open $U \subset \mathrm{Conf}(\mathbb{C}, n)$ and taking values in $\hat{\mathcal{U}}(\hat{\mathfrak{t}}_n)$.

When $n = 3$, the differential system's solutions $(\mathrm{KZ})_3$ define an element of $\mathbb{C}\langle\langle X, Y \rangle\rangle$ and the asymptotic behaviour of these equations when $n = 3, 4$ determines the relations that this

[1](1962-1987) Vadim Knizhnik.
[2](1952-) Alexander Zamolodchikov.

element satisfies. It is important to emphasize that this « two stages principle » is enough to fully define a Drinfeld associator. The importance of this remark is developed in the next section when we integrate the geometry of $\mathrm{Conf}(\mathbb{C}, n)$ into this story. For now let's restrain ourselves on the study of this differential system.

Definition of the KZ associator

Recall that a function f of a complex variable is analytic at a point x_0 if it is developable in entire series in any open neighborhood of x_0 inside its domain set. This means that, for any open neighborhood $\mathcal{D}_{x_0}$ of x_0 in the domain set of f, there is a sequence $(a_n)_{n\geqslant 0}$ such that, for all $x \in \mathcal{D}_{x_0}$, the function f is written in the form of a convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We can easily observe that the system $(\mathrm{KZ})_3$ is written in terms of the total differential

$$\mathrm{dW} = \frac{1}{2i\pi}[t_{12} d\log(z_2 - z_1) + t_{13} d\log(z_3 - z_1) + t_{23} d\log(z_3 - z_2)]W.$$

Proposition 2.2.2. *The solutions of the system* $(\mathrm{KZ})_3$ *are of the form*

$$(z_3 - z_1)^{\frac{c_3}{2i\pi}} G\left(\frac{z_2 - z_1}{z_3 - z_1}\right),$$

where $c_3 := t_{12} + t_{13} + t_{23}$ *and* G *is a formal series in the non commutative variables* t_{12}, t_{23}, *with as coefficients analitical functions in the complex variable* $z \in \mathbb{C} - \{0, 1\}$ *which are solutions of the linear differential equation*

$$G'(z) \quad = \quad \frac{1}{2i\pi}\left(\frac{t_{12}}{z} + \frac{t_{23}}{z-1}\right) G(z). \tag{2.7}$$

Proof. The proof consists in the following stages:

1. First notice that
$$(z_3 - z_1)^{\frac{u}{2i\pi}} = \exp\left(\frac{\log(z_3 - z_1)u}{2i\pi}\right) = \sum_{k=0}^{\infty} \frac{\log(k(z_3 - z_1))u^k}{(2i\pi)^k} \in \mathfrak{t}_3$$
belongs to the center of $\mathfrak{t}_3$.
2. In $(\mathrm{KZ})_3$ do the variable change $W = (z_3 - z_1)^{\frac{u}{2i\pi}} \times I$.
3. Write $z = \frac{(z_2 - z_1)}{(z_3 - z_1)}$ and conclude.

□

Let $\mathcal{U} = \mathbb{C} - (]-\infty, 0] \bigcup [1, \infty[)$ where $]-\infty, 0]$ and $[1, \infty[$ are straight half-lines in $\mathbb{R} \subset \mathbb{C}$. Notice that $\mathcal{U}$ is simply connected.

Remark 2.2.3. *As a consequence of the fundamental theorem of linear differential equations, the equation (2.7) has analytic solutions in* $\mathcal{U}$ *which are unique once a value has been specified at any point of* $\mathcal{U}$.

Equation (2.7) has two unique singularities in $\mathbb{C}$ which are $z = 0$ and $z = 1$. By setting $w = 1/z$, we observe that this equation also has a singularity at ∞. These three singularities are regular.

Let's analyze the asymptotic behavior of the equation (2.7) as we approach our two unique singularities in $\mathbb{C}$ which are $z = 0$ and $z = 1$.

Proposition 2.2.4. *Equation (2.7) has two unique solutions G_0 and G_1 such that*

$$G_0(z) \quad \sim_0 \quad z^{\frac{t_{12}}{2i\pi}} \tag{2.8}$$

$$G_1(z) \quad \sim_1 \quad (1-z)^{\frac{t_{23}}{2i\pi}}. \tag{2.9}$$

In particular, G_0 and G_1 are not zero and therefore differ from each other by an invertible element. The KZ associator is the quotient $\Phi_{\mathrm{KZ}} := G_1^{-1}G_0 \in \mathbb{C}\langle\langle X, Y\rangle\rangle$.

Remark 2.2.5. *The equations (2.8) and (2.9) mean that $G_0(z)z^{-\frac{t_{12}}{2i\pi}}$ (resp. $G_1(z)(1-z)^{-\frac{t_{23}}{2i\pi}}$) have analogous continuations in a neighborhood of 0 (resp 1) taking at 0 (resp. at 1) the value 1. We observe in the same way that $z^{\frac{t_{12}}{2i\pi}}$ and $(1-z)^{\frac{t_{23}}{2i\pi}}$ are well defined in $\mathcal{U}$.*

Proof. The reader can consult the proof of Proposition 2.2.4 in [67].

□

One can show that Φ_{KZ} is independent of z calculating the derivative $\Phi'_{\mathrm{KZ}}(z)$.

Remark 2.2.6. *This definition is valid for all non-commutative symbols A and B. For each pair (A, B), we have two functions $G_0(z; A, B)$ and $G_1(z; A, B)$. We can then define*

$$\phi_{\mathrm{KZ}}(A, B) := G_1(-; A, B)^{-1}G_0(-; A, B).$$

In particular, $\phi_{\mathrm{KZ}}(t_{12}, t_{23}) = \Phi_{\mathrm{KZ}}$.

Let us reformulate Theorem A in the following way:

Theorem A. *The pair $(1, \Phi_{\mathrm{KZ}})$ is a Drinfeld $\mathbb{C}$-associator.*

Analytic proof of Theorem A

Below we reproduce Drinfeld's original proof of Theorem A.

1) Φ_{KZ} belongs to $\exp(\hat{\mathfrak{t}}_3)$:

Let us give the big steps of this part of the proof: first the universal enveloping algebra $\hat{\mathcal{U}}(\hat{\mathfrak{t}}_3(\mathbf{k}))$ has a structure of a filtered and completed Hopf $\mathbf{k}$-algebra. In particular, the coproduct Δ is given by the completed tensor product $\hat{\otimes}$. The elements of $\exp(\hat{\mathfrak{t}}_3)$ are identified with the group-like elements (i.e. elements that verify $\Delta(g) = g\hat{\otimes}g$) of $\hat{\mathcal{U}}(\hat{\mathfrak{t}}_3)$. Therefore, it suffices to show that $\Delta(\Phi_{\mathrm{KZ}}) = \Phi_{\mathrm{KZ}}\hat{\otimes}\Phi_{\mathrm{KZ}}$. This is obtained by using remark

2.2.6: notice that if G_0 and G_1 are group-like, then we can (in the case of G_0) use function $G_+(z) = G_0(z; t_{12} \otimes 1, t_{23} \otimes 1)G_0(z; 1 \otimes t_{12}, 1 \otimes t_{23})$ to conclude.

2) Antisymmetry relation:

If we replace z by $1 - z$ in equation (2.7), Φ_{KZ} is replaced by its inverse which is equivalent to swap t_{12} with t_{23} ie apply the permutation (123).

3) Pentagon relation:

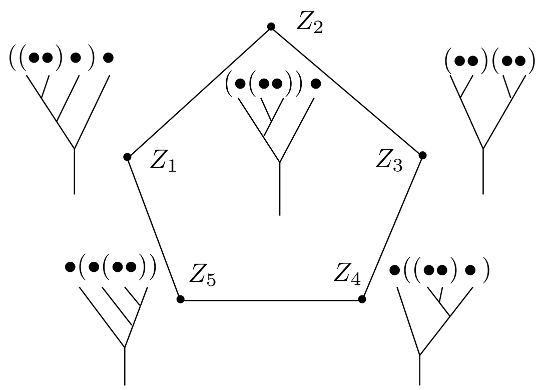
Let's start by describing the asymptotic behaviors of the solutions of the system $(\mathrm{KZ})_4$. Let

$$U := \{(z_1, \dots, z_4) \in \mathbb{R}^4 | z_1 < z_2 < z_3 < z_4\} \subset \mathrm{Re}(\mathrm{Conf}(\mathbb{C}, 4))$$

be an open subset in the real part of the 4 point configuration space of the complex plane. Consider the following 5 zones in U:

$$\begin{array}{llll} (Z_1) & z_2 - z_1 \ll & z_3 - z_1 & \ll z_4 - z_1; \\ (Z_2) & z_3 - z_2 \ll & z_3 - z_1 & \ll z_4 - z_1; \\ (Z_3) & z_3 - z_2 \ll & z_4 - z_2 & \ll z_4 - z_1; \\ (Z_4) & z_4 - z_3 \ll & z_4 - z_2 & \ll z_4 - z_1; \end{array}$$
$$(Z_5) \qquad z_4 - z_3 \ll z_4 - z_1 \qquad \text{and} \qquad z_2 - z_1 \ll z_4 - z_1.$$

How to represent these areas and how to relate them to each other? It's here where one of Drinfeld's brilliant ideas intervenes: they correspond to a pentagon where each edge corresponds to parenthesis arrangement: V_i and V_j are in the same parenthesis and V_k out of it if $|z_i - z_j| \ll |z_i - z_k|$. This way, $z_2 - z_1 \ll \; z_3 - z_1 \; \ll z_4 - z_1$ corresponds to the pair $((\bullet\bullet)\bullet)$. We can also say that it corresponds to a trivalent tree with four leafs as summarized in the following image:



Lemma 2.2.7. *There are five unique solutions $W_1, \dots, W_5$ to the system* $(\mathrm{KZ})_4$ *having the*

following asymptotic behaviors in the corresponding zones:

$$\begin{array}{rcl} W_1 & \sim & (z_2-z_1)^{\frac{t_{12}}{2i\pi}}(z_3-z_1)^{\frac{t_{13}+t_{23}}{2i\pi}}(z_4-z_1)^{\frac{t_{14}+t_{24}+t_{34}}{2i\pi}}; \\ W_2 & \sim & (z_3-z_2)^{\frac{t_{23}}{2i\pi}}(z_3-z_1)^{\frac{t_{12}+t_{13}}{2i\pi}}(z_4-z_1)^{\frac{t_{14}+t_{24}+t_{34}}{2i\pi}}; \\ W_3 & \sim & (z_3-z_2)^{\frac{t_{23}}{2i\pi}}(z_4-z_2)^{\frac{t_{24}+t_{34}}{2i\pi}}(z_4-z_1)^{\frac{t_{12}+t_{13}+t_{14}}{2i\pi}}; \\ W_4 & \sim & (z_4-z_3)^{\frac{t_{34}}{2i\pi}}(z_4-z_2)^{\frac{t_{23}+t_{34}}{2i\pi}}(z_4-z_1)^{\frac{t_{12}+t_{13}+t_{14}}{2i\pi}}; \\ W_5 & \sim & (z_2-z_1)^{\frac{t_{12}}{2i\pi}}(z_4-z_3)^{\frac{t_{34}}{2i\pi}}(z_4-z_1)^{\frac{t_{13}+t_{14}+t_{23}+t_{24}}{2i\pi}}. \end{array}$$

That is, we have for example

$$W_2 = f(u,v)(z_3-z_2)^{\frac{t_{23}}{2i\pi}}(z_3-z_1)^{\frac{t_{12}+t_{13}}{2i\pi}}(z_4-z_1)^{\frac{t_{14}+t_{24}+t_{34}}{2i\pi}},$$

where $u = \frac{(z_3-z_2)}{(z_4-z_1)}, v = \frac{(z_3-z_1)}{(z_4-z_1)}$ *and* f *is an analytic function on a neighborhood of* $(0,0)$ *with* $f(0,0) = 1$.

Proof. Let us give the steps to perform the calculation for W_5:

1. Demonstrate that, in this case, one can reduce the system $(\mathrm{KZ})_4$ to a three-variable system.
2. Make the substitution $W = g \cdot (z_4 - z_1)^{T/2i\pi}$ and reduce our system to a system with two variables. Deduce that g is a function in u and v.
3. Deduce that the system $(\mathrm{KZ})_4$ is now written

$$dg = \frac{1}{2i\pi}[t_{12}d\log(u) + t_{34}d\log(v) + dR(u,v)] \cdot g$$

   where R is an analytic function on a neighborhood of $(0,0)$. Conclude.
4. Use the technique of successive approximations to show that there is one, and only one solution to this equation of the form

$$\phi(u,v)u^{\frac{t_{12}}{2i\pi}}v^{\frac{t_{34}}{2i\pi}},$$

   where ϕ is an analytic function on a neighborhood of $(0,0)$ such that $\phi(0,0) = 1$
5. Use the principle of analytic continuation to show that the W_i functions are extended analytically to U.

□

Lemma 2.2.8. *The asymptotic expansions* $W_1, \ldots, W_5$ *satisfy the following relations:*

$$\begin{array}{rcl} W_1 & = & W_2 \cdot \Phi_{\mathrm{KZ}}(t_{12}, t_{23}); \\ W_2 & = & W_3 \cdot \Phi_{\mathrm{KZ}}(t_{12}+t_{13}, t_{24}+t_{34}); \\ W_3 & = & W_4 \cdot \Phi_{\mathrm{KZ}}(t_{23}, t_{34}); \\ W_4 & = & W_5 \cdot \Phi_{\mathrm{KZ}}(t_{13}+t_{23}, t_{34}); \\ W_5 & = & W_1 \cdot \Phi_{\mathrm{KZ}}(t_{12}, t_{23}+t_{34}). \end{array}$$

Proof. Let's prove the first identity. Let $V_1 = W_1 \cdot (z_4 - z_1)^{-\frac{1}{2i\pi}(t_{14}+t_{24}+t_{34})}$. We have

$$\begin{aligned} V_2 &= W_2 \cdot \Phi_{\mathrm{KZ}}(t_{12}, t_{23}) \cdot (z_4 - z_1)^{-\frac{1}{2i\pi}(t_{14}+t_{24}+t_{34})} \\ &= W_2 \cdot (z_4 - z_1)^{-\frac{1}{2i\pi}(t_{14}+t_{24}+t_{34})} \cdot d_4\Phi_{\mathrm{KZ}}. \end{aligned}$$

Indeed, $t_{14} + t_{24} + t_{34}$ commutes with all t_{ij} for $i, j < 4$ and therefore with $\Phi_{\mathrm{KZ}}(t_{12}, t_{23})$. We have

$$(z_4 - z_1)^{-\frac{1}{2i\pi}(t_{14}+t_{24}+t_{34})} = e^{-\frac{1}{2i\pi}(t_{14}+t_{24}+t_{34})\log(z_4-z_1)}$$

which has a series expansion and we obtain the required commutation.

On the other hand, we have $V_1 = V_2$. Indeed, if $z_1 < z_2 < z_3 < z_4$, then V_1 and V_2 are analytic (and z_4 can be eventually infinite). Additionally, V_1 and V_2 verify

$$\frac{\partial V}{\partial z_i} = \begin{cases} \frac{1}{2i\pi}\sum\limits_{j\neq 1}\frac{t_{ij}}{z_1-z_j} \\ \frac{1}{2i\pi}\sum\limits_{j\neq i}\frac{t_{ij}}{z_i-z_j} \cdot V - \frac{1}{2i\pi} \cdot \frac{t_{14}+t_{24}+t_{34}}{z_1-z_4} & \text{if } i = 2, 3 \\ \frac{1}{2i\pi}\sum\limits_{j\neq 4}\frac{[t_{14},V]}{z_4-z_j} \end{cases}$$

The first two equations and the asymptotic developments of V_1 and V_2 show that the two functions match for $z_4 = \infty$. As a consequence, from the above equation one gets $V_1 = V_2$.

The rest of the equations are found in the same way. □

Finally, in light of these relations, we obtain

$$\Phi_{\mathrm{KZ}}(t_{13}+t_{23}, t_{34})\Phi_{\mathrm{KZ}}(t_{12}, t_{23}+t_{24}) = \Phi_{\mathrm{KZ}}(t_{12}, t_{23})\Phi_{\mathrm{KZ}}(t_{12}+t_{13}, t_{24}+t_{34})\Phi_{\mathrm{KZ}}(t_{23}, t_{34}).$$

We conclude that Φ_{KZ} satisfies the pentagon relation.

4) Two Hexagons relations:

Applying the permutation (123), we find that the relations of the two hexagons are satisfied by Φ_{KZ} if, and only if, only one of them is satisfied by Φ_{KZ}. To demonstrate that Φ_{KZ} satisfies one of the two hexagons one proceeds in an analogous way to that we used to demonstrate the pentagon relation: find six solutions of $(\mathrm{KZ})_3$ in different regions with standard asymptotic behaviors corresponding to the edges of an hexagon and show that these solutions have relations that imply the required hexagon relation.

We leave the detail of this proof to the reader's care.

2.2.2 Reminders on flat connections

We recall very quickly some definitions of the theory of vector bundles. The reader interested in a detailed introduction illustrated on the subject may consult [72].

Flat connections

Let X be a complex manifold and $E \longrightarrow X$ a vector $\mathbb{C}$-bundle X. Recall that $\Omega^0(X,E) = \Gamma(X,E)$ and that $\Omega^1(X,E) = \Gamma(T^*X \otimes E)$.

Definition 2.2.9. *A holomorphic* connection ∇ on $E \longrightarrow X$ *is a linear map*

$$\nabla : \Gamma(X,E) \longrightarrow \Omega^1(X,E)$$

verifying, for all $f \in \mathcal{O}(X), s \in \Gamma(X,E)$, the Leibniz relation *:*

$$\nabla(f \cdot s) = (df) \otimes s + f \cdot \nabla(s).$$

Remark 2.2.10.

- *Be ∇_1 and ∇_2 two connections over $E \longrightarrow X$. The difference $\nabla_1 - \nabla_2$ is $\mathcal{O}(X)$-linear.*
- *Locally, a section s is written in the form*

$$s = f_1 e_1 + \cdots + f_d e_d$$

where $f_1, \dots, f_d$ are complex analytic functions on X and $\{e_1, \dots, e_d\}$ is a basis of the fiber.

- *All connections ∇ over $E \longrightarrow X$ can be written locally under the form*

$$\nabla s = \mathrm{d}_{\mathrm{dR}}\, s - \Gamma s,$$

where d_{dR} is the de Rham differential and Γ is a differential 1-form on X taking values in the ring $\mathrm{End}(E)$ of endomorphisms of E.

- *A section s of $E \longrightarrow X$ is horizontal with respect to a connection ∇ if $\nabla s = 0$ that is, if locally s is solution of the differential system*

$$ds = \Gamma s.$$

Let's move on to present the notion of parallel transport for a connection on a vector bundle $E \longrightarrow X$. Let

$$\begin{array}{rcl} \gamma : [0,1] & \longrightarrow & X \\ t & \longmapsto & \gamma(t) \end{array}$$

be a continuous path in X. One can perform the pullback of the matrix Γ of differential forms over X along γ into a matrix

$$A(t)dt = \gamma^* \Gamma$$

of differential forms over the interval $[0,1]$. In light of the theory of ordinary differential equations, there is a unique smooth map $A_\gamma : [0,1] \longrightarrow \mathrm{Aut}^{\mathrm{lin}}(E,X)$, where $\mathrm{Aut}^{\mathrm{lin}}(E,X)$ is the group of linear automorphisms of the bundle $E \longrightarrow X$, such that $A_\gamma(0) = \mathrm{id}$ and $w(t) = A_\gamma(t) w(0)$ is a solution of the differential equation

$$\frac{dw(t)}{dt} = A(t) w(t).$$

Definition 2.2.11. *The parallel transport of the connection* ∇ *along* γ *is the linear isomorphism* $A_\gamma(1)$ *between the fiber at the initial point* γ *and the end point of* γ*. We will denote it by*

$$T_\gamma : F_{\gamma(0)} \simeq F_{\gamma(1)}.$$

In particular, we have a map

$$(\gamma : [0,1] \to X) \quad \longmapsto \quad \left(T_\gamma : F_{\gamma(0)} \xrightarrow{\simeq} F_{\gamma(1)}\right)$$

so that if $\gamma' : [0,1] \to X$ is such that $\gamma(1) = \gamma'(0)$ (in which case we say that the continuous paths γ and γ' are juxtaposable and the path $\gamma \cdot \gamma'$ is then continuous) Then

$$T_{\gamma\cdot\gamma'} = T_\gamma \circ T_{\gamma'}.$$

Definition 2.2.12. *The holonomy group of* ∇ *based at a point* $x_0 \in X$ *is the subgroup of* $\mathrm{Aut}(F_{\gamma(0)})$ *generated by* T_γ *for all loops* γ *based at* $x_0 \in X$*.*

Let ∇ be a connection over a vector bundle $E \longrightarrow X$. We can extend ∇ into a covariant derivative

$$\Gamma(E) \xrightarrow{\nabla} \Omega^1(X,E) \xrightarrow{\nabla} \Omega^2(X,E) \longrightarrow \cdots$$

by means of the formula

$$\nabla(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^{|\omega|} \omega \wedge \nabla\omega'$$

Definition 2.2.13. *The curvature of the connection* ∇ *is the map*

$$\nabla^2 := \nabla \circ \nabla : \Gamma(E) \longrightarrow \Omega^2(X,E).$$

Remark 2.2.14.

- *The curvature is a map which is* $\mathcal{O}(X)$*-linear.*
- *Locally, the curvature is expressed in terms of* Γ *by*

$$\nabla^2 = -d_{\mathrm{dR}}\omega + \omega \wedge \omega.$$

Before constructing explicitely the parallel transport application, let's modify the proposed framework a little bit by extending it to the case of the G-principal bundles, where G is a Lie group.

G-principal bundles and associated connections

Let G be a topological group.

Definition 2.2.15. *A* G*-principal bundle is a fiber bundle* $\pi : P \longrightarrow M$ *together with a continuous free and transitive right action of* G *on* P*, denoted*

$$\begin{aligned} R : G &\longrightarrow \mathrm{End}(P) \\ g &\longmapsto (R_g : p \mapsto g \cdot p) \end{aligned}$$

such that G *preserves the fibers of* P *(i.e. if* $y \in \pi^{-1}(\{x\})$ *then* $y \cdot g \in \pi^{-1}(\{x\})$ *for all* $g \in G$*).*

Remark 2.2.16.

1. *This implies that each fiber is homeomorphic to the G group.*
2. *A principal bundle is trivial if, and only if, it admits a global section.*

We can extend this definition to the case where G is a Lie group, with associated Lie $\mathbf{k}$-algebra $\mathfrak{g}$, and M is a differientable manifold by demanding that π to be differentiable and that the action
G on P is also differentiable. In this way, we will demand that the notion of connection in this setting to be « compatible » with the action of G as follows:

Definition 2.2.17. *Let $P \longrightarrow M$ a G-principal bundle. A G-principal connection is defined by a differential 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ taking values in the Lie $\mathbf{k}$-algebra $\mathfrak{g}$ associated to G such that*

1. *ω is G -equivariant i.e. $\mathrm{ad}_g(R_g^*\omega) = \omega$, where ad_g is the adjoint representation;*
2. *if $\gamma \in \mathfrak{g}$ and $X\gamma$ is the fundamental vector field associated with γ by differentiation of the action of G on P, then $\omega(X_\gamma) = \gamma$ (identically over P).*

Remark 2.2.18. *Let G be a Lie group with associated Lie $\mathbf{k}$-algebra $\mathfrak{g}$, let $P \longrightarrow M$ be a trivial G-principal bundle and let $\omega \in \Omega^1(M, \mathfrak{g})$ be a differential 1-form that defines a connection on P. In this case the curvature is given by the differential 2-form with values in $\mathfrak{g}$ defined by*

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega^2(M, \mathfrak{g}),$$

where d is the external differential, $[- \wedge -]$ is the operation $\Omega^1(M, \mathfrak{g}) \times \Omega^1(M, \mathfrak{g}) \longrightarrow \Omega^2(M, \mathfrak{g})$ defined, for all pairs of tangent vectors v_1 and v_2 a M, by

$$[\omega \wedge \eta](v_1, v_2) = [\omega(v_1), \eta(v_2)] - [\omega(v_2), \eta(v_1)]$$

so that we get

$$\Omega(v_1, v_2) = d\omega(v_1, v_2) + \frac{1}{2}[\omega \wedge \omega](v_1, v_2) = d\omega(v_1, v_2) + [\omega(v_1), \omega(v_2)].$$

We will denote in the future $[\omega, \omega]$ for the 1-form bracket.

regularized holonomy and regularized iterated integrals

Let us quickly explain the formulation of parallel transport in terms of path ordered exponentials.

Remark 2.2.19. *Let G be a Lie group with associated Lie $\mathbf{k}$-algebra $\mathfrak{g}$. Consider the following general Cauchy problem:*

$$\left\{ \begin{array}{l} d\varphi = \alpha\varphi \\ \varphi(0) = 1_G \end{array} \right. , \tag{2.10}$$

where $\varphi : [0,1] \longrightarrow G$ is a function and $\alpha \in \Omega^1([0,1],\mathfrak{g})$ is a differential 1*-form taking values in* $\mathfrak{g}$*. Then there is a unique solution ϕ of (2.10) and we can define the* path ordered exponential

$$\mathcal{P}\exp\left(\int_0^1 \alpha\right) := \phi(1) \in G.$$

As a consequence of Picard succesive iterations method, we can explicitely develop this element into the so-called Dyson series*:*

$$\mathcal{P}\exp\left(\int_0^1 f(t)dt\right) = 1+\left(\int_0^1 f(t_1)dt_1\right)+\cdots+\left(\int_{0\leqslant t_n\leqslant\cdots\leqslant t_1\leqslant 1} dt_1\ldots dt_n f(t_1)\ldots f(t_n)\right)+\cdots$$

We can similarly extend this definition for every differientable manifld M considering, for $\alpha \in \Omega^1(M,\mathfrak{g})$ and $\gamma : [0,1] \to M$, the path ordered exponential

$$\mathcal{P}\exp\left(\int_\gamma \alpha\right) = \mathcal{P}\exp\left(\int_{[0,1]} \gamma^*\alpha\right).$$

In this case, considering the trivial G-principal bundle over M, the parallel transport of the connection $\nabla = d - \alpha$ along the path γ is precisely

$$\mathcal{P}\exp\left(\int_\gamma \alpha\right).$$

If γ is a piece-wise smooth path on M, then the iterated integral of the differential 1-forms $\omega_1,\ldots,\omega_n \in \Omega^1(M,G)$ is

$$\int_\gamma \omega_1\cdots\omega_n := \int_{0\leqslant t_n\leqslant\cdots\leqslant t_1\leqslant 1} dt_1\ldots dt_n f(t_1)\ldots f(t_n)$$

Proposition 2.2.20. *Let $P \longrightarrow M$ be the trivial G-principal over M and let ∇ be a connection on this bundle. It is said that a connection ∇ is flat if, equivalently:*

1. *The curvature $\nabla\circ\nabla$ of the connection is zero;*
2. *the 1-form ω associated to ∇ satisfies the Maurer-Cartan equation :*

$$d\omega + \frac{1}{2}[\omega,\omega] = 0;$$

3. *For each pair (γ_1,γ_2) of homotopic paths in X we have $T_{\gamma_1} = T_{\gamma_2}$.*

Remark 2.2.21. *If this is the case, then the parallel transport of ∇ along a loop based on a point $x_0 \in X$ induces a group morphism*

$$\rho : \pi_1(X,x_0) \longrightarrow \mathrm{Aut}(E_{x_0})$$

called monodromy morphism or a monodromy representation of the fundamental group of X with respect to its action on the fiber of x_0.

Proof. We will only show the first equivalence:

- Step 1 $\Longrightarrow$ 2: A horizontal section of this connection satisfies $df = -\omega f$. If the connection is flat, then

$$0 = -d^2 f = d(\omega f) = (d\omega)f - \omega \wedge df = (d\omega + w \wedge w)f = \left(d\omega + \frac{1}{2}[\omega, \omega]\right) f$$

  for any horizontal section. As locally there is a flat frame of the bundle, this implies that $d\omega + w \wedge w = 0$.

- The step 2 $\Longrightarrow$ 1: this follows from the Frobenius theorem.

$\square$

We will assume the following:

Proposition 2.2.22. *Let ω a differential 1-form over a Riemann surface M with logarithmic singularities over a finite subset S of M. Then, for all z_1, z_2 in $M - S$, the following limits exist:*

$$\lim_{t \longrightarrow 0} t^{\nabla(\omega)_s} T_\omega(\gamma_t^{z_1}) \qquad \lim_{t \longrightarrow 0} T_\omega(\gamma_t^{z_2}) t^{-\nabla(\omega)_s}.$$

In the next section we will give a particular example of a flat connection that is naturally associated to the $(\mathrm{KZ})_n$ system. We will dicover how to retrieve multizeta values from the parallel transport of this connection and find new relations for these numbers

2.2.3 The universal KZ connection

The objective of this section is to convince the reader of the fact that, using basic results on the geometry of the configuration spaces, the proof of the fact that the KZ associator is a Drinfeld associator is a consequence of the flatness of a certain connection defined on this space and therefore, in a certain way, the manipulations of the KZ differential equations becomes visible. This allows to have a better understanding of the architecture of the Drinfeld associators.

The differential system $(\mathrm{KZ})_n$ leads to an associated connection, the universal KZ connection, which is flat in the configuration space of n points in the complex plane. Regardless of its application to the understanding of Drinfeld associators and multizeta values, this connection has several fields of application: for example, it provides a monodromy representation of the fundamental group of its basis space, that is to say of the pure braid group on the plane. This implies, in particular, the formality of this group, as we will explain below.

Let $\mathcal{P} := \mathrm{Conf}(\mathbb{C}, n) \times \exp(\hat{\mathfrak{t}}_n)$ be the trivial $\exp(\hat{\mathfrak{t}}_n)$-bundle over $\mathrm{Conf}(\mathbb{C}, n)$.

Definition 2.2.23. *The universal KZ connection is $\nabla_n^{\mathrm{KZ}} := d - \omega_n^{\mathrm{KZ}}$, where ω_n^{KZ} is the differential 1-form over $\mathrm{Conf}(\mathbb{C}, n)$ with values in the Kohno-Drinfeld Lie $\mathbb{C}$-algebra $\mathfrak{t}_n$ given by the following formula:*

$$\omega_n^{\mathrm{KZ}} \quad := \quad \sum_{1 \leqslant i < j \leqslant n} d\log(z_i - z_j) t_{ij}.$$

Remark 2.2.24. *A function* $\sigma : \mathrm{Conf}(\mathbb{C}, n) \longrightarrow \mathfrak{t}_n$ *is a horizontal section of* ∇_n^{KZ} *if, and only if,* σ *is a solution of the system* $(\mathrm{KZ})_n$. *Indeed, as* ∇_n^{KZ} *is a connection defined on a trivial* $\exp(\mathfrak{t}_n)$*-bundle, its horizontal sections are functions* $\mathrm{Conf}(\mathbb{C}, n) \longrightarrow \mathfrak{t}_n$ *so that* σ *is well defined as a section.*

Why is it universal? The explanation of the exact meaning of the word « universal » goes through several points. Let us begin by defining the holonomy Lie algebra of a smooth variety and its de Rham fundamental group following [26] and [37].

2.2.4 Reminders on the Riemann-Hilbert correspondence

Let X be a complex smooth variety. Let $H^{\bullet}_{\mathrm{dR}}(X)$ be the de Rham cohomology complex of X, let

$$\mu : \wedge^2 H^1_{\mathrm{dR}}(X) \longrightarrow H^2_{\mathrm{dR}}(X)$$

be the multiplication map, let us denote $H_1(X)$ for the dual of $H^1_{\mathrm{dR}}(X)$ and let $K^{\perp} \subset \wedge^2 H_1(X)$ be the dual subspace of $K := \ker(\mu) \subset \wedge^2 H^1_{\mathrm{dR}}(X)$.

Let $\bar{X}$ be the smooth compactification of X with $D = \bar{X} - X$ a normal crossings divisor. For simplicity we suppose that $H^1_{\mathrm{dR}}(X)$ is pure of weight 2, implying that $H^1(X)$ is isomorphic to $H^0(\bar{X}, \Omega^1_{\bar{X}}(\log(D))$.

Deligne established in [26] an equivalence of tensor categories between:

- the category $\mathrm{VBFC}(X)$ of vector bundles with a flat connection on X with regular singularities,
- the category $\mathrm{LS}(X)$ of topological local systems on X.

This is known as the Riemann-Hilbert corrrespondence.

Notice that here, for a vector space E, $\wedge^2 E$ identifies with the degree 2 component of the free Lie algebra generated by E.

Now, one can attach to these tensor categories its unipotent part (see [37] for details). The R-H correspondence then induces an equivalence between the unipotent parts of these categories:

$$\mathrm{RH}^{uni} : \mathrm{VBFC}(X)^{uni} \xrightarrow{\sim} \mathrm{LS}(X)^{uni}. \tag{2.11}$$

This map associates to each object of $\mathrm{VBFC}(X)$ the local system of its horizontal sections.

Any point $x \in X$ gives rise to two fiber functors $F^{ls}_x : \mathrm{LS}(X) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ and $F^{vb}_x : \mathrm{VBFC}(X) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ and to a canonical isomorphism $F^{ls}_x \circ \mathrm{RH} \simeq F^{vb}_x$.

Definition 2.2.25. *Let* X *be a complex smooth variety.*

- *The holonomy Lie* $\mathbb{C}$*-algebra* $\mathfrak{hol}(X)$ *of* X *is the free Lie* $\mathbb{C}$*-algebra over* $H_1^{\mathrm{dR}}(X)$ *modulo relations in* $K^{\perp}$.

- *The de Rham fundamental group of X is the unipotent Tannakian fundamental group of the category of vector bundles with flat unipotent connections on X with regular singularities at infinity :*
$$\pi_1^{\mathrm{dR}}(X,x)^{uni} := \mathrm{Aut}^{\otimes}(F_x^{vb}),$$
for the choice of a base point $x \in X$.

- *The Betti fundamental group of X with base point x is the Tannakian group corresponding to F_x^{ls} :*
$$\pi_1^B(X,x) := \mathrm{Aut}^{\otimes}(F_x^{ls}).$$

Remark 2.2.26.
- *The relations in $K^{\perp}$ are all in degree 2, so $\mathfrak{hol}(X)$ is provided with a natural graduation.*
- *The R-H correspondence then provides us with a map $\pi_1^{\mathrm{dR}}(X,x)^{uni} \longrightarrow \pi_1^B(X,x)$.*

Deligne then proved the following result.

Theorem 2.2.27. *The Lie algebra of $\pi_1^{\mathrm{dR}}(X)$ coincides with the degree completion $\hat{\mathfrak{hol}}(X)$ of the holonomy Lie algebra of X.*

In practice it can be convenient to characterise $\mathfrak{hol}(X)$ the following way.

Proposition 2.2.28. *If $H^*_{\mathrm{dR}}(X)$ is generated by $H^1_{\mathrm{dR}}(X)$, then $\mathfrak{hol}(X)$ is Koszul dual to $H^*_{\mathrm{dR}}(X)$ as commutative algebras.*

Let us finish this reminder on some comments on Gauss-Manin connections in the complex analytic context.

Let $f : X \longrightarrow S$ be a smooth family of complex manifolds. We have a local system $\mathrm{R}^n\, f_*\mathbb{C}$ of complex vector spaces on S, defining a holomorphic vector bundle $\mathcal{V} := \mathrm{R}^n\, f_*\mathbb{C} \otimes \mathcal{O}_S$ on S with an integrable connection $\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_S$ of the family, so we get a connection on the latter. We have a map
$$DR : D^b_{hr}(\mathcal{D}_X) \longrightarrow D^b_c(\mathbb{C}_X),$$
so that $\mathcal{M} \longmapsto DR(\mathcal{M}) := \omega_X \otimes^L_{D_X} \mathcal{M}$ is the analytic de Rham complex. By the R-H correspondence the map DR is an equivalence. DR sends a $\mathcal{O}$-coherent $\mathcal{D}$-module (i.e. a vector bundle with an integrable connexion) to a local system (i.e. a locally constant sheaf). The inverse functor sends a locally constant V to the vector bundle $\mathcal{O}_X \otimes_C V$ together with the only connexion such that V is the local system of horizontal sections in $(\mathcal{O}_X \otimes_C V, \nabla)$.

The Gauss-Manin connection is then defined as $DR^{-1}(Rf_*\mathbb{C}_X)$.

2.2.5 Universality of the KZ connection

Let us compute the holonomy Lie algebra of the configuration spaces $\mathrm{Conf}(\mathbb{C}, n)$.

1. Suppose that $X = \mathrm{Conf}(\mathbb{C}, 2)$. As $\mathrm{Conf}(\mathbb{C}, 2) \cong \mathbb{C}^2 - \{z_1 = z_2\}$, we can take the following coordinates:
$$\begin{aligned} x &= z_1 + z_2 \\ y &= z_1 - z_2 \end{aligned}.$$
In these coordinates, the only differential 1-form with logarithmic singularities on X is $d\log(y)$. In this way, we find
$$\mathfrak{hol}(X) = \mathfrak{f}_1(\mathbb{C}) \cong \mathfrak{t}_2(\mathbb{C})$$
where $\mathfrak{f}_1$ is the free Lie $\mathbf{k}$-algebra on 1 generator (which is of dimension equal to 1).
2. Suppose that $X = \mathrm{Conf}(\mathbb{C}, n)$. Then we have
   - $H^1_{\mathrm{dR}}(X)$ is generated by the 1-forms
   $$\omega_{ij} = d\log(z_i - z_j),$$
   where $1 \leqslant i < j \leqslant n$.
   - (Arnold) K is generated by
   $$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij},$$
   where $1 \leqslant i < j < k \leqslant n$.
   - If $\{t_{ij}\}_{i<j} \in H_1^{\mathrm{dR}}(X)$ is the dual basis to the basis $\{\omega_{ij}\}_{i<j}$ of $H^1_{\mathrm{dR}}(X)$, then $K^{\perp}$ is generated by elements
   $$t_{ij} \wedge t_{kl}, t_{ij} \wedge (t_{ik} + t_{jk}),$$
   where $\mathrm{card}(i, j, k, l) = 4$.

   In conclusion, the holonomy Lie $\mathbb{C}$-algebra of $\mathrm{Conf}(\mathbb{C}, n)$ is the Kohno-Drinfeld Lie $\mathbb{C}$-algebra $\mathfrak{t}_n$.

In this way, we see that the system $d\varphi = \sum\limits_{1\leqslant i<j\leqslant n} t_{ij} d\log(z_i - z_j)$ is defined in a natural way in the $\exp(\widehat{\mathfrak{hol}}(X))$-trivial bundle over $X = \mathrm{Conf}(\mathbb{C}, n)$.

In addition, this system contains the smallest amount of information necessary to be well defined:

- Let W be a vector space and consider the trivial vector bundle $\mathrm{Conf}(\mathbb{C}, n) \times W \longrightarrow \mathrm{Conf}(\mathbb{C}, n)$. Let's consider the connection
$$\tilde{\nabla} = d - \sum_{1\leqslant i<j\leqslant n} d\log(z_i - z_j) A_{ij}.$$
defined over the above bundle, where A_{ij} are endomorphisms of W. In this case, a sufficient condition for $\tilde{\nabla}$ to be flat is that A_{ij} satisfy the three infinitesimal braid relations. In this sense, the Kohno-Drinfeld Lie algebra is the « simplest » possible so that the connection satisfies these relations.

- Consider the connection

$$\nabla = d - \sum_{1\leqslant i\neq j\leqslant n} A_{ij}(z_i - z_j)d(z_i - z_j),$$

where the matrices $A_{ij}(z_i - z_j)$ act in the i-th and the j-th entries of $V = V_1 \otimes \cdots \otimes V_n$. In this case, the connection is flat if, and only if, the family $\{A_{ij}(z_i - z_j)\}$ satisfies the Yang-Baxter equation

$$[A_{ik}(z_i - z_k), A_{kj}(z_k - z_j)] + [A_{ik}(z_i - z_k), A_{ij}(z_i - z_j)] + [A_{ij}(z_i - z_j), A_{kj}(z_k - z_j)] = 0.$$

In particular, if we consider the simplest possible choice of r-matrix, that is, if we consider

$$A_{ij}(z_i - z_j) := \frac{t_{ij}}{z_i - z_j},$$

where t_{ij} are formal symbols, then we have

$$\begin{aligned} \nabla \text{ is flat} \quad &\Longleftrightarrow \quad \{A_{ij}\} \text{ satisfies the Yang-Baxter equation} \\ &\Longleftrightarrow \quad \{t_{ij}\} \text{ satisfies the infinitesimal braid relations.} \end{aligned}$$

2.2.6 Reminders on semi-simple Lie algebras.

Let $\mathfrak{g}$ be a Lie $\mathbf{k}$-algebra. Its adjoint representation is the $\mathbf{k}$-vector space map given by

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathrm{End}(\mathfrak{g}). \\ x & \longmapsto & (\mathrm{ad}_x : y \mapsto [x, y]) \end{array}$$

If $\mathfrak{g}$ is finite dimensional, then :

- There is a well defined bilinear symmetric form

$$B(x, y) := \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$$

called Killing form, which is $\mathfrak{g}$-invariant under the action of $\mathrm{Aut}(\mathfrak{g})$ and such that

$$B([x, y], z) = B(x, [y, z]),$$

for all $x, y, z \in \mathfrak{g}$.

- If $\{X_i\}_{i\leqslant n}$ is a basis of $\mathfrak{g}$ and $\{X^i\}_{i\leqslant n}$ is its dual basis with respect to B, the Casimir element is

$$\Omega = \sum_{i=1}^{n} X_i X^i \in Z(\mathcal{U}(\mathfrak{g}))$$

i.e. commutes with all elements in $\mathfrak{g}$ and is independent of the choice of the basis.

- If $\mathrm{char}(\mathbf{k}) = 0$ then:

$$\mathfrak{g} \text{ is semi-simple} \Longleftrightarrow B \text{ is non-degenerate.}$$

Now let $\mathfrak{g}$ be a finite dimensional Lie $\mathbb{C}$-algebra, let

$$S^2(\mathfrak{g}) := T^2(\mathfrak{g})/(x \otimes y - y \otimes x)$$

be the symmetric algebra associated to $\mathfrak{g}$. Then, for an orthonormal basis of $\mathfrak{g}$ with respect to B, we have that $T(v_1, v_2) = \Sigma_u a_u \otimes b_u \in S^2(\mathfrak{g})$ satisfies $T(v_1, v_2) = T(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)})$. We have $\mathcal{O}_{\mathfrak{g}^*} = S(\mathfrak{g})$.

An element y in $S^2(\mathfrak{g})$ is said to be $\mathfrak{g}$ invariant if $[x \otimes x, y] = 0$, for all $x \in \mathfrak{g}$. The set of such elements will be denoted $S^2(\mathfrak{g})^{\mathfrak{g}}$. Then $t_{\mathfrak{g}} = \Sigma_u e_u \otimes f_u \in S^2(\mathfrak{g})^{\mathfrak{g}}$. By choosing a basis we get

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

where $c_{ij}^k = -c_{ji}^k$ are the structure constants. In particular

$$\begin{aligned} (a \odot b)(c \odot d) &= \mathrm{ac} \otimes \mathrm{bd}; \\ [a \otimes b, c \otimes d] &= \mathrm{ac} \otimes \mathrm{bd} - \mathrm{ca} \otimes \mathrm{db} \neq [a, c] \odot [b, d]. \end{aligned}$$

Let G be a connected Lie group with associated Lie algebra $\mathfrak{g}$. If G acts on a differientable manifold M, then $x \in \mathfrak{g}$ is represented by a first order differential operator over M and this representation ρ is in $C^\infty(M)$. If G and G' are n dimensional and have the same structure constants, then they are locally isomorphic. This means that the structure constants are related to the second order partial derivatives in a neighborhood of the identity but give local properties over the whole group : for instance, they tell if locally the multiplication is contractible.

2.2.7 Realizations of the universal KZ connection

The universal KZ connection « has realizations »: consider

- a (semi-)simple Lie $\mathbb{C}$-algebra $\mathfrak{g}$;
- a symmetric $\mathfrak{g}$-invariant 2-tensor $\Omega = \sum_r x_r \otimes y_r \in \mathfrak{g} \otimes \mathfrak{g}$ (which is constructed from the Casimir, coming from the Killing form associated with $\mathfrak{g}$),
- a non-zero integer $n \in \mathbb{N}^{\geqslant 1}$,
- a finite dimensional $\mathfrak{g}$-module V,
- a formal parameter $\hbar = \frac{h}{2i\pi} \in \mathbb{C}$.

Let's define

$$t^{ij} := \sum_r \alpha_r^{(1)} \otimes \cdots \otimes \alpha_r^{(n)} \in (\mathcal{U}(\mathfrak{g}))^{\otimes n},$$

where $\alpha_r^{(i)} = x_r$, $\alpha_r^{(j)} = y_r$ and $\alpha_r^{(k)} = 1$, where $k \neq i, j$. Then

1. Every t^{ij} induces an endomorphism of $V^{\otimes n}$ that satisfies the infinitesimal braid relations.
   - This fact is a consequence of the construction of t^{ij} and the $\mathfrak{g}$-invariance of $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$.
2. We have $t^{ij} = t^{ji}$.

- This fact is a consequence of the symmetry of the 2-tensor t^{ij}.

We conclude that we have a morphism

$$\begin{array}{rcl} \exp(\hat{\mathfrak{t}}_n(\mathbb{C})) & \longrightarrow & \mathrm{End}(V^{\otimes n})[[\hbar]] \\ e^{t_{ij}} & \longmapsto & \hbar t^{ij} \end{array}$$

and the system

$$(\mathrm{KZ}')_n \qquad dw = \hbar \sum_{1 \leqslant i < j \leqslant n} d\log(z_i - z_j) t^{ij} w$$

is called realization of the universal system $(\mathrm{KZ})_n$ associated to $(\mathfrak{g}, V)$.

2.2.8 Holonomy of the connection ∇_3^{KZ} and geometric definition of the KZ associator

Let $\varepsilon > 0$. Denote $X_0 = t_{12}$ and $X_1 = t_{23}$. Let $\Phi_\varepsilon(X_0, X_1)$ be the parallel transport of the universal KZ connection with respect to the path $\varphi : [0,1] \longrightarrow \mathbb{C} - \{0,1\}$ such that $\gamma(0) = \varepsilon$, $\gamma(1) = 1 - \varepsilon$ and $\gamma(t) \in \mathbb{R}$, that is, given by the path ordered exponential

$$\begin{array}{rcl} \Phi_\varepsilon(X_0, X_1) & := & \mathcal{P}\exp\left(\int_{\gamma(0)}^{\gamma(1)} \left(\frac{X_0}{z} + \frac{X_1}{z-1}\right) dz\right) \\ & = & \sum_{\omega \text{word in} X_0, X_1} c_\omega(\varepsilon) \cdot \omega \end{array}$$

where, for $j_0, \ldots, j_n \in \{0,1\}$, $\omega = x_{j_0} \cdots x_{j_n} \in \mathbb{Q}\langle X_0, X_1 \rangle$, and

$$c_\omega(\varepsilon) = \int_{\gamma(0)}^{\gamma(1)} \frac{dt_1}{t_1 - z_{j_1}} \int_{\gamma(0)}^{t_1} \frac{dt_2}{t_2 - z_{j_2}} \cdots \int_{\gamma(0)}^{t-n-1} \frac{dt_n}{t_n - z_{j_n}}. \tag{2.12}$$

Recall that the polylogarithm function is given, for $s, z \in \mathbb{C}$, by

$$\mathrm{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

and that multizeta values are the real numbers

$$\zeta(k_1, \ldots, k_r) := \sum_{n_1 > n_2 > \ldots > n_r > 0} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}} = \sum_{n_1 > n_2 > \ldots > n_r > 0} \left(\prod_{i=1}^{r} \frac{1}{n_i^{k_i}}\right)$$

where $k_1, \ldots, k_{r-1} \in \mathbb{N}^{\geqslant 1}$, $k_r \geq 2$.

We are going to admit the following proposition, which we will explain in the next subsection.

Proposition 2.2.29. *For each word ω in X_0 and X_1, the scalar $c_\omega(\varepsilon)$ is a polynomial in polylogarithm functions of the form $\mathrm{Li}_n(\varepsilon)$ and in the function $\log(\varepsilon)$. In particular, if the word ω ends in X_1 (in particular ω can be written in the form $\omega = X_0^{n_1-1} X_1 X_0^{n_2-1} X_1 \ldots X_0^{n_k-1} X_1$, where $n_i \geqslant 2$, for all $k \geqslant 1$), then the function $c_\omega(\varepsilon)$ converges when ε tends to 0 and we have*

$$\lim_{\varepsilon \to 0} c_\omega(\varepsilon) = (-1)^k \zeta(n_1, \ldots, n_k).$$

Corollary 2.2.30. *$\Phi_\varepsilon(X_0, X_1)$ has an asymptotic expansion into a homogenous polynomial $\Phi(\log(\varepsilon))$, that is,*

$$\Phi_\varepsilon(X_0, X_1) - \Phi(\log(\varepsilon)) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Proposition 2.2.31. *The Drinfeld KZ associator is, equivalently, defined by:*

1. *the quotient $\Phi^{\mathrm{KZ}} := G_1^{-1}G_0 \in \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$ of Proposition 2.2.4 of the first section;*
2. *the regularized holonomy of the connection ∇_3^{KZ} between 0 and 1 (following the real part of $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$) i.e. the limit*

$$\Phi_{\mathrm{KZ}}(X_0, X_1) := \lim_{\varepsilon\to 0} \varepsilon^{X_1}\Phi\varepsilon(X_0, X_1)\varepsilon^{-X_0};$$

3. *the regularization $\Phi_{\mathrm{KZ}} := \Phi(0)$ of the polynomial $\Phi(\log(\varepsilon))$ by formally setting $\log(\varepsilon) = 0$.*

Proof. First, the three definitions make sense in light of the above paragraphs. Let's prove that these definitions are equivalent. We have the expression

$$\Phi_\varepsilon(X_0, X_1) = G_1(1-\varepsilon)G_1^{-1}G_0G_0^{-1}(\varepsilon)$$

so that the following limit exists

$$\lim_{\varepsilon\to 0} \varepsilon^{X_1}\Phi_\varepsilon(X_0, X_1)\varepsilon^{-X_0} = \lim_{\varepsilon\to 0}(\varepsilon^{X_1}G_1(1-\varepsilon))G_1^{-1}G_0(G_0^{-1}(\varepsilon)\varepsilon^{-X_0}) = G_1^{-1}G_0$$

Therefore, the asymptotic expansion of $\Phi_\varepsilon(X_0, X_1)$ is $\varepsilon^{X_1}G_1^{-1}G_0\varepsilon^{-X_0}$. □

Monodromy of the KZ connection and geometric proof of Theorem A.

Let's start with a crucial result that will allow us to work with the connection.

Proposition 2.2.32. *The universal KZ connection is flat, that is: $(\nabla_{\mathrm{KZ}})^2 = 0$.*

Proof. We compute:

$$\begin{aligned}
\nabla^2 &= \sum_{\substack{i<j\\k<l}} \frac{[t_{ij}, t_{kl}]d(z_i - z_j)d(z_k - z_l)}{(z_i - z_j)(z_k - z_l)} \\
&= \sum_{i\neq j} dz_i dz_j \left(\sum_{\substack{i\neq k\\j\neq l}} \frac{[t_{ij}, t_{kl}]}{(z_i - z_j)(z_k - z_l)} \right) \\
&= \sum_{i\neq j} dz_i dz_j \left(\sum_{j\neq l} \frac{[t_{ij}, t_{kl}]}{(z_i - z_j)(z_j - z_l)} + \sum_{i\neq k} \frac{[t_{ik}, t_{ji}]}{(z_i - z_k)(z_j - z_i)} \right) \\
&= \sum_{i\neq j} dz_i dz_j \left(\sum_{k\neq i,j} \frac{-[t_{ik}, t_{jk}]}{(z_i - z_j)(z_j - z_k)} + \sum_{k\neq i,j} \frac{[t_{ik}, t_{jk}]}{(z_i - z_k)(z_j - z_i)} \right) \\
&= -\sum_{i\neq j} dz_i dz_j \sum_{k\neq i,j} \frac{[t_{ik}, t_{jk}]}{(z_i - z_j)(z_j - z_k)} = 0.
\end{aligned}$$

We conclude that the connection is flat. □

In this way, we can talk about *monodromy* of the connection and we will see how the Drinfeld associator relations arise, in the case of the KZ associator, precisely from this monodromy. We reproduce the proof of Theorem A. The element

$$\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) \quad := \quad \lim_{\varepsilon \to 0} \varepsilon^{t_{23}} \mathcal{P} \exp \left(\int_{\varepsilon}^{1-\varepsilon} \left(\frac{t_{12}}{z} + \frac{t_{23}}{z-1} \right) dz \right) \varepsilon^{-t_{12}}$$

is the regularized holonomy between 0 and 1 of the universal KZ connection, seen in the complex projective line minus three points. Let's prove that the pair $(2i\pi, \Phi_{\mathrm{KZ}})$ is a Drinfeld associator.

The case $n = 2$. First of all, we have

$$\begin{array}{rcl} \mathrm{Conf}(\mathbb{C}, n) & \cong & \mathbb{C} \times \mathbb{C} - \{0\} \\ (z_1, z_2) & \longmapsto & (t, w) := (z_2, z_1 - z_2) \end{array}$$

In this way, the KZ connection is written

$$\nabla_2^{\mathrm{KZ}} = d - \frac{t_{12}}{w} dw.$$

The associated KZ equation is the system

$$\begin{cases} \frac{\partial}{\partial w} F = \frac{t_{12}}{w} F \\ \frac{\partial}{\partial t} F = 0 \end{cases} .$$

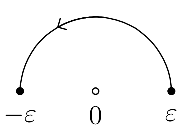
Therefore, the solutions are given by

$$F(z_1, z_2) = C(z_1 - z_2)^{12},$$

for a certain constant C. Let

$$\begin{array}{rcl} \gamma : [0,1] & \longrightarrow & \mathbb{C} - \{0\} \\ t & \longmapsto & \varepsilon e^{i\pi t} \end{array}$$

be the continuous path that draws a closed semi-circle from ε to $-\varepsilon$ in $\mathbb{C} - \{0\}$:



We immediately find that the regularized holonomy of the connection ∇_2^{KZ} is $e^{i\pi t_{12}} = e^{\frac{\lambda}{2} t_{12}}$ for $\lambda = 2i\pi$.

The case $n = 3$: First, we have an isomorphism

$$\begin{array}{rcl} \mathrm{Conf}(\mathbb{C}, 3) & \cong & \mathbb{C} \times \mathbb{C}^{\times} \times (\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}) \\ (z_1, z_2, z_3) & \longmapsto & (t, w, z) := \left(z_3, z_1 - z_3, \dfrac{z_1 - z_2}{z_1 - z_3} \right) \end{array}$$

By a change of coordinates, the equations $(\mathrm{KZ})_3$ become:

$$\begin{cases} \frac{\partial}{\partial w}F = \frac{t_{12}+t_{13}+t_{23}}{w}F \\ \frac{\partial}{\partial t}F = 0 \\ \frac{\partial}{\partial z}F = \frac{t_{12}}{z}F + \frac{t_{23}}{z-1}F \end{cases}$$

Remark 2.2.33. *Notice that we are rephrasing the results of previous section. Indeed, the solution in this case is*

$$F(z_1, z_2, z_3) = (z_1 - z_3)^t + t_{13} + t_{23}G\left(\frac{z_1 - z_2}{z_1 - z_3}\right),$$

where $G(z)$ solves the equation

$$\frac{\partial}{\partial z}G = \frac{t_{12}}{z}G + \frac{t_{23}}{z-1}G.$$

We are ready to start the proof:

$\longrightarrow$ **$\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) \in \exp(\hat{\mathfrak{f}}(t_{12}, t_{23}))$:** For now, we only know that $\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) \in \mathbb{C}\langle\langle t_{12}, t_{23}\rangle\rangle$. To show that $\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) \in \exp(\hat{\mathfrak{f}}(t_{12}, t_{23})$ we have to prove that $\Phi_{\mathrm{KZ}}(t_{12}, t_{23})$ is group-like, meaning $\Delta\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) = \Phi_{\mathrm{KZ}}(t_{12}, t_{23})\hat{\otimes}\Phi_{\mathrm{KZ}}(t_{12}, t_{23})$. On the one hand,

$$\Delta\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) = \Phi_{\mathrm{KZ}}(\Delta t_{12}, \Delta t_{23}) = \Phi_{\mathrm{KZ}}(t_{12} \otimes 1 + 1 \otimes t_{12}, t_{23} \otimes 1 + 1 \otimes t_{23}).$$

On the other hand, $\Phi_{\mathrm{KZ}}(t_{12} \otimes 1 + 1 \otimes t_{12}, t_{23} \otimes 1 + 1 \otimes t_{23})$ is the holonomy of the connection

$$\begin{aligned} \nabla &= d - \left(\frac{t_{12} \otimes 1 + 1 \otimes t_{12}}{z} + \frac{t_{23} \otimes 1 + 1 \otimes t_{23}}{z-1}\right)dz \\ &= d - \frac{t_{12} \otimes 1 + 1 \otimes t_{12}}{z}dz - \frac{t_{23} \otimes 1 + 1 \otimes t_{23}}{z-1}dz, \end{aligned}$$

which can also be seen as the sum of two connections in two different bundles. In this way, the holonomy can be calculated separately. Finally, we get

$$\Delta\Phi_{\mathrm{KZ}}(t_{12}, t_{23}) = \Phi_{\mathrm{KZ}}(t_{12} \otimes 1 + 1 \otimes t_{12}, t_{23} \otimes 1 + 1 \otimes t_{23}) = \Phi_{\mathrm{KZ}}(t_{12}, t_{23})\hat{\otimes}\Phi^{\mathrm{KZ}}(t_{12}, t_{23}).$$

$\longrightarrow$ **Antisymmetry:** Taking the change of variables $z = 1 - y$, the connection is written

$$d - \left(\frac{t_{12}}{y-1} + \frac{t_{23}}{y}\right)dy,$$

whose holonomy between ε and $1 - \varepsilon$ is $\Phi_\varepsilon(t_{23}, t_{12})$. By symmetry, this is also the holonomy from $1 - \varepsilon$ to ε of the original connection i.e. the inverse of the holonomy from ε and $1 - \varepsilon$ of the same connection. In this way, $\Phi_\varepsilon(t_{23}, t_{12}) = \Phi_\varepsilon(t_{12}, t_{23})^{-1}$. Automatically, we verify that the same equation is preserved after asymptotic expansion and regularization.

$\longrightarrow$ **Two hexagons:** Using the monodromy calculation in the case $n = 2$, we easily see that the regularized holonomy of ∇_3^{KZ} around the singularity $z = 0$ (in the counterclockwise direction) in $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ is $e^{i\pi t_{12}}$.

One can easily prove the following facts

1. If we take the path γ in the clockwise direction, we get a holonomy equal to $e^{-i\pi t_{12}}$.
2. The regularized holonomy of ∇_3^{KZ} around the singularity $z=1$ (counterclockwise direction) in $\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\}$ is $e^{2i\pi t_{23}}$.
3. Making a change of variables to be determined, the regularized holonomy of ∇_3^{KZ} around the singularity $z=\infty$ (counterclockwise direction) in $\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\}$ is $e^{2i\pi t_{13}}$.

In this way, we can consider the paths

$$\gamma^+ := \gamma_1^+\gamma_2^+\gamma_3^+\gamma_4^+\gamma_5^+\gamma_6^+ \text{ and } \gamma^- := \gamma_1^-\gamma_2^-\gamma_3^-\gamma_4^-\gamma_5^-\gamma_6^-,$$

formed by the juxtaposition of the following 6 paths:

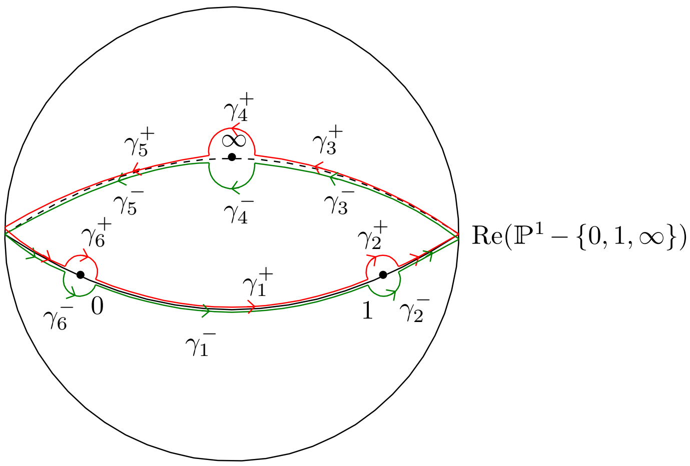


Figure 2.1: Paths in $\mathcal{M}_{0,4}=\mathbb{P}^1-\{0,1,\infty\}$.

We have calculated the holonomy for each of these paths. Notice that the path γ^+ is contractible and the connection is flat so the parallel transport along γ^+ is $T_{\gamma^+}=1$. Also, as γ^+ is composed of 6 terms we get an equation

$$R_{12}\Phi_\varepsilon(t_{13},t_{12})R_{13}\Phi_\varepsilon(t_{23},t_{13})R_{23}\Phi_\varepsilon(t_{12},t_{23})=1.$$

Using the asymptotic expansion of Φ_ε, we can see that $R_{12}\Phi_\varepsilon(t_{13},t_{12})R_{13}\Phi_\varepsilon(t_{23},t_{13})R_{23}\Phi_\varepsilon(t_{12},t_{23})$ has an asymptotic expansion which is a polynomial in ε in each degree. In this way, this equation must be preserved for the part in the constant term of the expansion, that is, when we formally establish $\log(\varepsilon)=0$.

On the other hand, by using exercise ??? of the first section, we know that $\mathfrak{t}_3(\mathbb{C})\simeq\mathbb{C}c_3\oplus\mathfrak{f}(t_{12},t_{23})(\mathbb{C})$. This way, we obtain

$$e^{\frac{-\lambda}{2}t_{12}}\Phi(t_{13},t_{12})e^{\frac{-\lambda}{2}t_{13}}\Phi(t_{23},t_{13})e^{\frac{-\lambda}{2}t_{23}}\Phi(t_{12},t_{23})=1 \text{ en } \exp(\hat{\mathfrak{t}}_3(\mathbb{C})),$$

for $\lambda=2i\pi$.

Remark 2.2.34. *One has to read the juxtaposition of paths from the left to the right (i.e. in the path $\gamma_1^+\gamma_2^+$ we first travel γ_1^+ and then γ_2^+). The composition of holonomies is read from the right to the left as for the composition of functions.*

One can easily show the following facts

1. The holonomy of the path γ^- gives the relation of the remaining hexagon.
2. Let $h \in \mathbb{C}^\times$. If we consider the connection
$$\nabla^{\mathrm{KZ}}_{n,h} = d - \frac{h}{2i\pi} \sum_{1 \leqslant i < j \leqslant n} t_{ij} d\log(z_i - z_j),$$
and we denote Φ^h_{KZ} the regularized holonomy between 0 and 1 of $\nabla^{\mathrm{KZ}}_{3,h}$, find $\lambda = h$ is such that $(\lambda, \Phi^h_{\mathrm{KZ}})$ is a $\mathbb{C}$-associator.

The case $n = 4$: We will present the main steps of the proof of the pentagon relation, leaving the detail to the care of the reader.

$\longrightarrow$ **The pentagon:** After identifying $\mathrm{Conf}(\mathbb{C}, 4)$ with a product of spaces involving the space $(\mathbb{P}^1 - \{0, 1, \infty\})^2 - \{(z, z)\}$, one can interpret the KZ associator as the regularized holonomy from 0 to 1 of the KZ connection over the space $(\mathbb{P}^1 - \{0, 1, \infty\})^2 - \{(z, z)\}$. The path corresponding to the pentagon in
$$\mathrm{Re}((\mathbb{P}^1 - \{0, 1, \infty\})^2 - \{(z, z)\}),$$
presented in the last subsection corresponding to the regions $Z_1, \ldots, Z_5$ of $\mathrm{Re}(\mathrm{Conf}(\mathbb{C}, 4))$, is precisely the path below.

As for the two hexagons, this path is contractible so that its holonomy is equal to 1. By noticing that we can indeed take regularized holonomy, we obtain the required pentagon relation.

2.2.9 Application I : Associator relations for multizeta values

Integral formulation of multizeta values

Recall that the multizeta values are the real numbers
$$\zeta(k_1, \ldots, k_r) := \sum_{n_1 > n_2 > \ldots > n_r > 0} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}}$$
where $(k_1, \ldots, k_r) \in (\mathbb{N}^{\geqslant 2})^r$. These numbers have been studied since Euler (1775). The nature (transcendence/irrationality) of these numbers is a field of much mystery and of which we do not know much.

Proposition 2.2.35 (Kontsevich-Zagier). *The multizeta values can be written as the integrals:*
$$\zeta(k_1, \ldots, k_r) = (-1)^r \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n - \epsilon_n} \cdot \frac{dt_{n-1}}{t_{n-1} - \epsilon_{n-1}} \cdots \frac{dt_1}{t_1 - \epsilon_1}$$
where
$$(\epsilon_1, \ldots, \epsilon_n) = \Bigg(\underbrace{0, \ldots, 0}_{k_1 - 1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{k_2 - 1 \text{ times}}, 1, \ldots, \underbrace{0, \ldots, 0}_{k_r - 1 \text{ times}}, 1 \Bigg).$$

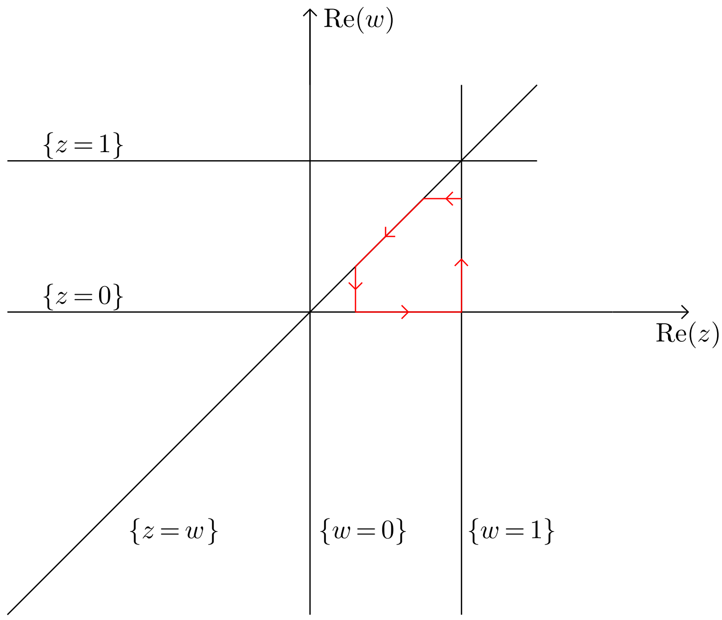


Figure 2.2: Paths in $\mathrm{Re}((\mathbb{P}^1-\{0,1,\infty\})^2-\{(z,z)\})$.

Example 2.2.36. *We have*

$$\begin{aligned}\zeta(2) &= \int_0^1\int_0^{t_1}\frac{dt_2}{1-t_2}\frac{dt_1}{t_1} = \int_0^1\int_0^{t_1}\frac{dt_1}{t_1}\sum_{n\geqslant 1}t_2^{n-1}dt_2 = \int_0^1\sum_{n\geqslant 1}\frac{t_1^n}{n}\frac{dt_1}{t_1}\\ &= \sum_{n\geqslant 1}\frac{1}{n}\int_0^1 t_1^{n-1}dt_1 = \sum_{n\geqslant 1}\frac{1}{n^2}\end{aligned}$$

so we find the original definition of $\zeta(2)$.

Proposition 2.2.37. *The Knizhnik-Zamolodchikov associator is a generating series of all (regularized) multizeta values i.e. we have:*

$$\Phi_{\mathrm{KZ}}(X,Y) = \sum_{w word\ in X,Y}\zeta_w\cdot w.$$

where ζ_w *is the (regularized) multizeta value associated with the word* w.

Example 2.2.38. *In particular, we have a computation in low degree of this series:*

$$\begin{aligned}\Phi_{\mathrm{KZ}}(A,B) &= 1+\zeta(2)[A,B]+\zeta(3)[A,[A,B]]+\zeta(1,2)[[A,B],B]\\ &\quad +\zeta(4)[A,[A,[A,B]]]+\zeta(1,3)[A,[[A,B],B]]+\zeta(1,1,2)[[[A,B],B],B]\\ &\quad +1\zeta(2)^2[A,B]^2+\dots\end{aligned}$$

MZVs and admissible words

How are the MZVs distributed in the series of Proposition 2.2.37? To answer this question we need to introduce the notion of admissible words. Let's start by calculating the iterated integrals involved in the KZ associator for two different kinds of words.

Remark 2.2.39. *To calculate the integrals (2.12), we can use the relations*

$$\begin{aligned} \frac{1}{t-1} &= -\sum_{j=0}^{\infty} t^j \\ \int \frac{\log^n(t)dt}{t} &= \frac{1}{n+1}\log^{n+1}(t) \\ \int \log^n(t)t^m dt &= \sum_{j=0}^{n} \frac{(-1)^j}{m+1}\frac{n!}{(N-j)!}\log^{n-j}(t)t^{m+1} \end{aligned}$$

Example 2.2.40. *Suppose that $\omega = X_0X_0X_1$. We are going to simplify the computations by omitting the terms that tend toward 0 when ε tends to 0. In that case, the triple integral in c_ω is*

$$\begin{aligned} c_\omega(\varepsilon) &= \int_{\gamma(0)}^{\gamma(1)} \frac{dt_1}{t_1}\int_{\gamma(0)}^{t_1}\frac{dt_2}{t_2}\int_{\gamma(0)}^{t_2}\frac{dt_3}{t_3-1} \\ &= \int_{\varepsilon}^{1-\varepsilon}\frac{dt_1}{t_1}\int_{\varepsilon}^{t_1}\frac{dt_2}{t_2}\left(-\sum_{j\geqslant 0}\frac{t_2^{j+1}}{j+1}\right) \\ &= -\int_{\varepsilon}^{1-\varepsilon}\frac{dt_1}{t_1}\left(\sum_{j\geqslant 0}\frac{t_1^{j+1}}{(j+1)^2}\right) \\ &= -\sum_{j\geqslant 0}\frac{(1-\varepsilon)^{j+1}}{(j+1)^3} \xrightarrow{\varepsilon\to 0} -\zeta(3). \end{aligned}$$

Notice that, in this case, c_ω converges when c_w equals 0.

Example 2.2.41. *Suppose this time that $\omega = X_0X_1X_0$. We calculate in this case:*

$$\begin{aligned} c_\omega(\varepsilon) &= \int_{\gamma(0)}^{\gamma(1)}\frac{dt_1}{t_1}\int_{\gamma(0)}^{t_1}\frac{dt_2}{t_2-1}\int_{\gamma(0)}^{t_2}\frac{dt_3}{t_3} \\ &= \int_{\varepsilon}^{1-\varepsilon}\frac{dt_1}{t_1}\int_{\varepsilon}^{t_1}\frac{dt_2}{t_2-1}(\log(t_2)-\log(\varepsilon)) \\ &= \int_{\varepsilon}^{1-\varepsilon}\frac{dt_1}{t_1}\left(\sum_{j\geqslant 0}\frac{t_1^{j+1}}{j+1}\log\left(\frac{t_1}{\varepsilon}\right)-\sum_{j\geqslant 0}\frac{t_1^{j+1}}{(j+1)^2}-\sum_{j\geqslant 0}\frac{\varepsilon^{j+1}}{j+1}\log(\varepsilon)-\sum_{j\geqslant 0}\frac{\varepsilon^{j+1}}{(j+1)^2}\right). \end{aligned}$$

Omitting the terms that tend towards 0 we obtain a term in

$$-\sum_{j\geqslant 0}\frac{(1-\varepsilon)^{j+1}}{(j+1)^3}-\sum_{j\geqslant 0}\frac{(1-\varepsilon)^{j+1}}{(j+1)^3}-\sum_{j\geqslant 0}\frac{(1-\varepsilon)^{j+1}}{(j+1)^2}\log(\varepsilon)\sim -2\zeta(3)-\zeta(2)\log(\varepsilon).$$

This expression diverges logarithmically with ε. This is one of the reasons why we are forced to renormalize the holonomy: to be able to eliminate these divergent terms.

What are the words ω for which $c_\omega(\varepsilon)$ converges? To answer this question we have to talk about admissible words.

Definition 2.2.42. *An admissible (or convergent) word in letters X, Y is a word $\omega \in \mathbb{Q}\langle X, Y\rangle$ starting with X and ending for Y of the form $\omega = XvY$ where v is any word in X and Y.*

We are ready to characterize multizeta values with respect to convergent words.

Proposition 2.2.43. *We have a bijective map*

$$\begin{array}{rcl} (\mathbb{N}^{\geqslant 2})^r & \longleftrightarrow & \{\textit{admissible words in } x, y\} \\ (k_1, \ldots, k_r) & \longleftrightarrow & x^{k_1-1}yx^{k_2-1}y\cdots x^{k_r-1}y \end{array}$$

and the value $c_\omega(\varepsilon)$ converges towards

$$\zeta(k_1, \ldots, k_r) := \zeta_{x^{k_1-1}yx^{k_2-1}y\cdots x^{k_r-1}y}.$$

precisely when the word w is admissible.

Remark 2.2.44.

- *This explains Proposition 2.2.29.*
- *There is a way to associate to the rest of the words (those that are not admissible) a slightly more general notion of multizeta values called regularized multizeta values which we will not present in here.*

Calculation of the KZ Associator in low degree

Let's calculate the terms in degree up to 2 of the associator Φ_{KZ}. We have

$$\begin{aligned} \Phi_\varepsilon(t_{12}, t_{23}) &= \mathcal{P}\exp\left(\int_\varepsilon^{1-\varepsilon}\left(\frac{t_{12}}{z} + \frac{t_{23}}{1-z}\right)dz\right) \\ &= 1 + \int_\varepsilon^{1-\varepsilon}\left(\frac{t_{12}}{t_1} + \frac{t_{23}}{1-t_1}\right)dt_1 \\ &\quad + \int_\varepsilon^{1-\varepsilon}\left(\int_\varepsilon^{t_1}\left(\frac{t_{12}^2}{t_1t_2} + \frac{t_{12}t_{23}}{t_1(1-t_2)} + \frac{t_{23}t_{12}}{t_2(1-t_1)} + \frac{t_{23}^2}{(1-t_2)(1-t_1)}\right)dt_2\right)dt_1 \\ &\quad + \ldots \end{aligned}$$

The degree 1 term is

$$t_{12}\log\left(\frac{1-\varepsilon}{\varepsilon}\right) + t_{23}\log\left(\frac{\varepsilon}{1-\varepsilon}\right).$$

The degree 2 terms are:

$$\begin{aligned} \int_\varepsilon^{1-\varepsilon}\left(\int_\varepsilon^{t_1}\left(\frac{t_{12}^2}{t_1t_2}\right)dt_2\right)dt_1 &= \int_\varepsilon^{1-\varepsilon}\left(t_{12}^2\frac{\log(t_1)-\log(\varepsilon)}{t_1}\right)dt_1 \\ &= \frac{t_{12}^2}{2}(\log(1-\varepsilon)^2 - \log(\varepsilon)^2) - t_{12}^2\log(\varepsilon)(\log(1-\varepsilon)-\log(\varepsilon)) \\ &= t_{12}^2\left(\frac{\log(1-\varepsilon)^2}{2} + \frac{\log(\varepsilon)^2}{2} - \log(\varepsilon)\log(1-\varepsilon)\right), \end{aligned}$$

and

$$\int_{\varepsilon}^{1-\varepsilon}\left(\int_{\varepsilon}^{t_1}\left(\frac{t_{12}t_{23}}{t_1(1-t_2)}\right)dt_2\right)dt_1 = \int_{\varepsilon}^{1-\varepsilon}\left(t_{12}t_{23}\frac{\log(1-t_1)-\log(1-\varepsilon)}{t_1}\right)dt_1$$
$$= t_{12}t_{23}(\mathrm{Li}_2(\varepsilon)-\mathrm{Li}_2(1-\varepsilon)\log(1-\varepsilon)(\log(1-\varepsilon)-\log(\varepsilon))).$$

One can show that

$$\int_{\varepsilon}^{1-\varepsilon}\left(\int_{\varepsilon}^{t_1}\left(\frac{t_{23}t_{12}}{t_2(1-t_1)}\right)dt_2\right)dt_1 = t_{23}t_{12}(\mathrm{Li}_2(1-\varepsilon)-\mathrm{Li}_2(\varepsilon)-\log(\varepsilon)(\log(\varepsilon)-\log(1-\varepsilon)))$$

and

$$\int_{\varepsilon}^{1-\varepsilon}\left(\int_{\varepsilon}^{t_1}\left(\frac{t_{23}^2}{(1-t_2)(1-t_1)}\right)dt_2\right)dt_1 = t_{23}^2\left(\frac{\log(\varepsilon)^2}{2}+\frac{\log(1-\varepsilon)^2}{2}-\log(\varepsilon)\log(1-\varepsilon)\right).$$

Using the Taylor expansions

$$\varepsilon^{-t_{23}} = 1-t_{23}\log(\varepsilon)+t_{23}^2\frac{\log(\varepsilon)^2}{2}+\cdots,$$

and

$$\varepsilon^{t^{12}} = 1+t_{12}\log(\varepsilon)+t_{12}^2\frac{\log(\varepsilon)^2}{2}+\cdots,$$

and noticing that $\mathrm{Li}_2(0)=0$ and $\mathrm{Li}_2(1)=\zeta(2)$, we can simplify:

$$\begin{aligned}
\Phi_{\mathrm{KZ}}(t_{12},t_{23}) &= \lim_{\varepsilon\to0} \quad \varepsilon^{-t_{23}}\mathcal{P}\exp\left(\int_{\varepsilon}^{1-\varepsilon}\left(\frac{t_{12}}{z}+\frac{t_{23}}{1-z}\right)dz\right)t^{t_{12}}\\
&= \lim_{\varepsilon\to0} \quad 1-t_{23}\log(\varepsilon)-t_{12}\log(\varepsilon)+t_{23}\log(\varepsilon)+t_{12}\log(\varepsilon)+t_{23}^2\frac{\log(\varepsilon)^2}{2}+t_{12}^2\frac{\log(\varepsilon)^2}{2}\\
&\qquad +[t_{12},t_{23}](\mathrm{Li}_2(\varepsilon)-\mathrm{Li}_2(1-\varepsilon))-t_{23}t_{12}\log(\varepsilon)^2+t_{23}^2\frac{\log(\varepsilon)^2}{2}+t_{12}^2\frac{\log(\varepsilon)^2}{2}\\
&\qquad --t_{23}\log(t)^2(t_{23}-t_{12})+\log(t)^2(t_{23}-t_{12})t_{12}-t_{23}t_{12}\log(\varepsilon)^2+\cdots\\
&= \lim_{\varepsilon\to0} \quad (1+[t_{12},t_{23}](\mathrm{Li}_2(\varepsilon)-\mathrm{Li}_2(1-\varepsilon))+\cdots)\\
&= 1- \quad \zeta(2)[t_{12},t_{23}]+\cdots
\end{aligned}$$

In conclusion, $\Phi_{\mathrm{KZ}}(t_{12},t_{23})$ is a generating series of all multizeta values. As a corollary, we obtain new relations between the different multizeta values coming from the pentagon and hewagons relations of the associators:

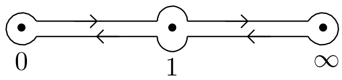
Corollary 2.2.45. *Multizeta values satisfy the Drinfeld associator relations.*

Not only that, but thanks to the geometric definition of the associator KZ, we can find old relations that go back to Euler's works, as illustrated by the following theorem shown by Pierre Deligne:

Theorem 2.2.46 (Deligne, section 18 of [27]). *The relation*

$$\zeta(2n) = (-1)^{n-1}\frac{\mathcal{B}2n}{2\times(2n)!}(2\pi)2n$$

comes from the relations of antisymmetry and pentagon of the contractible path in $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ *given by*



2.2.10 Application II : Formality of the pure braid group

Definition 2.2.47. *Let G be a finitely generated group. It is called formal if there is a Lie algebra isomorphism* $\mathrm{Lie}(\hat{G}(\mathbf{k})) \longrightarrow \hat{\mathrm{gr}}\,\mathrm{Lie}(\hat{G}(\mathbf{k}))$*, whose associated graded morphism is the identity.*

One can then retrieve from the flatness of the universal KZ connection such an isomorphism for $G = \mathrm{PB}_n$. Namely, the monodromy representation morphism

$$\rho_{\mathrm{KZ}} : \mathrm{PB}_n \longrightarrow \exp(\hat{\mathfrak{t}}_n)$$

factors through the $\mathbb{C}$-prounipotent completion $\widehat{\mathrm{PB}}_n(\mathbb{C})$ of PB_n and one can show the following

Proposition 2.2.48. *The map*

$$\tilde{\rho} : \widehat{\mathrm{PB}}_n(\mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}_n)$$

is an isomorphism of $\mathbb{C}$-prounipotent groups.

Remark 2.2.49. *Returning to the consideration of the holonomy Lie algebra and the de Rham fundamental group of* $\mathrm{Conf}(\mathbb{C}, n)$*, this result establishes an isomorphism*

$$\begin{tikzcd}
\pi_1^{\mathrm{Top}}(\mathrm{Conf}(\mathbb{C}, n)) \arrow[d] \arrow[dr] & \\
\pi_1^{\mathrm{B}}(\mathrm{Conf}(\mathbb{C}, n)) \arrow[r, "\simeq"] & \pi_1^{\mathrm{dR}}(\mathrm{Conf}(\mathbb{C}, n))
\end{tikzcd}$$

where $\pi_1^{\mathrm{B}}(\mathrm{Conf}(\mathbb{C}, n))$ *is the Betti fundamental group of* $\mathrm{Conf}(\mathbb{C}, n)$*, which identifies to the $\mathbb{C}$-prounipotent completion of the topological fundamental group* $\pi_1^{\mathrm{Top}}(\mathrm{Conf}(\mathbb{C}, n))$*. This provides an inverse morphism to the map* $\pi_1^{\mathrm{dR}}(\mathrm{Conf}(\mathbb{C}, n)) \longrightarrow \pi_1^{\mathrm{B}}(\mathrm{Conf}(\mathbb{C}, n))$ *given by the R-H correspondence.*

This conceptual interpretation of the formality of PB_n will be translated to the cyclotomic (easily) and genus 1 (with a lot more of work) cases.

2.3 The cyclotomic KZ associator

2.3.1 The universal cyclotomic KZ connection

Let $\Gamma = \mathbb{Z}/N\mathbb{Z}$ and let $\mathfrak{t}_n^{\Gamma}(\mathbf{k})$ be the Lie $\mathbf{k}$-algebra with generators t_{0i}, $(1 \leq i \leq n)$, and t_{ij}^{α}, $(1 \leq i \neq j \leq n,\ \alpha \in \mathbb{Z}/N\mathbb{Z})$, and relations:

(NS) $t_{ij}^{\alpha} = t_{ji}^{-\alpha}$,

(NL) $[t_{0i}, t_{jk}^{\alpha}] = 0$ and $[t_{ij}^{\alpha}, t_{kl}^{\beta}] = 0$,

(N4T) $[t_{ij}^{\alpha}, t_{ik}^{\alpha+\beta} + t_{jk}^{\beta}] = 0$,

(NT1) $[t_{0i}, t_{0j} + \sum_{\alpha\in\Gamma} t_{ij}^{\alpha}] = 0$,

(NT2) $[t_{0i} + t_{0j} + \sum_{\beta\in\Gamma} t_{ij}^{\beta}, t_{ij}^{\alpha}] = 0$,

where $1 \leq i, j, k, l \leq n$ are pairwise distinct and $\alpha, \beta \in \Gamma$. We will call it the $\mathbf{k}$-Lie algebra of infinitesimal cyclotomic braids.

The universal cyclotomic KZ connection on the trivial $\exp(\hat{\mathfrak{t}}_{n,N}(\mathbb{C}))$-bundle over

$$\mathrm{Conf}(\mathbb{C}^{\times}, n, \Gamma) := (\mathbb{C}^{\times})^n - \{\mathbf{z} = (z_1, \ldots, z_n) | z_i^N = z_j^N \text{ for some } i \neq j\}$$

is defined by the differential 1-form

$$\omega_{n,N}^{\mathrm{KZ}} := \sum_{i=1}^{n} \left(\frac{t_{0i}}{z_i} + \sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}, 1\leq i\neq j\leq n} \frac{t_{ij}^{\alpha}}{z_i - \zeta^{\alpha} z_j} \right) \mathrm{d}\, z_i, \tag{2.13}$$

where ζ is a primitive Nth root of unity. It is a fact that this connection is flat.

2.3.2 Reminders on partial prounipotent completions

Let us recall the Enriquez' notion of partial prounipotent completion that we will use later in Chapter 7.

Let $\varphi : G \longrightarrow H$ be a surjective group morphism such that $G_G := \operatorname{Ker} \varphi$ is finitely generated.

Definition 2.3.1. *There is a non-connected pro-algebraic group $G(\varphi, \mathbf{k})$, fitting in an exact sequence $1 \longrightarrow G_0(\mathbf{k}) \longrightarrow G(\varphi, \mathbf{k}) \longrightarrow H \longrightarrow 1$, and a group morphism $G \longrightarrow G(\varphi, \mathbf{k})$, such that the diagram*

$$\begin{tikzcd}
1 \arrow[r] & G_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_0(\mathbf{k}) \arrow[r] & G(\varphi, \mathbf{k}) \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

commutes. The group $G(\varphi, \mathbf{k})$ is called relative $\mathbf{k}$-prounipotent completion of G with respect to φ.

We direct the reader to the article [33] for more details on this definition as well as for the following one.

Definition 2.3.2. *We say that the group morphism $\varphi : G \longrightarrow H$ is formal if there exists a group isomorphism $G(\mathbf{k}, \varphi) \simeq \exp(\operatorname{gr}\operatorname{Lie} G_0(\mathbf{k})) \rtimes H$, restricting to a formality isomorphism for G_0, and such that the diagram*

$$\begin{tikzcd}
1 \arrow[r] & G_0(\mathbf{k}) \arrow[r] \arrow[d] & G(\mathbf{k},\varphi) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \exp(\hat{\operatorname{gr}}\operatorname{Lie} G_0(\mathbf{k})) \arrow[r] & \exp(\hat{\operatorname{gr}}\operatorname{Lie} G_0(\mathbf{k})) \rtimes H \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

commutes.

Example 2.3.3.

- *The morphism $B_n \longrightarrow \mathfrak{S}_n$ is formal, where B_n is the fundamental group of $\operatorname{Conf}(\mathbb{C}, n)/\mathfrak{S}_n$. It is interesting to say that this result follows from [74] when $\mathbf{k} = \mathbb{C}$, and from [31] for $\mathbf{k} = \mathbb{Q}$.*
- *Denote*
  - $G_0 = \mathrm{PB}_n^{\Gamma} := \pi_1(\operatorname{Conf}(\mathbb{C}^{\times}, n, \Gamma))$,
  - $G = B_n^1 = \pi_1(\operatorname{Conf}(\mathbb{C}^{\times}, n)/\mathfrak{S}_n)$ *and*
  - $\varphi_{n,N} : B_n^1 \longrightarrow \Gamma^n \rtimes \mathfrak{S}_n$.

  One can show that the monodromy of the universal cyclotomic KZ connection gives us vertical isomorphisms

$$\begin{tikzcd}
1 \arrow[r] & \hat{\mathrm{PB}}_n^{\Gamma}(\mathbb{C}) \arrow[r] \arrow[d] & \hat{\mathrm{B}}_n(\varphi_n, \mathbb{C}) \arrow[r] \arrow[d] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] \arrow[d] & 1\ . \\
1 \arrow[r] & \exp(\hat{\mathfrak{t}}_n^{\Gamma}(\mathbb{C})) \arrow[r] & \exp(\hat{\mathfrak{t}}_n^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n) \arrow[r] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] & 1
\end{tikzcd}$$

2.3.3 Realisations

Let $\mathfrak{g}$ be a Lie $\mathbf{k}$-algebra and let $t_{\mathfrak{g}} = \Sigma_u e_u \otimes f_u \in S^2(\mathfrak{g})^{\mathfrak{g}}$. Suppose that we have a morphism

$$\Gamma \longrightarrow \operatorname{Aut}(\mathfrak{g}, t_{\mathfrak{g}}); \alpha \mapsto \alpha$$

i.e. $\alpha^N = \mathrm{id}$. Then we have a decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l} = \mathfrak{g}^{\Gamma}$ and $\mathfrak{u} = \bigoplus_{\chi \in \hat{\Gamma} - \{0\}} \mathfrak{g}_{\chi}$. Take a decomposition $t_{\mathfrak{g}} = t_{\mathfrak{l}} + t_{\mathfrak{u}}$ where $t_{\mathfrak{l}} \in S^2(\mathfrak{l})^{\mathfrak{l}}$ and $t_{\mathfrak{u}} \in S^2(\mathfrak{u})^{\mathfrak{l}}$. Let $\bar{\sigma}$ be a generator of $\Gamma \subset \mathcal{U}(\mathfrak{g}) \rtimes \Gamma$.

Theorem 2.3.4. *There is a unique Lie algebra morphism*

$$\begin{array}{rcl}
\mathcal{U}(\hat{\mathfrak{t}}_{n,N}) \rtimes \Gamma^n & \longrightarrow & \mathcal{U}(\mathfrak{l}) \otimes (\mathcal{U}(\mathfrak{g}) \rtimes \Gamma)^{\otimes n} \\
t_{0i} & \longmapsto & N\left(t_{\mathfrak{l}}^{(0i)} + \dfrac{1}{2} t_{\mathfrak{l}}^{(ii)}\right) \otimes 1 \\
t_{ij}^{\alpha} & \longmapsto & 1 \otimes (\sigma^{\alpha} \otimes \mathrm{id})(t_{\mathfrak{g}}^{(ij)}) \\
s_i & \longmapsto & \bar{\sigma}^{(i)}.
\end{array}$$

2.4 The elliptic KZB associator

In this section we introduce the basic tools that were used on constructing the universal elliptic KZB connection and which will be used in the second part of this thesis. We will profit this occasion to rely all conventions for theta functions that different authors (at our knowledge) that work on the KZB connection use at present.

2.4.1 Quick reminder on Eisenstein series and theta functions

In what follows $G_k(\tau)$ are the Eisenstein series defined for all $k \geq 2$, by

$$G_k(\tau) := \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{m=-\infty \\ m\neq 0 \text{ if } n=0}}^{\infty} \frac{1}{(m+n\tau)^k} \right) = 2\zeta(k) + \frac{2\cdot(2\pi \mathrm{i})^k}{(k-1)!} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m,$$

where $\sigma_\alpha(k) = \sum_{d|k} d^\alpha$.

Enriquez Approach: Let $\mathfrak{H} := \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$ be the Poincaré half-plane. The theta function we will use is denoted $(z,\tau) \mapsto \theta_\tau(z)$, for $(z,\tau) \in \mathbb{C} \times \mathfrak{H}$, where

$$\theta(z,\tau) := \theta_\tau(z) := \frac{e^{\pi i z} - e^{-\pi i z}}{2i\pi} \prod_{n\geqslant 1} \frac{(1-e^{2\pi i(z+n\tau)})(1-e^{2\pi i(-z+n\tau)})}{(1-e^{2\pi i n\tau})^2}.$$

and it is the unique holomorphic function $\mathbb{C}\times\mathfrak{H} \longrightarrow \mathbb{C}$ such that $\theta_\tau(z+1) = -\theta_\tau(z) = \theta_\tau(-z)$, $\theta_\tau(z+\tau) = -e^{-i\pi\tau} e^{-2\pi \mathrm{i}z}\theta_\tau(z)$, $\frac{\partial}{\partial z}\theta_\tau(z)_{|z=0} = 1$, and $(\theta_\tau(-))^{-1}(0) = \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. Furthermore, we have $\theta(z|\tau+1) = \theta(z|\tau)$ and $\theta(z/\tau|1/\tau) = (1/\tau)e^{(\pi i/\tau)z^2}\theta(z|\tau)$. Recall that the Dedekind η-function is given by $\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n)$ where $q = e^{2\pi \mathrm{i}\tau}$.

The classical odd Jacobi theta function is, for $q = e^{2i\pi\tau}$,

$$\begin{aligned} \vartheta_1(z,\tau) &:= -\sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi\tau n^2 + 2i\pi n(z+\frac{1}{2})} \\ &= -\sum_{n\in\mathbb{Z}} e^{i\pi\tau\left(n+\frac{1}{2}\right)^2 + 2i\pi\left(n+\frac{1}{2}\right)(z+\frac{1}{2})} \end{aligned}$$

and we have $\vartheta_1(z,\tau) = 2\pi\eta^3(\tau)\theta_\tau(z)$. Set $\hat{\vartheta}(z,\tau) = \frac{\vartheta_1(z,\tau)}{2\pi}$. This also gives a heat equation for ϑ:

$$\partial_\tau \hat{\vartheta} = (1/4\pi i)\partial_z^2 \hat{\vartheta}$$

Brown-Levin-Racinet-Zagier approach: The standard odd elliptic theta functions are

$$\vartheta_1^{\mathrm{Std}}(u,\tau) := \sum_{n\in\mathbb{Z}} (-1)^{n-\frac{1}{2}} e^{2i\pi u\left(n+\frac{1}{2}\right) + i\pi\tau\left(n+\frac{1}{2}\right)^2}$$

$$\vartheta_{11}^{\mathrm{Std}}(u,\tau) := i\sum_{n\in\mathbb{Z}} (-1)^n e^{2i\pi u\left(n+\frac{1}{2}\right) + i\pi\tau\left(n+\frac{1}{2}\right)^2}$$

$$\begin{aligned}
\vartheta^{\mathrm{Zag}}(u,\tau) &:= \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{(n+\frac{1}{2})u} \\
&= \sum_{n\in\mathbb{Z}}(-1)^n e^{u(n+\frac{1}{2})+i\pi\tau(n+\frac{1}{2})^2} \\
&= \frac{1}{i}\vartheta_{11}(\frac{u}{2i\pi},\tau),
\end{aligned}$$

and we can express $\vartheta^{\mathrm{Zag}}(u,\tau)$ as a product via the Jacobi triple product formula (in Zagier's paper):

$$\vartheta^{\mathrm{Zag}}(u,\tau) = q^{\frac{1}{8}}\left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^u)(1-q^n e^{-u}).$$

Finally, the theta function used by Brown-Levin is

$$\theta^{\mathrm{BL}}(\xi,\tau) = \frac{\vartheta_{11}(\xi,\tau)}{\eta(\tau)} = q^{1/12}(z^{1/2} - z^{1/2})\prod_{j\geqslant 1}(1zq^j)(1z^1q^j),$$

and the one used by Levin-Racinet is

$$\theta^{\mathrm{LR}}(\xi,\tau) = iq^{1/8}(z^{1/2}z^{1/2})\prod_{j\geqslant 0}(1zq^j)(1z^1q^j)(1q^j).$$

We have

$$\begin{aligned}
\vartheta_1(z,\tau) &= -\sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi\tau n^2+2i\pi n(z+\frac{1}{2})} \\
&= -\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(u+\frac{1}{2}\right)^2} e^{z(n+\frac{1}{2})} e^{\pi \mathrm{i}n+\frac{\pi\mathrm{i}}{2}} \\
&= -\mathrm{i}\vartheta^{\mathrm{Zag}}(z,\tau).
\end{aligned}$$

Kronecker series. The Kronecker series used by Zagier is the meromorphic function $\mathbb{C}\times\mathbb{C}\times\mathfrak{H}\longrightarrow\mathbb{C}$ defined by

$$F^{\mathrm{Zag}}(u,v,\tau) := \frac{\vartheta^{\mathrm{Zag}\,\prime}(0,\tau)\vartheta^{\mathrm{Zag}}(u+v,\tau)}{\vartheta^{\mathrm{Zag}}(u,\tau)\vartheta^{\mathrm{Zag}}(v,\tau)},$$

and the Kronecker series used by Enriquez is

$$F^{\mathrm{En}}(x,z,\tau) = \frac{\theta'(0,\tau)\theta(z+x,\tau)}{\theta(z,\tau)\theta(x,\tau)} = \frac{\theta(z+x,\tau)}{\theta(z,\tau)\theta(x,\tau)}.$$

Thus, as $\vartheta^{\mathrm{Zag}}(z) = 2\pi\mathrm{i}\eta(\tau)^3\theta(z,\tau)$, we get

$$F^{\mathrm{Zag}}(z,x,\tau) = F^{\mathrm{En}}(z,x,\tau).$$

Next, the one used by Brown-Levin is

$$F^{\mathrm{BL}}(u,v,\tau) := \frac{\theta^{\mathrm{BL}\,\prime}(0,\tau)\theta^{\mathrm{BL}}(u+v,\tau)}{\theta^{\mathrm{BL}}(u,\tau)\theta^{\mathrm{BL}}(v,\tau)},$$

and is related to the one used by Levin-Racinet, denoted

$$F^{\mathrm{LR}}(\xi,\eta,\tau) := \frac{\theta^{\mathrm{LR}\prime}(0,\tau)\theta^{\mathrm{LR}}(\xi+\eta,\tau)}{\theta^{\mathrm{LR}}(\xi,\tau)\theta^{\mathrm{LR}}(\eta,\tau)},$$

by the formula

$$F^{\mathrm{LR}}(\xi,\eta,\tau) = 2i\pi F^{\mathrm{Zag}}(2i\pi\xi, 2i\pi\eta,\tau).$$

Finally, we have $\vartheta_1(z,\tau) = 2\pi\eta^3(\tau)\theta(z,\tau)$ and $\vartheta_{11}(z,\tau) = \eta(\tau)\theta^{\mathrm{BL}}(z,\tau)$ and $\vartheta_{11}(z,\tau) = i\vartheta^{\mathrm{Zag}}(2i\pi z,\tau)$. We have

$$\eta(\tau)\theta^{\mathrm{BL}}(z,\tau) \quad = \quad i\vartheta^{\mathrm{Zag}}(2i\pi z,\tau)$$

In conclusion we get

- $F^{\mathrm{BL}}(\xi,\eta,\tau) = F^{\mathrm{LR}}(\xi,\eta,\tau)$,
- $F^{\mathrm{Zag}}(\xi,\eta,\tau) = F^{\mathrm{En}}(\xi,\eta,\tau)$, and
- $F^{\mathrm{LR}}(\xi,\eta,\tau) = 2i\pi F^{\mathrm{Zag}}(2i\pi\xi, 2i\pi\eta,\tau)$.

In what follows we take Enriquez' convention for the theta function.

2.4.2 The universal elliptic KZB connection

For $\tau \in \mathfrak{h}$, denote $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$ and denote, for $n \geq 1$,

$$\mathrm{Diag}_{1,n} := \{(\mathbf{z},\tau) \in \mathbb{C}^n \times \mathfrak{H} | z_{ij} \in \Lambda_\tau, \text{ for some } i \neq j\}.$$

The semidirect product $((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \mathrm{SL}_2(\mathbb{Z})$ acts on $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{1,n}$ by

- $(\mathbf{n},\mathbf{m},u) * (\mathbf{z},\tau) := (\mathbf{z} + \mathbf{n} + \tau\mathbf{m} + u(\sum_i \delta_i), \tau)$ for $(\mathbf{n},\mathbf{m},u) \in (\mathbb{Z}^n)^2 \times \mathbb{C}$,
- $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) * (\mathbf{z},\tau) := (\frac{\mathbf{z}}{\gamma\tau+\delta}, \frac{\alpha\tau+\beta}{\gamma\tau+\delta})$ for $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

The moduli space $\mathcal{M}_{1,n}$ of elliptic curves with n marked points is defined as the quotient

$$\mathcal{M}_{1,n} := (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{1,n} / ((\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})),$$

and its reduced version is

$$\bar{\mathcal{M}}_{1,n} := (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{1,n} / (((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \mathrm{SL}_2(\mathbb{Z})).$$

Remark 2.4.1.
- *In [24], $\mathcal{M}_{1,n}$ is denoted $\tilde{\mathcal{M}}_{1,n}$ and $\bar{\mathcal{M}}_{1,n}$ is denoted $\mathcal{M}_{1,n}$. We shifted the notations of [24] for compatibility with our conventions for Chapter 6.*
- *The space $\mathcal{M}_{1,1}$ is the universal curve over $\bar{\mathcal{M}}_{1,1} = \mathfrak{h}/\mathrm{SL}_2(\mathbb{Z})$ and for $n = 2$ the moduli space $\bar{\mathcal{M}}_{1,2}$ is the punctured universal elliptic curve over $\bar{\mathcal{M}}_{1,1}$. This is a fibration with, as fibers at (equivalence classes of) τ, (equivalence classes of) the punctured elliptic curves $E_\tau^\times := E_\tau - \{0\}$.*
- *Remark that if*

$$\mathrm{C}(E_\tau, n) := \mathrm{Conf}(E_\tau, n)/E_\tau$$

are the reduced configuration spaces of E_τ, then $\mathrm{C}(E_\tau, 2) = E_\tau^\times$.

- *More generally, the fibers of the fibration* $\bar{\mathcal{M}}_{1,n+1} \longrightarrow \bar{\mathcal{M}}_{1,1}$ *are (the equivalence classes of) the spaces* $\mathrm{Conf}(E_\tau^\times, n)$.

For any $n \geq 0$, recall that $\mathfrak{t}_{1,n}(\mathbf{k})$ is defined as the bigraded Lie $\mathbf{k}$-algebra freely generated by $x_1, \dots, x_n$ in degree $(1,0)$, $y_1, \dots, y_n$ in degree $(0,1)$ (for $i = 1, ..., n$), and t_{ij} in degree $(1,1)$ (for $1 \leq i \neq j \leq n$), together with the relations (S), (L), (4T), and the following additional elliptic relations as well:

($\mathrm{S}_{e\ell\ell}$) $[x_i, y_j] = t_{ij}$ for $i \neq j$,

($\mathrm{N}_{e\ell\ell}$) $[x_i, x_j] = [y_i, y_j] = 0$ for $i \neq j$,

($\mathrm{T}_{e\ell\ell}$) $[x_i, y_i] = -\sum_{j|j\neq i} t_{ij}$,

($\mathrm{L}_{e\ell\ell}$) $[x_i, t_{jk}] = [y_i, t_{jk}] = 0$ if $\#\{i,j,k\} = 3$,

($4\mathrm{T}_{e\ell\ell}$) $[x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0$ for $i \neq j$.

The $\sum_i x_i$ and $\sum_i y_i$ are central in $\mathfrak{t}_{1,n}(\mathbf{k})$, and we also consider the quotient

$$\bar{\mathfrak{t}}_{1,n}(\mathbf{k}) := \mathfrak{t}_{1,n}(\mathbf{k})/(\sum_i x_i, \sum_i y_i)\,.$$

Example 2.4.2. $\bar{\mathfrak{t}}_{1,2}(\mathbf{k})$ *is equal to the free Lie* $\mathbf{k}$*-algebra* $\mathfrak{f}_2(\mathbf{k})$ *on two generators* $x = x_1$ *and* $y = y_2$.

Let $\mathfrak{d}_+$ be the free Lie algebra with generators δ_{2m} $(m \geq 1)$. Denote the standard generatons e, f, h of $\mathfrak{sl}_2$ by $d := h$, $X := e$ and $\Delta_0 := f$. Denote $\mathfrak{d} := d_+ \rtimes \mathfrak{sl}_2$ their semi-direct product, the δ_{2m} acting as highest weight elements (see [24] for details).

Proposition 2.4.3 ([24])**.** *There is a Lie algebra morphism* $\mathfrak{d} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n})$ *inducing a Lie algebra morphism* $\mathfrak{d} \longrightarrow \mathrm{Der}(\bar{\mathfrak{t}}_{1,n})$.

An easy consequence is that we can then form the semi-direct products

$$\mathbf{G}_n := \exp((\mathfrak{t}_{1,n} \rtimes \mathfrak{d}_+)^\wedge) \rtimes \mathrm{SL}_2(\mathbb{C}) \qquad \bar{\mathbf{G}}_n := \exp((\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+)^\wedge) \rtimes \mathrm{SL}_2(\mathbb{C})$$

Theorem 2.4.4 ([24])**.** *There is a unique* $\mathbf{G}_n$*-bundle* $\mathcal{P}_n$ *over* $\mathcal{M}_{1,n}$ *with a flat universal KZB connection, locally defined by*

$$\nabla^{\mathrm{KZB}}_{1,n} := d - \Delta(\mathbf{z}|\tau)d\tau - \sum_{i=1}^n K_i(\mathbf{z}|\tau)dz_i,$$

where $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$*, for* $1 \leq i \leq n$*, we have*

$$K_i(\mathbf{z}|\tau) := -y_i + \sum_{j:j\neq i} k(\mathrm{ad}x_i, z_i - z_j|\tau)(t_{ij}),$$

with $k(x, z|\tau) := \frac{\theta(z+x|\tau)}{\theta(z|\tau)\theta(x|\tau)} - \frac{1}{x}$*, and*

$$\Delta(\mathbf{z}|\tau) := -\frac{1}{2\pi \mathrm{i}}\Big(\Delta_0 + \sum_{n\geq 1}(2n+1)G_{2n+2}(\tau)\delta_{2n} - \sum_{i<j}\partial_x k(\mathrm{ad}x_i, z_i - z_j|\tau)(t_{ij})\Big)\,.$$

This induces a unique $\bar{\mathbf{G}}_n$*-bundle* $\bar{\mathcal{P}}_n$ *over* $\bar{\mathcal{M}}_{1,n}$ *with a flat connection denoted* $\bar{\nabla}^{\mathrm{KZB}}_{1,n}$.

Remark 2.4.5. • *When we say the connection is locally defined as so, we mean that there is a unique such connection such that the pull-back to* $X := (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{1,n}$ *is the connection* $\nabla^{\mathrm{KZB}}_{1,n}$ *on the trivial* $\mathbf{G}_n$*-bundle over* X*.*

- *There is also an unordered marked points version of this connection that will not be recalled in here.*
- *By fixing* τ *and choosing a section* $\mathrm{Conf}(E_\tau, n)$ *of a representative in the equivalence class* $[(E_\tau, z_1, \ldots, z_n)] \in \mathcal{M}_{1,n}$*, this connection restricts to a flat connection*

$$\nabla^{\mathrm{KZB}}_{1,n,\tau} := d - \sum_{i=1}^{n} K_i(\mathbf{z}|\tau) dz_i,$$

on the (unique) principal $\exp(\hat{\mathfrak{t}}_{1,n})$*-bundle over* $\mathrm{Conf}(E_\tau, n)$*.*

Let us fix $\tau \in \mathfrak{h}$. Recall that the Lie algebra $\bar{\mathfrak{t}}_{1,2}(\mathbb{C})$ is isomorphic to the free Lie algebra $\mathfrak{f}_2(\mathbb{C})$ generated by two elements $x := x_1$ and $y := y_1$. We define the elliptic KZB associators $A(\tau), B(\tau)$ as the regularized holonomies from 0 to 1 and 0 to τ of the differential equation

$$G'(z) = -\frac{\theta_\tau(z + \operatorname{ad} x) \operatorname{ad} x}{\theta_\tau(z)\theta_\tau(\operatorname{ad} x)}(y) \cdot G(z), \tag{2.14}$$

with values in the group $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}(\mathbb{C}))$ More precisely, this equation has a unique solution $G(z)$ defined over $\{a + b\tau,\ \text{for } a, b \in]0, 1[\}$ such that $G(z) \simeq (-2\pi \mathrm{i}\, z)^{-[x,y]}$ at $z \longrightarrow 0$. In this case,

$$A(\tau) := G(z)^{-1}G(z + 1), \quad B(\tau) := G(z)^{-1}e^{2\pi \mathrm{i}\, x}G(z + \tau).$$

These are elements of the group $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}(\mathbb{C}))$. A recollection of the main features of elliptic associators is done in the first part of [35] and will not be reproduced here.

2.4.3 Universality

As in the genus 0 case, one can ask in what manner this connection is universal and now it will be of great importance to distinguish the case where the connection is defined over the moduli space to the one that is defined only in the configuration space. Indeed, in the genus 0 case, the moduli space $\mathcal{M}_{0,n+1}$ of rational curves with $n + 1$ marked points is isomorphic to the quotient of the configuration space of n points in the plane modulo the action of $Aut(\mathbb{C})$ by homographies:

$$\mathcal{M}_{0,n+1} \simeq \mathrm{Conf}(\mathbb{C}, n)/(\mathbb{C}^* \rtimes \mathbb{C}).$$

In the genus 1 case, however, such a relation is not true. Another issue here is that the de Rham complex in this setup (either for the configuration space setting or the moduli space setting) is not generated by the first cohomology group so we will not be able to apply Proposition 2.2.28.

Towards the Gauss-Manin connection $\nabla^{\mathrm{KZB}}_{1,n}$ over $\mathcal{M}_{1,n}$

In this section we give an insight on the fact that the connection $\nabla^{\mathrm{KZB}}_{1,n}$ is the universal by explaining that it is (conjecturally) the Gauss-Manin connection over $\mathcal{M}_{1,n}$. Let us start with the restriction of this connection to that over $\mathrm{Conf}(E_\tau, n)$, following [37].

Set $X := \mathrm{Conf}(E_\tau, n)$ and $x \in \mathrm{Conf}(E_\tau, n)$. Then (ommiting to explicit the basis point for simplicity as X is arc-wise connected) we have $\pi_1^{Top}(X) = \mathrm{PB}_{1,n}$ and $\pi_1^B(X) = \widehat{\mathrm{PB}}_{1,n}(\mathbb{C})$.

Theorem 2.4.6 ([37]). • *There is*

- *an explicit tensor functor*

$$F : \mathrm{VBFC}(\mathrm{Conf}(E_\tau, n))^{uni} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

- *a natural isomorphism*

$$\begin{array}{rcl} \mathrm{VBFC}(\mathrm{Conf}(E_\tau, n))^{uni} & \longrightarrow & \mathrm{Iso}_{\mathrm{Vec}_{\mathbb{C}}}(F(\mathcal{E}, \nabla), F_x^{vb}(\mathcal{E}, \nabla)) \\ (\mathcal{E}, \nabla) & \longmapsto & i_{(\mathcal{E}, \nabla)} \end{array}$$

between the functors F and F_x^{vb},

- *a canonical isomorphism $\mathrm{Aut}^{\otimes}(F) \simeq \exp(\hat{\mathfrak{t}}^{\mathbb{C}}_{1,n})$.*

• *The composed isomorphim*

$$\exp(\hat{\mathfrak{t}}_{1,n}) \xrightarrow{\sim} \mathrm{Aut}^{\otimes}(F) \xrightarrow{\sim} \mathrm{Aut}^{\otimes}(F_x^{vb}) \xrightarrow[\mathrm{RH}]{\sim} \mathrm{Aut}^{\otimes}(F_x^{ls}) \xrightarrow{\sim} \widehat{\mathrm{PB}}_{1,n}(\mathbb{C})$$

coincides with the inverse of the completed monodromy representation map

$$\widehat{\mathrm{PB}}_{1,n}(\mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n})$$

induced by the universal KZB connection $\nabla^{\mathrm{KZB}}_{1,n,\tau}$ over $\mathrm{Conf}(E_\tau, n)$.

Now, following [63], let us show that the bundle $\mathcal{P}_n$ with the KZB connection is the de Rham realization of a topological local system $\mathcal{P}_n^{\mathrm{Top}}$.

Denote by Y the universal covering space of $\bar{\mathcal{M}}_{1,n+1}$. This is also the universal covering space of $\bar{\mathcal{M}}^{\mathfrak{h}}_{1,n+1} = \big(\mathbb{C}^{n+1} \times \mathfrak{h}\big) - \Delta_{n+1}$. Choose a base point $[E_\tau, 0, \mathbf{z}]$ of $\bar{\mathcal{M}}_{1,n+1}$, where $\mathbf{z} = (z_1, \dots, z_n)$, and $z_i \neq 0$ for all $1 \leq i \leq n$. Choose a lift y of it to Y. This determines an isomorphism of $\mathrm{Aut}(Y/\bar{\mathcal{M}}_{1,n+1})$ with $\pi_1(\bar{\mathcal{M}}_{1,n+1}, [E_\tau, 0, \mathbf{z}])$.

Denote the unipotent completion of $\pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z})$ over $\mathbb{C}$ by $\mathcal{P}_o$. The natural action

$$\begin{array}{rcl} \pi_1(\bar{\mathcal{M}}_{1,n+1}, [E_\tau, 0, \mathbf{z}]) \times \pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z}) & \longrightarrow & \pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z}), \\ (g, \gamma) & \longmapsto & g\gamma g^{-1} \end{array}$$

determines a left action of $\pi_1(\bar{\mathcal{M}}_{1,n+1}, [E_\tau, 0, \mathbf{z}])$ on $\mathcal{P}_o$. We can therefore form the quotient

$$\big(\mathcal{P}_o \times Y\big)/\pi_1(\bar{\mathcal{M}}_{1,n+1}, [E_\tau, 0, \mathbf{z}])$$

by the diagonal $\pi_1(\bar{\mathcal{M}}_{1,n+1}, [E_\tau, 0, \mathbf{z}])$-action. This is a flat right principal $\mathcal{P}_o$-bundle which we shall denote by $\mathcal{P}_n^{\mathrm{Top}} \longrightarrow \bar{\mathcal{M}}_{1,n+1}$. Its fiber over $[E_\tau, 0, \mathbf{z}]$ is naturally isomorphic to the unipotent completion of $\pi_1(\mathrm{Conf}(E_\tau^\times, n))$.

Since the Lie algebra $\mathfrak{p}_o$ of $\mathcal{P}_o$ can be viewed as a group with multiplication defined by the Baker-Campbell-Hausdorff formula, we can (and will) view $\mathcal{P}_n^{\mathrm{Top}}$ as a local system of Lie algebras.

Choose a base point $[E_\tau, 0, \mathbf{z}]$ of $\bar{\mathcal{M}}_{1,n+1}$, where $\mathbf{z} = (z_1, \ldots, z_n)$, and $z_i \neq 0$ for all $1 \leq i \leq n$. There is a natural isomorphism

$$\pi_1(\bar{\mathcal{M}}_{1,n+1}, [E_\tau, 0, \mathbf{z}]) \simeq \Gamma_{1,n+1},$$

where $\Gamma_{1,n}$ is the mapping class group of a genus 1 curve with n marked points (see [13]).

The restriction of the universal elliptic KZB connection to $\mathrm{Conf}(E_\tau^\times, n)$ defines a homomorphism $\pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z}) \longrightarrow \mathrm{Aut}(\hat{\bar{\mathfrak{t}}}_{1,n+1})$ whose image lies in the subgroup $\exp(\hat{\bar{\mathfrak{t}}}_{1,n+1})$ which acts on $\hat{\bar{\mathfrak{t}}}_{1,n+1}$ via the adjoint action. From the formality morphism [24, Proposition2.2], we conclude that it induces an isomorphism $\hat{\pi}_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z})(\mathbb{C}) \longrightarrow \exp(\hat{\bar{\mathfrak{t}}}_{1,n+1})$.

Identify $\exp(\hat{\bar{\mathfrak{t}}}_{1,n+1})$ with $\hat{\pi}_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z})(\mathbb{C})$ via this isomorphism. Then one has the monodromy representations

$$\rho^{\mathrm{KZB}} : \Gamma_{1,n+1} \longrightarrow \mathrm{Aut}(\exp(\hat{\bar{\mathfrak{t}}}_{1,n+1})) \text{ and } \rho^{\mathrm{Top}} : \Gamma_{1,n+1} \longrightarrow \mathrm{Aut}(\exp(\hat{\bar{\mathfrak{t}}}_{1,n+1}))$$

of $\mathcal{P}_n$ and $\mathcal{P}_n^{\mathrm{Top}}$. To prove that $\mathcal{P}_n^{\mathrm{Top}}$ and $\mathcal{P}_n$ are isomorphic (seen here as principal bundles), we have to prove that $\rho^{\mathrm{KZB}} = \rho^{\mathrm{Top}}$. Observe that if $\gamma \in \pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z})$, then $\rho^{\mathrm{Top}}(\gamma)$ and $\rho^{\mathrm{KZB}}(\gamma)$ are both conjugation by the image of γ in $\mathcal{P}_n$ as the restriction of $\mathcal{P}_n$ and $\mathcal{P}_n^{\mathrm{Top}}$ to $\mathrm{Conf}(E_\tau^\times, n)$ are isomorphic.

As explained below, rigidity explains that if the restriction to each fiber (that is, to each configuration space) is the correct local system, then it is the correct local system over the whole moduli space $\bar{\mathcal{M}}_{1,n+1}$. More precisely, the marked points version of [63], Theorem 14.2 is then

Theorem 2.4.7. *The exponential map induces an isomorphism of the local system over $\bar{\mathcal{M}}_{1,n+1}$ of flat sections of the universal elliptic KZB connection on $\mathcal{P}$ with the locally constant sheaf $\mathcal{P}_n^{\mathrm{Top}}$ over $\bar{\mathcal{M}}_{1,n+1}$. Equivalently, the diagram*

$$\begin{tikzcd}
\pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z}) \arrow[r, "\rho^{\mathrm{KZB}}"] \arrow[d, equal] & \mathrm{Aut}(\hat{\bar{\mathfrak{t}}}_{1,n+1}) \arrow[d, "\simeq"] \\
\pi_1(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z}) \arrow[r, "\rho^{\mathrm{Top}}"] & \mathrm{Aut}(\exp(\hat{\bar{\mathfrak{t}}}_{1,n+1}))
\end{tikzcd}$$

commutes.

Proof. One can apply [63, Lemma 14.1] to

- $\Gamma = \Gamma_{1,n+1} = \pi_1^{\mathrm{Top}}(\bar{\mathcal{M}}_{1,n+1})$,
- $N = \pi_1^{\mathrm{Top}}(\mathrm{Conf}(E_\tau^\times, n), \mathbf{z}) \simeq \bar{\mathrm{PB}}_{1,n+1}$, which has trivial center,
- $\mathcal{N} = \exp(\hat{\bar{\mathfrak{t}}}_{1,n+1})$,
- $\phi = \rho^{\mathrm{Top}}$

to establish the equality of ρ^{KZB} and ρ^{Top}. □

Remark 2.4.8. *By combining Hain's and Enriquez-Etingof's results one should be able to conclude that the universal elliptic KZB connection $\bar{\nabla}_{1,n+1}^{\mathrm{KZB}}$ is the Gauss-Manin connection on $\bar{\mathcal{M}}_{1,n+1}$.*

2.4.4 Reminders on Hecke algebras

Differential operators. The algebra of differential operators $\mathrm{Diff}(\mathfrak{g})$ on $\mathfrak{g}$ is generated by linear forms over $\mathfrak{g}$ denoted $x^* \in \mathfrak{g}^*$ and differential operators denoted ∂_x, for $x \in \mathfrak{g}$. By choosing a basis we a family $(x_\alpha, \partial_\alpha)$ where $x_\alpha := x_\alpha^*$ is a degree 1 polynomial and ∂_α is the derivative in the direction x_α. These elements have relations

- $[x^*, y^*] = 0$,
- $[\partial_v, \partial_w] = 0$,
- $[\partial_w, v^*] = v^*(w)$.

Remark 2.4.9. $\mathrm{Diff}(\mathfrak{g}^*)$ *is a quantization of* $T^*\mathfrak{g}^* = \mathfrak{g} \times \mathfrak{g}^*$ *and, by identifying* $\mathfrak{g}$ *with its dual, we denote* $x := x^* \in \mathfrak{g}$ *and* $\mathrm{Diff}(\mathfrak{g}) = \mathrm{Diff}(\mathfrak{g}^*)$*.*

In conclusion,

$$\mathrm{Diff}(\mathfrak{g}) = \langle x_a, \partial_a; a \in \mathfrak{g}\rangle / \begin{pmatrix} a \mapsto x_a, a \mapsto \partial_a \text{ are linear} \\ [x_a, x_b] = [\partial_a, \partial_b] = 0 \\ [\partial_a, x_b] = \langle a, b\rangle_{\mathfrak{g}} \end{pmatrix}.$$

Quantum Hamiltonian reduction. Let us briefly recall what Hamiltonian reduction is about. Let X be a symplectic variety and let G be a Lie group acting on X with associated Lie algebra $\mathfrak{g}$. The moment map is a G-equivariant map $\mu : X \longrightarrow \mathfrak{g}^*$ such that $\mu^* : \mathfrak{g} \subset C^\infty(\mathfrak{g}^*) \longrightarrow C^\infty(X)$ satisfies that for all $x \in \mathfrak{g}$, $f \in C^\infty(X)$,

$$\{\mu^* x, f\} = \vec{X}(f) \Longrightarrow \{\mu^* x, \mu^* y\} = \vec{X}(\mu^* y) = \mu^* \vec{X}(y) = \mu^*[X, Y].$$

Then $\mu^{-1}(0)/G$ is Poisson. Thus,

$$C^\infty((\mu^*)^{-1}(0)) = C^\infty(X)/(C^\infty(X)\mu^*(\mathfrak{g}))^{\mathfrak{g}}.$$

In conclusion, let A_0 be a Poisson algebra and $\mu_0^* : \mathfrak{g} \longrightarrow A_0$ be a Lie algebra morphism. If $\mathfrak{g}$ acts on A_0 by means of $\{\mu_0^* X, -\}$, then the Hamiltonian reduction of A_0 is the Poisson algebra

$$A_0^{\mathfrak{g}}/(A_0^{\mathfrak{g}}\mu_0^*(\mathfrak{g}))^{\mathfrak{g}}.$$

Now let A be an associative algebra which is a quantization of A_0, that is, $A \simeq A_0[\![h]\!]$. Let $\mu^* : \mathfrak{g} \longrightarrow A$ be a Lie algebra morphism which is a quantization of μ_0^* i.e. we have

- $\mu^* = \mu_0^* + \circ(h)$,
- $a * b = a.b + h\{a, b\} + \circ(h)$.

Then $\mathfrak{g}$ acts on A by the commutator $[\mu^* X, -]$ and it can be shown that $(A\mu^*\mathfrak{g})^{\mathfrak{g}}$ is a two-sided ideal of $A^{\mathfrak{g}}$ so

$$A^{\mathfrak{g}}/(A^{\mathfrak{g}}\mu^*(\mathfrak{g}))^{\mathfrak{g}}$$

is an associative algebra called quantum Hamiltonian reduction, as it is a quantization of the above Hamiltonian reduction.

Hecke algebras. Let $n \geqslant 1$ be a natural number. As we saw earlier, $\mathrm{Diff}(\mathfrak{g})$ is a quantization of $T^*\mathfrak{g}^*$ and $\mathcal{U}(\mathfrak{g})$ is a quantization of $\mathfrak{g}^*$. Thus, the moment map is just the coadjoint action

$$\mathrm{Diff}(\mathfrak{g}) \longrightarrow \mathfrak{g}^*$$

i.e. induces a Lie algebra morphism

$$\begin{array}{rcl} \mathfrak{g} & \longrightarrow & \mathrm{Diff}(\mathfrak{g}) \\ a & \longmapsto & X_a \end{array}$$

called quantum moment map, or infinitesimal adjoint action. We also have a Lie algebra map $\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})^{\otimes n}$ so we get a map

$$\begin{array}{rcl} \varphi : \mathfrak{g} & \longrightarrow & \mathrm{Diff}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})^{\otimes n} := A_n \\ a & \longmapsto & Y_a := X_a \otimes 1 + 1 \otimes \sum_{i=1}^{n} a^{(i)} \end{array}$$

where $a^{(i)} = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ and $X_a = \sum_{\alpha} x_{[a,e_\alpha]}\partial_\alpha$.

Proposition 2.4.10. *Denote $\mathfrak{g}^{\mathrm{diag}} := \mathrm{im}(\varphi)$. Then the vector subspace $A_n\mathfrak{g}^{\mathrm{diag}}$ is two-sided ideal.*

Proof. If $x, y \in H := \{x \in A_n; \mathfrak{g}^{\mathrm{diag}}x \subset A_n\mathfrak{g}^{\mathrm{diag}}\} \supset A_n\mathfrak{g}^{\mathrm{diag}}$, then

- $\mathfrak{g}^{\mathrm{diag}}(x+y) \subset A_n\mathfrak{g}^{\mathrm{diag}}$;
- $\mathfrak{g}^{\mathrm{diag}}(xy) \subset A_n\mathfrak{g}^{\mathrm{diag}}$;
- it is stable by left and right multiplication ($A_n\mathfrak{g}^{\mathrm{diag}}x \subset A_n\mathfrak{g}^{\mathrm{diag}}$).

□

We conclude that the quotient $H/A_n\mathfrak{g}^{\mathrm{diag}}$ is an associative algebra.

Definition 2.4.11. *The Hecke algebra of $(A_n, \mathfrak{g}^{\mathrm{diag}})$ is (the quantum Hamiltonian reduction):*

$$H_n(\mathfrak{g}) = \{x \in A_n; \forall a \in \mathfrak{g}, Y_a x \in A_n\mathfrak{g}^{\mathrm{diag}}\}/A_n\mathfrak{g}^{\mathrm{diag}}.$$

Remark 2.4.12. *The name " Hecke algebra " here is justified because this situation is in perfect analogy to that where usual Hecke algebras appear. If $H \subset G$ are simple groups, one can ask about the representations, which are modules over $\mathbb{C}[H]$ and $\mathbb{C}[G]$ respectively. One then constructs $H(G,H) = \mathbb{C}[h\mathfrak{G}/h]]$. If V is a $\mathbb{C}[G]$-module, then Hecke showed that V^H is a $H(G,H)$-module. In other words, $H_n(\mathfrak{g})$ is the Hecke algebra associated to the quantum moment map $\mathfrak{g} \longrightarrow A$.*

Classical dynamical Yang-Baxter equations. The classical dynamical Yang-Baxter equation was introduced in [44] by Felder whose construction we now recall. Suppose we have a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ together with an element $Z \in (\wedge^2\mathfrak{g})^{\mathfrak{g}}$. A (non-modified) classical dynamical r-matrix for the pair $(\mathfrak{g}, \mathfrak{h})$ is a regular $\mathfrak{h}$-equivariant map $\rho : \mathfrak{h}^\vee \longrightarrow \wedge^2\mathfrak{g}$ which satisfies the (non-modified) classical dynamical Yang-Baxter equation (CDYBE)

$$\mathrm{CYB}(\rho) - \mathrm{Alt}(d\rho) = 0$$

where

- $\mathrm{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$,
- $\mathrm{Alt}(d\rho) := \sum_i h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i}$.

and where (h_i) and (λ^i) are basis dual to each other in $\mathfrak{h}$ and $\mathfrak{h}^\wedge$ respectively.

Remark 2.4.13. *Here regular means C^∞, meromorphic, formal etc. depending on the context.*

Assume that $\mathfrak{g}$ is finite dimensional and that we have a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$, i.e., $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra and $\mathfrak{n} \subset \mathfrak{g}$ is a vector subspace such that $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$; assume also that $t_{\mathfrak{g}} = t_{\mathfrak{h}} + t_{\mathfrak{n}}$, where $t_{\mathfrak{h}} \in S^2(\mathfrak{h})^{\mathfrak{h}}$ and $t_{\mathfrak{n}} \in S^2(\mathfrak{n})^{\mathfrak{h}}$.

We assume that for a generic $h \in \mathfrak{h}$, $\mathrm{ad}(h)_{|\mathfrak{n}} \in \mathrm{End}(\mathfrak{n})$ is invertible (i.e. that the decomposition is non-degenerate). This condition is equivalent to the nonvanishing of $P(\lambda) := \det(\mathrm{ad}(\lambda^\vee)_{|\mathfrak{n}}) \in S^{\dim \mathfrak{n}}(\mathfrak{h})$, where $\lambda \mapsto \lambda^\vee$ is the map $\mathfrak{h}^* \longrightarrow \mathfrak{h}$, with $\lambda^\vee := (\lambda \otimes \mathrm{id})(t_{\mathfrak{h}})$. If G is a Lie group with Lie algebra $\mathfrak{g}$, an equivalent condition is that a generic element of $\mathfrak{g}^*$ is conjugate to some element in $\mathfrak{h}^*$ (see [38]).

Let us set, for $\lambda \in \mathfrak{h}^*$,

$$r(\lambda) := (\mathrm{id} \otimes (\mathrm{ad}\, \lambda^\vee)_{|\mathfrak{n}}^{-1})(t_{\mathfrak{n}})$$

and denote $\mathfrak{h}^*_{\mathrm{reg}} = \{\lambda \in \mathfrak{h}^* | P(\lambda) \neq 0\}$. Then $r : \mathfrak{h}^*_{\mathrm{reg}} \longrightarrow \wedge^2(\mathfrak{n})$ is a classical dynamical r-matrix for the pair $(\mathfrak{g}, \mathfrak{h})$ (see [38]).

2.4.5 Realizations of the universal elliptic KZB connection

As in the genus 0 case, the universal KZB connection has realizations.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field $\mathbf{k}$ of characteristic equal to 0 and let $H_n(\mathfrak{g})$ be its associated Hecke algebra.

Proposition 2.4.14. *There is a unique Lie algebra morphism $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d} \longrightarrow \mathcal{H}_n(\mathfrak{g})$, defined by*

- $\bar{x}_i \longmapsto \sum_\alpha \mathrm{x}_\alpha \otimes e_\alpha^{(i)}$,
- $\bar{y}_i \longmapsto -\sum_\alpha \partial_\alpha \otimes e_\alpha^{(i)}$,
- $\bar{t}_{ij} \longmapsto 1 \otimes t_{\mathfrak{g}}^{(ij)}$.
- $\Delta_0 \longmapsto -\frac{1}{2}(\sum_\alpha \partial_\alpha^2) \otimes 1$,
- $X \longmapsto \frac{1}{2}(\sum_\alpha \mathrm{x}_\alpha^2) \otimes 1$,

- $d \longmapsto \frac{1}{2}(\sum_\alpha \mathrm{x}_\alpha \, \partial_\alpha + \partial_\alpha \, \mathrm{x}_\alpha) \otimes 1$,
- $\delta_{2m} \longmapsto \frac{1}{2} \sum_{\alpha_1,\ldots,\alpha_{2m},\alpha} \mathrm{x}_{\alpha_1} \cdots \mathrm{x}_{\alpha_{2m}} \otimes (\sum_{i=1}^n (\mathrm{ad}(e_{\alpha_1}) \cdots \mathrm{ad}(e_{\alpha_{2m}})(e_\alpha) \cdot e_\alpha)^{(i)})$

for $m \geq 1$.

This morphism also extends to a morphism $U(\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}) \rtimes S_n \longrightarrow \mathcal{H}_n(\mathfrak{g}) \rtimes S_n$ *by the assignment* $\sigma \longmapsto \sigma$.

Under the assumptions of the above subsection, one can show that the universal KZB connection induces a classical dynamical r-matrix which is the realization of the universal KZB connection associated to the pair $(\mathfrak{g}, \mathfrak{h})$.

If moreover we assume that $\mathfrak{g}$ is simple and $\mathfrak{h}$ is Cartan, then it can be shown that the universal KZB connection realizes to the former KZB connection constructed by Bernard in [9] in the context of Wess-Zumino-Witten models.

2.5 Reminders on operads, operadic modules and moperads

In this section we fix a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ having small colimits and such that $\otimes$ commutes with these.

2.5.1 $\mathfrak{S}$-modules

An $\mathfrak{S}$-*module* (in $\mathcal{C}$) is a functor $S : \mathbf{Bij} \longrightarrow \mathcal{C}$, where $\mathbf{Bij}$ denotes the category of finite sets with bijections as morphisms. It can also be defined as a collection $(S(n))_{n\geq 0}$ of objects of $\mathcal{C}$ such that $S(n)$ is endowed with a right action of the symmetric group $\mathfrak{S}_n$ for every n; one has $S(n) := S(\{1, \ldots, n\})$. A *morphism* of $\mathfrak{S}$-modules $\varphi : S \longrightarrow T$ is a natural transformation. It is determined by the data of a collection $\varphi(n) : S(n) \longrightarrow T(n)$ of $\mathfrak{S}_n$-equivariant morphisms in $\mathcal{C}$.

The category $\mathfrak{S}$-mod of $\mathfrak{S}$-modules is naturally endowed with a symmetric monoidal product $\otimes$ defined as follows:

$$(S \otimes T)(n) := \coprod_{p+q=n} (S(p) \otimes T(q))_{\mathfrak{S}_p \times \mathfrak{S}_q}^{\mathfrak{S}_n} .$$

Here, if $H \subset G$ is a group inclusion, then $(-)_H^G$ is left adjoint to the restriction functor from the category of objects carrying a G-action to the category of objects carrying an H-action.

We let the reader check that the symmetric sequence $\mathbf{1}_\otimes$ defined by

$$\mathbf{1}_\otimes(n) := \begin{cases} \mathbf{1} & \text{if } n = 0 \\ \emptyset & \text{else} \end{cases}$$

is a monoidal unit.

There is another (non-symmetric) monoidal product $\circ$ on $\mathfrak{S}$-mod, defined as follows:

$$(S \circ T)(n) := \coprod_{k \geq 0} T(k) \underset{\mathfrak{S}_k}{\otimes} \left(S^{\otimes k}(n)\right) .$$

Here, if H is a group and X, Y are objects carrying an H-action, then

$$X \underset{H}{\otimes} Y := \operatorname{coeq}\left(\coprod_{h \in H} X \otimes Y \overset{h \otimes \mathrm{id}}{\underset{\mathrm{id} \otimes h}{\rightrightarrows}} X \otimes Y \right) .$$

We let the reader check that the symmetric sequence $\mathbf{1}_{\mathfrak{S}}$ defined by

$$\mathbf{1}_{\circ}(n) := \begin{cases} \mathbf{1} & \text{if } n = 1 \\ \emptyset & \text{else} \end{cases}$$

is a monoidal unit for $\circ$.

2.5.2 Operads

An *operad* (in $\mathcal{C}$) is a unital monoid in $(\mathfrak{S}\text{-mod}, \circ, \mathbf{1}_{\circ})$. The category of operads in $\mathcal{C}$ will be denoted $\mathrm{Op}\mathcal{C}$.

More explicitly, an operad structure on a $\mathfrak{S}$-module $\mathcal{O}$ is the data:

- of a unit map $e : \mathbf{1} \longrightarrow \mathcal{O}(\{1\})$.
- for every sets I, J and any element $i \in I$, of a *partial composition*

$$\circ_i : \mathcal{O}(I) \otimes \mathcal{O}(J) \longrightarrow \mathcal{O}\left(J \sqcup I - \{i\}\right)$$

satisfying the following constraints:

- if we have sets I, J, K, and elements $i \in I$, $j \in J$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(I) \otimes \mathcal{O}(J) \otimes \mathcal{O}(K) \arrow[r, "\circ_i \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \circ_j"] & \mathcal{O}\left(J \sqcup I - \{i\}\right) \otimes \mathcal{O}(K) \arrow[d, "\circ_j"] \\ \mathcal{O}(I) \otimes \mathcal{O}\left(K \sqcup J - \{j\}\right) \arrow[r, "\circ_i"] & \mathcal{O}\left(K \sqcup J \sqcup I - \{i, j\}\right) \end{tikzcd}$$

- if we have sets I, J_1, J_2 and elements $i_1, i_2 \in I$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(I) \otimes \mathcal{O}(J_1) \otimes \mathcal{O}(J_2) \arrow[r, "\circ_{i_1} \otimes \mathrm{id}"] \arrow[d, "(\circ_{i_2} \otimes \mathrm{id})(23)"] & \mathcal{O}\left(J_1 \sqcup I - \{i_1\}\right) \otimes \mathcal{O}(J_2) \arrow[d, "\circ_{i_2}"] \\ \mathcal{O}\left(J_2 \sqcup I - \{i_2\}\right) \otimes \mathcal{O}(J_1) \arrow[r, "\circ_{i_1}"] & \mathcal{O}\left(J_2 \sqcup J_1 \sqcup I - \{i_1, i_2\}\right) \end{tikzcd}$$

- if we have sets I, I', J, $i \in I$ and a bijection $\sigma : I \longrightarrow I'$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(I) \otimes \mathcal{O}(J) \arrow[r, "\mathcal{O}(\sigma)"] \arrow[d, "\circ_i"] & \mathcal{O}(I') \otimes \mathcal{O}(J) \arrow[d, "\circ_{\sigma(i)}"] \\ \mathcal{O}\left(J \sqcup I - \{i\}\right) \arrow[r, "\mathcal{O}(\mathrm{id} \sqcup \sigma_{|I - \{i\}})"] & \mathcal{O}\left(J \sqcup I' - \{\sigma(i)\}\right) \end{tikzcd}$$

- if we have a set I and $i \in I$, then the following diagrams commute:

$$\begin{tikzcd} \mathbf{1} \otimes \mathcal{O}(I) \arrow[r, "e\otimes\mathrm{id}"] \arrow[dr, "\simeq"'] & \mathcal{O}(\{1\}) \otimes \mathcal{O}(I) \arrow[d, "\circ_1"] \\ & \mathcal{O}(I) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(I) \otimes \mathbf{1} \arrow[r, "\mathrm{id}\otimes e"] \arrow[d, "\simeq"'] & \mathcal{O}(I) \otimes \mathcal{O}(\{1\}) \arrow[d, "\circ_i"] \\ \mathcal{O}(I) \arrow[r, "\simeq", "i\mapsto 1"'] & \mathcal{O}\left(I \sqcup \{1\} - \{i\}\right) \end{tikzcd}$$

Example 2.5.1. *Let X be an object of $\mathcal{C}$. Then we define, for any finite set I, the set $\underline{\mathrm{End}}(X)(I) := \mathrm{Hom}_{\mathcal{C}}(X^{\otimes I}, X)$. Composition of tensor products of maps provide $\underline{\mathrm{End}}(X)$ with the structure of an operad in sets.*

Given an operad in sets $\mathcal{O}$, an $\mathcal{O}$*-algebra in* $\mathcal{C}$ is an object X of $\mathcal{C}$ together with a morphism of operads $\mathcal{O} \longrightarrow \underline{\mathrm{End}}(X)$.

2.5.3 Example of an operad: Stasheff polytopes

To any finite set I we associate the configuration space $\mathrm{Conf}(\mathbb{R}, I) = \{\mathbf{x} = (x_i)_{i\in I} \in \mathbb{R}^I | x_i \neq x_j \text{ if } i \neq j\}$ and its reduced version

$$\mathrm{C}(\mathbb{R}, I) := \mathrm{Conf}(\mathbb{R}, I)/\mathbb{R} \rtimes \mathbb{R}_{>0}\,.$$

The Fulton-MacPherson compactification $\overline{\mathrm{C}}(\mathbb{R}, I)$ of $\mathrm{C}(\mathbb{R}, I)$ (see [48]) is a disjoint union of $|I|$-th Stasheff polytopes [96], indexed by $\mathfrak{S}_I$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{R}, I) := \overline{\mathrm{C}}(\mathbb{R}, I) - \mathrm{C}(\mathbb{R}, I)$ is the union, over all partitions $I = J_1 \coprod \cdots \coprod J_k$, of

$$\partial_{J_1,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{R}, I) := \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{R}, J_i) \times \overline{\mathrm{C}}(\mathbb{R}, k)\,.$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{R}, -)$ with the structure of an operad in topological spaces (where the monoidal structure is given by the cartesian product).

One can see that $\overline{\mathrm{C}}(\mathbb{R}, I)$ is actually a manifold with corners, and that, considering only zero-dimensional strata of our configuration spaces, we get a suboperad $\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -)$ that can be shortly described as follows:

- $\mathbf{Pa}(I)$ is the set of pairs (σ, p) with σ is a linear order on I and p a maximal parenthesization of $\underbrace{\bullet \cdots \bullet}_{|I| \text{ times}}$,
- the operad structure is given by substitution.

Notice that $\mathbf{Pa}$ is actually an operad in sets, and that $\mathbf{Pa}$-algebras are nothing else than *magmas*.

2.5.4 Modules over an operad: Bott-Taubes polytopes

A *module* over an operad $\mathcal{O}$ (in $\mathcal{C}$) is a left $\mathcal{O}$-module in $(\mathfrak{S}\text{-mod}, \circ, \mathbf{1}_\circ)$. Notice that any operad is a module over itself. We let the reader find the very explicit description of a module in terms of partial compositions, as for operads.

To any finite set I we associate the configuration space $\mathrm{Conf}(\mathbb{S}^1, I) = \{\mathbf{x} = (x_i)_{i\in I} \in (\mathbb{S}^1)^I | x_i \neq x_j \text{ if } i \neq j\}$ and its reduced version

$$\mathrm{C}(\mathbb{S}^1, I) := \mathrm{Conf}(\mathbb{S}^1, I)/\mathbb{S}^1 .$$

The Fulton–MacPherson compactification $\overline{\mathrm{C}}(\mathbb{S}^1, I)$ of $\mathrm{C}(\mathbb{S}^1, I)$ is a disjoint union of $|I|$-th Bott–Taubes polytopes [15], indexed by $\mathfrak{S}_I$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{S}^1, I) := \overline{\mathrm{C}}(\mathbb{S}^1, I) - \mathrm{C}(\mathbb{S}^1, I)$ is the union, over all partitions $I = J_1 \coprod \cdots \coprod J_k$, of

$$\partial_{J_1,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{S}^1, I) := \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{R}, J_i) \times \overline{\mathrm{C}}(\mathbb{S}^1, k) .$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{S}^1, -)$ with the structure of a module over the operad $\overline{\mathrm{C}}(\mathbb{R}, -)$ in topological spaces.

One can see that $\overline{\mathrm{C}}(\mathbb{S}^1, I)$ is actually a manifold with corners, and that, considering only zero-dimensional strata of our configuration spaces, we get $\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{S}^1, -)$, which is a module over $\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -)$.

2.5.5 Moperads over an operad

Let $\mathcal{O}$ be an operad. A *moperad* over an operad $\mathcal{O}$ is an $\mathfrak{S}$-module $\mathcal{P}$ carrying

- a unital monoid structure for the monoidal product $\otimes$,
- and a left $\mathcal{O}$-module structure for the monoidal product $\circ$, that are compatible in the following sense:
  - One first observes that there is a natural map $(\mathcal{O} \circ \mathcal{P}) \otimes \mathcal{Q} \longrightarrow \mathcal{O} \circ (\mathcal{P} \otimes \mathcal{Q})$.
  - Then the compatibility means that the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{O} \circ \mathcal{P}) \otimes \mathcal{P} \arrow[r] \arrow[d] & \mathcal{P} \otimes \mathcal{P} \arrow[dr] & \\
\mathcal{O} \circ (\mathcal{P} \otimes \mathcal{P}) \arrow[r] & \mathcal{O} \circ \mathcal{P} \arrow[r] & \mathcal{P}
\end{tikzcd}$$

The map $(\mathcal{O} \circ \mathcal{P}) \otimes \mathcal{P} \longrightarrow \mathcal{P}$ one obtains decomposes into maps

$$\mathcal{P}(k) \otimes \mathcal{P}(m_0) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_k) \longrightarrow \mathcal{P}(m_0 + \cdots + m_k)$$

satisfying certain associativity, unit and $\mathfrak{S}$-equivariance relations. We leave it as an exercise to check that, within the symmetric monoidal category of differential graded vector spaces, this definition coincides with Willwacher's one from [103] (from which we borrowed the name "moperad"). Note that the monoid structure for the monoidal product $\otimes$ encodes precisely the partial composition with respect to the second colour. We will denote this partial composition by $\circ_0$.

2.5.6 Example of a moperad over an operad: coloured Stasheff polytopes

To any finite set I we associate the configuration space $\mathrm{Conf}(\mathbb{R}_{>0}, I) = \{\mathbf{x} = (x_i)_{i\in I} \in (\mathbb{R}_{>0})^I | x_i \neq x_j \text{ if } i \neq j\}$ and its reduced version

$$\mathrm{C}(\mathbb{R}_{>0}, I) := \mathrm{Conf}(\mathbb{R}_{>0}, I)/\mathbb{R}_{>0}\,.$$

The Fulton–MacPherson compactification $\overline{\mathrm{C}}(\mathbb{R}_{>0}, I)$ of $\mathrm{C}(\mathbb{R}_{>0}, I)$ is a disjoint union of $|I|$-th Stasheff polytopes with two kinds of colours, indexed by $\mathfrak{S}_I$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{R}_{>0}, I) := \overline{\mathrm{C}}(\mathbb{R}_{>0}, I) - \mathrm{C}(\mathbb{R}_{>0}, I)$ is the union, over all partitions $I = J_0 \coprod J_1 \coprod \cdots \coprod J_k$, of

$$\partial_{J_0,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{R}_{>0}, I) := \overline{\mathrm{C}}(\mathbb{R}_{>0}, k) \times \overline{\mathrm{C}}(\mathbb{R}_{>0}, J_0) \times \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{R}, J_i)\,.$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{R}_{>0}, -)$ with the structure of a $\overline{\mathrm{C}}(\mathbb{R}, -)$-moperad in topological spaces.

One can see that $\overline{\mathrm{C}}(\mathbb{R}_{>0}, I)$ is a manifold with corners, and that considering only zero-dimensional strata of our configuration spaces we get a sub-moperad $\mathbf{Pa_0} \subset \overline{\mathrm{C}}(\mathbb{R}_{>0}, -)$ that can be shortly described as follows:

- $\mathbf{Pa_0}(I)$ is the set of pairs (σ, p) with σ is a linear order on I and p a maximal parenthesization of $\left(0\underbrace{\bullet\cdots\bullet}_{|I|\text{ times}}\right)$ such that there is no action of $\mathfrak{S}_n$ on 0, but this element can be inside a parenthesis. This means that we allow points to be near the origin.
- The $\overline{\mathrm{C}}(\mathbb{R}, -)$-moperad structure is given by substitution as above.

Forgetting the $\overline{\mathrm{C}}(\mathbb{R}, -)$-moperad structure on $\overline{\mathrm{C}}(\mathbb{R}_{>0}, -)$ and considering a $\overline{\mathrm{C}}(\mathbb{R}, -)$-module structure on it amounts to forbidding points to be close to the origin (i.e. the 0-strand cannot be inside a parenthesis in this case).

2.5.7 Prounipotent completion and fake pull-back of operads in groupoids

Let $\mathbf{k}$ be a $\mathbb{Q}$-ring. We denote by $\mathbf{CoAlg_k}$ the symmetric monoidal category of complete filtered topological coassociative cocommutative counital $\mathbf{k}$-coalgebras, where the monoidal product is given by the completed tensor product $\hat{\otimes}_{\mathbf{k}}$ over $\mathbf{k}$.

Let $\mathbf{Cat}(\mathbf{CoAlg_k})$ be the category of small $\mathbf{CoAlg_k}$-enriched categories. It is symmetric monoidal as well, with monoidal product $\otimes$ defined as follows:

- $\mathrm{Ob}(C \otimes C') := \mathrm{Ob}(C) \times \mathrm{Ob}(C')$.
- $\mathrm{Hom}_{C\otimes C'}\big((c, c'), (d, d')\big) := \mathrm{Hom}_C(c, d)\hat{\otimes}_{\mathbf{k}} \mathrm{Hom}_{C'}(c', d')$.

Let us now consider the symmetric monoidal category $\mathbf{Grpd}$ of groupoids, with symmetric monoidal structure given by the cartesian product. We have a symmetric monoidal functor

$$\begin{array}{rcl} \mathbf{Grpd} & \longrightarrow & \mathbf{Cat}(\mathbf{CoAlg_k}) \\ \mathcal{G} & \longmapsto & \mathcal{G}(\mathbf{k}) \end{array}$$

defined as follows:

- Objects of $\mathcal{G}(\mathbf{k})$ are objects of $\mathcal{G}$.
- For $a, b \in Ob(\mathcal{G})$,
$$\mathrm{Hom}_{\mathcal{G}(\mathbf{k})}(a,b) = \mathbf{k}\cdot\widehat{\mathrm{Hom}_{\mathcal{G}}}(a,b)\,.$$
Here $\mathbf{k}\cdot\mathrm{Hom}_{\mathcal{G}}(a,b)$ is equipped with the unique coalgebra structure such that the elements of $\mathrm{Hom}_{\mathcal{G}}(a,b)$ are grouplike (meaning that they are diagonal for the coproduct and that their counit is 1), and the "$\widehat{\ }$" refers to the completion with respect to the topology defined by the sequence $\big(\mathrm{Hom}_{\mathcal{I}^k}(a,b)\big)_{k\geq 0}$, where:
  - $\mathcal{I}^k$ is the category having the same objects as $\mathcal{G}$ and morphisms lying in the k-th power (for the composition of morphisms) of kernels of the counits of $\mathbf{k}\cdot\mathrm{Hom}_{\mathcal{G}}(a,b)$'s.
- For a functor $F : \mathcal{G} \longrightarrow \mathcal{H}$, $F(\mathbf{k}) : \mathcal{G}(\mathbf{k}) \longrightarrow \mathcal{H}(\mathbf{k})$ is the functor given by F on objects and by $\mathbf{k}$-linearly extending F on morphisms.

Being symmetric monoidal, this functor sends operads in groupoids to operads in $\mathbf{Cat}(\mathbf{CoAlg_k})$.

Example 2.5.2. *For instance, viewing* $\mathbf{Pa}$ *as an operad in groupoid (with only identities as morphisms), then* $\mathbf{Pa}(\mathbf{k})$ *is the operad in* $\mathbf{Cat}(\mathbf{CoAlg_k})$ *with same objects as* $\mathbf{Pa}$*, and whose morphisms are*
$$\mathrm{Hom}_{\mathbf{Pa}(\mathbf{k})(n)}(a,b) = \begin{cases} \mathbf{k} & \text{if } a = b \\ 0 & \text{else} \end{cases}$$
with $\mathbf{k}$ *being equipped with the obvious coproduct* $\Delta(1) = 1\otimes 1$ *and counit* $\epsilon(1) = 1$*.*

The functor we have just defined has a right adjoint
$$G : \mathbf{Cat}(\mathbf{CoAlg_k}) \longrightarrow \mathbf{Grpd}\,,$$
that we can describe as follows:

- For C in $\mathbf{Cat}(\mathbf{CoAlg_k})$, objects of $G(C)$ are objects of C.
- For $a, b \in Ob(\mathcal{G})$, $\mathrm{Hom}_{G(C)}(a,b)$ is the subset of grouplike elements in $\mathrm{Hom}_C(a,b)$.

Being right adjoint to a symmetric monoidal functor, it is lax symmetric monoidal, and thus it sends operads (resp. modules, resp. moperad) to operads (resp. modules, resp. moperad).

We thus get a $\mathbf{k}$-*prounipotent completion* functor $\mathcal{G} \mapsto \hat{\mathcal{G}}(\mathbf{k}) := G\big(\mathcal{G}(\mathbf{k})\big)$ for operads (resp. modules, resp. moperad) in groupoids.

Finally, let $\mathcal{P}, \mathcal{Q}$ be two operads (resp. modules, resp. moperad) in groupoids. If we are given a morphism $f : \mathrm{Ob}(\mathcal{P}) \longrightarrow \mathrm{Ob}(\mathcal{Q})$ between the operads (resp. operad modules, resp. moperads) of objects of $\mathcal{P}$ and $\mathcal{Q}$, then (following [47]) we can define an operad (resp. operad module, resp. moperad) $f^\star\mathcal{Q}$ in the following way:

- $\mathrm{Ob}(f^\star\mathcal{Q}) := \mathrm{Ob}(\mathcal{P})$,

- $\mathrm{Hom}_{(f^\star\mathcal{Q})(n)}(p,q) := \mathrm{Hom}_{\mathcal{Q}(n)}(f(p),f(q))$.

In particular, $f^\star\mathcal{Q}$ inherits the operad structure of $\mathcal{P}$ for its operad of objects and that of $\mathcal{Q}$ for the morphisms.

Remark 2.5.3. *Notice that this is not a pull-back in the category of operads in groupoids.*

2.5.8 Pointed versions

Observe that there is an obvious operad *Unit* defined by

$$Unit(n) = \begin{cases} \mathbf{1} & \text{if } n = 0,1 \\ \emptyset & \text{else} \end{cases}$$

By convention, all our operads $\mathcal{O}$ will be pointed in the sense that they will come equipped with a specific operad morphism $Unit \longrightarrow \mathcal{O}$. Morphisms of operads are required to be compatible with this pointing. Actually, all operads appearing in this paper are such that $\mathcal{O}(n) \simeq \mathbf{1}$ if $n = 0,1$.

Now, if $\mathcal{P}$ is an $\mathcal{O}$-module, then it naturally becomes a *Unit*-module as well, by restriction. By convention, all our modules will be pointed as well, in the sense that they will come equipped with a specific *Unit*-module morphism $Unit \longrightarrow \mathcal{P}$. Morphisms of modules are required to be compatible with the pointing. Again, all modules appearing in this paper are such that $\mathcal{P}(n) \simeq \mathbf{1}$ if $n = 0,1$.

Finally, there is a nice moperad *Minut* over *Unit*, which is such that $Minut(n) = \mathbf{1}$ for all $n \geq 0$. By convention, all our moperads will be pointed, in the sense that they will come equipped with a specific *unit*-moperad morphism $Minut \longrightarrow \mathcal{Q}$. Morphisms of moperads are required to be compatible with the pointing.

Remark 2.5.4. *In the category of sets, Minut is the sub-Unit-moperad of* $\mathbf{Pa_0}$ *that consists only of the left-most maximal parenthesization.*

The main reason for these rather strange conventions is that we need the following features, that we have in the case of compactified configuration spaces:

- For operads, modules and moperads, we want to have "deleting operations" $\mathcal{O}(n) \longrightarrow \mathcal{O}(n-1)$ that decrease arity.
- For modules and moperads, we want to be able to see the operad "inside" them, i.e. we want to have distinguished morphism $\mathcal{O} \longrightarrow \mathcal{P}$ of $\mathfrak{S}$-modules.

Example 2.5.5. *For instance, being a* $\mathbf{Pa}$*-moperad,* $\mathbf{Pa_0}$ *comes together with a morphism of* $\mathfrak{S}$*-modules* $\mathbf{Pa} \longrightarrow \mathbf{Pa_0}$. *We let the reader check that it sends a parenthesized permutation* $\mathbf{p}$ *to* $0(\mathbf{p})$.

2.5.9 Group actions

Let G be a group and $\mathcal{O}$ be an operad. We say an $\mathcal{O}$-module $\mathcal{P}$ carry a G-action if

- for every $n \geq 0$, G^n acts $\mathfrak{S}_n$-equivariantly on $\mathcal{P}(n)$, from the left.
- for every $m \geq 0$, $n \geq 0$, and $1 \leq i \leq n$, the partial composition

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{O}(m) \longrightarrow \mathcal{P}(n+m-1)$$

is equivariant along the group morphism

$$\begin{array}{ccc} G^n & \longrightarrow & G^{n+m-1} \\ (g_1, \ldots, g_n) & \longmapsto & (g_1, \ldots, g_{i-1}, \underbrace{g_i, \ldots, g_i}_{m \text{ times}}, g_{i+1}, \ldots, g_n) \end{array}$$

If $\mathcal{P}$ is a moperad, we additionally require that the partial composition

$$\circ_0 : \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m)$$

is G^{n+m}-equivariant.

A morphism $\mathcal{P} \longrightarrow \mathcal{Q}$ of $\mathcal{O}$-modules (or $\mathcal{O}$-moperads) with G-action is said *G-equivariant* if, for every $n \geq 0$, the map $\mathcal{P}(n) \longrightarrow \mathcal{Q}(n)$ is G^n-equivariant.

2.6 Grothendieck-Teichmüller groups

Initially, Grothendieck-Teichmüller groups and associators were, in the genus 0, cyclotomic and genus 1 cases, constructed by using braided monoidal categories, braided modules categories and elliptic structures over braided monoidal categories respectively. Already in V. Drinfeld's work, associators had an implicit operadic nature (made explicit in [5]) which permits to define associators as formality isomorphisms between operads closely related to the little disks operad. More specifically, there is an operad in groupoids **PaB** encapsulating the combinatorics of parenthesized braidings and an operad in groupoids G**PaCD** encapsulating the combinatorics of parenthesized chord diagrams. The former is obtained (roughly) by considering a parenthesized version of the (pure) braid group on the torus. The latter is obtained from the collection $\mathfrak{t}(\mathbf{k})$ of Lie $(\mathbf{k})$-algebras $\mathfrak{t}_n(\mathbf{k})$, for $n \geq 1$, which has a natural operad structure. In this scope, the (naive) Grothendieck-Teichmüller group consists on the group of automorphisms of **PaB** which are the identity on objects, the graded Grothendieck-Teichmüller group is the group of automorphisms of G**PaCD** which are the identity on objects, and, by denoting $\widehat{\mathbf{PaB}}(\mathbf{k})$ the $\mathbf{k}$-prounipotent completion of **PaB**, then the set of $\mathbf{k}$-associators consists on the set of isomorphisms $\widehat{\mathbf{PaB}}(\mathbf{k}) \longrightarrow G\mathbf{PaCD}$ of operads in $\mathbf{k}$-prounipotent groupoids which are the identity on objects. It can be shown that these operadic point of view is compatible with the classic one, namely that there is a one-to-one correspondence between the operadic definition of these objects and the objects defined in the literature in terms of elements satisfying certain equations.

Let us mention that in [47], B. Fresse developped a very general rational homotopy theory for operads in order to understand from a homotopical viewpoint, a deep relationship between operads and Grothendieck-Teichmüller groups which was first foreseen by M. Kontsevich in his work on deformation quantization process in mathematical physics.

More specifically, after developing a general theory permitting to endow the category of operads in simplicial sets (and, further, of Hopf cooperads) with a (nice enough) model category structure, the author uses an application of homotopy spectral sequences to show that the Grothendieck-Teichmüller group has a topological interpretation as a group of homotopy automorphisms associated to the little 2-disc operad. A similar characterisation of the set of associators is also done in the author's work.

2.6.1 Compactified configuration space of the plane

To any finite set I we associate a configuration space

$$\mathrm{Conf}(\mathbb{C}, I) = \{\mathbf{z} = (z_i)_{i\in I} \in \mathbb{C}^I | z_i \neq z_j \text{ if } i \neq j\}\,.$$

We also consider its reduced version

$$\mathrm{C}(\mathbb{C}, I) := \mathrm{Conf}(\mathbb{C}, I)/\mathbb{C} \rtimes \mathbb{R}_{>0}.$$

We then consider the Fulton–MacPherson compactification $\overline{\mathrm{C}}(\mathbb{C}, I)$ of $\mathrm{C}(\mathbb{C}, I)$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{C}, I) = \overline{\mathrm{C}}(\mathbb{C}, I) - \mathrm{C}(\mathbb{C}, I)$ is made of the following irreducible components: for any partition $I = J_1 \coprod \cdots \coprod J_k$ there is a component

$$\partial_{J_1,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{C}, I) \cong \overline{\mathrm{C}}(\mathbb{C}, k) \times \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{C}, J_i)\,.$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{C}, -)$ with the structure of an operad in topological spaces.

2.6.2 The operad of parenthesized braids

We have inclusions of topological operads

$$\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -) \subset \overline{\mathrm{C}}(\mathbb{C}, -)\,.$$

Then it makes sense to define

$$\mathbf{PaB} := \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}, -), \mathbf{Pa}\right)\,,$$

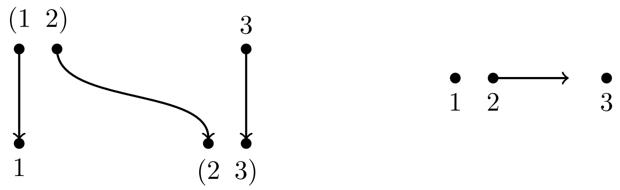
which is an operad in groupoids.

Example 2.6.1 (Description of $\mathbf{PaB}(2)$)**.** *Let us first recall that* $\mathbf{Pa}(2) = \mathfrak{S}_2$*, and that* $\overline{C}(\mathbb{C}, 2) \simeq S^1$*. Besides the identity morphism in* $\mathbf{PaB}(2)$ *going from* (12) *to* (12)*, we have an arrow* $R^{1,2}$ *in* $\mathbf{PaB}(2)$ *going from* (12) *to* (21) *which can be depicted as follows*[3]*:*

[3]We actually have another arrow, that can be obtained from the first one as $(R^{2,1})^{-1}$ according to the notation that is explained after Theorem 2.6.3, and which can be depicted as an undercrossing braid.

Two incarnations of $R^{1,2}$

Example 2.6.2 (Notable arrows in $\mathbf{PaB}(3)$)**.** *Let us first recall that* $\mathbf{Pa}(3) = \mathfrak{S}_3 \times \{(\bullet\bullet)\bullet, \bullet(\bullet\bullet)\}$ *and that* $\overline{C}(\mathbb{R}, 3) \cong \mathfrak{S}_3 \times [0, 1]$*. Therefore, we have an arrow* $\Phi^{1,2,3}$ *(the identity path in* $[0, 1]$*) from* $(12)3$ *to* $1(23)$ *in* $\mathbf{PaB}(3)$*. It can be depicted as follows:*



Two incarnations of $\Phi^{1,2,3}$

The following result is borrowed from [47, Theorem 6.2.4], even though it perhaps already appeared in [5] in a different form.

Theorem 2.6.3. *As an operad in groupoids having* **Pa** *as operad of objects,* **PaB** *is freely generated by* $R := R^{1,2}$ *and* $\Phi := \Phi^{1,2,3}$ *together with the following relations:*

(H1) $R^{1,2}\Phi^{2,1,3}R^{1,3} = \Phi^{1,2,3}R^{1,23}\Phi^{2,3,1}$*, as arrows from* $(12)3$ *to* $2(31)$ *in* $\mathbf{PaB}(3)$*,*

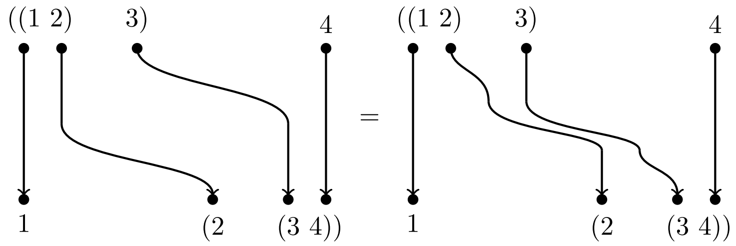
(H2) $(R^{2,1})^{-1}\Phi^{2,1,3}(R^{3,1})^{-1} = \Phi^{1,2,3}(R^{23,1})^{-1}\Phi^{2,3,1}$*, as arrows from* $(12)3$ *to* $2(31)$ *in* $\mathbf{PaB}(3)$*,*

(P) $\Phi^{12,3,4}\Phi^{1,2,34} = \Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}$*, as arrows from* $((12)3)4$ *to* $1(2(34))$ *in* $\mathbf{PaB}(4)$*.*

We now briefly explain the notation we have been using in the above statement, which is quite standard. In this article, we write the composition of paths from left to right (and we draw the braids from top to bottom). If X is an arrow from $\mathbf{p}$ to $\mathbf{q}$ in $\mathbf{PaB}(n)$, then

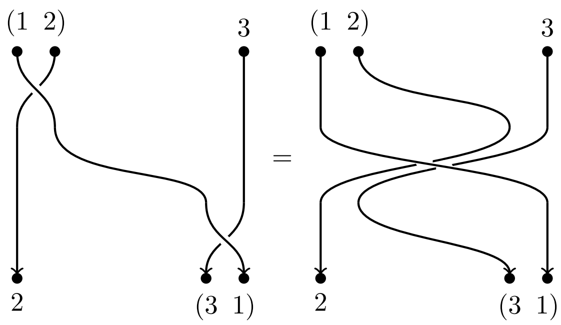
- for any $\mathbf{r} \in \mathbf{Pa}(k)$, the identity of $\mathbf{r}$ in $\mathbf{PaB}(k)$ is also denoted $\mathbf{r}$.
- for any $\mathbf{r} \in \mathbf{Pa}(k)$, we write $X^{1,\ldots,n}$ for $\mathbf{r} \circ_1 X \in \mathbf{PaB}(n + k - 1)$.
- for any $\sigma \in \mathfrak{S}_{n+k-1}$ we define $X^{\sigma_1,\ldots,\sigma_n} := (X^{1,\ldots,n}) \cdot \sigma$.
- for any $\mathbf{r} \in \mathbf{Pa}(k)$, $X^{\mathbf{r},k+1,\ldots,k+n-1} := X \circ_1 \mathbf{r} \in \mathbf{PaB}(n + k - 1)$.
- we allow ourselves to combine these in an obvious way.

We let the reader figuring out that this notation is unambiguous as soon as we specify the starting object of our arrows. For example, the pentagon (P) and the first hexagon (H1) relations can be respectively depicted as



(P)

and



(H1)

or, as commuting diagrams (giving the name of the relations)

$$
\begin{tikzcd}
 & (12)(34) \arrow[dr, "\Phi^{12,3,4}"] & \\
1(2(34)) \arrow[ur, "\Phi^{1,2,34}"] \arrow[d, "\Phi^{2,3,4}"] & & ((12)3)4 \\
1((23)4) \arrow[rr, "\Phi^{1,23,4}"] & & (1(23))4 \arrow[u, "\Phi^{1,2,3}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & (12)3 \arrow[r, "\Phi^{1,2,3}"] \arrow[dl, "R^{1,2}"'] & 1(23) \arrow[dr, "R^{1,23}"] & \\
(21)3 \arrow[dr, "\Phi^{2,1,3}"'] & & & (23)1 \arrow[dl, "\Phi^{2,3,1}"] \\
 & 2(13) \arrow[r, "R^{1,3}"] & 2(31) &
\end{tikzcd}
$$

2.6.3 The operad of chord diagrams

In [5, 47] it is shown[4] that the collection of Kohno-Drinfeld Lie $\mathbf{k}$-algebras $\mathfrak{t}_n(\mathbf{k})$ defined in the introduction is provided with the structure of an operad in the category $grLie_{\mathbf{k}}$ of positively graded finite dimensional Lie algebras over $\mathbf{k}$, with symmetric monoidal strucure is given by the direct sum $\oplus$. Partial compositions are described as follows:

$$
\begin{array}{rccc}
\circ_k : & \mathfrak{t}_I(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) & \longrightarrow & \mathfrak{t}_{J \sqcup I - \{i\}}(\mathbf{k}) \\
 & (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha\beta} \\
 & (t_{ij}, 0) & \longmapsto & \left\{ \begin{array}{ccc} t_{ij} & \text{if} & k \notin \{i,j\} \\ \sum\limits_{p \in J} t_{pj} & \text{if} & k = i \\ \sum\limits_{p \in J} t_{ip} & \text{if} & j = k \end{array} \right.
\end{array}
$$

[4] Even though the author of [5] does not use the concept of an operad.

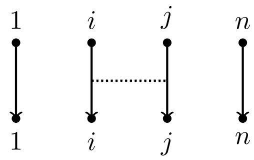
Observe that we have a lax symmetric monoidal functor

$$\hat{\mathcal{U}} : grLie_{\mathbf{k}} \longrightarrow \mathbf{Cat}(\mathbf{CoAlg_k})$$

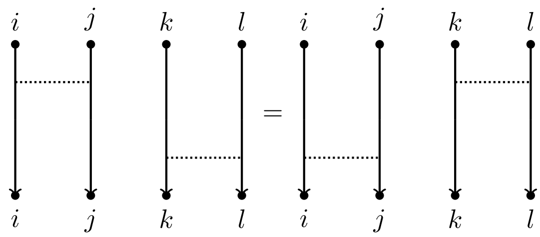
sending a positively graded Lie algebra to the degree completion of its universal envelopping algebra, which is a complete filtered cocommutative Hopf algebra, viewed as a $\mathbf{CoAlg_k}$-enriched category with only one object.

We then consider the operad of *chord diagrams* $\mathbf{CD}(\mathbf{k}) := \hat{\mathcal{U}}(\mathfrak{t}(\mathbf{k}))$ in $\mathbf{Cat}(\mathbf{CoAlg_k})$.

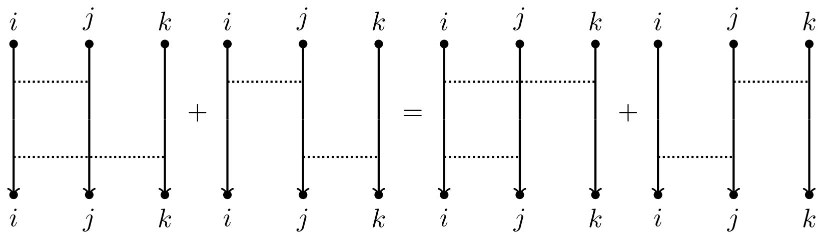
Remark 2.6.4. *This denomination comes from the fact that morphisms in $\mathbf{CD}(\mathbf{k})(n)$ can be represented as linear combinations of diagrams of chords on n vertical strands, where the chord diagram corresponding to t_{ij} can be represented as*



and the composition is given by vertical concatenation of diagrams. Partial compositions can easily be understood as "cabling and removal operations" on strands (see [5, 47]). Relations (L) and (4T) defining each $\mathfrak{t}_n(\mathbf{k})$ can be represented as follows:



(L)



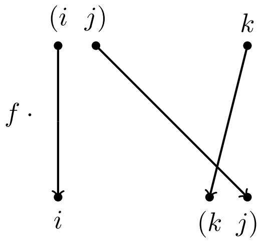
(4T)

2.6.4 The operad of parenthesized chord diagrams

Recall that the operad $\mathbf{CD}(\mathbf{k})$ has only one object in each arity. Hence we have an obvious terminal morphism of operads $\omega_1 : \mathbf{Pa} = \mathrm{Ob}(\mathbf{Pa}(\mathbf{k})) \longrightarrow \mathrm{Ob}(\mathbf{CD}(\mathbf{k}))$, and thus we can consider the operad

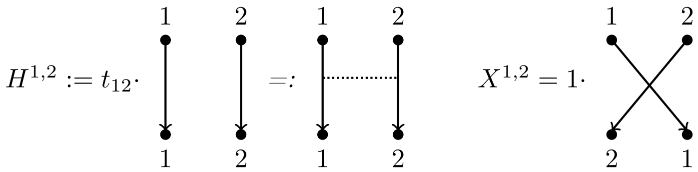
$$\mathbf{PaCD}(\mathbf{k}) := \omega_1^{\star}\mathbf{CD}(\mathbf{k})$$

of *parenthesized chord diagrams*. Here is a self-explanatory example of how to depict a morphism in $\mathbf{PaCD}(\mathbf{k})(3)$:

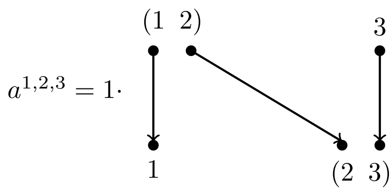


where $f \in \mathbf{CD}(\mathbf{k})(3)$.

Example 2.6.5 (Notable arrows of $\mathbf{PaCD}(\mathbf{k})$)**.** *We have the following arrows in* $\mathbf{PaCD}(\mathbf{k})(2)$*:*



We also have the following arrow in $\mathbf{PaCD}(\mathbf{k})(3)$*:*



Remark 2.6.6. *The elements* $H^{1,2}, X^{1,2}$ *and* $a^{1,2,3}$ *are generators of the operad* $\mathbf{PaCD}(\mathbf{k})$ *and satisfy the following relations:*

(P) $a^{12,3,4}a^{1,2,34} = a^{1,2,3}a^{1,23,4}a^{2,3,4}$,

(H) $X^{12,3} = a^{1,2,3}X^{2,3}(a^{1,3,2})^{-1}X^{1,3}a^{3,1,2}$,

(Inv) $H^{2,1} = X^{1,2}H^{1,2}(X^{1,2})^{-1}$,

(SH) $H^{12,3} = (a^{1,2,3})^{-1}H^{2,3}a^{1,2,3} + (X^{2,1})^{-1}(a^{2,1,3})^{-1}H^{1,3}a^{2,1,3}X^{2,1}$.

In particular, even if $\mathbf{PaCD}(\mathbf{k})$ *does not have a presentation in terms of generators and relations (as is the case fot* $\mathbf{PaB}$*), one can shown that* $\mathbf{PaCD}(\mathbf{k})$ *has a universal property with respect to the generators* $H^{1,2}, X^{1,2}$ *and* $a^{1,2,3}$ *and the above relations (see [47, Theorem 10.3.4] for details).*

2.6.5 Drinfeld associators

Let us first introduce some terminology that we use in this paragraph, as well as later in the paper:

- $\mathbf{Grpd_k}$ denote the (symmetric monoidal) category of $\mathbf{k}$-prounipotent groupoids (which is the image of the completion functor $\mathcal{G} \mapsto \hat{\mathcal{G}}(\mathbf{k})$).
- for $\mathcal{C}$ being $\mathbf{Grpd}$, $\mathbf{Grpd_k}$, or $\mathbf{Cat}(\mathbf{CoAlg_k})$, the notation

$$\mathrm{Aut}^+_{\mathrm{Op}\,\mathcal{C}} \quad (\text{resp. } \mathrm{Iso}^+_{\mathrm{Op}\,\mathcal{C}})$$

refers to those automorphisms (resp. isomorphisms) which are the identity on objects.

In the remainder if this section we recall some well known results on the operadic point of view on associators and Grothendieck-Teichmüller groups, which will be useful later on. Even though the statements and proofs of all the results in this subsection can be found in [47], it is worth mentionning that a "pre-operadic" approach was initiated by Bar-Natan in [5].

Definition 2.6.7. *A Drinfeld* $\mathbf{k}$*-associator is an isomorphism between the operads* $\widehat{\mathbf{PaB}}(\mathbf{k})$ *and* $G\mathbf{PaCD}(\mathbf{k})$ *in* $\mathbf{Grpd_k}$*, which is the identity on objects. We denote by*

$$\mathbf{Ass}(\mathbf{k}) := \mathrm{Iso}^+_{\mathbf{Grpd_k}}(\widehat{\mathbf{PaB}}(\mathbf{k}), G\mathbf{PaCD}(\mathbf{k}))$$

the set of $\mathbf{k}$*-associators.*

Theorem 2.6.8. *There is a one-to-one correspondence between the set of Drinfeld* $\mathbf{k}$*-associators and the set* $\mathrm{Ass}(\mathbf{k})$ *of couples* $(\mu, \varphi) \in \mathbf{k}^\times \times \exp(\hat{\mathfrak{f}}_2(\mathbf{k}))$*, such that*

(S) $\varphi^{3,2,1} = (\varphi^{1,2,3})^{-1}$,

(H) $\varphi^{1,2,3} e^{\mu t_{23}/2} \varphi^{2,3,1} e^{\mu t_{31}/2} \varphi^{3,1,2} e^{\mu t_{12}/2} = e^{\mu(t_{12}+t_{13}+t_{23})/2}$,

(P) $\varphi^{1,2,3} \varphi^{1,23,4} \varphi^{2,3,4} = \varphi^{12,3,4} \varphi^{1,2,34}$,

where $\varphi^{1,2,3} = \varphi(t_{12}, t_{23})$ *is viewed as an element of* $\exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$ *via the inclusion* $\hat{\mathfrak{f}}_2(\mathbf{k}) \subset \hat{\mathfrak{t}}_3(\mathbf{k})$ *sending* x *to* t_{12} *and* y *to* t_{23}*.*

Two observations are in order:

- the free Lie $\mathbf{k}$-algebra $\mathfrak{f}_2(\mathbf{k})$ in two generators x, y is graded, with generators having degree 1, and its degree completion is denoted by $\hat{\mathfrak{f}}_2(\mathbf{k})$.
- the $\mathbf{k}$-prounipotent group $\exp(\hat{\mathfrak{f}}_2(\mathbf{k}))$ is thus isomorphic to the $\mathbf{k}$-prounipotent completion $\widehat{\mathrm{F}}_2(\mathbf{k})$ of the free group F_2 on two generators.

This Theorem was first implicitely shown by Drinfeld in [31]. An explicit proof can be found in [47, Theorem 10.2.9], and relies on the universal property of $\mathbf{PaB}$ from Theorem 2.6.3. In particular, a morphism $F : \widehat{\mathbf{PaB}}(\mathbf{k}) \longrightarrow G\mathbf{PaCD}(\mathbf{k})$ is uniquely determined by a scalar parameter $\mu \in \mathbf{k}$ and $\varphi \in \exp(\hat{\mathfrak{f}}_2(\mathbf{k}))$ such that we have the following assignment in the morphism sets of the parenthesized chord diagram operad $\mathbf{PaCD}$:

- $F(R) = e^{\mu t_{12}/2}$,
- $F(\Phi) = \varphi(t_{12}, t_{23})$,

where R and Φ are the ones from Examples 2.6.1 and 2.6.2.

Example 2.6.9 (The KZ Associator)**.** *The first associator was constructed by Drinfeld with the help of the monodromy of the KZ connection and is known as the KZ associator* Φ_{KZ}*. It is defined as the the renormalized holonomy from* 0 *to* 1 *of* $G'(z) = (\frac{t_{12}}{z} + \frac{t_{12}}{z-1})G(z)$*, i.e.,* $\Phi_{\mathrm{KZ}} := G_{0^+}^{-1}G_{1^-} \in \exp(\hat{\mathfrak{t}}_3(\mathbb{C}))$*, where* G_{0^+}, G_{1^-} *are the solutions such that* $G_{0^+}(z) \sim z^{t_{12}}$ *when* $z \longrightarrow 0^+$ *and* $G_{1^-}(z) \sim (1-z)^{t_{23}}$ *when* $z \longrightarrow 1^-$*. We have*

$$\Phi_{\mathrm{KZ}}(V,U) = \Phi_{\mathrm{KZ}}(U,V)^{-1},\ \Phi_{\mathrm{KZ}}(U,V)e^{\pi \mathrm{i} V}\Phi_{\mathrm{KZ}}(V,W)e^{\pi \mathrm{i} W}\Phi_{\mathrm{KZ}}(W,U)e^{\pi \mathrm{i} U} = 1,$$

where $U = \bar{t}_{12} \in \mathfrak{f}_2(\mathbb{C}) \simeq \bar{\mathfrak{t}}_3(\mathbb{C}) := \mathfrak{t}_3(\mathbb{C})/(t_{12}+t_{13}+t_{23})$*,* $V = \bar{t}_{23} \in \bar{\mathfrak{t}}_3(\mathbb{C})$ *and* $U + V + W = 0$*, and*

$$\Phi_{\mathrm{KZ}}^{12,3,4}\Phi_{\mathrm{KZ}}^{1,2,34} = \Phi_{\mathrm{KZ}}^{1,2,3}\Phi_{\mathrm{KZ}}^{1,23,4}\Phi_{\mathrm{KZ}}^{2,3,4}$$

(relation in $\exp(\hat{\mathfrak{t}}_4(\mathbb{C}))$*) so* $(2\pi \mathrm{i}, \Phi_{\mathrm{KZ}})$ *is an element of* $\mathrm{Ass}(\mathbb{C})$*.*

2.6.6 Grothendieck–Teichmuller group

Definition 2.6.10. *The Grothendieck–Teichmüller group is defined as the group*

$$\mathbf{GT} := \mathrm{Aut}^+_{\mathrm{Op}\,\mathbf{Grpd}}(\mathbf{PaB})$$

of automorphisms of the operad in groupoids **PaB** *which are the identity of objects. One defines similarly the* **k***-pro-unipotent version*

$$\widehat{\mathbf{GT}}(\mathbf{k}) := \mathrm{Aut}^+_{\mathrm{Op}\,\mathbf{Grpd}_{\mathbf{k}}}\left(\widehat{\mathbf{PaB}}(\mathbf{k})\right)$$

There are also pro-ℓ *and profinite versions, denoted* $\mathbf{GT}_\ell$ *and* $\widehat{\mathbf{GT}}$*, that we will not consider in this paper.*

We can also characterize elements of **GT** and $\widehat{\mathbf{GT}}(\mathbf{k})$ as solutions of certain explicit algebraic equations. This characterization proves that the above operadic definition of **GT** coincides with the one given by Drinfeld in his original paper [31]. In this article we will focus on the **k**-pro-unipotent version of this group in genus 0 and 1, and twisted situations.

Definition 2.6.11. *Drinfeld's Grothendieck–Teichmüller group* $\widehat{\mathrm{GT}}(\mathbf{k})$ *consists of pairs*

$$(\lambda, f) \in \mathbf{k}^\times \times \widehat{\mathrm{F}}_2(\mathbf{k})$$

which satisfy the following equations:

(BS) $f(x,y) = f(y,x)^{-1}$,

(BH) $x_1^\nu f(x_1,x_2)x_2^\nu f(x_2,x_3)x_3^\nu f(x_3,x_1) = 1$,

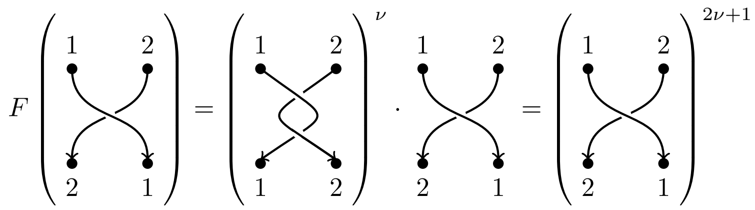
(BP) $f(x_{13}x_{23},x_{34})f(x_{12},x_{23}x_{24}) = f(x_{12},x_{23})f(x_{12}x_{13},x_{23}x_{34})f(x_{23},x_{34})$ *in* $\widehat{\mathrm{PB}}_4(\mathbf{k})$,

where x_1, x_2, x_3 *are 3 variables subject only to* $x_1x_2x_3 = 1$*,* $\nu = \frac{\lambda-1}{2}$*, and* x_{ij} *is the elementary pure braid* P_{ij} *from the introduction. The multiplication law is given by*

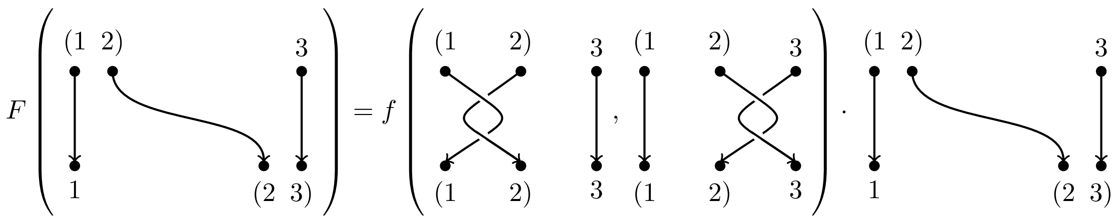
$$(\lambda_1, f_1)(\lambda_2, f_2) = (\lambda_1\lambda_2, f_2(f_1(x,y)x^{\lambda_1}f_1(x,y)^{-1}, y^{\lambda_1})f_1(x,y)).$$

Theorem 2.6.12. *There is an isomorphism between the groups* $\widehat{\mathbf{GT}}(\mathbf{k})$ *and* $\widehat{\mathrm{GT}}(\mathbf{k})$.

This was first implicitely shown by Drinfeld in [31]. An explicit proof of this theorem can be found for example in [47, Theorem 11.1.7]. In particular, one obtains the couple (λ, f) from an automorphism $F \in \widehat{\mathbf{GT}}(\mathbf{k})$ as follows. We have



(2.15)



(2.16)

In other words, if we set $\lambda = 2\nu + 1$, we get the assignment

- $F(R^{1,2}) = (R^{1,2})^{\lambda}$,
- $F(\Phi^{1,2,3}) = f(x_{12}, x_{23}) \cdot \Phi^{1,2,3}$.

Remark 2.6.13. *It is important to notice that the profinite, pro-ℓ, $\mathbf{k}$-pro-unipotent versions of the Grothendieck–Teichmüller group do not coincide with the profinite, pro-ℓ, $\mathbf{k}$-pro-unipotent completions of the"thin" Grothendieck–Teichmüller group* $\mathbf{GT}$ *which only consists of the pairs* $(1,1)$ *and* $(-1,1)$. *We have morphisms*

$$\mathbf{GT} \longrightarrow \widehat{\mathbf{GT}} \twoheadrightarrow \mathbf{GT}_{\ell} \hookrightarrow \widehat{\mathbf{GT}}(\mathbb{Q}_{\ell}) \quad \text{and} \quad \mathbf{GT} \longrightarrow \widehat{\mathbf{GT}}(\mathbf{k})\,.$$

2.6.7 Graded Grothendieck–Teichmuller group

Definition 2.6.14. *The graded Grothendieck–Teichmüller group is the group*

$$\mathbf{GRT}(\mathbf{k}) := \mathrm{Aut}^{+}_{\mathrm{Op}\,\mathbf{Grpd}_{\mathbf{k}}}(G\mathbf{PaCD}(\mathbf{k}))$$

of automorphisms of $G\mathbf{PaCD}(\mathbf{k})$ *that are the identity on objects.*

Remark 2.6.15. *When restricted to the full subcategory* $\mathbf{Cat}(\mathbf{CoAlg}^{\mathbf{conn}}_{\mathbf{k}})$ *of* $\mathbf{CoAlg}_{\mathbf{k}}$*-enriched categories for which the hom-coalgebras are connected, the functor G leads to an equivalence between* $\mathbf{Cat}(\mathbf{CoAlg}^{\mathbf{conn}}_{\mathbf{k}})$ *and* $\mathbf{Grpd}_{\mathbf{k}}$. *Hence there is an isomorphism*

$$\mathbf{GRT}(\mathbf{k}) \simeq \mathrm{Aut}^{+}_{\mathrm{Op}\,\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})}(\mathbf{PaCD}(\mathbf{k}))\,.$$

Again, the operadic definition of $\mathbf{GRT}(\mathbf{k})$ happens to coincide with the one originaly given by Drinfeld.

Definition 2.6.16. *Let* GRT_1 *be the set of elements in* $g \in \exp(\hat{\mathfrak{f}}_2(\mathbf{k})) \subset \exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$ *such that*

- $g^{3,2,1} = g^{-1}$ *and* $g^{1,2,3}g^{2,3,1}g^{3,1,2} = 1$*, in* $\exp(\hat{\mathfrak{t}}_3(\mathbf{k})))$*,*
- $t_{12} + \mathrm{Ad}(g^{1,2,3})^{-1}(t_{23}) + \mathrm{Ad}(g^{2,1,3})^{-1}(t_{13}) = t_{12} + t_{13} + t_{23}$*, in* $\hat{\mathfrak{t}}_3(\mathbf{k}))$*,*
- $g^{1,2,3}g^{1,23,4}g^{2,3,4} = g^{12,3,4}g^{1,2,34}$*, in* $\exp(\hat{\mathfrak{t}}_4(\mathbf{k})))$*,*

One has the following multiplication law on GRT_1*:*

$$(g_1 * g_2)(A, B) := g_1(\mathrm{Ad}(g_2(A, B))(A), B)g_2(A, B)\,.$$

Drinfeld showed in [31] that the above GRT_1 is stable under $*$, that it defines a group structure on it, and that rescaling transformations $g(x, y) \mapsto \lambda \cdot g(x, y) = g(\lambda x, \lambda y)$ define an action of $\mathbf{k}^\times$ of $\mathbf{GRT}_1$ by automorphisms.

Theorem 2.6.17. *There is a group isomorphism* $\mathbf{GRT}(\mathbf{k}) \cong \mathbf{k}^\times \rtimes \mathrm{GRT}_1$*.*

This was first implicitely shown by Drinfeld in [31]. An explicit proof of this theorem can be found for example in [47, Theorem 10.3.10]. In particular, we obtain the couple (λ, g) from an automorphism $G \in \mathbf{GRT}(\mathbf{k})$ by the assignment

- $G(X^{1,2}) = X^{1,2}$,
- $G(H^{1,2}) = \lambda H^{1,2}$,
- $G(a^{1,2,3}) = g(t_{12}, t_{23}) \cdot a^{1,2,3}$.

2.6.8 Torsors

Recall first that there is a left free and transitive group action of $\widehat{\mathrm{GT}}(\mathbf{k})$ on $\mathrm{Ass}(\mathbf{k})$, defined by

$$(\lambda, f) * (\mu, \Phi) := (\lambda\mu, \Phi(A, B)f(e^{\mu A}, \Phi(A, B)^{-1}e^{\mu B}\Phi(A, B))) = (\mu', \Phi').$$

Recal also that there is a right free and transitive group action of $\mathrm{GRT}(\mathbf{k})$ on $\mathrm{Ass}(\mathbf{k})$ defined as follows: for $g \in \mathrm{GRT}_1(\mathbf{k})$ and $(\mu, \Phi) \in \underline{M}(\mathbf{k})$, $(\mu, \Phi) * g := (\mu, \tilde{\Phi})$, where

$$\tilde{\Phi}(t_{12}, t_{23}) = \Phi((t_{12}, \mathrm{Ad}(g)t_{23}))g,$$

and for $c \in \mathbf{k}^\times$,$(\mu, \Phi) * c := (c\mu, c * \Phi)$, where $(c * \Phi)(A, B) = \Phi(cA, cB)$. This makes $(\widehat{\mathrm{GT}}(\mathbf{k}), \mathrm{Ass}(\mathbf{k}), \mathrm{GRT}(\mathbf{k}))$ into a torsor.

Proposition 2.6.18. *There is a torsor isomorphism*

$$(\widehat{\mathbf{GT}}(\mathbf{k}), \mathbf{Ass}(\mathbf{k}), \mathbf{GRT}(\mathbf{k})) \longrightarrow (\widehat{\mathrm{GT}}(\mathbf{k}), \mathrm{Ass}(\mathbf{k}), \mathrm{GRT}(\mathbf{k})) \tag{2.17}$$

Proof. On the one hand, in [47, Theorem 10.3.13] it is shown that the natural left free and transitive action of $\widehat{\mathbf{GT}}(\mathbf{k})$ over $\mathbf{Ass}(\mathbf{k})$ coincides with the action of $\mathrm{GT}(\mathbf{k})$ over $\mathrm{Ass}(\mathbf{k})$ via the correspondence of Theorem 2.6.12. Thus, both actions are compatible. On the other hand, in [47, Theorem 11.2.1], it is shown that the natural right free and transitive action of $\mathbf{GRT}(\mathbf{k})$ over $\mathbf{Ass}(\mathbf{k})$ coincides with the action of $\mathrm{GRT}(\mathbf{k})$ over $\mathrm{Ass}(\mathbf{k})$ via the correspondence of Theorem 2.6.17. Thus, both actions are also compatible. $\square$

Chapter 3

Results

The contributions below focus on questions related to the higher genus and the twisted elliptic avatars of the V. Drinfeld's story of KZ equations, associators and the group **GT**.

One the one hand, in Part I we make use of the theory of the Fulton-MacPherson compactification, combined with operads, moperads ([103]) and operadic modules ([46]) to describe in a conceptual manner twisted and higher genus versions of associators, Grothendieck-Teichmüller groups and their graded versions.

On the other hand, in part II we focus on the twisted elliptic case to show the existence of a so-called twisted elliptic $\mathbb{C}$-associator arising from a flat universal KZB connection defined on a principal bundle over the moduli space of elliptic curves with a level structure. The theory of such a connection has immediate applications as to establishing the formality of some subgroups of the pure braid group on the torus and producing representations of Cherednik algebras. Analogously to the elliptic case, the coefficients of the generating series of the twisted elliptic KZB associator will then be called twisted elliptic multiple zeta values (teMZVs for short).

3.1 Operadic structures on associators and Grothendieck-Teichmüller groups

As said, the set of $\mathbf{k}$-associators is in a one-to-one correspondence with the set of isomorphisms $\widehat{\mathbf{PaB}}(\mathbf{k}) \longrightarrow G\mathbf{PaCD}$ of operads in $\mathbf{k}$-prounipotent groupoids which are the identity on objects. More generally, to any orientable compact surface Σ_g of genus $g \geq 2$, one can associate a (framed) configuration space of n points on Σ_g from which to obtain arbitrary genus definitions of Grothendieck-Teichmüller groups and associators. More specifically, one can consider the operad $\mathbf{PaB}_g$ of genus g parenthesized braidings associated to the fundamental groupoids of the Fulton-MacPherson compactified (framed) configuration spaces $\mathrm{Conf}(\Sigma_g, n)$ of Σ_g, based on the collection of sets of parenthesized permutations. Next, the "holonomy" Lie algebra $\mathfrak{t}_{g,n}$ of $\mathrm{Conf}(\Sigma_g, n)$ became available ([33]) and can be naturally endowed with the structure of a $\mathfrak{t}$-module. Then, a version of this Lie algebra (taking into account the framing of the configuration spaces) will permit us to define a $G\mathbf{PaCD}$-module $G\mathbf{PaCD}_g$ of genus g parenthesized chord diagrams. The genus g Grothendieck-Teichmüller group $\mathbf{GT}_g$ will consist of group of automorphisms of the $\mathbf{PaB}$-module $\mathbf{PaB}_g$, the genus g graded Grothedieck-Teichmüller group $\mathbf{GRT}_g$ will consist of group of automorphisms of the $G\mathbf{PaCD}$-module $G\mathbf{PaCD}_g$ and the set Ass_g of genus g associators will consist of the isomorphisms of modules $\widehat{\mathbf{PaB}}_g(\mathbf{k}) \longrightarrow G\mathbf{PaCD}_g$ which are the identity on objects. The main result of these constructions is that, seen as a $\mathbf{PaB}$-module, $\mathbf{PaB}_g$ has a nice presentation and we extract from it a characterisation of the set Ass_g of genus g associators in terms of elements satisfying some equations in Theorem 5.3.13.

Further results are obtained in the elliptic, cyclotomic and twisted elliptic cases. In [25], we give yet a new version of these operadic point of view on associators by taking a purely topological point of view. Starting with the (reduced) twisted configuration spaces of the complex cylinder and the torus, denoted respectively $\mathrm{Conf}(\mathbb{C}^\times, n, N)$ and $\mathrm{Conf}(\mathbb{T}, n, \Gamma)$, for $M, N \geq 1$ and $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, one can construct the Fulton-MacPherson compactification of these spaces. Then by considering, for all $n \geq 1$, the collection of their fundamental groupoids based on well chosen versions of collections of parenthesized permutations, they will be endowed with a $\mathbf{PaB}$-moperad (see [103] for the definition of a moperad over an operad) and a $\mathbf{PaB}$-operadic module structure respectively, denoted $\mathbf{PaB}^N$ and $\mathbf{PaB}^\Gamma_{e\ell\ell}$. Both $\mathbf{PaB}^N$ and $\mathbf{PaB}^\Gamma_{e\ell\ell}$ have nice presentations by generators and relations. Similarly to the genus 1 case, one can construct from the Lie algebras $\mathfrak{t}_{n,N}$ and $\mathfrak{t}^\Gamma_{1,n}$, a $G\mathbf{PaCD}$-moperad and a $G\mathbf{PaCD}$-module denoted $G\mathbf{PaCD}^N$ and $G\mathbf{PaCD}^\Gamma_{e\ell\ell}$ respectively. Then Grothendieck-Teichmüller groups and associators in this scope will be constructed as above[1]. We eventually get the following theorem.

Theorem 3.1.1. *The following maps are bitorsor isomorphisms*

$$(\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{e\ell\ell}(\mathbf{k})) \longrightarrow (\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k}), \mathrm{Ell}(\mathbf{k}), \mathrm{GRT}_{e\ell\ell}(\mathbf{k})) \tag{3.1}$$

$$(\widehat{\mathbf{GT}}^\Gamma(\mathbf{k}), \mathbf{Ass}^\Gamma(\mathbf{k}), \mathbf{GRT}^\Gamma(\mathbf{k})) \longrightarrow (\widehat{\mathrm{GT}}^\Gamma(\mathbf{k}), \mathrm{Ass}^\Gamma(\mathbf{k}), \mathrm{GRT}^\Gamma(\mathbf{k})). \tag{3.2}$$

Moreover, there is a torsor $(\widehat{\mathbf{GT}}^\Gamma_{e\ell\ell}(\mathbf{k}), \mathbf{Ell}^\Gamma(\mathbf{k}), \mathbf{GRT}^\Gamma_{e\ell\ell}(\mathbf{k}))$ *which allows us to define twisted elliptic counterparts* $\widehat{\mathrm{GT}}^\Gamma_{e\ell\ell}(\mathbf{k})$*,* $\mathrm{Ell}^\Gamma(\mathbf{k})$*, and* $\mathrm{GRT}^\Gamma_{e\ell\ell}(\mathbf{k})$ *of Grothendieck-Teichmüller groups and associators in their non-operadic characterization.*

3.2 The twisted elliptic KZB associator

We define a twisted version of the genus one KZB connection introduced in [24]. This is a flat connection on a principal bundle over the moduli space of elliptic curves with a level structure and n marked points.

Consider the group $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. and consider the following (finite index) subgroup of $\mathrm{SL}_2(\mathbb{Z})$:

[1] Let us remark that a very interesting continuation of the exploration of these operadic structures should be to adapt Fresse's model category structures to operadic modules to give a homotopical characterisation of $\mathbf{GT}_{e\ell\ell}(\mathbb{Q})$ in terms of homotopy automorphisms associated to little disks on the torus.

$$\mathrm{SL}_2^{\Gamma}(\mathbb{Z}) := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a \equiv 1 \bmod M,\, d \equiv 1 \bmod N,\, b \equiv 0 \bmod N,\, c \equiv 0 \bmod M \Big\}.$$

The quotient $Y(\Gamma) := \mathfrak{H}/\mathrm{SL}_2^{\Gamma}(\mathbb{Z})$ is a complex orbifold whose points classify isomorphism classes of pairs (E, ϕ) where E is an elliptic curve and $\phi : \Gamma \longrightarrow E$ is an injective group morphism that is orientation preserving. Such an elliptic curve with additional structure will be called Γ-structured elliptic curve. More generally, one can construct the *moduli space* $\mathcal{M}_{1,n}^{\Gamma}$ *of* Γ*-structured elliptic curves with* n *ordered marked points.*

Let E be an elliptic curve over $\mathbb{C}$ and consider the connected unramified Γ-covering $p : \tilde{E} \longrightarrow E$ corresponding to the canonical surjective group morphism $\rho : \pi_1(E) = \mathbb{Z}^2 \longrightarrow \Gamma$ sending the generators of $\mathbb{Z}^2$ to their respective classes in Γ. By choosing an uniformization of E, we define the Γ-twisted configuration space associated to $\tilde{E}$ as

$$\mathrm{Conf}(E, n, \Gamma) = (\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma})/(\mathbb{Z} + \tau\mathbb{Z})^n$$

where $\mathrm{Diag}_{\tau,n,\Gamma} := \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_{ij} := z_i - z_j \in (1/M)\mathbb{Z} + (\tau/N)\mathbb{Z}$ for some $i \neq j\}$. Then, the spaces $\mathrm{Conf}(E, n, \Gamma)$ are (roughly) fibers at τ of fibrations $\mathcal{M}_{1,n}^{\Gamma} \longrightarrow Y(\Gamma)$. The holonomy Lie algebra of $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ will be denoted $\mathfrak{t}_{1,n}^{\Gamma}$ and has generators $x_1, \dots, x_n$, $y_1, \dots, y_n$ and t_{ij}^{α} ($\alpha \in \Gamma$, $1 \leq i \neq j \leq n$).

As in the elliptic case, one can define a Lie algebra $\mathfrak{d}^{\Gamma}$, which has two components: the first is $\mathfrak{sl}_2$ and the second is a free (bigraded) Lie algebra $\mathfrak{d}_+^{\Gamma}$ generated by $\delta_{s,\gamma}$'s ($s \geq 0$, $\gamma \in \Gamma$) with relations $\delta_{s,\gamma} = (-1)^s \delta_{s,-\gamma}$. The $\delta_{s,\gamma}$ also act as highest weight elements for $\mathfrak{sl}_2$.

Proposition 3.2.1. *We have a bigraded Lie algebra morphism* $\rho : \mathfrak{d}^{\Gamma} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$.

We can then construct a group $\mathbf{G}_n^{\Gamma}$ whose Lie algebra has as components the holonomy Lie algebra $\mathfrak{t}_{1,n}^{\Gamma}$ and the so called twisted derivation algebra $\mathfrak{d}^{\Gamma}$.

Let e, h and f form the standard basis of $\mathfrak{sl}_2$ and write $\xi_{s,\gamma} := \rho(\delta_{s,\gamma})$ and $\Delta_0 := \rho(f)$. Let $\tilde{\gamma} = (c_0, c) \in (1/M)\mathbb{Z} + (\tau/N)\mathbb{Z}$ be any lift of $\gamma \in \Gamma$ and consider an element $[(\mathbf{z}, \tau)] \in \mathcal{M}_{1,n}^{\Gamma}$.

Theorem 3.2.2. *There is a unique* $\mathbf{G}_n^{\Gamma}$*-bundle* $P_{n,\Gamma}$ *over* $\mathcal{M}_{1,n}^{\Gamma}$ *(given locally by sections) with a flat universal KZB connection, locally defined by*

$$\nabla_{n,\Gamma}^{\mathrm{KZB}} := d - \Delta(\mathbf{z}|\tau)d\tau - \sum_{i=1}^{n} K_i(\mathbf{z}|\tau)dz_i$$

where, for $1 \leq i \leq n$*, we have*

$$K_i(\mathbf{z}|\tau) := -y_i + \sum_{j : j \neq i} \sum_{\gamma \in \Gamma} k_{\gamma}(\mathrm{ad}x_i, z|\tau)(t_{ij}^{\gamma})$$

with $k_{\gamma}(x, z|\tau) := e^{-2\pi \mathrm{i} cx} \frac{\theta(z - \tilde{\gamma} + x|\tau)}{\theta(z - \tilde{\gamma}|\tau)\theta(x|\tau)} - \frac{1}{x}$*, and*

$$\Delta(\mathbf{z}|\tau) := -\frac{1}{2\pi \mathrm{i}} \Big(\Delta_0 + \sum_{s \geq 0, \gamma \in \Gamma} \frac{1}{2} A_{s,\gamma}(\tau)\xi_{s,\gamma} - \sum_{i<j} \partial_x k_{\gamma}(\mathrm{ad}x_i, z|\tau)(t_{ij}^{\gamma}) \Big),$$

with $\partial_x k_{-\gamma}(x, 0|\tau) := \sum_{s \geq 0} A_{s,\gamma}(\tau) x^s$.

Some facts about the construction of the connection in the above theorem:

1. The connection $\nabla^{\mathrm{KZB}}_{n,\Gamma}$ restricts to a flat connection $\nabla^{\mathrm{KZB}}_{n,\tau,\Gamma} := d - \sum_i K_i(\mathbf{z}|\tau)dz_i$ locally defined on a principal $\exp(\hat{\mathfrak{t}}^{\Gamma}_{1,n})$-bundle $P_{\tau,n,\Gamma}$ over $\mathrm{Conf}(E, n, \Gamma)$. This will allow us to establish the formality of the fundamental group of $\mathrm{Conf}(E, n, \Gamma)$.
2. One can see that the term $\partial_x k_0(x, 0|\tau) = (\theta'/\theta)'(x) + 1/x^2$ permits to retrieve classical Eisenstein series and that for any $\gamma \in \Gamma - \{0\}$, the expansion of the term $\partial_x k_{-\gamma}(x, 0|\tau)$ will also be given in terms of (a slightly different version of) Eisenstein series.
3. The universal twisted elliptic KZB connection realizes as the usual KZB connection associated to elliptic dynamical r-matrices with spectral parameter [42, 44] and produces representations of Cherednik algebras related with cyclotomic double affine Hecke algebras ([16]).

Let $\bar{\mathfrak{t}}^{\Gamma}_{1,2}$ be the Lie $\mathbb{C}$-algebra generated by x, y and t^{α}, for $\alpha \in \Gamma$, such that $[x, y] = \sum_{\alpha\in\Gamma} t^{\alpha}$. We define the twisted elliptic KZB associator as the couple $e^{\Gamma}(\tau) := (A^{\Gamma}(\tau), B^{\Gamma}(\tau)) \in \exp(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2}) \times \exp(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})$ consisting in the renormalized holonomies from 0 to $1/M$ and 0 to τ/N respectively as paths in $E - \{\text{torsion points}\}$, of the differential equation

$$J'(z) = F^{\Gamma}(z) \cdot J(z) \text{ for } F^{\Gamma}(z) := -\sum_{\alpha\in\Gamma} e^{-2\pi \mathrm{i} a x} \frac{\theta(z - \tilde{\alpha} + ad(x)|\tau)}{\theta(z - \tilde{\alpha}|\tau)\theta(ad(x)|\tau)}(t^{\alpha}). \tag{3.3}$$

with values in the group $\exp(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})$, where $\tilde{\alpha} = (a_0, a) \in (1/M)\mathbb{Z} + (\tau/N)\mathbb{Z}$ is a lift of $\alpha \in \Gamma$. In [25], after giving a general definition of the set $\mathrm{Ell}^{\Gamma}(\mathbf{k})$ of twisted elliptic $\mathbf{k}$-associators (with the use of the theory of operads, see below), we show the following result:

Theorem 3.2.3. *Let* $\mathrm{Ell}^{\Gamma}_{\mathrm{KZB}} := \mathrm{Ell}^{\Gamma}(\mathbb{C}) \times_{\mathrm{Ass}(\mathbb{C})} \{2\pi\mathrm{i}, \Phi_{\mathrm{KZ}}\}$. *There is an analytic map*

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathrm{Ell}^{\Gamma}_{\mathrm{KZB}} \\ \tau & \longmapsto & e^{\Gamma}(\tau) \end{array}.$$

This means that, for each $\tau \in \mathfrak{h}$, *the element* $(2\pi\mathrm{i}, \Phi_{\mathrm{KZ}}, A^{\Gamma}(\tau), B^{\Gamma}(\tau))$ *is a twisted elliptic* $\mathbb{C}$-*associator.*

As a consequence, the set $\mathbf{Ell}^{\Gamma}(\mathbb{C})$ is non-empty and there is an action of the twisted version $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ of the elliptic prounipotent Grothendieck-Teichmüller group on it. Finally, we establish a differential equation in the direction of τ for the ellipsitomic KZB associators. Namely, if we denote $\bar{\bar{\xi}}^{(2)}_{s,\gamma}$ for the derivation given by

- $\bar{\bar{\xi}}^{(2)}_{s,\gamma}(x) = -(\operatorname{ad} x)^{s+1}(t^{-\gamma}) + (-\operatorname{ad} x)^{s+1}(t^{\gamma})$,
- $\bar{\bar{\xi}}^{(2)}_{s,\gamma}(t^{\alpha}) = [-((\operatorname{ad} x)^{s} t^{\alpha-\gamma} + (-\operatorname{ad} x)^{s} t^{\alpha+\gamma}) + (\operatorname{ad} x)^{s} t^{-\gamma} + (-\operatorname{ad} x)^{s} t^{\gamma}, t^{\alpha}]$,

then we have the following result.

Theorem 3.2.4. *We have*

$$2\pi\mathrm{i}\frac{\partial}{\partial\tau} A^{\Gamma}(\tau) = \left(-\Delta_0 - \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{s\geqslant 0} A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma} \right) A^{\Gamma}(\tau),$$

$$2\pi\mathrm{i}\frac{\partial}{\partial\tau}B^{\Gamma}(\tau)=\left(-\Delta_0-\frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{s\geqslant 0}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma}\right)B^{\Gamma}(\tau).$$

Notice that this differential equation only involves the Eisenstein-Hurwitz series that we defined in Section 8.3.

3.3 Perspectives

This section presents an overview of the possible continuations of the results of this thesis.

The first goal is to pursue the study of the general theory of twisted elliptic associators and elliptic multiple zeta values at torsion points. Two complementary directions of this goal are detailed in a separate manner. The first one involves a complete study of the (prounipotent) twisted elliptic Grothendieck-Teichmüller group, its graded version and their actions on the set of twisted elliptic associators. The second consists of a full study of the coefficients arising from the twisted elliptic KZB associator, namely what we call twisted elliptic MZVs (teMZVs in short).

The second goal is to study the rational homotopy of operadic **PaB**-modules and elliptic Grothendieck-Teichmüller groups.

3.3.1 Twisted elliptic (graded) Grothendieck-Teichmüller groups

In [25] we mainly expressed twisted (graded) Grothendieck-Teichmüller groups and associators in their operadic versions (we also gave definitions of these objects in terms of elements satisfying some equations). Nevertheless, one needs to understand the intrinsic nature of these two groups and this set in order to study for example the decomposition of twisted elliptic MZVs. Indeed, as we will see in chapter 8 where we establish the differential equation satisfied by the twisted elliptic KZB associator, one needs to isolate some components of the twisted elliptic Grothendieck-Teichmüller group and have an explicit formula for the action of this group on the set of twisted elliptic $\mathbf{k}$-associators.

The action of the twisted elliptic Grothendieck-Teichmüller group $\mathbf{GT}^{\Gamma}_{e\ell\ell}$ and its graded version on $\mathrm{Ell}^{\Gamma}(\mathbf{k})$. Based on the definition of $\mathbf{GT}^{\Gamma}_{e\ell\ell}$ and its profinite, pro-ℓ and proalgebraic variants, defined by considering different versions of the **PaB**-module $\mathbf{PaB}^{\Gamma}_{e\ell\ell}$, we study the relations between these groups and their corresponding versions in the genus 0, cyclotomic and elliptic cases. In the proalgebraic case, we aim to obtain a semidirect product structure for $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$, analog to that obtained in the elliptic case. We will then fully describe the action of this group on twisted elliptic $\mathbf{k}$-associators. We hope to construct a morphism of torsors from the scheme of (cyclotomic) associators to its twisted elliptic analogue, which will permit us to establish the existence of twisted elliptic associators at extensions of $\mathbb{Q}$ by roots of unity. Next, we concentrate on the graded version $\mathbf{GRT}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ of the twisted elliptic Grothendieck-Teichmüller group. In particular, we will aim to establish the existence of the prounipotent radical $\mathrm{R}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ of $\mathbf{GRT}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ whose associated Lie algebra should be isomorphic to the twisted version of the special derivation algebra which will be constructed in chapter 8 from the definition of the twisted derivation algebra $\mathfrak{d}^{\Gamma}$ constructed in chapter 6. Special attention will be taken on the relation between this Lie algebra and the Lie algebra of the prounipotent radical of $\pi_1^{geom}(\mathrm{MEM})$.

Further investigations on the twisted elliptic KZB associator. Once we haveexplicitely constructed the action of $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ and $\mathbf{GRT}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}^{\Gamma}(\mathbf{k})$, we will be able to fully establish the differential equation for the twisted elliptic KZB associator in terms of the Eisenstein-Hurwitz series found in chapter 7. Next, combined with a full study of the genus, cusps (by using the Riemann-Hurwitz theorem) and mapping class group of the moduli space of once punctured Γ-structured elliptic curves for different choices of finite abelian groups Γ, we should be able to study the modular properties and asymptotic behaviour of the twisted elliptic KZB associator at all cusps of this moduli space. This will be of great importance when attacking the study of teMZVs as we will explain below.

Zariski closures, distribution relations and Galois groups actions for $\mathrm{Ell}^{\Gamma}(\mathbf{k})$. With a good understanding of the twisted elliptic mapping class group $\pi_1(\mathcal{M}^{\Gamma}_{1,n})$ at hand, we will aim to compute its Zariski closure in the automorphism groups of the prounipotent completions of some subgroups of the (pure) braid groups on the torus by studying the relation between the action of the group $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ on these prounipotent completions and the action of its graded counterpart. Next, if we take $\Gamma' = \mathbb{Z}/M'\mathbb{Z} \times \mathbb{Z}/N'\mathbb{Z}$ such that M' divides M and N' divides N, one should be able to study distribution relations satisfied by $\mathrm{Ell}^{\Gamma'}(\mathbf{k})$ and $\mathrm{Ell}^{\Gamma}(\mathbf{k})$ and show that, when imposing these distribution relations, one obtains a subset of twisted elliptic associators which will be a torsor under the action of some subgroups of $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ and $\mathbf{GRT}^{\Gamma}_{e\ell\ell}(\mathbf{k})$. Special importance will be given to study the relation between these subgroups and the (geometric) fundamental group of the once punctured Γ-structured elliptic curve. Finally, we sketch some relations between the twisted versions of Teichmüller groupoids in genus one, the arithmetic fundamental group $\pi_1((\mathcal{M}^{\Gamma}_{1,1})^L)$ (for different kinds of congruence subgroups and for L an extension of $\mathbb{Q}$ by roots of unity) and the profinite twisted elliptic Grothendieck-Teichmüller group $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}$.

3.3.2 Further investigations on elliptic MZVs at torsion points

The twisted elliptic KZB associator $e^{\Gamma}(\tau)$ has an expression in terms of iterated integrals. The twisted elliptic MZVs $I^{\Gamma}\begin{pmatrix} n_1 & n_2 & ,\dots, & n_r \\ \alpha_1 & \alpha_2 & ,\dots, & \alpha_r \end{pmatrix};\tau\Big)$ and $J^{\Gamma}\begin{pmatrix} n_1 & n_2 & ,\dots, & n_r \\ \alpha_1 & \alpha_2 & ,\dots, & \alpha_r \end{pmatrix};\tau\Big)$, for $n_1,\dots,n_r \geqslant 0$ and $\alpha_1,...,\alpha_r \in \Gamma$, are defined equivalently as the coefficients of the (modified) ellipsitomic KZB associators and as regularized iterated integrals of the function F^{Γ} defined above.

A first remark is that our approach to teMZVs is somewhat different to that in the recent work [19], where the authors use iterated integrals and the functions $F^{\Gamma}(z)$ to construct teMZVs and generalises to the case of any surjective morphism $\mathbb{Z}^2 \longrightarrow \Gamma$ sending the generators of $\mathbb{Z}^2$ to their respective classes modulo M and N.

Relations of teMZVs with the twisted special derivation algebra. In a joint effort with N. Matthes, we aim to investigate the relation of our teMZVs with those defined in ([19]) related to the non-planar part of the four-point one-loop open-string amplitude. In particular, by using the twisted version of H. Tsunogai's special derivation algebra, by relating it to the untwisted special derivation algbra, and by representing teMZVs as iterated integrals over well adapted Eisenstein series, we aim to derive the number of indecomposable elements of given weight and length for teMZVs. We also hope to get new interesting relations in the twisted special derivation algebra. Then, together with J. Broedel and O. Schlotterer, we will provide relations for teMZVs over a wide range of weights and lengths by computational methods.

Modularity properties and asymptotic behaviour of teMZVs. By combining the results on the asymtotic behaviour at cusps and the differential equation for the twisted elliptic KZB associator done in Project 1, we will deduce the asymptotic behaviour of teMZVs. We will aim to retrieve μ_N-MZVs and multiple Hurwitz values when degenerating teMZVs to the cusp $\mathrm{i}\infty$ and all other cusps of our modular curve. By the results in [19], we know this will be the case. We hope that by taking special cases of the group Γ, for instance $M = 5$ and different choices of N, we will retrieve some of the remaining periods of $\mathbb{P}^1 - \{0, \mu_5, \infty\}$ which are known not to be μ_5-MZVs.

Motivic aspects of the twisted elliptic KZB connection and teMZVs. In a broader sense, we aim to study some of the Hodge-de Rham theoretic aspects of $\mathcal{M}_{1,1}^{\Gamma}$. One can see $\mathcal{M}_{1,2}^{\Gamma}$ is the Γ-punctured universal curve over $\mathcal{M}_{1,1}^{\Gamma}$. The Lie algebra $\mathfrak{t}_{1,n}^{\Gamma}$ should be closely related to the local system over the moduli space of Γ-structured elliptic curves with a non-zero tangent vector at the origin. With this in mind, an interesting task to do is to explicit the $\mathbb{Q}$-de-Rham structure of this local system as was done in R. Hain's notes [63]. We aim to compute the restriction of the twisted elliptic KZB connection to various loci, such as the punctured first order neighbourhood of the Tate curve and a punctured formal neighbourhood of the identity section. We then explore Hodge theoretic aspects of this connection such as computing limit mixed Hodge structures relevant regions of $\mathcal{M}_{1,1}^{\Gamma}$. We hope to relate in the mid-term these constructions to motivic aspects of teMZVs and to universal mixed elliptic (and modular) motives.

3.3.3 Rational homotopy of operadic PaB-modules and elliptic Grothendieck-Teichmüller groups

Following the operadic point of view on elliptic associators and Grothendieck-Teichmüller groups, it is natural enough to study the homotopy aspects of these objets. The motivation to do this comes from the fact that, by Bezrukavnikov's results in [11], the configuration spaces $\mathrm{Conf}(\Sigma_g, n)$ of a genus g orientable surface Σ_g are 1-formal but not formal in general. In other words, they have non trivial higher homotopies. Now, for some years now, a way of studying higher homotopies on spaces has come with the introduction of higher categorical structures. The link between these two realms of mathematics has been straightened in particular by P. Safronov, who has studied in [91] the relation between shifted Poisson structures and classical (dynamical) r-matrixes. A natural quesiton to ask is then if a homotopical characterisation of $\mathbf{GT}_{e\ell\ell}$ will shed some light on the study of higher homotopies of the operadic module (over the little disks operad D_2) of little disks on the torus, denoted $D_{1,2}$.

A rationalization of the module of little disks on the torus. The first goal for achieving this study will consist on constructing a good rationalization of the module of little disks on the torus. First of all, as $\mathrm{Conf}(\mathbb{T}, n)$ is not formal (see [Bezr]), we have to work with the de Rham algebra $\Omega^*(\mathrm{Conf}(\mathbb{T}, n))$ instead of $H^*(\mathrm{Conf}(\mathbb{T}, n))$. We hope to be able to overcome this issue by stuying the de Rham algebra $\Omega^*(\mathrm{Conf}(\mathbb{T}, n))$ given in [20] and relating it with that contained in Kriz work [78] together with recent work by C. Sibilia in his PhD thesis. Let $C^*_{\mathrm{CE}}(\mathfrak{t}_{1,n})$ be the Chevalley-Eilenberg cochain complex of $\mathfrak{t}_{1,n}$. The first step is to obtain a quasi-isomorphism

$$C^*_{\mathrm{CE}}(\mathfrak{t}_{1,n}) \longrightarrow \Omega^*(\mathrm{Conf}(\mathbb{T}, n))$$

which would be enhanced into a Hopf dg-comodule quasi-isomorphism $C^*_{\mathrm{CE}}(\mathfrak{t}_1) \longrightarrow \Omega^*(\mathrm{Conf}(\mathbb{T}, -))$. This will lead to a rationalization of the module of little disks on the torus.

Homotopy theory of Hopf comodules. Next, it will be necessary to build a general homotopy theory for Hopf cooperadic comodules. Operadic modules are easier to work with than operads by their intrinsic linear nature (oposed as to that of operads). By this reason, the construction of model category structures on operadic modules in simplicial sets and their Λ-operadic versions should be within reach in the mid-term. The next step would be to use homotopy spectral sequences techniques in this scope to get a homotopical interpretation of $\mathbf{GT}_{e\ell\ell}$ in terms of the fundamental group (so in terms of the 1-truncation of the full homotopy theory) of $\mathrm{Conf}(\mathbb{T}, n)$. The final outcome of this study will then be constructing injective mappings

$$\mathrm{Ell}(\tau)_{\mathbb{Q}} \quad \longrightarrow \quad \mathrm{Iso}_{\mathrm{Ho}(\mathrm{Mod}_{D_2})}((D_{1,2})^{\wedge}_{\mathbb{Q}}, LG_{\bullet}\Omega^*(D_{1,2}))$$

and

$$(\mathrm{GT}_{ell})_{\mathbb{Q}} \quad \longrightarrow \quad \mathrm{Aut}_{\mathrm{Ho}(\mathrm{Mod}_{D_2})}((D_{1,2})^{\wedge}_{\mathbb{Q}}).$$

where $\mathrm{Ho}(\mathrm{Mod}_{D_2})$ is the homotopy category of D_2-modules, $(D_{1,2})^{\wedge}_{\mathbb{Q}}$ is a rationalization of the D_2-module $D_{1,2}$ related with Sullivan's models and $LG_{\bullet}\Omega^*(D_{1,2})$ is a module obtained from the de Rham complex of the D_2-module $D_{1,2}$.

Part I

Associators and Grothendieck-Teichmüller groups

Chapter 4

Operad structures on associators and Grothendieck-Teichmüller groups

4.1 Modules associated with configuration spaces (elliptic associators)

4.1.1 Compactified configuration space of the torus

Let $\mathbb{T}$ be the topological torus. To any finite set I we associate a configuration space

$$\mathrm{Conf}(\mathbb{T}, I) = \{\mathbf{z} = (z_i)_{i \in I} \in \mathbb{T}^I | z_i \neq z_j \text{ if } i \neq j\}.$$

We also consider its reduced version

$$\mathrm{C}(\mathbb{T}, I) := \mathrm{Conf}(\mathbb{T}, I)/\mathbb{T}.$$

We then consider the Fulton–MacPherson compactification $\overline{\mathrm{C}}(\mathbb{T}, I)$ of $\mathrm{C}(\mathbb{T}, I)$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{T}, I) = \overline{\mathrm{C}}(\mathbb{T}, I) - \mathrm{C}(\mathbb{T}, I)$ is made of the following irreducible components: for any partition $I = J_1 \coprod \cdots \coprod J_k$ there is a component

$$\partial_{J_1,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{T}, I) \cong \overline{\mathrm{C}}(\mathbb{T}, k) \times \prod_{i=1}^{k} \overline{\mathrm{C}}(\mathbb{C}, J_i).$$

The inclusion of boundary components provide $\overline{\mathrm{C}}(\mathbb{T}, -)$ with the structure of a module over the operad $\overline{\mathrm{C}}(\mathbb{C}, -)$ in topological spaces.

4.1.2 The PaB-module $\mathbf{PaB}_{e\ell\ell}$ of parenthesized elliptic (or beak) braids

In a similar manner as in §2.6.2, we have inclusions of topological modules[1]

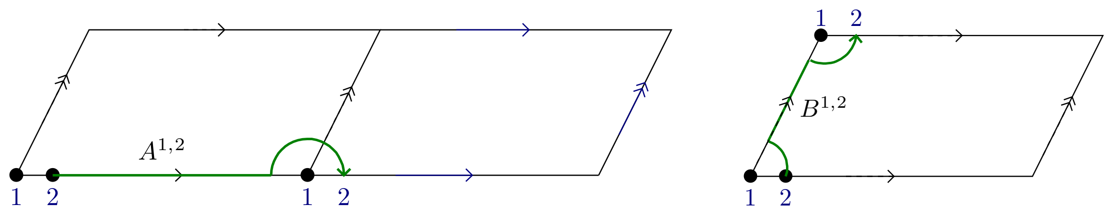
$$\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{S}^1, -) \subset \overline{\mathrm{C}}(\mathbb{T}, -)\,.$$

Then it makes sense to define

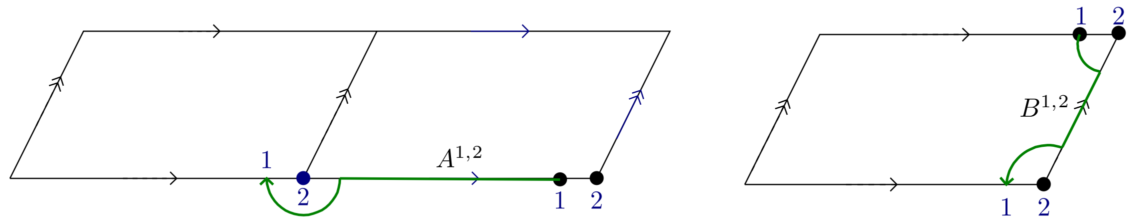
$$\mathbf{PaB}_{e\ell\ell} := \pi_1\left(\overline{\mathrm{C}}(\mathbb{T}, -), \mathbf{Pa}\right)\,,$$

which is a **PaB**-module in groupoids.

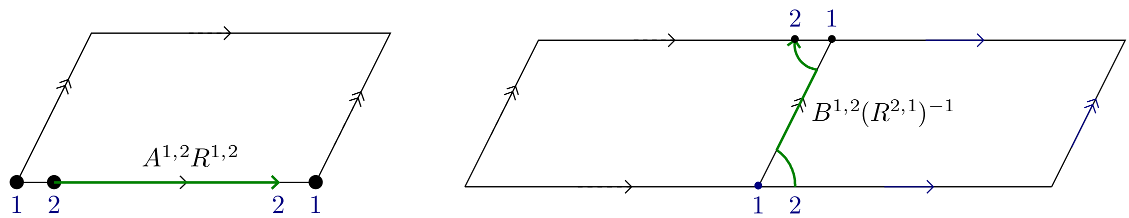
Example 4.1.1 (Structure of $\mathbf{PaB}_{e\ell\ell}(2)$)**.** *As in Example 2.6.1 we have an arrow $R^{1,2}$ going from (12) to (21). Additionnally, we also have two automorphisms of (12), denoted $A^{1,2}$ and $B^{1,2}$, corresponding to the following loops on $\overline{\mathrm{C}}(\mathbb{T}, 2)$:*



By global translation of the torus, these are the same loops as the following



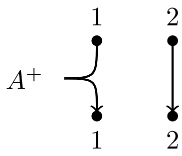
In particular, $A^{1,2}R^{1,2}$ and $B^{1,2}(R^{2,1})^{-1}$, which are morphisms from (12) to (21), correspond to the following paths $\overline{\mathrm{C}}(\mathbb{T}, 2)$:

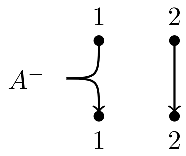


Remark 4.1.2. *The arrows $A^{1,2}$ and $B^{1,2}$ correspond to $A^{\pm}_{1,2}$ in [34, §1.3].*

Thus as $A^{1,2}$ can be depicted with the point indexed by 1 going to the left we will also formally depict $A^{1,2}$ and $B^{1,2}$ as follows:

[1] The second one depends on the choice of an embedding $\mathbb{S}^1 \hookrightarrow \mathbb{T}$: we choose by convention the "horinzontal" one.





One can rephrase [34, Proposition 1.3] in the following way:

Theorem 4.1.3. *As a* **PaB***-module in groupoids having* **Pa** *as* **Pa***-module of objects,* $\mathbf{PaB}_{e\ell\ell}$ *is freely generated by* $A := A^{1,2}$ *and* $B := B^{1,2}$*, together with the following relations:*

(N1) $\Phi^{1,2,3}A^{1,23}R^{1,23}\Phi^{2,3,1}A^{2,31}R^{2,31}\Phi^{3,1,2}A^{3,12}R^{3,12} = \mathrm{Id}_{(12)3}$,

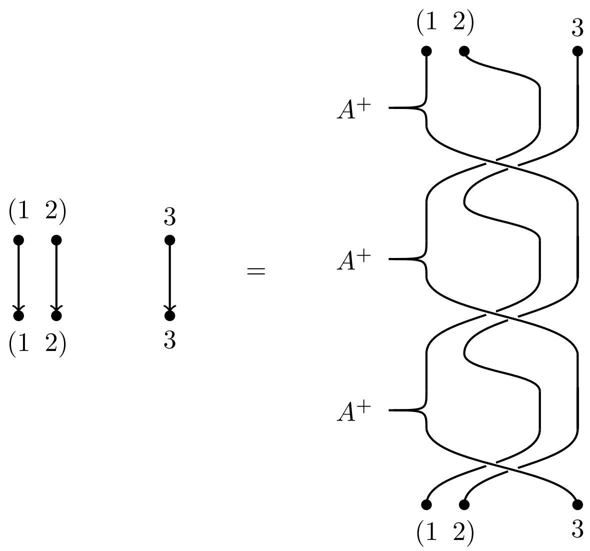
(N2) $\Phi^{1,2,3}B^{1,23}(R^{23,1})^{-1}\Phi^{2,3,1}B^{2,31}(R^{31,2})^{-1}\Phi^{3,1,2}B^{3,12}(R^{12,3})^{-1} = \mathrm{Id}_{(12)3}$,

(E) $R^{1,2}R^{2,1} = \big(\Phi^{1,2,3}B^{1,23}(\Phi^{1,2,3})^{-1}, (R^{2,1})^{-1}\Phi^{2,1,3}(A^{2,13})^{-1}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}\big)$,

as automorphisms of $(12)3$ *in* $\mathbf{PaB}_{e\ell\ell}(3)$.

Proof. Let $\mathcal{Q}$ be the **PaB**-module with the above presentation. We first show that there is a morphism of **PaB**-modules $\mathcal{Q} \longrightarrow \mathbf{PaB}_{e\ell\ell}$. We have already seen that there are two automorphisms A, B of (12) in $\mathbf{PaB}_{e\ell\ell}(2)$ (see Example 4.1.1). We have to prove that they indeed satisfy the relations (N1), (N2) and (E).

Relations (N1) and (N2) are satistfied: the first *nonagon relation* (N1) can be depicted as follows:

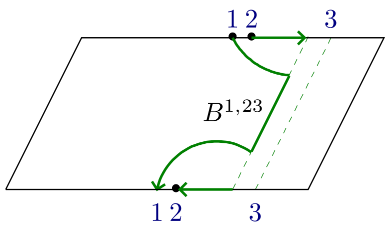


(N1)

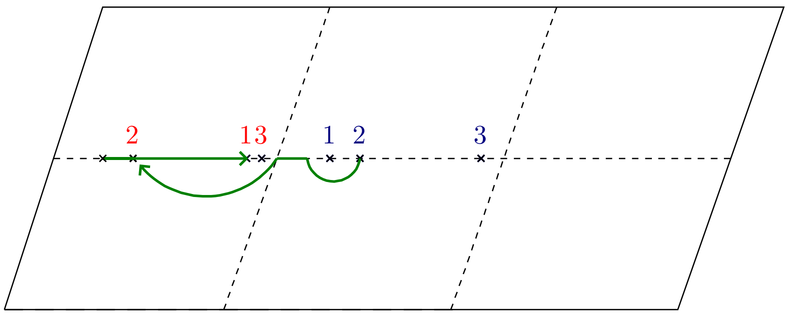
It is satisfied in $\mathbf{PaB}_{e\ell\ell}$, expressing the fact that when all (here, three) points move in the same direction on the torus, this corresponds to a constant path in the reduced configuration space of points on the torus. The same is true with the second nonagon relation (N2).

Relation (E) is satisfied: below one sees the path that is obtained from the right-hand-side of the *mixed relation* (E):

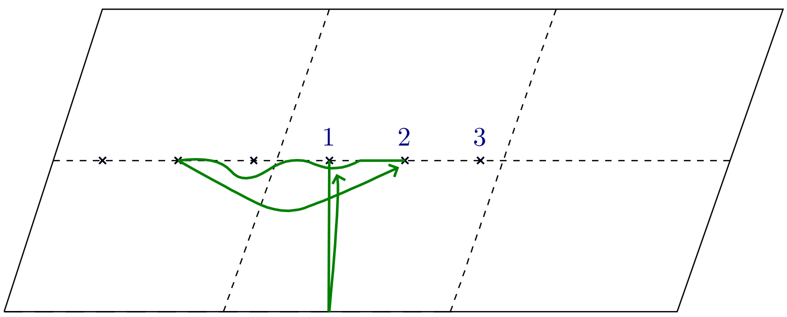
- $\Phi^{1,2,3}B^{1,23}(\Phi^{1,2,3})^{-1}$ is the path



- $(R^{2,1})^{-1}\Phi^{2,1,3}(A^{2,13})^{-1}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}$ is the path



One easily sees on the picture that the path is homotopic to the pure braiding of the first two points, that is $R^{1,2}R^{2,1}$, by means of the following picture



Thus, by the universal property of $\mathcal{Q}$, there is a morphism of **PaB**-modules $\mathcal{Q} \longrightarrow \mathbf{PaB}_{e\ell\ell}$, which is the identity on objects. To show that this map is in fact an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of objects arity-wise, as all groupoids are connected. Let $n \geq 0$, and p be the object $(\cdots((12)3)\cdots\cdots)n$ of $\mathcal{Q}(n)$ and $\mathbf{PaB}_{e\ell\ell}(n)$. We want to show that the induced morphism

$$\mathrm{Aut}_{\mathcal{Q}(n)}(p) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}_{e\ell\ell}(n)}(p) = \pi_1\left(\overline{\mathrm{C}}(\mathbb{T}, n), p\right)$$

is an isomorphism.

On the one hand, as $\bar{\mathrm{C}}(\mathbb{T},n)$ is a manifold with corners, we are allowed to move the basepoint p to a point p_{reg} which is included in the simply connected subset obtained as the image of[2]

$$D_{n,\tau} := \{\mathbf{z}\in\mathbb{C}^n | z_j = a_j+b_j\tau, a_j, b_j\in\mathbb{R}, 0<a_1<a_2<...<a_n<a_1+1, 0<b_1<b_2<...<b_n<b_1+1\}$$

in $\mathrm{C}(\mathbb{T},n)$, where $\mathbb{T}=\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. We then have an isomorphism of fundamental groups $\pi_1(\bar{\mathrm{C}}(\mathbb{T},n),p)\simeq\pi_1(\mathrm{C}(\mathbb{T},n),p_{reg})$.

On the other hand, in [34, Proposition 1.4], Enriquez constructs a universal elliptic structure $\mathbf{PaB}^{En}_{e\ell\ell}$, that by definition carries an action of the (algebraic version of the) reduced braid group on the torus $\overline{\mathrm{B}}_{1,n}$ in the following sense:

- $\mathbf{PaB}^{En}_{e\ell\ell}$ is a category.
- for every object p of $\mathbf{Pa}(n)$, there is a corresponding object $[p]$ in $\mathbf{PaB}^{En}_{e\ell\ell}$, and $[p]=[q]$ if p and q only differ by a permutation (but have the same underlying parenthesization).
- there are group morphisms $\overline{\mathrm{B}}_{1,n}\xrightarrow{\sim}\mathrm{Aut}_{\mathbf{PaB}^{En}_{e\ell\ell}}(p)\longrightarrow\mathfrak{S}_n$.

Moreover, one has by constuction of $\mathbf{PaB}^{En}_{e\ell\ell}$ that $\mathrm{Aut}_{\mathcal{Q}(n)}(p)$ is the kernel of the map $\mathrm{Aut}_{\mathbf{PaB}^{En}_{e\ell\ell}}([p])\longrightarrow\mathfrak{S}_n$. One can actually show that we have a commuting diagram

$$\begin{tikzcd}
\overline{\mathrm{PB}}_{1,n} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Aut}_{Q(n)}(p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{T},n),p\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{T},n),p_{reg}\right) \arrow[l, "\simeq"'] \arrow[d] \\
\overline{\mathrm{B}}_{1,n} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Aut}_{\mathbf{PaB}^{En}_{e\ell\ell}}(p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{T},n)/\mathfrak{S}_n,[p]\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{T},n)/\mathfrak{S}_n,[p_{reg}]\right) \arrow[l, "\simeq"'] \arrow[d] \\
\mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n
\end{tikzcd}$$

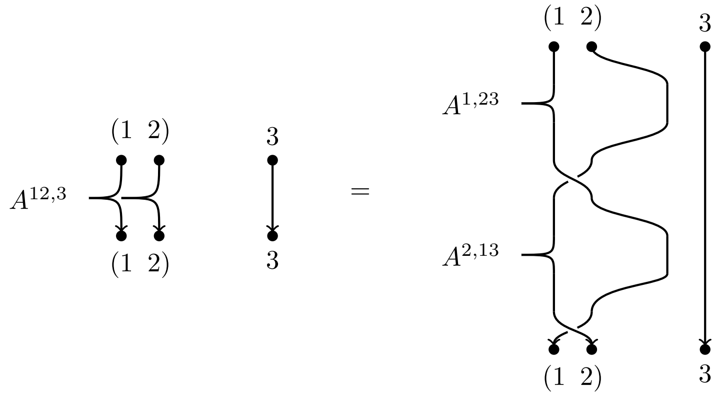
where all vertical sequences are short exact sequences. Thus, in order to show that the map $\mathrm{Aut}_{\mathcal{Q}(n)}(p)\longrightarrow\pi_1\left(\overline{\mathrm{C}}(\mathbb{T},n),p\right)$ is an isomorphism, we are left to show that

$$\overline{\mathrm{B}}_{1,n}\longrightarrow\pi_1(\mathrm{C}(\mathbb{T},n),p_{reg})$$

is indeed an isomorphism. But this map is nothing else than the map constructed in [12, Theorem 5], identifying the algebraic and topological versions of the braid group on the torus. □

Remark 4.1.4. *It is probably best to picture the nonagon relation by means of the following relation (this is relation 25 in [24]), which is equivalent to (N1), and that expresses a kind of ribbon description for* $A^{12,3}$*:*

[2] We have already done so for theproof of relation (E).



(N1bis)

4.1.3 The CD(k)-module of elliptic chord diagrams

For any $n \geq 0$, recall that $\mathfrak{t}_{1,n}(\mathbf{k})$ is defined as the bigraded Lie $\mathbf{k}$-algebra freely generated by $x_1, \ldots, x_n$ in degree $(1,0)$, $y_1, \ldots, y_n$ in degree $(0,1)$ (for $i = 1, ..., n$), and t_{ij} in degree $(1,1)$ (for $1 \leq i \neq j \leq n$), together with the relations (S), (L), (4T), and the following additional elliptic relations as well:

($\mathrm{S}_{e\ell\ell}$) $[x_i, y_j] = t_{ij}$ for $i \neq j$.

($\mathrm{N}_{e\ell\ell}$) $[x_i, x_j] = [y_i, y_j] = 0$ for $i \neq j$.

($\mathrm{T}_{e\ell\ell}$) $[x_i, y_i] = -\sum_{j|j\neq i} t_{ij}$.

($\mathrm{L}_{e\ell\ell}$) $[x_i, t_{jk}] = [y_i, t_{jk}] = 0$ if $\#\{i,j,k\} = 3$.

($4\mathrm{T}_{e\ell\ell}$) $[x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0$ for $i \neq j$.

The $\sum_i x_i$ and $\sum_i y_i$ are central in $\mathfrak{t}_{1,n}(\mathbf{k})$, and we also consider the quotient

$$\bar{\mathfrak{t}}_{1,n}(\mathbf{k}) := \mathfrak{t}_{1,n}(\mathbf{k})/(\sum_i x_i, \sum_i y_i)\,.$$

Example 4.1.5. $\bar{\mathfrak{t}}_{1,2}(\mathbf{k})$ *is equal to the free Lie* $\mathbf{k}$*-algebra* $\mathfrak{f}_2(\mathbf{k})$ *on two generators* $x = x_1$ *and* $y = y_2$.

Both $\mathfrak{t}_{1,n}$ and $\bar{\mathfrak{t}}_{1,n}$ are acted on by the symmetric group $\mathfrak{S}_n$, and one can show that the $\mathfrak{S}$-modules in $grLie_{\mathbf{k}}$

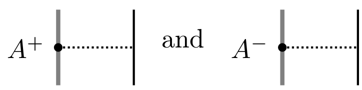
$$\mathfrak{t}_{e\ell\ell}(\mathbf{k}) := \{\mathfrak{t}_{1,n}(\mathbf{k})\}_{n\geq 0} \quad \text{and} \quad \bar{\mathfrak{t}}_{e\ell\ell}(\mathbf{k}) := \{\bar{\mathfrak{t}}_{1,n}(\mathbf{k})\}_{n\geq 0}$$

actually are $\mathfrak{t}(\mathbf{k})$-modules in $grLie_{\mathbf{k}}$. Partial compositions are defined as follows:

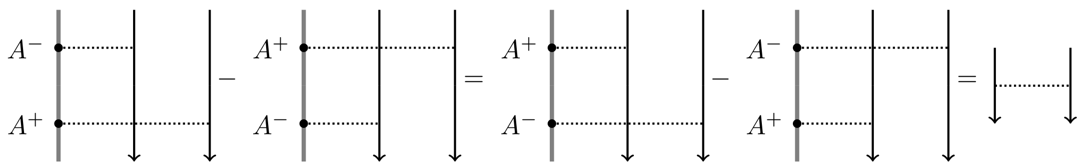
$$\begin{array}{rccc}
\circ_k : & \mathfrak{t}_{1,I}(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) & \longrightarrow & \mathfrak{t}_{1,J\sqcup I-\{i\}}(\mathbf{k}) \\
& (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha\beta} \\
& (t_{ij}, 0) & \longmapsto & \begin{cases} t_{ij} & \text{if} \quad k \notin \{i,j\} \\ \sum\limits_{p\in J} t_{pj} & \text{if} \quad k = i \\ \sum\limits_{p\in J} t_{ip} & \text{if} \quad j = k \end{cases} \\
& (x_i, 0) & \longmapsto & \begin{cases} x_i & \text{if} \quad k \neq i \\ \sum\limits_{p\in J} x_p & \text{if} \quad k = i \end{cases} \\
& (y_i, 0) & \longmapsto & \begin{cases} y_i & \text{if} \quad k \neq i \\ \sum\limits_{p\in J} y_p & \text{if} \quad k = i \end{cases}
\end{array}$$

We call $\mathfrak{t}_{e\ell\ell}(\mathbf{k})$, resp. $\bar{\mathfrak{t}}_{e\ell\ell}(\mathbf{k})$, the module of *infinitesimal elliptic braids*, resp. of *infinitesimal reduced elliptic braids*.

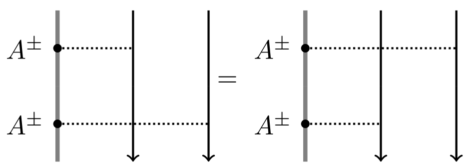
We finally define the $\mathbf{CD}(\mathbf{k})$-module $\mathbf{CD}_{e\ell\ell}(\mathbf{k}) := \hat{\mathcal{U}}(\bar{\mathfrak{t}}_{e\ell\ell}(\mathbf{k}))$ of *elliptic chord diagrams*. As in the genus 0 situation, morphisms in $\mathbf{CD}_{e\ell\ell}(\mathbf{k})(n)$ can be represented as chords on n vertical strands with extra chords correponding to the generators x_i and y_i as in the following picture:



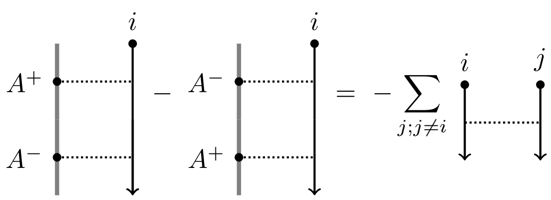
The relations elliptic relations introduced above can be represented as follows, analogously as for the genus 0 case:



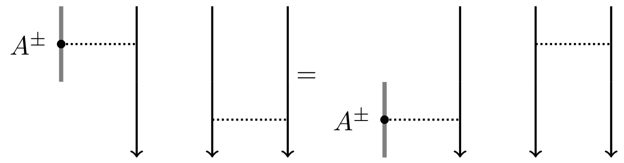
(S$_{e\ell\ell}$)



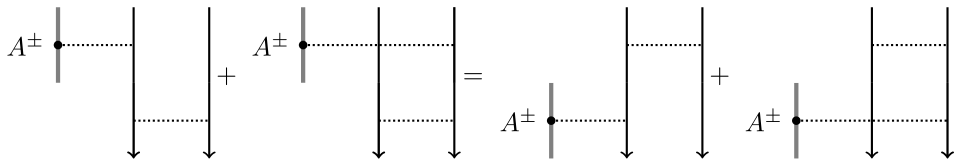
(N$_{e\ell\ell}$)



(T$_{e\ell\ell}$)



(L$_{e\ell\ell}$)



(4T$_{e\ell\ell}$)

Remark 4.1.6. *The relation between (a closely related version of)* $\mathbf{CD}_{e\ell\ell}(\mathbf{k})$ *and the elliptic Kontsevich integral was studied in Philippe Humbert's thesis [66].*

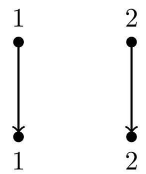
4.1.4 The PaCD(k)-module of parenthesized elliptic chord diagrams

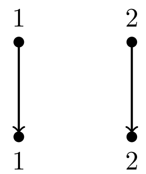
As in the genus zero case, the module of objects $\mathrm{Ob}(\mathbf{CD}_{e\ell\ell}(\mathbf{k}))$ of $\mathbf{CD}_{e\ell\ell}(\mathbf{k})$ is terminal. Hence we have a morphism of modules $\omega_2 : \mathbf{Pa} = \mathrm{Ob}(\mathbf{Pa}(\mathbf{k}) \longrightarrow \mathrm{Ob}(\mathbf{CD}_{e\ell\ell}(\mathbf{k}))$ over the morphism of operads ω_1 from §2.6.4, and thus we can define the $\mathbf{PaCD}(\mathbf{k})$-module[3]

$$\mathbf{PaCD}_{e\ell\ell}(\mathbf{k}) := \omega_2^{\star}\mathbf{CD}_{e\ell\ell}(\mathbf{k})\,,$$

in $\mathbf{Cat}(\mathbf{CoAss}_{\mathbf{k}})$, of so-called *parenthesized elliptic chord diagrams.*

Example 4.1.7 (Notable arrows in $\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})(2)$)**.** *We have the following arrows* $X^{1,2}_{e\ell\ell}$, $Y^{1,2}_{e\ell\ell}$ *in* $\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})(2)$

$X^{1,2}_{e\ell\ell} = x_1 \cdot$ 

$Y^{1,2}_{e\ell\ell} = y_1 \cdot$ 

Remark 4.1.8. *The elements* $X^{1,2}_{e\ell\ell}, Y^{1,2}_{e\ell\ell}$ *are generators of the* $\mathbf{PaCD}(\mathbf{k})$*-module* $\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})$ *and satisfy the following relations in* $\mathrm{End}_{\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})(3)}((12)3)$*:*

(Inv) $X^{2,1}_{e\ell\ell} = (X^{1,2})^{-1}X^{1,2}_{e\ell\ell}X^{1,2}$, $Y^{2,1}_{e\ell\ell} = (X^{1,2})^{-1}Y^{1,2}_{e\ell\ell}X^{1,2}$,

(Red) $X^{1,\emptyset}_{e\ell\ell} = Y^{1,\emptyset}_{e\ell\ell} = 0$,

(IN1) $X^{12,3}_{e\ell\ell} + a^{1,2,3}X^{1,23}X^{23,1}_{e\ell\ell}(a^{1,2,3}X^{1,23})^{-1} + X^{12,3}(a^{3,1,2})^{-1}X^{31,2}_{e\ell\ell}(X^{12,3}(a^{3,1,2})^{-1})^{-1} = 0$,

(IN2) $Y^{12,3}_{e\ell\ell} + a^{1,2,3}X^{1,23}Y^{23,1}_{e\ell\ell}(a^{1,2,3}X^{1,23})^{-1} + X^{12,3}(a^{3,1,2})^{-1}Y^{31,2}_{e\ell\ell}(X^{12,3}(a^{3,1,2})^{-1})^{-1} = 0$,

(IE) $H^{1,2} = [a^{1,2,3}X^{1,23}_{e\ell\ell}(a^{1,2,3})^{-1}, X^{1,2}a^{2,1,3}Y^{2,13}_{e\ell\ell}(a^{2,1,3})^{-1}(X^{1,2})^{-1}]$.

[3]Recall that $\mathbf{PaCD}(\mathbf{k})$ is defined as $\omega_1^{\star}\mathbf{CD}(\mathbf{k})$.

4.1.5 Elliptic associators

Let us introduce some terminology, complementing the one of §2.6.5. If $\mathcal{P} \longrightarrow \mathcal{Q}$ is a morphism between operads in $\mathcal{C}$, $\mathcal{M}$ is a module over $\mathcal{P}$, and $\mathcal{N}$ is a module over $\mathcal{Q}$, then we will consider operadic module mophisms $\mathcal{M} \longrightarrow \mathcal{N}$ in the category of $\mathcal{P}$-modules (*via* the restriction functor), and will simply refer to them as module morphisms if the context is clear.

For an operad $\mathcal{O}$ in $\mathcal{C}$, we denote $\mathrm{Mod}(\mathcal{O})$ the category of $\mathcal{O}$-modules.

Given the choice of an automorphism g of $\mathcal{O}$, we will denote by $\mathrm{Aut}^+_{(\mathrm{Mod}(\mathcal{O}),g)}(\mathcal{M})$ the group of automorphisms of the $\mathcal{O}$-module $\mathcal{M}$ with respect to the automorphism g and $\mathrm{Iso}_{(\mathrm{Mod}(\mathcal{P},\mathcal{Q}),\Phi)}(\mathcal{M},\mathcal{N})$, for the set of isomorphisms beween modules $\mathcal{M}$ and $\mathcal{N}$ with respect to an operad isomorphism Φ between $\mathcal{P}$ and $\mathcal{Q}$.

The superscript "+" still indicates that we consider morphisms that are the identity on objects.

Definition 4.1.9. *An elliptic associator over* $\mathbf{k}$ *is a couple* (F,G) *where* F *is a* $\mathbf{k}$*-associator and* G *is an isomorphism between the* $\widehat{\mathbf{PaB}}(\mathbf{k})$*-module* $\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})$ *and the* $G\mathbf{PaCD}(\mathbf{k})$*-module* $G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})$ *which is the identity on objects and which is compatible with* F*:*

$$\mathbf{Ell}(\mathbf{k}) := \mathrm{Iso}^+_{(\widehat{\mathbf{PaB}}(\mathbf{k}),G\mathbf{PaCD}(\mathbf{k}))}(\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k}), G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})).$$

Let us denote by $\{-\}$ the Lie algebra morphism $\mathfrak{t}_n(\mathbf{k}) \longrightarrow \bar{\mathfrak{t}}_{1,n}(\mathbf{k})$ sending $t_{ij} \in \mathfrak{t}_n(\mathbf{k})$ to $t_{ij} \in \bar{\mathfrak{t}}_{1,n}(\mathbf{k})$. Its induced group morphism $\exp(\bar{\mathfrak{t}}_n(\mathbf{k})) \longrightarrow \exp(\hat{\bar{\mathfrak{t}}}_{1,n}(\mathbf{k}))$ will be denoted the same way.

The following theorem identifies our definition of elliptic associators to the original one defined by Enriquez in [34].

Theorem 4.1.10. *There is a one-to-one correspondence between the set* $\mathbf{Ell}(\mathbf{k})$ *and the set* $\mathrm{Ell}(\mathbf{k})$ *of quadruples* (μ,Φ,A_+,A_-)*, where* $(\mu,\Phi) \in \mathrm{Ass}(\mathbf{k})$ *and* $A_\pm \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2}(\mathbf{k}))$*, such that:*

$$\alpha_\pm^{1,2,3}\alpha_\pm^{2,3,1}\alpha_\pm^{3,1,2} = 1, \ \textit{where } \alpha_\pm = \{\Phi^{1,2,3}\}A_\pm^{1,23}\{e^{\pm\mu(t_{12}+t_{13})/2}\}, \tag{4.1}$$

$$\{e^{\mu t_{12}}\} = \Big(\{\Phi\}A_-^{1,23}\{\Phi\}^{-1}, \{e^{-\mu t_{12}/2}\Phi^{2,1,3}\}(A_+^{2,13})^{-1}\{(\Phi^{2,1,3})^{-1}e^{-\mu t_{12}/2}\}\Big). \tag{4.2}$$

Proof. An associator F corresponds uniquely to a couple $(\mu,\Phi) \in \mathrm{Ass}(\mathbf{k})$ and an isomorphism G between $\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})$ and $G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})$ sends the arrows $A^{1,2}$ and $B^{1,2}$ of $\mathrm{End}_{\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})(2)}(12)$ to $A_+ \cdot X^{1,2}_{e\ell\ell}$ and $A_- \cdot Y^{1,2}_{e\ell\ell}$ with $A_\pm \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2})$ (recall that $\hat{\bar{\mathfrak{t}}}_{1,2}$ is the completed free Lie algebra in two generators). The image of relations (N1), (N2) and (E) are precisely the relations (4.1) and (4.2). □

Example 4.1.11 (Elliptic KZB Associators)**.** *Let us fix* $\tau \in \mathfrak{h}$*. Recall that the Lie algebra* $\bar{\mathfrak{t}}_{1,2}(\mathbb{C})$ *is isomorphic to the free Lie algebra* $\mathfrak{f}_2(\mathbb{C})$ *generated by two elements* $x := x_1$ *and* $y := y_1$*. We define the elliptic KZB associators* $A(\tau), B(\tau)$ *as the renormalized holonomies from* 0 *to* 1 *and* 0 *to* τ *of the differential equation*

$$G'(z) = -\frac{\theta_\tau(z+\mathrm{ad}\,x)\,\mathrm{ad}\,x}{\theta_\tau(z)\theta_\tau(\mathrm{ad}\,x)}(y)\cdot G(z), \tag{4.3}$$

with values in the group $\exp(\hat{\hat{\mathfrak{t}}}_{1,2}(\mathbb{C}))$ *More precisely, this equation has a unique solution* $G(z)$ *defined over* $\{a+b\tau,\ \text{for } a,b \in]0,1[\}$ *such that* $G(z) \simeq (-2\pi\,\mathrm{i}\,z)^{-[x,y]}$ *at* $z \longrightarrow 0$*. In this case,*

$$A(\tau) := G(z)^{-1}G(z+1), \quad B(\tau) := G(z)^{-1}e^{2\pi\,\mathrm{i}\,x}G(z+\tau).$$

These are elements of the group $\exp(\hat{\hat{\mathfrak{t}}}_{1,2}(\mathbb{C}))$*. More precisely, Enriquez showed in [34] that the element* $(2\pi\mathrm{i}, \Phi_{\mathrm{KZ}}, A(\tau), B(\tau))$ *is in* $\mathrm{Ell}(\mathbb{C})$*.*

4.1.6 Elliptic Grothendieck–Teichmüller group

Definition 4.1.12. *The (**k**-prounipotent version of the) elliptic Grothendieck–Teichmüller group is defined as the group*

$$\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k}) := \mathrm{Aut}^{+}_{(\mathrm{Mod}(\widehat{\mathbf{PaB}}(\mathbf{k})))}(\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k}))$$

of automorphisms of the $\widehat{\mathbf{PaB}}(\mathbf{k})$*-module* $\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})$ *which are the identity on objects.*

Again, we now show that our definition coincides with the original one defined by Enriquez in [34]. Recall that the set $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ is the set of tuples $(\lambda, f, g_\pm)$, where $(\lambda, f) \in \widehat{\mathrm{GT}}(\mathbf{k})$, $g_\pm \in \widehat{F}_2(\mathbf{k})$ such that

$$(f(\sigma_1^2,\sigma_2^2)g_\pm(X,Y)(\sigma_1\sigma_2^2\sigma_1)^{\pm\frac{\lambda-1}{2}}\sigma_1^{\pm1}\sigma_2^{\pm1})^3 = 1, \tag{4.4}$$

$$u^2 = (g_-, u^{-1}g_+^{-1}u^{-1}) \tag{4.5}$$

(identities in $\hat{\hat{B}}_{1,3}(\mathbf{k})$) where $u = f(\sigma_1^2,\sigma_2^2)^{-1}\sigma_1^\lambda f(\sigma_1^2,\sigma_2^2)$, and $g_\pm = g_\pm(X,Y)$.
For $(\lambda, f, g_\pm), (\lambda', f', g'_\pm) \in \widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$, we set

$$(\lambda, f, g_\pm)(\lambda', f', g'_\pm) := (\lambda'', f'', g''_\pm),$$

where $g''_\pm(X,Y) = g_\pm(g'_+(X,Y), g'_-(X,Y))$. This gives $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ a group structure. Moreover, for $(\lambda, f, g_+, g_-) \in \widehat{\mathrm{GT}}_{ell}(\mathbf{k})$ and $(\mu, \Phi, A_+, A_-) \in \mathrm{Ell}(\mathbf{k})$, we set

$$(\lambda, f, g_+, g_-) * (\mu, \Phi, A_+, A_-) := (\mu', \Phi', A'_+, A'_-)$$

where $A'_\pm := g_\pm(A_+, A_-)$. In [34], it is shown that this defines a left free and transitive group action of $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$.

Proposition 4.1.13. *There is a group isomorphism between* $\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k})$ *and* $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$*.*

Proof. Suppose that we have an automorphism G of $\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})$ which is the identity on objects. Then, by Theorem 4.1.3, such an automorphism is given by the data of an automorphism of the operad $\widehat{\mathbf{PaB}}(\mathbf{k})$, given by the pair $(\lambda, f) \in \widehat{\mathbf{GT}}(\mathbf{k})$, and the images of the two generators $A, B \in \mathrm{Aut}_{\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})(2)}(12)$. Let us denote $G(A) = g_+(X,Y)A$ and $G(B) = g_-(X,Y)B$, where $g_\pm \in \widehat{\mathrm{PB}}_{1,2}(\mathbf{k}) \simeq \widehat{F}_2(\mathbf{k})$. Then the obtained tuple $(\lambda, f, g_\pm)$ satisfies relations (4.4) and (4.5). Next, we show that this map is a group morphism. For this we show that the composition of automorphisms in $\mathrm{Aut}^{+}_{\mathrm{Mod}(\widehat{\mathbf{PaB}}(\mathbf{k}))}(\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k}))$ corresponds to the composition law of the group $\mathrm{GT}_{e\ell\ell}(\mathbf{k})$. We already know that the composition of automorphisms F_1 and F_2 in $\mathrm{Aut}^+_{\mathrm{Op}\hat{\mathcal{G}}}(\widehat{\mathbf{PaB}}(\mathbf{k}))$ corresponds to the composition law in $\mathrm{GT}(\mathbf{k})$, that is, the associated couples (λ, f_1) and (μ, f_2) in $\mathbf{k}^\times \times \hat{F}_2(\mathbf{k})$ satisfy

$$(F_1 \circ F_2)(R^{1,2}) = (R^{1,2})^{\lambda\mu}$$

$$\begin{aligned} (F_1 \circ F_2)(\Phi^{1,2,3}) &= F_1(F_2(\Phi^{1,2,3})) = F_1(f_2(x,y) \cdot \Phi^{1,2,3}) \\ &= F_1(f_2(x,y))F_1(\Phi^{1,2,3}) \\ &= (f_2(x^\lambda, f_1(x,y)y^\lambda f_1(x,y)^{-1})f_1(x,y)) \cdot \Phi^{1,2,3}, \end{aligned}$$

(here F_2 is generated by $x := \sigma_1^2$ and $y := \sigma_2^2$). We also already showed that any two automorphisms G and H in the group $\mathrm{Aut}^+_{\mathrm{Mod}(\widehat{\mathbf{PaB}}(\mathbf{k}))}(\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k}))$, depending on F_1 and F_2 respectively, are associated to couples $(g_+(X,Y), g_-(X,Y))$ and $(h_+(X,Y), h_-(X,Y))$ which represent automorphisms of the parenthesized word (12) in the groupoid $\widehat{\mathbf{PaB}}_{e\ell\ell}(\mathbf{k})(2)$ i.e. in $\hat{F}_2(\mathbf{k})$) (recall that $\hat{F}_2(\mathbf{k}) \simeq \widehat{\mathrm{PB}}_{1,2}(\mathbf{k})$ is nothing but the $\mathbf{k}$-prounipotent completion of the free group with generators X and Y). We then have

$$(H \circ G)(A) = H(g_+(X,Y)) = g_+(H(X), H(Y)) = g_+(h_+(X,Y), h_-(X,Y)).$$

Likewise, we find $(G \circ H)(B) = g_-(h_+(X,Y), h_-(X,Y))$ which concludes the proof, as the composite of operadic module morphisms $F \circ G$ is compatible with the composition of operad morphisms $F_1 \circ F_2$. The fact that that the underlying sets of $\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k})$ and $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ are isomorphic is a consequence of the fact that the set of elliptic associators is non empty, that there are free and transitive left actions of $\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathbf{Ell}(\mathbf{k})$ and of $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$ and the fact that there is a one-to-one correspondence between $\mathbf{Ell}(\mathbf{k})$ and $\mathrm{Ell}(\mathbf{k})$ so we get a composite of bijections

$$\mathbf{GT}_{e\ell\ell}(\mathbf{k}) \longrightarrow \mathbf{Ell}(\mathbf{k}) \longrightarrow \mathrm{Ell}(\mathbf{k}) \longrightarrow \widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k}).$$

This finishes the proof. □

4.1.7 Graded elliptic (graded) Grothendieck–Teichmüller group

Definition 4.1.14. *The graded elliptic Grothendieck-Teichmüller group is the group*

$$\mathbf{GRT}_{e\ell\ell}(\mathbf{k}) := \mathrm{Aut}^+_{(\mathrm{Mod}(\mathbf{PaCD}(\mathbf{k}))}(\mathbf{PaCD}_{e\ell\ell}(\mathbf{k}))$$

of automorphism group of the $\mathbf{PaCD}(\mathbf{k})$*-module* $\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})$ *which are the identity on objects.*

Notice that there is an isomorphism

$$\mathrm{Aut}^+_{(\mathrm{Mod}(\mathbf{PaCD}(\mathbf{k}))}(\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})) \simeq \mathrm{Aut}^+_{(\mathrm{Mod}(G\mathbf{PaCD}(\mathbf{k}))}(G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})).$$

Define $\mathrm{GRT}_1^{ell}(\mathbf{k})$ to be the set of tuples (g, u_+, u_-), such that $g \in \mathrm{GRT}_1(\mathbf{k})$, $u_\pm \in \hat{\mathfrak{t}}_{1,2}(\mathbf{k})$, satisfying

$$\mathrm{Ad}(g^{1,2,3})(u_\pm^{1,23}) + \mathrm{Ad}(g^{2,1,3})(u_\pm^{2,13}) + u_\pm^{3,12} = 0, \tag{4.6}$$

$$[\mathrm{Ad}(g^{1,2,3})(u_\pm^{1,23}), u_\pm^{3,12}] = 0, \tag{4.7}$$

$$[\mathrm{Ad}(g^{1,2,3})(u_+^{1,23}), \mathrm{Ad}(g^{2,1,3})(u_-^{2,13})] = t_{12}, \tag{4.8}$$

as relations in $\hat{\bar{\mathfrak{t}}}_{1,3}(\mathbf{k})$. Set $(g_1, u_+^1, u_-^1) * (g_2, u_+^2, u_-^2) := (g, u_+, u_-)$, where

$$u_\pm(x_1, y_1) := u_\pm^1(u_+^2(x_1, y_1), u_-^2(x_1, y_1)) \tag{4.9}$$

The group $\mathbf{k}^\times$ acts on $\mathrm{GRT}_1^{ell}(\mathbf{k})$ by rescaling

$$c \cdot (g, u_\pm) := (c \cdot g, c \cdot u_\pm),$$

where $c \cdot g$ is as above and

- $(c \cdot u_+)(x_1, y_1) := u_+(x_1, c^{-1}y_1)$,
- $(c \cdot u_-)(x_1, y_1) := cu_-(x_1, c^{-1}y_1)$.

We then set $\mathrm{GRT}_{ell}(\mathbf{k}) := \mathrm{GRT}_1^{ell}(\mathbf{k}) \rtimes \mathbf{k}^\times$. This defines a group structure on $\mathrm{GRT}_{ell}(\mathbf{k})$.

Moreover, there is an right group action of $\mathrm{GRT}_1^{ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$ given as follows : for $(g, u_\pm) \in \mathrm{GRT}_1^{ell}(\mathbf{k})$ and $(\mu, \Phi, A_\pm)\,\mathrm{Ell}(\mathbf{k})$, we set $(\mu, \Phi, A_\pm) * (g, u_\pm) := (\mu, \tilde{\Phi}, \tilde{A}_\pm)$, where

$$\tilde{A}_\pm(x_1, y_1) := A_\pm(u_+(x_1, y_1), u_-(x_1, y_1))$$

and, for $c \in \mathbf{k}^\times$, we set $(\mu, \Phi, A_\pm) * c := (\mu, c * \Phi, c\sharp A_\pm)$, where $(c\sharp A_\pm)(x_1, y_1) := A_\pm(x_1, y_1)$. In [34] this action is shown to be free and transitive. Notice that $\tilde{A}_\pm = \theta(A_\pm)$, where $\theta \in \mathrm{Aut}(\hat{\bar{\mathfrak{t}}}_{1,2}^{\mathbf{k}})$ is $x_1 \mapsto u_+(x_1, y_1)$ and $y_1 \mapsto u_-(x_1, y_1)$.

Proposition 4.1.15. *There is a group isomorphism between* $\mathbf{GRT}_{e\ell\ell}(\mathbf{k})$ *and* $\mathrm{GRT}_{e\ell\ell}(\mathbf{k})$.

Proof. The map $\mathbf{GRT}_{e\ell\ell}(\mathbf{k}) \longrightarrow \mathrm{GRT}_{e\ell\ell}(\mathbf{k})$ is constructed as follows. Let F be an automorphism in $\mathrm{Aut}^+_{\mathrm{Mod}(\mathbf{PaCD}(\mathbf{k}))}(\mathbf{PaCD}_{e\ell\ell}(\mathbf{k}))$ depending on an operad automorphism Ψ in $\mathrm{GRT}(\mathbf{k})$. We have

- $\Psi(X^{1,2}) = X^{1,2}$,
- $\Psi(H^{1,2}) = \lambda H^{1,2}$,
- $\Psi(a^{1,2,3}) = g(t_{12}, t_{23})a^{1,2,3}$,
- $F(X^{1,2}_{e\ell\ell}) = u_+(x, y) \cdot \mathrm{Id}_{1,2}$,
- $F(Y^{1,2}_{e\ell\ell}) = u_-(x, y) \cdot \mathrm{Id}_{1,2}$.

where $(\lambda, g) \in \mathrm{GRT}(\mathbf{k})$, $u_\pm \in \hat{\bar{\mathfrak{t}}}_{1,2}(\mathbf{k})$. In light of relations of Remark 4.1.8, we obtain that the tuple $(\lambda, g(t_{12}, t_{23}), u_+(x, y), u_-(x, y))$ satisfies relations (4.6), (4.7) and (4.8). The assignment $(\Psi, F) \mapsto (\lambda, g(t_{12}, t_{23}), u_+(x, y), u_-(x, y))$ defines a map $\mathbf{GRT}_{e\ell\ell}(\mathbf{k}) \longrightarrow \mathrm{GRT}_{e\ell\ell}(\mathbf{k})$. First we show that this map is a group morphism. For this we show that the composition of automorphisms in $\mathrm{Aut}^+_{\mathrm{Mod}(G\mathbf{PaCD}(\mathbf{k}))}(G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k}))$ corresponds to the composition law of the group $\mathrm{GRT}_{e\ell\ell}(\mathbf{k})$. We already know that the composition of automorphisms Φ and Ψ in $\mathrm{Aut}^+_{\mathrm{Op}\,\hat{\mathcal{G}}}(G\mathbf{PaCD}(\mathbf{k}))$ corresponds to the composition law in $\mathrm{GRT}(\mathbf{k})$, that is, the associated couples (λ, f_1) and (μ, f_2) in $\mathbf{k}^\times \times \exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$ satisfy

$$(\Phi \circ \Psi)(H^{1,2}) = \lambda\mu H^{1,2}$$

$$(\Phi \circ \Psi)(a^{1,2,3}) = f_2(\lambda t_{12}, f_1(t_{12}, t_{23}) \cdot \lambda t_{23} \cdot f_1(t_{12}, t_{23})^{-1}) f_1(t_{12}, t_{23}) \cdot a^{1,2,3}.$$

We also already showed that any two automorphisms G and H in the group $\mathrm{Aut}^+_{\mathrm{Mod}(G\mathbf{PaCD}(\mathbf{k}))}(G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k}))$, depending on Φ and Ψ respectively, are associated to couples $(g_+(x,y), g_-(x,y))$ and $(h_+(x,y), h_-(x,y))$ which represent automorphisms of the parenthesized word (12) in the groupoid $G\mathbf{PaCD}_{e\ell\ell}(\mathbf{k})(2)$ i.e. in $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}(\mathbf{k}))$ where $x = x_1$ and $y = y_1$ (recall that $\bar{\mathfrak{t}}_{1,2}(\mathbf{k})$ is nothing but the free Lie algebra over $\mathbf{k}$ with generators x and y). We then have

$$(H \circ G)(X^{1,2}_{e\ell\ell}) = H(g_+(x,y) \cdot \mathrm{Id}_{1,2}) = g_+(H(x), H(y)) \cdot \mathrm{Id}_{1,2} = g_+(h_+(x,y), h_-(x,y)) \cdot \mathrm{Id}_{1,2}\,.$$

Likewise, we find $(G \circ H)(Y^{1,2}_{e\ell\ell}) = g_-(h_+(x,y), h_-(x,y)) \cdot \mathrm{Id}_{1,2}$ which concludes the proof, as the composite of operadic module morphisms $F \circ G$ is compatible with the composition of operad morphisms $\Phi \circ \Psi$.

Next, this morphism is a bijection. This is a consequence of the fact that there exists a composite of bijections

$$\mathbf{GRT}_{e\ell\ell}(\mathbf{k}) \longrightarrow \mathbf{Ell}(\mathbf{k}) \longrightarrow \mathrm{Ell}(\mathbf{k}) \longrightarrow \mathrm{GRT}_{e\ell\ell}(\mathbf{k}).$$

□

4.1.8 Torsors

Finally, we enhance the above bijections into a torsor result.

Theorem 4.1.16. *There is a torsor isomorphism*

$$(\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{e\ell\ell}(\mathbf{k})) \longrightarrow (\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k}), \mathrm{Ell}(\mathbf{k}), \mathrm{GRT}_{e\ell\ell}(\mathbf{k})) \tag{4.10}$$

Proof. This is a summary of most of the above results. First of all, we know that $(\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{e\ell\ell}(\mathbf{k}))$ has a natural torsor structure and that $(\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k}), \mathrm{Ell}(\mathbf{k}), \mathrm{GRT}_{e\ell\ell}(\mathbf{k}))$ is a torsor by [34]. Next, we proved in Proposition 4.1.13 that there are group isomorphisms between $\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k})$ and $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ and in Proposition 4.1.15 that there are group isomorphisms between $\mathbf{GRT}_{e\ell\ell}(\mathbf{k})$ and $\mathrm{GRT}_{e\ell\ell}(\mathbf{k})$. Thus, it is sufficient to show that the actions of $\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathbf{Ell}(\mathbf{k})$ and of $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$ are compatible and that the actions of $\widehat{\mathbf{GRT}}_{e\ell\ell}(\mathbf{k})$ on $\mathbf{Ell}(\mathbf{k})$ and of $\mathrm{GRT}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$ are compatible. Under the correspondence of Theorem 4.1.13, the image of the natural action of $\widehat{\mathbf{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathbf{Ell}(\mathbf{k})$ is exactly the action of $\widehat{\mathrm{GT}}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$. Both actions are then compatible. Under the correspondence of Theorem 4.1.15, the image of the natural action of $\mathbf{GRT}_{e\ell\ell}(\mathbf{k})$ on $\mathbf{Ell}(\mathbf{k})$ is exactly the action of $\mathrm{GRT}_{e\ell\ell}(\mathbf{k})$ on $\mathrm{Ell}(\mathbf{k})$. Both actions are then compatible. □

4.2 Moperads associated with twisted configuration spaces (cyclotomic associators)

4.2.1 Compactified configuration space of the annulus

For each finite set I, let us consider the configuration space of $\mathbb{C}^\times$:

$$\mathrm{Conf}(\mathbb{C}^\times, I) := \left\{\mathbf{z} = (z_i)_{i\in I} \in (\mathbb{C}^\times)^I | z_i \neq z_j, \forall i \neq j\right\}.$$

Now consider its reduced version

$$\mathrm{C}(\mathbb{C}^\times, I) := \mathrm{Conf}(\mathbb{C}^\times, I)/\mathbb{R}_{>0}.$$

We clearly have an isomorphism between $\mathrm{C}(\mathbb{C}^\times, n)$ and $\mathrm{C}(\mathbb{C}, n+1)$. We then consider the Fulton-MacPherson compactification $\overline{\mathrm{C}}(\mathbb{C}^\times, n)$ of $\mathrm{C}(\mathbb{C}^\times, n)$. The boundary $\partial\overline{\mathrm{C}}(\mathbb{C}^\times, n) = \overline{\mathrm{C}}(\mathbb{C}^\times, n) - \mathrm{C}(\mathbb{C}^\times, n)$ is made of the following irreducible components: for any partition $[\![0, n]\!] = J_0 \coprod \cdots \coprod J_k$ such that $0 \in J_m$, for some $0 \leq m \leq k$, there is a component

$$\partial_{J_1,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{C}^\times, n) \cong \overline{\mathrm{C}}(\mathbb{C}^\times, k) \times \overline{\mathrm{C}}(\mathbb{C}^\times, J_m) \times \prod_{i=1;i\neq m}^{k} \overline{\mathrm{C}}(\mathbb{C}, J_i).$$

The inclusion of boundary components for which $m = 0$ provides $\overline{\mathrm{C}}(\mathbb{C}^\times, -)$ with the structure of a moperad over the operad $\overline{\mathrm{C}}(\mathbb{C}, -)$ in topological spaces.

4.2.2 The PaB-moperad of parenthesized braids with a frozen strand

We have inclusions of topological moperads

$$\mathbf{Pa}_0 \subset \overline{\mathrm{C}}(\mathbb{R}_{>0}, -) \subset \overline{\mathrm{C}}(\mathbb{C}^\times, -).$$

over

$$\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -) \subset \overline{\mathrm{C}}(\mathbb{C}, -).$$

We then define

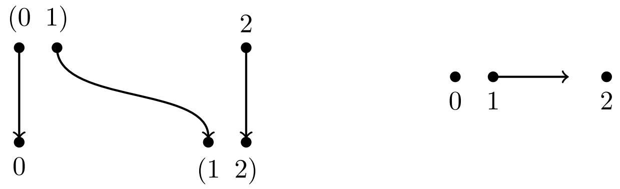
$$\mathbf{PaB}^1 := \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, -), \mathbf{Pa}_0\right),$$

which is a moperad over the operad in groupoids **PaB**.

Example 4.2.1 (Description of $\mathbf{PaB}^1(1)$)**.** *First observe that $\overline{C}(\mathbb{C}^\times, 1) \simeq \overline{C}(\mathbb{C}, 2) \simeq S^1$. Moreover, $\mathbf{Pa}_0 = \{(01)\}$. Hence $\mathbf{PaB}^1(1) \simeq \mathbb{Z}$: it has only one object (01) and is freely generated by an automorphism $E^{0,1}$ of (01), and can be depicted as an elementary pure braid:*

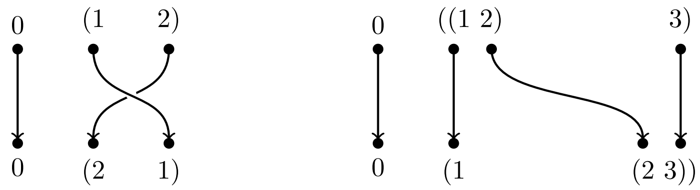
Two incarnations of $E^{0,1}$

Example 4.2.2 (Notable arrow in $\mathbf{PaB}^1(2)$)**.** *Let us first recall that* $\mathbf{Pa_0}(2) = \mathfrak{S}_2 \times \{(\bullet\bullet)\bullet, \bullet(\bullet\bullet)\}$ *and that* $\overline{C}(\mathbb{R}_{>0}, 2) \cong \mathfrak{S}_2 \times [0,1]$*. Hence we have an arrow* $\Psi^{0,1,2}$ *(the identity path in* $[0,1]$*) from* $(01)2$ *to* $0(12)$ *in* $\mathbf{PaB}^1(2)$*, which can be depicted as follows:*



Two incarnations of $\Psi^{0,1,2}$

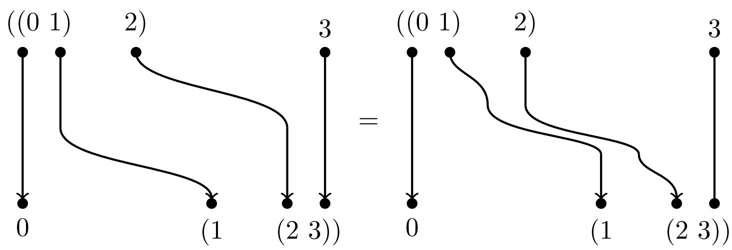
Remark 4.2.3. *Recall from §2.5.8 that, being a* $\mathbf{PaB}$*-moperad,* $\mathbf{PaB}^1$ *comes together with a morphism of* $\mathfrak{S}$*-modules* $\mathbf{PaB} \longrightarrow \mathbf{PaB}^1$*. In pictorial terms, this morphism sends a parentesized braid with* n *strands to a parenthesized braid with* $n+1$ *strands by adding a frozen stand labelled by* 0 *on the left. For instance, the images of* $R^{1,2}$ *(a morphism in* $\mathbf{PaB}(2)$*) and of* $\Phi^{1,2,3}$ *(a morphism in* $\mathbf{PaB}(3)$*) can be respectively depicted as follows:*



Theorem 4.2.4. *As a* $\mathbf{PaB}$*-moperad having* $\mathbf{Pa_0}$ *as* $\mathbf{Pa}$*-moperad of objects,* $\mathbf{PaB}^1$ *is freely generated by* $E := E^{0,1} \in \mathbf{PaB}^1(1)$ *and* $\Psi := \Psi^{0,1,2} \in \mathbf{PaB}^1(2)$ *together with the following relations:*

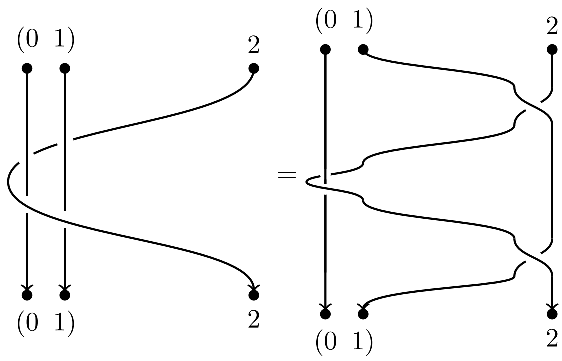
- *(MP)* $\Psi^{01,2,3}\Psi^{0,1,23} = \Psi^{0,1,2}\Psi^{0,12,3}\Phi^{1,2,3}$*, as arrows from* $((01)2)3$ *to* $0(1(23))$ *in* $\mathbf{PaB}^1(3)$*,*
- *(O)* $E^{01,2} = \Psi^{0,1,2}R^{1,2}(\Psi^{0,2,1})^{-1}E^{0,2}\Psi^{0,2,1}R^{2,1}(\Psi^{0,1,2})^{-1}$*, as arrows from* $(01)2$ *to* $(01)2$ *in* $\mathbf{PaB}^1(2)$*.*

Proof. We proceed in a similar way as in the elliptic case, using this time the results of [33, §4.4]. Let $\mathcal{Q}^1$ be the $\mathbf{PaB}$-moperad with the above presentation. From Examples 4.2.1 and 4.2.2 we deduce that, as a $\mathbf{PaB}$-moperad in groupoid, $\mathbf{PaB}^1$ contains two morphisms $E = E^{0,1}$ (in $\mathbf{PaB}^1(1)$) and $\Psi = \Psi^{0,1,2}$ (in $\mathbf{PaB}^1(2)$). One easily shows, using the following pictures, that they satisfy *mixed pentagon* and *octogon* relations, (MP) and (O):



(MP)

and



(O)

Therefore, by the universal property of $\mathcal{Q}^1$, there is a morphism of **PaB**-moperads $\mathcal{Q}^1 \longrightarrow \mathbf{PaB}^1$, which is the identity on objects. In order to show that this is an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of an object arity-wise because all groupoids involved are connected. Let $n \geq 0$, let p be the object $(\cdots(01)2\cdots\cdots)n$ of $\mathcal{Q}^1(n)$ and $\mathbf{PaB}^1(n)$. We want to show that the induced group morphism

$$\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^1(n)}(p) = \pi_1\big(\bar{\mathrm{C}}(\mathbb{C}^\times, n), p\big)$$

is an isomorphism.

On the one hand, we can replace the base-point p with $p_{reg} = (1, 2, \ldots, n) \in \mathrm{C}(\mathbb{C}^\times, n)$, as they are in the same path-connected component. Moreover, since the Fulton–MacPherson compactification does not change the homotopy type of our configuration spaces, we get an isomorphism

$$\pi_1(\bar{\mathrm{C}}(\mathbb{C}^\times, n), p) \simeq \pi_1(\mathrm{C}(\mathbb{C}^\times, n), p_{reg})\,.$$

On the other hand, in [33, §4.4], Enriquez proves several useful facts:

- Given a braided module category $\mathcal{M}$ over a braided monoidal category $\mathcal{C}$, an object X of $\mathcal{C}$, and an object M of $\mathcal{M}$, there is a group morphism

$$\mathrm{B}^1_n \longrightarrow \mathrm{Aut}_{\mathcal{M}}(M \otimes X^{\otimes n})\,,$$

  where, by convention, $M \otimes X^{\otimes n}$ comes equipped with the left-most parenthesization $((M \otimes X) \otimes ...) \otimes X$, and $\mathrm{B}^1_n = \mathrm{B}_{n+1} \times_{\mathfrak{S}_{n+1}} \mathfrak{S}_n$.

- There is a universal braided module category $\mathbf{PaB}^{1,Enr}$ generated by a single object 0, over the universal braided monoidal category $\mathbf{PaB}^{Enr}$ generated by a single object $\bullet$. Hence objects of $\mathbf{PaB}^{1,Enr}$ are parenthesizations of $0 \bullet \cdots \bullet$, and thus p determines an object (which we abusively still denote p).

- the morphism $\mathrm{B}^1_n \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^{1,Enr}}(p)$ is an isomorphism.

One can moreover see that, by construction, $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p)$ is exactly the kernel subgroup

$$\ker\left(\mathrm{Aut}_{\mathbf{PaB}^{1,Enr}(n)}(p) \longrightarrow \mathfrak{S}_n\right) \simeq \mathrm{PB}_{n+1}\ .$$

Hence we have a commuting diagram

$$\begin{tikzcd}
\mathrm{PB}^1_n \arrow[r, "\simeq"] \arrow[d] & \mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n), p\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{C}^\times, n), p_{reg}\right) \arrow[l, "\simeq"'] \arrow[d] \\
\mathrm{B}^1_n \arrow[r, "\simeq"] \arrow[d] & \mathrm{Aut}_{\mathbf{PaB}^{1,Enr}}(p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n)/\mathfrak{S}_n, [p]\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{C}^\times, n)/\mathfrak{S}_n, [p_{reg}]\right) \arrow[l, "\simeq"'] \arrow[d] \\
\mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n
\end{tikzcd}$$

where all vertical sequences are short exact sequences. Thus, in order to get that the map $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \longrightarrow \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n), p\right)$ is an isomorphism, we are left to prove that the composite map $\mathrm{B}^1_n \longrightarrow \pi_1(\mathrm{C}(\mathbb{C}^\times, n), p_{reg})$ is indeed an isomorphism. But this map is, by its very construction, the isomorphism (from [95, 102]) exhibiting a presentation by generators and relations of the braid group of a handlebody. $\square$

4.2.3 Compactified twisted configuration space of the annulus

Consider, for $N \geq 1$, the additive group $\Gamma = \mathbb{Z}/N\mathbb{Z}$. To every finite set I let us associate the so-called Γ-*twisted configuration space*

$$\mathrm{Conf}(\mathbb{C}^\times, I, \Gamma) = \{\mathbf{z} = (z_i)_{i\in I} \in (\mathbb{C}^\times)^I | z_i \neq \zeta z_j, \forall i \neq j, \forall \zeta \in \mu_N\}$$

(μ_N is the set of complex Nth roots of unity) and its reduced version

$$\mathrm{C}(\mathbb{C}^\times, I, \Gamma) := \mathrm{Conf}(\mathbb{C}^\times, I, \Gamma)/\mathbb{R}_{>0}\,.$$

Remark 4.2.5. *Observe that* $\mathrm{Conf}(\mathbb{C}^\times, I, \Gamma)$*, resp.* $\mathrm{C}(\mathbb{C}^\times, I, \Gamma)$*, is an* Γ^I*-covering space of* $\mathrm{Conf}(\mathbb{C}^\times, I)$*, resp.* $\mathrm{C}(\mathbb{C}^\times, I)$*, the covering map being given by* $(z_i)_{i\in I} \mapsto (z_i^N)_{i\in I}$*.*

There are also inclusions

$$\mathrm{Conf}(\mathbb{C}^\times, I, \Gamma) \hookrightarrow \mathrm{Conf}(\mathbb{C}^\times, I \times \mu_N) \qquad \text{and} \qquad \mathrm{C}(\mathbb{C}^\times, I, \Gamma) \hookrightarrow \mathrm{C}(\mathbb{C}^\times, I \times \mu_N)$$

given by $(z_i)_{i\in I} \mapsto (\zeta z_i)_{(i,\zeta)\in I\times\mu_N}$. This allows us to define the compactification $\overline{\mathrm{C}}(\mathbb{C}^\times, I, \Gamma)$ of $\mathrm{C}(\mathbb{C}^\times, I, \Gamma)$, as the closure of $\mathrm{C}(\mathbb{C}^\times, I, \Gamma)$ inside $\overline{\mathrm{C}}(\mathbb{C}^\times, I \times \mu_N)$. The irreducible components of its boundary $\partial\overline{\mathrm{C}}(\mathbb{C}^\times, I, \Gamma) = \overline{\mathrm{C}}(\mathbb{C}^\times, I, \Gamma) - \mathrm{C}(\mathbb{C}^\times, I, \Gamma)$ can be described as follows. For an arbitrary partition $J_0 \coprod \cdots \coprod J_k$ of $\{0\} \sqcup I$ there is a component

$$\partial_{J_1,\cdots,J_k}\overline{\mathrm{C}}(\mathbb{C}^\times, I, \Gamma) \cong \overline{\mathrm{C}}(\mathbb{C}^\times, k, \Gamma) \times \overline{\mathrm{C}}(\mathbb{C}^\times, J_m, \Gamma) \times \prod_{i=1;i\neq m}^{k} \overline{\mathrm{C}}(\mathbb{C}, J_i)\,,$$

where $m \in \{0, \ldots, k\}$ is the index such that $0 \in J_m$. The inclusion of boundary components such that $m = 0$ provides $\overline{\mathrm{C}}(\mathbb{C}^\times, -, \Gamma)$ with the structure of a moperad over the operad $\overline{\mathrm{C}}(\mathbb{C}, -)$ in topological spaces.

We let the reader check that the covering map $\mathrm{C}(\mathbb{C}^\times, I, \Gamma) \longrightarrow \mathrm{C}(\mathbb{C}^\times, I)$ from Remark 4.2.5 extends to a continuous map $\phi_n : \overline{\mathrm{C}}(\mathbb{C}^\times, I, \Gamma) \longrightarrow \overline{\mathrm{C}}(\mathbb{C}^\times, I)$ between their compactifications, and thus leads to a morphism of moperads.

Finally, one observes that the natural action of Γ^I on each $\mathrm{C}(\mathbb{C}^\times, I \times \mu_N)$, given by

$$(\alpha \cdot \mathbf{z})_{(j,\zeta)} := \mathbf{z}_{\left(j, e^{\frac{2i\pi\alpha_j}{N}}\zeta\right)}$$

induces an action of Γ on the moperad $\overline{\mathrm{C}}(\mathbb{C}^\times, -, \Gamma)$, in the sense of §2.5.9.

4.2.4 The Pa-moperad of labelled parenthesized permutations

Borrowing the notation from the previous subsection, we define $\mathbf{Pa_0^\Gamma}(n) := \phi_n^{-1}\big(\mathbf{Pa_0}(n)\big)$. Explicitly, $\mathbf{Pa_0^\Gamma}(n)$ is the set of parenthesized permutations of $\{0, 1, \ldots, n\}$ that fix 0 and that are equipped with a label $\{1, \ldots, n\} \longrightarrow \Gamma$.

Notation. As a matter of notation, we will write the label as an index attached to each $1, \ldots, n$. For instance, $(02_\alpha)1_0$ belongs to $\mathbf{Pa_0^\Gamma}(2)$ for every $\alpha \in \Gamma$.

Observe that the $\mathfrak{S}$-module (in sets) $\mathbf{Pa_0^\Gamma}$ carries the structure of a $\mathbf{Pa}$-moperad. Indeed, it is a fiber product

$$\mathbf{Pa_0^\Gamma} = \mathbf{Pa_0} \underset{\overline{\mathrm{C}}(\mathbb{C}^\times, -)}{\times} \overline{\mathrm{C}}(\mathbb{C}^\times, -, \Gamma)$$

in the category of $\mathbf{Pa}$-moperads (in topological spaces). Here are two self-explanatory examples of partial compositions:

$$(02_\alpha)1_\beta \circ_2 (12)3 = (0((2_\alpha 3_\alpha)4_\alpha))1_\beta \qquad \text{and} \qquad (02_\alpha)1_\beta \circ_0 (02_\alpha)1_0 = (((02_\alpha)1_0)4_\alpha)3_\beta\,.$$

Remark 4.2.6. *As we have seen in §2.5.8 of the previous Section, our conventions are such that the $\mathbf{Pa}$-moperad structure on $\mathbf{Pa_0^\Gamma}$ gives in particular a morphism of $\mathbf{Pa}$-modules $\mathbf{Pa} \longrightarrow \mathbf{Pa_0^\Gamma}$. One can see that it is the map that sends a parenthezised permutation $\mathbf{p}$ to $0(\mathbf{p})$ together with the trivial label function $i \mapsto 0$.*

Finally, $\mathbf{Pa_0^\Gamma}$ is acted on by Γ in the following way: for $n \geq 0$, Γ^n only acts on the labellings, *via* the group law of Γ. For instance, if $f : \{1, \ldots, n\} \longrightarrow \Gamma$ and $\alpha \in \Gamma^n$, then $(\alpha \cdot f)(i) = f(i) + \alpha_i$.

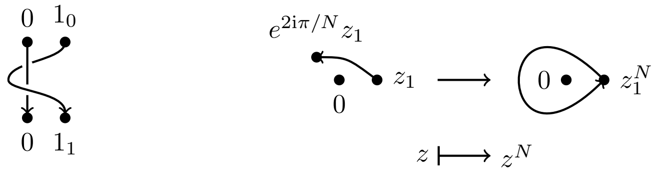
4.2.5 The PaB-moperad of twisted parenthesized braids

We define

$$\mathbf{PaB}^\Gamma := \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, -, \Gamma), \mathbf{Pa_0^\Gamma}\right)\,.$$

It is a $\mathbf{PaB}$-moperad (in groupoids), that carries an action of the group Γ. The maps $\phi_n : \overline{\mathrm{C}}(\mathbb{C}^\times, n, \Gamma) \longrightarrow \overline{\mathrm{C}}(\mathbb{C}^\times, n)$ induce a $\mathbf{PaB}$-moperad morphism $\mathbf{PaB}^\Gamma \longrightarrow \mathbf{PaB}^1$.

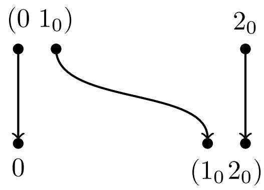
Example 4.2.7 (Description of $\mathbf{PaB}^\Gamma(1)$). *First observe that $\mathbf{Pa_0^\Gamma}(1) \longrightarrow \mathbf{Pa_0}(1)$ is the terminal map $\mu_N \simeq \{01_\alpha | \alpha \in \Gamma\} \longrightarrow \{01\} = *$. Then observe that the map $\overline{\mathrm{C}}(\mathbb{C}^\times, 1, \Gamma) \longrightarrow \overline{\mathrm{C}}(\mathbb{C}^\times, 1)$ is nothing but the path-connected Γ-cover $S^1 \longrightarrow S^1$. Hence we in particular have morphisms $E^{0,1}_\alpha$, $\alpha \in \Gamma$ from 01_α to $01_{\alpha+1}$ in $\mathbf{PaB}^\Gamma(1)$, being the unique lift of $E^{0,1}$ that starts at $01_\alpha \in \mathbf{Pa_0^\Gamma}(1)$. Pictorially:*



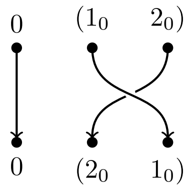
Two incarnations of $E_0^{0,1}$

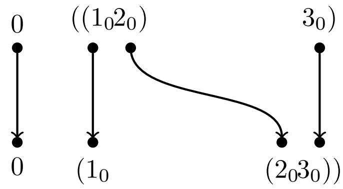
In the above picture, on the right we have pictured a path in the twisted configuration space, together with its image under the covering map, which is a loop. Diagrammatically (see the left of the above picture), we depict it as a pure braid (a loop in the base configuration space) together with appropriate base points (which uniquely determines the lift in the covering twisted configuration space).

Example 4.2.8 (Notable arrow in $\mathbf{PaB}^\Gamma(2)$)**.** *Let* $\Psi_0^{0,1,2}$ *be the unique lift of* $\Psi^{0,1,2}$ *(a morphism in* $\mathbf{PaB}^1(2)$*) starting at* $(01_0)2_0$*. It can be depicted as follows:*



Remark 4.2.9. *As in Remark 4.2.3, one sees from §2.5.8 there is a morphism of* $\mathfrak{S}$*-modules* $\mathbf{PaB} \longrightarrow \mathbf{PaB}^\Gamma$*. In pictorial terms, it sends a parentesized braid with* n *strands to a labelled parenthesized braid with* $n+1$ *strands by adding a frozen stand labelled by* 0 *on the left and choosing the trivial label. For instance, the images* $R_0^{1,2}$ *of* $R^{1,2}$ *and* $\Phi_0^{1,2,3}$ *of* $\Phi^{1,2,3}$ *can be respectively depicted as follows:*





We are now ready to provide an explicit presentation for the **PaB**-moperad $\mathbf{PaB}^\Gamma$:

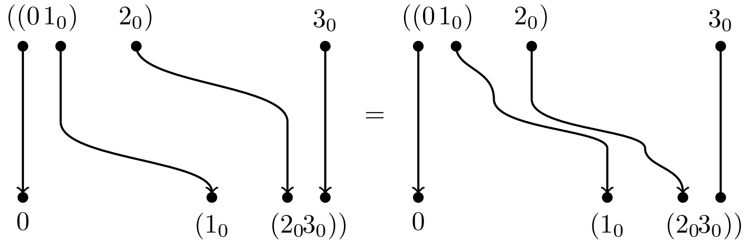
Theorem 4.2.10. *As a* **PaB***-moperad in groupoids with a* Γ*-action having* $\mathbf{Pa}_0^\Gamma$ *as* $\mathbf{Pa}_\mathbf{0}^\mathbf{\Gamma}$*-moperad of objects,* $\mathbf{PaB}^\Gamma$ *is freely generated by* $E_0 := E_0^{0,1}$ *and* $\Psi_0 := \Psi_0^{0,1,2}$ *together with the following relations:*

(MP) $\Psi_0^{01,2,3}\Psi_0^{0,1,23} = \Psi_0^{0,1,2}\Psi_0^{0,12,3}\Phi^{1,2,3}$*, as arrows from* $((01_0)2_0)3_0$ *to* $0(1_0(2_03_0)))$ *in* $\mathbf{PaB}^\Gamma(3)$*,*

(tO) $E_0^{01,2} = \Psi_0^{0,1,2}R^{1,2}(\Psi_0^{0,2,1})^{-1}E_0^{0,2}\alpha \cdot (\Psi_0^{0,2,1}R^{2,1}(\Psi_0^{0,1,2})^{-1})$*, as arrows from* $(01_0)2_0$ *to* $(01_0)2_1$ *in* $\mathbf{PaB}^\Gamma(2)$*, and where* $\alpha = (0,1) \in \Gamma^2$*.*

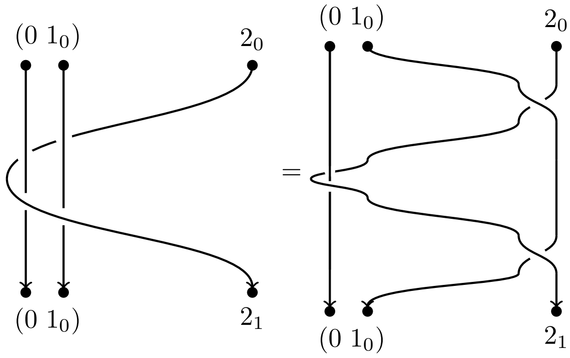
Proof. Let $\mathcal{Q}^\Gamma$ be the **PaB**-moperad with the above presentation, and recall that $\mathcal{Q}^1$ is the **PaB**-moperad with the presentation of Theorem 4.2.4. Our first goal is to show that there is a morphism $\mathcal{Q}^\Gamma \longrightarrow \mathbf{PaB}^\Gamma$ of **PaB**-moperads, commuting with the Γ-action. We have already seen in the Examples above that there are morphisms $E_0 := E_0^{0,1}$ and $\Psi_0 := \Psi_0^{0,1,2}$, in $\mathbf{PaB}^\Gamma(1)$ and $\mathbf{PaB}^\Gamma(2)$, respectively. We have to prove that they satisfy the *mixed pentagon* and *twisted octogon* relation, (MP) and (tO).

These relations are the unique lifts of the similar relations (MP) and (O) in $\mathbf{PaB}^1$ from Theorem 4.2.4, starting at $((01_0)2_0)3_0$ and $(01_0)2_0$, respectively. They can be depicted as follows:



(MP)

and



(tO)

By universal property of $\mathcal{Q}^\Gamma$ there is a Γ-equivariant morphism of **PaB**-moperads $Q^\Gamma \longrightarrow \mathbf{PaB}^\Gamma$, which is the identity on objects. As before, in order to show that this is an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of an object arity-wise (because all groupoids involved are connected). Let $n \geq 0$, let $\tilde{p}$ be the object $(\cdots(01_0)2_0\cdots\cdots)n_0$ of $\mathcal{Q}^\Gamma(n)$ and $\mathbf{PaB}^\Gamma(n)$, which lifts the object $p = (\cdots(01)2\cdots\cdots)n$ of $\mathcal{Q}^1(n) \simeq \mathbf{PaB}^1(n)$. We want to show that the induced group morphism

$$\mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde{p}) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^\Gamma(n)}(p_0) = \pi_1\big(\bar{\mathrm{C}}(\mathbb{C}^\times, n), \tilde{p}\big)$$

is an isomorphism.

We claim that it fits into a commuting diagram

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n, \Gamma), \tilde p\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{C}^\times, n), \tilde p_{reg}\right) \arrow[l, "\simeq"'] \arrow[d] \\
\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \arrow[r, "\simeq"] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n), p\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{C}^\times, n)), f(p_{reg})\right) \arrow[l, "\simeq"'] \arrow[d] \\
\Gamma^n \arrow[r, equal] & \Gamma^n \arrow[r, equal] & \Gamma^n
\end{tikzcd}$$

where only the left-most vertical arrows remain to be described.

<u>The morphism $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \longrightarrow \Gamma^n$.</u> Let $*$ be the terminal operad in groupoids. We have a $*$-moperad structure on the following $\mathfrak{S}$-module in groupoids: $\underline{\Gamma} = \{\Gamma^n\}_{\mathfrak{n}\geq 0}$, where we view a group as a groupoid with only one object, and where the action of the symmetric group is by permutation. The moperad structure is described as follows:

- $\circ_0 : \Gamma^n \times \Gamma^m \longrightarrow \Gamma^{n+m}$ is the concatenation of sequences.
- for every $i \neq 0$, $\circ_i : \Gamma^n \longrightarrow \Gamma^{n+m-1}$ is the partial diagonal

$$(\alpha_1, \dots, \alpha_n) \longmapsto (\alpha_1, \dots, \alpha_{i-1}, \underbrace{\alpha_i, \dots, \alpha_i}_{m \text{ times}}, \alpha_{i+1}, \dots, \alpha_n)\,.$$

We let the reader check that sending E to $1 \in \Gamma$ and Ψ to $(0,0) \in \Gamma^2$ defines a moperad morphism $\mathbf{PaB}^1 \longrightarrow \underline{\Gamma}$ along the terminal operad morphism $\mathbf{PaB} \longrightarrow *$. This in particular induces a group morphism

$$\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \longrightarrow \Gamma^n$$

for every $n \geq 0$. Heuristically, this morphism counts, for every i, and modulo N, the number of times that $E^{0,i}$ appears in an element of $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p)$. It is obviously surjective, and we let the reader check that the following triangle commutes:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \arrow[r, "\simeq"] \arrow[dr] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n), p\right) \arrow[d] \\
& \Gamma^n
\end{tikzcd}$$

<u>The morphism $\mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde p) \longrightarrow \mathrm{Aut}_{\mathcal{Q}^1(n)}(p)$.</u> We have a Γ-equivariant morphism of **PaB**-moperads $\mathcal{Q}^\Gamma \longrightarrow \mathcal{Q}^1$, where Γ acts trivially on $\mathcal{Q}^1$, which forgets the label on objects, and sends the generators E_0 and Ψ_0 to E and Ψ, respectively. It obviously fits into a commmuting square

$$\begin{tikzcd}
\mathcal{Q}^\Gamma \arrow[r] \arrow[d] & \mathbf{PaB}^\Gamma \arrow[d] \\
\mathcal{Q}^1 \arrow[r] & \mathbf{PaB}^1
\end{tikzcd}$$

of **PaB**-moperads. This induces in particular a group morphism

$$\mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde p) \longrightarrow \mathrm{Aut}_{\mathcal{Q}^1(n)}(p)$$

for every $n \geq 0$, that fits into a commuting square

$$\begin{tikzcd} \mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde{p}) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n, \Gamma), \tilde{p}\right) \arrow[d] \\ \mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \arrow[r, "\simeq"] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{C}^\times, n), p\right) \end{tikzcd}$$

We now turn to the proof of the fact that the left-most vertical sequence is a short exact sequence, which shows that

$$\mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde{p}) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^\Gamma(n)}(p_0) = \pi_1\left(\bar{\mathrm{C}}(\mathbb{C}^\times, n), \tilde{p}\right)$$

is an isomorphism.
This morphism is injective. Indeed, an automorphism of $\tilde{p}$ in $\mathcal{Q}^\Gamma(n)$ can be represented by a finite sequence $\tilde{S}$ of R's, Φ's, E_0's, Ψ_0's, and their images under the action of Γ^n. The image of such an automorphism under $\mathcal{Q}^\Gamma \longrightarrow \mathcal{Q}^1$ is represented by the corresponding finite sequence S of R's, Φ's, E's and Ψ's. Every modification of S using the relations (MP) and (O) can be lifted (uniquely) to a modification of $\tilde{S}$ using (MP), (tO), or their images under the action of Γ^n. Hence, if an automorphism has trivial image, then it must be trivial.
The sequence is exact. We already know from the commuting diagram that the image of $\mathrm{Aut}_{\mathcal{Q}^\Gamma(n)}(\tilde{p})$ in $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p)$ lies in the kernel of $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \longrightarrow \Gamma^n$. We finally can show that the image is exactly the kernel. Indeed:

- Using (O), every element g in $\mathrm{Aut}_{\mathcal{Q}^1(n)}(p)$ can be written represented by a product of Φ's, R's, Ψ's and E's, where the only E's appearing are of the form $E^{0,i}$.
- Such an element admits a unique lift to a morphism $\tilde{g}$ in $\mathcal{Q}^\Gamma(n)$, with source being $\tilde{p}$ (one just replace Φ's, R's, Ψ's and E's in the expression for g by Φ's, R's, Ψ_0's and E_0's, maybe acted on by Γ^n in order to get the correct starting objects).
- An element g as above lies in

$$\ker\left(\mathrm{Aut}_{\mathcal{Q}^1(n)}(p) \longrightarrow \Gamma^n\right)$$

  if and only if for every i, the number of occurence of $E^{0,i}$ (counted in an algebraic way) is a multiple of N. This tells us in particular that the target of the lifted morphism shall be the same as its source, so that $\tilde{g}$ lies in the kernel.

This ends the proof of the Proposition. □

4.2.6 Infinitesimal cyclotomic braids

Let $\Gamma = \mathbb{Z}/N\mathbb{Z}$, I a finte set, and let $\mathfrak{t}_I^\Gamma(\mathbf{k})$ be the Lie $\mathbf{k}$-algebra with generators t_{0i}, $(i \in I)$, and t_{ij}^α, $(i \neq j \in I$, $\alpha \in \mathbb{Z}/N\mathbb{Z})$, and relations:

(NS) $t_{ij}^\alpha = t_{ji}^{-\alpha}$,

(NL) $[t_{0i}, t_{jk}^\alpha] = 0$ and $[t_{ij}^\alpha, t_{kl}^\beta] = 0$, (N4T) $[t_{ij}^{\alpha}, t_{ik}^{\alpha+\beta} + t_{jk}^{\beta}] = 0$,

(NT1) $[t_{0i}, t_{0j} + \sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} t_{ij}^{\alpha}] = 0$,

(NT2) $[t_{0i} + t_{0j} + \sum_{\beta\in\mathbb{Z}/N\mathbb{Z}} t_{ij}^{\beta}, t_{ij}^{\alpha}] = 0$,

where $i, j, k, l \in I$ are pairwise distinct and $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$. We will call it the $\mathbf{k}$-Lie algebra of infinitesimal cyclotomic braids.

The above definition is obviously functorial with respect to bijections, exhibiting $\mathfrak{t}^{\Gamma}(\mathbf{k})$ as an $\mathfrak{S}$-module. It moreover also has the structure of a $\mathfrak{t}(\mathbf{k})$-moperad, where partial compositions are defined as follows: for $i \in I$,

$$
\begin{array}{rccc}
\circ_k: & \mathfrak{t}_I^{\Gamma}(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) & \longrightarrow & \mathfrak{t}^{\Gamma}_{J\sqcup I-\{i\}}(\mathbf{k}) \\
& (0, t_{pq}) & \longmapsto & t_{pq}^{0} \\
& (t_{jk}^{\alpha}, 0) & \longmapsto & \begin{cases} t_{jk}^{\alpha} & \text{if} \quad i \notin \{j,k\} \\ \sum\limits_{r\in J} t_{rk}^{\alpha} & \text{if} \quad j = i \\ \sum\limits_{r\in J} t_{jr}^{\alpha} & \text{if} \quad k = i \end{cases} \\
& (t_{0i}, 0) & \longmapsto & \begin{cases} t_{0j} & \text{if} \quad j \neq i \\ \sum\limits_{p\in J} t_{0p} & \text{if} \quad j = i \end{cases}
\end{array}
$$

and

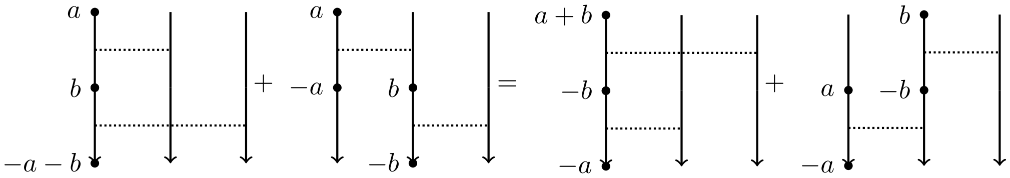
$$
\begin{array}{rccc}
\circ_0: & \mathfrak{t}_I^{\Gamma}(\mathbf{k}) \oplus \mathfrak{t}_J^{\Gamma}(\mathbf{k}) & \longrightarrow & \mathfrak{t}^{\Gamma}_{J\sqcup I}(\mathbf{k}) \\
& (0, t_{0p}) & \longmapsto & t_{0p} \\
& (0, t_{pq}^{\alpha}) & \longmapsto & t_{pq}^{\alpha} \\
& (t_{jk}^{\alpha}, 0) & \longmapsto & t_{jk}^{\alpha} \\
& (t_{0i}, 0) & \longmapsto & t_{0i} + \sum\limits_{j\in J} t_{ji}^{0}
\end{array}
$$

We will call $\mathfrak{t}^{\Gamma}(\mathbf{k})$ the moperad of *infinitesimal cyclotomic braidings.*

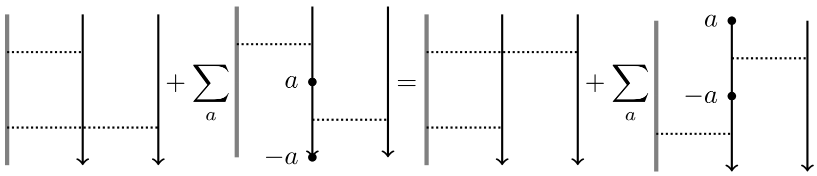
We then consider the $\mathbf{CD}(\mathbf{k})$-moperad of *cyclotomic chord diagrams* $\mathbf{CD}^{\Gamma}(\mathbf{k}) := \hat{\mathcal{U}}(\mathfrak{t}^{\Gamma}(\mathbf{k}))$ in $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$.

Remark 4.2.11. *Morphisms in $\mathbf{CD}^{\Gamma}(\mathbf{k})(n)$ can be represented as linear combinations of diagrams of chords on $n+1$ vertical strands, together with a labelling of the last n strands by elements of Γ. Thus, borrowing the representation of such chord diagrams from [17] (where the relation to Vassiliev invariants has been explored), the infinitesimal cyclotomic braid relations can be depicted as follows:*

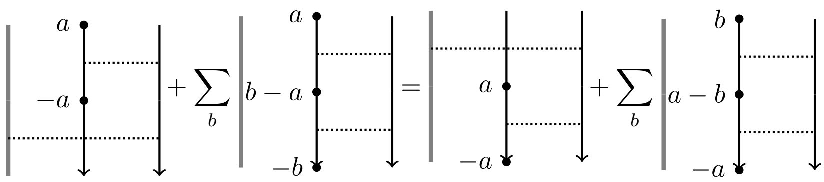
(NS)



(N4T)



(NT1)



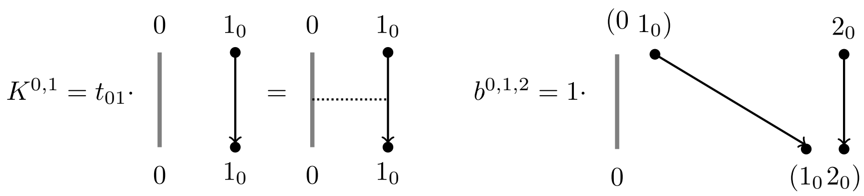
(NT2)

Since $\mathbf{CD}^\Gamma(\mathbf{k})$ has only one object in each arity, then we have an obvious terminal morphism of moperads $\omega_3 : \mathbf{Pa}_0^\Gamma \longrightarrow \mathrm{Ob}(\mathbf{CD}^\Gamma(\mathbf{k}))$, over the operad morphism $\omega_1 : \mathbf{Pa} \longrightarrow \mathrm{Ob}(\mathbf{CD}(\mathbf{k}))$ from §2.6.4. Hence we can consider the moperad

$$\mathbf{PaCD}^\Gamma(\mathbf{k}) := \omega_3^\star \mathbf{CD}^\Gamma(\mathbf{k})$$

of *parenthesized cyclotomic chord diagrams*, over the operad $\mathbf{PaCD}(\mathbf{k}) = \omega_1^\star \mathbf{CD}(\mathbf{k})$ in $\mathbf{Cat}(\mathbf{CoAss_k})$.

Example 4.2.12 (Notable arrows of $\mathbf{PaCD}^\Gamma(\mathbf{k})$)**.** *We have the following arrows in* $\mathbf{PaCD}^\Gamma(\mathbf{k})(1)$ *and* $\mathbf{PaCD}^\Gamma(\mathbf{k})(2)$*, respectively:*



Remark 4.2.13. *Again, there is an action of* Γ *on* $\mathbf{PaCD}^\Gamma(\mathbf{k})$ *and the elements* $K^{0,1}$ *and* $b^{0,1,2}$ *are generators of the* $\mathbf{PaCD}(\mathbf{k})$*-moperad* $\mathbf{PaCD}^\Gamma(\mathbf{k})$ *and satisfy the following relations*

- $b^{01,2,3}b^{0,1,23} = b^{0,1,2}b^{0,12,3}a^{1,2,3}$,
- $K^{01,2} = b^{0,1,2}X^{1,2}(b^{0,2,1})^{-1}K^{0,2}\alpha \cdot \left(b^{0,2,1}X^{2,1}(b^{0,1,2})^{-1}\right)$, *for* $\alpha = (0,1) \in \Gamma^2$,
- $b^{0,1,2}X^{1,2}(b^{0,2,1})^{-1}\alpha \cdot (b^{0,2,1}X^{2,1}(b^{0,1,2})^{-1}) = 1$,
- $K^{0,1} + \sum_{\alpha=1}^{N} \alpha \cdot (\mathrm{Ad}(b^{0,1,2})(H_0^{1,2})) + \mathrm{Ad}\left(b^{0,1,2}X^{1,2}(b^{0,2,1})^{-1}\right)(K^{0,2}) = 0.$

4.2.7 Cyclotomic associators

We borrow an expand the terminology from §2.6.5 and §4.1.5.

If $\mathcal{P} \longrightarrow \mathcal{Q}$ is a morphism between operads in $\mathcal{C}$, $\mathcal{M}$ is a $\mathcal{P}$-moperad, and $\mathcal{N}$ a $\mathcal{Q}$-moperad, then we will consider moperad mophisms $\mathcal{M} \longrightarrow \mathcal{N}$ in the category of $\mathcal{P}$-moperads (*via* the restriction functor), and will simply refer to them as moperad morphisms if the context is clear.

For an operad $\mathcal{O}$ in $\mathcal{C}$, we denote $\mathrm{Mop}(\mathcal{O})$ the category of $\mathcal{O}$-moperads. Given the choice of an automorphism g of $\mathcal{O}$, we will denote by $\mathrm{Aut}^{+}_{(\mathrm{Mop}(\mathcal{O}),g)}(\mathcal{M})$ the group of automorphisms of the $\mathcal{O}$-moperad $\mathcal{M}$ with respect to the automorphism g and $\mathrm{Iso}^{+}_{(\mathrm{Mop}(\mathcal{P},\mathcal{Q}),\Phi)}(\mathcal{M},\mathcal{N})$, for the set of isomorphisms beween moperads $\mathcal{M}$ and $\mathcal{N}$ with respect to an operad isomorphism Φ between $\mathcal{P}$ and $\mathcal{Q}$.

In addition to the superscript "+", we may also add a superscript "Γ" when only considering morphisms that are Γ-equivariant.

The rest of this section can be seen as an operadic reformulation of (some parts of) [33].

Definition 4.2.14. *A cyclotomic associator is a couple (F,G) where F is in $\mathbf{Ass}(\mathbf{k})$ and G is a Γ-equivariant isomorphism between the $\widehat{\mathbf{PaB}}(\mathbf{k})$-moperad $\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k})$ and the $G\mathbf{PaCD}(\mathbf{k})$-moperad $G\mathbf{PaCD}^{\Gamma}(\mathbf{k})$ which is the identity on objects and which is compatible with F. Denote by*

$$\mathbf{Ass}^{\Gamma}(\mathbf{k}) := \mathrm{Iso}^{+}_{(\widehat{\mathbf{PaB}}(\mathbf{k}),G\mathbf{PaCD}(\mathbf{k}))}(\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k}), G\mathbf{PaCD}^{\Gamma}(\mathbf{k}))^{\Gamma}$$

the set of cyclotomic associators.

Denote $\Psi^{0,1,2} := \Psi(t_{01}, t_{12}^{0}, ..., t_{12}^{N-1})$, $\Psi_a^{0,1,2} := \theta(a)\cdot\Psi^{0,1,2} = \Psi(t_{01}, t_{12}^{a}, ..., t_{12}^{a+N-1})$ and $\Psi_a^{0,2,1} := (12)\cdot\Psi_a^{0,1,2} = \Psi(t_{02}, t_{21}^{a}, ..., t_{21}^{a+N-1}) = \Psi(t_{02}, t_{12}^{a}, ..., t_{12}^{a+1-N})$. Denote $\mathfrak{t}_{2,N}^{0}(\mathbf{k})$ for the free Lie algebra $\mathfrak{f}(\mathbf{k})(t_{01}^{0}, t_{12}^{0}, ..., t_{12}^{N-1})$. We have the following theorem:

Theorem 4.2.15. *There is a one-to-one correspondence between elements of $\mathbf{Ass}^{\Gamma}(\mathbf{k})$ and those of the set $\mathrm{Ass}_1^{\Gamma}(\mathbf{k})$ consisting on triples $(\lambda,\Phi,\Psi) \in \times\mathbf{k}^{\times} \times \exp(\hat{\mathfrak{t}}_3^{0}(\mathbf{k})) \times \exp(\hat{\mathfrak{t}}_{2,N}^{0}(\mathbf{k}))$, such that $(\lambda,\Phi) \in \mathrm{Ass}(\mathbf{k})$ and Ψ satisfies*

(MP) $\Psi^{01,2,3}\Psi^{0,1,23} = \Psi^{0,1,2}\Psi^{0,12,3}\{\Phi^{1,2,3}\}$,

(O) $\{e^{\frac{\lambda}{N}t_{01}}\}\Psi_0^{0,1,2}\{e^{\frac{\lambda}{2}t_{12}^{0}}\}(\Psi_0^{0,2,1})^{-1}\{e^{\frac{\lambda}{N}t_{02}}\}\Psi_a^{0,2,1}\{e^{\frac{\lambda}{2}t_{12}^{a}}\}\Psi_a^{0,1,2} = 1$,

where $a = \bar{1} \in \mathbb{Z}/N\mathbb{Z}$.

Proof. Let $\tilde{F}$ be a $\mathbf{k}$-associator $\widehat{\mathbf{PaB}}(\mathbf{k}) \longrightarrow G\mathbf{PaCD}(\mathbf{k})$ and let $\tilde{G}$ be an isomorphism

$$\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k}) \longrightarrow G\mathbf{PaCD}^{\Gamma}(\mathbf{k})$$

of $(\widehat{\mathbf{PaB}}(\mathbf{k}), G\mathbf{PaCD}(\mathbf{k}))$-moperads which is the identity on objects and which is compatible with $\tilde{F}$. It corresponds to a unique morphism $G : \mathbf{PaB}^{\Gamma} \longrightarrow G\mathbf{PaCD}^{\Gamma}(\mathbf{k})$. From the presentation of $\mathbf{PaB}^{\Gamma}$, we know that $\tilde{G}$ is uniquely determined by the images of $E_0^{0,1} \in \mathrm{Hom}_{\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k})(1)}(01_0, 01_1)$ and $\Psi_0^{0,1,2} \in \mathrm{Hom}_{\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k})(2)}((01_0)2_0, 0(1_02_0))$ at the morphisms level. Thus, there are elements $u \in \exp(\hat{\mathfrak{t}}_1^{\Gamma}(\mathbf{k}))$ and $v \in \exp(\hat{\mathfrak{t}}_2^{\Gamma}(\mathbf{k}))$ such that $G(E_0^{0,1}) = u \cdot E_0^{0,1}$

and $G(\Phi_0^{0,1,2}) = v \cdot \Phi_0^{0,1,2}$. Now, we have a Lie algebra isomorphism $\mathfrak{t}_2^\Gamma(\mathbf{k}) \simeq \mathbf{k}(c) \oplus \mathfrak{f}(\mathbf{k})(t_{01}, t_{12}^0, ..., t_{12}^{N-1})$ where $c = t_{01}^0 + t_{02}^0 + \sum_{a\in\Gamma} t_{12}^a$. Thus, u is of the form $e^{\lambda_1 c}$ and v is of the form $e^{\lambda_2 c} f(t_{01}, t_{12}^0, ..., t_{12}^{N-1})$. Now, we know that the image of $E_0^{0,1}$ in $\mathbf{PaB}^1$ induced by the projection $z \longrightarrow z^N$ is $E^{0,1}$. Thus, we can identify $\lambda_1 = \frac{\lambda}{N}$ and then $u = e^{\frac{\lambda}{N} t_{01}}$. Finally, the fact that $\Phi_0^{0,1,2}$ is Γ-invariant ensures that v is of the form $f(t_{01}, t_{12}^0, ..., t_{12}^{N-1})$. Once we simplified this way u and v, the images of the Octogon and Mixed Pentagon relation in $G\mathbf{PaCD}^\Gamma(\mathbf{k})$ imply relations (MP) and (O) in the above theorem. $\square$

Example 4.2.16 (Cyclotomic KZ Associator)**.** *Consider the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}\, z} H(z) = \left(\frac{t_{01}}{z} + \sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} \frac{t_{12}^\alpha}{z-\zeta^\alpha} \right) H(z), \tag{4.11}$$

where ζ is a primitive Nth root of unity, and let H_{0^+}, H_{1^-} be the solutions such that $H_{0^+}(z) \sim z^{t_{01}}$ when $z \longrightarrow 0^+$ and $H_{1^-}(z) \sim z^{t_{12}^1}$ when $z \longrightarrow 1^-$. Then the renormalized holonomy $\Psi_{\mathrm{KZ}} = H_{1^-}^{-1} H_{0^+} \in \exp(\hat{\mathfrak{t}}_{2,N}^0)$ from 0 to 1 of the above differential equation is the cyclotomic KZ associator constructed by Enriquez in [33]. More precisely, Enriquez showed that the quadruple $(-\overline{1}, 2i\pi, \Phi_{\mathrm{KZ}}, \Psi_{\mathrm{KZ}})$ is in $\mathrm{Ass}^\Gamma(\mathbb{C})$.

4.2.8 Cyclotomic Grothendieck-Teichmüller groups

Definition 4.2.17. *The ($\mathbf{k}$-pro-unipotent version of the) cyclotomic Grothendieck-Teichmüller group is defined as the group*

$$\widehat{\mathbf{GT}}^\Gamma(\mathbf{k}) := \mathrm{Aut}^+_{(\mathrm{Mop}(\widehat{\mathbf{PaB}}(\mathbf{k})))}(\widehat{\mathbf{PaB}}^\Gamma(\mathbf{k}))^\Gamma$$

of automorphisms of the $\widehat{\mathbf{PaB}}(\mathbf{k})$-moperad $\widehat{\mathbf{PaB}}^\Gamma(\mathbf{k})$ which are Γ-equivariant and which are the identity on objects.

Notice that such an automorphism depends on an automorphism of $\widehat{\mathbf{PaB}}(\mathbf{k})$ i.e. on an element Φ of $\widehat{\mathbf{GT}}(\mathbf{k})$. Let $\widehat{F}_2(\phi_N, \mathbf{k})$ be the partial $\mathbf{k}$-pro-unipotent completion of the free group F_2 with respect to the surjective group morphism $\phi_N : F_2 \longrightarrow \mathbb{Z}/N\mathbb{Z}$ sending x to $\overline{1}$ and y to $\overline{0}$ and $\widehat{P}_4(\phi_{3,N}, \mathbf{k})$ the partial $\mathbf{k}$-pro-unipotent completion of P_4 with respect to the map $\phi_{3,N} : B_4 \times_{\mathfrak{S}_4} \mathfrak{S}_3 \longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathfrak{S}_3$ induced by the $(\mathbb{Z}/N\mathbb{Z} \times \mathfrak{S}_3)$-fold map $\mathrm{Conf}(\mathbb{C}^\times, 3, \Gamma) \longrightarrow \mathrm{Conf}(\mathbb{C}, 3)/\mathfrak{S}_3$ where $\mathfrak{S}_3$ is interpreted as the subgroup of the group $\mathfrak{S}_4$ of permutations of $0, ..., 3$ which fix 0. Denote $\mathbf{k}(N)^\times = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbf{k}^\times$. See [33] for more details on the subject of partial pro-unipotent completions. Finally, recall that $\mathrm{PB}_{n,N}$ has generators $x_{0,i}^N$ and $x_{ij}^\alpha := x_{0,i}^{-\alpha} x_{ij} x_{0,i}^\alpha$. In particular, the generators of $\mathrm{PB}_{2,N}$ will be denoted by $X := x^N$ and $y(\alpha) := x^{-\alpha} y x^\alpha$ for $0 \le \alpha \le N-1$.

In [33], the author constructed a cyclotomic version of the Grothendieck-Teichmüller group which we now recall. Define $\widehat{\mathrm{GT}}^\Gamma(\mathbf{k})$ to be the set of elements $(\lambda, \mu, f, g) \in \mathbf{k}^\times \times \mathbf{k}(N)^\times \times \widehat{F}_2(\mathbf{k}) \times \widehat{F}_2(\phi_N, \mathbf{k})$, satisfying $(\lambda, f) \in \widehat{\mathrm{GT}}(\mathbf{k})$ and

(O) $x^\mu g(x,y) y^{\frac{\lambda+1}{2}} g(x^{-1}y^{-1})^{-1} (x^{-1}y^{-1})^\mu g(x^{-1}y^{-1}, y) y^{\frac{\lambda-1}{2}} g(x,y)^{-1} = 1$ in $\widehat{F}_2(\phi_N, \mathbf{k})$, (MP) $g(x_{02}x_{12}, x_{23})g(x_{01}, x_{12}x_{13}) = g(x_{01}, x_{12})g(x_{01}x_{02}, x_{13}x_{23})f(x_{12}, x_{23})$ in $\widehat{P}_4(\phi_{3,N}, \mathbf{k})$.

The set $\widehat{\mathbf{GT}}^\Gamma(\mathbf{k})$ has a internal composition law defined by

$$(\lambda_1, \mu_1, f_1, g_1) * (\lambda_2, \mu_2, f_2, g_2) = (\lambda, \mu, f, g),$$

given as follows. Write $y(\alpha) = x^\alpha y x^{-\alpha}$ and identify (λ, μ, f, g) with (a, k, f, g) where $\mu = (a, k) \in \mathbf{k}(N)$ so $\lambda = \tilde{a} + Nk$. Then $(a_1, k_1, f_1, g_1)(a_2, k_2, f_2, g_2) = (a, k, f, g)$, where $a = a_1 a_2$, k is such that $\tilde{a} + Nk = (\tilde{a}_1 + Nk_1)(\tilde{a}_2 + Nk_2)$, $f(x, y)$ is given by

$$f(x, y) = f_2(x^{\lambda_1}, f_1(x, y)y^{\lambda_1} f_1(x, y)^{-1}) \cdot f_1(x, y),$$

and

$$\begin{aligned}
&g(X|y(0), \dots, y(N-1)) = g_1(X|y(0), \dots, y(N-1)) \cdot \\
&g_2\Big(X^{\tilde{a}_1 + Nk_1} |\, \mathrm{Ad}(g_1(X|y(0), \dots, y(N-1)))(y(0)^{\tilde{a}_1 + Nk_1}), \\
&\ \mathrm{Ad}\left(X^{k_1} g_1(X|y(\tilde{a}_1), \dots, y(\tilde{a}_1 + N - 1))\right)(y(\tilde{a}_1)^{\tilde{a}_1 + Nk_1}), \dots, \\
&\ \mathrm{Ad}\left(X^{(N-1)k_1} g_1(X|y((N-1)\tilde{a}_1), \dots, y((N-1)\tilde{a}_1 + N - 1))\right)((y(N-1)\tilde{a}_1)^{\tilde{a}_1 + Nk_1})\Big).
\end{aligned}$$

The group $\widehat{\mathrm{GT}}^\Gamma(\mathbf{k})$ acts on $\mathrm{Ass}^\Gamma(\mathbf{k})$ on the left as follows:

$$(\lambda, \mu, f, g) * (a', \lambda', \Phi', \Psi') = (\bar{\mu}a', [\mu]\lambda', \Phi'', \Psi''), \tag{4.12}$$

where

$$\Phi''(t_{12}, t_{23}) := \Phi'(t_{12}, t_{23}) f(e^{\lambda' t_{12}}, \mathrm{Ad}(\Phi'(t_{12}, t_{23}))(e^{\lambda' t_{23}})),$$

$$\begin{aligned}
\Psi''(t^0_{12}|t^0_{23}, \dots, t^{N-1}_{23}) \ :=\ & \Psi'(t^0_{12}|t^0_{23}, \dots, t^{N-1}_{23}) \\
& g\Big(\lambda' t^0_{12} |\, \mathrm{Ad}\left(\Psi'(t^0_{12}|t^0_{23}, \dots, t^{N-1}_{23})\right)(\lambda' t^0_{23}), \\
& \mathrm{Ad}\left((\lambda'/N)t^0_{12} \Psi'(t^0_{12}|t^{a'}_{23}, \dots, t^{a'+N-1}_{23})\right)(\lambda' t^{a'}_{23}), \dots, \\
& \mathrm{Ad}\left((N-1)(\lambda'/N)t^0_{12} \Psi'(t^0_{12}|t^{(N-1)a'}_{23}, \dots, t^{(N-1)a'+N-1}_{23})\right)(\lambda' t^{(N-1)a'}_{23})\Big)
\end{aligned}$$

(recall that $\lambda = [\mu]$, so if $\mu = (a, k)$, then $\lambda = \tilde{a} + Nk$; also $\bar{\mu} = a$). It was shown in [33] that this action is free and transitive.

Proposition 4.2.18. *There is a group isomorphism between $\widehat{\mathbf{GT}}^\Gamma(\mathbf{k})$ and $\widehat{\mathrm{GT}}^\Gamma(\mathbf{k})$.*

Proof. The map $\mathbf{GT}^\Gamma(\mathbf{k}) \longrightarrow \mathrm{GT}^\Gamma(\mathbf{k})$ is constructed as follows. Suppose that we have an automorphism G of $\widehat{\mathbf{PaB}}^\Gamma(\mathbf{k})$ which is the identity on objects and which is compatible with an automorphism F of the operad $\widehat{\mathbf{PaB}}(\mathbf{k})$. F is given by the pair $(\lambda, f) \in \widehat{\mathbf{GT}}(\mathbf{k})$, and G is determined by the images of the two generators E_0 and Ψ_0, in $\mathbf{PaB}^\Gamma(1)$ and $\mathbf{PaB}^\Gamma(2)$, respectively. Thus, an automorphism (F, G) in $\mathbf{GT}^\Gamma(\mathbf{k})$ is uniquely determined by elements $(\lambda, \mu, f, g) \in \mathbf{k}^\times \times \mathbf{k}(N)^\times \times \widehat{F}_2(\mathbf{k}) \times \widehat{F}_2(\phi_N, \mathbf{k})$ such that

- $F(R^{1,2}) = (R^{1,2})^\lambda$,

- $F(\Phi^{1,2,3}) = f(x_{12}, x_{23}) \cdot \Phi^{1,2,3}$,
- $G(\Psi_0^{0,1,2}) = g(x^N|y(0), \ldots, y(N-1)) \cdot \Psi_0^{0,1,2} \cdot$,
- $G(E_0^{0,1}) = \mu \cdot E_0^{0,1}$.

The relation between a and λ was explained in the proof of Theorem 4.2.15. Then, the defining relations in the presentation of $\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k})$ imply that the tuple (λ, μ, f, g) satisfies relations (O) and (MP). The assignment $(\Psi, F) \mapsto (\lambda, \mu, f, g)$ then defines a map $\mathbf{GT}^{\Gamma}(\mathbf{k}) \longrightarrow \mathrm{GT}^{\Gamma}(\mathbf{k})$.

Let's now prove that this map is a group morphism. We will show that the composition of automorphisms in $\mathrm{Aut}^+_{\mathrm{Mop}(\widehat{\mathbf{PaB}}(\mathbf{k}))}(\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k}))$ corresponds to the composition law of the group $\mathrm{GT}^{\Gamma}(\mathbf{k})$. As before, the composition of automorphisms F_1 and F_2 in $\mathrm{Aut}^+_{\mathrm{Op}\,\hat{\mathcal{G}}}(\widehat{\mathbf{PaB}}(\mathbf{k}))$ corresponds to the composition law in $\mathrm{GT}(\mathbf{k})$, that is, the associated couples (λ, f_1) and (μ, f_2) in $\mathbf{k}^{\times} \times \hat{F}_2(\mathbf{k})$ satisfy

$$(F_1 \circ F_2)(R^{1,2}) = (R^{1,2})^{\lambda\mu}$$
$$(F_1 \circ F_2)(\Phi^{1,2,3}) = \Phi^{1,2,3} \cdot (f_2(x^{\lambda}, f_1(x,y)y^{\lambda}f_1(x,y)^{-1}) \cdot f_1(x,y)),$$

(here F_2 is generated by $x := \sigma_1^2$ and $y := \sigma_2^2$). We also already showed that any two automorphisms G and H in the group $\mathrm{Aut}^+_{\mathrm{Mop}(\widehat{\mathbf{PaB}}(\mathbf{k}))}(\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k}))$, depending on Ψ_1 and Ψ_2 respectively, are associated to couples $(\mu_1, g_1(x^N|y(0), \ldots, y(N-1)))$ and $(\mu_2, g_2(x^N|y(0), \ldots, y(N-1)))$ where g_1 and g_2 are elements of in $\widehat{F}_2(\phi_N, \mathbf{k})$. Analogously to relation (2.15), as $E_0^{0,1}$ is an arrow from $(01_0)2_0$ to $(01_\alpha)2_0$ for some primitive element $\alpha \in \Gamma$, then $E_0^{0,1}$ is sent via G to $(E_0^{0,1})^{kN} \cdot E_0^{0,1}$ for some $k \in \mathbb{Z}$.

Let us now place ourselves in the group $A = \mathrm{Aut}_{\widehat{\mathbf{PaB}}^{\Gamma}(\mathbf{k})(2)}((01_0)2_0, (01_0)2_0)$. In A, we have

$$x_{01}^N = ((E_0^{0,1})^N)2 = \mu((E_0^{0,1})^N, 2) = \mu \circ_0 (E_0^{0,1})^N$$

We then have $F(x_{01}^N) = (x_{01}^N)^{\lambda}$ for some invertible $\lambda \in \mathbf{k}^{\times}$. Next, let us compute $F(x_{12}^0)$. Again, analogously to relation (2.16), in A, the element $(x_{12}^0)^2$ can be decomposed as

$$(01_0)2_0 \xrightarrow{\Phi^{0,1,2}} 0(1_0 2_0) \xrightarrow{\mu(0,(R_0^{1,2})^2)} 0(1_0 2_0) \xrightarrow{(\Phi^{0,1,2})^{-1}} (01_0)2_0.$$

Then, as

$$F(\Phi^{0,1,2}) = \Phi^{0,1,2} \cdot g_1(x_{01}|x_{12}^0, \ldots, x_{12}^{N-1})$$

and

$$F(0(R_{12}^0)^2) = F(\mu(0,(R_{12}^0)^2) = \mu(0, F((R_{12}^0)^2)) = (x_{12}^0)^{2\lambda'},$$

we obtain, for $\lambda = 2\lambda' + 1$

$$F(x_{12}^0) = g_1(x_{01}|x_{12}^0, \ldots, x_{12}^{N-1}) \cdot (x_{12}^0)^{\lambda} \cdot g_1^{-1}(x_{01}|x_{12}^0, \ldots, x_{12}^{N-1})$$

Next, as $x_{12}^{\alpha} = \alpha \cdot x_{12}^0$ for $\alpha \in \Gamma$, by Γ-equivariance we wave

$$\begin{aligned}
F(x_{12}^{\alpha}) &= \alpha \cdot F(x_{12}^0) \\
&= \alpha \cdot (g_1(x_{01}^N|x_{12}^0, \ldots, x_{12}^{N-1}) \cdot (x_{12}^0)^{\lambda} \cdot g_1^{-1}(x_{01}^N|x_{12}^0, \ldots, x_{12}^{N-1})) \\
&= g_1(\alpha \cdot x_{01}^N|\alpha \cdot x_{12}^0, \ldots, \alpha \cdot x_{12}^{N-1}) \cdot \alpha \cdot (x_{12}^0)^{\lambda} \cdot g_1^{-1}(\alpha \cdot x_{01}^N|\alpha \cdot x_{12}^0, \ldots, \alpha \cdot x_{12}^{N-1}) \\
&= g_1(x_{01}^{\alpha N}|x_{12}^{\alpha}, \ldots, x_{12}^{\alpha+N-1}) \cdot (x_{12}^{\alpha})^{\lambda} \cdot g_1^{-1}(x_{01}^N|x_{12}^{\alpha}, \ldots, x_{12}^{\alpha+N-1}) \\
&= \mathrm{Ad}(g_1(x_{01}^{\alpha N}|x_{12}^{\alpha}, \ldots, x_{12}^{\alpha+N-1}))(x_{12}^{\alpha})^{\lambda}).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
(F\circ G)(\Psi^{0,1,2}) &= F(\Psi^{0,1,2}\cdot g_2(x_{01}^N|x_{12}^0,\dots,x_{12}^{N-1}))\\
&= \Psi^{0,1,2}\cdot g_1(x_{01}^N|x_{12}^0,\dots,x_{12}^{N-1})\cdot g_2(F(x_{01}^N)|F(x_{12}^0),\dots,F(x_{12}^{N-1}))\\
&= \Psi^{0,1,2}\cdot g_1(x_{01}^N|x_{12}^0,\dots,x_{12}^{N-1})\\
&\cdot g_2(\lambda\cdot x_{01}^N|\lambda\cdot g_1(x_{01}|x_{12}^0,\dots,x_{12}^{N-1})\cdot(x_{12}^0)^\lambda\cdot\lambda\cdot g_1^{-1}(x_{01}^N|x_{12}^0,\dots,x_{12}^{N-1}),\\
&\dots,\lambda\cdot g_1(x_{01}^N|x_{12}^{N-1},\dots,x_{12}^{2N-2})\cdot x_{12}^{(N-1)\lambda}\cdot\lambda\cdot g_1^{-1}(x_{01}^N|x_{12}^{N-1},\dots,x_{12}^{2N-2}))\\
&= \Psi^{0,1,2}\cdot\lambda\cdot g_1(x_{01}^N|x_{12}^0,\dots,x_{12}^{N-1})\\
&\cdot g_2(x_{01}^N|g_1(x_{01}|x_{12}^0,\dots,x_{12}^{N-1})\cdot x_{12}^0\cdot g_1^{-1}(x_{01}^N|x_{12}^0,\dots,x_{12}^{N-1}),\\
&\dots,g_1(x_{01}^N|x_{12}^{N-1},\dots,x_{12}^{2N-2})\cdot x_{12}^{N-1}\cdot g_1^{-1}(x_{01}^N|x_{12}^{N-1},\dots,x_{12}^{2N-2})))
\end{aligned}$$

which is nothing but the composition law in the group $\mathrm{GT}^\Gamma(\mathbf{k})$. This concludes the proof, as the composite of moperad morphisms $F\circ G$ is compatible with the composition of operad morphisms $\Phi\circ\Psi$. Now, the fact that the defining sets in $\mathbf{GT}^\Gamma(\mathbf{k})$ and $\mathrm{GT}^\Gamma(\mathbf{k})$ are isomorphic is a straightforward consequence of the composite of bijections

$$\mathbf{GT}^\Gamma(\mathbf{k})\longrightarrow\mathbf{Ass}^\Gamma(\mathbf{k})\longrightarrow\mathrm{Ass}^\Gamma(\mathbf{k})\longrightarrow\mathrm{GT}^\Gamma(\mathbf{k}).$$

This finishes the proof. □

Definition 4.2.19. *The graded cyclotomic Grothendieck–Teichmüller group is the group*

$$\mathbf{GRT}^\Gamma(\mathbf{k}):=\mathrm{Aut}^+_{(\mathrm{Mop}(\mathbf{PaCD}(\mathbf{k}),\Phi)}(\mathbf{PaCD}^\Gamma(\mathbf{k}))^\Gamma$$

of Γ-equivariant automorphisms of the $\mathbf{PaCD}(\mathbf{k})$-moperad $\mathbf{PaCD}^\Gamma(\mathbf{k})$ which are the indentity on objects.

Definition 4.2.20. *Define $\mathrm{GRT}^\Gamma_{(\bar 1,1)}(\mathbf{k})$ as the set of pairs (Φ,Ψ) with $\Phi\in\mathrm{GRT}_1(\mathbf{k})$ and $\Psi\in\exp(\hat{\mathfrak{t}}_3^\Gamma(\mathbf{k}))$, such that*

$$\Psi^{0,1,2}(\Psi^{0,2,1})^{-1}\Psi(t_{02}|t_{12}^1,t_{12}^0,\dots,t_{12}^{2-N})\Psi(t_{01}|t_{12}^1,\dots,t_{12}^N)^{-1}=1,\tag{4.13}$$

$$t_{01}+\sum_{\alpha=1}^N\mathrm{Ad}(\Psi(t_{01}|t_{12}^\alpha,\dots,t_{12}^{\alpha+N-1}))(t_{12}^\alpha)+\mathrm{Ad}\left(\Psi_{0,1,2}\Psi_{0,2,1}^{-1}\right)(t_{02})=0,\tag{4.14}$$

as equalities in $\hat{\mathfrak{t}}_2^\Gamma(\mathbf{k})$, where $t_{01}+\sum_{\alpha=0}^n t_{12}^\alpha+t_{02}=0$, and

$$\Psi^{01,2,3}\Psi^{0,1,23}=\Psi^{0,1,2}\Psi^{0,12,3}\Phi^{1,2,3},\tag{4.15}$$

as an equality in $\exp(\hat{\mathfrak{t}}_3^\Gamma(\mathbf{k}))$. $\mathrm{GRT}^\Gamma_{(\bar 1,1)}(\mathbf{k})$ is a group when equipped with the product

$$(\Phi_1,\Psi_1)*(\Phi_2,\Psi_2)=(\Phi,\Psi),$$

where

- $\Phi(t_{12},t_{23})=\Phi_2(t_{12},t_{23})\Phi_1(t_{12},\mathrm{Ad}\,\Phi_2(t_{12},t_{23})(t_{23}))$,
- $\Psi^{0,1,2}=\Psi_1\left(t_{01}|\,\mathrm{Ad}((\Psi_2^{0,1,2}))(t_{12}^0),\dots,\mathrm{Ad}(\Psi_2(t_{01}|t_{12}^{N-1},\dots,t_{12}^{2N-2}))(t_{12}^{N-1})\right)\cdot\Psi_2^{0,1,2}\cdot.$

The action of $(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbf{k}^\times$ *by automorphisms of* $\mathfrak{t}_3^\Gamma$ *(resp.* $\mathfrak{t}_3$*) given by* $(c,\gamma)\cdot t_{0i} = \gamma t_{0i}$, $(c,\gamma)\cdot t_{ij}^\alpha = \gamma t_{ij}^{c\alpha}$ *(resp.* $(c,\gamma)\cdot t_{ij} = \gamma t_{ij}$*) induces its action by automorphisms of* $\mathrm{GRT}^\Gamma_{(\bar{1},1)}(\mathbf{k})$*. We denote by* $\mathrm{GRT}^\Gamma(\mathbf{k})$ *the corresponding semidirect product.*

$\mathrm{GRT}^\Gamma_{(\bar{1},1)}(\mathbf{k})$ acts on $\mathrm{Ass}^\Gamma(\mathbf{k})$ from the right by $(\Phi,\Psi) * (h,k) = (\Phi',\Psi')$, where

$$\Phi'(t_{12},t_{23}) = h(t_{12},t_{23})\Phi(t_{12},\mathrm{Ad}(h(t_{12},t_{23}))(t_{23})), \tag{4.16}$$

$$\begin{aligned}&\Psi'(t_{01}|t_{12}^0,\dots,t_{12}^{N_1}) = k(t_{01}|t_{12}^0,\dots,t_{12}^{N_1}) \\ &\Psi\Big(t_{01}|\,\mathrm{Ad}\left(k(t_{01}|t_{12}^0,\dots,t_{12}^{N-1})\right)(t_{12}^0),\dots,\mathrm{Ad}\left(k(t_{01}|t_{12}^{N-1},\dots,t_{12}^{2(N-1)})\right)(t_{12}^{N-1})\Big).\end{aligned} \tag{4.17}$$

This action preserves each $\mathrm{Ass}^\Gamma_{(a,\lambda)}(\mathbf{k})$, and it extends to an action of $\mathrm{GRT}^\Gamma(\mathbf{k})$ on $\mathrm{Ass}^\Gamma(\mathbf{k})$, which is compatible with the action of $(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbf{k}^\times$ on $(\mathbb{Z}/N\mathbb{Z})\times\mathbf{k}$ and commutes with the left action of $\mathrm{GT}^\Gamma(\mathbf{k})$ on $\mathrm{Ass}^\Gamma(\mathbf{k})$.

Proposition 4.2.21. *There is a group isomorphism between* $\mathbf{GRT}^\Gamma(\mathbf{k})$ *and* $\mathrm{GRT}^\Gamma(\mathbf{k})$.

Proof. The map $\mathbf{GRT}^\Gamma(\mathbf{k}) \longrightarrow \mathrm{GRT}^\Gamma(\mathbf{k})$ is constructed as follows. Let F be an automorphism in $\mathrm{Aut}^+_{\mathrm{Mop}(G\mathbf{PaCD}(\mathbf{k}))}(G\mathbf{PaCD}^\Gamma(\mathbf{k}))$ depending on an operad automorphism Ψ in $\mathbf{GRT}(\mathbf{k})$. We have

- $\Psi(X^{1,2}) = X^{1,2}$,
- $\Psi(H^{1,2}) = \lambda H^{1,2}$,
- $\Psi(a^{1,2,3}) = f(t_{12},t_{23})\cdot a^{1,2,3}$,
- $F(b^{0,1,2}) = g(t_{01}|t_{12}^0,\dots,t_{12}^{N-1})\cdot b^{0,1,2}$,
- $F(K^{0,1}) = \mu K^{0,1}$.

where $(\lambda,f)\in\mathrm{GRT}(\mathbf{k})$ and $(\mu,g)\in\mathbf{k}(N)^\times\times\exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$. In light of relations of Remark 4.2.13, the tuple (λ,f,g) satisfies relations (4.13), (4.14) and (4.15). The assignment $(\Psi,F)\mapsto(\lambda,g(t_{12},t_{23}),u_+(x,y),u_-(x,y))$ then defines a map $\mathbf{GRT}^\Gamma(\mathbf{k})\longrightarrow\mathrm{GRT}^\Gamma(\mathbf{k})$.

Let's now prove that the composition of automorphisms in $\mathrm{Aut}^+_{\mathrm{Mop}(G\mathbf{PaCD}(\mathbf{k}))}(G\mathbf{PaCD}^\Gamma(\mathbf{k}))$ corresponds to the composition law of the group $\mathrm{GRT}^\Gamma(\mathbf{k})$. We already know that the composition of automorphisms Φ and Ψ in $\mathrm{Aut}^+_{\mathrm{Op}\,\hat{\mathcal{G}}}(G\mathbf{PaCD}(\mathbf{k}))$ corresponds to the composition law in $\mathrm{GRT}(\mathbf{k})$, that is, the associated couples (λ,f_1) and (μ,f_2) in $\mathbf{k}^\times\times\exp(\hat{\hat{\mathfrak{t}}}_3(\mathbf{k}))$ satisfy

$$(\Phi\circ\Psi)(H_{1,2}) = \lambda\mu H_{1,2}$$

$$(\Phi\circ\Psi)(\mathbb{1}_{1,2,3}) = \mathbb{1}_{1,2,3}\cdot f_1(t_{12},t_{23})\cdot f_2(\lambda t_{12}, f_1(t_{12},t_{23})\cdot\lambda t_{23}\cdot f_1(t_{12},t_{23})^{-1}).$$

We also already showed that any two automorphisms F and G in the group $\mathrm{Aut}^+_{\mathrm{Mop}(G\mathbf{PaCD}(\mathbf{k}))}(G\mathbf{PaCD}^\Gamma(\mathbf{k}))$, depending on Φ and Ψ respectively, are associated to elements $\phi_1(t_{01}|t_{12}^0,\dots,t_{12}^{N-1})$ and $\phi_2(t_{01}|t_{12}^0,\dots,t_{12}^{N-1})$ which represent automorphisms of the parenthesized word $(01_0)2_0$ in the groupoid $G\mathbf{PaCD}^\Gamma(\mathbf{k})(2)$ i.e. in $\exp(\hat{\hat{\mathfrak{t}}}_2^\Gamma(\mathbf{k}))$. Let us now place ourselves in the group $A = \mathrm{Aut}_{G\mathbf{PaCD}^\Gamma(\mathbf{k})(3)}((01_0)2_0)$. In A, we have

$$t_{01} = (K^{0,1})2 = \mu(K^{0,1},2) = \mu\circ_0 K^{0,1}$$

We then have $F(t_{01}) = \lambda t_{01}$ for some invertible $\lambda \in \mathbf{k}^\times$. Next, let us compute $F(t_{12}^0)$. Again in the group A, the element t_{12}^0 can be decomposed as

$$(01_0)2_0 \xrightarrow{\mathbb{1}_{0,1,2}} 0(1_0 2_0) \xrightarrow{\mu(0,H_{12}^0)} 0(1_0 2_0) \xrightarrow{\mathbb{1}_{0,1,2}^{-1}} (01_0)2_0.$$

Then, as

$$F(\mathbb{1}_{0,1,2}) = \mathbb{1}_{0,1,2} \cdot \phi_1(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1})$$

and

$$F(0, H_{12}^0) = F(\mu(0, H_{12}^0) = \mu(0, F(H_{12}^0)) = \lambda t_{12}^0,$$

we obtain

$$F(t_{12}^0) = \phi_1(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1}) \cdot \lambda t_{12}^0 \cdot \phi_1^{-1}(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1})$$

Next, as $t_{12}^\alpha = \alpha \cdot t_{12}^0$ for $\alpha \in \Gamma$, by Γ-equivariance we wave

$$\begin{aligned}
F(t_{12}^\alpha) &= \alpha \cdot F(t_{12}^0) \\
&= \alpha \cdot (\phi_1(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1}) \cdot \lambda t_{12}^0 \cdot \phi_1^{-1}(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1})) \\
&= \phi_1(\alpha \cdot t_{01}|\alpha \cdot t_{12}^0, \ldots, \alpha \cdot t_{12}^{N-1}) \cdot \lambda\alpha \cdot t_{12}^0 \cdot \phi_1^{-1}(\alpha \cdot t_{01}|\alpha \cdot t_{12}^0, \ldots, \alpha \cdot t_{12}^{N-1}) \\
&= \phi_1(t_{01}|t_{12}^\alpha, \ldots, t_{12}^{\alpha+N-1}) \cdot \lambda t_{12}^\alpha \cdot \phi_1^{-1}(t_{01}|t_{12}^\alpha, \ldots, t_{12}^{\alpha+N-1})
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
(F \circ G)(b^{0,1,2}) &= F(b^{0,1,2} \cdot \phi_2(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1})) \\
&= b^{0,1,2} \cdot \phi_2(F(t_{01})|F(t_{12}^0), \ldots, F(t_{12}^{N-1})) \\
&= b^{0,1,2} \cdot \phi_2(\lambda \cdot t_{01}|\lambda \cdot \phi_1(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1}) \cdot \lambda t_{12}^0 \cdot \lambda \cdot \phi_1^{-1}(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1}), \\
&\ldots, \lambda \cdot \phi_1(t_{01}|t_{12}^{N-1}, \ldots, t_{12}^{2N-2}) \cdot \lambda t_{12}^{N-1} \cdot \lambda \cdot \phi_1^{-1}(t_{01}|t_{12}^{N-1}, \ldots, t_{12}^{2N-2})) \\
&= b^{0,1,2} \cdot \lambda \cdot \phi_2(t_{01}|\phi_1(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1}) \cdot t_{12}^0 \cdot \phi_1^{-1}(t_{01}|t_{12}^0, \ldots, t_{12}^{N-1}), \\
&\ldots, \phi_1(t_{01}|t_{12}^{N-1}, \ldots, t_{12}^{2N-2}) \cdot t_{12}^{N-1} \cdot \phi_1^{-1}(t_{01}|t_{12}^{N-1}, \ldots, t_{12}^{2N-2})))
\end{aligned}$$

which is nothing but the composition law in the group $\mathrm{GRT}^\Gamma(\mathbf{k})$. This concludes the proof, as the composite of moperad morphisms $F \circ G$ is compatible with the composition of operad morphisms $\Phi \circ \Psi$. Now, the fact that the defining sets in $\mathbf{GRT}^\Gamma(\mathbf{k})$ and $\mathrm{GRT}^\Gamma(\mathbf{k})$ are isomorphic is a straightforward consequence of the composite of bijections

$$\mathbf{GRT}^\Gamma(\mathbf{k}) \longrightarrow \mathbf{Ass}^\Gamma(\mathbf{k}) \longrightarrow \mathrm{Ass}^\Gamma(\mathbf{k}) \longrightarrow \mathrm{GRT}^\Gamma(\mathbf{k}).$$

This finishes the proof. □

4.2.9 Torsors

Finally, we promote this correspondence into a torsor isomorphism.

Theorem 4.2.22. *There is a torsor isomorphism*

$$(\widehat{\mathbf{GT}}^\Gamma(\mathbf{k}), \mathbf{Ass}^\Gamma(\mathbf{k}), \mathbf{GRT}^\Gamma(\mathbf{k})) \longrightarrow (\widehat{\mathrm{GT}}^\Gamma(\mathbf{k}), \mathrm{Ass}^\Gamma(\mathbf{k}), \mathrm{GRT}^\Gamma(\mathbf{k})). \tag{4.18}$$

Proof. This is a summary of most of the above results. First of all, we know that $(\widehat{\mathbf{GT}}^{\Gamma}(\mathbf{k}), \mathbf{Ass}^{\Gamma}(\mathbf{k}), \mathbf{GRT}^{\Gamma}(\mathbf{k}))$ has a natural torsor structure and that $(\widehat{\mathrm{GT}}^{\Gamma}(\mathbf{k}), \mathrm{Ass}^{\Gamma}(\mathbf{k}), \mathrm{GRT}^{\Gamma}(\mathbf{k}))$ is a torsor by [33]. Next, we proved in Proposition 4.2.18 that there are group isomorphisms between $\widehat{\mathbf{GT}}^{\Gamma}(\mathbf{k})$ and $\widehat{\mathrm{GT}}^{\Gamma}(\mathbf{k})$ and in Proposition 4.2.21 that there are group isomorphisms between $\mathbf{GRT}^{\Gamma}(\mathbf{k})$ and $\mathrm{GRT}^{\Gamma}(\mathbf{k})$. Thus, it is sufficient to show that the actions of $\widehat{\mathbf{GT}}^{\Gamma}(\mathbf{k})$ on $\mathbf{Ass}^{\Gamma}(\mathbf{k})$ and of $\widehat{\mathrm{GT}}^{\Gamma}(\mathbf{k})$ on $\mathrm{Ass}^{\Gamma}(\mathbf{k})$ are compatible and that the actions of $\widehat{\mathbf{GRT}}^{\Gamma}(\mathbf{k})$ on $\mathbf{Ass}^{\Gamma}(\mathbf{k})$ and of $\mathrm{GRT}^{\Gamma}(\mathbf{k})$ on $\mathrm{Ass}^{\Gamma}(\mathbf{k})$ are compatible. Under the correspondence of Theorem 4.2.18, the image of the natural action of $\widehat{\mathbf{GT}}^{\Gamma}(\mathbf{k})$ on $\mathbf{Ass}^{\Gamma}(\mathbf{k})$ is exactly the action of $\widehat{\mathrm{GT}}^{\Gamma}(\mathbf{k})$ on $\mathrm{Ass}^{\Gamma}(\mathbf{k})$. Both actions are then compatible. Under the correspondence of Theorem 4.2.21, the image of the natural action of $\mathbf{GRT}^{\Gamma}(\mathbf{k})$ on $\mathbf{Ass}^{\Gamma}(\mathbf{k})$ is exactly the action of $\mathrm{GRT}^{\Gamma}(\mathbf{k})$ on $\mathrm{Ass}^{\Gamma}(\mathbf{k})$. Both actions are then compatible. □

4.3 Modules associated with twisted configuration spaces (ellipsitomic associators)

4.3.1 Compactified twisted configuration space of the torus

Consider the group $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, let $\mathbb{T}$ be the topological torus and consider the connected Γ-covering $p : \tilde{\mathbb{T}} \longrightarrow \mathbb{T}$ corresponding to the canonical surjective group morphism $\rho : \pi_1(\mathbb{T}) = \mathbb{Z}^2 \longrightarrow \Gamma$ senging the generators of Z^2 to their corresponding reduction in Γ. To any finite set I with cardinality n we associate the Γ-twisted configuration space

$$\mathrm{Conf}(\mathbb{T}, I, \Gamma) := \{\mathbf{z} = (z_1, \ldots, z_n) \in \tilde{\mathbb{T}}^I | p(z_i) \neq p(z_j) \text{ if } i \neq j\},$$

and let $\mathrm{C}(\mathbb{T}, I, \Gamma) := \mathrm{Conf}(\mathbb{T}, I, \Gamma)/\tilde{\mathbb{T}}$ be its reduced version. We then consider the Fulton-MacPherson compactification $\overline{\mathrm{C}}(\mathbb{T}, n, \Gamma)$ of $\mathrm{C}(\mathbb{T}, n, \Gamma)$ in the same way as before by means of the well-defined map

$$\mathrm{C}(\mathbb{T}, n, \Gamma) \hookrightarrow \overline{\mathrm{C}}(\mathbb{T}, (MN)^n).$$

The boundary $\partial\overline{\mathrm{C}}(\mathbb{T}, n, \Gamma) = \overline{\mathrm{C}}(\mathbb{T}, n, \Gamma) - \mathrm{C}(\mathbb{T}, n, \Gamma)$ is made of the following irreducible components: for any partition $J_1 \coprod \cdots \coprod J_k$ of $\{1, ..., n\}$ there is a component

$$\partial_{J_1, \cdots, J_k}\overline{\mathrm{C}}(\mathbb{T}, n, \Gamma) \cong \prod_{i=1}^{k} (\overline{\mathrm{C}}(\mathbb{C}, J_i)) \times \overline{\mathrm{C}}(\mathbb{T}, k, \Gamma).$$

The inclusion of boundary components provides $\overline{\mathrm{C}}(\mathbb{T}, -, \Gamma)$ with the structure of a module over the operad $\overline{\mathrm{C}}(\mathbb{C}, -)$ in topological spaces.

4.3.2 The PaB-module parenthesized twisted elliptic braids

We have inclusions of topological modules

$$\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{S}^1, -) \subset \overline{\mathrm{C}}(\mathbb{T}, -)$$

over

$$\mathbf{Pa} \subset \overline{\mathrm{C}}(\mathbb{R}, -) \subset \overline{\mathrm{C}}(\mathbb{C}, -)\,.$$

Denote by $i : \mathbf{Pa} \longrightarrow \overline{\mathrm{C}}(\mathbb{S}^1, -)$ the inclusion morphism. Let $\Gamma = \mathbb{Z}/M\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}$ and for every set I_n of cardinality n, consider the collection of all $(N\times M)^n$-fold maps $\phi_n : \overline{\mathrm{C}}(\mathbb{T}, \Gamma, n) \longrightarrow \overline{\mathrm{C}}(\mathbb{T}, n)$. We get a collection of diagrams

$$\begin{tikzcd}
\mathbf{Pa}_n^{\Gamma} \arrow[r] \arrow[d] & \overline{\mathrm{C}}(\mathbb{T}, \Gamma, n) \arrow[d, "\phi_n"] \\
\mathbf{Pa}_n \arrow[r, "i_n"] & \overline{\mathrm{C}}(\mathbb{T}, n)
\end{tikzcd}$$

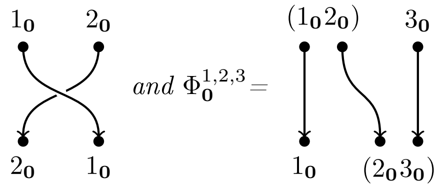
where we define $\mathbf{Pa}_n^{\Gamma} := i_n^{\star}\phi_n$ i.e. as the pull-back of the fold map along the inclusion map. For example, elements of $\mathbf{Pa}_n^{\Gamma}$ are Γ-labelled parenthesized permutations of length n and $\mathbf{Pa}^{\Gamma}$ is an operad module over $\mathbf{Pa}$. Then it makes sense to define

$$\mathbf{PaB}_{e\ell\ell}^{\Gamma} := \pi_1\left(\overline{\mathrm{C}}(\mathbb{T}, \Gamma, -), \mathbf{Pa}^{\Gamma}\right)\,,$$

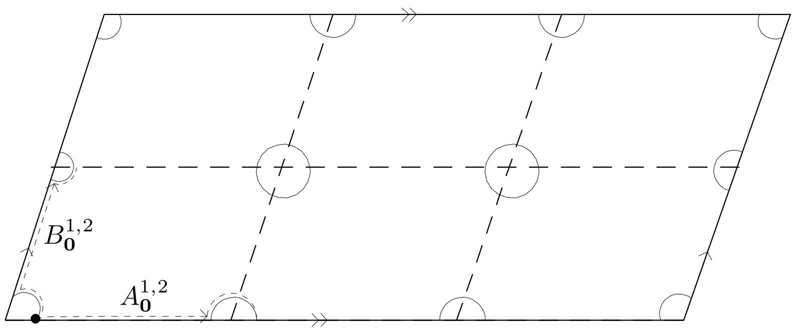
which is a **PaB**-module.

Example 4.3.1 (Notable arrows in $\mathbf{PaB}_{e\ell\ell}^{\Gamma}(2)$)**.** *Write* $\mathbf{0} := (\bar{0}, \bar{0})$*. Let* $R_{\mathbf{0}}^{1,2}$ *and* $\Phi_{\mathbf{0}}^{1,2,3}$ *be the unique lifts of* $R^{1,2}$ *and* $\Phi^{1,2,3} \in \mathbf{PaB}$ *starting at* $1_{\mathbf{0}}2_{\mathbf{0}}$ *and* $(1_{\mathbf{0}}2_{\mathbf{0}})3_{\mathbf{0}}$ *respectively. These paths can be depicted as follows:*

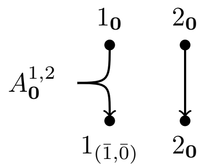
$R_{\mathbf{0}}^{1,2} =$ *and* $\Phi_{\mathbf{0}}^{1,2,3} =$

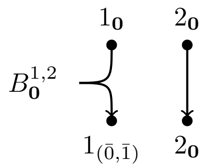


Next, for $1 \leq i \neq j \neq k \leq n$ *and* $\alpha \in \Gamma$*, let* $\theta(\alpha_i) \cdot R_{\mathbf{0}}^{i,j}$ *and* $\theta(\alpha_i) \cdot \Phi_{\mathbf{0}}^{i,j,k}$ *be the unique lifts of* $R^{i,j}$ *and* $\Phi^{i,j,k} \in \mathbf{PaB}$ *starting at* $i_{\alpha}j_{\mathbf{0}}$ *and* $(i_{\alpha}j_{\mathbf{0}})k_{\mathbf{0}}$ *respectively. Additionnally, we also have two morphisms,* $A_{\mathbf{0}}^{1,2}$ *and* $B_{\mathbf{0}}^{1,2}$ *from* $(1_{\mathbf{0}}2_{\mathbf{0}})$ *to* $(1_{(\bar{1},\bar{0})}2_{\mathbf{0}})$ *and from* $(1_{\mathbf{0}}2_{\mathbf{0}})$ *to* $(1_{(\bar{0},\bar{1})}2_{\mathbf{0}})$ *respectively which are the following paths*



They can be alternatively depicted as follows:





Now let $p, q \geq 1$. We introduce the following notation:

$$(A_{\mathbf{0}}^{1,2})^{(p,0)_1} := \prod_{k=0,\ldots,p-1}^{\rightarrow} (\theta((\bar{k},\bar{0})_1)\cdot A_{\mathbf{0}}^{1,2}) = A_{\mathbf{0}}^{1,2}(\theta((\bar{1},\bar{0})_1)\cdot A_{\mathbf{0}}^{1,2})(\theta((\bar{2},\bar{0})_1)\cdot A_{\mathbf{0}}^{1,2})\cdots(\theta((\overline{p-1},\bar{0})_1)\cdot A_{\mathbf{0}}^{1,2}),$$

which is an element in $\mathrm{Hom}_{\mathbf{PaB}^{\Gamma}_{e\ell\ell}(2)}((1_{\mathbf{0}}, 2_{\mathbf{0}}), (1_{(\bar{p},\bar{0})}, 2_{\mathbf{0}}))$ and

$$(B_{\mathbf{0}}^{1,2})^{(0,q)_1} := \prod_{k=0,\ldots,q-1}^{\rightarrow} (\theta((\bar{0},\bar{k})_1)\cdot B_{\mathbf{0}}^{1,2}) = B_i(\theta((\bar{0},\bar{1})_1)\cdot B_{\mathbf{0}}^{1,2})(\theta((\bar{0},\bar{2})_1)\cdot B_{\mathbf{0}}^{1,2})\cdots(\theta((\bar{0},\overline{q-1})_1)\cdot B_{\mathbf{0}}^{1,2})$$

which is an element in $\mathrm{Hom}_{\mathbf{PaB}^{\Gamma}_{e\ell\ell}(2)}((1_{\mathbf{0}}, 2_{\mathbf{0}}), (1_{(\bar{0},\bar{q})}, 2_{\mathbf{0}}))$.

Theorem 4.3.2. *As a* **PaB***-module (in groupoid) having* $\mathbf{Pa}^{\Gamma}$ *as* **Pa***-module of objects,* $\mathbf{PaB}^{\Gamma}_{e\ell\ell}$ *is freely generated by* $A_{\mathbf{0}} := A_{\mathbf{0}}^{1,2}$ *and* $B_{\mathbf{0}} := B_{\mathbf{0}}^{1,2}$ *together with the following relations:*

(tN1) $(\mathcal{A})^{(M,0)} = \mathrm{Id}_{1_{\mathbf{0}}2_{\mathbf{0}},3_{\mathbf{0}}}$, *where*

$$\mathcal{A} := \Phi_{\mathbf{0}}^{1,2,3} A_{\mathbf{0}}^{1,23} \theta((\bar{1},\bar{0})_1)(R_{\mathbf{0}}^{1,23}\Phi_{\mathbf{0}}^{2,3,1} A_{\mathbf{0}}^{2,31}\theta((\bar{1},\bar{0})_2)(R_{\mathbf{0}}^{2,31}\Phi^{3,1,2}A_{\mathbf{0}}^{3,12}\theta((\bar{1},\bar{0})_3 R_{\mathbf{0}}^{3,12})),$$

(tN2) $(\mathcal{B})^{(0,N)} = \mathrm{Id}_{1_{\mathbf{0}}2_{\mathbf{0}},3_{\mathbf{0}}}$, *where*

$$\mathcal{B} := \Phi_{\mathbf{0}}^{1,2,3} B_{\mathbf{0}}^{1,23} \theta((\bar{0},\bar{1})_1)(R_{\mathbf{0}}^{1,23}\Phi^{2,3,1} B_{\mathbf{0}}^{2,31}\theta((\bar{0},\bar{1})_2)(R_{\mathbf{0}}^{2,31}\Phi^{3,1,2}B_{\mathbf{0}}^{3,12}\theta((\bar{0},\bar{1})_3 R_{\mathbf{0}}^{3,12})),$$

(tE) $R_{(\bar{1},\bar{1})}^{1,2} R_{(\bar{1},\bar{1})}^{2,1} = \Phi^{1,2,3} B_{\mathbf{0}}^{1,23}\theta((\bar{0},\bar{1})_1)((\Phi_{\mathbf{0}}^{1,2,3})^{-1}(R_{\mathbf{0}}^{1,2})^{-1}\Phi_{\mathbf{0}}^{2,1,3}(A_{\mathbf{0}}^{2,13})^{-1}\theta((\overline{-1},\bar{0})_2)X)$ *where*

$$X = (\Phi_{\mathbf{0}}^{2,1,3})^{-1}(R_{\mathbf{0}}^{2,1})^{-1}\Phi_{\mathbf{0}}^{1,2,3}(B_{\mathbf{0}}^{1,23})^{-1}\theta((\bar{0},\overline{-1})_1)((\Phi_{\mathbf{0}}^{1,2,3})^{-1}R_{\mathbf{0}}^{1,2}\Phi_{\mathbf{0}}^{2,1,3}A_{\mathbf{0}}^{2,13}Y)$$

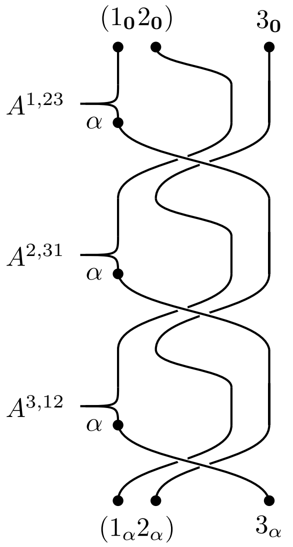
and

$$Y = \theta((\bar{1},\bar{0})_2)((\Phi_{\mathbf{0}}^{2,1,3})^{-1}R_{\mathbf{0}}^{2,1})$$

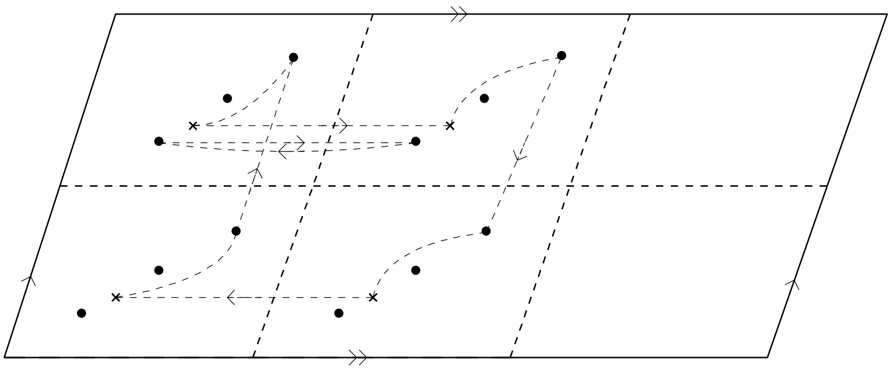
as arrows from $(1_0 2_0)3_0$ *to* $(1_0 2_0)3_0$ *in* $\mathbf{PaB}^{\Gamma}_{e\ell\ell}(3)$.

Proof. Let Q^{Γ} be the **PaB**-module with the above presentation, Q be the **PaB**-module with the presentation in Theorem 4.1.3, let $n \geq 1$ and let $p \in Q^{\Gamma}(n)$. By universal property of Q^{Γ}, there is a morphism of **PaB**-modules $Q^{\Gamma} \longrightarrow \mathbf{PaB}^{\Gamma}_{e\ell\ell}$ which is the identity on objects. Indeed, relations (tN1), (tN2), (tE) are satisfied by $\mathbf{PaB}^{\Gamma}_{e\ell\ell}$.

For instance, $\mathcal{A}$ can be depicted as follows



and the right hand side of relation (tE) can be pictured as follows in the open twisted configuration space:



As before, we are left to prove that the morphism $\mathrm{Aut}_{Q^\Gamma(n)}(p) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^\Gamma_{e\ell\ell}(n)}(p)$ is a group isomorphism.

On the one hand, by definition of $\mathbf{PaB}^\Gamma_{e\ell\ell}$, we know that $\mathrm{Aut}_{\mathbf{PaB}^\Gamma_{e\ell\ell}(n)}(p)$ is exactly the fundamental group $\pi_1(\bar{\mathrm{C}}(\mathbb{T}, n, \Gamma), p)$, where p is in the boundary of $\bar{\mathrm{C}}(\mathbb{T}, n, \Gamma)$. By the same argument as before, we have isomorphisms $\pi_1(\bar{\mathrm{C}}(\mathbb{T}, n, \Gamma), p) \simeq \pi_1(\mathrm{C}(\mathbb{T}, n, \Gamma), p_{reg})$ and $\pi_1(\bar{\mathrm{C}}(\mathbb{T}, [n], \Gamma), [p]) \simeq \pi_1(\mathrm{C}(\mathbb{T}, [n], \Gamma), [p_{reg}])$. Consider the Γ^{n-1}-cover map $f : \overline{\mathrm{C}}(\mathbb{T}, n, \Gamma) \longrightarrow \overline{\mathrm{C}}(\mathbb{T}, n)$. Now, one can identify $\mathrm{Aut}_{\mathbf{PaB}^\Gamma_{e\ell\ell}(n)}(p)$ with the kernel of the surjective map $\mathrm{Aut}_{\mathbf{PaB}_{e\ell\ell}(n)}(f(p)) \longrightarrow \Gamma^n/\Gamma$ and the isomorphism $\mathrm{Aut}_{Q(n)}(f(p)) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}_{e\ell\ell}(n)}(f(p))$ commutes with the projections to

Γ^n/Γ. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}_{Q^\Gamma(n)}(p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{T},n,\Gamma),p\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{T},n,\Gamma),p_{reg}\right) \arrow[l] \arrow[d] \\
\mathrm{Aut}_{Q(n)}(f(p)) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{C}}(\mathbb{T},n),f(p)\right) \arrow[d] & \pi_1\left(\mathrm{C}(\mathbb{T},n),f(p_{reg})\right) \arrow[l] \arrow[d] \\
\Gamma^n/\Gamma \arrow[r, equal] & \Gamma^n/\Gamma \arrow[r, equal] & \Gamma^n/\Gamma
\end{tikzcd}$$

Thus, in order to show that $\mathrm{Aut}_{Q^\Gamma(n)}(p) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^\Gamma_{e\ell\ell}(n)}(p)$ is an isomorphism, it suffices to show that $\mathrm{Aut}_{Q^\Gamma(n)}(p)$ is isomorphic to the kernel of the projection $\mathrm{Aut}_{Q(n)}(f(p)) \longrightarrow \Gamma^n/\Gamma$.

Let us first show that the map $\phi : \mathrm{Aut}_{Q^\Gamma(n)}(p) \longrightarrow \mathrm{Aut}_{Q(n)}(f(p))$ is injective. By definition, Q^Γ is generated in the morphisms level by $A^{1,2}_{\mathbf{0}}$ and $B^{1,2}_{\mathbf{0}}$. The map ϕ sends $A^{1,2}_{\mathbf{0}}$ and $B^{1,2}_{\mathbf{0}}$ to the generators A and B in $\mathbf{PaB}_{e\ell\ell}(2)$.

An element of $\mathrm{Aut}_{Q^\Gamma(n)}(p)$ will be given by some string, which we will denote g, in the generators $A_\pm$ of $\mathbf{PaB}^\Gamma_{e\ell\ell}$ and the liftings of $R, \Phi \in \mathbf{PaB}$. Let g be the image by ϕ of some string h in $\mathrm{Aut}_{Q^\Gamma(n)}(p)$. Now, to ask g to be trivial means that there is a finite number of operations involving only relations (N1), (N2),and (E) in $\mathbf{PaB}_{e\ell\ell}$ taking the string g to the identity map. But these relations in $\mathbf{PaB}_{e\ell\ell}$ are the images of the corresponding relations, seen as relations in $\mathbf{PaB}^\Gamma_{e\ell\ell}$. Thus, we conclude that the procedure that takes f to the identity map is in fact the image of a procedure taking h to the identity map in $\mathrm{Aut}_{Q^\Gamma(n)}(p)$. This shows the injectivity of ϕ.

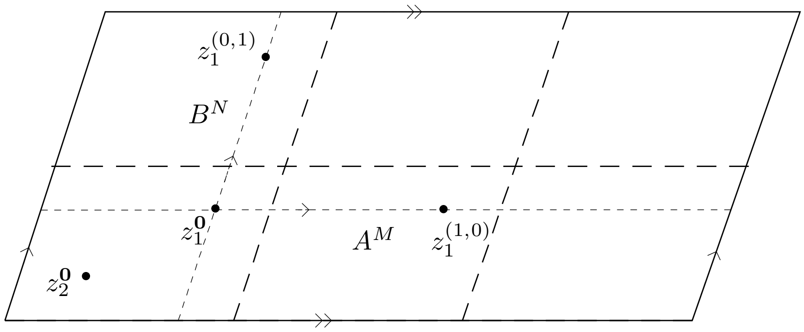
Finally, the map ϕ is surjective in the kernel of the projection $\phi_1 : \mathrm{Aut}_{Q(n)}(f(p)) \longrightarrow \Gamma^n/\Gamma$. Recall the presentation of $\bar{\mathrm{B}}_{1,n}$: its generators are σ_i $(i = 1, ..., n-1)$, A_i, B_i $(i = 1, ..., n)$, C_{jk} $(1 \le j < k \le n)$ and its relations are:

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, ..., n-2$,
- $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $1 \le i < j \le n$,
- $\sigma_i^{-1}X_i\sigma_i^{-1} = X_{i+1}, \sigma_iY_i\sigma_i = Y_{i+1}$, for $i = 1, ..., n-1$,
- $(\sigma_i, X_j) = (\sigma_i, Y_j) = 1$, for $i \in \{1, ..., n-1\}, j \in \{1, ..., n\}, j \neq i, i+1$,
- $\sigma_i^2 = C_{i,i+1}C_{i+1,i+2}C_{i,i+2}^{-1}$, for $i = 1, ..., n-1$,
- $(A_i, A_j) = (B_i, B_j) = 1$, for any i, j, $\quad A_1 = B_1 = 1$,
- $(B_k, A_kA_j^{-1}) = (B_kB_j^{-1}, A_k) = C_{jk}$, for $1 \le j < k \le n$,
- $(A_i, C_{jk}) = (B_i, C_{jk}) = 1$, for $1 \le i \le j < k \le n$,

with $X_i = A_iA_{i+1}^{-1}$, $Y_i = B_iB_{i+1}^{-1}$ for $i = 1, ..., n$ (we set $A_{n+1} = B_{n+1} = C_{i,n+1} = 1$). In particular, these relations imply

- $C_{jk} = \sigma_{j,j+1...k}...\sigma_{j+n-k,j+n-k+1...n}\sigma_{j,j+1...n-k+j+1}...\sigma_{k-1,k...n}$,

where $\sigma_{i,i+1...j} = \sigma_{j-1}...\sigma_i$. Recall that $\mathrm{Aut}_{Q(n)}(f(p))$ is nothing but the kernel of $B_{1,n} \longrightarrow \Gamma$ sending X_i to the class of $(1,0)$, Y_i to the class of $(0,1)$ and σ_i to the class of $(0,0)$. Thus, the kernel $ker\phi_1$ is generated by elements A_i^M, B_i^N and $R_\alpha^{1,2}$. If we denote z_i^0 for the marked points of the form $z_i = a_i + \tau b_i$, where $0 < a_n < \cdots < a_1 < 1/M$ and $0 < b_n < \cdots < b_1 < 1/N$ and z_i^α for $z_i^0 + \tilde{\alpha}$ with $\alpha \in \Gamma$, then the orbit of z_i^0 is $\Gamma \cdot z_i^0 = \{z_i^0 + \alpha; \alpha \in \Gamma\}$. Then, we can represent the elements $A^M := A_1^M$ and $B^N = B_1^N$ in the open twisted configuration space as follows



These elements A^M and B^N are precisely the images of the generators $A_{\mathbf{0}}^{1,2}$ and $B_{\mathbf{0}}^{1,2}$ in Q^Γ. Thus, any string in $\mathrm{Aut}_{Q(n)}(f(p))$ contained in the kernel of ϕ_1 is the image of some string in $\mathrm{Aut}_{Q^\Gamma(n)}(p)$. In conclusion, the map $\phi : \mathrm{Aut}_{Q^\Gamma(n)}(p) \longrightarrow \mathrm{Aut}_{Q(n)}(f(p))$ is a bijection in the kernel of ϕ_1. So, by commutativity of the above diagram, we obtain an isomorphism $\mathrm{Aut}_{Q^\Gamma(n)}(p) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}^\Gamma_{e\ell\ell}(n)}(p)$ which lead us to the fact that the morphism $Q^\Gamma \longrightarrow \mathbf{PaB}^\Gamma_{e\ell\ell}$ of **PaB**-modules is an isomorphism.

$\square$

We obtain a $\mathbf{PaB}(\mathbf{k})$-module in $\mathbf{Cat}(\mathbf{CoAss}_\mathbf{k})$ denoted $\mathbf{PaB}^\Gamma_{e\ell\ell}(\mathbf{k}) := \Delta_\mathbf{k}(\mathbf{PaB}^\Gamma_{e\ell\ell})$. Now consider its associated inverse system of $\mathbf{PaB}^{(m)}(\mathbf{k})$-modules given, for all $m \in \mathbb{N}$, by

$$(\mathbf{PaB}^\Gamma_{e\ell\ell})^{(m)}_\mathbf{k} := \mathbf{PaB}^\Gamma_{e\ell\ell}(\mathbf{k})/(\mathcal{I}^m(\mathbf{k}) \cdot \mathbf{PaB}^\Gamma_{e\ell\ell}(\mathbf{k})).$$

By taking the inverse limit over m of these inverse system, we get a $\widehat{\mathbf{PaB}}(\mathbf{k})$-module in $\mathbf{Cat}(\mathbf{CoAss}_\mathbf{k})$

$$\widehat{\mathbf{PaB}}^\Gamma_{e\ell\ell}(\mathbf{k}) := \varprojlim((\mathbf{PaB}^\Gamma_{e\ell\ell})^{(m)}_\mathbf{k}).$$

4.3.3 The Lie algebras $\mathfrak{t}^\Gamma_{1,n}(\mathbf{k})$ and $\bar{\mathfrak{t}}^\Gamma_{1,n}(\mathbf{k})$ of infinitesimal twisted elliptic braidings

In this paragraph, Γ can be replaced by any finite abelian group (with the additive notation).

Definition 4.3.3. *For any integer $n \geq 1$ we define $\mathfrak{t}^\Gamma_{1,n}(\mathbf{k})$ to be the bigraded $\mathbf{k}$-Lie algebra with generators $x_1, \dots, x_n$ in degree $(1,0)$, $y_1, \dots, y_n$ in degree $(0,1)$, t^α_{ij} ($\alpha \in \Gamma$, $1 \leq i \neq j \leq n$) in degree $(1,1)$, and relations*

(NS) $t^\alpha_{ij} = t^{-\alpha}_{ji}$, *for* $i \neq j$,

(NL) $[t^\alpha_{ij}, t^\beta_{kl}] = 0$, *for* $\mathrm{card}\{i,j,k,l\} = 4$,

(N4T) $[t^\alpha_{ij}, t^{\alpha+\beta}_{ik} + t^\beta_{jk}] = 0$, *for* $\mathrm{card}\{i,j,k\} = 3$, *(Ell1)* $[x_i, y_j] = [x_j, y_i] = \sum_{\alpha\in\Gamma} t_{ij}^{\alpha}$, *for* $i \neq j$

(Ell2) $[x_i, x_j] = [y_i, y_j] = 0$

(NEll1) $[x_i, y_i] = -\sum_{j:j\neq i}\sum_{\alpha\in\Gamma} t_{ij}^{\alpha}$,

(NEll2) $[x_i, t_{jk}^{\alpha}] = [y_i, t_{jk}^{\alpha}] = 0$, *for* $\mathrm{card}\{i,j,k\} = 3$,

(NEll3) $[x_i + x_j, t_{ij}^{\alpha}] = [y_i + y_j, t_{ij}^{\alpha}] = 0$, *for* $i \neq j$,

for all $\alpha, \beta \in \Gamma$. *We will call* $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ *the* $\mathbf{k}$*-Lie algebra of infinitesimal twisted elliptic braidings.*

Observe that $\sum_i x_i$ and $\sum_i y_i$ are central in $\mathfrak{t}_{1,n}^{\Gamma}$. Then we denote by $\overline{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})$ the quotient of $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ by $\sum_i x_i$ and $\sum_i y_i$, and the natural morphism $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k}) \longrightarrow \overline{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})\,;\, u \mapsto \bar{u}$. There is an action $\theta : \Gamma^n \longrightarrow \mathrm{Aut}(\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k}))$ given by $\theta(\alpha_i) : t_{ij}^{\beta} \mapsto t_{ij}^{\beta+\alpha}$, and with t_{kl}^{β}, for $k, l \neq i$), x_k and y_k invariant for arbitrary k arbitrary. It restricts to an action on $\overline{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})$.

Proposition 4.3.4. *For any group morphism* $\rho : \Gamma_1 \longrightarrow \Gamma_2$ *we have a comparison morphism* $\phi : \mathfrak{t}_{1,n}^{\Gamma_1}(\mathbf{k}) \longrightarrow \mathfrak{t}_{1,n}^{\Gamma_2}(\mathbf{k})$ *defined by* $x_i \mapsto x_i$, $y_i \mapsto y_i$, *and*

$$t_{ij}^{\alpha} \longmapsto \frac{1}{\#\ker(\rho)} \sum_{\beta\in\mathrm{coker}(\rho)} t_{ij}^{\rho(\alpha)+\beta}.$$

Proof. Let us prove that relation $[x_i, y_j] = \sum_{\alpha\in\Gamma} t_{ij}^{\alpha}$, where $i \neq j$, is preserved by ϕ. On the one hand $[\phi(x_i), \phi(y_j)] = \sum_{\alpha\in\Gamma_2} t_{ij}^{\alpha}$. On the other hand

$$\phi([x_i, y_j]) = \sum_{\alpha\in\Gamma_1} \phi(t_{ij}^{\alpha}) = \sum_{\alpha\in\Gamma_1} \frac{1}{\#\ker(\rho)} \sum_{\beta\in\mathrm{coker}(\rho)} t_{ij}^{\rho(\alpha)+\beta} = \sum_{\alpha\in\Gamma_2} t_{ij}^{\alpha}.$$

The last equality holds because $\rho(\alpha)$ is in the image of ρ and β is not. The fact that remaining relations are preserved is immediate. □

When ρ is not surjective it depends on the choice of a section $\mathrm{coker}(\rho) \longrightarrow \Gamma_2$. Comparison morphisms commute with insertion-corpoduct morphisms. Moreover, both are bigraded and pass to the quotient by $\sum_i x_i$, $\sum_i y_i$. When $\mathbf{k} = \mathbb{C}$ we write $\mathfrak{t}_{1,n}^{\Gamma} := \mathfrak{t}_{1,n}^{\Gamma}(\mathbb{C})$ and $\overline{\mathfrak{t}}_{1,n}^{\Gamma} := \overline{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbb{C})$.

Lemma 4.3.5. $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ *admits the following presentation : generators are* x_i, y_i $(i = 1, \dots, n)$ t_{ij}^{α} $(\alpha \in \Gamma)$ *and relations are*

- $t_{ij}^{\alpha} = t_{ji}^{-\alpha} \quad (i \neq j)\,;$,
- $[t_{ij}^{\alpha}, t_{kl}^{\beta}] = 0 \quad (\mathrm{card}\{i,j,k,l\} = 4)$,
- $[t_{ij}^{\alpha}, t_{ik}^{\alpha+\beta} + t_{jk}^{\beta}] = 0 \quad (\mathrm{card}\{i,j,k\} = 3)$,
- $[x_i, y_j] = [x_j, y_i] = \sum_{\alpha\in\Gamma} t_{ij}^{\alpha} \quad (i \neq j)$
- $[x_i, x_j] = [y_i, y_j] = 0\,;$
- $[\sum_j x_j, y_i] = [\sum_j y_j, x_i] = 0$ *(for any i)*
- $[x_i, t_{jk}^{\alpha}] = [y_i, t_{jk}^{\alpha}] = 0 \quad (\mathrm{card}\{i,j,k\} = 3)$,

Proof. If x_i, y_i and t_{ij}^{α} satisfy the initial relations, then

$$[\sum_j x_j, y_i] = [x_i, y_i] + [\sum_{j \neq i} x_j, y_i] = -\sum_{j: j \neq i} \sum_{\alpha \in \Gamma} t_{ij}^{\alpha} + \sum_{j: j \neq i} \sum_{\alpha \in \Gamma} t_{ij}^{\alpha} = 0.$$

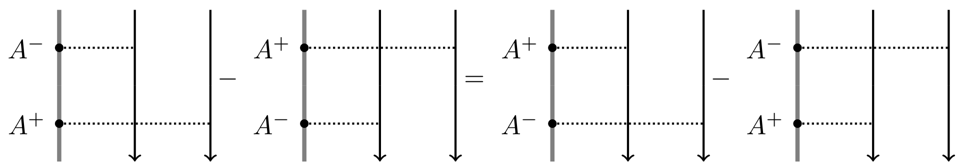
Now, if x_i, y_i and t_{ij}^{α} satisfy the above relations, then relations $[\sum\limits_j x_j, y_i] = 0$ and $[x_j, y_i] = \sum_{\alpha \in \Gamma} t_{ij}^{\alpha}$, for $i \neq j$, imply that $[x_i, y_i] = -\sum_{j: j \neq i} \sum_{\alpha \in \Gamma} t_{ij}^{\alpha}$. Now, relations $[\sum\limits_k x_k, y_j] = 0$ and $[\sum\limits_k x_k, x_i] = 0$ imply that $[\sum\limits_k x_k, \sum_{\alpha \in \Gamma} t_{ij}^{\alpha}] = 0$. Thus, as $[x_i, t_{jk}^{\alpha}] = 0$ if $\mathrm{card}\{i, j, k\} = 3$, we obtain relation $[x_i + x_j, t_{ij}^{\alpha}] = 0$, for $i \neq j$. In the same way we obtain $[y_i + y_j, t_{ij}^{\alpha}] = 0$, for $i \neq j$. □

The $\mathfrak{t}(\mathbf{k})$-module $\mathfrak{t}_1^{\Gamma}(\mathbf{k})$ of infinitesimal twisted elliptic braidings

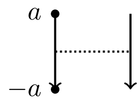
The collection $\mathfrak{t}_1^{\Gamma}(\mathbf{k})$ of the Lie algebras $\mathfrak{t}_{1,n}^{\Gamma}$, for $n \geq 1$ is provided with the structure of a $\mathfrak{t}(\mathbf{k})$-module in $Lie_{\mathbf{k}}$ when endowed with the partial operadic module composition structures given as follows.

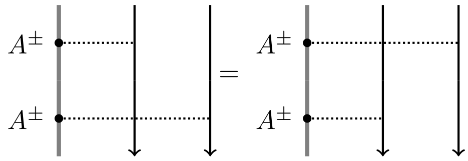
$$\begin{array}{rccc}
\circ_k : & \mathfrak{t}_{1,I}^{\Gamma}(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) & \longrightarrow & \mathfrak{t}_{1,J \sqcup I - \{i\}}^{\Gamma}(\mathbf{k}) \\
& (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha\beta} \\
& (t_{ij}^{\alpha}, 0) & \longmapsto & \begin{cases} t_{ij}^{\alpha} & \text{if} \quad k \notin \{i, j\} \\ \sum\limits_{p \in J} t_{pj}^{\alpha} & \text{if} \quad k = i \\ \sum\limits_{p \in J} t_{ip}^{\alpha} & \text{if} \quad j = k \end{cases} \\
& (x_i, 0) & \longmapsto & \begin{cases} x_i & \text{if} \quad k \neq i \\ \sum\limits_{p \in J} x_p & \text{if} \quad k = i \end{cases} \\
& (y_i, 0) & \longmapsto & \begin{cases} y_i & \text{if} \quad k \neq i \\ \sum\limits_{p \in J} y_p & \text{if} \quad k = i \end{cases}
\end{array}$$

These operadic compositions also induce an operad module structure on the collection of the Lie algebras $\bar{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})$. We will call $\bar{\mathfrak{t}}_1^{\Gamma}(\mathbf{k})$ the module of *infinitesimal twisted elliptic braidings* and, for $\mathbf{CD}_{e\ell\ell}^{\Gamma}(n) := \hat{\mathcal{U}}(\hat{\bar{\mathfrak{t}}}_{1,n}^{\Gamma}(\mathbf{k}))$, the corresponding module in associative algebras $\mathbf{CD}_{e\ell\ell}^{\Gamma} := \{\mathbf{CD}_{e\ell\ell}^{\Gamma}(n)\}_{n \geq 1}$ will be called the module of Γ-labelled elliptic chord diagrams. The elements of the module $\mathbf{CD}_{e\ell\ell}^{\Gamma}$ can be depicted as Γ-labelled elliptic chords on n vertical strands. Thus, by combining the different representations we used in the cyclotomic and elliptic cases, we can depict the labelled elliptic chord relations as follows (we denote $A^+ = x$ and $A^- = y$):

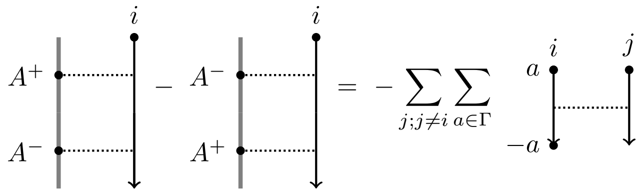


(Ell1b)

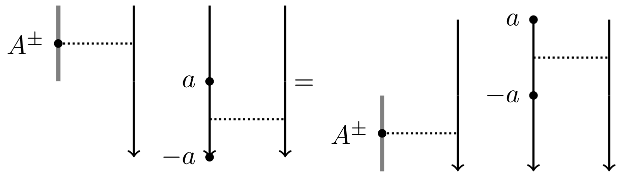
$= \sum_{a\in\Gamma}$ 



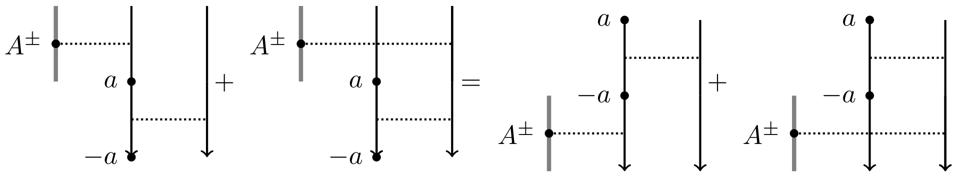
(Ell2b)



(NEll1)



(NEll2)



(NEll3)

Remark 4.3.6. *We expect to study the relation between $\mathbf{CD}^{\Gamma}_{e\ell\ell}$ and Vassiliev invariants in the near future.*

Let $\widehat{\mathbf{CD}}^{\Gamma}_{e\ell\ell}(n)$ be the I-adic completion of $\mathbf{CD}^{\Gamma}_{e\ell\ell}(n)$ with respect to the augmentation ideal I. Since we are in possession of a $\mathbf{Pa}(\mathbf{k})$-module $\mathbf{Pa}^{\Gamma}(\mathbf{k})$, a $\widehat{\mathbf{CD}}(\mathbf{k})$-module $\widehat{\mathbf{CD}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ in $\mathbf{Cat}(\mathbf{CoAss_k})$ and of an operad module morphism $\omega_4 : \mathbf{Pa}^{\Gamma} \longrightarrow \mathrm{Ob}(\widehat{\mathbf{CD}}^{\Gamma}_{e\ell\ell}(\mathbf{k}))$, we are ready to define the $\mathbf{PaCD}(\mathbf{k})$-module

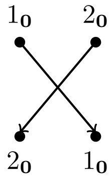
$$\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k}) := \omega_4^{\star}\widehat{\mathbf{CD}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$$

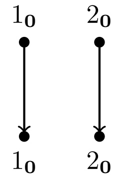
in $\mathbf{Cat}(\mathbf{CoAss_k})$ of parenthesized Γ-labelled elliptic chord diagrams.

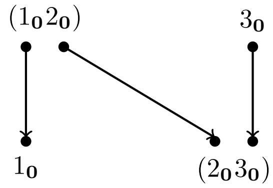
We have $\mathrm{Ob}(\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})) := \mathbf{Pa}^{\Gamma}$ and

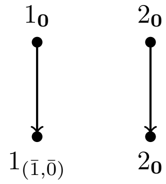
$$\mathrm{Mor}_{\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})(n)}(p,q) := \mathrm{Mor}_{\widehat{\mathbf{CD}}^{\Gamma}_{e\ell\ell}(\mathbf{k})(n)}(pt,pt) = \hat{\mathcal{U}}(\hat{\mathfrak{t}}^{\Gamma}_{1}(\mathbf{k})).$$

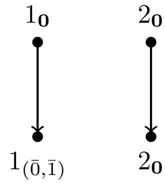
Example 4.3.7 (Notable arrows in $\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})(2)$ and $\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})(3)$)**.** *We have the following arrows in $\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})(2)$ and $\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})(3)$*

$X_{\mathbf{0}}^{1,2} = 1\cdot$ 

$H_{\mathbf{0}}^{1,2} = t_{12}^{\mathbf{0}}\cdot$ 

$a_{\mathbf{0}}^{1,2,3} =$ 

$X_{\mathbf{0},e\ell\ell}^{1,2} = x_1\cdot$ 

$Y_{\mathbf{0},e\ell\ell}^{1,2} = y_1\cdot$ 

Remark 4.3.8. *The elements $X_{1,2}^{e\ell\ell}, Y_{1,2}^{e\ell\ell}$ are generators of the $\mathbf{PaCD}(\mathbf{k})$-module $\mathbf{PaCD}_{e\ell\ell}^{\Gamma}(\mathbf{k})$ and satisfy the following relations*

(tN1) $\tilde{\mathcal{A}}^{(M,0)} = 1$, *where*

$$\tilde{\mathcal{A}} = a_{\mathbf{0}}^{1,2,3} X_{e\ell\ell}^{1,23} \theta((\bar{1},\bar{0})_1)(X_{\mathbf{0}}^{1,23} a_{\mathbf{0}}^{2,3,1} X_{e\ell\ell}^{2,31} \theta((\bar{1},\bar{0})_2)(X_{\mathbf{0}}^{2,31} Z_1))$$

and

$$Z_1 = a_{\mathbf{0}}^{3,1,2} X_{e\ell\ell}^{3,12} \theta((\bar{1},\bar{0})_3) X_{\mathbf{0}}^{3,12}$$

(tN2) $\tilde{\mathcal{B}}^{(0,N)} = 1$, *where*

$$\tilde{\mathcal{B}} = a_{\mathbf{0}}^{1,2,3} Y_{e\ell\ell}^{1,23} \theta((\bar{1},\bar{0})_1)(X_{\mathbf{0}}^{1,23} a_{\mathbf{0}}^{2,3,1} Y_{e\ell\ell}^{2,31} \theta((\bar{1},\bar{0})_2)(X_{\mathbf{0}}^{2,31} Z_2)),$$

and

$$Z_2 = a_{\mathbf{0}}^{3,1,2} Y_{e\ell\ell}^{3,12} \theta((\bar{1},\bar{0})_3) X_{\mathbf{0}}^{3,12}$$

(tM) $X_{\mathbf{0}}^{1,2} X_{\mathbf{0}}^{2,1} = a_{\mathbf{0}}^{1,2,3} Y_{\mathbf{0},e\ell\ell}^{1,23} \theta((\bar{0},\bar{1})_1)((a_{\mathbf{0}}^{1,2,3})^{-1} X_{\mathbf{0}}^{1,2} a_{\mathbf{0}}^{2,1,3} (X_{\mathbf{0},e\ell\ell}^{2,13})^{-1} \theta((\overline{-1},\bar{0})_2) X)$, *where*

$$X = (a_{\mathbf{0}}^{2,1,3})^{-1} X_{\mathbf{0}}^{1,2} a_{\mathbf{0}}^{1,2,3} (Y_{\mathbf{0},e\ell\ell}^{1,23})^{-1} \theta((\bar{0},\overline{-1})_1)((a_{\mathbf{0}}^{1,2,3})^{-1} X_{\mathbf{0}}^{1,2} a_{\mathbf{0}}^{2,1,3} X_{\mathbf{0},e\ell\ell}^{2,13} Y),$$

and

$$Y = \theta((\bar{1},\bar{0})_2)((a_{\mathbf{0}}^{2,1,3})^{-1} X_{\mathbf{0}}^{1,2})$$

as arrows from $(1_0 2_0)3_0$ to $(1_0 2_0)3_0$ in $\mathbf{PaB}_{e\ell\ell}^{\Gamma}(3)$.

4.3.4 Twisted elliptic associators

Fix $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Definition 4.3.9. *A twisted elliptic $\mathbf{k}$-associator is a couple (F, G) where F is in $\mathbf{Ass}(\mathbf{k})$ and G is a Γ-equivariant isomorphism between the $\widehat{\mathbf{PaB}}(\mathbf{k})$-module $\widehat{\mathbf{PaB}}_{e\ell\ell}^{\Gamma}(\mathbf{k})$ and the $G\mathbf{PaCD}(\mathbf{k})$-module $G\mathbf{PaCD}_{e\ell\ell}^{\Gamma}(\mathbf{k})$ which is the identity on objects and which is compatible with F. We denote the set of twisted elliptic $\mathbf{k}$-associators by*

$$\mathbf{Ell}^{\Gamma}(\mathbf{k}) := \mathrm{Iso}^{+}_{(\widehat{\mathbf{PaB}}(\mathbf{k}), G\mathbf{PaCD}(\mathbf{k}))}(\widehat{\mathbf{PaB}}_{e\ell\ell}^{\Gamma}(\mathbf{k}), G\mathbf{PaCD}_{e\ell\ell}^{\Gamma}(\mathbf{k}))^{\Gamma}.$$

Theorem 4.3.10. *There is a one-to-one correspondence between elements of* $\mathbf{Ell}^\Gamma(\mathbf{k})$ *and those the set* $\mathrm{Ell}^\Gamma(\mathbf{k})$ *consisting on quadruples* (μ,Φ,A_+,A_-)*, where* $(\mu,\Phi)\in\mathrm{Ass}(\mathbf{k})$ *and* $A_\pm\in\exp(\hat{\mathfrak{t}}^\Gamma_{1,2}(\mathbf{k}))$*, such that:*

(tN1) $(\tilde{\mathcal{A}}_+)^{(M,0)}=1$ *where*

$$\tilde{\mathcal{A}}_+=\{\Phi^{1,2,3}\}A_+^{1,23}\theta((\bar{1},\bar{0})_1)(\{e^{\mu(t^{\mathbf{0}}_{12}+t^{\mathbf{0}}_{13})/2}\}\{\Phi^{2,3,1}\}A_+^{2,31}\theta((\bar{1},\bar{0})_2)(\{e^{\mu(t^{\mathbf{0}}_{23}+t^{\mathbf{0}}_{12})/2}\}Z))$$

and

$$Z=\{\Phi^{3,1,2}\}A_+^{3,12}\theta((\bar{1},\bar{0})_3\{e^{\mu(t^{\mathbf{0}}_{31}+t^{\mathbf{0}}_{32})/2}\})$$

(tN2) $(\tilde{\mathcal{A}}_-)^{(0,N)}=1$ *where*

$$\tilde{\mathcal{A}}_-=\{\Phi^{1,2,3}\}A_-^{1,23}\theta((\bar{0},\bar{1})_1)(\{e^{-\mu(t^{\mathbf{0}}_{12}+t^{\mathbf{0}}_{13})/2}\}\{\Phi^{2,3,1}\}A_-^{2,31}\theta((\bar{0},\bar{1})_2)(\{e^{-\mu(t^{\mathbf{0}}_{23}+t^{\mathbf{0}}_{12})/2}\}Z))$$

and

$$Z=\{\Phi^{3,1,2}\}A_-^{3,12}\theta((\bar{0},\bar{1})_3\{e^{-\mu(t^{\mathbf{0}}_{31}+t^{\mathbf{0}}_{32})/2}\})$$

(tM) $\{e^{\mu t^{\mathbf{0}}_{12}}\}=\{\Phi\}A_-^{1,23}\theta((\bar{0},\bar{1})_1)(\{\Phi\}^{-1}\{e^{-\mu t^{\mathbf{0}}_{12}/2}\}\{\Phi^{2,1,3}\}(A_+^{2,13})^{-1}\theta((\overline{-1},\bar{0})_2X))$*, where*

$$X=\{(\Phi^{2,1,3})^{-1}\}\{e^{-\mu t^{\mathbf{0}}_{12}/2}\}\{\Phi\}(B_-^{1,23})^{-1}\theta((\bar{0},\overline{-1})_1)(\{\Phi\}^{-1}\{e^{\mu t_{12}/2}(\Phi^{2,1,3})\}(A_+^{2,13}Y))$$

and

$$Y=\theta((\bar{1},\bar{0})_2)(\{(\Phi^{2,1,3})^{-1}e^{\mu t_{12}/2}\})$$

Proof. This fact is a consequence of Theorem 4.3.2. Indeed, any morphism from $\widehat{\mathbf{PaB}}^\Gamma_{e\ell\ell}(\mathbf{k})$ to an operad Q is determined completely by the images of the generators of $\widehat{\mathbf{PaB}}^\Gamma_{e\ell\ell}(\mathbf{k})$ satisfying the images in Q of relations (tN1), (tN2) and (tE), which, for the case $Q=G\mathbf{PaCD}^\Gamma_{e\ell\ell}(\mathbf{k})$, are precisely the relations in the above theorem. □

In Section 7.2 we will give an example of such mathematical object.

Definition 4.3.11. *The (*$\mathbf{k}$*-pro-unipotent version of the) twisted elliptic Grothendieck–Teichmüller group is defined as the group*

$$\widehat{\mathbf{GT}}^\Gamma_{e\ell\ell}(\mathbf{k}):=\mathrm{Aut}^+_{\mathrm{Mod}(\widehat{\mathbf{PaB}}(\mathbf{k}))}(\widehat{\mathbf{PaB}}^\Gamma_{e\ell\ell}(\mathbf{k}))^\Gamma$$

of automorphisms of the $\widehat{\mathbf{PaB}}(\mathbf{k})$*-module* $\widehat{\mathbf{PaB}}^\Gamma_{e\ell\ell}(\mathbf{k})$ *which are* Γ*-equivariant and which are the identity on objects.*

Definition 4.3.12. *The graded twisted elliptic Grothendieck-Teichmüller group is the group*

$$\mathbf{GRT}^\Gamma_{e\ell\ell}(\mathbf{k}):=\mathrm{Aut}^+_{\mathrm{Mod}(\mathbf{PaCD}(\mathbf{k}))}(\mathbf{PaCD}^\Gamma_{e\ell\ell}(\mathbf{k}))^\Gamma$$

of automorphisms the $\mathbf{PaCD}(\mathbf{k})$*-module* $\mathbf{PaCD}^\Gamma_{e\ell\ell}(\mathbf{k})$ *which are* Γ*-equivariant and which are the identity on objects.*

Theorem 4.3.13. *The set* $\mathbf{Ell}^\Gamma(\mathbb{C})$ *is non empty.*

Proof. In Section 9 we will construct an element in this set. □

Now, any automorphism (F,G) in $\mathbf{GT}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ is defined as follows

- $F(R^{1,2}_{\mathbf{0}}) = (R^{1,2}_{\mathbf{0}})^{\lambda}$,
- $F(\Phi^{1,2,3}_{\mathbf{0}}) = f(x_{12},x_{23})\cdot\Phi^{1,2,3}_{\mathbf{0}}$,
- $G(A_{\mathbf{0}}) = g_+(X,Y,P^{\alpha};\alpha\in\Gamma)$,
- $G(B_{\mathbf{0}}) = g_-(X,Y,P^{\alpha};\alpha\in\Gamma)$.

where $(\lambda,\mu,g_+,g_-)\in\mathbf{k}^{\times}\times\widehat{F}_2(\mathbf{k})\times\widehat{\mathrm{PB}}^{\Gamma}_{1,2}(\mathbf{k})\times\widehat{\mathrm{PB}}^{\Gamma}_{1,2}(\mathbf{k})$. The fact that $(\lambda,f)\in\widehat{\mathrm{GT}}(\mathbf{k})$ is clear and $g_{\pm}\in\widehat{\mathrm{PB}}^{\Gamma}_{1,2}(\mathbf{k})$ satisfy the following relations:

$$\Big(f(\sigma_1^2,\sigma_2^2)g_+(X,Y,P^{\alpha};\alpha\in\Gamma)\theta((\bar{1},\bar{0})_1)\cdot(\sigma_1\sigma_2(\sigma_1\sigma_2^2\sigma_1)^{\frac{\lambda-1}{2}})\Big)^{(3M,0)}=1, \tag{btN1}$$

$$\Big(f(\sigma_1^2,\sigma_2^2)g_-(X,Y,P^{\alpha};\alpha\in\Gamma)\theta((\bar{0},\bar{1})_1)\cdot(\sigma_1^{-1}\sigma_2^{-1}(\sigma_1\sigma_2^2\sigma_1)^{-\frac{\lambda-1}{2}})\Big)^{(0,3N)}=1, \tag{btN2}$$

$$u^2 = g_-\theta((\bar{0},\bar{1})_1)(u^{-1}g_+^{-1}\theta((\bar{-1},\bar{0})_1)(u^{-1}g_-^{-1}\theta((\bar{0},\bar{-1})_1)((ug_+\theta((\bar{1},\bar{0})_1)u))), \tag{btE}$$

as identities in $\widehat{\bar{B}}^{\Gamma}_{1,3}(\mathbf{k})$, where $u=f(\sigma_1^2,\sigma_2^2)^{-1}\sigma_1^{\lambda}f(\sigma_1^2,\sigma_2^2)$, $g_{\pm}=g_{\pm}(X,Y,P^{\alpha};\alpha\in\Gamma)$.

Let us define $\widehat{\mathrm{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ as the set of all $(\lambda,\mu,g_+,g_-)\in\mathbf{k}^{\times}\times\widehat{F}_2(\mathbf{k})\times\widehat{\mathrm{PB}}^{\Gamma}_{1,2}(\mathbf{k})\times\widehat{\mathrm{PB}}^{\Gamma}_{1,2}(\mathbf{k})$ satisfying relations (btN1), (btN2) and (btE).

The image of the categorical composition of $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ and endows $\widehat{\mathrm{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ with a group structure which can explicitely be described as follows.
For $(\lambda,f,g_{\pm}),(\lambda',f',g'_{\pm})\in\widehat{\mathrm{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$, we set

$$(\lambda,f,g_{\pm})(\lambda',f',g'_{\pm}) := (\lambda'',f'',g''_{\pm})$$

where (λ'',f'') is as in (2.6.11) and

$$g''_{\pm}(X,Y,P^{\alpha};\alpha\in\Gamma) = g_{\pm}(g'_+(X,Y,P^{\alpha};\alpha\in\Gamma),g'_-(X,Y,P^{\alpha};\alpha\in\Gamma),(P^{\alpha})^{\lambda};\alpha\in\Gamma).$$

Proposition 4.3.14. *There is a group isomorphism* $\widehat{\mathbf{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ *and* $\widehat{\mathrm{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$.

Proof. This is a consequence of Theorem 4.3.2. Indeed, from the presentation of $\widehat{\mathbf{PaB}}^{\Gamma}_{e\ell\ell}$ (induced by the presentation of $\mathbf{PaB}^{\Gamma}_{e\ell\ell}$ via the morphism $\mathbf{PaB}^{\Gamma}_{e\ell\ell}\longrightarrow\widehat{\mathbf{PaB}}^{\Gamma}_{e\ell\ell}$) we know that an automorphism F of $\widehat{\mathbf{PaB}}^{\Gamma}_{e\ell\ell}$ which is the identity on objects is completely determined by the images of its generators satisfying relations (tN1), (tN2) and (tE), which are precisely the defining relations of $\widehat{\mathrm{GT}}^{\Gamma}_{e\ell\ell}(\mathbf{k})$. □

Recall that the image of the action of $\widehat{\mathbf{GT}}(\mathbf{k})$ on $\mathbf{Ass}(\mathbf{k})$ under correspondences 2.6.8 and 2.6.12 yields an action of $\widehat{\mathrm{GT}}(\mathbf{k})$ on $\mathrm{Ass}(\mathbf{k})$, defined as in 2.6.8. For $(\lambda,f,g_+,g_-)\in\widehat{\mathrm{GT}}^{\Gamma}_{ell}(\mathbf{k})$ and $(\mu,\Phi,A_+,A_-)\in\mathrm{Ell}^{\Gamma}(\mathbf{k})$, we set

$$(\lambda,f,g_+,g_-)*(\mu,\Phi,A_+,A_-) := (\mu',\Phi',A'_+,A'_-)$$

where $A'_\pm := g_\pm(A_+, A_-, (P^\alpha)^\lambda; \alpha \in \Gamma)$. This is precisely the image of the action of $\widehat{\mathbf{GT}}^\Gamma_{\ell\ell}(\mathbf{k})$ on $\mathbf{Ell}^\Gamma(\mathbf{k})$ under the above correspondence.

The image of the group law in $\mathbf{GRT}^\Gamma_{e\ell\ell}(\mathbf{k})$ is described as follows.

Define $(\mathrm{GRT}^\Gamma_{ell})_1(\mathbf{k})$ as the set of all (g, u_+, u_-), such that $g \in \mathrm{GRT}_1(\mathbf{k})$, $u_\pm \in \hat{\bar{\mathfrak{t}}}^\Gamma_{1,2}(\mathbf{k})$, satisfying the following relations:

$$\sum_{\mathrm{i}=0}^{M-1} (\theta((\bar{i},\bar{0})_{123}) \cdot \Big(g^{1,2,3} u_+^{1,23} \theta((\bar{1},\bar{0})_1)(g^{1,2,3})^{-1} + g^{2,1,3} u_+^{2,13} \theta((\bar{1},\bar{0})_2)(g^{2,1,3})^{-1} + u_+^{3,12} \Big) = 0, \tag{tN1}$$

$$\sum_{\mathrm{i}=0}^{N-1} (\theta((\bar{0},\bar{i})_{123}) \cdot \Big(g^{1,2,3} u_-^{1,23} \theta((\bar{0},\bar{1})_1)(g^{1,2,3})^{-1} + g^{2,1,3} u_-^{2,13} \theta((\bar{0},\bar{1})_2)(g^{2,1,3})^{-1} + u_-^{3,12} \Big) = 0, \tag{tN2}$$

$$g^{1,2,3} u_+^{1,23} \theta((\bar{1},\bar{0})_1)(g^{1,2,3})^{-1} u_+^{3,12} - u_+^{3,12} \theta((\bar{1},\bar{0})_3) \cdot \Big(g^{1,2,3} u_+^{1,23} \theta((\bar{1},\bar{0})_1)(g^{1,2,3})^{-1} \Big) = 0, \tag{tL1}$$

$$g^{1,2,3} u_-^{1,23} \theta((\bar{0},\bar{1})_1)(g^{1,2,3})^{-1} u_-^{3,12} - u_-^{3,12} \theta((\bar{0},\bar{1})_3) \cdot \Big(g^{1,2,3} u_-^{1,23} \theta((\bar{0},\bar{1})_1)(g^{1,2,3})^{-1} \Big) = 0, \tag{tL2}$$

$$\begin{aligned} t_{12}^{\mathbf{0}} =& g^{1,2,3} u_+^{1,23} \theta((\bar{1},\bar{0})_1) \cdot \Big((g^{1,2,3})^{-1} g^{2,1,3} u_-^{2,13} \theta((\bar{0},\bar{1})_2)(g^{2,1,3})^{-1} \Big) \\ &- g^{2,1,3} u_-^{2,13} \theta((\bar{0},\bar{1})_2) \cdot \Big(g^{2,1,3})^{-1} g^{1,2,3} u_+^{1,23} \theta((\bar{1},\bar{0})_1)(g^{1,2,3})^{-1} \Big) \end{aligned} \tag{tE}$$

(relations in $\hat{\bar{\mathfrak{t}}}^\Gamma_{1,3}(\mathbf{k})$). Set $(g_1, u_+^1, u_-^1) * (g_2, u_+^2, u_-^2) := (g, u_+, u_-)$, where

$$u_\pm(x, y, t^\alpha; \alpha \in \Gamma) := u_\pm^1(u_+^2(x, y, t^\alpha; \alpha \in \Gamma), u_-^2(x, y, t^\alpha; \alpha \in \Gamma), t^\alpha; \alpha \in \Gamma), \tag{4.19}$$

where $\bar{\mathfrak{t}}^\Gamma_{1,2}(\mathbf{k})$ is viewed as the Lie algebra generated by x, y, t^α, for $\alpha \in \Gamma$, with relation $[x, y] = \sum_{\alpha\in\Gamma} t^\alpha$.

The group $\mathbf{k}^\times$ acts on $(\mathrm{GRT}^\Gamma_{ell})_1(\mathbf{k})$ by

$$u_\pm(x, y, t^\alpha; \alpha \in \Gamma) := u_\pm^1(u_+^2(x, y, t^\alpha; \alpha \in \Gamma), u_-^2(x, y, t^\alpha; \alpha \in \Gamma), t^\alpha; \alpha \in \Gamma), \tag{4.20}$$

where

- $c \cdot g$ is as above,
- $(c \cdot u_+)(x, y, t^\alpha; \alpha \in \Gamma) := u_+(x, c^{-1}y, ct^\alpha; \alpha \in \Gamma)$,
- $(c \cdot u_-)(x, y, t^\alpha; \alpha \in \Gamma) := cu_-(x, c^{-1}y, ct^\alpha; \alpha \in \Gamma)$.

We then set $\mathrm{GRT}^\Gamma_{ell}(\mathbf{k}) := (\mathrm{GRT}^\Gamma_{ell})_1(\mathbf{k}) \rtimes \mathbf{k}^\times$. The image in $\mathrm{GRT}^\Gamma_{ell}(\mathbf{k})$ of the group law in $\mathbf{GRT}^\Gamma_{e\ell\ell}(\mathbf{k})$ is exactly the group law defined by (4.19) and (4.20).

One can establish then the following torsor conjecture.

Conjecture 4.3.15. *The triple* $(\widehat{\mathrm{GT}}^\Gamma_{e\ell\ell}(\mathbf{k}), \mathrm{Ell}^\Gamma(\mathbf{k}), \mathrm{GRT}^\Gamma_{e\ell\ell}(\mathbf{k}))$ *is a torsor.*

If the above conjecture is true, then a consequence is that there is torsor isomorphism

$$(\widehat{\mathbf{GT}}^\Gamma_{e\ell\ell}(\mathbf{k}), \mathbf{Ell}^\Gamma(\mathbf{k}), \mathbf{GRT}^\Gamma_{e\ell\ell}(\mathbf{k})) \longrightarrow (\widehat{\mathrm{GT}}^\Gamma_{e\ell\ell}(\mathbf{k}), \mathrm{Ell}^\Gamma(\mathbf{k}), \mathrm{GRT}^\Gamma_{e\ell\ell}(\mathbf{k})). \tag{4.21}$$

Chapter 5

Operads and higher genus associators

This chapter consists of the first part of a study devoted to the rational homotopy theory of modules over (framed) E_2-operads associated to genus g oriented surfaces. On the one hand, we aim to study the characterization of the elliptic Grothendieck-Teichmüller group as the group of homotopy automorphisms in the homotopy category of D_2-modules of some rationalization of the module $\mathrm{D}_{1,2}$ of little 2-disks on a torus. On the other hand, we aim to study the characterization of higher genus Grothendieck-Teichmüller groups as groups of homotopy automorphisms in the homotopy category of D_2-modules of some rationalization of the module $\mathrm{D}^f_{g,2}$ of framed little 2-disks on a compact orientable genus g topological surface Σ_g.

In this chapter we will concentrate on the higher genus story. After briefly recalling framed Fulton-MacPherson compactifications and their associated operadic structures, we introduce a full suboperad $\mathbf{PaB}^f \subset \pi_1(\mathrm{D}^f_2)$ of framed parenthesized braidings by restricting the object sets of the groupoid so that $B(\mathbf{PaB}^f) \xrightarrow{\sim} B(\pi_1(\mathrm{D}^f_2))$. We then construct the corresponding operad $\mathbf{PaCD}^f$ of parenthesized framed chord diagrams, framed associators and framed Grothendieck-Teichmüller groups in terms of $\mathbf{PaB}^f$ and $\mathbf{PaCD}^f$.

We then turn to the genus g situation and we introduce a full submodule $\mathbf{PaB}^f_g \subset \pi_1(\mathrm{D}^f_{g,2})$ of genus g framed parenthesized braidings by restricting the object sets of the groupoid so that $B(\mathbf{PaB}^f_g) \xrightarrow{\sim} B(\pi_1(\mathrm{D}^f_{g,2}))$.

Next, we define the $\mathbf{PaCD}^f$-module $\mathbf{PaCD}^f_g$ of genus g parenthesized framed chord diagrams.

Finally, we give operadic definitions of genus g associators and (graded) Grothendieck-Teichmüller groups, extract from them explicit equations for this objects and conjecture the existence of such an associator by means of the framed genus g universal KZB connection yet to be defined.

It should be interesting to relate the Lie algebra of our genus g graded Grothendieck-Teichmüller group to the higher genus Kashiwara-Vergne Lie algebra $\mathfrak{krv}^{(g,n+1)}$ which is being studied in the recent work [4].

5.1 Operad structures on framed FM compactifications

Let $n \geqslant 1$ and consider the Fulton-MacPherson compactification $\mathrm{FM}_k(n)$ of $\mathrm{Conf}(\mathbb{R}^k, n)$. These spaces assemble into an operad $\mathrm{FM}_k := \mathrm{FM}(\mathrm{Conf}(\mathbb{R}^k, -))$, which is known to be weakly equivalent to the little k-disks operad D_k. The interior of $\mathrm{FM}_k(n)$ is the reduced configuration space $\mathrm{C}(\mathbb{R}^k, n)$.

Now, let M be a closed smooth manifold of dimension k. Consider the configuration space of M

$$\mathrm{Conf}(M, n) = \{(x_1, \dots, x_n) \in M^n; x_i \neq x_j \text{ if } i \neq j\}.$$

The spaces $\mathrm{Conf}(M, n)$ are weakly equivalent to their Fulton-MacPherson compactification $\mathrm{FM}_M(n) := \mathrm{FM}(\mathrm{Conf}(M, n))$. When M is parallelizable, the spaces $\mathrm{FM}_M(n)$ form a right FM_k-module FM_M. Otherwise, we need to introduce the framed versions of all the above geometric objects. This consists on seting a choice of trivialization of the tangent bundle of M in order to specify in which direction we will insert the disks on M constructed by the Fulton-MacPherson compactification.

Let M be a Riemannian closed oriented[1] compact k-manifold and consider the bundle projection $\pi_M : \mathrm{SO}(M) \to M$, where $\mathrm{SO}(M)$ is the principal GL_k-bundle of special orthogonal linear frames on M. The framed configuration space $\mathrm{Conf}^f(M, n)$ of n distinct points in M is

$$\mathrm{Conf}^f(M, n) := \{(\boldsymbol{x}, f_1, \dots, f_n) \in \mathrm{Conf}(M, n) \times \mathrm{SO}(M)^{\times n} | f_i \in \pi_M^{-1}(x_i)\}.$$

This is the same to define $\mathrm{Conf}^f(M, n)$ as the pullback of the diagram

$$\begin{tikzcd} & \mathrm{SO}(M)^{\times n} \arrow[d] \\ \mathrm{Conf}(M, n) \arrow[r] & M^{\times n} \end{tikzcd}$$

so $\mathrm{Conf}^f(M, n) \longrightarrow \mathrm{Conf}(M, n)$ is a principal $\mathrm{SO}(k)^{\times n}$-bundle. If M is parallelizable, $\mathrm{Conf}^f(M, n)$ is isomorphic to $\mathrm{Conf}(M, n) \times \mathrm{SO}(k)^{\times n}$. For instance, this is the case when $M = \mathbb{R}^k$ or $M = \mathbb{T}$. The symmetric group $\mathfrak{S}_n$ acts on $\mathrm{Conf}^f(\mathbb{C}, [n])$ by relabelling the indexes of the marked points. The map $\mathrm{Conf}^f(\mathbb{C}, [n]) := \mathrm{Conf}^f(\mathbb{C}, n)/\mathfrak{S}_n \longrightarrow \mathrm{Conf}(\mathbb{C}, [n])$ is a locally trivial bundle with fiber $\mathrm{SO}(2)^{\times n}$.

We have framed versions of the little k-disks spaces which are $\mathfrak{S}_n$-equivariant homotopy equivalent to framed configuration spaces of $\mathbb{R}^k$:

$$\mathrm{D}_k^f(n) \xrightarrow{\sim} \mathrm{Conf}^f(\mathbb{R}^k, n).$$

There is a $\mathfrak{S}_n$-equivariant homotopy equivalence similar to the one above in the case for manifolds but with very restrictive assumptions (see [96] for more details).

Let G be a topological group and $(\mathcal{O}, \{\circ_i^{m,n}\}_{m,n})$ be an operad in left G-spaces and suppose that the partial operadic compositions $\circ_i^{m,n}$ in $\mathcal{O}$ are G-equivariant. The semidirect-product

[1] In the case of non-oriented manifolds one can only consider the bundle projection $\mathrm{O}(M) \to M$.

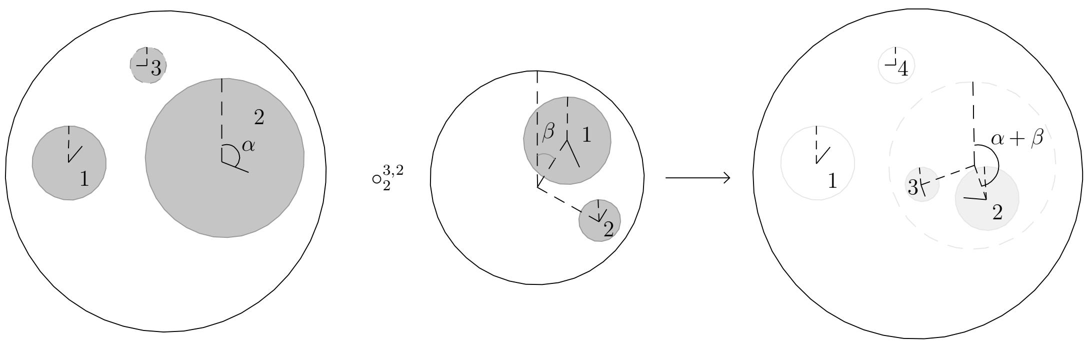
operad $G \ltimes \mathcal{O}$ is the topological operad defined by $(G \ltimes \mathcal{O})(n) := G^n \times \mathcal{O}(n)$ and with partial operadic compositions denoted $\tilde{\circ}_i^{m,n}$ and given, for $\boldsymbol{g} = (g_1, \ldots, g_m), \boldsymbol{g}' = (g_1', \ldots, g_n')$ and $1 \leqslant i \leqslant m$ by

$$(\boldsymbol{g}, x_1)\tilde{\circ}_i^{m,n}(\boldsymbol{g}', x_2) := (\boldsymbol{g}'', x_1 \circ_i^{m,n} (g_i \cdot x_2)) \in G^{n+m-1} \times \mathcal{O}(n+m-1),$$

where $\boldsymbol{g}'' = (g_1, \ldots, g_{i-1}, g_i g_1', \ldots, g_i g_m', g_{i+1}, \ldots, g_n)$. Consider the framed Fulton-MacPherson compactified configuration spaces

$$\mathrm{FM}_k^f(n) := \mathrm{SO}(k) \times \mathrm{FM}_k(n).$$

The interior of $\mathrm{FM}_k^f(n)$ is $\mathrm{Conf}^f(\mathbb{R}^k, n)$. The $\mathrm{SO}(k)$-action is compatible with the operad structure of $\mathrm{FM}_k(n)$. Thus, these spaces form an operad $\mathrm{FM}_k^f := \mathrm{SO}(k) \ltimes \mathrm{FM}_k$ called framed Fulton-MacPherson operad, which turns out to be weakly equivalent to the framed little k-disks operad. The partial composition morphisms can be pictured as follows:



Summarizing the above results, we get

$$\begin{tikzcd}
\mathrm{D}_k^f(n) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Conf}^f(\mathbb{R}^k, n) \arrow[d] & \mathrm{FM}_k^f(n) \arrow[l, "\simeq"'] \arrow[d] \\
\mathrm{D}_k(n) \arrow[r, "\simeq"] & \mathrm{Conf}(\mathbb{R}^k, n) & \mathrm{FM}_k(n) \arrow[l, "\simeq"']
\end{tikzcd}$$

where the horizontal arrows are $\mathfrak{S}_n$-equivariant homotopy equivalences and the vertical arrows are $\mathrm{SO}(k)^{\times n}$-principal bundles. This diagram does not enhance into an operad map.

Nevertheless, in [39], an operad morphism $\phi : \mathrm{FM}_k \longrightarrow \mathrm{D}_k$ was constructed and it is easy to verify that ϕ is ewuivariant for the action of $\mathrm{SO}(k)$ on these two operads and by construction, the data of the framings are compatible with this map (since the rotation of a disk will preserve that disk). Thus, we can construct a square

$$\begin{tikzcd}
\mathrm{D}_k^f \arrow[d] & \mathrm{FM}_k^f \arrow[l, "\simeq"'] \arrow[d] \\
\mathrm{D}_k & \mathrm{FM}_k \arrow[l, "\simeq"']
\end{tikzcd} \tag{5.1}$$

where the horizontal arrows are weak equivalences of operads in topological spaces (see [39] for details).

Now, if M is an oriented k-manifold, then the collection of its framed Fulton-MacPherson compactifications forms a right FM_k^f-module denoted FM_M^f where each space $\mathrm{FM}_M^f(n)$ is a principal $\mathrm{SO}(k)^{\times n}$-bundle over $\mathrm{FM}_M(n)$. Then we also have

$$\begin{tikzcd} \mathrm{D}_M^f(n) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Conf}^f(M,n) & \mathrm{FM}_M^f(n) \arrow[l, "\simeq"'] \arrow[d] \\ \mathrm{D}_M(n) \arrow[r, "\simeq"] & \mathrm{Conf}(M,n) & \mathrm{FM}_M(n) \arrow[l, "\simeq"'] \end{tikzcd} \tag{5.2}$$

where again the horizontal maps are $\mathfrak{S}_n$-equivariant homotopy equivalences. If M is parallelizable, then the semi-direct product in the below spaces becomes an usual product and we get a square

$$\begin{tikzcd} \mathrm{D}_M^f \arrow[d] & \mathrm{FM}_M^f \arrow[l, "\simeq"'] \arrow[d] \\ \mathrm{D}_M & \mathrm{FM}_M \arrow[l, "\simeq"'] \end{tikzcd} \tag{5.3}$$

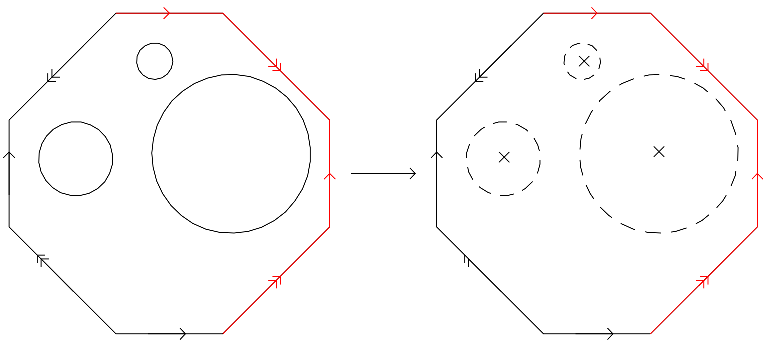
If M is not parallelizable the first line of this square does not hold but we still have a weak equivalence $\mathrm{FM}_M^f \xrightarrow{\simeq} \mathrm{D}_M^f$ of modules over $\mathrm{FM}_k^f \xrightarrow{\simeq} \mathrm{D}_k^f$.

The case of genus g orientable surfaces

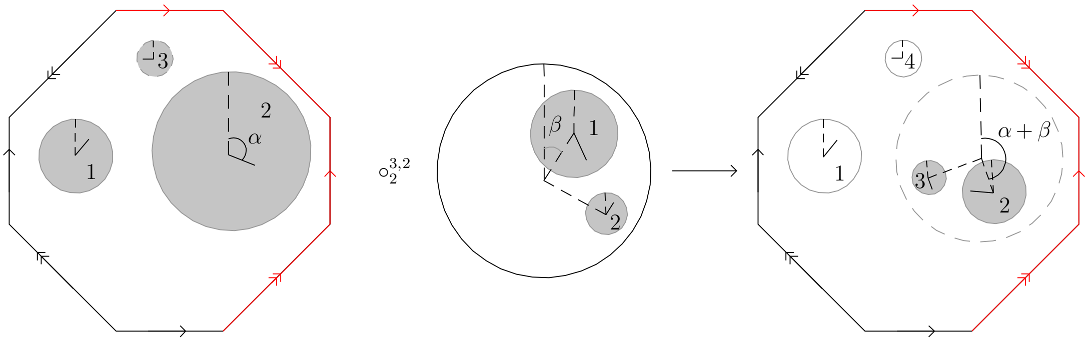
We now concentrate in the case $k = 2$ (i.e. compact oriented topological surfaces). Let $g \geq 0$ and $n > 0$ be integers. For a compact topological oriented surface Σ_g of genus g without boundary, we consider the space $\mathrm{Conf}(\Sigma_g, n)$ of configurations of n points in Σ_g. It is homotopy equivalent to the space $\mathrm{D}_{2,g}(n)$ of n little 2-disks with disjoint interiors on Σ_g

$$\mathrm{D}_{2,g}(n) \xrightarrow{\sim} \mathrm{Conf}(\Sigma_g, n).$$

This map can be represented as follows (in the case $g = 2$)



The surfaces Σ_g are not parallelizable for $g > 1$ so we consider the framed versions of the above spaces. Namely, the collection $\mathrm{D}_{2,g}^f$ of spaces of framed little 2-disks on Σ_g has the structure of an operadic module over the framed little 2-disks operad D_2^f. We can represent the action of D_2^f on $\mathrm{D}_{2,g}^f$ as follows (in the case $g = 2$):



In particular, if $g = 1$, as $\mathbb{T}$ is pararellizable so the space $\mathrm{D}_{2,1}^f(n)$ is isomorphic to $\mathrm{D}_{2,1}(n) \times \mathrm{SO}(2)^{\times n}$. Now let Σ_g be a genus g closed connected oriented surface with a smooth and semi-algebraic manifold structure and consider its framed Fulton-MacPherson compactification $\mathrm{FM}_{2,g}^f(n)$. The space $\mathrm{FM}_{2,g}^f(n)$ is a manifold with corners whose interior is $\mathrm{Conf}^f(\Sigma_g, n)$ and the insertion of boundary components of $\mathrm{FM}_{2,g}^f$ with respect to the direction of the frame endows the collection $\mathrm{FM}_{2,g}^f$ of these spaces with the structure of a FM_2^f-module. More explicitely, $\mathrm{FM}_{2,g}^f(n)$ is obtained from the pullback

$$\begin{tikzcd} & SO(2)^{\times n} \arrow[d] \\ \mathrm{FM}_{\Sigma_g}(n) \arrow[r] & \Sigma_g^{\times n} \end{tikzcd}$$

where $\mathrm{SO}(2) \longrightarrow \Sigma_g$ is the frame bundle over Σ_g for some specified Riemannian metric.

5.2 Operads associated to framed configuration spaces (framed associators)

5.2.1 Framed configuration spaces on $\mathbb{C}$

The fundamental group of the unordered framed configuration space $\mathrm{Conf}^f(\mathbb{C}, [n])$ was studied in [71] and is isomorphic to the framed braid group B_n^f generated by elements $\sigma_1, \sigma_2, \dots, \sigma_{n-1}, f_1, f_2, \dots, f_n$ together with relations

(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ if $i \in [n-2]$,

(B2) $(\sigma_i, \sigma_j) = 1$ if $|i-j| > 1$,

(FB1) $f_i f_j = f_j f_i$ for all i, j,

(FB2) $\sigma_i f_j = f_{\sigma_i(j)}\sigma_i$ for all i, j.

The space $\mathrm{Conf}^f(\mathbb{C},[n])$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathrm{B}_n^f,1)$and the group B_n^f is a semidirect product $\mathbb{Z}^n \ltimes \mathrm{B}_n$ where the action of B_n on $\mathbb{Z}^n$ is given by $a(r_i,....,r_n) = (r_{\sigma(1)}, r_{\sigma(2)}, ..., r_{\sigma(n)})$. If $f_1^{r_1}, f_2^{r_2}, \cdots, f_n^{r_n}, \alpha \in \mathrm{B}_n^f$ with $\alpha \in \mathrm{B}_n$ then the r_i's are called framings. The product in this notation is given by

$$(f_1^{r_1} f_2^{r_2} \cdots f_n^{r_n}\alpha)(f_1^{s_1} f_2^{s_2} \cdots f_n^{s_n}\beta) = f_1^{r_1+s_{\alpha(1)}} f_2^{r_2+s_{\alpha(2)}} \cdots f_n^{r_n+s_{\alpha(n)}}\alpha\beta$$

The fundamental group PB_n^f of $\mathrm{Conf}^f(\mathbb{C},n)$ at any basepoint is the direct product $\mathrm{PB}_n^f = \mathbb{Z}^n \times \mathrm{PB}_n$. One can represent such braids as ribbon braids as we will see in the following subsection.

5.2.2 The operad $\mathbf{PaB}^f$ of framed parenthesized braidings

The boundary $\partial\,\mathrm{FM}_2^f(n) = \mathrm{FM}_2^f(n) - \mathrm{Conf}^f(\mathbb{R}^2,n)$ of $\mathrm{FM}_2^f(n)$ is made of the following irreducible components: for any decomposition $n = n_1 + \cdots + n_k$ there is a component

$$\partial_{n_1,\cdots,n_k}\mathrm{FM}_2^f(n) \cong \prod_{i=1}^k \mathrm{FM}_2^f(n_i) \times \mathrm{FM}_2^f(n)\,.$$

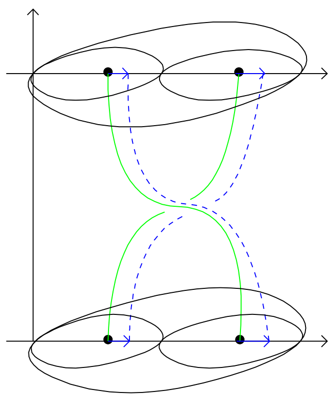
The inclusion of boundary components provide FM_2^f with the structure of an operad FM_2^f in topological spaces and we have inclusions inclusions of topological operads

$$\mathrm{Pa} \subset \mathrm{Conf}^f(\mathbb{R},-) \subset \mathrm{FM}_2^f\,.$$

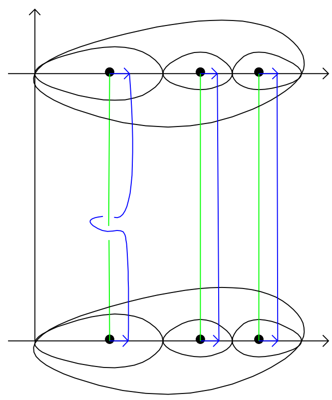
The operad in groupoids of framed parenthesized braidings is defined as

$$\mathbf{PaB}^f := \pi_1(\mathrm{FM}_2^f, \mathbf{Pa}).$$

Notable arrows in $\mathbf{PaB}^f(1)$, $\mathbf{PaB}^f(2)$ and $\mathbf{PaB}^f(3)$. We have an arrow $R^{1,2} \in \mathrm{Hom}_{\mathbf{PaB}^f(2)}(12,21)$ and an arrow $\Phi^{1,2,3} \in \mathrm{Hom}_{\mathbf{PaB}^f(3)}((12)3, 1(23))$ which correspond to the very same paths as in the unframed case. In particular, $R^{1,2}$ can be represented as follows



There is also a braid $F^1 \in \mathrm{End}_{\mathbf{PaB}^f(1)}(1)$ corresponding to the framing. In $\mathbf{PaB}^f(3)$ it can be represented as follows

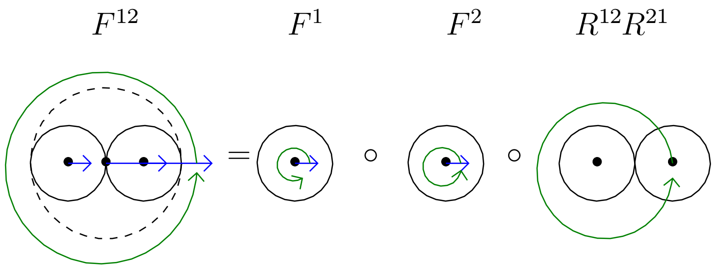


This should be considered as a single ribbon braid being twisted 360 degrees and the blue strand is the transport of a point lying in the surface of this ribbon braid.

Recall the definition of the operad **CoB** of coloured braids from [47, Subsection 5.2.8]. As in the case of the operad **PaB**, the operad $\mathbf{PaB}^f$ can be defined as the fake pullback of the framed version $\mathbf{CoB}^f$ of **CoB** and we have a presentation of $\mathbf{PaB}^f$ in terms of generators and relations. Namely, as an operad in groupoids having **Pa** as operad of objects, $\mathbf{PaB}^f$ is generated by $F := F^1 \in \mathbf{PaB}^f(1)$, $R := R^{1,2} \in \mathbf{PaB}^f(2)$ and $\Phi =: \Phi^{1,2,3} \in \mathbf{PaB}^f(3)$ together with relations (H1), (H2), (P) and the following relation:

(F) $R^{1,2}R^{2,1}F^1F^2 = F^{12}$ as arrows from (12) to (12) in $\mathbf{PaB}^f(2)$.

The proof of this result can be found in [14, Lemma 7.4]. In particular, one can represent relation (F) by means of the following picture:



5.2.3 The non-symmetric operad PB^f of framed braidings

Let us now introduce two non-symmetric operads that will be of use in Lemma 5.3.7.

The collection $\mathrm{PB}^f := \{\mathrm{PB}^f_n\}_{n\geq 1}$ can be endowed with the structure of a non-symmetric operad given by partial compositions

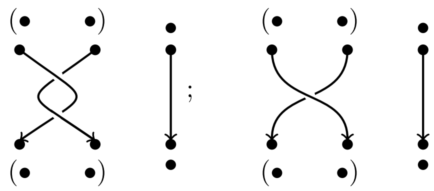
$$\circ_i : \mathrm{PB}^f_n \times \mathrm{PB}^f_m \longrightarrow \mathrm{PB}^f_{n+m-1} \tag{5.4}$$
$$(b, b') \longmapsto b \circ_i b' \tag{5.5}$$

where $b \circ_i b'$ is defined by replacing the i-labelled strand in b by the braid b' made very thin. Via the homotopy equivalence between framed little disks and framed configuration spaces we presented in the last section, one checks that the above operadic composition for PB^f is induced by that on $\mathcal{D}_2^f$. In the same way, one can construct an non-symmetric operad in groupoids B^f in the following way :

- The objects of $B^f(n)$ are unnumbered maximal parenthesizations of lenght n. In particular, this means that for every object p of $\mathbf{Pa}(n)$, there is a corresponding object $[p]$ in $B^f(n)$, and $[p] = [q]$ if p and q only differ by a permutation (but have the same underlying parenthesization).
- B^f is freely generated by $F := F^1 \in B^f(1)$, $R := R^{1,2} \in B^f(2)$ and $\Phi := \Phi^{1,2,3} \in B^f(3)$ together with relations (H1), (H2), (P) and the following relation:

  (F) $R^{1,2}R^{2,1}F^1F^2 = F^{12}$ as arrows from $(\bullet\bullet)$ to $(\bullet\bullet)$ in $B^f(2)$.
- B^f is the image of $\mathbf{PaB}^f$ via the forgetful map $\mathbf{Op} \longrightarrow \mathbf{NsOp}$ sending an operad to a non-symmetric operad.
- It follows that there are group morphisms $\mathrm{B}_n^f \xrightarrow{\sim} \mathrm{Aut}_{B^f(n)}(p) \longrightarrow \mathfrak{S}_n$, the left one being an isomorphism.

For example, arrows in $\mathrm{Aut}_{B^f(3)}((\bullet\bullet)\bullet)$ can be depicted as follows:

 (5.6)

We let the reader depict the generators $F \in B^f(1)$, $R \in B^f(2)$ and $\Phi \in B^f(3)$ accordingly.

5.2.4 The operad $\mathbf{PaCD}^f(\mathbf{k})$ of parenthesized framed chord diagrams

Let $\mathfrak{t}_n^f(\mathbf{k})$ denote the graded Lie algebra over $\mathbf{k}$ generated by t_{ij}, $1 \le i, j \le n$ with relations

(FT1) $t_{ij} = t_{ji}$,

(FT2) $[t_{ij}, t_{kl}] = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$,

(FT3) $[t_{ij}, t_{ik} + t_{jk}] = 0$ if $\{i, j\} \cap \{k\} = \emptyset$.

This means we have a decomposition $\mathfrak{t}_n^f(\mathbf{k}) = \bigoplus_{i=1}^n \mathbf{k}t_{ii} \oplus \mathfrak{t}_n(\mathbf{k})$. In other words, this translates into insertion-coproduct morphisms as for each map $\phi : \{1, ..., m\} \longrightarrow \{1, ..., n\}$, there exists a Lie algebra morphism $\mathfrak{t}_n^f \longrightarrow \mathfrak{t}_m^f$, defined by $(t_{ij})^\phi := \sum_{i'\in\phi^{-1}(i), j'\in\phi^{-1}(j)} t_{i'j'}$.

Remark 5.2.1. *The above definition coincides with that appearing in [8], indeed it is isomorphic to the graded Lie algebra over* $\mathbf{k}$ *generated by* t_{ij}, $1 \le i \ne j \le n$ *and* t_k, $1 \le k \le n$, *with relations*

(T1,T2,T3) $t_{ij} = t_{ji}$; $[t_{ij}, t_{kl}] = 0$ *if* $\#\{i, j, k, l\} = 4$; $[t_{ij}, t_{ik} + t_{jk}] = 0$ *if* $\#\{i, j, k\} = 3$,

(FT2') $[t_i, t_j] = 0$ *for* $1 \leq i, j \leq n$,

(FT3') $[t_i, t_{jk}] = 0$ *for all* i, j, k.

The collection of the framed Lie algebras $\mathfrak{t}_n^f(\mathbf{k})$, for $n \geq 1$ is provided with the structure of an operad in (positively graded finite dimensional) Lie algebras over $\mathbf{k}$, denoted $\mathfrak{t}^f(\mathbf{k})$ and given by the following operadic partial compositions:

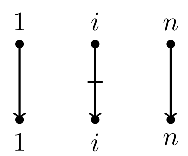
$$\begin{array}{cccc}
\circ_k : & \mathfrak{t}_I^f(\mathbf{k}) \oplus \mathfrak{t}_J^f(\mathbf{k}) & \longrightarrow & \mathfrak{t}^f_{J \sqcup I - \{i\}}(\mathbf{k}) \\
& (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha\beta} \\
& (t_{ij}, 0) & \longmapsto & \begin{cases} t_{ij} & \text{if} \quad k \notin \{i, j\} \\ \sum\limits_{p \in J} t_{pj} & \text{if} \quad k = i \\ \sum\limits_{p \in J} t_{ip} & \text{if} \quad j = k \end{cases}
\end{array}$$

In other words, under the correspondence of Remark 5.2.1, this is the same as the following composition:

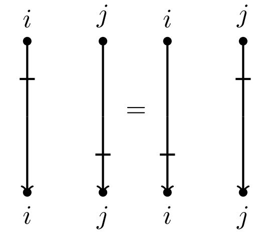
$$\begin{array}{cccc}
\circ_k : & \mathfrak{t}_m^f(\mathbf{k}) \oplus \mathfrak{t}_n^f(\mathbf{k}) & \longrightarrow & \mathfrak{t}^f_{n+m-1}(\mathbf{k}) \\
& (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha+k-1\beta+k-1} \\
& (0, t_{\alpha}) & \longmapsto & t_{\alpha+k-1} \\
& (t_{ij}, 0) & \longmapsto & \begin{cases} t_{i+n-1j+n-1} & \text{if} \quad k < i < j \\ \sum\limits_{p=i}^{i+n-1} t_{pj+n-1} & \text{if} \quad k = i < j \\ t_{ij+n-1} & \text{if} \quad i < k < j \\ \sum\limits_{p=j}^{j+n-1} t_{ip} & \text{if} \quad i < j = k \\ t_{ij} & \text{if} \quad i < j < k \end{cases} \\
& (t_i, 0) & \longmapsto & \begin{cases} t_{i+n-1} & \text{if} \quad k < i \\ \sum\limits_{p=i}^{i+n-1} t_p & \text{if} \quad k = i \\ t_i & \text{if} \quad i < k \end{cases}
\end{array}$$

for $1 \leq i, j, k \leq m$ with $i < j$ and $1 \leq \alpha, \beta \leq n$. We can then construct the operad $\mathbf{CD}^f(\mathbf{k}) := \hat{\mathcal{U}}(\hat{\mathfrak{t}}^f(\mathbf{k}))$ in $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$ called the operad of framed chord diagrams.

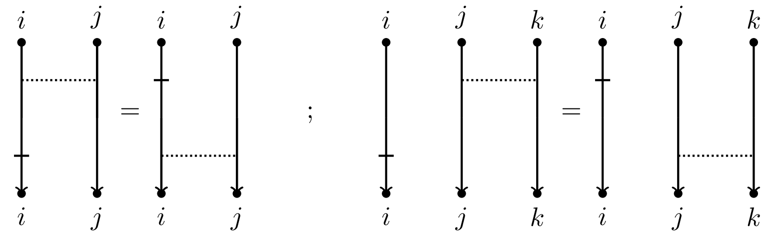
Remark 5.2.2. *This denomination comes from the fact that morphisms in* $\mathbf{CD}^f(\mathbf{k})(n)$ *can be represented as linear combinations of diagrams of chords on* n *vertical strands, where the chord diagram corresponding to* t_{ij} *can be represented as in the unframed case, the chord corresponding to* t_i *as*



and the composition is given by vertical concatenation of diagrams. Partial compositions can easily be understood as "cabling and removal operations" on strands (see [5, 47]). Relations (T1,T2,T3) can be described as in the in the unframed case and the remaining relations defining each $\mathfrak{t}_n(\mathbf{k})$ *can be represented as follows:*



(FT1)



(FT2)

Let $\widehat{\mathbf{CD}}^f(n)$ be the I-adic completion of $\mathbf{CD}^f(n)$ with respect to the augmentation ideal I. Since we are in possession of operads $\mathbf{Pa}(\mathbf{k})$ and $\widehat{\mathbf{CD}}^f(\mathbf{k})$ in $\mathbf{Cat}(\mathbf{CoAss_k})$ and of an operad morphism $\omega : \mathbf{Pa} \longrightarrow \mathrm{Ob}(\widehat{\mathbf{CD}}^f(\mathbf{k}))$, we are ready to define the operad

$$\mathbf{PaCD}^f(\mathbf{k}) := \omega^\star \widehat{\mathbf{CD}}^f(\mathbf{k})$$

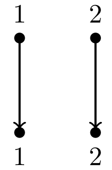
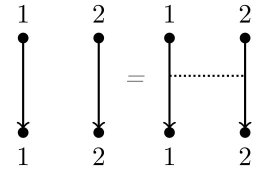
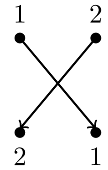
in $\mathbf{Cat}(\mathbf{CoAss_k})$ of parenthesized framed chord diagrams. We have

- $\mathrm{Ob}(\mathbf{PaCD}^f(\mathbf{k})) := \mathbf{Pa}$,
- $\mathrm{Mor}_{\mathbf{PaCD}^f(\mathbf{k})(n)}(p,q) := \mathrm{Mor}_{\widehat{\mathbf{CD}}^f(\mathbf{k})(n)}(pt,pt) = \hat{\mathcal{U}}(\hat{\mathfrak{t}}_n^f(\mathbf{k}))$.

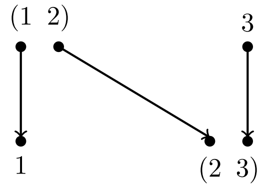
Example 5.2.3 (Notable arrows in $\mathbf{PaCD}^f(\mathbf{k})(1)$, $\mathbf{PaCD}^f(\mathbf{k})(2)$ and $\mathbf{PaCD}^f(\mathbf{k})(3)$)**.** *We have the following arrow* P^1*, in* $\mathbf{PaCD}^f(\mathbf{k})(1)$

$P^1 = t_{11}\cdot$ 

as well as the following arrows in $\mathbf{PaCD}^f(\mathbf{k})(2)$

$P^{1,2} := t_{11}\cdot$  $H^{1,2} := t_{12}\cdot$  $X^{1,2} = 1\cdot$ 

We also have the following arrow in $\mathbf{PaCD}(\mathbf{k})(3)$*:*

$a^{1,2,3} = 1\cdot$ 

Remark 5.2.4. *The elements* $a^{1,2,3}$, $X^{1,2}$, $H^{1,2}$ *and* P_i *are generators of* $\mathbf{PaCD}(\mathbf{k})$*, satisfy the pentagon and the two hexagons relations and the following relation:*

(iF) $P^{1,2}H^{1,2}X^{1,2}P^{2,1}(X^{1,2})^{-1}H^{1,2} = P^{12}$ *as arrows from* (12) *to* (12) *in* $\mathbf{PaB}^f(2)$.

5.2.5 Framed associators

Definition 5.2.5. *We define the set of framed* $\mathbf{k}$*-associators to be the set*

$$\mathbf{Ass}^f(\mathbf{k}) := \mathrm{Iso}^+_{\mathrm{Op}\,\mathbf{Grpd}_\mathbf{k}}(\widehat{\mathbf{PaB}}^f(\mathbf{k}), G\mathbf{PaCD}^f(\mathbf{k}))$$

if isomorphisms between $\widehat{\mathbf{PaB}}^f(\mathbf{k})$ *and* $G\mathbf{PaCD}^f(\mathbf{k})$ *which are the identity on objects.*

An immediate consequence of [14, Lemma 7.4] is then

Proposition 5.2.6. *There is a one-to-one correspondence between the set of framed* $\mathbf{k}$*-associators* $\mathbf{Ass}^f(\mathbf{k})$ *and the set* $\mathrm{Ass}^f(\mathbf{k})$ *of triples* (λ, μ, φ) *where* $(\mu, \varphi) \in \mathrm{Ass}(\mathbf{k})$ *and* $\lambda \in \mathbf{k}^\times$ *such that*

(F) $e^{\lambda(t_1+t_2+2t_{12})} = e^{\lambda(t_1+t_2)+\mu t_{12}}$.

Corollary 5.2.7. *By taking* $\mu = 2\lambda$*, on can establish a bijection between the set of framed associators and the set of associators.*

Moreover, by [14, Lemma 7.7], the there is a group isomorphism

$$\widehat{\mathbf{GT}}(\mathbf{k}) \simeq \widehat{\mathbf{GT}}^f(\mathbf{k}) := \mathrm{Aut}^+_{\mathrm{Op}\,\mathbf{Grpd}_\mathbf{k}}(\widehat{\mathbf{PaB}}^f(\mathbf{k}))$$

and the fact that $\mathfrak{t}^f_n(\mathbf{k}) = \bigoplus_{i=1}^n \mathbf{k}t_i \oplus \mathfrak{t}_n(\mathbf{k})$ gives us a further isomorphism

$$\mathbf{GRT}(\mathbf{k}) \simeq \mathbf{GRT}^f(\mathbf{k}) := \mathrm{Aut}^+_{\mathrm{Op}\,\mathbf{Grpd}_\mathbf{k}}(\mathbf{PaCD}^f(\mathbf{k})).$$

Proposition 5.2.8. *The set* $\mathbf{Ass}^f(\mathbb{C})$ *is non empty.*

We will prove this statement in the following subsection.

5.2.6 The framed universal KZ connection

Define the framed universal KZ connection on the trivial $\exp(\widehat{\mathfrak{t}}_n^f)$-principal bundle over $\mathrm{Conf}^f(\mathbb{C}, n)$ as the connection given by the holomorphic 1-form

$$w_n^{f\,\mathrm{KZ}} := \sum_{1\leqslant i\leqslant n} t_{ii}\,\mathrm{d}\,log(\lambda_i) + \sum_{1\leqslant i<j\leqslant n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij} \in \Omega^1(\mathrm{Conf}^f(\mathbb{C}, n), \mathfrak{t}_n^f),$$

which takes its values in $\mathfrak{t}_n^f$ and where $\lambda_i \in \mathbb{C}^\times$ is a fiber coordinate, for all $1 \le i \le n$.

Theorem 5.2.9. *The connection* $\nabla_n^{f\,\mathrm{KZ}} := \mathrm{d} - w_n^{f\,\mathrm{KZ}}$ *is flat.*

Proof. Let $w_1 := \sum\limits_{1\leqslant i\leqslant n} t_i\,\mathrm{d}\,log(\lambda_i)$ and $w_2 := \sum\limits_{1\leqslant i<j\leqslant n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij}$. We want to show that $[w_1 + w_2, w_1 + w_2] = 0$. We have

$$\begin{aligned}[w_1 + w_2, w_1 + w_2] &= [w_1, w_1] + [w_2, w_2] + [w_1, w_2] + [w_2, w_1] \\ &= 2[w_1, w_2]\end{aligned}$$

since $[w_1, w_1] = 0$ because the relation (FT1), $[w_2, w_2] = 0$ because of flatness of the unframed KZ connection, and $[w_2, w_1] + [w_2, w_1] = 2[w_1, w_2]$. Next, because of relation (FT2), we have

$$[w_1, w_2] = [t_i\,\mathrm{d}\,log(\lambda), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}] + \sum_{1\leqslant i<j\leqslant n} [t_j\,\mathrm{d}\,log(\lambda), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}].$$

And finally,

$$\sum_{1\leqslant i<j\leqslant n} [t_i\,\mathrm{d}\,log(\lambda), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}] + \sum_{1\leqslant i<j\leqslant n} [t_j\,\mathrm{d}\,log(\lambda), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}] = 0$$

□

In particular, by sending f_k to t_{kk}, we get morphism of splitting short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{k}^n \arrow[r] \arrow[d] & \widehat{\mathrm{PB}}_n^f(\mathbf{k}) \arrow[r] \arrow[d] & \widehat{\mathrm{PB}}_n(\mathbf{k}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbf{k}^n \arrow[r] & \exp(\widehat{\mathfrak{t}}_n^f(\mathbf{k})) \arrow[r] & \exp(\hat{\mathfrak{t}}_n(\mathbf{k})) \arrow[r] & 1 \end{tikzcd} \tag{5.7}$$

showing that $\widehat{\mathrm{PB}}_n^f(\mathbf{k}) \longrightarrow \exp(\hat{\mathfrak{t}}_n^f(\mathbf{k}))$ is a $\mathbf{k}$-pro-unipotent group isomorphism. Similarly we get an isomorphism

$$\widehat{\mathrm{B}}_n^f(\mathbf{k}) \longrightarrow \exp(\widehat{\mathfrak{t}}_n^f(\mathbf{k})) \rtimes \mathfrak{S}_n.$$

Proof of Proposition 5.2.8. Let $x \in \mathrm{Conf}^f(\mathbb{C}, n)$ and let $T_x^{f,\mathrm{KZ}}$ be the parallel transport morphism associated to $\omega_{f,n}^{\mathrm{KZ}}$. Then

$$T_x^{f,\mathrm{KZ}}(f_i) = e^{2i\pi\lambda_i} \in \exp(\hat{\mathfrak{t}}_n^f).$$

□

5.3 Modules associated to framed configuration spaces (genus g associators)

5.3.1 Configuration spaces of surfaces

Define the pure braid group with n strands in genus g as the fundamental group of $\mathrm{Conf}(\Sigma_g, n)$, $\mathrm{PB}_{g,n} := \pi_1(\mathrm{Conf}(\Sigma_g, n))$. The corresponding braid group is then $\mathrm{B}_{g,n} = \pi_1(\mathrm{Conf}(\Sigma_g, [n]))$, where $\mathrm{Conf}(\Sigma_g, [n]) = \mathrm{Conf}(\Sigma_g, n)/\mathfrak{S}_n$. Algebraically, according to [7], $\mathrm{B}_{g,n}$ is presented by generators X_a, Y_a, σ_i ($1 \le a \le g$, $1 \le i \le n-1$) and relations

(B1),(B2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ if $i \in [n-2]$, $\quad (\sigma_i, \sigma_j) = 1$ if $|i-j| > 1$,

(BG1) $(X_a, \sigma_i) = (Y_a, \sigma_i) = 1$ if $i > 1, 1 \le a \le g$,

(BG2) $(\sigma_1^{-1}X_a\sigma_1^{-1}, X_a) = (\sigma_1^{-1}Y_a\sigma_1^{-1}, Y_a) = 1$ if $1 \le a \le g$,

(BG3) $(\sigma_1^{-1}X_a\sigma_1^{-1}, X_b) = (\sigma_1^{-1}X_a\sigma_1^{-1}, Y_b) = (\sigma_1^{-1}Y_a\sigma_1^{-1}, X_b) = (\sigma_1^{-1}Y_a\sigma_1^{-1}, Y_b) = 1$ if $a < b$,

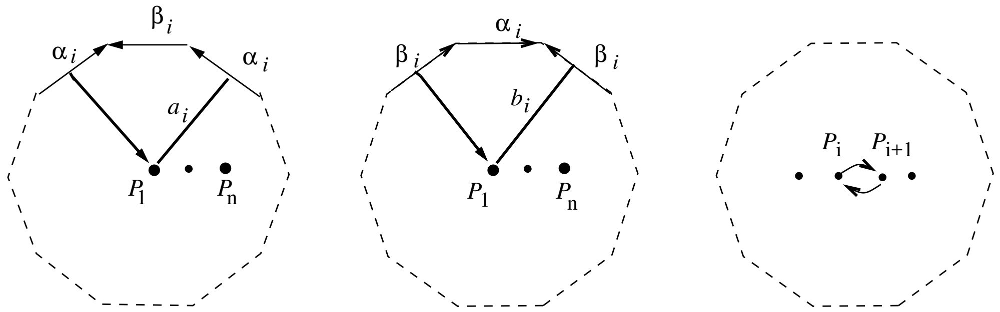
(BG4) $(\sigma_1(X_a)^{-1}\sigma_1, (Y_a)^{-1}) = \sigma_1^2$ if $1 \le a \le g$,

(BG5) $\prod_{1 \le a \le g}(X_a, (Y_a)^{-1}) = \sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1$.

The morphism $\mathrm{B}_{g,n} \longrightarrow \mathfrak{S}_n$ is given by $X_a, Y_a \mapsto 1$, $\sigma_i \mapsto s_i := (i, i+1)$. It is proved in [7] that $\mathrm{PB}_{g,n}$ is the kernel of this map and is generated by X_a^i, Y_a^i ($1 \le i \le n$, $1 \le a \le g$), where $Z_a^i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1} Z_a \sigma_1^{-1} \cdots \sigma_{i-1}^{-1}$ for Z any of the letters X, Y.

The geometric interpretation of the presentation of $\mathrm{B}_{g,n}$ for $g \ge 1$ is constructed as follows[2]

- **Generators :** We represent Σ_g as a polygon L of $4g$ sides with the standard identification of edges. We can consider braids as paths on L, which we draw with the usual "over and under" information at the crossing points. and we represent the generators of $\mathrm{B}_{g,n}$ realized as braids on L.



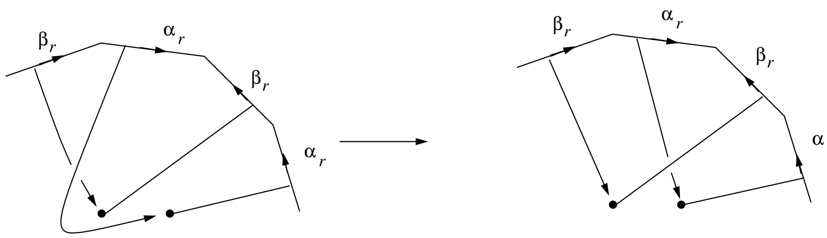
a_i $\qquad$ b_i $\qquad$ σ_i

Notice that in the braid a_i (respectively b_i) the only non trivial string is the first one, which goes through the the wall α_i (the wall β_i). Remark also that $\sigma_1 \ldots, \sigma_{n-1}$ are the classical braid generators on the disk so relations (B1), (B2) hold.

[2]We borrow the drawings from [7].

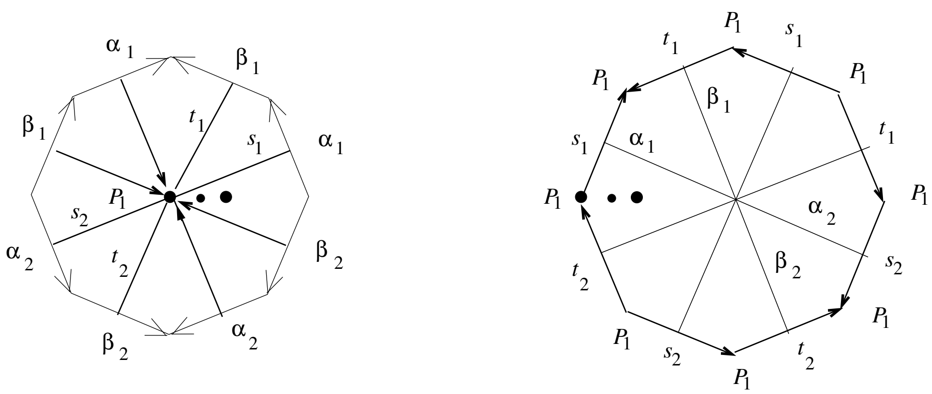
- **Relations (BG1-BG3) :** The fact that these relations hold is trivial and is explained in [7].

- **Relation (BG4) :** Indeed, there is a homotopy between $\sigma_1^{-1}a_r\sigma_1^{-1}b_r$ and $b_r\sigma_1^{-1}a_r\sigma_1$ represented in the following picture:

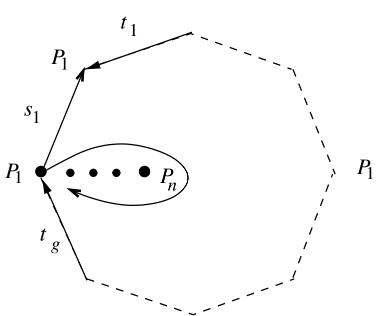


- **Alternative fundamental domain and relation (BG5) :** Let s_r and t_r be the first string of a_r and b_r respectively, where $1 \leq r \leq 2g$.

  We can obtain a new fundamental domain, denoted L_1 with vertex P_1, by cutting L along the paths $s_1, t_1, \ldots, s_g, t_g$ and by glueing the pieces along the edges of L as we can see in the following picture, for $g = 2$:



On L_1 it is clear that $[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]$ is equivalent to the braid represented as follows



This braid is equivalent to the braid $\sigma_1\sigma_2 \ldots \sigma_{n-1}^2 \ldots \sigma_2\sigma_1$ so (BG5) in $\mathrm{PB}_{g,n}$.

5.3.2 Framed configuration spaces on surfaces

In this section we assume $g > 1$. In [8], the authors showed that the fundamental group $\mathrm{PB}^f_{g,n}$ of $\mathrm{Conf}^f(\Sigma_g, n)$ can be exhibited as a non-splitting central extension

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathrm{PB}^f_{g,n} \xrightarrow{\beta_n} \mathrm{PB}_{g,n} \longrightarrow 1 \tag{5.8}$$

where β_n is the morphism induced by the projection map $\mathrm{Conf}^f(\Sigma_g, n) \longrightarrow \mathrm{Conf}(\Sigma_g, n)$ (*i.e.* β_n consists in forgetting the framing). $\mathrm{Conf}^f(\Sigma_g, n)$ is an Eilenberg-Maclane space of type $(\mathrm{PB}^f_{g,n}, 1)$. This short exact sequence extends to the following non-split short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathrm{B}^f_{g,n} \xrightarrow{\widehat{\beta}_n} \mathrm{B}_{g,n} \longrightarrow 1\ , \tag{5.9}$$

where $\widehat{\beta}_n$ consists in forgetting the framing. $\mathrm{Conf}^f(\Sigma_g, [n])$ is an Eilenberg-Maclane space of type $(\mathrm{B}^f_{g,n}, 1)$.

The framed pure braid group $\mathrm{PB}^f_{g,n}$ is generated by $A_{i,j}$ and f_k where $1 \le i \le 2g+n-1, 2g+1 \le j \le 2g+n,\ i < j,\ 1 \le k \le n$ together with the following relations

(PR1) $A^{-1}_{i,j} A_{r,s} A_{i,j} = A_{r,s}$ if $(i < j < r < s)$ or $(r+1 < i < j < s)$ or ($i = r+1 < j < s$ for even $r < 2g$ or $r > 2g$),

(PR2) $A^{-1}_{i,j} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A^{-1}_{i,s}$ if $(i < j < s)$;

(PR3) $A^{-1}_{i,j} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A^{-1}_{j,s} A^{-1}_{i,s}$ if $(i < j < s)$;

(PR4) $A^{-1}_{i,j} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A^{-1}_{i,s} A^{-1}_{j,s} A_{r,s} A_{j,s} A_{i,s} A^{-1}_{j,s} A^{-1}_{i,s}$ if $(i+1 < r < j < s)$ or ($i+1 = r < j < s$ for odd $r < 2g$ or $r > 2g$)

(ER1) $A^{-1}_{r+1,j} A_{r,s} A_{r+1,j} = A_{r,s} A_{r+1,s} A^{-1}_{j,s} A^{-1}_{r+1,s}$ if r odd and $r < 2g$;

(ER2) $A^{-1}_{r-1,j} A_{r,s} A_{r-1,j} = A_{r-1,s} A_{j,s} A^{-1}_{r-1,s} A_{r,s} A_{j,s} A_{r-1,s} A^{-1}_{j,s} A^{-1}_{r-1,s}$ if r even and $r < 2g$,

(C) the f_k are central

(FTR) $[A^{-1}_{2g,2g+k}, A_{2g-1,2g+k}] \cdots [A^{-1}_{2,2g+k}, A_{1,2g+k}] =$
$A_{2g+1,2g+k} \cdots A_{2g+k-1,2g+k} A_{2g+k,2g+k+1} \cdots A_{2g+k,2g+n}\, f_k^{2(g-1)}$
where $1 \le k \le n$, and where we set $A_{2g+1,2g+1} = A_{2g+n,2g+n} = 1$.

The group $B^f_{g,n}$ is generated by $A_1, B_1, \ldots, A_g, B_g, \sigma_1, \ldots, \sigma_{n-1}, f_1, \ldots, f_n$ together with the following relations (B1), (B2), (FB1), (FB2) and

(FBG1) $c_i \sigma_j = \sigma_j c_i$ for all $j \ge 2$, $c_i = A_i$ or B_i and $i = 1, \ldots, g$

(FBG2) $c_i \sigma_1 c_i \sigma_1 = \sigma_1 c_i \sigma_1 c_i$ for $c_i = A_i$ or B_i and $i = 1, \ldots, g$

(FBG3) $A_i \sigma_1 B_i = \sigma_1 B_i \sigma_1 A_i \sigma_1$ for $i = 1, \ldots, g$

(FBG4) $c_i \sigma_1^{-1} c_j \sigma_1 = \sigma_1^{-1} c_j \sigma_1 c_i$ for $c_i = A_i$ or B_i, $c_j = A_j$ or B_j and $1 \le j < i \le g$

(FBG5) $\prod_{i=1}^{g} [A_i^{-1}, B_i] = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 f_1^{2(g-1)}$

5.3.3 The $\mathbf{PaB}^f$-module of parenthesized framed genus g braids

Consider the framed Fulton-MacPherson compactification $\mathrm{FM}^f_{2,g}(n)$ of $\mathrm{Conf}^f(\Sigma_g, n)$.

The boundary $\partial\,\mathrm{FM}^f_{2,g}(n) = \mathrm{FM}^f_{2,g}(n) - \mathrm{Conf}^f(\Sigma_g, n)$ is made of the following irreducible components: for any decomposition $n = n_1 + \cdots + n_k$ there is a component

$$\partial_{n_1,\cdots,n_k}\,\mathrm{FM}^f_{2,g}(n) \cong \prod_{i=1}^{k} \mathrm{FM}^f_2(n_i) \times \mathrm{FM}^f_{2,g}(n)\,.$$

The inclusion of boundary components provide $\mathrm{FM}^f_{2,g}$ with the structure of a module over the operad FM^f_2 in topological spaces. Given a choice of an embedding $\mathbb{S}^1 \hookrightarrow \Sigma_g$, we have inclusions

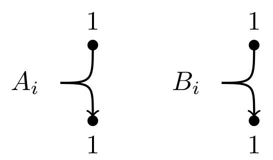
$$\mathbf{Pa}(n) \subset \overline{\mathrm{C}}^f(\mathbb{S}^1, n) \subset \mathrm{FM}^f_{2,g}(n).$$

We then define

$$\mathbf{PaB}^f_g := \pi_1(\mathrm{FM}^f_{2,g}, \mathbf{Pa})\,,$$

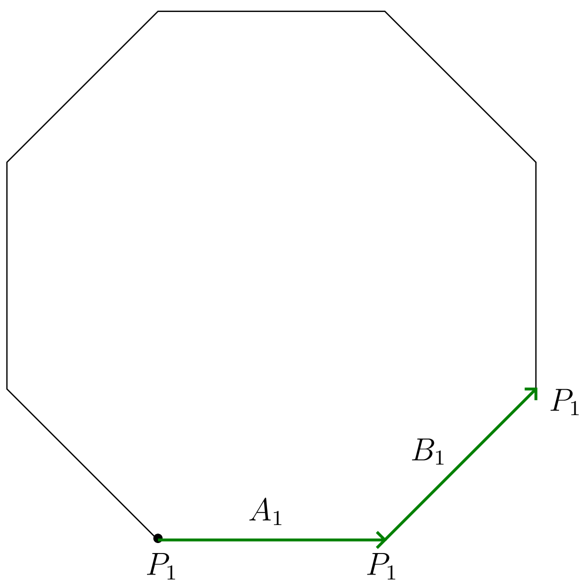
which is a $\mathbf{PaB}^f$-module in groupoids.

Example 5.3.1. Structure of $\mathbf{PaB}^f_g(1)$. *As opposed to the unframed reduced genus 1 case, we have non trivial arrows in arity 1. More precisely, we have $2g$ automorphisms, A_i and $B_i \in \mathrm{End}_{\mathbf{PaB}^f_g(1)}(1)$, for all $1 \leqslant i \leqslant g$, that can be depicted as follows:*



(5.10)

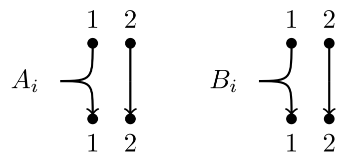
and correspond to the following paths in Σ_g. We fix the marked points in the first A-cycle, thus A_1 and B_1 correspond to the paths:



All other A_i and B_i are depicted in the same way.

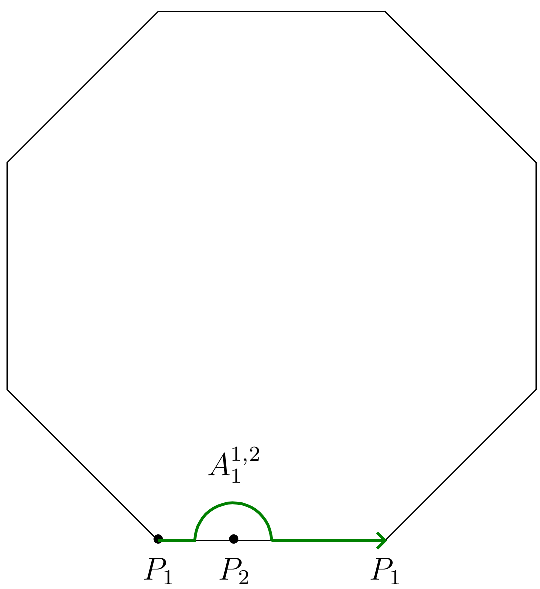
Example 5.3.2. Notable arrows in $\mathbf{PaB}_g^f(2)$.

We have $2g$ automorphisms, $A_i^{1,2}$ and $B_i^{1,2} \in \mathrm{End}_{\mathbf{PaB}_g^f(2)}(12)$, for all $1 \leqslant i \leqslant g$, that can be depicted as follows:

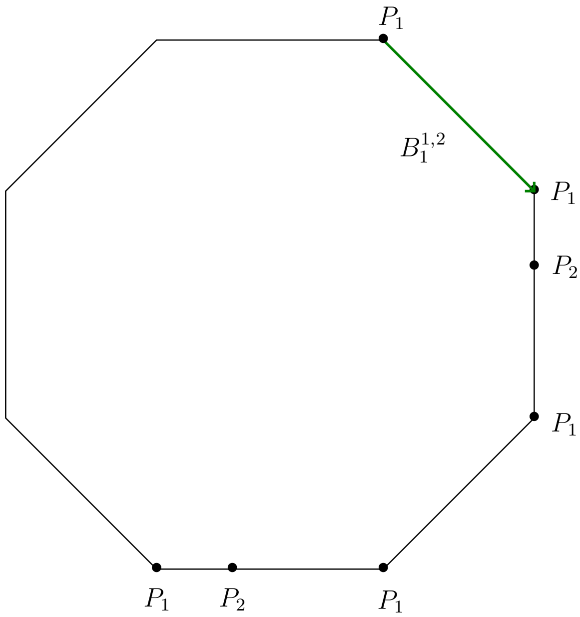


(5.11)

and correspond to the following paths in $\mathrm{Conf}^f(\Sigma_g, 2)$ Again, we fix the marked points in the first A-cycle, thus $A_1^{1,2}$ correspond to the path:

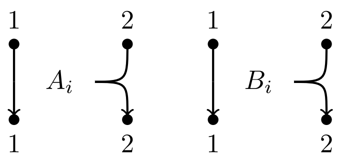


Next, the map $B_1^{1,2}$ *corresponds to the path:*



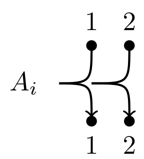
All other $A_i^{1,2}$ *and* $B_i^{1,2}$ *are depicted along the same representation as that for* $B_1^{1,2}$.

Moreover, we also have arrows



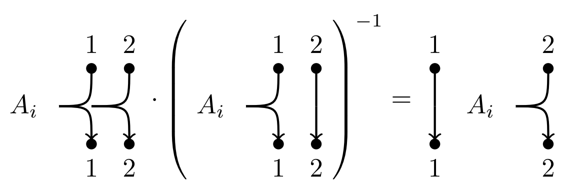
We let the reader draw the corresponding paths in $\mathrm{Conf}^f(\Sigma_g, 2)$.

Remark 5.3.3. *By doubling the only braid in* $A_i \in \mathbf{PaB}_g^f(1)$, *which amounts to taking* $\circ_1(A_i, \mathrm{id}_{12}) \in \mathbf{PaB}_g^f(2)$, *we get an arrow* A_i^{12} *depicted as follows:*



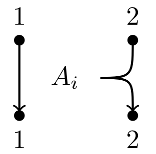
It is then a fact that

$$A_i^{12}(A_i^{1,2})^{-1} =$$



(5.12)

This means that even if, contrary to the reduced genus 1 case, $A_i^{1,2}$ is not equal to



one can retrieve the latter arrow from the composite $A_i^{12}(A_i^{1,2})^{-1}$.

Definition 5.3.4. *Let $\mathbf{CoB}_g^f$ the $\mathbf{CoB}^f$-module in groupoids with $\mathfrak{S}$-module of objects $\mathfrak{S}$ and where, for $n \geqslant 1$, the morphisms of $\mathbf{CoB}_g^f(n)$ consists of isotopy classes of genus g framed braids (i.e. elements of the braid group $B_{g,n}^f$) α together with a colouring bijection $i \mapsto \alpha_i$ between the index set $i \in \{1, \dots, n\}$ which leaves the last strand uncoloured and the strands $\alpha_i \in \{\alpha_1, \dots, \alpha_n\}$ of our braid α and the data of a special braid corresponding to the framing.*

The following theorem can be undestood as a rephrasing of the MacLane-Joyal-Street coherence theorem for framed genus g D_2-modules.

Theorem 5.3.5. *As a $\mathbf{PaB}^f$-module in groupoids having $\mathbf{Pa}$ as $\mathbf{Pa}$-module of objects, $\mathbf{PaB}_g^f$ is isomorphic freely generated by $A_i^{1,2}$ and $B_i^{1,2}$, for all $1 \leqslant i \leqslant g$, in $\mathbf{PaB}_g^f(2)$, together with relations*

(Red) $A_i^{1,\emptyset} := A_i$, $B_i^{1,\emptyset} := B_i$, $A_i^{\emptyset,2} := \mathrm{Id}^1$, $B_i^{\emptyset,2} := \mathrm{Id}^1$ *in* $\mathbf{PaB}_g^f(1)$,

(D1) $\Phi^{1,2,3}A_i^{1,23}R^{1,23}\Phi^{2,3,1}A_i^{2,31}R^{2,31}\Phi^{3,1,2}A_i^{3,12}R^{3,12} = A_i^{(12)3}$,

(D2) $\Phi^{1,2,3}B_i^{1,23}(R^{23,1})^{-1}\Phi^{2,3,1}B_i^{2,31}(R^{31,2})^{-1}\Phi^{3,1,2}B_i^{3,12}(R^{12,3})^{-1} = B_i^{(12)3}$,

for all $1 \leq i \leq g$, and the following relation:

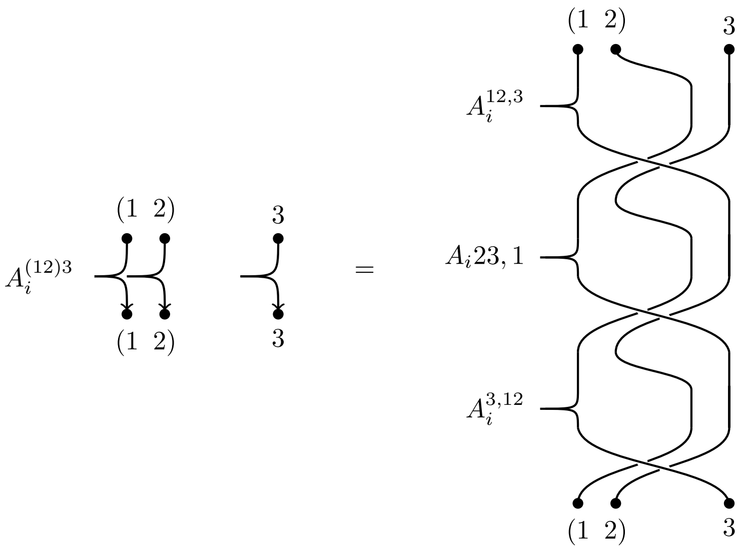
(gE) $R^{1,2}R^{2,1}(F^1)^{2(g-1)} = \prod_{i=1}^g \left(\Phi^{1,2,3}B_i^{1,23}(\Phi^{1,2,3})^{-1}, (R^{2,1})^{-1}\Phi^{2,1,3}(A_i^{2,13})^{-1}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}\right)$

as arrows from $(12)3$ to $(12)3$ in $\mathbf{PaB}_g^f(3)$.

Remark 5.3.6. *An easy consequence of the above theorem is that $\mathbf{PaB}_g^f$ identifies with the fake pullback $\omega^\star\mathbf{CoB}_g^f$ of the $\mathbf{CoB}^f$-module $\mathbf{CoB}_g^f$ along the forgetful functor $\omega : \mathbf{Pa} \longrightarrow \mathfrak{S}$,*

Proof. Let $\mathcal{Q}$ be the $\mathbf{PaB}^f$-module with the above presentation. We first show that there is a morphism of $\mathbf{PaB}^f$-modules $\mathcal{Q} \longrightarrow \mathbf{PaB}_g^f$. We have already seen that there are $2g$ automorphisms A_i, B_i of (1) in $\mathbf{PaB}_g^f(1)$ (see Example 5.3.1) and $2g$ automorphisms $A_i^{1,2}, B_i^{1,2}$ of (12) in $\mathbf{PaB}_g^f(2)$ (see Example 5.3.2). We have to prove that they indeed satisfy the relations (D1), (D2) and (gE).

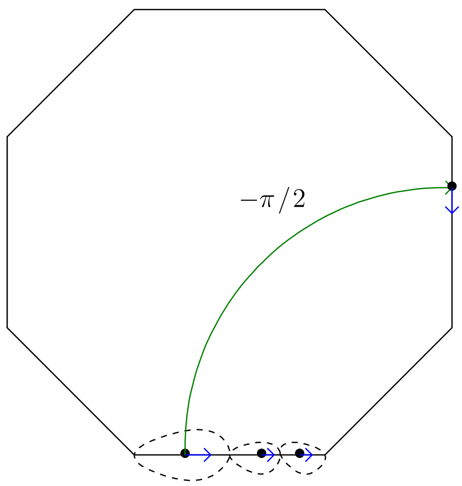
Relations (D1) and (D2) are satistfied: the first *decagon relation* (D1) can be depicted as follows:



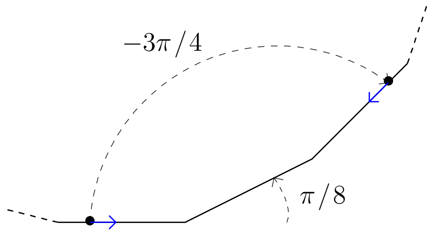
(D1)

It is satisfied in $\mathbf{PaB}_g^f$, expressing the fact that when all (here, three) points move along a generating generating loop on Σ_g, this corresponds to the path in the framed configuration space of points on Σ_g twisting the three points. The same is true with the second decagon relation (D2).

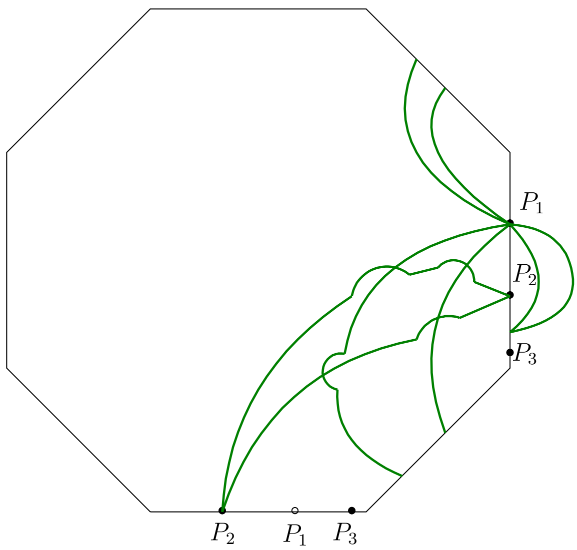
Relation (gE) is satisfied: Relation (gE) is more difficult to draw so we sketch the way to think of the right-hand-side. Align the points in a generating cycle of the genus g surface (this means that they are in the boundary of the compactified framed configuration space). Then if a point travels through a cycle, its corresponding framing will naturally start to spin as one can see in the following picture, for $g = 2$



and for $g = 4$

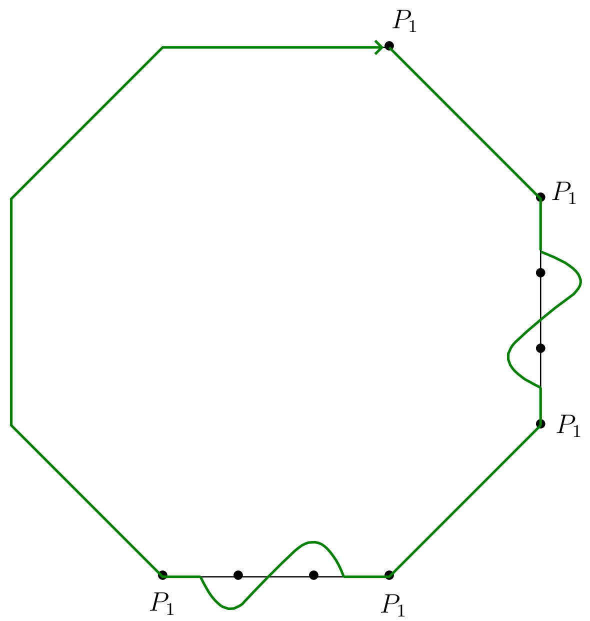


If we consider a polygon with $4g$ sides corresponding to a genus g surface, then for each marked point travelling through the generating cycles, the framing attached to that point will be twisted by an angle of $\pi - \frac{\pi}{g}$. Next, one can interpret the path on the right hand side of (gE) as the following path. As we already took care of the behaviour of the framing we will neglect this information in the picture. $\left(\Phi^{1,2,3}B_1^{1,23}(\Phi^{1,2,3})^{-1}, (R^{2,1})^{-1}\Phi^{2,1,3}(A_1^{2,13})^{-1}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}\right)$ corresponds to the following picture



One can see that, if $i \neq j$, then the paths corresponding to a A_i cycle and a B_j cycle do not intersect.

Another possible way to interpret this goes as follows: if we suppose that the marked points were chosen to be in the A_1-cycle of Σ_g, the right hand side of (gE) can be drawn as follows:



In conclusion, one can then easily see that if we take a point and make it travel around all the generating cycles concerned in the right-hand-side of relation (gE), the corresponding framing will make $2g \times \frac{(g-1)}{g} = 2(g-1)$ complete spins and the first point P_1 will have done a complete loop around the second point P_2. This is exactly the left-hand-side of equation (gE).

Thus, by the universal property of $\mathcal{Q}$, there is a morphism of $\mathbf{PaB}^f$-modules $\mathcal{Q} \longrightarrow \mathbf{PaB}_g^f$, which is the identity on objects. To show that this map is in fact an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of objects arity-wise, as all groupoids are connected. Let $n \geq 0$, and p be the object $(\cdots((12)3)\cdots\cdots)n$ of $\mathcal{Q}(n)$ and $\mathbf{PaB}_g^f(n)$. We want to show that the induced morphism

$$\mathrm{Aut}_{\mathcal{Q}(n)}(p) \longrightarrow \mathrm{Aut}_{\mathbf{PaB}_g^f(n)}(p) = \pi_1\left(\overline{\mathrm{Conf}}^f(\Sigma_g, n), p\right)$$

is an isomorphism.

On the one hand, as $\overline{\mathrm{Conf}}^f(\Sigma_g, n)$ is a manifold with corners, we are allowed to move the basepoint p to a point p_{reg} which is included in the fundamental domain L_1 described in subsection 5.3.1. We then have an isomorphism of fundamental groups $\pi_1(\overline{\mathrm{Conf}}^f(\Sigma_g, n), p) \simeq \pi_1(\mathrm{Conf}^f(\Sigma_g, n), p_{reg})$.

On the other hand, one can construct a non-symetric module $\tilde{Q}$ in groupoids over B^f carrying an action of the (algebraic version of the) framed braid group $\mathrm{B}_{g,n}^f$ on Σ_g in the following sense:

- for each $n \geq 1$, $\tilde{Q}(n)$ is a groupoid with maximal parenthesizations of unnumbered elements as objects.
- $\tilde{Q}$ is freely generated by $A_i^{1,2} := A_i^{\bullet,\bullet}$ and $B_i^{1,2} := B_i^{\bullet,\bullet}$ in $\tilde{Q}(2)$, for all $1 \leqslant i \leqslant g$, satisfying relations (Red), (D1), (D2) and (gE).

- in Lemma 5.3.7 we show that there are group morphisms $\mathrm{B}^f_{g,n} \xrightarrow{\sim} \mathrm{Aut}_{\tilde{Q}(n)}(p) \longrightarrow \mathfrak{S}_n$, the left one being an isomorphism.

In the same way the collection $\{\mathrm{PB}^f_{g,n}\}_{n\geq 1}$ of pure genus g braids owns a non-symmetric PB^f-module structure denoted PB^f_g.

Moreover, one the forgetful map **Op** $\longrightarrow$ **NsOp** between the category of operads and the category of non-symmetric operads induces a map $Q \longrightarrow \tilde{Q}$. Then, one has by constuction of $\tilde{Q}$ that $\mathrm{Aut}_{\mathcal{Q}(n)}(p)$ is the kernel of the map $\mathrm{Aut}_{\tilde{Q}(n)}([p]) \longrightarrow \mathfrak{S}_n$. One can actually show that we have a commuting diagram

$$\begin{tikzcd}
\mathrm{PB}^f_{g,n} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Aut}_{Q(n)}(p) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{Conf}}^f(\Sigma_g, n), p\right) \arrow[d] & \pi_1\left(\mathrm{Conf}^f(\Sigma_g, n), p_{reg}\right) \arrow[l, "\simeq"'] \arrow[d] \\
\mathrm{B}^f_{g,n} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Aut}_{\tilde{Q}(n)}([p]) \arrow[r] \arrow[d] & \pi_1\left(\overline{\mathrm{Conf}}^f(\Sigma_g, n)/\mathfrak{S}_n, [p]\right) \arrow[d] & \pi_1\left(\mathrm{Conf}^f(\Sigma_g, n)/\mathfrak{S}_n, [p_{reg}]\right) \arrow[l, "\simeq"'] \arrow[d] \\
\mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n \arrow[r, equal] & \mathfrak{S}_n
\end{tikzcd}$$

where all vertical sequences are short exact sequences. Thus, in order to show that the map $\mathrm{Aut}_{\mathcal{Q}(n)}(p) \longrightarrow \pi_1\left(\overline{\mathrm{Conf}}^f(\Sigma_g, n), p\right)$ is an isomorphism, we are left to show that

$$\mathrm{B}^f_{g,n} \longrightarrow \pi_1\left(\mathrm{Conf}^f(\Sigma_g, n)/\mathfrak{S}_n, [p_{reg}]\right)$$

is indeed an isomorphism. But this map is nothing else than the map constructed in [8, Theorem 13], identifying the algebraic and topological versions of the framed braid group on Σ_g.

□

Lemma 5.3.7. *Let $\tilde{Q}$ be the operadic B^f-module with unnumbered maximal paranthesizations as objects and with generators $A_i^{1,2} := A_i^{\bullet,\bullet}$ and $B_i^{1,2} := B_i^{\bullet,\bullet}$, for all $1 \leqslant i \leqslant g$, in $\tilde{Q}(2)$ satisfying relations (Red), (D1), (D2) and (gE).*

Let p be the object in $\tilde{Q}(n)$ given by right parenthesization $p := (\bullet(\bullet(\bullet(\ldots((\bullet\bullet))\ldots)$. Then there is a unique group isomorphism

$$\phi_n : B^f_{g,n} \longrightarrow \mathrm{Aut}_{\tilde{Q}(n)}(p),$$

such that

- $A_i \mapsto A_i^{1,2\ldots n}$, *for all* $1 \leqslant i \leqslant g$;
- $B_i \mapsto B_i^{1,2\ldots n}$, *for all* $1 \leqslant i \leqslant g$;
- $\sigma_i \mapsto R^{i,i+1}$; *for all* $1 \leqslant i \leqslant n-1$;
- $f_i \mapsto F^i$, *for all* $1 \leqslant i \leqslant n$;

where $A^{1,2\ldots n} \in \mathrm{Aut}_{\tilde{Q}(n)}(p)$ is obtained from $A^{1,2}$, F^i is obtained from F^1 and $R^{i,i+1} \in \mathrm{Aut}_{\tilde{Q}(n)}(p)$ is obtained from $R^{1,2}$ by some finite sequences of arrows involving the associator and the operadic module morphisms since the parenthesizations are unmarked.

In particular, by applying a finite sequence of associators one can show that the above lemma remains true for all possible choices of base points $p \in \tilde{Q}(n)$.

Let us sketch the proof of this Lemma (a complete proof will be done un subsequent works).

Proof. For simplicity, we omit the associativity constraints. One can show by induction that the image of $A_i^k := \sigma_{k-1} A_i^{k-1} \sigma_{k-1}$ is

$$R^{12\ldots(k-1),k} A_i^{1,2\ldots(n-1)} R^{k,12\ldots(k-1)}$$

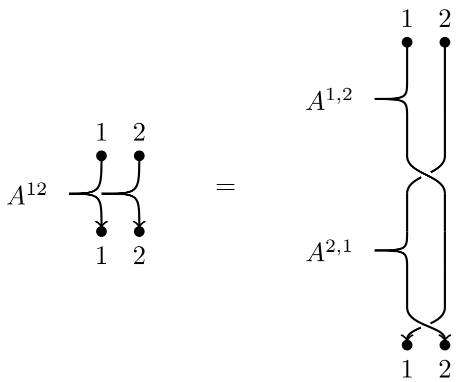
therefore the image of $A_i^1 \cdots A_i^k$ is $A^{\pm}_{X^{\otimes k}, X^{\otimes n-k}}$. We will thus reduce to the cases n=2,3 in the rest of the proof.

<u>ϕ_n is a well-defined group morphism:</u> Let us first show that there is indeed such a group morphism. First of all, the braid relations are preserved as there are morphisms from B_3 to both groups (the first one is classic, the second one is induced by the fact that $\tilde{Q}$ is a B^f-module.

Notice that, by removing the third braid in relation (D1), we obtain relation

$$A_i^{1,2} R^{1,2} A_i^{2,1} R^{2,1} = A_i^{12}$$

which can be depicted as follows:



(D1bis)

Then, one shows that relations (FBG1-4) are satisfied by the same reasoning that [34, Proposition 1.3] in the following way: for each $1 \leqslant i \leqslant g$, take $X_1^+ := A_i$ and $X_1^- := (B_i)^{-1}$. Then relations (FBG1-3) are equivalent to

$$(\sigma_1^{\pm 1} X_1^{\pm})^2 = (X_1^{\pm} \sigma_1^{\pm 1})^2, \quad (X_1^{\pm}, \sigma_i) = 1 \text{ for } i = 2, \ldots, n-1, \quad (X_1^-, (X_2^+)^{-1}) = \sigma_1^2,$$

and are thus preserved by ϕ_n. Relation (FBG4) is preserved by naturality in $\mathrm{Aut}_{\tilde{Q}(n)}(p)$.

Thus, we have a group morphism

<u>ϕ_n is surjective:</u> The fact that the map ϕ_n is surjective is a consequence of the fact that all the defining relations in $\tilde{Q}(n)$ come from the defining relations of $B^f_{g,n}$ and the oepradic module partial compositions.

ϕ_n is injective: Let us now show the injectivity of this map. Let $\bar{Q}$ be the oeprad module with same objects as $\tilde{Q}$ and; for every object p of $\bar{Q}(n)$, we define $Aut_{\bar{Q}(n)}(p) := B^f_{g,n}$. Next we have a map $\tilde{Q} \longrightarrow \bar{Q}$ sending the generations $A_i^{1,2}$ to A_i and $B_i^{1,2}$ to B_i in $B^f_{g,2}$. Indeed, if we denote $X_1^+ := A_i$ and $X_1^- := (B_i)^{-1}$, then we have relations $(\sigma_2^{\pm 1}\sigma_1^{\pm 1}X_1^{\pm})^3 = X_{123}^{\pm}$, $(X_1^-, (\sigma_1 X_1^+ \sigma_1)^{-1}) = \sigma_1^2$ and $\prod_{i=1}^g (B_i, (\sigma_1 A_i \sigma_1)^{-1}) = \sigma_1^2 f_1^{2(g-1)}$ show that relations (Red), (D1), (D2) and (gE) are preserved.

Then, as $\mathbf{PaB}^f$ acts on both of these operadic modules we conclude that there is a map $Aut_{\tilde{Q}(n)}(p) \longrightarrow Aut_{\bar{Q}(n)}(p)$. In order to prove the injectivity of ϕ, we are left to prove that the composite

$$B^f_{g,n} \longrightarrow Aut_{\tilde{Q}(n)}(p) \longrightarrow Aut_{\bar{Q}(n)}(p)$$

is the identity morphism, which is true as, by construction of both maps. □

This means that any $\mathfrak{f}\mathrm{D}_2$-module morphism $\phi : \mathbf{PaB}^f_g \longrightarrow P$, is determined (up to isomorphism) by A_i, B_i and the above three relations. As in the framed genus 0 situation, we have a $\mathbf{PaB}^f(\mathbf{k})$-module in $\mathbf{Cat}(\mathbf{CoAss_k})$ denoted $\mathbf{PaB}^f_g(\mathbf{k}) := \Delta_{\mathbf{k}}(\mathbf{PaB}^f_g)$. Now consider its associated inverse system of $(\mathbf{PaB}^f)^{(m)}(\mathbf{k})$-modules given, for all $m \in \mathbb{N}$, by

$$(\mathbf{PaB}^f_g)^{(m)}(\mathbf{k}) := \mathbf{PaB}^f_g(\mathbf{k})/(\mathcal{I}^m(\mathbf{k}) \cdot \mathbf{PaB}^f_g(\mathbf{k})).$$

By taking the inverse limit over m of these inverse system, we get a $\widehat{\mathbf{PaB}}^f(\mathbf{k})$-module in $\mathbf{Cat}(\mathbf{CoAss_k})$

$$\widehat{\mathbf{PaB}}^f_g(\mathbf{k}) := \varprojlim((\mathbf{PaB}^f_g)^{(m)}(\mathbf{k})).$$

5.3.4 The PaCD(k)-module of parenthesized genus g chord diagrams

Let us consider $g > 0$ and $n \geq 0$ and define $\mathfrak{t}_{g,n}(\mathbf{k})$ as the $\mathbf{k}$-Lie algebra with generators x_a^i, y_a^i, t_{ij} for $i \neq j \in [n], 1 \leq a \leq g$ satisfying relations (T1), (T2), (T3) and

(G1,G2) $[x_a^i, x_b^j] = 0$ and $[y_a^i, y_b^j] = 0$ if $i \neq j$

(G3) $[x_a^i, y_b^j] = \delta_{ab} t_{ij}$ if $i \neq j$;

(G4) $[x_a^i + x_a^j, t_{ij}] = [x_a^k, t_{ij}] = 0$ if $\{i,j\} \cap \{k\} = \emptyset$;

(G5) $[y_a^i + y_a^j, t_{ij}] = [y_a^k, t_{ij}] = 0$ if $\{i,j\} \cap \{k\} = \emptyset$;

(G6) $\sum\limits_{a=1}^{g} [x_a^i, y_a^i] + \sum_{j:j\neq i} t_{ij} = 0$;

The Lie algebra $\mathfrak{t}_{g,n}(\mathbf{k})$ is equipped with a grading given by $deg(x_i^a) = (1,0)$, $deg(y_i^a) = (0,1)$. The total degree defines a positive grading on $\mathfrak{t}_{g,n}(\mathbf{k})$; we denote by $\hat{\mathfrak{t}}_{g,n}(\mathbf{k})$ the corresponding completion. If $\mathbf{k} = \mathbb{C}$, we will denote $\mathfrak{t}_{g,n}(\mathbf{k}) := \mathfrak{t}_{g,n}$.

Theorem 5.3.8. *(Bezrukavnikov, Enriquez) There is a monodromy morphism* $\mathrm{PB}_{g,n} \longrightarrow \exp(\hat{\mathfrak{t}}_{g,n})$ *inducing an isomorphism of Lie algebras* $\mathrm{Lie}(\mathrm{PB}_{g,n})^{\mathbb{C}} \xrightarrow{\sim} \hat{\mathfrak{t}}_{g,n}$.

The collection $\mathfrak{t}_g(\mathbf{k}) := \{\mathfrak{t}_{g,n}(\mathbf{k})\}_{n\geq 1}$ is provided by the structure of a $\mathfrak{t}_g(\mathbf{k})$-module in $Lie_{\mathbf{k}}$ as follows. The $\mathfrak{S}$-module $\mathfrak{t}_g(\mathbf{k})$ inherits the structure of a module over the operad $\mathfrak{t}$ in $Lie_{\mathbf{k}}$ with respect to the collection of maps given on the generators as follows:

$$\begin{array}{rccc}
\circ_k : & \mathfrak{t}_{g,I}(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) & \longrightarrow & \mathfrak{t}_{g,J\sqcup I-\{i\}}(\mathbf{k}) \\
& (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha\beta} \\
& (t_{ij}, 0) & \longmapsto & \begin{cases} t_{ij} & \text{if} \quad k \notin \{i,j\} \\ \sum\limits_{p\in J} t_{pj} & \text{if} \quad k = i \\ \sum\limits_{p\in J} t_{ip} & \text{if} \quad j = k \end{cases} \\
& (x_i^a, 0) & \longmapsto & \begin{cases} x_i^a & \text{if} \quad k \neq i \\ \sum\limits_{p\in J} x_p^a & \text{if} \quad k = i \end{cases} \\
& (y_i^a, 0) & \longmapsto & \begin{cases} y_i^a & \text{if} \quad k \neq i \\ \sum\limits_{p\in J} y_p^a & \text{if} \quad k = i \end{cases}
\end{array}$$

Since we are in possession of operad modules $\mathbf{Pa}(\mathbf{k})$ and $\widehat{\mathbf{CD}}_g(\mathbf{k})$ in $\mathbf{Cat}(\mathbf{CoAss_k})$ and of an operad module morphism $f : \mathbf{Pa} \longrightarrow \mathrm{Ob}(\widehat{\mathbf{CD}}_g(\mathbf{k}))$, we are ready to define the $\mathbf{PaCD}(\mathbf{k})$-module

$$\mathbf{PaCD}_g(\mathbf{k}) := f^\star \widehat{\mathbf{CD}}_g(\mathbf{k})$$

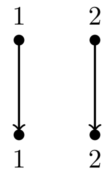
in $\mathbf{Cat}(\mathbf{CoAss_k})$ of parenthesized genus g chord diagrams. We have $\mathrm{Ob}(\mathbf{PaCD}_g(\mathbf{k})) := \mathbf{Pa}$ and $\mathrm{Mor}_{\mathbf{PaCD}_g(\mathbf{k})(n)}(p,q) := \mathrm{Mor}_{\widehat{\mathbf{CD}}_g(\mathbf{k})(n)}(pt,pt) = \hat{\mathcal{U}}(\hat{\mathfrak{t}}_{g,n}(\mathbf{k}))$.

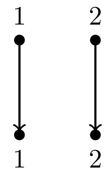
Example 5.3.9 (Notable arrows in $\mathbf{PaCD}_g(\mathbf{k})(2)$)**.** *We have the following arrows X_i, Y_i in* $\mathbf{PaCD}_g(\mathbf{k})(1)$

$X_i = x_i^1 \cdot$ 

$Y_i = y_i^1 \cdot$ 

and $X_i^{1,2}$, $Y_i^{1,2}$ in $\mathbf{PaCD}_g(\mathbf{k})(2)$

$X_i^{1,2} = x_i^1 \cdot$ 

$Y_i^{1,2} = y_i^1 \cdot$ 

Remark 5.3.10. *The elements $X_i^{1,2}, Y_i^{1,2}$ are generators of the $\mathbf{PaCD}(\mathbf{k})$-module $\mathbf{PaCD}_g(\mathbf{k})$ and satisfy the following relations*

(Red) $X_i^{\emptyset,2} = Y_i^{\emptyset,2} = 0$, $X_i^{1,\emptyset} := X_i$, $Y_i^{1,\emptyset} := Y_i$,

(D1) $a^{1,2,3}X_i^{1,23}X^{1,23}a^{2,3,1}X_i^{2,31}X^{2,31}a^{3,1,2}X_i^{3,12}X^{3,12} = X_i^{(12)3}$,

(D2) $a^{1,2,3}Y_i^{1,23}X^{1,23}a^{2,3,1}Y_i^{2,31}X^{2,31}a^{3,1,2}Y_i^{3,12}X^{3,12} = Y_i^{(12)3}$,

(gE) $X^{1,2}X^{2,1}P_1^{2(g-1)} = \Big(a^{1,2,3}Y_i^{1,23}(a^{1,2,3})^{-1}, X^{2,1}a^{2,1,3}(X_i^{2,13})^{-1}(a^{2,1,3})^{-1}X^{1,2}\ \Big)$.

5.3.5 The $\mathbf{PaCD}^f(\mathbf{k})$-module of parenthesized genus g framed chord diagrams

Let $\mathfrak{t}^f_{g,n}(\mathbf{k})$ denote the graded Lie algebra over $\mathbf{k}$ generated by t_{ij}, $1 \leq i,j \leq n$, x^i_a, y^i_a for $1 \leq i \leq n, 1 \leq a \leq g$ with relations (FT1), (FT2), (FT3), (G1), (G2), (G3) and the following relation

(FG4) $[x^i_a + x^j_a, t_{ij}] = [x^k_a, t_{ij}] = 0$ if $\{i,j\} \cap \{k\} = \emptyset$, for $1 \leq i \leq n, 1 \leq a \leq g$;

(FG5) $[y^i_a + y^j_a, t_{ij}] = [y^k_a, t_{ij}] = 0$ if $\{i,j\} \cap \{k\} = \emptyset$, for $1 \leq i \leq n, 1 \leq a \leq g$;

(FG6) $\sum\limits_{a=1}^{g} [x^i_a, y^i_a] + \sum_{j:j\neq i} t_{ij} + 2(g-1)t_{ii} = 0$, for $1 \leq i \leq n, 1 \leq a \leq g$;

The map $\mathrm{PB}^f_{g,n} \longrightarrow \exp(\hat{\mathfrak{t}}^f_{g,n}(\mathbf{k}))$ sends the f_k to t_{kk} and all other generators as in the unframed case. It induces a morphism of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{k}^n \arrow[r] \arrow[d] & \widehat{\mathrm{PB}}^f_{g,n}(\mathbf{k}) \arrow[r] \arrow[d] & \widehat{\mathrm{PB}}_{g,n}(\mathbf{k}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbf{k}^n \arrow[r] & \exp(\hat{\mathfrak{t}}^f_{g,n}(\mathbf{k})) \arrow[r] & \exp(\hat{\mathfrak{t}}_{g,n}(\mathbf{k})) \arrow[r] & 1
\end{tikzcd} \tag{5.13}$$

This shows that the map $\widehat{\mathrm{PB}}^f_{g,n}(\mathbf{k}) \longrightarrow \exp(\hat{\mathfrak{t}}^f_{g,n}(\mathbf{k}))$ is a $\mathbf{k}$-pro-unipotent group isomorphism. Later on we will derive this result from the flatness of a connection defined over $\mathrm{Conf}^f(\Sigma_g, n)$.

The $\mathfrak{S}$-module $\mathfrak{t}^f_g(\mathbf{k}) := \{\mathfrak{t}^f_{g,n}(\mathbf{k})\}_{n\geq 1}$ inherits the structure of a module over the operad $\mathfrak{t}^f$ in $Lie_{\mathbf{k}}$ with respect to the collection of maps given on the generators as follows:

$$\begin{array}{rccc}
\circ_k : & \mathfrak{t}^f_{g,I}(\mathbf{k}) \oplus \mathfrak{t}^f_J(\mathbf{k}) & \longrightarrow & \mathfrak{t}^f_{g,J\sqcup I-\{i\}}(\mathbf{k}) \\
& (0, t_{\alpha\beta}) & \longmapsto & t_{\alpha\beta} \\
& (t_{ij}, 0) & \longmapsto & \begin{cases} t_{ij} & \text{if} \quad k \notin \{i,j\} \\ \sum\limits_{p\in J} t_{pj} & \text{if} \quad k = i \\ \sum\limits_{p\in J} t_{ip} & \text{if} \quad j = k \end{cases} \\
& (x^a_i, 0) & \longmapsto & \begin{cases} x^a_i & \text{if} \quad k \neq i \\ \sum\limits_{p\in J} x^a_p & \text{if} \quad k = i \end{cases} \\
& (y^a_i, 0) & \longmapsto & \begin{cases} y^a_i & \text{if} \quad k \neq i \\ \sum\limits_{p\in J} y^a_p & \text{if} \quad k = i \end{cases}
\end{array}$$

Let $1 \leq i, j, k \leq m$ with $i < j$, then

$$\begin{array}{rccc}
\mathfrak{t}_{\mathfrak{g}}^{\mathfrak{f}}(\mathbf{k})\circ_k^{m,n}: & \mathfrak{t}_{g,n}^f(\mathbf{k}) \oplus \mathfrak{t}_m^f(\mathbf{k}) & \longrightarrow & \mathfrak{t}_{g,m-1+n}^f(\mathbf{k}) \\
& (0, t_{\alpha\beta})) & \longmapsto & t_{\alpha+k-1\beta+k-1} \\
& (0, t_{\alpha})) & \longmapsto & t_{\alpha+k-1} \\
& (x_i^a, 0) & \longmapsto & \begin{cases} x_{i+n-1}^a & \text{if} \quad k < i \\ \sum_{p=i}^{i+n-1} x_p^a & \text{if} \quad i = k \\ x_i^a & \text{if} \quad i < k \end{cases} \\
& (y_i^a, 0) & \longmapsto & \begin{cases} y_{i+n-1}^a & \text{if} \quad k < i \\ \sum_{p=i}^{i+n-1} y_p^a & \text{if} \quad i = k \\ y_i^a & \text{if} \quad i < k \end{cases} \\
& (t_{ij}, 0) & \longmapsto & \begin{cases} t_{i+n-1j+n-1} & \text{if} \quad k < i < j \\ \sum_{p=i}^{i+n-1} t_{pj+n-1} & \text{if} \quad k = i < j \\ t_{ij+n-1} & \text{if} \quad i < k < j \\ \sum_{p=j}^{j+n-1} t_{ip} & \text{if} \quad i < j = k \\ t_{ij} & \text{if} \quad i < j < k \end{cases} \\
& (t_i, 0) & \longmapsto & \begin{cases} t_{i+n-1} & \text{if} \quad k < i \\ \sum\limits_{p=i}^{i+n-1} t_p & \text{if} \quad k = i \\ t_i & \text{if} \quad i < k \end{cases}
\end{array}$$

We can then construct the $\mathbf{CD}^f(\mathbf{k})$-module $\mathbf{CD}_g^f(\mathbf{k}) := \hat{\mathcal{U}}(\hat{\mathfrak{t}}_g^f(\mathbf{k}))$ of genus g framed chord diagrams.

Let $\widehat{\mathbf{CD}}_g^f(n)$ be the I-adic completion of $\mathbf{CD}_g^f(n)$ with respect to the augmentation ideal I. Since we are in possession of operad modules $\mathbf{Pa}(\mathbf{k})$ and $\widehat{\mathbf{CD}}_g^f(\mathbf{k})$ in $\mathbf{Cat}(\mathbf{CoAss_k})$ and of an operad module morphism $\omega : \mathbf{Pa} \longrightarrow \mathrm{Ob}(\widehat{\mathbf{CD}}^f(\mathbf{k}))$, we are ready to define the $\mathbf{PaCD}^f(\mathbf{k})$-module

$$\mathbf{PaCD}_g^f(\mathbf{k}) := \omega^\star \widehat{\mathbf{CD}}_g^f(\mathbf{k})$$

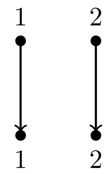
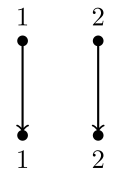
in $\mathbf{Cat}(\mathbf{CoAss_k})$ of parenthesized framed genus g chord diagrams. We have $\mathrm{Ob}(\mathbf{PaCD}_g^f(\mathbf{k})) := \mathbf{Pa}$ and $\mathrm{Mor}_{\mathbf{PaCD}_g^f(\mathbf{k})(n)}(p, q) := \mathrm{Mor}_{\widehat{\mathbf{CD}}_g^f(\mathbf{k})(n)}(pt, pt) = \hat{\mathcal{U}}(\hat{\mathfrak{t}}_g^f(\mathbf{k}))$.

Example 5.3.11 (Notable arrows in $\mathbf{PaCD}_g^f(\mathbf{k})(1)$ and $\mathbf{PaCD}_g^f(\mathbf{k})(2)$)**.** *We have the following arrows X_i, Y_i in $\mathbf{PaCD}_g^f(\mathbf{k})(1)$*

$X_i = x^i \cdot$ 

$Y_i = y^i \cdot$ 

and $X_i^{1,2}$, $Y_i^{1,2}$ in $\mathbf{PaCD}_g^f(\mathbf{k})(2)$

$X_i^{1,2} = x^i \cdot$  $\qquad Y_i^{1,2} = y^i \cdot$ 

We leave the reader the care of drawing the chord diagrams corresponding to the relations (FG4-6) accordingly.

5.3.6 Genus g associators

Definition 5.3.12. *A genus g associator over* $\mathbf{k}$ *is couple* (F,G) *where* $F \in \mathbf{Ass}^f(\mathbf{k})$ *is a* $\mathbf{k}$*-associator and* G *is an isomorphism between the* $\widehat{\mathbf{PaB}}^f(\mathbf{k})$*-module* $\widehat{\mathbf{PaB}}_g^f(\mathbf{k})$ *and the* $G\mathbf{PaCD}^f(\mathbf{k})$*-module* $G\mathbf{PaCD}_g^f(\mathbf{k})$ *which is the identity on objects and which is compatible with* F*. We denote its set by*

$$\mathbf{Ass}_g(\mathbf{k}) := \mathrm{Iso}^+_{(\widehat{\mathbf{PaB}}^f(\mathbf{k}), G\mathbf{PaCD}^f(\mathbf{k}))}(\widehat{\mathbf{PaB}}_g^f(\mathbf{k}), G\mathbf{PaCD}_g^f(\mathbf{k})).$$

Theorem 5.3.13. *There is a one-to-one correspondence between elements of* $\mathbf{Ass}_g(\mathbf{k})$ *and elements of the set* $\mathrm{Ass}_g(\mathbf{k})$ *consisting on tuples* $(\mu, \Phi, A_1, \dots, A_g, B_1, \dots, B_g)$ *where* $(\mu, \Phi) \in \mathrm{Ass}(\mathbf{k})$ *and* $A_i, B_i \in \exp(\hat{\mathfrak{t}}_{g,2}^f)$*, for* $i = 1, ..., g$*, such that, for* $1 \leq i \leq g$ *we have*

$$\alpha_i^{1,2,3}\alpha_i^{2,3,1}\alpha_i^{3,1,2} = A_i^{(12)3}, \text{ where } ,\alpha_i = \{\Phi^{1,2,3}\}A_i^{1,23}\{e^{\mu(t_{12}+t_{13})/2}\}, \tag{5.14}$$

$$\beta_i^{1,2,3}\beta_i^{2,3,1}\beta_i^{3,1,2} = B_i^{(12)3}, \text{ where } \beta_i = \{\Phi^{1,2,3}\}B_i^{1,23}\{e^{-\mu(t_{12}+t_{13})/2}\}, \tag{5.15}$$

$$\{e^{\mu t_{12}+2(g-1)\mu t_1}\} = \prod_{i=1}^{g} \left(\{\Phi\}B_i^{1,23}\{\Phi\}^{-1}, \{e^{-\mu t_{12}/2}\Phi^{2,1,3}\}(A_i^{2,13})^{-1}\{(\Phi^{2,1,3})^{-1}e^{-\mu t_{12}/2}\}\right). \tag{5.16}$$

Proof. Let $(F,G) \in \mathbf{Ass}_g(\mathbf{k})$. An automorphism F of $\mathbf{PaB}^f$ corresponds uniquely to a couple $(\mu, \Phi) \in \mathrm{Ass}(\mathbf{k})$ as, by setting $\mu = 2\lambda$, one can neglect the term λ intervening in $\mathrm{Ass}^f(\mathbf{k})$. An automorphism G of $\mathbf{PaB}_g^f$ is uniquely given as follows. The generators $A_i^{1,2}$ and $B_i^{1,2}$ in $Aut_{\widehat{\mathbf{PaB}}_g(\mathbf{k})(2)}(12)$ are sent via G to A_+^i and A_-^i respectively, with $A_\pm \in \exp(\hat{\mathfrak{t}}_{g,2})$. The image of relations (D1), (D2) and (gE) are precisely the relations (5.14, (5.15)) and (5.16) under this correspondence. □

Conjecture 5.3.14. *The set of genus g $\mathbb{C}$-associators* $\mathrm{Ass}_g^f(\mathbb{C})$ *is not empty.*

We will give some comments on this conjecture in the following subsection

5.3.7 Towards the genus g KZB associator

Let us recall the construction from [36] of the universal genus g KZB connection (defined over the configuration spaces). Endow the surface Σ_g with a complex structure and denote C the resulting smooth closed complex curve. We have an isomorphism

$$\pi_1(C,x) \xrightarrow{\sim} \pi_g := \langle A_a, B_a, 1 \le a \le g | \prod_{a=1}^{g} (A_a, B_a) = 1 \rangle.$$

and each path from x to y in C induces an isomorphism $\pi_1(C,x) \longrightarrow \pi_1(C,y)$ We have

$$\mathrm{PB}_{g,n} = \pi_1(\mathrm{Conf}(C,n),x)$$

where $x := (x_1, \dots, x_n) \in \mathrm{Conf}(C,n)$.

Define the map $\rho_0 : \mathrm{PB}_{g,n} \longrightarrow \exp(\hat{\mathfrak{f}}_g^{\oplus n})$ by means of the following composite

$$\mathrm{PB}_{g,n} = \pi_1(\mathrm{Conf}(C,n),x) \longrightarrow \pi_1(C^n,x) = \prod_{i\in[n]} \pi_1(C,x_i) \longrightarrow \pi_g^n \longrightarrow F_g^n \longrightarrow \exp(\hat{\mathfrak{f}}_g)^n,$$

where F_g is the free group with generators $\gamma_a, 1 \le a \le g$, $\pi_g \longrightarrow F_g$ is the composite

$$\pi_g \longrightarrow \pi_g/N \longrightarrow F_g$$

where $\pi_g \longrightarrow \pi_g/N$ is the quotient morphism, where N is the normal subgroup generated by the A_a, $1 \le a \le g$ and $\pi_g/N \longrightarrow F_g$, $\bar{B}_a \mapsto \gamma_a$ is the isomorphism induced from the presentation of π_g/N, where $F_g \longrightarrow \exp(\hat{\mathfrak{f}}_g)$ is the assignment $\gamma_a \mapsto \exp(x_a)$.

According to [36], the principal G-bundle with flat connection on $X = \mathrm{Cf}_n(C)$ corresponding to ρ_0 is then $i^*(\mathcal{P}_n)$, where $i : X \longrightarrow C^n$ is the inclusion and

$$(\mathcal{P}_n \longrightarrow C^n) = (\mathcal{P}_1^0 \longrightarrow C)^n \times_{\exp(\hat{\mathfrak{f}}_g)^n} \exp(\hat{\mathfrak{t}}_{g,n}),$$

where $(\mathcal{P}_1^0 \longrightarrow C)$ is the principal $\exp(\hat{\mathfrak{f}}_g)$-bundle with flat connection corresponding to the above morphism $\pi_g \longrightarrow F_g \longrightarrow \exp(\hat{\mathfrak{f}}_g)$.

Denote the set of flat connections of degree 1 by

$$F_1 = \{\alpha \in \Omega^1(C^n - (\text{diagonals}), \mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1]) | d\alpha = \alpha \wedge \alpha = 0\}$$

and denote its subset of holomorphic flat connections by

$$F_1^{hol} = \{\alpha \in H^0(C^n, \Omega^{1,0}_{C^n} \otimes (\mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1])(*\,\mathrm{Diag})) | d\alpha = \alpha \wedge \alpha = 0\}$$

with $\mathrm{Diag} = \sum_{i<j} \mathrm{Diag}_{ij}$ and $\mathrm{Diag}_{ij} \subset C^n$ is the diagonal corresponding to $z_i = z_j$. Then Enriquez showed the following:

Theorem 5.3.15. *There is an element $\alpha_{KZ} \in F_1^{hol}$ given by*

$$\alpha_{g,n}^{KZB} = \sum_{i=1}^{n} \alpha_i, \tag{5.17}$$

where $\alpha_i \in H^0(C, K_C^{(i)} \otimes (\mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1])(\sum_{j:j\neq i} \Delta_{ij}))$ expands as $\alpha_i \equiv \sum_{1\le a\le g} \omega_a^{(i)} y_a^i$ modulo $\hat{\oplus}_{q\ge 2} \mathfrak{t}_{g,n}[1,q]$.

As in [36], $K_C^{(i)} = O_C^{\boxtimes i-1} \boxtimes K_C \boxtimes O_C^{\boxtimes n-i}$, $\omega_a^{(i)} = 1^{\otimes i-1} \otimes \omega_a \otimes 1^{\otimes n-i}$, where $(\omega_a)_{1\leq i\leq g}$ are the holomorphic differentials such that $\int_{A_a} \omega_b = \delta_{ab}$ and A_a, B_a are the images of A_a, B_a under $\pi_g \longrightarrow \pi_g^{ab} \simeq H_1(C, \mathbb{Z})$.

Recall the universal g-KZB connection over the configuration space $\mathrm{Conf}(\Sigma_g, n)$ is a particular explicit element $\alpha_{\mathrm{KZ}} \in F_1^{hol}$ can be constructed as a sum

$$\alpha_{g,n}^{\mathrm{KZB}} = \sum_{i=1}^{n} \alpha_i, \tag{5.18}$$

where $\alpha_i \in H^0(C, K_C^{(i)} \otimes (\mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1])(\sum_{j:j\neq i} \Delta_{ij}))$ expands as $\alpha_i \equiv \sum_{1\leq a\leq g} \omega_a^{(i)} y_a^i$ modulo $\hat{\oplus}_{q\geq 2}\mathfrak{t}_{g,n}[1,q]$.

Consider integers (g,n) in hyperbolic position (i.e. $2-2g-n<0$) and let S be a genus g topological compact oriented surface, $x_1, ..., x_n$ n marked points on it. Now let X be a Riemann surface modeled on S with genus g and n marked points. As X is hyperbolic, the Uniformisation Theorem says that X is isomorphic to a quotient $\mathfrak{h}/\Gamma$ of the Poincaré half-plane $\mathfrak{h}$ by a discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$. Fix $\tau \in \mathfrak{h}$ and consider a uniformization Σ_g of X. This corresponds to a point κ in the moduli space $\mathcal{M}_{g,n}$. Such a point can be described by $3g+n-3$ parameters. Enriquez chowed that, under this uniformization, the one form α_{KZ} induces a flat connection

$$\nabla_{g,n,\kappa}^{\mathrm{KZB}} := \mathrm{d} - \alpha_{g,n,\kappa}^{\mathrm{KZB}}$$

over $\mathrm{Conf}(\Sigma_{g,\kappa}, n)$. Now, the fundamental group $\pi_1(\Sigma_{g,\kappa}^{\times}, \mathbf{z}_0)$ of $\Sigma_{g,\kappa}^{\times} := \Sigma_{g,\kappa} - 0$ is the nothing but the free group $F(x^1, y^1, x^2, y^2, ..., x^g, y^g)$ on $2g$ generators. Now choose a non-zero tangent vector $\overrightarrow{v}_0$ of $\Sigma_{g,\kappa}$ at 0. Then, flatness of $\nabla_{g,n,\kappa}^{KZB}$ implies the existence of a $\mathbb{Q}$-algebra map

$$\begin{array}{rcl} T^{g,\mathrm{KZB}}_{-\overrightarrow{v}_0, \overrightarrow{v}_0} : \mathbb{Q}[\pi_1(\Sigma_{g,\tau}^{\times}, -\overrightarrow{v}_0, \overrightarrow{v}_0)] & \longrightarrow & \mathbb{Q}\langle\langle x^1, y^1, x^2, y^2, ..., x^g, y^g \rangle\rangle \\ \gamma & \longmapsto & T^{\mathrm{g,KZB}}_{-\overrightarrow{v}_0, \overrightarrow{v}_0}(\gamma) := \displaystyle\sum_{k=0}^{\infty} Reg \int_{\gamma} \alpha_{g,n,\kappa}^{\mathrm{KZB}} \end{array}$$

Definition 5.3.16. *The non-framed genus g KZB associator is the tuple*

$$e_g(\kappa) := (A_1(\kappa), B_1(\kappa), \ldots, A_g(\kappa), B_g(\kappa))$$

where

$$\begin{array}{rcl} A_i(\kappa) & := & T^{g,\mathrm{KZB}}_{-\overrightarrow{v}_0, \overrightarrow{v}_0}(\gamma_i^a) \\ B_i(\kappa) & := & T^{g,\mathrm{KZB}}_{-\overrightarrow{v}_0, \overrightarrow{v}_0}(\gamma_i^b) \end{array}$$

where γ_i^a and γ_i^b are the generating loops in $\pi_1^B(\Sigma_{g,\kappa})$.

We do not know what kind of monodromy relations these associators may have. In particular, if we want to relate them to our operadic definition of genus g associators we need to extend the universal KZB connection to its framed version.

We then have

Conjecture 5.3.17. *There is a flat universal framed KZB connection* $\nabla^{f\,\mathrm{KZB}}_{g,n,\kappa}$ *defined on the principal* $\exp(\bar{\mathfrak{t}}^f_{g,n})$*-bundle over* $\mathrm{Conf}^f(C,n)$ *constructed as above such that*

- *its pullback of* $\nabla^{f\,\mathrm{KZB}}_{g,n,\kappa}$ *to the associated* $\exp(\bar{\mathfrak{t}}^f_{g,n})$*-bundle over* C^n *is*

$$\nabla^{f\,\mathrm{KZB}}_{g,n,\kappa} := \mathrm{d} - \alpha^{fKZB}_{g,n}$$

  where

$$\alpha^{fKZB}_{g,n} := \alpha^{KZB}_{g,n} + \sum_{1\leqslant i\leqslant n} t_i \,\mathrm{d}\, log(\lambda_i);$$

- *the 1-form* $\alpha^{fKZB}_{g,n}$ *is* $(\mathbb{C}^\times)^n$*-basic and the induced connection on the* $\exp(\bar{\mathfrak{t}}_{g,n})$*-bundle over* $\mathrm{Conf}(C,n)$ *given above coincides with the universal genus g KZB connection in theorem 5.3.15.*

Let κ represent a point in the moduli space $\mathcal{M}_{g,n}$. In the case $g=2$ i.e. the hyperelliptic case, we can write $\kappa=(\tau_1,\tau_2)$. Let $(2i\pi,\Phi^f_{\mathrm{KZ}})$ be the framed KZ associator coming from the framed universal KZ connection defined above.

If this conjecture holds, then a consequence should be that $(2i\pi,\Phi^f_{\mathrm{KZ}},e^f_g(\kappa))$, where $e^f_g(\kappa) = (A^f_1(\kappa),B^f_1(\kappa),\ldots,A^f_g(\kappa),B^f_g(\kappa))$ is the framed version of the above genus g KZB associator, is a genus g framed $\mathbb{C}$-associator.

5.3.8 Genus g Grothendieck-Teichmüller groups

Let us finish this chapter by quickly giving definitions of Grothendieck-Teichmüller groups in genus g by means of the operadic point of view of these objects.

Definition 5.3.18. *The (*$\mathbf{k}$*-prounipotent version of the) genus g Grothendieck–Teichmüller group is defined as the group*

$$\widehat{\mathbf{GT}}^f_g(\mathbf{k}) := Aut^+_{(\mathrm{Mod}(\widehat{\mathbf{PaB}}^{\{}(\mathbf{k})))}(\widehat{\mathbf{PaB}}^f_g(\mathbf{k}))$$

of automorphisms of the $\widehat{\mathbf{PaB}}^f(\mathbf{k})$*-module* $\widehat{\mathbf{PaB}}^f_g(\mathbf{k})$ *which are the identity on objects.*

The presentation of $\mathbf{PaB}^f_g$ then implies the following: each automorphism F of $\mathbf{PaB}^f_g$ compatible with an automorphism G of $\mathbf{PaB}^f$ is uniquely defined by

- $G(R^{1,2}) = (R^{1,2})^\lambda$,
- $G(\Phi^{1,2,3}) = \Phi^{1,2,3}\cdot f(x,y)$,
- $F(A^{1,2}_i) = g^+_i(x_1,y_1,\ldots,x_g,y_g)$,
- $F(B^{1,2}_i) = g^-_i(x_1,y_1,\ldots,x_g,y_g)$,

where $(\lambda,f)\in\widehat{\mathbf{GT}}^f(\mathbf{k})$ and $g^i_\pm\in\widehat{\mathrm{PB}}_{g,2}(\mathbf{k})$. These elements satisfy relations induced by (Red), (D1), (D2) and (gE) which will be left to be studied in a subsequent work.

Definition 5.3.19. *The graded genus g Grothendieck-Teichmüller group is the group*

$$\mathbf{GRT}_g(\mathbf{k}) := Aut^+_{(Mod(\mathbf{PaCD}(\mathbf{k}))}(\mathbf{PaCD}_g(\mathbf{k}))$$

of automorphisms of the $\mathbf{PaCD}^f(\mathbf{k})$*-module* $\mathbf{PaCD}^f_g(\mathbf{k})$ *which are the identity on objects.*

Notice that there is an isomorphism

$$Aut^+_{(Mod(\mathbf{PaCD}^f(\mathbf{k}))}(\mathbf{PaCD}^f_g(\mathbf{k})) \simeq Aut^+_{(Mod(G\mathbf{PaCD}^f(\mathbf{k}))}(G\mathbf{PaCD}^f_g(\mathbf{k})).$$

Part II

On the twisted elliptic KZB associator

Chapter 6

On the universal twisted elliptic KZB connection

6.1 Bundles with flat connections on Γ-twisted configuration spaces

6.1.1 Principal bundles over Γ-twisted configuration spaces

Let $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and let E be an elliptic curve over $\mathbb{C}$ and consider the connected unramified Γ-covering $p : \tilde{E} \longrightarrow E$ corresponding to the canonical surjective group morphism $\rho : \pi_1(E) \cong \mathbb{Z}^2 \longrightarrow \Gamma$ where $\pi_1(E) \cong \mathbb{Z}^2$ is the natural choice of such an isomorphism. Let us then define the *twisted configuration space*

$$\mathrm{Conf}(E, n, \Gamma) := \{\mathbf{z} = (z_1, \ldots, z_n) \in \tilde{E}^n | p(z_i) \neq p(z_j) \text{ if } i \neq j\},$$

and $\mathrm{C}(E, n, \Gamma) := \mathrm{Conf}(E, n, \Gamma)/\tilde{E}$ its reduced version. Notice that $\mathrm{C}(E, n, \Gamma)$ is just the inverse image of $\mathrm{C}(E, n)$ under the surjection $p^n : \tilde{E}^n \longrightarrow E^n$.

Let us fix a uniformization $\tilde{E} \simeq E_\tau$, where $\tau \in \mathfrak{H}$: $E_\tau = \mathbb{C}/\Lambda_\tau$, with $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. Then $E \simeq E_{\tau,\Gamma}$, where $E_{\tau,\Gamma} = \mathbb{C}/\Lambda_{\tau,\Gamma}$ and $\Lambda_{\tau,\Gamma} := (1/M)\mathbb{Z} \times (\tau/N)\mathbb{Z}$. Therefore

$$\mathrm{Conf}(E, n, \Gamma) \simeq (\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma})/\Lambda_\tau^n,$$

where

$$\mathrm{Diag}_{\tau,n,\Gamma} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_{ij} := z_i - z_j \in \Lambda_{\tau,\Gamma} \text{ for some } i \neq j\}.$$

We now define a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$-bundle $P_{\tau,n,\Gamma}$ over $\mathrm{Conf}(E, n, \Gamma)$ as the quotient

$$\big((\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)\big)/\Lambda_\tau^n.$$

In other words, it is the restriction on $\mathrm{Conf}(E, n, \Gamma)$ of the bundle over $\mathbb{C}^n/\Lambda_\tau^n$ for which a section on $U \subset \mathbb{C}^n/\Lambda_\tau^n$ is a regular map $f : \pi^{-1}(U) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$ such that

- $f(\mathbf{z}+\delta_i) = f(\mathbf{z})$,
- $f(\mathbf{z}+\tau\delta_i) = e^{-2\pi \mathrm{i} x_i} f(\mathbf{z})$.

Here $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n/\Lambda_\tau^n$ is the canonical projection and δ_i is the ith vector of the canonical basis of $\mathbb{C}^n$.

Since the $e^{-2\pi \mathrm{i} \bar{x}_i}$'s in $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma)$ pairwise commute and their product is 1, then the image of $P_{\tau,n,\Gamma}$ under the natural morphism $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \longrightarrow \exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma)$ is the pull-back of a principal $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma)$-bundle $\bar{P}_{\tau,n,\Gamma}$ over $\mathrm{C}(E,n,\Gamma)$.

6.1.2 Variations

The first variation we are interested in concerns *unordered configuration spaces.*

The symmetric group $\mathfrak{S}_n$ acts freely by automorphisms of $\mathrm{Conf}(E,n,\Gamma)$ by $\sigma * (z_1,\dots,z_n) := (z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(n)})$. This descends to a free action of $\mathfrak{S}_n$ on $\mathrm{C}(E,n,\Gamma)$. We then defined the unordered twisted configuration spaces

$$\mathrm{Conf}(E,[n],\Gamma) := \mathrm{Conf}(E,n,\Gamma)/\mathfrak{S}_n \text{ and } \mathrm{C}(E,[n],\Gamma) := \mathrm{C}(E,n,\Gamma)/\mathfrak{S}_n\,.$$

The symmetric group $\mathfrak{S}_n$ also obviously acts on the Lie algebra $\mathfrak{t}_{1,n}^\Gamma$. One can then define, keeping the notation of the previous paragraph, a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$-bundle $P_{\tau,[n],\Gamma}$ over $\mathrm{Conf}(E,[n],\Gamma)$: it is the restriction on $\mathrm{Conf}(E,[n],\Gamma)$ of the bundle over $\mathbb{C}^n/\Lambda_\tau^n \rtimes \mathfrak{S}_n$ for which a section on $U \subset \mathbb{C}^n/\Lambda_\tau^n \rtimes \mathfrak{S}_n$ is a regular map $f : \pi^{-1}(U) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$ such that

- $f(\mathbf{z}+\delta_i) = f(\mathbf{z})$,
- $f(\mathbf{z}+\tau\delta_i) = e^{-2\pi \mathrm{i} x_i} f(\mathbf{z})$,
- $f(\sigma * \mathbf{z}) = \sigma f(\mathbf{z})$.

In more compact form:

$$P_{\tau,[n],\Gamma} = \big((\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n\big)/(\Lambda_\tau^n \rtimes \mathfrak{S}_n)\,.$$

Remark 6.1.1. *As before, $P_{\tau,[n],\Gamma}$ descends to a principal $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$-bundle $\bar{P}_{\tau,[n],\Gamma}$ over the reduced unordered twisted configuration space $C(E,[n],\Gamma)$.*

The second variation concerns ordinary configuration spaces of the base $E = E_{\tau,\Gamma}$ of the covering map $E_\tau \longrightarrow E_{\tau,\Gamma}$.

Recall from §4.3.3 that the group Γ^n acts on $\hat{\mathfrak{t}}_{1,n}^\Gamma$ *via* θ. Hence one has a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \Gamma^n$-bundle

$$P_{(\tau,\Gamma),n} := \big((\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \Gamma^n\big)/\Lambda_{\tau,\Gamma}^n$$

over $\mathrm{Conf}(E,n) \simeq \mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}/\Lambda_{\tau,\Gamma}^n$. Here the action of Λ_τ^n on $\hat{\mathfrak{t}}_{1,n}^\Gamma$ is given by the morphism

$$\Lambda_\tau \longrightarrow \Gamma\,, \quad a + b\tau \mapsto (\bar{a},\bar{b})\,.$$

Remark 6.1.2. *In a similar way as before, the above bundle obviously descends to a principal $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma) \rtimes (\Gamma^n/\Gamma)$-bundle $\bar{P}_{(\tau,\Gamma),n}$ over the reduced ordinary configuration space $C(E,n)$.*

In concrete terms, a section over $U \subset \mathbb{C}^n/\Lambda_{\tau,\Gamma}$ of $P_{(\tau,\Gamma),n}$ is a regular map $f : \pi^{-1}(U) \longrightarrow \exp(\hat{\mathfrak{t}}^{\Gamma}_{1,n}) \rtimes \Gamma^n$ such that

- $f(\mathbf{z} + \delta_i/M) = (\bar{1}, \bar{0})_i f(\mathbf{z})$,
- $f(\mathbf{z} + \tau\delta_i/N) = (\bar{0}, \bar{1})_i e^{\frac{-2\pi \mathrm{i} x_i}{N}} f(\mathbf{z})$.

Remark 6.1.3. *We leave to the reader the task of combining the two variations.*

6.1.3 Flat connections on $P_{\tau,n,\Gamma}$ and its variants

A flat connection $\nabla_{\tau,n,\Gamma}$ on $P_{\tau,n,\Gamma}$ is the same as an equivariant flat connection on the trivial $\exp(\hat{\mathfrak{t}}^{\Gamma}_{1,n})$-bundle over $\mathbb{C}^n - \mathrm{Diag}_{\tau,\mathrm{n},\Gamma}$, i.e., a connection of the form

$$\nabla_{\tau,n,\Gamma} := d - \sum_{i=1}^{n} K_i(\mathbf{z}|\tau) dz_i \,,$$

where $K_i(-|\tau) : \mathbb{C}^n \longrightarrow \hat{\mathfrak{t}}^{\Gamma}_{1,n}$ are meromorphic with only poles at $\mathrm{Diag}_{\tau,n,\Gamma}$, and such that for any i, j:

(a) $K_i(\mathbf{z} + \delta_j|\tau) = K_i(\mathbf{z}|\tau)$,

(b) $K_i(\mathbf{z} + \tau\delta_j|\tau) = e^{-2\pi \mathrm{i}\,\mathrm{ad}(x_j)} K_i(\mathbf{z}|\tau)$,

(c) $[\partial_i - K_i(\mathbf{z}|\tau), \partial_j - K_j(\mathbf{z}|\tau)] = 0$.

Moreover, the image of $\nabla_{\tau,n,\Gamma}$ under $\hat{\mathfrak{t}}^{\Gamma}_{1,n} \longrightarrow \hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,n}$ is the pull-back of a (necessarily flat) connection $\bar{\nabla}_{\tau,n,\Gamma}$ on $\bar{P}_{\tau,n,\Gamma}$ if and only if:

(d) $\bar{K}_i(\mathbf{z}|\tau) = \bar{K}_i(\mathbf{z} + u \sum_i \delta_i|\tau)$ for any $u \in \mathbb{C}$ and $\sum_i \bar{K}_i(\mathbf{z}|\tau) = 0$.

Similarly, the image of $\nabla_{\tau,n,\Gamma}$ under $\hat{\mathfrak{t}}^{\Gamma}_{1,n} \longrightarrow \hat{\mathfrak{t}}^{\Gamma}_{1,n} \rtimes \Gamma^n$ is the pull-back of a (necessarily flat) connection $\nabla_{(\tau,\Gamma),n}$ on $P_{(\tau,\Gamma),n}$ if and only if:

(e) $K_i(\mathbf{z} + \frac{\delta_j}{M}|\tau) = \theta((\bar{1}, \bar{0})_j) K_i(\mathbf{z}|\tau)$,

(f) $K_i(\mathbf{z} + \frac{\tau\delta_j}{N}|\tau) = \theta((\bar{0}, \bar{1})_j) e^{\frac{-2\pi \mathrm{i}}{N}\mathrm{ad}(x_j)} K_i(\mathbf{z}|\tau)$,

Remark 6.1.4. *Observe that (e) implies (a), and that (f) implies (b).*

Finally, the image of $\nabla_{\tau,n,\Gamma}$ under $\hat{\mathfrak{t}}^{\Gamma}_{1,n} \longrightarrow \hat{\mathfrak{t}}^{\Gamma}_{1,n} \rtimes \mathfrak{S}_n$ is the pull-back of a (necessarily flat) connection $\nabla_{\tau,[n],\Gamma}$ on $\bar{P}_{\tau,[n],\Gamma}$ if and only if:

(g) $K_i((ij) * \mathbf{z}) = (ij) \cdot K_i(\mathbf{z})$.

6.1.4 Constructing the connection

We now construct a connection satisfying properties (d) to (g). Let us take the same conventions for theta functions as in [24]. Observe that for any $\tilde{\alpha} = (a_0, a) \in \Lambda_{\tau,\Gamma}$, the term $e^{-2\pi \mathrm{i} a x}(\theta(z - \tilde{\alpha}) + x)/\left(\theta(z - \tilde{\alpha})\theta(x)\right)$ only depends on the class $\alpha = (\bar{a}_0, \bar{a}) \in \Gamma$ of $\tilde{\alpha}$ mod Λ_τ. The we set

$$k_\alpha(x, z|\tau) := e^{-2\pi \mathrm{i} a x} \frac{\theta(z - \tilde{\alpha} + x|\tau)}{\theta(z - \tilde{\alpha}|\tau)\theta(x|\tau)} - \frac{1}{x} = e^{-2\pi \mathrm{i} a x} k(x, z - \tilde{\alpha}|\tau) + \frac{e^{-2\pi \mathrm{i} a x} - 1}{x} \,,$$

where $k(x,z|\tau) := \frac{\theta(x+z)}{\theta(x)\theta(z)} - \frac{1}{x}$ (as in [24]), and

$$K_{ij}(z|\tau) := \sum_{\alpha\in\Gamma} k_\alpha(\mathrm{ad}x_i, z|\tau)(t_{ij}^\alpha), \quad K_i(\mathbf{z}|\tau) := -y_i + \sum_{j:j\neq i} K_{ij}(z_{ij}|\tau).$$

In the rest of the section we fix $\tau \in \mathfrak{H}$ and drop it from the notation. Recall from [24] that $k(x, z \pm 1) = k(x,z)$ and

$$k(x, z\pm\tau) = e^{\mp 2\pi \mathrm{i} x} k(x,z) + \frac{e^{\mp 2\pi\mathrm{i}x} - 1}{x}.$$

Proposition 6.1.5. *The $K_{ij}(z)$'s have the following equivariance properties:*

$$K_{ij}(z+1/M) = \theta((\bar{1},\bar{0})_i)(K_{ij}(z)), \tag{6.1}$$

$$K_{ij}(z-\tau/N) = e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_i)}\theta((\bar{0},\bar{-1})_i)(K_{ij}(z)) + \theta((\bar{0},\bar{-1})_i)(\sum_{\alpha\in\Gamma}\frac{e^{-2\pi\mathrm{i}\mathrm{ad}x_i}-1}{\mathrm{ad}x_i}(t_{ij}^\alpha)). \tag{6.2}$$

Proof. The first equation comes from a straightforward verification. Let us show the second relation. On the one hand, we have

$$\begin{aligned}
K_{ij}\left(z-\frac{\tau}{N}\right) &= \sum_{\alpha\in\Gamma} k_\alpha\left(\mathrm{ad}(x_i), z-\frac{\tau}{N}\right)(t_{ij}^\alpha) \\
&= \left(\sum_{\alpha\in\Gamma} e^{\frac{-2i\pi a}{N}\mathrm{ad}(x_i)} k\left(\mathrm{ad}(x_i), z-\frac{\tau}{N}-\tilde{\alpha}\right) + \frac{e^{\frac{-2i\pi a}{N}\mathrm{ad}(x_i)}-1}{\mathrm{ad}(x_i)}\right)(t_{ij}^\alpha) \\
&= \left(\sum_{\alpha\in\Gamma} e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)} k(\mathrm{ad}(x_i), z-\tilde{\alpha}) + \frac{e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)}-1}{\mathrm{ad}(x_i)}\right)(t_{ij}^{\alpha-(\bar{0},\bar{1})}) \\
&= \theta(\bar{0},\overline{-1})\left(\sum_{\alpha\in\Gamma} e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)} k(\mathrm{ad}(x_i), z-\tilde{\alpha}) + \frac{e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)}-1}{\mathrm{ad}(x_i)}\right)(t_{ij}^\alpha)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
e^{\frac{-2i\pi}{N}\mathrm{ad}(x_j)} K_{ij}(z) &= e^{\frac{-2i\pi}{N}\mathrm{ad}(x_j)}\left(\sum_{\alpha\in\Gamma} k_\alpha(\mathrm{ad}(x_i), z)\right)(t_{ij}^\alpha) \\
&= e^{\frac{2i\pi}{N}\mathrm{ad}(x_i)}\left(\sum_{\alpha\in\Gamma} e^{\frac{-2i\pi a}{N}\mathrm{ad}(x_i)} k(\mathrm{ad}(x_i), z-\tilde{\alpha}) + \frac{e^{\frac{-2i\pi a}{N}\mathrm{ad}(x_i)}-1}{\mathrm{ad}(x_i)}\right)(t_{ij}^\alpha) \\
&= \left(\sum_{\alpha\in\Gamma} e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)} k(\mathrm{ad}(x_i), z-\tilde{\alpha}) + \frac{e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)} - e^{\frac{2i\pi}{N}\mathrm{ad}(x_i)}}{\mathrm{ad}(x_i)}\right)(t_{ij}^\alpha)
\end{aligned}$$

so

$$\sum_{\alpha\in\Gamma} e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)} k(\mathrm{ad}(x_i), z-\tilde{\alpha}) = e^{\frac{-2i\pi}{N}\mathrm{ad}(x_j)} K_{ij}(z) - \sum_{\alpha\in\Gamma}\frac{e^{\frac{-2i\pi(a-1)}{N}\mathrm{ad}(x_i)} - e^{\frac{2i\pi}{N}\mathrm{ad}(x_i)}}{\mathrm{ad}(x_i)}(t_{ij}^\alpha)$$

By putting these two equations together we finally get

$$\begin{aligned} K_{ij}\left(z - \frac{\tau}{N}\right) &= \theta(\overline{0}, \overline{-1}) e^{\frac{-2i\pi}{N}\,\mathrm{ad}(x_j)} K^{\Gamma}_{ij}(z) \\ &\quad + \sum_{\alpha\in\Gamma} \frac{-e^{\frac{-2i\pi(a-1)}{N}\,\mathrm{ad}(x_i)} + e^{\frac{2i\pi}{N}\,\mathrm{ad}(x_i)} + e^{\frac{-2i\pi(a-1)}{N}\,\mathrm{ad}(x_i)} - 1}{\mathrm{ad}(x_i)}(t^{\alpha}_{ij}) \\ &= \theta(\overline{0}, \overline{-1}) e^{\frac{-2i\pi}{N}\,\mathrm{ad}(x_j)} K_{ij}(z) + \theta(\overline{0}, \overline{-1}) \left(\sum_{\alpha\in\Gamma} \frac{e^{\frac{2i\pi}{N}\,\mathrm{ad}(x_i)} - 1}{\mathrm{ad}(x_i)}(t^{\alpha}_{ij}) \right). \end{aligned}$$

□

Now recall that $\frac{e^{\frac{2i\pi}{N}\mathrm{ad}(x_i)}-1}{\mathrm{ad}(x_i)} = \frac{1-e^{\frac{-2i\pi}{N}\mathrm{ad}(x_j)}}{\mathrm{ad}(x_j)}$ and $\frac{1-e^{\frac{-2i\pi}{N}\mathrm{ad}(x_j)}}{\mathrm{ad}(x_j)}(t_{ij}) = \left(1 - e^{\frac{-2i\pi}{N}\mathrm{ad}(x_j)}\right)(y_i)$. We thus have

$$K_i\left(\mathbf{z} + \frac{\tau}{N}\delta_j\right) = -y_i + \sum_{j'\neq i,j} K_{ij'}(z_{ij'}) + K_{ij}\left(z_{ij} - \frac{\tau}{N}\right)$$

and therefore we get the announced relation

$$K_i\left(\mathbf{z} + \frac{\tau}{N}\delta_j\right) = \theta((\overline{0}, \overline{1})_j) e^{\frac{-2i\pi}{N}\,\mathrm{ad}(x_j)} K_i(\boldsymbol{z}).$$

Consequently the $K_i(\mathbf{z})$'s satisfy conditions (e) and (f) above (and thus also (a) and (b)).

Moreover, the $K_i(\mathbf{z})$'s also satisfy conditions (d). Indeed, the first part of (d) is immediate and $k_\alpha(x, z) + k_{-\alpha}(-x, -z) = 0$, therefore $K_{ij}(z) + K_{ji}(-z) = 0$, and thus $\sum_i K_i(\mathbf{z}) = -\sum_i y_i$.

Finally, from their very definition, the $K_i(\mathbf{z})$'s also satisfy condition (g).

In the next paragraph we show that the flatness condition (c) is satisfied.

6.1.5 Flatness of the connection

Proposition 6.1.6. $[\partial_i - K_i(\mathbf{z}), \partial_j - K_j(\mathbf{z})] = 0$, *i.e., condition (c) is satisfied.*

Proof. First we have

$$\partial_i(K_j(\mathbf{z})) - \partial_j(K_i(\mathbf{z})) = \partial_i K_{ji}(z_{ji}) - \partial_j K_{ij}(z_{ij}) = \partial_i(K_{ij}(z_{ij}) + K_{ji}(z_{ji})) = 0$$

since $K_{ij}(z) + K_{ji}(-z) = 0$. Therefore we have to prove that $[K_i(\mathbf{z}), K_j(\mathbf{z})] = 0$. As in [24] it follows from the universal classical dynamical Yang-Baxter equation:

$$-[y_i, K_{jk}] + [K_{ji}, K_{ki}] + c.p.(i, j, k) = 0\,, \qquad \text{(CDYBE)}$$

which we now prove (here $K_{ij} := K_{ij}(z_{ij})$). For any $f(x) \in \mathbb{C}[[x]]$ we have

$$[y_k, f(\mathrm{ad}x_i)(t^{\alpha}_{ij})] = \sum_{\beta\in\Gamma} \frac{f(\mathrm{ad}x_i) - f(-\mathrm{ad}x_j)}{\mathrm{ad}x_i + \mathrm{ad}x_j}[-t^{\beta}_{ki}, t^{\alpha}_{ij}],$$

$$[y_i, f(\mathrm{ad}x_j)(t^{\alpha}_{jk})] = \sum_{\beta\in\Gamma} \frac{f(\mathrm{ad}x_j) - f(\mathrm{ad}x_i + \mathrm{ad}x_j)}{-\mathrm{ad}x_i}[-t^{\beta}_{ij}, t^{\alpha}_{jk}],$$

$$[y_j, f(\mathrm{ad}x_k)(t^\alpha_{ki})] = \sum_{\beta\in\Gamma} \frac{f(-\mathrm{ad}x_i - \mathrm{ad}x_j) - f(-\mathrm{ad}x_i)}{-\mathrm{ad}x_j}[-t^\beta_{jk}, t^\alpha_{ki}].$$

It follows that the l.h.s. of (CDYBE) is now

$$\sum_{\alpha,\beta\in\Gamma} \big(k_\alpha(-\mathrm{ad}x_j, z_{ij})k_\beta(-\mathrm{ad}x_k, z_{ik}) - k_\alpha(\mathrm{ad}x_i, z_{ij})k_{\beta-\alpha}(-\mathrm{ad}x_k, z_{jk})$$
$$+k_\beta(\mathrm{ad}x_i, z_{ik})k_{\beta-\alpha}(\mathrm{ad}x_j, z_{jk}) + \frac{k_{\beta-\alpha}(\mathrm{ad}x_j, z_{jk}) - k_{\beta-\alpha}(\mathrm{ad}x_i + \mathrm{ad}x_j, z_{jk})}{\mathrm{ad}x_i}$$
$$+\frac{k_\beta(\mathrm{ad}x_i, z_{ik}) - k_\beta(\mathrm{ad}x_i + \mathrm{ad}x_j, z_{ik})}{\mathrm{ad}x_j} - \frac{k_\alpha(\mathrm{ad}x_i, z_{ij}) - k_\alpha(-\mathrm{ad}x_j, z_{ij})}{\mathrm{ad}x_i + \mathrm{ad}x_j}\big)[t^\alpha_{ij}, t^\beta_{ik}]\,,$$

and thus (CDYBE) follows from the identity

$$k_\alpha(-v,z)k_\beta(u+v,z') - k_\alpha(u,z)k_{\beta-\alpha}(u+v,z'-z) + k_\beta(u,z')k_{\beta-\alpha}(v,z'-z)$$
$$+\frac{k_{\beta-\alpha}(v,z'-z) - k_{\beta-\alpha}(u+v,z'-z)}{u} + \frac{k_\beta(u,z') - k_\beta(u+v,z')}{v}$$
$$-\frac{k_\alpha(u,z) - k_\alpha(-v,z)}{u+v} = 0\,.$$

This last identity can be written as

$$\left(k_\alpha(-v,z) - \frac{1}{v}\right)\left(k_\beta(u+v,z') + \frac{1}{u+v}\right) - \left(k_\alpha(u,z) + \frac{1}{u}\right)\left(k_{\beta-\alpha}(u+v,z'-z) + \frac{1}{u+v}\right)$$
$$+\left(k_\beta(u,z') + \frac{1}{u}\right)\left(k_{\beta-\alpha}(v,z'-z) + \frac{1}{v}\right) = 0\,,\tag{6.3}$$

which (taking into account that $k_\alpha(x,z) + (1/x) = e^{-2\pi \mathrm{i} a x}\left(k(x, z-\tilde\alpha) + (1/x)\right)$) is a consequence of equation (3) of [24]. □

We have therefore proved:

Theorem 6.1.7. *$\nabla_{\tau,n,\Gamma}$ is a flat connection on $P_{\tau,n,\Gamma}$, and its image under $\hat{\mathfrak{t}}^\Gamma_{1,n} \longrightarrow \hat{\bar{\mathfrak{t}}}^\Gamma_{1,n}$ is the pull-back of a flat connection $\bar\nabla_{\tau,n,\Gamma}$ on $\bar P_{\tau,n,\Gamma}$.* □

6.2 Lie algebras of derivations and associated groups

6.2.1 The Lie algebras $\tilde{\mathfrak{d}}^\Gamma_0$ and $\tilde{\mathfrak{d}}^\Gamma$

Let $\mathfrak{f}_\Gamma$ be the free Lie algebra with generators x, t^α ($\alpha\in\Gamma$). Let $p,q>0$. We define $\tilde{\mathfrak{d}}^{p,q}_0$ to be the subspace of $\mathfrak{f}_\Gamma \oplus (\mathfrak{f}_\Gamma)^{\oplus|\Gamma|}$ consisting of elements

$$(D, C),\ \text{where } C = (C_\alpha)_{\alpha\in\gamma},$$

such that $\deg_x(D) + \deg_t(D) = \deg_x(C_\alpha) + \deg_t(C_\alpha) = p$ and $\deg_t(D) - 1 = \deg_t(C_\alpha) = q$ for every $\alpha\in\Gamma$, and that satisfy the following of linear equations:

(i) $C_\alpha(x, t^\beta) = C_{-\alpha}(-x, t^{-\beta})$ in $\mathfrak{f}_\Gamma$,

(ii) $[x, D(x,t^\beta)] + \sum_\alpha [t^\alpha, C_\alpha(x,t^\beta)] = 0$ in $\mathfrak{f}_\Gamma$,

(iii) $[D(x_1, t_{13}^\beta), y_2] + c.p.(1,2,3) = 0$ in $\mathfrak{t}_{1,3}^\Gamma$,

(iv) $[D(x_1, t_{12}^\beta) + D(x_1, t_{13}^\beta) - [C_\alpha(x_2, t_{23}^\beta), y_1], t_{23}^\alpha] = 0$ in $\mathfrak{t}_{1,3}^\Gamma$,

(v) $[C_\alpha(x_1, t_{12}^\gamma), t_{13}^{\alpha+\beta} + t_{23}^\beta] + [t_{13}^{\alpha+\beta}, C_{\alpha+\beta}(x_1, t_{13}^\gamma)] + [t_{23}^\beta, C_\beta(x_2, t_{23}^\gamma)]$ commutes with t_{12}^α in $\mathfrak{t}_{1,3}^\Gamma$.

Remark that (i) and (ii) imply another relation

(vi) $D(x, t^\beta) = -D(-x, t^{-\beta})$,

which is very useful for computations. Then $\tilde{\mathfrak{d}}_0^\Gamma := \oplus_{p,q}(\tilde{\mathfrak{d}}_0^\Gamma)^{p,q}$.

We then define a Lie bracket $\langle , \rangle$ on $\mathfrak{f}_\Gamma \oplus (\mathfrak{f}_\Gamma)^{\oplus|\Gamma|}$ as follows:

$$\langle (D,C), (D',C') \rangle := (\delta_C(D') - \delta_{C'}(D), [C,C'] + \delta_C(C') - \delta_{C'}(C)),$$

where $\delta_C \in \mathrm{Der}(\mathfrak{f}_\Gamma)$ is the derivation

- $x \mapsto 0$, $t^\alpha \mapsto [t^\alpha, C_\alpha]$,
- δ_C acts on $(\mathfrak{f}_\Gamma)^{\oplus|\Gamma|}$ componentwise on a direct sum : $\delta_C(C')_\alpha = \delta_C(C'_\alpha)$,
- the bracket is understood componentwise as well: $[C,C']_\alpha = [C_\alpha, C'_\alpha]$.

We let the reader check that $\tilde{\mathfrak{d}}_0^\Gamma$ is stable under $\langle , \rangle$, and becomes a bigraded Lie algebra[1].

We now define $\tilde{\mathfrak{d}}^\Gamma$ as the quotient of the free product $\tilde{\mathfrak{d}}_0^\Gamma * \mathfrak{sl}_2$ by the relations $[\tilde{e}, (D,C)] = 0$, $[\tilde{h}, (D,C)] = (p-q)(D,C)$, and $(\mathrm{ad}^p \tilde{f})(D,C) = 0$ if $(D,C) \in \tilde{\mathfrak{d}}_0^\Gamma$ is homogeneous of bidegree (p,q). Here

$$\tilde{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tilde{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \tilde{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form the standard basis of $\mathfrak{sl}_2$. If we respectively give degree $(1,-1)$, $(0,0)$ and $(-1,1)$ to $\tilde{e}$, $\tilde{h}$ and $\tilde{f}$ then $\tilde{\mathfrak{d}}^\Gamma$ becomes $\mathbb{Z}^2$-graded.

We then define $\tilde{\mathfrak{d}}_+^\Gamma := \ker(\tilde{\mathfrak{d}}^\Gamma \longrightarrow \mathfrak{sl}_2)$, which is $(\mathbb{Z}_{>0})^2$-graded. One observes that it is positively graded and finite dimensional in each degree. Thus, it is a direct sum of finite dimensional $\mathfrak{sl}_2$-modules.

6.2.2 The Lie algebras $\mathfrak{d}_0^\Gamma$ and $\mathfrak{d}^\Gamma$

We write $\mathfrak{d}_0^\Gamma$ for the free bigraded Lie algebra generated by $\delta_{s,\gamma}$'s ($s \geq 0$, $\gamma \in \Gamma$) in degree $(s+1, s)$ with relations

$$\delta_{s,\gamma} = (-1)^s \delta_{s,-\gamma},$$

for all $s \geq 0$ and $\gamma \in \Gamma$.

We then define $\mathfrak{d}^\Gamma$ as the quotient of the free product $\mathfrak{d}_0^\Gamma * \mathfrak{sl}_2$ by the relations $[\tilde{e}, \delta_{s,\gamma}] = 0$, $[\tilde{h}, \delta_{s,\gamma}] = s\delta_{s,\gamma}$ and $\mathrm{ad}^{s+1}(\tilde{f})(\delta_{s,\gamma}) = 0$; and $\mathfrak{d}_+^\Gamma$ as the kernel of $\mathfrak{d}^\Gamma \longrightarrow \mathfrak{sl}_2$. As above, we have $\mathfrak{d}^\Gamma = \mathfrak{d}_+^\Gamma \rtimes \mathfrak{sl}_2$, and $\mathfrak{d}_+^\Gamma$ is positively graded (actually $(\mathbb{Z}_{>0})^2$-graded).

[1] The proof is straightforward but quite long. We do not give it since we do use another simpler Lie algebra below.

We now give examples of elements in $\tilde{\mathfrak{d}}_0^\Gamma$ that are of some use below. For any $s \in \mathbb{N}$ and $\gamma \in \Gamma$, we set

$$D_{s,\gamma} := \sum_{p+q=s-1} \sum_{\beta\in\Gamma} [(\mathrm{ad}x)^p t^{\beta-\gamma}, (-\mathrm{ad}x)^q t^\beta]$$

and

$$(C_{s,\gamma})_\alpha := (\mathrm{ad}x)^s t^{\alpha-\gamma} + (-\mathrm{ad}x)^s t^{\alpha+\gamma}.$$

Observe that $(D_{s,\gamma}, C_{s,\gamma}) = (-1)^s (D_{s,-\gamma}, C_{s,-\gamma})$.

The following result tells us that $\delta_{s,\gamma} \mapsto (D_{s,\gamma}, C_{s,\gamma})$ defines a bigraded Lie algebra morphism $\mathfrak{d}_0^\Gamma \longrightarrow \tilde{\mathfrak{d}}_0^\Gamma$, that obviously extends to $\mathfrak{d}^\Gamma \longrightarrow \tilde{\mathfrak{d}}^\Gamma$.

Proposition 6.2.1. $(D_{s,\gamma}, C_{s,\gamma}) \in (\tilde{\mathfrak{d}}_0^\Gamma)^{s+1,1}$.

Proof. First observe that relations (i) and (vi) are obviously satisfied.

To prove (ii) it suffices to notice that in the free Lie algebra with three generators x, t_1, t_2 we have

$$[t_1, (\mathrm{ad}\, x)^s t_2] + [t_2, (-\mathrm{ad}\, x)^s t_1] = \sum_{p+q=s-1} [x, [(-\mathrm{ad}\, x)^q t_1, (\mathrm{ad}\, x)^p t_2]]\,.$$

Let us prove (iii). In $\mathfrak{t}_{1,n}^\Gamma$ we compute for $\#\{i,j,k\} = 3$,

$$[y_k, (\mathrm{ad}\, x_i)^p t_{ij}^\alpha] = -\sum_{k+l=p-1} \sum_\beta (\mathrm{ad}\, x_i)^k [t_{ik}^\beta, (\mathrm{ad}\, x_i)^l t_{ij}^\alpha]$$

$$= \sum_{k+l=p-1} \sum_\beta (\mathrm{ad}\, x_i)^k (-\mathrm{ad}\, x_j)^l [t_{ik}^\beta, t_{kj}^{\alpha-\beta}] = \sum_{k+l=p-1} \sum_\beta [(\mathrm{ad}\, x_i)^k t_{ik}^\beta, (-\mathrm{ad}\, x_j)^l t_{kj}^{\alpha-\beta}]\,.$$

Therefore, in $\mathfrak{t}_{1,3}^\Gamma$, we have

$$[y_1, D(x_2, t_{23}^\beta)] = \sum_{k+l+m=s-2} \sum_{\alpha,\beta} [[(\mathrm{ad}\, x_2)^k t_{21}^\beta, (-\mathrm{ad}\, x_3)^l t_{13}^{\alpha-\beta-\gamma}], (-\mathrm{ad}\, x_2)^m t_{23}^\alpha]$$

$$+ \sum_{k+l+m=s-2} \sum_{\alpha,\beta} (-1)^{l+m+1} [(\mathrm{ad}\, x_2)^k t_{23}^{\alpha-\gamma}, [(\mathrm{ad}\, x_2)^l t_{21}^\beta, (-\mathrm{ad}\, x_3)^m t_{13}^{\alpha-\beta}]]\,.$$

Then $[y_1, D(x_2, t_{23}^\beta)] + c.p.(1,2,3) = 0$ follows from the Jacobi identity.

Let us prove (iv). On the one hand we have

$$[D(x_1, t_{12}^\beta) + D(x_1, t_{13}^\beta), t_{23}^\alpha] =$$

$$= \sum_{p+q=s-1} \sum_{\beta\in\Gamma} [[(\mathrm{ad}x_1)^p t_{12}^{\beta-\gamma}, (-\mathrm{ad}x_1)^q t_{12}^\beta] + [(\mathrm{ad}x_1)^p t_{13}^{\beta-\gamma}, (-\mathrm{ad}x_1)^q t_{13}^\beta], t_{23}^\alpha]$$

$$= -\sum_{p+q=s-1} \sum_{\beta\in\Gamma} \big([(\mathrm{ad}x_1)^p [t_{13}^{\alpha+\beta-\gamma}, t_{23}^\alpha], (-\mathrm{ad}x_1)^q t_{12}^\beta] + [(\mathrm{ad}x_1)^p t_{12}^{\beta-\gamma}, (-\mathrm{ad}x_1)^q [t_{13}^{\alpha+\beta}, t_{23}^\alpha]]$$

$$+[(\mathrm{ad}x_1)^p [t_{12}^{\beta-\gamma}, t_{23}^\alpha], (-\mathrm{ad}x_1)^q t_{13}^{\alpha+\beta}] + [(\mathrm{ad}x_1)^p t_{13}^{\alpha+\beta-\gamma}, (-\mathrm{ad}x_1)^q [t_{12}^\beta, t_{23}^\alpha]]\big)$$

$$= [t_{23}^\alpha, \sum_{p+q=s-1} \sum_{\beta\in\Gamma} (\mathrm{ad}x_1)^p [t_{13}^{\alpha+\beta-\gamma}, (-\mathrm{ad}x_1)^q t_{12}^\beta] + (\mathrm{ad}x_1)^p [t_{12}^\beta, (-\mathrm{ad}x_1)^q t_{13}^{\alpha+\beta+\gamma}]]$$

$$= [t_{23}^{\alpha}, \sum_{p+q=s-1} \sum_{\beta\in\Gamma} (\mathrm{ad}x_2)^p(-\mathrm{ad}x_3)^q [t_{13}^{\alpha+\beta-\gamma} + (-1)^s t_{13}^{\alpha+\beta+\gamma}, t_{12}^{\beta}]] \,.$$

On the other hand, we have

$$[C_\alpha(x_2, t_{23}^{\beta}), y_1] = [(\mathrm{ad}x_2)^s t_{23}^{\alpha-\gamma} + (-\mathrm{ad}x_2)^s t_{23}^{\alpha+\gamma}, y_1]$$

$$= - \sum_{p+q=s-1} \sum_{\beta\in\Gamma} (\mathrm{ad}\, x_2)^p (-\mathrm{ad}\, x_3)^q [t_{12}^{\beta}, t_{31}^{\alpha+\beta-\gamma} + (-1)^s t_{31}^{\alpha+\beta+\gamma}] \,.$$

Therefore (iv) is satisfied.

Let us prove (v). We have

$$[C_\alpha(x_1, t_{12}^{\gamma}), t_{13}^{\alpha+\beta} + t_{23}^{\beta}] = [(\mathrm{ad}\, x_1)^s t_{12}^{\alpha-\gamma} + (-\mathrm{ad}\, x_1)^s t_{12}^{\alpha+\gamma}, t_{13}^{\alpha+\beta} + t_{23}^{\beta}]$$

$$= (\mathrm{ad}\, x_2)^s [t_{12}^{\alpha+\gamma} + (-1)^s t_{12}^{\alpha-\gamma}, t_{13}^{\alpha+\beta}] + (\mathrm{ad}\, x_1)^s [t_{12}^{\alpha-\gamma} + (-1)^s t_{12}^{\alpha+\gamma}, t_{23}^{\beta}]$$

$$= (\mathrm{ad}\, x_2)^s [t_{13}^{\alpha+\beta}, t_{23}^{\beta-\gamma} + (-1)^s t_{23}^{\beta+\gamma}] + (\mathrm{ad}\, x_1)^s [t_{23}^{\beta}, t_{13}^{\alpha+\beta-\gamma} + (-1)^s t_{13}^{\alpha+\beta+\gamma}] \,.$$

Therefore, by defining $A = t_{23}^{\beta-\gamma} + (-1)^s t_{23}^{\beta+\gamma}$ and $B = t_{13}^{\alpha+\beta-\gamma} + (-1)^s t_{13}^{\alpha+\beta+\gamma}$ we have

$$[t_{12}^{\alpha}, [C_\alpha(x_1, t_{12}^{\gamma}), t_{13}^{\alpha+\beta} + t_{23}^{\beta}]] = [t_{12}^{\alpha}, [t_{13}^{\alpha+\beta}, (\mathrm{ad}\, x_2)^s A] + [t_{23}^{\beta}, (\mathrm{ad}\, x_1)^s B]]$$

$$\begin{aligned}
&= [[t_{12}^{\alpha}, t_{13}^{\alpha+\beta}], (-\mathrm{ad}\, x_3)^s A] + [t_{13}^{\alpha+\beta}, (-\mathrm{ad}\, x_3)^s [t_{12}^{\alpha}, A]] \\
&+ [[t_{12}^{\alpha}, t_{23}^{\beta}], (-\mathrm{ad}\, x_3)^s B] + [t_{23}^{\beta}, (-\mathrm{ad}\, x_3)^s [t_{12}^{\alpha}, B]] \\
&= [[t_{23}^{\beta}, t_{12}^{\alpha}], (-\mathrm{ad}\, x_3)^s A] + [t_{13}^{\alpha+\beta}, (-\mathrm{ad}\, x_3)^s [B, t_{12}^{\alpha}]] \\
&+ [[t_{13}^{\alpha+\beta}, t_{12}^{\alpha}], (-\mathrm{ad}\, x_3)^s B] + [t_{23}^{\beta}, (-\mathrm{ad}\, x_3)^s [A, t_{12}^{\alpha}]] \\
&= [[t_{23}^{\beta}, (\mathrm{ad}\, x_2)^s A] + [t_{13}^{\alpha+\beta}, (\mathrm{ad}\, x_1)^s B], t_{12}^{\alpha}] \,.
\end{aligned}$$

This finishes the proof. □

Remark 6.2.2. *We do not know if* $\mathfrak{d}_0^{\Gamma} \longrightarrow \tilde{\mathfrak{d}}_0^{\Gamma}$ *is injective or not.*

6.2.3 Derivations of $\mathfrak{t}_{1,n}^{\Gamma}$ and $\bar{\mathfrak{t}}_{1,n}^{\Gamma}$

Lemma 6.2.3. *We have a bigraded Lie algebra morphism* $\tilde{\mathfrak{d}}_0^{\Gamma} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$*, taking* $(D, C) \in \tilde{\mathfrak{d}}_0^{\Gamma}$ *to the derivation* $\xi_{(D,C)}$:

$$\begin{aligned}
x_i &\longmapsto 0, \\
y_i &\longmapsto \sum_{j:j\neq i} D(x_i, t_{ij}^{\beta}), \\
t_{ij}^{\alpha} &\longmapsto [t_{ij}^{\alpha}, C_\alpha(x_i, t_{ij}^{\beta})].
\end{aligned}$$

This induces a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}_0^{\Gamma} \longrightarrow \mathrm{Der}(\bar{\mathfrak{t}}_{1,n}^{\Gamma})$.

Proof. We have to prove that defining relations of $\mathfrak{t}_{1,n}^{\Gamma}$ are preserved by $\xi := \xi_{(D,C)}$. First observe that relations $[x_i, x_j] = [x_i + x_j, t_{ij}^{\alpha}] = [x_i, t_{jk}^{\alpha}] = [t_{ij}^{\alpha}, t_{kl}^{\alpha}] = 0$ are obviously preserved. Then conditions (i) and (ii) respectively imply that $t_{ij}^{\alpha} = t_{ji}^{-\alpha}$ and $[x_i, y_j] = \sum_{\alpha} t_{ij}^{\alpha}$ are preserved. Condition (vi) implies that $[x_i, y_j] = [x_j, y_i]$ is preserved, and (vi) together with (iii) imply that $[y_i, y_j] = 0$ is preserved. Therefore it follows from the centrality of $\sum_i x_i$ and $\xi(\sum_i x_i) = 0$ that

$$\xi([x_i, y_i]) = \xi(-\sum_{j:j\neq i} [x_j, y_i]) = \xi(\sum_{j;j\neq i} \sum_{\alpha} t_{ij}^{\alpha}).$$

Condition (iv) ensures that $[y_i, t_{jk}^{\alpha}] = 0$ is preserved, and together with (vi) it implies that $[y_i + y_j, t_{ij}^{\alpha}] = 0$ is preserved. Finally condition (v) implies that the twisted infinitesimal braid relations are preserved, and the first part of the statement follows.

For the second part of the statement it remains to prove that the centrality of $\sum_i y_i$ is preserved. This follows directly from the identity $\xi(\sum_i y_i) = 0$ that we now prove. Relation (vi) implies that for any $i \neq j$ one has $D(x_i, t_{ij}^{\beta}) = -D(-x_i, t_{ij}^{-\beta}) = -D(x_j, t_{ji}^{\beta})$ in $\mathfrak{t}_{1,n}^{\Gamma}$ (the last equality happens since $\deg_t(D) = \deg_t(C_\alpha) + 1 > 0$), and hence

$$\xi(\sum_i y_i) = \sum_{i\neq j} D(x_i, t_{ij}^{\beta}) = \sum_{i<j} D(x_i, t_{ij}^{\beta}) - \sum_{j<i} D(x_j, t_{ji}^{\beta}) = 0\,.$$

We are done (the compatibility with bracket and grading are easy to check).

The last part of the statementis a consequence of the fact that $\xi(\sum_i y_i) = \xi(\sum_i x_i) = 0$, that we have already proved. $\square$

We now prove that this morphism extends to a Lie algebra morphism $\tilde{\mathfrak{d}}^{\Gamma} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$:

Proposition 6.2.4. *We have a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}^{\Gamma} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$ taking $(D,C) \in \tilde{\mathfrak{d}}_0^{\Gamma}$ to $\xi_{(D,C)}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sl}_2$ to the derivation*

$$\xi_g : t_{ij}^{\alpha} \mapsto 0, \begin{pmatrix} x_i & y_i \end{pmatrix} \mapsto \begin{pmatrix} x_i & y_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This induces a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}^{\Gamma} \longrightarrow \mathrm{Der}(\bar{\mathfrak{t}}_{1,n}^{\Gamma})$.

In what follows we write $\mathbf{d} := \tilde{h}$, $\mathbf{X} := \tilde{e}$ and $\Delta_0 := \tilde{f}$ and $\tilde{\mathbf{d}} := \xi_{\tilde{h}}$, $\tilde{\mathbf{X}} := \xi_{\tilde{e}}$ and $\tilde{\Delta}_0 := \xi_{\tilde{f}}$.

Proof. It is obvious that for any $g, g' \in \mathfrak{sl}_2$, ξ_g defines a derivation of the same degree of $\mathfrak{t}_{1,n}^{\Gamma}$, and that $\xi_{[g,g']} = [\xi_g, \xi_{g'}]$. Hence we have a bigraded Lie algebra morphism $\mathfrak{sl}_2 * \tilde{\mathfrak{d}}_0^{\Gamma} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$. Let us prove that it factorizes through the quotient $\tilde{\mathfrak{d}}^{\Gamma}$.

It is relatively clear that $[\tilde{\mathbf{X}}, \xi_{(D,C)}] = 0$ and $[\tilde{\mathbf{d}}, \xi_{(D,C)}] = (p-q)(D,C)$ if $(D,C) \in (\tilde{\mathfrak{d}}_0^{\Gamma})^{p,q}$. Thus it remains to prove that $(\mathrm{ad}\,\tilde{\Delta}_0)^p(\xi_{(D,C)}) = 0$ if $(D,C) \in (\tilde{\mathfrak{d}}_0^{\Gamma})^{p,q}$. We do this now. Let us write $\xi := \xi_{(D,C)}$ and $A := (\mathrm{ad}\,\tilde{\Delta}_0)^p(\xi)$. Then after an easy computation one obtains on generators:

$$\begin{aligned} A(x_i) =& - p\tilde{\Delta}_0^{p-1}\xi(y_i) = -p\tilde{\Delta}_0^{p-1}(\sum_{j:j\neq i} D(x_i, t_{ij}^\beta)), \\ A(y_i) =&\tilde{\Delta}_0^p\xi(y_i) = \tilde{\Delta}_0^p(\sum_{j:j\neq i} D(x_i, t_{ij}^\beta)), \\ A(t_{ij}^\alpha) =&\tilde{\Delta}_0^p\xi(t_{ij}^\alpha) = \tilde{\Delta}_0^p([t_{ij}^\alpha, C_\alpha(x_i, t_{ij}^\beta)]). \end{aligned}$$

Finally remark that we have an increasing filtration on $\mathfrak{t}_{1,n}^\Gamma$ defined by $\deg(x_i) = 1$ and $\deg(t_{ij}^\alpha) = \deg(y_i) = 0$. Δ_0 decreases the degree by 1 and vanishes on degree zero elements. The result then follows from the fact that $\deg_x(C_\alpha) = p-q < p$ and $\deg_x(D) = p-q-1 < p-1$. $\square$

Now composing with $\mathfrak{d}_0^\Gamma \longrightarrow \tilde{\mathfrak{d}}_0^\Gamma$ (resp. $\mathfrak{d}^\Gamma \longrightarrow \tilde{\mathfrak{d}}^\Gamma$) one obtains a Lie algebra morphism $\mathfrak{d}_0^\Gamma \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$ (resp. $\mathfrak{d}^\Gamma \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$). We write $\xi_{s,\gamma} := \xi_{(D_{s,\gamma}, C_{s,\gamma})}$ for the image of $\delta_{s,\gamma}$. We then have $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}^\Gamma = (\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma) \rtimes \mathfrak{sl}_2$, with $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma$ positively graded (since both $\mathfrak{t}_{1,n}^\Gamma$ and $\mathfrak{d}_+^\Gamma$ are $(\mathbb{Z}_{\geq 0})^2$-graded) and a sum of finite dimensional $\mathfrak{sl}_2$-modules. Therefore we can construct the semi-direct product group

$$\mathbf{G}_n^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C}), \tag{6.4}$$

where $\exp(\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma)^\wedge$ is the exponential group associated to the degree completion of $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma$. Similarly, we define $\bar{\mathbf{G}}_n^\Gamma := \exp(\bar{\mathfrak{t}}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C})$.

Notice that one can also define semi-direct product groups $\tilde{\mathbf{G}}_n^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \rtimes \tilde{\mathfrak{d}}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C})$ and $\tilde{\bar{\mathbf{G}}}_n^\Gamma := \exp(\bar{\mathfrak{t}}_{1,n}^\Gamma \rtimes \tilde{\mathfrak{d}}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C})$. We therefore have the following commutative diagram:

$$\begin{tikzcd} \mathbf{G}_n^\Gamma \arrow[r] \arrow[d] & \tilde{\mathbf{G}}_n^\Gamma \arrow[d] \\ \bar{\mathbf{G}}_n^\Gamma \arrow[r] & \tilde{\bar{\mathbf{G}}}_n^\Gamma. \end{tikzcd} \tag{6.5}$$

Lemma 6.2.5. *The kernel of* $\tilde{\mathfrak{d}}_0^\Gamma \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$ *($n \geq 2$) is the space of elements* $(0, C)$ *for which* C_α *is proportional to* t^α*, and* $\ker(\mathfrak{d}_0^\Gamma \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)) = \mathbb{C}\delta_{0,0}$.

Proof. Let us first prove it for $n = 2$. Recall that $\bar{\mathfrak{t}}_{1,2}^\Gamma = \mathfrak{t}_{1,2}^\Gamma/(x_1 + x_2, y_1 + y_2)$, so it is the Lie algebra generated by x (the class of x_1), y (the class of y_1) and t^α's (classes of t_{12}^α's) with the relation $[x, y] = \sum_{\alpha\in\Gamma} t^\alpha$. Then the derivation $\xi_{(D,C)}$ associated to $(D, C) \in \tilde{\mathfrak{d}}_0^\Gamma$ is given by

$$x \mapsto 0, y \mapsto D(x, t^\beta), t^\alpha \mapsto [t^\alpha, C_\alpha(x, t^\beta)].$$

This derivation vanishes if and only if $D = 0$ and C_α is proportional to t^α. Finally, the result for $n \geq 2$ follows from the fact that

$$\xi_{(D,C)}^{(2)} = (u \mapsto u^{1,2,\emptyset,\ldots,\emptyset}) \circ \xi_{(D,C)}^{(n)} \circ (u \mapsto u^{1,\ldots,n}),$$

where $\xi_{(D,C)}^{(n)}$ denotes the derivation of $\mathfrak{t}_{1,n}^\Gamma$ associated to (D, C). $\square$

6.2.4 Comparison morphisms

Let $\rho : \Gamma_1 \longrightarrow \Gamma_2$ a group morphism. We have a comparison morphism $\tilde{\mathfrak{d}}_0^{\Gamma_1} \longrightarrow \tilde{\mathfrak{d}}_0^{\Gamma_2}$, $(D, C) \mapsto (D^\rho, C^\rho)$ defined by

$$D^\rho := D\left(x, \sum_{\gamma \in \mathrm{coker}(\rho)} \frac{t^{\rho(\beta)+\gamma}}{\#\ker(\rho)}\right),\ (C^\rho)_\alpha := C_\alpha\left(x, \sum_{\gamma \in \mathrm{coker}(\rho)} \frac{t^{\rho(\beta)+\gamma}}{\#\ker(\rho)}\right).$$

When ρ is not surjective it depends on the choice of a section $\mathrm{coker}(\rho) \longrightarrow \Gamma_2$. It extends to $\tilde{\mathfrak{d}}^{\Gamma_1} \longrightarrow \tilde{\mathfrak{d}}^{\Gamma_2}$ by sending the generators of $\mathfrak{sl}_2$ to themselves. These comparison morphisms are compatible with the morphisms $\tilde{\mathfrak{d}}^{\Gamma_i} \longrightarrow \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma_i})$, for $i = 1, 2$. Namely, there is a commutative diagram

$$\begin{tikzcd} \tilde{\mathfrak{d}}^{\Gamma_1} \ltimes \mathfrak{t}_{1,n}^{\Gamma_1} \arrow[r] \arrow[d] & \mathfrak{t}_{1,n}^{\Gamma_1} \arrow[d] \\ \tilde{\mathfrak{d}}^{\Gamma_2} \ltimes \mathfrak{t}_{1,n}^{\Gamma_2} \arrow[r] & \mathfrak{t}_{1,n}^{\Gamma_2} \end{tikzcd}$$

Finally, we have comparison morphisms for the corresponding groups that fit into a commutative diagram

$$\begin{tikzcd} \tilde{\mathbf{G}}_n^{\Gamma_1} \arrow[r] \arrow[d] & \tilde{\mathbf{G}}_n^{\Gamma_2} \arrow[d] \\ \tilde{\bar{\mathbf{G}}}_n^{\Gamma_1} \arrow[r] & \tilde{\bar{\mathbf{G}}}_n^{\Gamma_2}. \end{tikzcd} \tag{6.6}$$

Notice that the image of $(D_{s,\gamma}, C_{s,\gamma})$ under a comparison morphism is no longer of this form except if ρ is injective. In this case (and in this case only) we have a comparison morphism $\mathfrak{t}_{1,n}^{\Gamma_1} \rtimes \mathfrak{d}^{\Gamma_1} \longrightarrow \mathfrak{t}_{1,n}^{\Gamma_2} \rtimes \mathfrak{d}^{\Gamma_2}$ taking x_i's, y_i's, $\mathbf{d}$, $\mathbf{X}$ and Δ_0 to themselves, and t_{ij}^α to $\sum_{\beta \in \mathrm{coker}(\rho)} t_{ij}^{\rho(\alpha)+\beta}$ and $\delta_{s,\gamma}$ to $\sum_{\beta \in \mathrm{coker}(\rho)} \delta_{s,\rho(\gamma)+\beta}$. In particular we have a canonical natural inclusion $\mathbf{G}_n^0 \longrightarrow \mathbf{G}_n^\Gamma$ (which descends to an inclusion $\bar{\mathbf{G}}_n^0 \longrightarrow \bar{\mathbf{G}}_n^\Gamma$).

6.3 Bundles with flat connections on moduli spaces

6.3.1 On some subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and moduli spaces

Consider the group $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and consider the following (finite index) subgroup of $\mathrm{SL}_2(\mathbb{Z})$:

$$\mathrm{SL}_2^\Gamma(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv 1 \bmod M, d \equiv 1 \bmod N, b \equiv 0 \bmod N \text{ and } c \equiv 0 \bmod M \right\}.$$

We write $Y(\Gamma)$ for the set of equivalences classes of pairs (E, ϕ) where E is an elliptic curve and $\phi : \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow E$ is an injective group morphism that is orientation preserving i.e. such that the basis $(\frac{d}{dt}_{|t=0}(t\phi(\bar{1}, \bar{0})), \frac{d}{dt}_{|t=0}(t\phi(\bar{0}, \bar{1}))$ of T_0E is direct. Then, one can see that $Y(\Gamma) = \mathfrak{H}/\mathrm{SL}_2^\Gamma(\mathbb{Z})$ and therefore inherits the structure of a complex orbifold.

Remark 6.3.1. *The biggest congruence subgroup on which the connection we will construct in this section is well defined and flat is the subgroup* $\tilde{\mathrm{SL}}_2^{\Gamma}(\mathbb{Z})$ *of* $\mathrm{SL}_2(\mathbb{Z})$ *consisting of matrices* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $Mb \equiv 0$ *mod* N *and* $Nc \equiv 0$ *mod* M*. Nevertheless, in order to retrieve the twisted elliptic KZB connection defined at the level of configuration spaces, it suffices to consider the usual congruence subgroup* $\mathrm{SL}_2^{\Gamma}(\mathbb{Z}) \subset \tilde{\mathrm{SL}}_2^{\Gamma}(\mathbb{Z})$*.*

Recall the following standard group actions:

- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{C}^n \times \mathfrak{H}$:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * (\mathbf{z}|\tau) := \left(\frac{\mathbf{z}}{c\tau + d} \Big| \frac{a\tau + b}{c\tau + d} \right) .$$
This obviously descends to an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n \times \mathfrak{H}/\mathbb{C}$, where $\mathbb{C}$ acts diagonally on $\mathbb{C}^n$: $u \cdot (\mathbf{z}|\tau) := (\mathbf{z} + u \sum_i \delta_i |\tau)$.
- The group $(\mathbb{Z}^n)^2$ acts on $\mathbb{C}^n \times \mathfrak{H}$:
$$(\mathfrak{m}, \mathfrak{n}) * (\mathbf{z}|\tau) := (\mathbf{z} + \mathfrak{m} + \tau\mathfrak{n}|\tau) .$$
It obvioulsy descends to an action of $(\mathbb{Z}^n)^2/\mathbb{Z}^2$ on $\mathbb{C}^n \times \mathfrak{H}/\mathbb{C}$, where $\mathbb{Z}^2$ is the diagonal subgroup in $(\mathbb{Z}^n)^2 = (\mathbb{Z}^2)^n$.
- Finally, there is a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $(m, n) \in \mathbb{Z}^2$ by automorphisms:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} n & m \end{pmatrix} \longrightarrow \begin{pmatrix} n & m \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$
We can thus form the semi-direct products $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ and $((\mathbb{Z}^n)^2/\mathbb{Z}^2) \rtimes \mathrm{SL}_2(\mathbb{Z})$

A few observations are then in order:

- The above actions are compatible in the sense that we have a left action of $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n \times \mathfrak{H}$, which descends to an action of $\big((\mathbb{Z}^n)^2/\mathbb{Z}^2\big) \rtimes \mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{C}^n \times \mathfrak{H})/\mathbb{C}$, where $\mathbb{Z}^2$ is embedded in $(\mathbb{Z}^n)^2$ via the diagonal map. One can think of translation by $\mathbb{C}$ as a left or right action as it commutes with the G-action.
- The action of $(\mathbb{Z}^n)^2$ preserves the subset
$$\mathrm{Diag}_{n,\Gamma} := \{(\mathbf{z}|\tau) \in \mathbb{C}^n \times \mathfrak{H} | \mathbf{z} \in \mathrm{Diag}_{\tau,n,\Gamma}\} .$$
- The action of the subgroup $\mathrm{SL}_2^{\Gamma}(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{Z})$ also preserves $\mathrm{Diag}_{n,\Gamma}$.

We are thus ready to define several variants of $Y(\Gamma)$ "with marked points":

- We define the quotient
$$\bar{\mathcal{M}}_{1,n}^{\Gamma} := (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^{\Gamma}(\mathbb{Z}) \setminus \big((\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}\big)/\mathbb{C}$$
and call it the *moduli space of* Γ*-structured elliptic curves with* n *ordered marked points.*

- It has a non-reduced variant

$$\mathbf{p} : \mathcal{M}^{\Gamma}_{1,n} := \left((\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}\right) / (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}^{\Gamma}_2(\mathbb{Z}) \twoheadrightarrow \bar{\mathcal{M}}^{\Gamma}_{1,n} \,.$$

- One can also define the *moduli space of* Γ*-structured elliptic curves with* n *unordered marked points*

$$\bar{\mathcal{M}}^{\Gamma}_{1,[n]} := \bar{\mathcal{M}}^{\Gamma}_{1,n} / \mathfrak{S}_n$$

  and its non-reduced variant

$$\mathcal{M}^{\Gamma}_{1,[n]} := \mathcal{M}^{\Gamma}_{1,n} / \mathfrak{S}_n \,.$$

Remark 6.3.2. *We have* $\bar{\mathcal{M}}^{\Gamma}_{1,1} = \bar{\mathcal{M}}^{\Gamma}_{1,[1]} = Y(\Gamma)$*, and* $\mathcal{M}^{\Gamma}_{1,1} = \mathcal{M}^{\Gamma}_{1,[1]}$ *is the universal curve over it. The fiber of* $\mathcal{M}^{\Gamma}_{1,n} \longrightarrow Y(\Gamma)$ *(resp.* $\bar{\mathcal{M}}^{\Gamma}_{1,n} \longrightarrow Y(\Gamma)$*) at (the class of)* τ *is precisely the twisted (resp. reduced twisted) configuration space* $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ *(resp.* $C(E_{\tau,\Gamma}, n, \Gamma)$*). Moreover, the map*

$$h : \bar{\mathcal{M}}^{\Gamma}_{1,2} \longrightarrow \bar{\mathcal{M}}^{\Gamma}_{1,1}$$

factors through (and is open in) $\mathcal{M}^{\Gamma}_{1,1}$*. We can interpret* $\bar{\mathcal{M}}^{\Gamma}_{1,2}$ *as the* Γ*-punctured universal curve over* $Y(\Gamma)$*.*

6.3.2 Principal bundles over $\mathcal{M}^{\Gamma}_{1,n}$ and $\bar{\mathcal{M}}^{\Gamma}_{1,n}$

In this §, $\mathbf{G}^{\Gamma}_n$ is defined as in (6.4) and we define a principal $\mathbf{G}^{\Gamma}_n$-bundle $P_{n,\Gamma}$ over $\mathcal{M}^{\Gamma}_{1,n}$ whose image under the natural morphism $\mathbf{G}^{\Gamma}_n \longrightarrow \bar{\mathbf{G}}^{\Gamma}_n$ is the pull-back of a principal $\bar{\mathbf{G}}^{\Gamma}_n$-bundle $\bar{P}_{n,\Gamma}$ over $\bar{\mathcal{M}}^{\Gamma}_{1,n}$. Let us fix the notation first: for $u \in \mathbb{C}^{\times}$ and $v, w_i \in \mathbb{C}$ $(i = 1, \ldots, n)$,

$$u^{\mathbf{d}} := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, e^{v\mathbf{X}} := \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Since $[\mathbf{X}, x_i] = 0$ then it makes sense to define $e^{v\mathbf{X} + \sum_i w_i x_i} := e^{v\mathbf{X}} e^{\sum_i w_i x_i}$. In particular, we have $\mathrm{Ad}(u^{\mathbf{d}})(x_i) = u x_i$ and $\mathrm{Ad}(u^{\mathbf{d}})(y_i) = y_i/u$ $(\forall i)$, $\mathrm{Ad}(u^{\mathbf{d}})(\mathbf{X}) = u^2 \mathbf{X}$ and $\mathrm{Ad}(u^{\mathbf{d}})(\Delta_0) = \Delta_0/u^2$. Let $\pi : \mathbb{C}^n \times \mathfrak{H} \longrightarrow \mathcal{M}_{1,n}$ be the canonical projection.

Proposition 6.3.3. *There exists a unique principal* $\mathbf{G}^{\Gamma}_n$*-bundle* $P_{n,\Gamma}$ *over* $\mathcal{M}^{\Gamma}_{1,n}$ *for which a section on* $U \subset \mathcal{M}^{\Gamma}_{1,n}$ *is a function* $f : \pi^{-1}(U) \longrightarrow \mathbf{G}^{\Gamma}_n$ *such that*

$$\begin{aligned} f(\mathbf{z} + \delta_i | \tau) &= f(\mathbf{z}|\tau), \\ f(\mathbf{z} + \tau\delta_i | \tau) &= e^{\frac{-2\pi \mathrm{i} x_i}{N}} f(\mathbf{z}|\tau), \\ f(\mathbf{z}, \tau + 1) &= f(\mathbf{z}|\tau), \\ f(\frac{\mathbf{z}}{\tau} | -\frac{1}{\tau}) &= \tau^{\mathbf{d}} e^{\frac{2\pi \mathrm{i}}{\tau}(\mathbf{X} + \sum_i z_i x_i)} f(\mathbf{z}|\tau). \end{aligned}$$

Moreover, the image of $P_{n,\Gamma}$ *under* $\mathbf{G}^{\Gamma}_n \longrightarrow \bar{\mathbf{G}}^{\Gamma}_n$ *is the pull-back of a unique principal* $\bar{\mathbf{G}}^{\Gamma}_n$*-bundle* $\bar{P}_{n,\Gamma}$ *over* $\bar{\mathcal{M}}^{\Gamma}_{1,n}$ *for which a section on* $U \subset \bar{\mathcal{M}}^{\Gamma}_{1,n}$ *is a function* $f : (\mathbf{p} \circ \pi)^{-1}(U) \longrightarrow \bar{\mathcal{M}}^{\Gamma}_{1,n}$ *satisfying the above conditions (with* x_i*'s replaced by* $\bar{x}_i$*'s) and such that* $f(\mathbf{z} + v \sum_i \delta_i | \tau) = f(\mathbf{z}|\tau)$ *for any* $v \in \mathbb{C}$*.*

Proof. First recall that for $\Gamma = 0$ this is precisely [24, Proposition 3.4]. Then observe that we have an obvious map $\iota : \mathcal{M}_{1,n}^{\Gamma} \longrightarrow \mathcal{M}_{1,n}^{0}$. Therefore we define $P_{n,\Gamma}$ (resp. $\bar{P}_{n,\Gamma}$) to be the image under the natural inclusion $\mathbf{G}_n^0 \longrightarrow \mathbf{G}_n^{\Gamma}$ (resp. $\bar{\mathbf{G}}_n^0 \longrightarrow \bar{\mathbf{G}}_n^{\Gamma}$) of $\iota^* P_{n,0}$ (resp. $\iota^* \bar{P}_{n,0}$).

We thus proved existence. Unicity is obvious. □

In other words, there exists a unique non-abelian 1-cocycle $(c_g)_{g\in(\mathbb{Z}^n)^2\rtimes \mathrm{SL}_2(\mathbb{Z})}$ on $\mathbb{C}^n \times \mathfrak{H}$ with values in $\mathbf{G}_n^{\Gamma}$ such that $c_{(\delta_i,0)} = 1$, $c_{(0,\delta_i)} = e^{-2\pi \mathrm{i} x_i}$, $c_S = 1$ and

$$c_T(\mathbf{z}|\tau) = \tau^{\mathbf{d}} e^{(2\pi\mathrm{i}/\tau)(\mathbf{X}+\sum_j z_j x_j)} = e^{2\pi\mathrm{i}(\tau\mathbf{X}+\sum_j z_j x_j)}\tau^{\mathbf{d}},$$

where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the generators of $\mathrm{SL}_2(\mathbb{Z})$. Here *cocycle* means (as in [24]) that c_g's are holomorphic functions $\mathbb{C}^n \times \mathfrak{H} \longrightarrow \mathbf{G}_n^{\Gamma}$ satisfying the cocycle condition $c_{gg'}(\mathbf{z}|\tau) = c_g(g' * (\mathbf{z},\tau))c_{g'}(\mathbf{z}|\tau)$.

Remark 6.3.4. *Notice that we do have a $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$-cocycle (since our bundle is define as the pull-back of a bundle on $\mathcal{M}_{1,1}^0$) but the cocycle defining $P_{n,\Gamma}$ is its restriction to $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^{\Gamma}(\mathbb{Z})$.*

6.3.3 Connections on $P_{n,\Gamma}$ and $\bar{P}_{n,\Gamma}$

A connection on $P_{n,\Gamma}$ is the same as an equivariant connection on the trivial $\mathbf{G}_n^{\Gamma}$-bundle over $\mathbb{C}^n \times \mathfrak{H} - \mathrm{Diag}_{n,\Gamma}$. Namely, it is of the form $\nabla_{n,\Gamma} := d - \eta(\mathbf{z}|\tau)$, where η is a $\mathfrak{t}_{1,n}^{\Gamma} \rtimes \mathfrak{d}^{\Gamma}$-valued meromorphic one-form on $\mathbb{C}^n \times \mathfrak{H}$ with only poles on $\mathrm{Diag}_{n,\Gamma}$, and the equivariance condition reads: for any $g \in (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^{\Gamma}(\mathbb{Z})$,

$$g^*\eta = (dc_g(\mathbf{z}|\tau))c_g(\mathbf{z}|\tau)^{-1} + \mathrm{Ad}(c_g(\mathbf{z}|\tau))(\eta(\mathbf{z}|\tau)). \tag{6.7}$$

We now construct such a connection. For any $\gamma \in \Gamma$ we define $g_\gamma(x,z|\tau) := \partial_x k_\gamma(x,z|\tau)$,

$$\varphi_\gamma(x|\tau) = \sum_{s\geq 0} A_{s,\gamma}(\tau)x^s := g_{-\gamma}(x,0|\tau).$$

Then we set

$$\Delta(\mathbf{z}|\tau) := -\frac{1}{2\pi\mathrm{i}}\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}(\tau)\delta_{s,\gamma} - \sum_{i<j} g_{ij}(z_{ij}|\tau)\right),$$

where $g_{ij}(z|\tau) := \sum_{\alpha\in\Gamma} g_\alpha(\mathrm{ad}x_i, z|\tau)(t_{ij}^{\alpha})$. And finally, with $K_i(\mathbf{z}|\tau)$'s as in §6.1.3, we define

$$\eta(\mathbf{z}|\tau) := \Delta(\mathbf{z}|\tau)d\tau + \sum_i K_i(\mathbf{z}|\tau)dz_i.$$

Remark 6.3.5. *One can see that $\varphi_{\mathbf{0}}(x) = (\theta'/\theta)'(x) + 1/x^2$ and that for any $\gamma \in \Gamma - \{0\}$*

$$\varphi_\gamma(x) = \partial_x\left(e^{2\pi\mathrm{i}cx}\frac{\theta(\tilde{\gamma}+x)}{\theta(\tilde{\gamma})\theta(x)} - \frac{1}{x}\right),$$

where $\tilde{\gamma} = (c_0, c) \in \Lambda_{\tau,\Gamma} - \Lambda_\tau$ is any lift of γ.

Proposition 6.3.6. *The equivariance identity* (6.7) *is satisfied for any* $g \in (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

Before proving this statement, let us notice that the $\mathrm{SL}_2(\mathbb{Z})$-equivariance is stronger than what we need (the $\mathrm{SL}_2^\Gamma(\mathbb{Z})$-equivariance), but easier to prove. The action of $\mathrm{SL}_2(\mathbb{Z})$ moves the poles while $\mathrm{SL}_2^\Gamma(\mathbb{Z})$ fixes them. In both cases, it makes sense to prove this proposition for meromorphic forms on $\mathbb{C}^n \times \mathfrak{h}$.

Proof. For $g = (\delta_j, 0)$, the identity translates into $K_i(\mathbf{z} + \delta_j|\tau) = K_i(\mathbf{z}|\tau)$ $(i = 1, \dots, n)$ and $\Delta(\mathbf{z} + \delta_j|\tau) = \Delta(\mathbf{z}|\tau)$, which are immediate.

For $g = (0, \delta_j)$, the identity translates into $K_i(\mathbf{z} + \tau\delta_j|\tau) = e^{-2\pi \mathrm{i}\,\mathrm{ad}(x_j)} K_i(\mathbf{z}|\tau)$ $(\forall i)$ and

$$\Delta(\mathbf{z} + \tau\delta_j|\tau) + K_j(\mathbf{z} + \tau\delta_j|\tau) = e^{-2\pi \mathrm{i}\,\mathrm{ad}(x_j)} \Delta(\mathbf{z}|\tau). \tag{6.8}$$

The first equality is proved in §6.1.3, and we prove the second one now. First remember that for any $\tau \in \mathfrak{H}$, $z \in \mathbb{C} - (\frac{1}{M}\mathbb{Z} + \frac{\tau}{N}\mathbb{Z}))$ and $\alpha \in \Gamma$, we have the following identity in $\mathbb{C}[[x]]$:

$$e^{-2\pi \mathrm{i}x}(g_\alpha(x, z) - 1/x^2) + 1/x^2 - 2\pi\mathrm{i}(k_\alpha(x, z + \tau) + 1/x) = g_\alpha(x, z + \tau)\,. \tag{6.9}$$

Then we can compute $2\pi\mathrm{i}\left(K_j(\mathbf{z} + \tau\delta_j|\tau) - e^{-2\pi \mathrm{i}\,\mathrm{ad}(x_j)}\Delta(\mathbf{z}|\tau)\right)$: it is equal to

$$2\pi\mathrm{i}\left(\sum_{k:k\neq j} k_\alpha(\mathrm{ad}x_j, z_{jk} + \tau) - y_j\right) + \Delta_0 + \frac{1 - e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}}{\mathrm{ad}x_j}(y_j) + \frac{1}{2}\sum_{\substack{s\geq 0,\\ \gamma\in\Gamma}} A_{s,\gamma}\delta_{s,\gamma} - e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}\sum_{k<l} g_{kl}(z_{kl})\,,$$

and therefore using

$$\frac{1 - e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}}{\mathrm{ad}x_j}(y_j) - 2\pi\mathrm{i}y_j = \left(\frac{e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j} - 1}{(\mathrm{ad}x_j)^2} + \frac{2\pi\mathrm{i}}{\mathrm{ad}x_j}\right)\left(\sum_{\alpha\in\Gamma}\sum_{k:k\neq j} t_{jk}^\alpha\right)$$

together with (6.9) we obtain

$$\Delta_0 + \frac{1}{2}\sum_{s\geq 0, \gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\substack{k<l\\ k,l\neq j}} g_{kl}(z_{kl}) - \sum_{\substack{k:k\neq j\\ \alpha\in\Gamma}} g_\alpha(\mathrm{ad}x_j, z_{jk} + \tau)(t_{jk}^\alpha)\,,$$

which is precisely equal to $-2\pi\mathrm{i}\Delta(\mathbf{z} + \tau\delta_j)$.

For $g = S$, the identity translates into $K_i(\mathbf{z}|\tau + 1) = K_i(\mathbf{z})$ $(\forall i)$ and $\Delta(\mathbf{z}|\tau + 1) = \Delta(\mathbf{z})$. Both equalities obviously follow from $\theta(z|\tau + 1) = \theta(z|\tau)$.

For $g = T$, the identity translates into

$$\frac{1}{\tau}K_i(\frac{\mathbf{z}}{\tau}| - \frac{1}{\tau}) = \mathrm{Ad}\left(c_T(\mathbf{z}|\tau)\right)\left(K_i(\mathbf{z}|\tau)\right) + 2\pi\mathrm{i}x_i \tag{6.10}$$

for all $i \in \{1, \dots, n\}$ and

$$\frac{1}{\tau^2}\left(\Delta(\frac{\mathbf{z}}{\tau}| - \frac{1}{\tau}) - \sum_i z_i K_i(\frac{\mathbf{z}}{\tau}| - \frac{1}{\tau})\right) = \mathrm{Ad}\left(c_T(\mathbf{z}|\tau)\right)\left(\Delta(\mathbf{z}|\tau)\right) + \frac{\mathbf{d}}{\tau} - 2\pi\mathrm{i}\mathbf{X}\,. \tag{6.11}$$

Let us check (6.10) first. $\mathrm{Ad}(e^{2\pi\mathrm{i}(\sum_j z_jx_j+\tau\mathbf{X})}\tau^{\mathbf{d}})(-y_i)+2\pi\mathrm{i}x_i$ equals

$$-\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_j z_jx_j})(y_i/\tau)=-\frac{y_i}{\tau}-\frac{e^{2\pi\mathrm{i}\mathrm{ad}(\sum_j z_jx_j)}-1}{\mathrm{ad}(\sum_j z_jx_j)}([\sum_j z_jx_j,\frac{y_i}{\tau}])$$

$$=-\frac{y_i}{\tau}-\frac{e^{2\pi\mathrm{i}\sum_j z_j\mathrm{ad}x_j}-1}{\sum_j z_j\mathrm{ad}x_j}(\sum_{\substack{j:j\neq i\\ \alpha\in\Gamma}}\frac{z_{ji}}{\tau}t^\alpha_{ij})=-\frac{y_i}{\tau}-\sum_{j:j\neq i}\frac{e^{2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i}}{z_{ij}\mathrm{ad}x_i}(\sum_{\alpha\in\Gamma}\frac{z_{ji}}{\tau}t^\alpha_{ij}).$$

Therefore we have

$$-\frac{y_i}{\tau}=\mathrm{Ad}(c_T(\mathbf{z}|\tau))(-y_i)+2\pi\mathrm{i}x_i-\sum_{j:j\neq i}\frac{e^{2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i}}{\mathrm{ad}x_i}(\sum_{\alpha\in\Gamma}\frac{t^\alpha_{ij}}{\tau}). \tag{6.12}$$

Now substituting $(x,z)=(\mathrm{ad}x_j,z_j)$ in

$$\frac{1}{\tau}(k_\alpha(x,\frac{z}{\tau}|-\frac{1}{\tau})=e^{2\pi\mathrm{i}zx}k_\alpha(\tau x,z|\tau)+\frac{e^{2\pi\mathrm{i}zx}-1}{\tau x}, \tag{6.13}$$

then applying to t^α_{ij}, summing over $j\neq i$ and $\alpha\in\Gamma$, and adding up (6.12) we obtain (6.10) by using that

$$e^{2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i}k_\alpha(\tau\mathrm{ad}x_i,z_{ij}|\tau)(t^\alpha_{ij})=\mathrm{Ad}(e^{2\pi\mathrm{i}(\tau\mathbf{X}+\sum_j z_jx_j)}\tau^{\mathbf{d}})(k_\alpha(\mathrm{ad}x_i,z_{ij}|\tau)(t^\alpha_{ij})).$$

We now check (6.11). Differentiating (6.13) w.r.t. x and dividing by τ, we get

$$\frac{1}{\tau^2}g_\alpha(x,\frac{z}{\tau}|-\frac{1}{\tau})=e^{2\pi\mathrm{i}zx}g_\alpha(\tau x,z|\tau)+\frac{2\pi\mathrm{i}z}{\tau^2}k_\alpha(x,\frac{z}{\tau}|-\frac{1}{\tau})+\frac{1+2\pi\mathrm{i}zx-e^{2\pi\mathrm{i}zx}}{\tau^2x^2}.$$

Now substituting $(x,z)=(\mathrm{ad}x_i,z_{ij})$, applying to t^α_{ij}, and summing over $\alpha\in\Gamma$ we obtain

$$\begin{aligned}\frac{1}{\tau^2}g_{ij}(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) \quad&=\quad \mathrm{Ad}(c_T(\mathbf{z}|\tau))\,(g_{ij}(\mathbf{z}|\tau))+\frac{2\pi\mathrm{i}z_{ij}}{\tau^2}K_{ij}(\frac{z_{ij}}{\tau}|-\frac{1}{\tau})\\ &\quad+\left(\frac{1+2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i-e^{2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i}}{\tau^2(\mathrm{ad}x_i)^2}\right)(\sum_{\alpha\in\Gamma}t^\alpha_{ij}).\end{aligned}$$

Then taking the sum over $i<j$ one gets

$$\frac{1}{\tau^2}\sum_{i<j}g_{ij}(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau})=\mathrm{Ad}(c_T(\mathbf{z}|\tau))\left(\sum_{i<j}g_{ij}(\mathbf{z}|\tau)\right)+\frac{2\pi\mathrm{i}}{\tau^2}\sum_i z_iK_i(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau})+B(\mathbf{z}), \tag{6.14}$$

where

$$B(\mathbf{z}):=\sum_i\frac{2\pi\mathrm{i}z_iy_i}{\tau^2}+\sum_{i<j}\left(\frac{1+2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i-e^{2\pi\mathrm{i}z_{ij}\mathrm{ad}x_i}}{\tau^2(\mathrm{ad}x_i)^2}\right)(\sum_\alpha t^\alpha_{ij}).$$

Lemma 6.3.7. $\mathrm{Ad}\,(c_T(\mathbf{z}|\tau))\,(\Delta_0)=\frac{\Delta_0}{\tau^2}+\frac{2\pi\mathrm{i}\mathbf{d}}{\tau}-(2\pi\mathrm{i})^2(\frac{1}{\tau}\sum_i z_ix_i+\mathbf{X})+B(\mathbf{z})$.

Proof of the lemma. We first compute

$$\begin{aligned}\mathrm{Ad}\,(c_T(\mathbf{z}|\tau))\,(\Delta_0)\quad&=\quad\mathrm{Ad}(e^{2\pi\mathrm{i}(\tau\mathbf{X}+\sum_i z_ix_i)})(\frac{\Delta_0}{\tau^2})=\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_ix_i})(\frac{\Delta_0}{\tau^2}+\frac{2\pi\mathrm{i}\mathbf{d}}{\tau}-(2\pi\mathrm{i})^2\mathbf{X})\\ &=\quad\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_ix_i})(\frac{\Delta_0}{\tau^2})+\frac{2\pi\mathrm{i}\mathbf{d}}{\tau}-(2\pi\mathrm{i})^2(\frac{1}{\tau}\sum_i z_ix_i+\mathbf{X}).\end{aligned}$$

It remains to show that $\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_ix_i})(\frac{\Delta_0}{\tau^2})=\frac{\Delta_0}{\tau^2}+B(\mathbf{z})$. The proof of this fact goes along the same lines of computation as in [24, pp.16-17]. $\square$

Using the above lemma and equation (6.14), one sees that equation (6.11) follows from

$$\mathrm{Ad}(c_T(\mathbf{z}|\tau)(\sum_{s,\gamma} A_{s,\gamma}(\tau)\delta_{s,\gamma}) = \sum_{s,\gamma} A_{s,\gamma}(-\frac{1}{\tau})\delta_{s,\gamma}\,.$$

This last equality is proved using $[x_i, \delta_{s,\gamma}] = 0 = [\mathbf{X}, \delta_{s,\gamma}]$, $[\mathbf{d}, \delta_{s,\gamma}] = s\delta_{s,\gamma}$, and, since $\varphi_\gamma(x|-\frac{1}{\tau}) = \tau^2\varphi_\gamma(\tau x|\tau)$, we get $A_{s,\gamma}(-\frac{1}{\tau}) = \tau^{s+2}A_{s,\gamma}(\tau)$. □

We therefore have:

Theorem 6.3.8. *$\nabla_{n,\Gamma}$ defines a connection on $P_{n,\Gamma}$. Moreover, its image under $\mathbf{G}_n^\Gamma \longrightarrow \bar{\mathbf{G}}_n^\Gamma$ is the pull-back of a connection $\bar\nabla_{n,\Gamma}$ on $\bar P_{n,\Gamma}$.*

Proof. The first part follows from Proposition 6.3.6 above. For the second part, we need to prove the three following identities:

- $\sum_i \bar K_i(\mathbf{z}|\tau) = 0$;
- $\bar K_i(\mathbf{z} + u\sum_j \delta_j|\tau) = \bar K_i(\mathbf{z}|\tau)$, for all i;
- $\bar\Delta(\mathbf{z} + u\sum_j \delta_j|\tau) = \bar\Delta(\mathbf{z}|\tau)$.

The first two equalities have already been proven, and the last one is obvious. □

6.3.4 Flatness

In this paragraph we prove the flatness of $\nabla_{n,\Gamma}$ (and thus of $\bar\nabla_{n,\Gamma}$).

Proposition 6.3.9. *For any $i \in \{1,\ldots,n\}$ we have $[\partial_\tau - \Delta(\mathbf{z}|\tau), \partial_i - K_i(\mathbf{z}|\tau)] = 0$.*

In what follows, we often drop τ from the notation when it does not lead to any confusion.

Proof. Let us first prove that $\partial_\tau K_i(\mathbf{z}) = \partial_i\Delta(\mathbf{z})$. This follows from the identity $\partial_z g_\alpha(x,z) = 2\pi\mathrm{i}\partial_\tau k_\alpha(x,z)$, which is proved as follows (here $\tilde\alpha = (a_0, a)$ is any lift of α):

$$\begin{aligned}
\partial_z g_\alpha(x,z) &= \partial_z\partial_x k_\alpha(x,z) = \partial_z\partial_x\left(e^{-2\pi\mathrm{i}ax}k(x,z-\tilde\alpha) + \frac{e^{-2\pi\mathrm{i}ax}-1}{x}\right)\\
&= e^{-2\pi\mathrm{i}ax}\partial_z\partial_x k(x,z-\tilde\alpha) - 2\pi\mathrm{i}ae^{-2\pi\mathrm{i}ax}\partial_z k(x,z-\tilde\alpha)\\
&= 2\pi\mathrm{i}e^{-2\pi\mathrm{i}ax}\partial_\tau k(x,z-\tilde\alpha) - 2\pi\mathrm{i}ae^{-2\pi\mathrm{i}ax}\partial_z k(x,z-\tilde\alpha)\\
&= 2\pi\mathrm{i}\partial_\tau\left(e^{-2\pi\mathrm{i}ax}k(x,z-\tilde\alpha)\right) = 2\pi\mathrm{i}\partial_\tau k_\alpha(x,z).
\end{aligned}$$

It remains to prove that $[\Delta(\mathbf{z}), K_i(\mathbf{z})] = 0$.

Let us first prove it in the case $n = 2$. Namely, we will prove that

$$[\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\alpha\in\Gamma} g_\alpha(\mathrm{ad}\,x_1, z)(t_{12}^\alpha)\,,\, y_2 + \sum_{\beta\in\Gamma} k_\beta(\mathrm{ad}\,x_1, z)(t_{12}^\beta)] = 0. \tag{6.15}$$

One the one hand,

$$
\begin{aligned}
&[\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\alpha\in\Gamma} g_\alpha(\operatorname{ad} x_1, z)(t_{12}^\alpha)\,,\, y_2] \\
&= [y_1, \sum_{\alpha\in\Gamma} g_\alpha(\operatorname{ad} x_1, z)(t_{12}^\alpha)] - \frac{1}{2}\sum_{\alpha,\gamma\in\Gamma}\sum_{p,q} a_{p,q}^\gamma [\operatorname{ad}^p x_1(t_{12}^{\alpha-\gamma}), \operatorname{ad}^q x_1(t_{12}^\alpha)]\,,
\end{aligned}
$$

where

$$
\frac{\varphi_\gamma(u) - \varphi_{-\gamma}(v)}{u+v} = \sum_{p,q} a_{p,q}^\gamma u^p v^q\,.
$$

On the other hand, we have

$$
[\Delta_0, \sum_\beta k_\beta(\operatorname{ad} x_1, z)(t_{12}^\beta)] = [y_1, \sum_\beta g_\beta(\operatorname{ad} x_1, z)(t_{12}^\beta)] + \sum_{p,q}\sum_{\alpha,\beta\in\Gamma} b_{p,q}^{\alpha,\beta}(z)[\operatorname{ad}^p x_1(t_{12}^\alpha), \operatorname{ad}^q x_1(t_{12}^\beta)]\,,
$$

where the series $\sum_{p,q} b_{p,q}^{\alpha,\beta}(z)u^p v^q$ is given by

$$
\frac{1}{2}\left(\frac{1}{v^2}\left(k_\beta(u+v,z) - k_\beta(u,z) - v\partial_u k_\beta(u,z)\right) - \frac{1}{u^2}\left(k_\alpha(u+v,z) - k_\alpha(v,z) - u\partial_v k_\alpha(v,z)\right)\right)\,.
$$

Therefore the l.h.s. of (6.15) equals

$$
\frac{1}{2}\left(\sum_{p,q}\sum_{\alpha,\beta\in\Gamma} c_{p,q}^{\alpha,\beta}(z)[\operatorname{ad}^p x_1(t_{12}^\alpha), \operatorname{ad}^q x_1(t_{12}^\beta)]\right)\,,
$$

where $\sum_{p,q} c_{p,q}^{\alpha,\beta} u^p v^q(z)$ is given by

$$
\begin{aligned}
\frac{1}{v^2}\left(k_\beta(u+v,z) - k_\beta(u,z) - v g_\beta(u,z)\right) &- \frac{1}{u^2}\left(k_\alpha(u+v,z) - k_\alpha(v,z) - u g_\alpha(v,z)\right) \\
+\frac{\varphi_{\beta-\alpha}(u) - \varphi_{\alpha-\beta}(v)}{u+v} &+ k_\alpha(u+v,z)\varphi_{\alpha-\beta}(v) - k_\beta(u+v,z)\varphi_{\beta-\alpha}(u) \\
&+ k_\beta(u,z)g_\alpha(v,z) - g_\beta(u,z)k_\alpha(v,z)\,,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&\left(g_{\beta-\alpha}(u,z-z') - \frac{1}{u^2}\right)\left(k_\alpha(u+v,z') + \frac{1}{u+v}\right) - \left(g_{\alpha-\beta}(v,z'-z) - \frac{1}{v^2}\right)\left(k_\beta(u+v,z) + \frac{1}{u+v}\right) \\
&+ \left(g_\alpha(v,z') - \frac{1}{v^2}\right)\left(k_\beta(u,z) + \frac{1}{u}\right) - \left(g_\beta(u,z) - \frac{1}{u^2}\right)\left(k_\alpha(v,z') + \frac{1}{v}\right)
\end{aligned}
\tag{6.16}
$$

with $z = z'$. Thus to end the proof of equation (6.15) the following lemma is sufficient:

Lemma 6.3.10. *Expression (6.16) equals zero.*

Proof of the lemma. The case $\alpha = \beta = 0$ follows from an explicit computation. Then we chose lifts $\tilde\alpha = (a_0, a)$ and $\tilde\beta = (b_0, b)$ of α and β, respectively. One has

$$
\begin{aligned}
&k_\alpha(x,z) + 1/x = e^{-2\mathrm{i}\pi a x}\left(k(x, z-\tilde\alpha) + 1/x\right) \quad \text{and} \\
&g_\alpha(x,z) - 1/x^2 = e^{-2\mathrm{i}\pi a x}\left(g(x,z-\tilde\alpha) - 1/x^2\right) - 2\mathrm{i}\pi b\left(k_\alpha(x,z) + 1/x\right)\,.
\end{aligned}
$$

Therefore (6.16) equals

$$-2\mathrm{i}\pi(a-b)\Big(\left(k_\alpha(v,z')+\frac{1}{v}\right)\left(k_\beta(u,z)+\frac{1}{u}\right)+\left(k_{\beta-\alpha}(u,z-z')+\frac{1}{u}\right)\left(k_\alpha(u+v,z')+\frac{1}{u+v}\right)$$
$$+\left(k_{\alpha-\beta}(v,z'-z)-\frac{1}{v}\right)\left(k_\beta(u+v,z)+\frac{1}{u+v}\right)\Big),$$

which vanishes because of (6.3). □

Let us now assume that $n>2$.

Let $\mathfrak{t}_{n,+}^\Gamma\subset\mathfrak{t}_{1,n}^\Gamma$ be the subalgebra generated by x_i, t_{jk}^α $(i,j,k=1,\dots,n,\ j\neq k,\ \alpha\in\Gamma)$.

We have functions $E_{ij}(\mathbf{z})$ with values in $\mathfrak{t}_{n,+}^\Gamma$ defined by $E_{ij}(\mathbf{z})=[\Delta_0,k_{ij}]-[y_i,g_{ij}]$, which decomposes as $e_{ij}(\mathbf{z})+\sum_{k\neq i,j}e_{ijk}(\mathbf{z})$, where $e_{ij}(\mathbf{z})$ takes its values in

$$\mathrm{Span}_{p,q,\alpha,\beta}[(\mathrm{ad}x_i)^p(t_{ij}^\alpha),(\mathrm{ad}x_j)^q(t_{ij}^\beta)]$$

and $e_{ijk}(\mathbf{z})$ takes its values in $\mathrm{Span}_{\alpha,\beta}\,\mathbb{C}[\mathrm{ad}x_i,\mathrm{ad}x_j][t_{ij}^\alpha,t_{jk}^\beta]$. Explicitly,

$$e_{ij}(\mathbf{z})=\sum_{\alpha,\beta}\sum_{p,q}b_{p,q}^{\alpha,\beta}(z_{ij})[\mathrm{ad}^px_i(t_{ij}^\alpha),\mathrm{ad}^qx_i(t_{ij}^\beta)]\,,$$

where $b_{p,q}^{\alpha,\beta}(z)$ is as before, and

$$e_{ijk}(\mathbf{z})=\sum_{\alpha,\beta}\left(\frac{k_\alpha(\mathrm{ad}x_i,z_{ij})-k_\alpha(-\mathrm{ad}x_j,z_{ij})}{(\mathrm{ad}x_i+\mathrm{ad}x_j)^2}-\frac{g_\alpha(-\mathrm{ad}x_j,z_{ij})}{\mathrm{ad}x_i+\mathrm{ad}x_j}\right)[t_{ij}^\alpha,t_{ik}^\beta].$$

On the other hand, we have $Y_{ijk}(\mathbf{z})\in\mathfrak{t}_{n,+}^\Gamma$ defined by $Y_{ijk}(\mathbf{z})=[y_i,g_{jk}]$. It takes its values in $\mathrm{Span}_{\alpha,\beta}\,\mathbb{C}[\mathrm{ad}x_i,\mathrm{ad}x_j][t_{ij}^\alpha,t_{jk}^\beta]$. Explicitly,

$$Y_{ijk}(\mathbf{z})=-\sum_{\alpha,\beta}\frac{g_\beta(\mathrm{ad}x_j,z_{jk})-g_{-\beta}(\mathrm{ad}x_k,-z_{jk})}{\mathrm{ad}x_j+\mathrm{ad}x_k}[t_{ij}^\alpha,t_{jk}^\beta]$$

(remember that $g_\alpha(u,z)=g_{-\alpha}(-u,-z)$). We have

$$\begin{aligned}[\Delta(\mathbf{z}),K_1(\mathbf{z})] &= \sum_{i>1}\left([\Delta_0,k_{1i}]-[y_1,g_{1i}]+[\frac{1}{2}\sum_\alpha\delta_{\varphi_\alpha},k_{1i}]-[g_{1i},k_{1i}]\right)-[\frac{1}{2}\sum_\alpha\delta_{\varphi_\alpha},y_1]\\ &\quad-\sum_{1<i<j}([g_{1i},k_{1j}]+[g_{1j},k_{1i}]+[g_{ij},k_{1i}+k_{1j}])\\ &= \sum_{i>1}\left(e_{12}+[\frac{1}{2}\sum_\alpha\delta_{\varphi_\alpha},k_{12}]-[g_{12},k_{12}]-[\frac{1}{2}\sum_\alpha\delta_{\varphi_\alpha},y_1]\right)_{1i} \qquad (6.17)\\ &\quad+\sum_{1<i<j}(e_{1ij}+e_{1ji}-Y_{1ij}-[g_{ij},k_{1i}+k_{1j}]-[g_{1i},k_{1j}]-[g_{1j},k_{1i}])\end{aligned}$$

where $\{-\}_{1i}$ is the natural morphism $\mathfrak{t}_{1,2}^\Gamma\longrightarrow\mathfrak{t}_{1,n}^\Gamma$, $u_1\mapsto u_1$, $u_2\mapsto u_i$ $(u=x,y)$, $t_{12}^\alpha\mapsto t_{1i}^\alpha$. It is easy to see that the line (6.17) equals $\sum_{i>1}([\Delta(z_{1i}),K_1(z_{1i})])_{1i}$ which is zero as we have seen before (case $n=2$).

Therefore $[\Delta(\mathbf{z}), K_1(\mathbf{z})]$ equals

$$\sum_{1<i<j}\sum_{\alpha,\beta}\Big(\frac{k_\alpha(\mathrm{ad}x_1, z_{1i}) - k_\alpha(-\mathrm{ad}x_i, z_{1i}) - g_\alpha(-\mathrm{ad}x_i, z_{1i})(\mathrm{ad}x_1+\mathrm{ad}x_i)}{(\mathrm{ad}x_1+\mathrm{ad}x_i)^2}[t^\alpha_{1i}, t^\beta_{1j}]$$
$$-\frac{k_\beta(\mathrm{ad}x_1, z_{1j}) - k_\beta(-\mathrm{ad}x_j, z_{1j}) - g_\beta(-\mathrm{ad}x_j, z_{1j})(\mathrm{ad}x_1+\mathrm{ad}x_j)}{(\mathrm{ad}x_1+\mathrm{ad}x_j)^2}[t^\alpha_{1i}, t^\beta_{1j}]$$
$$-\frac{g_{\beta-\alpha}(\mathrm{ad}x_i, z_{ij}) - g_{\alpha-\beta}(\mathrm{ad}x_j, -z_{ij})}{\mathrm{ad}x_i+\mathrm{ad}x_j}[t^\alpha_{1i}, t^\beta_{1j}]$$
$$-\left(k_\alpha(\mathrm{ad}x_1, z_{1i})g_{\beta-\alpha}(-\mathrm{ad}x_j, z_{ij}) - k_\beta(\mathrm{ad}x_1, z_{1j})g_{\beta-\alpha}(\mathrm{ad}x_i, z_{ij})\right)[t^\alpha_{1i}, t^\beta_{1j}]$$
$$-\left(k_\beta(-\mathrm{ad}x_j, z_{1j})g_\alpha(-\mathrm{ad}x_i, z_{1i}) - k_\alpha(-\mathrm{ad}x_i, z_{1i})g_\beta(-\mathrm{ad}x_j, z_{1j})\right)[t^\alpha_{1i}, t^\beta_{1j}]\Big)\,,$$

which is zero because of Lemma 6.3.10. □

We have therefore proved (Proposition 6.1.6 and Proposition 6.3.9 above):

Theorem 6.3.11. *The connection $\nabla_{n,\Gamma}$ is flat, and thus so is $\bar\nabla_{n,\Gamma}$.* □

Let us now show how the universal KZB connexion over moduli spaces coincides with the one defined over configuration spaces.

Remark 6.3.12. *The connection $\nabla_{n,\Gamma}$ defined above is an extension to the twisted moduli space $\mathcal{M}^\Gamma_{1,n}$ of the connection $\nabla_{n,\tau,\Gamma}$ defined over the twisted configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ from Section 6.1.3.*

Indeed, the pull-back of the principal $\mathbf{G}^\Gamma_n$-bundle with flat connection $(\mathcal{P}_{n,\Gamma}, \nabla_{n,\Gamma})$ along the inclusion

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma) \hookrightarrow \mathcal{M}^\Gamma_{1,n}$$

of the fiber at (the class of) τ in $Y(\Gamma)$ admits a reduction of structure group to

$$\exp(\mathfrak{t}^\Gamma_{1,n}) \subset \mathbf{G}^\Gamma_n$$

as we will now explain.

Let us first pull-back the principal $\mathbf{G}^\Gamma_n$-bundle with flat connection $(\mathcal{P}_{n,\Gamma}, \nabla_{n,\Gamma})$ along the projection

$$C^\Gamma(n) := \big((\mathbb{C}^n\times\mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}\big)/(\mathbb{Z}^n)^2 \twoheadrightarrow \mathcal{M}^\Gamma_{1,n}\,.$$

The resulting flat bundle admits a reduction of structure group to

$$\mathbf{N}^\Gamma_n := \exp(\mathfrak{t}^\Gamma_{1,n} \rtimes \mathfrak{d}^\Gamma_+)^\wedge \rtimes N_+ \subset \mathbf{G}^\Gamma_n\,,$$

where $N_+ \subset \mathrm{SL}_2(\mathbb{C})$ is the connected subgroup with Lie algebra $\mathbb{C}\Delta_0$.

Let us then further pull-back this principal $\mathbf{N}^\Gamma_n$-bundle to the fiber

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma) \hookrightarrow C^\Gamma(n)$$

at $\tau\in\mathfrak{H}$ of the projection $C^\Gamma(n) \longrightarrow \mathfrak{H}$. The resulting flat bundle admits a further restriction of structure group to $\exp(\mathfrak{t}^\Gamma_{1,n}) \subset \mathbf{N}^\Gamma_n$. One easily sees from our explicit formulæthat it coincides with $(P_{\tau,n,\Gamma}, \nabla_{\tau,n,\Gamma})$ constructed in Section 6.1.3.

Similarly, the connection $\bar\nabla_{n,\Gamma}$ is an extension to the twisted moduli space $\bar{\mathcal{M}}^\Gamma_{1,n}$ of the connection $\bar\nabla_{n,\tau,\Gamma}$ defined over the reduced twisted configuration space $\mathrm{C}(E_{\tau,\Gamma}, n, \Gamma)$.

6.3.5 Variations

Let us first consider the unordered variants

$$\mathcal{M}_{1,[n]}^{\Gamma} := \mathcal{M}_{1,n}^{\Gamma}/\mathfrak{S}_n \quad \text{and} \quad \bar{\mathcal{M}}_{1,[n]}^{\Gamma} := \bar{\mathcal{M}}_{1,n}^{\Gamma}/\mathfrak{S}_n\,,$$

where, as before, the action of $\mathfrak{S}_n$ is again by permutation on $\mathbb{C}^n$.

Proposition 6.3.13. *1. There exists a unique principal $\mathbf{G}_n^{\Gamma} \rtimes \mathfrak{S}_n$-bundle $\mathcal{P}_{[n],\Gamma}$ over $\mathcal{M}_{1,[n]}^{\Gamma}$, such that a section over $U \subset \mathcal{M}_{1,[n]}^{\Gamma}$ is a function*

$$f : \tilde{\pi}^{-1}(U) \longrightarrow \mathbf{G}_n^{\Gamma} \rtimes \mathfrak{S}_n$$

satisfying the conditions of Proposition 6.3.3 as well as $f(\sigma\mathbf{z}|\tau) = \sigma f(\mathbf{z}|\tau)$ for $\sigma \in \mathfrak{S}_n$ (here $\tilde{\pi} : (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma} \longrightarrow \mathcal{M}_{1,[n]}^{\Gamma}$ is the canonical projection).
2. There exists a unique flat connection $\nabla_{[n],\Gamma}$ on $\mathcal{P}_{[n],\Gamma}$, whose pull-back to $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}$ is the connection

$$\mathrm{d} - \Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$$

on the trivial $\mathbf{G}_n^{\Gamma} \rtimes \mathfrak{S}_n$-bundle.
3. The image of $(\mathcal{P}_{[n],\Gamma}, \nabla_{[n],\Gamma})$ under $\mathbf{G}_n^{\Gamma} \rtimes \mathfrak{S}_n \longrightarrow \bar{\mathbf{G}}_n^{\Gamma} \rtimes \mathfrak{S}_n$ is the pull-back of a flat principal $\bar{\mathbf{G}}_n^{\Gamma} \rtimes \mathfrak{S}_n$-bundle $(\bar{\mathcal{P}}_{[n],\Gamma}, \bar{\nabla}_{[n],\Gamma})$ on $\bar{\mathcal{M}}_{1,[n]}^{\Gamma}$.

Proof. For the proof of the first point, one easily checks that $\sigma c_{\tilde{g}}(\mathbf{z}|\tau)\sigma^{-1} = c_{\sigma\tilde{g}\sigma^{-1}}(\sigma^{-1}\mathbf{z})$, where $\tilde{g} \in (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^{\Gamma}(\mathbb{Z})$, $\sigma \in \mathfrak{S}_n$. It follows that there is a unique cocycle $c_{(\tilde{g},\sigma)} : \mathbb{C}^n \times \mathfrak{H} \longrightarrow \bar{\mathbf{G}}_n^{\Gamma} \rtimes \mathfrak{S}_n$ such that $c_{(\tilde{g},1)} = c_{\tilde{g}}$ and $c_{(1,\sigma)}(\mathbf{z}|\tau) = \sigma$.

For the proof of the second point, taking into account Theorem 6.3.11, one only has to show that this connection is $\mathfrak{S}_n$-equivariant. We have already mentioned that $\sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ is equivariant, and $\bar{\Delta}(\mathbf{z}|\tau)$ is also checked to be so.

The third point is obvious. □

For every (class of) τ in $Y(\Gamma)$, one has an action of Γ^n on the fiber $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ at τ of $\mathcal{M}_{1,n}^{\Gamma} \twoheadrightarrow Y(\Gamma)$, resp. an action of Γ^n/Γ on the fiber $\mathrm{C}(E_{\tau,\Gamma}, n, \Gamma)$ at τ of $\bar{\mathcal{M}}_{1,n}^{\Gamma} \twoheadrightarrow Y(\Gamma)$. Recall that

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)/\Gamma^n = \mathrm{Conf}(E_{\tau,\Gamma}, n) \quad \text{and} \quad \mathrm{C}(E_{\tau,\Gamma}, n, \Gamma)/(\Gamma^n/\Gamma) = \mathrm{C}(E_{\tau,\Gamma}, n)\,.$$

This action depends holomorphically of τ, so that we have an action of Γ^n on $\mathcal{M}_{1,n}^{\Gamma}$, resp. an action of Γ^n/Γ on $\bar{\mathcal{M}}_{1,n}^{\Gamma}$.

Proposition 6.3.14. *1. There exists a unique principal $\mathbf{G}_n^{\Gamma} \rtimes \Gamma^n$-bundle over $\mathcal{M}_{1,n}^{\Gamma}/\Gamma^n$, such that a section over $U \subset \mathcal{M}_{1,n}^{\Gamma}/\Gamma^n$ is a function*

$$f : \tilde{\pi}^{-1}(U) \longrightarrow \mathbf{G}_n^{\Gamma} \rtimes \Gamma^n$$

satisfying the following conditions:

$$f(\mathbf{z}+\frac{\delta_i}{M}|\tau) = (\bar{1},\bar{0})_i f(\mathbf{z}|\tau),$$
$$f(\mathbf{z}+\tau\frac{\delta_i}{N}|\tau) = e^{\frac{-2\pi \mathrm{i} x_i}{N}}(\bar{0},\bar{1})_i f(\mathbf{z}|\tau),$$
$$f(\mathbf{z},\tau+1) = f(\mathbf{z}|\tau),$$
$$f(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) = \tau^{\mathbf{d}} e^{\frac{2\pi \mathrm{i}}{\tau}(\mathbf{X}+\sum_i z_i x_i)} f(\mathbf{z}|\tau).$$

Here, $\tilde{\pi} : (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma} \longrightarrow \mathcal{M}^{\Gamma}_{1,n}/\Gamma^n$ *is the canonical projection.*
2. There exists a unique flat connection on this bundle whose pull-back to $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}$ *is the connection*

$$\mathrm{d} - \Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$$

on the trivial $\mathbf{G}^{\Gamma}_n \rtimes \Gamma^n$*-bundle.*
3. The image of the above flat bundle under $\mathbf{G}^{\Gamma}_n \rtimes \Gamma^n \longrightarrow \bar{\mathbf{G}}^{\Gamma}_n \rtimes (\Gamma^n/\Gamma)$ *is the pull-back of a flat principal* $\bar{\mathbf{G}}^{\Gamma}_n \rtimes (\Gamma^n/\Gamma)$*-bundle on* $\bar{\mathcal{M}}^{\Gamma}_{1,n}/(\Gamma^n/\Gamma)$.

Proof. The first assertion is left to the reader. Assertion 3 is evident. Let us prove assertion 2. By Proposition 6.1.5, we know that the K_i satisfy

(e) $K_i(\mathbf{z}+\frac{\delta_j}{M}|\tau) = \theta((\bar{1},\bar{0})_j)K_i(\mathbf{z}|\tau)$,
(f) $K_i(\mathbf{z}+\frac{\tau\delta_j}{N}|\tau) = \theta((\bar{0},\bar{1})_j)e^{\frac{-2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)}K_i(\mathbf{z}|\tau)$.

The fact that $\Delta(\mathbf{z}+\frac{\delta_j}{M}|\tau) = \theta((\bar{1},\bar{0})_j)\Delta(\mathbf{z}|\tau)$ is immediate. Thus, it remains to show that $\Delta(\mathbf{z}+\frac{\tau\delta_j}{N}|\tau) = e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}}\theta((\bar{0},\bar{1})_j)(\Delta(\mathbf{z}|\tau) - K_j(\mathbf{z}|\tau))$ which is proved in Lemma 6.3.15 below. □

Lemma 6.3.15. *We have*

$$\Delta(\mathbf{z}+\frac{\tau\delta_j}{N}|\tau) = e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}}\theta((\bar{0},\bar{1})_j)(\Delta(\mathbf{z}|\tau) - K_j(\mathbf{z}|\tau)). \tag{6.18}$$

Proof. On the one hand, we have

$$-2\pi\mathrm{i}\Delta(\mathbf{z}+\frac{\tau\delta_j}{N}) = \Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\substack{k<l\\ k,l\neq j}} g_{kl}(z_{kl}) - \sum_{\substack{k:k\neq j\\ \alpha\in\Gamma}} g_\alpha(\mathrm{ad}x_j, z_{jk}+\frac{\tau}{N})(t^\alpha_{jk}).$$

On the other hand, as

$$\begin{aligned} e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}}(\Delta_0) &= (1-(1-e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}})(\Delta_0) = (\Delta_0) + \frac{1-e^{\frac{-2\pi\mathrm{i}\mathrm{ad}x_j}{N}}}{\mathrm{ad}x_j}(y_j) \\ &= \frac{e^{\frac{-2\pi\mathrm{i}\mathrm{ad}x_j}{N}}-1}{(\mathrm{ad}x_j)^2}\left(\sum_{\alpha\in\Gamma}\sum_{k:k\neq j} t^\alpha_{jk}\right) \end{aligned}$$

and the $\delta_{s,\gamma}$ commute with the x_j, we compute

$$2\pi\mathrm{i}\left(K_j(\mathbf{z}+\frac{\tau}{N}\delta_j|\tau) - e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}}\theta((\bar{0},\bar{1})_j)\Delta(\mathbf{z}|\tau)\right)$$

$$= 2\pi\mathrm{i}\left(\theta((\bar{0},\overline{-1})_j)K_j(\mathbf{z}+\frac{\tau}{N}\delta_j|\tau) - e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}}\Delta(\mathbf{z}|\tau)\right)$$

$$= \quad 2\pi\mathrm{i}\theta((\bar{0},\overline{-1})_j)\left(\sum_{k:k\neq j} k_\alpha(\mathrm{ad}x_j, z_{jk}+\frac{\tau}{N}) - y_j\right) + \Delta_0 + \frac{1-e^{\frac{-2\pi\mathrm{i}\mathrm{ad}x_j}{N}}}{\mathrm{ad}x_j}(y_j)$$
$$+\frac{1}{2}\sum_{\substack{s\geq 0,\\ \gamma\in\Gamma}} A_{s,\gamma}\delta_{s,\gamma} - e^{\frac{-2\pi\mathrm{i}\mathrm{ad}x_j}{N}}\sum_{k<l} g_{kl}(z_{kl}).$$

Next, by combining

$$K_{ij}(z-\frac{\tau}{N}) = e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_i)}\theta((\bar{0},\bar{-1})_i)(K_{ij}(z)) + \theta((\bar{0},\bar{-1})_i)(\sum_{\alpha\in\Gamma}\frac{e^{-2\pi\mathrm{i}\mathrm{ad}x_i}-1}{\mathrm{ad}x_i}(t_{ij}^\alpha)),$$

and equations

$$g_\alpha(x,z) - 1/x^2 = e^{-2\mathrm{i}\pi a x}\left(g(x,z-\tilde{\alpha}) - 1/x^2\right) - 2\mathrm{i}\pi b\left(k_\alpha(x,z)+1/x\right) .$$

We can follow the same lines as in the proof of relation (6.8) to obtain the wanted equation. □

We also leave to the reader the task of combining several variants.

6.4 Realizations

6.4.1 Realizations of $\mathfrak{t}_{1,n}^\Gamma$, $\bar{\mathfrak{t}}_{1,n}^\Gamma$ and $\mathfrak{t}_{n,+}^\Gamma$

Let $\mathfrak{g}$ be a Lie algebra and $t_\mathfrak{g} \in S^2(\mathfrak{g})^\mathfrak{g}$ be nongenerate. Assume that we have a group morphism $\theta : \Gamma \longrightarrow \mathrm{Aut}(\mathfrak{g}, t_\mathfrak{g})$ and set $\mathfrak{l} := \mathfrak{g}^\Gamma$ and $\mathfrak{u} := \oplus_{\chi\in\widehat{\Gamma}-\{0\}}\mathfrak{g}_\chi$, where $\mathfrak{g}_\chi$ is the eigenspace of $\mathfrak{g}$ corresponding to the character $\chi : \Gamma \longrightarrow \mathbb{C}^*$. Then we have $\mathfrak{g} = \mathfrak{l}\oplus\mathfrak{u}$ with $[\mathfrak{l},\mathfrak{u}] \subset \mathfrak{u}$, and $\mathfrak{t} = \mathfrak{t}_\mathfrak{l} + \mathfrak{t}_\mathfrak{u}$ with $\mathfrak{t}_\mathfrak{l} \in S^2(\mathfrak{l})^\mathfrak{l}$ and $\mathfrak{t}_\mathfrak{u} \in S^2(\mathfrak{u})^\mathfrak{l}$. We denote by $(a,b) \mapsto \langle a,b\rangle$ the invariant pairing on $\mathfrak{l}$ corresponding to $t_\mathfrak{l}$ and write $t_\mathfrak{l} = \sum_\nu e_\nu \otimes e_\nu$.

Let $\mathrm{Diff}(\mathfrak{l}^*)$ be the algebra of algebraic differential operators on $\mathfrak{l}^*$. It has generators x_l, ∂_l $(l \in \mathfrak{l})$ and relations $\mathrm{x}_{tl+l'} = t\,\mathrm{x}_l + \mathrm{x}_{l'}$, $\partial_{tl+l'} = t\partial_l + \partial_{l'}$, $[\mathrm{x}_l, \mathrm{x}_{l'}] = 0 = [\partial_l, \partial_{l'}]$ and $[\partial_l, \mathrm{x}_{l'}] = \langle l, l'\rangle$. Moreover, one has a Lie algebra morphism $\mathfrak{l} \longrightarrow \mathrm{Diff}(\mathfrak{l}^*); l \mapsto X_l := \sum_\nu \mathrm{x}_{[l,e_\nu]}\,\partial_{e_\nu}$. We denote by $\mathfrak{l}^{\mathrm{diag}}$ the image of the induced morphism

$$\mathfrak{l} \ni l \mapsto Y_l := X_l \otimes 1 + 1 \otimes \sum_{i=1}^n l^{(i)} \in \mathrm{Diff}(\mathfrak{l}^*) \otimes U(\mathfrak{g})^{\otimes n} ,$$

and define $H_n(\mathfrak{g}, \mathfrak{l}^*)$ as the Hecke algebra of $A_n := \mathrm{Diff}(\mathfrak{l}^*) \otimes U(\mathfrak{g})^{\otimes n}$ with respect to $\mathfrak{l}^{\mathrm{diag}}$. Namely, $H_n(\mathfrak{g}, \mathfrak{l}^*) := (A_n)^\mathfrak{l}/(A_n\mathfrak{l}^{\mathrm{diag}})^\mathfrak{l}$. It acts in an obvious way on $(O_{\mathfrak{l}^*} \otimes (\otimes_{i=1}^n V_i))^\mathfrak{l}$ if $(V_i)_{1\leq i\leq n}$ is a collection of $\mathfrak{g}$-modules.

Let us set $\mathrm{x}_\nu := \mathrm{x}_{e_\nu}$ and $\partial_\nu := \partial_{e_\nu}$, and write $\alpha^{(i)}\cdot$ for the action of $\alpha \in \Gamma$ on the i-th component in $U(\mathfrak{g})^{\otimes n}$.

Finally, recall that the twisted elliptic Kohno-Drinfeld Lie algebra $\mathfrak{t}_{1,n}^\Gamma$ is defined in Definition 4.3.3.

Proposition 6.4.1. *There is a unique Lie algebra morphism* $\rho_{\mathfrak{g}} : \mathfrak{t}_{1,n}^{\Gamma} \longrightarrow H_n(\mathfrak{g}, \mathfrak{l}^*)$ *defined by*

$$x_i \longmapsto M \sum_{\nu} \mathrm{x}_\nu \otimes e_\nu^{(i)},$$
$$y_i \longmapsto -N \sum_{\nu} \partial_\nu \otimes e_\nu^{(i)},$$
$$t_{ij}^{\alpha} \longmapsto 1 \otimes (\alpha^{(1)} \cdot t_{\mathfrak{g}})^{(ij)}.$$

It induces a Lie algebra morphism $\bar{\rho}_{\mathfrak{g}} : \bar{\mathfrak{t}}_{1,n}^{\Gamma} \longrightarrow H_n(\mathfrak{g}, \mathfrak{l}^*)$.

Proof. Let us use the presentation of $\mathfrak{t}_{1,n}^{\Gamma}$ coming from Lemma 4.3.5. The only non trivial check is that the relation $[\sum_j x_j, y_i] = 0$ is preserved. We have

$$\begin{aligned}
\rho_{\mathfrak{g}}\left(\sum_{i=1}^{n} x_i\right) &= M \sum_{\nu} \mathrm{x}_\nu \otimes \sum_{i=1}^{n} e_\nu^{(i)} = M \sum_{\nu} (\mathrm{x}_\nu \otimes 1)\left(1 \otimes \sum_{i=1}^{n} e_\nu^{(i)}\right) \\
&\equiv M \sum_{\nu} (\mathrm{x}_\nu \otimes 1)(Y_\nu - X_\nu \otimes 1) \\
&\equiv M - \sum_{\nu} \mathrm{x}_\nu X_\nu \otimes 1 = M \sum_{\nu_1, \nu_2} \mathrm{x}_{e_{\nu_1}} \mathrm{x}_{[e_{\nu_1}, e_{\nu_2}]} \partial_{\nu_2} \otimes 1 = 0
\end{aligned}$$

as $\mathrm{x}_{e_{\nu_1}}$ commutes with $\mathrm{x}_{[e_{\nu_1}, e_{\nu_2}]}$ and $t_{\mathfrak{l}}$ is invariant. Here the sign $\equiv$ means that both terms define the same equivalence class in $H_n(\mathfrak{g}, \mathfrak{l})$. Thus,

$$[\rho_{\mathfrak{g}}\left(\sum_{j} x_j\right), \rho_{\mathfrak{g}}(y_i)] \equiv [0, \rho_{\mathfrak{g}}(y_i)] = 0.$$

The proof that $[\sum_j y_j, x_i] = 0$ is preserved is a consequence of the fact that $\rho\left(\sum_j y_j\right) = 0$, which was proven in [24, Proposition 6.1]. The fact that this induces a Lie algebra morphism $\bar{\rho}_{\mathfrak{g}} : \bar{\mathfrak{t}}_{1,n}^{\Gamma} \longrightarrow H_n(\mathfrak{g}, \mathfrak{l})$ is then clear. $\square$

Let $\mathfrak{t}_{n,+}^{\Gamma} \subset \mathfrak{t}_{1,n}^{\Gamma}$ be the Lie subalgebra generated by x_i's and t_{jk}^{α}'s. Then the restriction of $\rho_{\mathfrak{g}}$ to $\mathfrak{t}_{n,+}^{\Gamma}$ lifts to a Lie algebra morphism $\mathfrak{t}_{n,+}^{\Gamma} \longrightarrow (O_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}}$. Moreover, $(O_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}}$ is a subalgebra of $H_n(\mathfrak{g}, \mathfrak{l}^*)$ that is a Lie ideal for the commutator and one has a commutative diagram

$$\begin{tikzcd}
\mathfrak{t}_{1,n}^{\Gamma} \times \mathfrak{t}_{n,+}^{\Gamma} \arrow[r, "(u,v)\mapsto[u,v]"] \arrow[d] & \mathfrak{t}_{n,+}^{\Gamma} \arrow[d] \\
H_n(\mathfrak{g}, \mathfrak{l}^*) \times (O_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}} \arrow[r] & (O_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}} .
\end{tikzcd}$$

6.4.2 Realizations of $\mathfrak{t}_{1,n}^{\Gamma} \rtimes \mathfrak{d}^{\Gamma}$ and $\bar{\mathfrak{t}}_{1,n}^{\Gamma} \rtimes \mathfrak{d}^{\Gamma}$

Let us write $t_{\mathfrak{g}} = \sum_u a_u \otimes a_u$.

Proposition 6.4.2. *The Lie algebra morphism $\rho_{\mathfrak{g}}$ (resp. $\bar{\rho}_{\mathfrak{g}}$) of Proposition 6.4.1 extends to a Lie algebra morphism $\mathfrak{t}_{1,n}^{\Gamma} \rtimes \mathfrak{d}^{\Gamma} \longrightarrow H_n(\mathfrak{g}, \mathfrak{l}^*)$ (resp. $\bar{\mathfrak{t}}_{1,n}^{\Gamma} \rtimes \mathfrak{d}^{\Gamma} \longrightarrow H_n(\mathfrak{g}, \mathfrak{l}^*)$) defined by*

$$\begin{aligned}
\mathbf{d} &\longmapsto -\frac{1}{2}(\sum_{\nu} \mathrm{x}_\nu \, \partial_\nu + \partial_\nu \, \mathrm{x}_\nu) \otimes 1, \\
\mathbf{X} &\longmapsto \frac{1}{2}(\sum_{\nu} \mathrm{x}_\nu^2) \otimes 1, \\
\Delta_0 &\longmapsto -\frac{1}{2}(\sum_{\nu} \partial_\nu^2) \otimes 1, \\
\xi_{s,\gamma} &\longmapsto \frac{1}{|\Gamma|} \sum_{\nu_1, \cdots, \nu_s, u} \mathrm{x}_{\nu_1} \cdots \mathrm{x}_{\nu_s} \otimes \sum_{i=1}^{n} \left(\mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u)\right)^{(i)} .
\end{aligned}$$

Here $\odot$ denotes the symmetric product: $A \odot B := AB + BA$.

Proof. Since $t_{\mathfrak{g}}$ is invariant under the commuting actions of Γ and $\mathfrak{l}$ then the relation $\xi_{s,\gamma} = (-1)^s \xi_{s,-\gamma}$ is also preserved. This invariance argument also implies that $[\rho_{\mathfrak{g}}(\xi_{s,\gamma}), \rho_{\mathfrak{g}}(x_i)]$ equals

$$\frac{1}{|\Gamma|} \sum_{\nu_1, \cdots, \nu_s, \nu, u} \mathrm{x}_{\nu_1} \cdots \mathrm{x}_{\nu_s} \, \mathrm{x}_\nu \otimes \sum_{t=1}^{s} (\mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}([e_\nu, e_{\nu_t}]) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u))^{(i)}$$

which is zero since the first and second factor are respectively symmetric and antisymmetric in (ν, ν_t). Let us now prove that the relation $[\xi_{s,\gamma}, t_{ij}^{\alpha}] = [t_{ij}^{\alpha}, (\mathrm{ad} x_i)^s (t_{ij}^{\alpha-\gamma}) + (\mathrm{ad} x_j)^s (t_{ij}^{\alpha+\gamma})]$ is preserved. It is sufficient to do it for $n = 2$:

$$\rho_{\mathfrak{g}}(\xi_{s,\gamma} + (\mathrm{ad} x_1)^s (t_{12}^{\alpha-\gamma}) + (\mathrm{ad} x_2)^s (t_{12}^{\alpha+\gamma})) = \sum_{\nu_1, \cdots, \nu_s} \mathrm{x}_{\nu_1} \cdots \mathrm{x}_{\nu_s} \otimes (\alpha^{(1)} \cdot \Delta(B_{\nu_1, \cdots, \nu_s})) ,$$

where Δ is the standard coproduct of $U\mathfrak{g}$ and $B_{\nu_1, \cdots, \nu_s} := \sum_u \mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u)$; therefore $\rho_{\mathfrak{g}}(\xi_{s,\gamma} + (\mathrm{ad} x_1)^s (t_{12}^{\alpha-\gamma}) + (\mathrm{ad} x_2)^s (t_{12}^{\alpha+\gamma}))$ commutes with $\rho_{\mathfrak{g}}(t_{12}^{\alpha})$. Hence it remains to prove that the relation $[\xi_{s,\gamma}, \frac{y_i}{N}] = \sum_{j: j \neq i} D_{s,\gamma}(\frac{x_i}{M}, \frac{t_{ij}^{\beta}}{|\Gamma|})$ is preserved. For this we compute $[\rho_{\mathfrak{g}}(\xi_{s,\gamma}), \rho_{\mathfrak{g}}(\frac{y_i}{N})]$: it equals

$$\begin{aligned}
&\frac{1}{|\Gamma|} \sum_{\substack{\nu_1, \cdots, \nu_s \\ \nu, u}} \Big(\sum_{j=1}^{n} [\partial_\nu, \mathrm{x}_{\nu_1} \cdots \mathrm{x}_{\nu_s}] \otimes e_\nu^{(i)} \big(\mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u)\big)^{(j)} \\
&+ \mathrm{x}_{\nu_1} \cdots \mathrm{x}_{\nu_s} \, \partial_\nu \otimes [e_\nu, \mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u)]^{(i)} \Big) \\
&= \frac{1}{|\Gamma|} \sum_{l=1}^{s} \sum_{\nu_1, \ldots, \nu_s, \nu} \mathrm{x}_{\nu_1} \cdots \check{\mathrm{x}}_{\nu_l} \cdots \mathrm{x}_{\nu_s} \otimes \sum_{j=1}^{n} \Big(e_\nu^{(i)} \big(\mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u)\big)^{(j)} - (i \leftrightarrow j) \Big) .
\end{aligned}$$

The term corresponding to $j = i$ is the linear map $S^{s-1}(\mathfrak{l}) \longrightarrow U(\mathfrak{g})^{\otimes n}$ such that for $x \in \mathfrak{l}$

$$x^{s-1} \longmapsto \frac{1}{|\Gamma|} \sum_{\substack{p+q=s-1 \\ \nu, u}} [e_\nu, \mathrm{ad}(x)^p \mathrm{ad}(e_\nu) \mathrm{ad}(x)^q (a_u) \odot (\gamma \cdot a_u)]^{(i)} .$$

Using $\mathfrak{l}$-invariance of $\sum_u a_u \odot (\gamma \cdot a_u)$ one obtains that this last expression equals

$$= \frac{1}{|\Gamma|} \sum_{\substack{p+q+r=s-1 \\ \nu, u}} \big(\mathrm{ad}(x)^p \mathrm{ad}([e_\nu, x]) \mathrm{ad}(x)^q \mathrm{ad}(e_\nu) (\mathrm{ad} x)^r (a_u) \odot (\gamma \cdot a_u)$$

$$+\mathrm{ad}(x)^p\mathrm{ad}(e_\nu)\mathrm{ad}(x)^q\mathrm{ad}([e_\nu,x])\mathrm{ad}(x)^r(a_u)\odot(\gamma\cdot a_u)\Big)^{(i)},$$

which is zero from the $\mathfrak{l}$-invariance of $t_\mathfrak{l}=\sum_\nu e_\nu\otimes e_\nu$. The term corresponding to $j\neq i$ is the linear map $S^{s-1}(\mathfrak{l})\longrightarrow U(\mathfrak{g})^{\otimes n}$ such that for $x\in\mathfrak{l}$

$$x^{s-1}\longmapsto\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}\left(\mathrm{ad}(x)^p\mathrm{ad}(e_\nu)\mathrm{ad}(x)^q(a_u)\odot(\gamma\cdot a_u)\right)^{(j)}e_\nu^{(i)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}\left(\mathrm{ad}(x)^p([e_\nu,a_u])\odot(-\mathrm{ad}(x))^q(\gamma\cdot a_u)\right)^{(j)}e_\nu^{(i)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}(-1)^q\left(\mathrm{ad}(x)^p([e_\nu,a_u])\odot(\mathrm{ad}(x))^q(\gamma\cdot a_u)\right)^{(j)}e_\nu^{(i)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}(-1)^q\left(\mathrm{ad}(x)^p([e_\nu,a_u])\odot(\mathrm{ad}(x))^q(\gamma\cdot a_u)\right)^{(j)}e_\nu^{(i)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|^2}\sum_{\beta\in\Gamma}\sum_{\substack{p+q=s-1\\ v,u}}(-1)^q\left(\mathrm{ad}(x)^p([a_v,a_u])\odot(\mathrm{ad}(x))^q(\gamma\cdot a_u)\right)^{(j)}(\beta\cdot a_v)^{(i)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|^2}\sum_{\beta\in\Gamma}\sum_{p+q=s-1}(-1)^q\sum_{\nu,u}\left(\mathrm{ad}(x)^p(a_v)\odot\mathrm{ad}(x)^q(\gamma\cdot a_u)\right)^{(i)}(\beta\cdot[a_u,a_v])^{(j)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|^2}\sum_{\beta\in\Gamma}\sum_{p+q=s-1}(-1)^q\sum_{\nu,u}\left(\mathrm{ad}(x)^p(\beta\cdot a_v)\odot\mathrm{ad}(x)^q((\beta+\gamma)\cdot a_u)\right)^{(i)}[a_u,a_v]^{(j)}-(i\leftrightarrow j)$$

$$=\frac{1}{|\Gamma|^2}\sum_{\beta\in\Gamma}\sum_{p+q=s-1}(-1)^q\sum_{\nu,u}\left(\mathrm{ad}(x)^p((\beta-\gamma)\cdot a_v)\odot\mathrm{ad}(x)^q((\beta)\cdot a_u)\right)^{(i)}[a_u,a_v]^{(j)}-(i\leftrightarrow j)$$

which coincides with the image of

$$D_{s,\gamma}\left(\frac{x_i}{M},\frac{t_{ij}^\beta}{|\Gamma|}\right)=\sum_{p+q=s-1}\sum_{\beta\in\Gamma}\left[\left(\mathrm{ad}\frac{x_i}{M}\right)^p\left(\frac{t_{ij}^\beta}{|\Gamma|}\right),\left(-\mathrm{ad}\frac{x_i}{M}\right)^q\left(\frac{t_{ij}^\beta}{|\Gamma|}\right)\right]$$

under $\rho_\mathfrak{g}$. In conclusion we get the relation

$$\rho_\mathfrak{g}\left(\left[\xi_{s,\gamma},\frac{y_i}{N}\right]\right)=\left[\rho_\mathfrak{g}(\xi_{s,\gamma}),\rho_\mathfrak{g}\left(\frac{y_i}{N}\right)\right].$$

A direct computation shows that the commutation relations of $[\mathbf{X},\xi_{s,\gamma}]=0$, $[\mathbf{d},\xi_{s,\gamma}]=s\xi_{s,\gamma}$ and $\mathrm{ad}^{s+1}(\Delta_0)(\xi_{s,\gamma})=0$ are preserved, which finishes the proof. □

6.4.3 Reductions

Assume that $\mathfrak{l}$ is finite dimensional and we have a reductive decomposition $\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{m}$, i.e. $\mathfrak{h}\subset\mathfrak{l}$ is a subalgebra and $\mathfrak{m}\subset\mathfrak{l}$ is a vector subspace such that $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$. We also assume that $t_\mathfrak{l}=t_\mathfrak{h}+t_\mathfrak{m}$ with $t_\mathfrak{h}=\sum_{\bar\nu}e_{\bar\nu}\otimes e_{\bar\nu}\in S^2(\mathfrak{h})^\mathfrak{h}$ and $t_\mathfrak{m}\in S^2(\mathfrak{m})^\mathfrak{h}$, and that for a generic $h\in\mathfrak{h}$, $\mathrm{ad}(h)_{|\mathfrak{m}}\in\mathrm{End}(\mathfrak{m})$ is invertible. This last condition means that

$$P(\lambda):=\det(\mathrm{ad}(\lambda^\vee))_{|\mathfrak{m}})\in S^{\dim(\mathfrak{m})}(\mathfrak{h})$$

is nonzero, where $\lambda^\vee := (\lambda \otimes \mathrm{id})(t_{\mathfrak{h}})$ for any $\lambda \in \mathfrak{h}^*$.

We now define $H_n(\mathfrak{g}, \mathfrak{h}^*_{reg})$. As in the previous paragraph, $\mathrm{Diff}(\mathfrak{h}^*)$ has generators $\bar{\mathrm{x}}_h$, $\bar{\partial}_h$ ($h \in \mathfrak{h}$) and relations

$$\begin{aligned}\bar{\mathrm{x}}_{th+h'} &= t\bar{\mathrm{x}}_h + \bar{\mathrm{x}}_{h'},\\ \bar{\partial}_{th+h'} &= t\bar{\partial}_h + \bar{\partial}_{h'},\\ [\bar{\mathrm{x}}_h, \bar{\mathrm{x}}_{h'}] = 0 &= [\bar{\partial}_h, \bar{\partial}_{h'}],\\ [\bar{\partial}_h, \bar{\mathrm{x}}_{h'}] &= \langle h, h' \rangle,\end{aligned}$$

and $\mathrm{Diff}(\mathfrak{h}^*_{reg}) = \mathrm{Diff}(\mathfrak{h}^*)[\frac{1}{P}]$ with $[\bar{\partial}_l, \frac{1}{P}] = -\frac{[\bar{\partial}_l, P]}{P^2}$. One has a Lie algebra morphism

$$\mathfrak{h} \longrightarrow \mathrm{Diff}(\mathfrak{h}^*); h \longmapsto \bar{X}_h := \sum_{\bar{\nu}} \mathrm{x}_{[h, e_{\bar{\nu}}]}\, \partial_{e_{\bar{\nu}}}.$$

We denote by $\mathfrak{h}^{\mathrm{diag}}$ the image of the map

$$\mathfrak{h} \ni h \longmapsto \bar{Y}_h := \bar{X}_h + \sum_{i=1}^n l^{(i)} \in \mathrm{Diff}(\mathfrak{h}^*_{reg}) \otimes U(\mathfrak{g})^{\otimes n} =: B_n,$$

and define $H_n(\mathfrak{g}, \mathfrak{h}^*_{reg})$ as the Hecke algebra of B_n with respect to $\mathfrak{h}^{\mathrm{diag}}$:

$$H_n(\mathfrak{g}, \mathfrak{h}^*_{reg}) := (B_n)^{\mathfrak{h}} / (B_n \mathfrak{h}^{\mathrm{diag}})^{\mathfrak{h}}.$$

It acts in an obvious way on $(O_{\mathfrak{h}^*_{reg}} \otimes (\otimes_{i=1}^n V_i))^{\mathfrak{h}}$ if $(V_i)_{1 \le i \le n}$ is a collection of $\mathfrak{g}$-modules. Finally, let us set, for $\lambda \in \mathfrak{h}^*$,

$$r(\lambda) := (\mathrm{id} \otimes (\mathrm{ad}\, \lambda^\vee)^{-1}_{|\mathfrak{m}})(t_{\mathfrak{m}}).$$

Then, following [38], $r : \mathfrak{h}^*_{\mathrm{reg}} \longrightarrow \wedge^2(\mathfrak{m})$ is an $\mathfrak{h}$-equivariant map satisfying the classical dynamical Yang-Baxter equation (CDYBE)

$$\sum_{\bar{\nu}} e^{(1)}_{\bar{\nu}} \partial_{\bar{\nu}} r^{(23)} + [r^{(12)}, r^{(13)}] + c.p.(1,2,3) = 0\,,$$

and we write $r = \sum_\delta a_\delta \otimes b_\delta \otimes \ell_\delta \in (\mathfrak{m}^{\otimes 2} \otimes S(\mathfrak{h})[1/P])^{\mathfrak{h}}$.

Proposition 6.4.3. *There is a unique Lie algebra morphism $\rho_{\mathfrak{g},\mathfrak{h}} : \mathfrak{t}^\Gamma_{1,n} \longrightarrow H_n(\mathfrak{g}, \mathfrak{h}^*_{reg})$ given by*

$$\begin{aligned} x_i &\longmapsto M \sum_{\bar{\nu}} \bar{\mathrm{x}}_{\bar{\nu}} \otimes h^{(i)}_{\bar{\nu}},\\ y_i &\longmapsto -N \sum_{\bar{\nu}} \bar{\partial}_{\bar{\nu}} \otimes h^{(i)}_{\bar{\nu}} + \sum_j \sum_\delta \ell_\delta \otimes a^{(i)}_\delta b^{(j)}_\delta,\\ t^\alpha_{ij} &\longmapsto 1 \otimes (\alpha^{(1)} \cdot t_{\mathfrak{g}})^{(ij)}.\end{aligned}$$

Proof. First of all, the images of the above elements are all $\mathfrak{h}$-invariant. As in [24], we will imply summation over repeated indices, and adopt the following conventions: $\bar{\partial}_{e_{\bar{\nu}}} = \bar{\partial}_{\bar{\nu}}$, $\bar{\mathrm{x}}_{e_{\bar{\nu}}} = \bar{\mathrm{x}}_{\bar{\nu}}$, and $1 \otimes -$'s and $- \otimes 1$'s may be dropped from the notation.

In particular, $\rho_{\mathfrak{g},\mathfrak{h}}(\bar{x}_i) = h^{(i)}_{\bar\nu}\bar{\mathrm{x}}_{\bar\nu}$, $\rho_{\mathfrak{g},\mathfrak{h}}(\bar{y}_i) = -h^{(i)}_{\nu}\bar\partial_\nu + \sum_{j=1}^n r(\lambda)^{(ij)}$ (here, for $x\otimes y \in \mathfrak{g}^{\otimes 2}$, $(x\otimes y)^{(ii)} := x^{(i)}y^{(i)}$).

We will use the same presentation of $\bar{\mathfrak{t}}^{\Gamma}_{1,n}$ as in Lemma 4.3.5. The relations $[\bar{x}_i,\bar{x}_j]=0$ and $\bar{t}^{\alpha}_{ij} = \bar{t}^{-\alpha}_{ji}$ are obviously preserved.

Let us check that $[\bar{x}_i,\bar{y}_j] = \sum \bar{t}^{\alpha}_{ij}$ is preserved. We have for $i\neq j$,

$$\begin{aligned}\frac{1}{MN}[\rho_{\mathfrak{g},\mathfrak{h}}(\bar{x}_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar{y}_j)] &= -\sum_{\bar\nu_1,\bar\nu_2}[\bar{\mathrm{x}}_{\bar\nu_1},\partial_{\bar\nu_2}]h^{(i)}_{\bar\nu_1}h^{(j)}_{\bar\nu_2} + \sum \bar\nu,\delta,k\bar{\mathrm{x}}_{\bar\nu}[h^{(i)}_{\bar\nu},\ell_\delta\otimes a^{(j)}_{\delta}b^{(k)}_{\delta}] \\ &= t^{(ij)}_{\mathfrak{h}} + t^{(ij)}_{\mathfrak{m}} = t^{(ij)}_{\mathfrak{l}} = \frac{1}{MN}\sum_{\alpha\in\Gamma}\alpha^{(i)}\cdot t^{(ij)}_{\mathfrak{g}}\end{aligned}$$

by the same argument as in Proposition 6.4.1.

Let us check that $\sum_i \bar{x}_i = \sum_i \bar{y}_i = 0$ are preserved. We have $\sum_i \rho_{\mathfrak{g},\mathfrak{h}}(\bar{x}_i) = 0$ and $\sum_i \rho_{\mathfrak{g},\mathfrak{h}}(\bar{y}_i) = \sum_{\bar\nu,i} h^{(i)}_{\bar\nu}\partial_{\bar\nu}$ (by the antisymmetry of r), which equals zero as in as in Proposition 6.4.1.

The fact that the relation $[\bar{y}_i,\bar{y}_j]=0$ is satisfied for $i\neq j$ is a consequence of the dynamical Yang-Baxter equation (this follows from the exact same argument as in the proof of [24, Proposition 63]).

Next, $[\bar{x}_i,\bar{t}^{\alpha}_{jk}]=0$ is preserved (i,j,k distinct). Indeed, we have

$$[\rho_{\mathfrak{g},\mathfrak{h}}(\bar{x}_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar{t}^{\alpha}_{jk})] = \sum_{\bar\nu}\bar{\mathrm{x}}_{\bar\nu}[h^{(i)}_{\bar\nu},\alpha^{(i)}\cdot t^{(jk)}_{\mathfrak{g}}] = 0\,.$$

Finally $[\bar{y}_i,\bar{t}^{\alpha}_{jk}]=0$ is preserved (i,j,k distinct): we have

$$\begin{aligned}[\rho_{\mathfrak{g},\mathfrak{h}}(\bar{y}_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar{t}^{\alpha}_{jk})] =& [-\sum_{\bar\nu}h^{(i)}_{\bar\nu}\bar\partial_{\bar\nu} + \sum_l r^{(il)},\alpha^{(j)}\cdot t^{(jk)}_{\mathfrak{g}})] \\ =& [r(\lambda)^{(ij)} + r(\lambda)^{(ik)},\alpha^{(j)}\cdot t^{(jk)}_{\mathfrak{g}})] = 0\,,\end{aligned}$$

where the last equality follows the the $\mathfrak{g}$-invariance of $t_{\mathfrak{g}}$. □

Remark 6.4.4. *We expect that there is Lie algebra morphism* $\mathrm{red}_{\mathfrak{l},\mathfrak{h}} : H_n(\mathfrak{g},\mathfrak{l}^*) \longrightarrow H_n(\mathfrak{g},\mathfrak{h}^*_{reg})$ *such that the following diagram commutes*

$$\begin{tikzcd} \mathfrak{t}^{\Gamma}_{1,n} \arrow[r, "\rho_{\mathfrak{g}}"] \arrow[dr, "\rho_{\mathfrak{g},\mathfrak{h}}"'] & H_n(\mathfrak{g},\mathfrak{l}^*) \arrow[d, "\mathrm{red}_{\mathfrak{l},\mathfrak{h}}"] \\ & H_n(\mathfrak{g},\mathfrak{h}^*_{reg}) \end{tikzcd}$$

6.4.4 Elliptic dynamical r-matrix systems as realizations of the universal Γ-KZB system on twisted configuration spaces

Let $K(z)$ be a meromorphic function on $\mathbb{C}$ with values in the subalgebra $\widehat{\mathfrak{t}^{\Gamma}_{2,+}} \subset \widehat{\mathfrak{t}^{\Gamma}_{1,2}}$ generated by x_1, x_2, t^{α}_{12} ($\alpha\in\Gamma$), such that $K(-z) = -K(z)^{2,1}$ and satisfying the universal CDYBE with a spectral parameter

$$-[y_1,K(z_{23})^{2,3}] + [K(z_{12})^{1,2},K(z_{13})^{1,3}] + c.p.(1,2,3) = 0\,.$$

On the one hand, it follows from §6.4.1 that the image $r(\mathrm{x},z) := \rho_{\mathfrak{g}}(K(z))$ of $K(z)$ under $\rho_{\mathfrak{g}} : \widehat{\mathfrak{t}_{2,+}^{\Gamma}} \longrightarrow (\hat{O}_{\mathfrak{l}^*} \otimes \mathfrak{g}^{\otimes 2})^{\mathfrak{l}}$ is a dynamical r-matrix[2] with spectral parameter, i.e. a solution of the CDYBE with a spectral parameter for the pair $(\mathfrak{l},\mathfrak{g})$

$$\sum_{\nu} e_{\nu}^{(1)} \partial_{\nu} r(\mathrm{x}, z_{23})^{(23)} + [r(\mathrm{x}, z_{12})^{(12)}, r(\mathrm{x}, z_{13})^{(13)}] + c.p.(1,2,3) = 0\,,$$

which satisfies $r(\mathrm{x},-z) = -r(\mathrm{x},z)^{(21)}$. On the other hand, the image of $K(z)$ under $\rho_{\mathfrak{g},\mathfrak{h}} : \widehat{\mathfrak{t}_{2,+}^{\Gamma}} \longrightarrow (\hat{O}_{h^*_{reg}} \otimes \mathfrak{g}^{\otimes 2})^{\mathfrak{h}}$ is precisely equal to the restriction $\rho_{\mathfrak{g}}(K(z))|_{\mathfrak{h}^*} \in (\hat{O}_{h^*_{reg}} \otimes \mathfrak{g}^{\otimes 2})^{\mathfrak{h}}$ of $\rho_{\mathfrak{g}}(K(z))$ to $\mathfrak{h}^*$. Then applying [38, Proposition 0.1], we conclude that

$$\tilde{r}(\bar{\mathrm{x}}, z) := \rho_{\mathfrak{g},\mathfrak{h}}(K(z)) + r(\lambda)$$

is a solution of the CDYBE with spectral parameter for $(\mathfrak{h},\mathfrak{g})$:

$$\sum_{\bar{\nu}} e_{\bar{\nu}}^{(1)} \partial_{\bar{\nu}} \tilde{r}(\bar{\mathrm{x}}, z_{23})^{(23)} + [\tilde{r}(\bar{\mathrm{x}}, z_{12})^{(12)}, \tilde{r}(\bar{\mathrm{x}}, z_{13})^{(13)}] + c.p.(1,2,3) = 0\,.$$

Then for any n-tuple $\underline{V} = (V_1, \ldots, V_n)$ of $\mathfrak{g}$-modules one has a flat connection $\nabla^{(\underline{V})}_{\tau,n,\Gamma}$ on the trivial vector bundle over $\mathbb{C}^n - \mathrm{Diag}_{\tau,n\Gamma}$ with fiber $(O_{\mathfrak{h}^*_{reg}} \otimes (\otimes_i V_i))^{\mathfrak{h}}$, defined by the following compatible system of first order differential equations:

$$\partial_{z_i} F(\bar{\mathrm{x}}, \mathbf{z}) = \sum_{\bar{\nu}} e_{\bar{\nu}}^{(i)} \cdot \bar{\partial}_{\bar{\nu}} F(\bar{\mathrm{x}}, \mathbf{z}) + \sum_{j:j\neq i} \tilde{r}^{(ij)}(\bar{\mathrm{x}}, z_{ij}) \cdot F(\bar{\mathrm{x}}, \mathbf{z})\,. \tag{6.19}$$

Here $\mathbf{z} \mapsto F(\bar{\mathrm{x}}, \mathbf{z})$ is a function with values in $(O_{\mathfrak{h}^*_{reg}} \otimes (\otimes_i V_i))^{\mathfrak{h}}$.

Starting from $K(z) = K_{12}(z)$ as in §6.1.3, it would be interesting to know if one can recover (up to gauge equivalence), using the above realization morphisms, the generalization of Felder's elliptic dynamical r-matrices [44] constructed in [42, 43].

Letu develop a bit more this idea. Set $K(z) = K_{12}(z)$ like in §6.1.3 and focus on the case when $\mathfrak{g}$ is a simple Lie algebra. Let us introduce some standard notation: Δ^+ is the set of positive roots, $(h_i)_i$ is an orthonomal basis of $\mathfrak{h} = \mathfrak{g}_0$, and for any positive root α one has $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ and $\mathfrak{g}_{-\alpha} = \mathbb{C}f_\alpha$ with $\langle e_\alpha, f_\alpha\rangle = 1$. Then one has

$$t_{\mathfrak{g}} = \frac{1}{2}\sum_i h_i \otimes h_i + \sum_{\alpha\in\Delta^+} (e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha)\,.$$

Assume that $\theta(\bar{1},\bar{0}) = \mathrm{Ad}(e^{2\pi\mathrm{i}\rho/\kappa})$, where ρ is the half-sum of positive roots and κ the dual Coxeter number of $\mathfrak{g}$. Observe that this automorphisms can be defined alternatively by $h_i \mapsto h_i$, $e_\alpha \mapsto e^{2\pi\mathrm{i}|\alpha|/\kappa} e_\alpha$ and $f_\alpha \mapsto e^{-2\pi\mathrm{i}|\alpha|/\kappa} f_\alpha$ (here $|\alpha|$ is the lenght of the root α). Therefore $\mathfrak{l} \subset \mathfrak{h}$,

[2] Remember that $O_{\mathfrak{l}^*} := S(\mathfrak{l})$ and $\hat{O}_{\mathfrak{l}^*} := \hat{S}(\mathfrak{l})$.

and thus we can compute, writing $\beta := \theta(\bar 0, \bar 1)$,

$$\begin{aligned} r(\mathrm{x}, z) &= \frac{1}{\kappa N}\sum_{\gamma\in\Gamma} k_\gamma(\mathrm{ad}(\mathrm{x}^\vee)^{(1)}, z)(\gamma^{(1)}\cdot t_{\mathfrak{g}}) \\ &= \frac{1}{\kappa N}\sum_{\substack{k=0,\dots,\kappa-1\\ l=0,\dots,N-1}}\Big(\sum_{\alpha\in\Delta^+}\big(e^{-2\pi\mathrm{i}l\langle \mathrm{x},\alpha\rangle}\frac{\theta(z-\frac{k}{\kappa}-\frac{l\tau}{N}+\langle \mathrm{x},\alpha\rangle)}{\theta(z-\frac{k}{\kappa}-\frac{l\tau}{N})\theta(\langle \mathrm{x},\alpha\rangle)}e^{2\pi\mathrm{i}k|\alpha|/\kappa}\beta^l(e_\alpha)\otimes f_\alpha \\ &\quad + e^{2\pi\mathrm{i}l\langle \mathrm{x},\alpha\rangle}\frac{\theta(z-\frac{k}{\kappa}-\frac{l\tau}{N}-\langle \mathrm{x},\alpha\rangle)}{\theta(z-\frac{k}{\kappa}-\frac{l\tau}{N})\theta(-\langle \mathrm{x},\alpha\rangle)}e^{-2\pi\mathrm{i}k|\alpha|/\kappa}\beta^l(f_\alpha)\otimes e_\alpha\big) \\ &\quad + \sum_i \frac{\theta'}{\theta}(z-\frac{k}{\kappa}-\frac{l\tau}{N})\beta^l(h_i)\otimes h_i\Big). \end{aligned}$$

This should correspond to the generalization of Felder's elliptic dynamical r-matrices.

Example 6.4.5. *If $\mathfrak{g} = \mathfrak{sl}_n$ and $\theta(\bar 0, \bar 1)$ is the conjugation by the cyclic permutation $(1\cdots n)$ (hence we have $M = N = n$) then $\mathfrak{h} = \{0\}$ and $r(z)$ is Belavin's elliptic solution of the classical (non dynamical) Yang-Baxter equation [6]. In this case the Γ-KZB system realizes as the elliptic KZ system [40] (see also [76, 79]).*

6.4.5 Elliptic structures

Let H, B, D be algebras with units and morphisms $\Delta_H : H \longrightarrow H\otimes H$, $\Delta_B : B \longrightarrow B\otimes H$ and $a : B \longrightarrow D$. We also assume that H and B are augmented. We defines two sets of nonassociative words: W_n is the set of words in the free magma generated by $0, 1, \dots, n$, containing $0, 1, \dots, n$ exactly once, and starting with 0, and V_n is the set of words in the free magma generated by $1, \dots, n$ and containing $1, \dots, n$ exactly once. For any $w \in W_n$ (resp. $v \in V_n$) we can obviously associate an iterated coproduct $\Delta_B^{(w)} : B \longrightarrow B\otimes H^{\otimes n}$ (resp. $\Delta_H^{(v)} : H \longrightarrow H^{\otimes n}$). By convention $\Delta_B^{(\emptyset)} = \epsilon_B$ and $\Delta_H^{(\emptyset)} = \epsilon_H$.

Define J_w to be the left ideal in $D\otimes H^{\otimes n}$ generated by the image of $B_+ := \ker(\epsilon_B)$ by $(a\otimes \mathrm{id}^{\otimes n})\circ\Delta_B^{(w)}$.

We then consider Hecke bimodules $H^{w,w'} := (D\otimes H^{\otimes n}/J_{w'})^{J_w}$, and write $H^w := H^{w,w}$. There is an obvious composition morphism $H^{w,w'}\otimes H^{w',w''} \longrightarrow H^{w,w''}$. The symmetric group $\mathfrak{S}_n$ acts on the disjoint union of bimodules by permuting the last n factors. In other words for any $\sigma\in\mathfrak{S}_n$ on has an isomorphism $\sigma* : H^{w,w'} \longrightarrow H^{\sigma(w),\sigma(w')}$, where $\sigma(w)$ is obtained from w by replacing $i \neq 0$ by $\sigma(i)$. In particular for any fixed w_0 we can define an algebra structure on $\cup_{\sigma\in\mathfrak{S}_n} H^{w_0,\sigma(w_0)}\sigma$ by $(h\sigma)(h'\sigma') = h(\sigma * h')\sigma\sigma'$.

Given words $w \in W_n$, $w_0 \in W_{k_0}$, and $v_i \in V_{k_i}$ $(i = 1, \dots, n)$, one can construct a new word $w(w_0, v_1, \dots, v_n)$ by making the following substitutions: $0 \leftrightarrow w_0$, $0 \neq i \leftrightarrow \tilde v_i$, where $\tilde v_i$ is obtain from v_i by replacing $j \in \{1, \dots, n\}$ with $\sum_{l<i} k_l + j$. We then define a morphism $H^w \longrightarrow H^{w(w_0,v_1,\dots,v_n)}$ by

$$D\otimes H^{\otimes n} \ni h \mapsto h^{w_0,\tilde v_1,\dots,\tilde v_n} := (\mathrm{id}_D\otimes\Delta_H^{(v_1)}\otimes\cdots\otimes\Delta_H^{(v_n)})(h).$$

If (H, Δ_H, Φ_H) is a quasibialgebra then Φ_H gives rise to an invertible (w.r.t. the composition of Hecke bimodules) element of $H^{0(1(23)),0((12)3)}$ that induces an isomorphism $H^{0((12)3)} \longrightarrow$

$H^{0(1(23))}$ defined by $X \mapsto \Phi_H X \Phi_H^{-1}$. Then given two words $w = 0(v)$ and $w' = 0(v')$ having the same underlying associative word, one can use Φ_H to construct an isomorphism $H^w \longrightarrow H^{w'}$; moreover, all possible isomorphisms constructed in this way are equal, providing that Φ_H satisfies the pentagon identity.

If (B, Δ_B, Ψ_B) is a quasicomodule algebra over H then Ψ_B gives rise to an invertible element of $H^{0(12),(0(1)2}$ that induces an isomorphism $H^{(01)2} \longrightarrow H^{0(12)}$ defined by $X \mapsto \Psi_B X \Psi_B^{-1}$. Then given two words w and w' having the same underlying associative word, one can use Φ_H and Ψ_B to construct an isomorphism $H^w \longrightarrow H^{w'}$; moreover, all possible isomorphisms constructed in this way are equal, providing that the pair (Φ_H, Ψ_B) satisfies the pseudotwist equation.

Recall ([33]) that if $(H, \Delta_H, R_H, \Phi_H)$ is a QTQBA and (B, E_B, Ψ_B) is a QRA quasireflection algebra (QRA) over it then R_H (resp. E_B) gives rise to an element in $H^{0(21),0(12)}$ (resp. $H^{01,01}$) that induces an isomorphism $H^{0(12)} \longrightarrow H^{0(21)}$ (resp. $H^{01} \longrightarrow H^{01}$). Therefore one obtains naturally a group morphism

$$B_n^{(1)} \longrightarrow \cup_{\sigma \in \mathfrak{S}_n} (H^{w_0, \sigma(w_0)})^\times \sigma\,,$$

where $B_n^{(1)}$ is the braid group with n strands on $\mathbb{C}^\times$ and w_0 any fixed word in W_n.

Chapter 7

Applications

7.1 Formality of subgroups of the pure braid group on the torus

7.1.1 Relative formality

Let G and S be two affine groups over $\mathbf{k}$ and let $\varphi : G \longrightarrow S$ be a surjective group morphism with finitely generated kernel $\operatorname{Ker}\varphi$. We then consider the category of pro-algebraic groups G' under G, together with a surjective morphism $\varphi' : G' \longrightarrow S$ with $\mathbf{k}$-prounipotent kernel. This category has an initial object, denoted $\varphi(\mathbf{k}) : G \longrightarrow G(\varphi, \mathbf{k})$, which we call the *relative ($\mathbf{k}$-prounipotent) completion* of G with respect to φ. One can easily check that the kernel $\operatorname{Ker}\big(\varphi(\mathbf{k})\big)$ of $\varphi(\mathbf{k})$ is the usual $\mathbf{k}$-prounipotent completion $\big(\operatorname{Ker}\varphi\big)(\mathbf{k})$ of the kernel of φ, which we can therefore unambiguously denote $\operatorname{Ker}\varphi(\mathbf{k})$.

Observe that this coincides with the partial completion defined [33, §1.1], and with the relative completion defined in [62] (which is somehow slightly more general).

Lemma 7.1.1. *If S is finite then the extension*

$$1 \longrightarrow \operatorname{Ker}\varphi(\mathbf{k}) \longrightarrow G(\varphi, \mathbf{k}) \longrightarrow S \longrightarrow 1$$

splits.

Proof. We consider the filtration $(F_i)_i$ given by the lower central series of $\operatorname{Ker}\varphi(\mathbf{k})$, and prove by induction by induction that

$$1 \longrightarrow \operatorname{Ker}\varphi(\mathbf{k})/F_i \longrightarrow G(\varphi, \mathbf{k})/F_i \longrightarrow S \longrightarrow 1$$

splits.
Initial step ($i = 2$): Recall that $F_1 = \operatorname{Ker}\varphi(\mathbf{k})$, and that F_1/F_2 is abelian and finitely generated, so that

$$1 \longrightarrow \operatorname{Ker}\varphi(\mathbf{k})/F_2 \longrightarrow G(\varphi, \mathbf{k})/F_2 \longrightarrow S \longrightarrow 1$$

splits as every extension of a finite group by a finite dimensional representation splits (this is because the cohomology of a finite group with coefficients in a divisible module vanishes).
Induction step: We have a (surjective) morphism of extensions

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Ker}\,\varphi(\mathbf{k})/F_{i+1} \arrow[r] \arrow[d] & G(\varphi,\mathbf{k})/F_{i+1} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{Ker}\,\varphi(\mathbf{k})/F_i \arrow[r] & G(\varphi,\mathbf{k})/F_i \arrow[r] & S \arrow[r] & 1 \end{tikzcd}$$

Assuming (by induction) that the bottom extension splits, we have that the corresponding obstruction class in the first non-abelian cohomology $H^1\big(S, \mathrm{Ker}\,\varphi(\mathbf{k})/F_i\big)$ is trivial. Hence, by exactness of

$$H^1\big(S, F_i/F_{i+1}\big) \longrightarrow H^1\big(S, \mathrm{Ker}\,\varphi(\mathbf{k})/F_{i+1}\big) \longrightarrow H^1\big(S, \mathrm{Ker}\,\varphi(\mathbf{k})/F_i\big)$$

we get that the obstruction class for the splitting of the top extension lies in the image of

$$H^1\big(S, F_i/F_{i+1}\big) \longrightarrow H^1\big(S, \mathrm{Ker}\,\varphi(\mathbf{k})/F_{i+1}\big)\,.$$

We conclude by using the vanishing of group cohomology of a finite group in a finite dimensional representation. □

The above Lemma tells us in particular that $G(\varphi,\mathbf{k}) \simeq \mathrm{Ker}(\varphi)(\mathbf{k}) \rtimes S$, and justifies the following definition from [33, §1.2].

Definition 7.1.2. *If S is finite, we say that the surjective group morphism $\varphi : G \longrightarrow S$ with finitely generated kernel is relatively formal if there exists a group isomorphism*

$$G(\mathbf{k},\varphi) \xrightarrow{\sim} \exp\big(\hat{\mathrm{gr}}\,\mathrm{Lie}\,\mathrm{Ker}\,\varphi(\mathbf{k})\big) \rtimes S$$

over S. This is equivalent to having an S-equivariant formality isomorphism

$$\mathrm{Ker}\,\varphi(\mathbf{k}) \xrightarrow{\sim} \hat{\mathrm{gr}}\,\mathrm{Lie}\,\mathrm{Ker}\,\varphi(\mathbf{k})\,.$$

Example 7.1.3. *The surjective morphism $\mathrm{B}_n \twoheadrightarrow \mathfrak{S}_n$ is formal, where B_n is the standard n strands braid group. This morphism, or rather the exact sequence*

$$1 \longrightarrow \mathrm{PB}_n \longrightarrow \mathrm{B}_n \longrightarrow \mathfrak{S}_n \longrightarrow 1\,,$$

can be deduced from the covering map $\mathrm{Conf}(\mathbb{C},n) \longrightarrow \mathrm{Conf}(\mathbb{C},n)/\mathfrak{S}_n$. It is interesting to say that this relative formality result follows from [75] when $\mathbf{k}=\mathbb{C}$, and from [31] for $\mathbf{k}=\mathbb{Q}$. We also refer to [62, Example 1.5] for interesting considerations about this example. More precisely, one has an $\mathfrak{S}_n$-equivariant isomorphism $\mathrm{PB}_n(\mathbf{k}) \xrightarrow{\sim} \exp(\hat{\mathfrak{t}}_n)$.

Example 7.1.4. *Let $G=\mathbb{Z}/N\mathbb{Z}$. From the covering map $\mathrm{Conf}(\mathbb{C}^\times,n,G) \longrightarrow \mathrm{Conf}(\mathbb{C}^\times,n)/\mathfrak{S}_n$ one also gets an exact sequence*

$$1 \longrightarrow \mathrm{PB}_n^G \longrightarrow \mathrm{B}_n^1 \longrightarrow G^n \rtimes \mathfrak{S}_n \longrightarrow 1\,.$$

It follows from [33, §1.3–1.6] that the surjective morphism $\mathrm{B}_n^1 \twoheadrightarrow G^n \rtimes \mathfrak{S}_n$ is formal. More precisely, Enriquez proves the existence of a $G^n \rtimes \mathfrak{S}_n$-equivariant isomorphism $\mathrm{PB}_n^G(\mathbf{k}) \xrightarrow{\sim} \exp(\hat{\mathfrak{t}}_n^\Gamma)$.

7.1.2 Relation between relative completion and completion of groupoids

In this paragraph we briefly compare the notion of relative **k**-prounipotent completion with the **k**-prounipotent completion for groupoids defined in §2.5.7.

There is a functor that goes

- from the category of surjective morphisms $G \longrightarrow S$ with finitely generated kernel and with S a finite group.
- to the category of groupoids.

This functor sends $\varphi : G \longrightarrow S$ to the groupoid $\mathcal{G}(\varphi)$ defined as follows:

- the set of objects of of $\mathcal{G}(\varphi)$ is S.
- for $s, s' \in S$,
$$\mathrm{Hom}_{\mathcal{G}(\varphi)}(s, s') := \{g \in G | \varphi g = s^{-1}s'\}$$
- the multiplication of arrows in $\mathcal{G}(\varphi)$ is the multiplication in G.

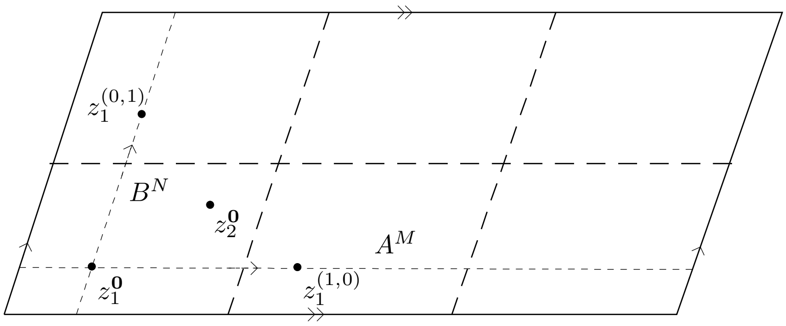
Example 7.1.5. *It is easy to check that* $\mathcal{G}(\mathrm{B}_n \longrightarrow \mathfrak{S}_n)$ *is the colored braid groupoid* $\mathbf{CoB}(n)$ *from [47, §5.2.8], which is an unparenthesized variant of* $\mathbf{PaB}(n)$*. Similarly:*

- *the groupoid*
$$\mathbf{CoB}^N(n) := \mathcal{G}\big(\mathrm{B}_n^1 \longrightarrow (\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n\big)$$
is an unparenthezised variant of the twisted parenthesized braid groupoid $\mathbf{PaB}^N(n)$ *from §4.2.5.*
- *the groupoid*
$$\mathbf{CoB}_{e\ell\ell}(n) := \mathcal{G}\big(\overline{\mathrm{B}}_{1,n} \longrightarrow \mathfrak{S}_n\big)$$
is an unparenthezised variant of the parenthezised elliptic braid groupoid $\mathbf{PaB}_{e\ell\ell}(n)$ *from §4.1.2.*
- *the groupoid*
$$\mathbf{CoB}_{e\ell\ell}^{\Gamma}(n) := \mathcal{G}\big(\overline{\mathrm{B}}_{1,n} \longrightarrow (\Gamma^n/\Gamma) \rtimes \mathfrak{S}_n\big)$$
is an unparenthezised variant of the twisted parenthezised elliptic braid groupoid $\mathbf{PaB}_{e\ell\ell}^{\Gamma}(n)$ *from §4.3.2.*

We let the reader prove that the following is true:
$$\mathcal{G}(\hat{\varphi})(\mathbf{k}) \simeq \mathcal{G}\big(\varphi(\mathbf{k})\big)\,.$$

7.1.3 Subgroups of $\mathrm{B}_{1,n}$

For $\tau \in \mathfrak{H}$ and $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, let $U_{\tau,n,\Gamma} \subset \mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}$ be the open subset of all $\mathbf{z} = (z_1, \ldots, z_n)$ of the form $z_i = a_i + \tau b_i$, where $0 < a_1 < \cdots < a_n < 1/M$ and $0 < b_1 < \cdots < b_n < 1/N$. If $\mathbf{z}_0 \in U_{\tau,n,\Gamma}$ then it both defines a point in the Γ-twisted configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ and in the (non twisted) unordered configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, [n])$:



Recall that the map

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma) \twoheadrightarrow \mathrm{Conf}(E_{\tau,\Gamma}, [n])$$

is a a covering map with structure group $\Gamma^n \rtimes \mathfrak{S}_n$. Hence we get a short exact sequence

$$1 \longrightarrow \mathrm{PB}_{1,n}^{\Gamma} \longrightarrow \mathrm{B}_{1,n} \xrightarrow{\varphi_n} \Gamma^n \rtimes \mathfrak{S}_n \longrightarrow 1\,,$$

where $\mathrm{PB}_{1,n}^{\Gamma} := \pi_1(\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma), \mathbf{z}_0)$ and $\mathrm{B}_{1,n} = \pi_1\big(\mathrm{Conf}(E_{\tau,\Gamma}, [n]), \mathbf{z}_0\big)$.

We will also consider $\mathrm{PB}_{1,n} = \pi_1\big(\mathrm{Conf}(E_{\tau,\Gamma}, n), \mathbf{z}_0\big)$, and the short exact sequence

$$1 \longrightarrow \mathrm{PB}_{1,n}^{\Gamma} \longrightarrow \mathrm{PB}_{1,n} \longrightarrow \Gamma^n \longrightarrow 1$$

associated with the Γ^n-covering map

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma) \twoheadrightarrow \mathrm{Conf}(E_{\tau,\Gamma}, n)\,.$$

Our main aim in this Section is to prove that the surjective morphism

$$\mathrm{B}_{1,n} \twoheadrightarrow \Gamma^n \rtimes \mathfrak{S}_n$$

is relatively formal, which in turns implies the relative formality of $\mathrm{PB}_{1,n} \longrightarrow \Gamma^n$, and the formality of $\mathrm{PB}_{1,n}^{\Gamma}$.

Moreover, we will have an explicit description of the relative completion in terms of the Lie algebra $\mathfrak{t}_{1,n}^{\Gamma}$.

7.1.4 The monodromy morphism $\mathrm{B}_{1,n} \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$

The monodromy of the flat $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$-bundle $(P_{(\tau,\Gamma),[n]}, \nabla_{(\tau,\Gamma),[n]})$ on $\mathrm{Conf}(E_{\tau,\Gamma}, [n])$ provides us with a group morphism

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),[n]} : \mathrm{B}_{1,n} \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)\,.$$

This actually fits into a morphism of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{PB}_{1,n}^{\Gamma} \arrow[r] \arrow[d] & \mathrm{B}_{1,n} \arrow[r] \arrow[d] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] \arrow[d, equal] & 1\ , \\ 1 \arrow[r] & \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \arrow[r] & \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n) \arrow[r] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] & 1 \end{tikzcd}$$

where the first vertical morphism is the monodromy morphism

$$\mu_{\mathbf{z}_0,\tau,n,\Gamma}\ :\ \mathrm{PB}_{1,n}^{\Gamma} \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$$

of associated with the flat $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$-bundle $(P_{\tau,n,\Gamma}, \nabla_{\tau,n,\Gamma})$ on $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$.

Indeed, this comes from the fact that $\nabla_{(\tau,\Gamma),[n]}$ is obtained by descent, from $\nabla_{\tau,n,\Gamma}$ and using its equivariance properties (see §6.1.2). More precisely, the monodromy of $\nabla_{(\tau,\Gamma),[n]}$ along a loop γ based at $\mathbf{z}_0$ in $\mathrm{Conf}(E_{\tau,\Gamma}, [n])$ can be computed along the following steps:

- First consider the unique lift $\tilde{\gamma}$ of γ departing from $\mathbf{z}_0 \in \mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$. Note that it ends at $g \cdot \mathbf{z}_0$, $g \in \Gamma^n \rtimes \mathfrak{S}_n$.
- Then compute the holonomy of $\nabla_{\tau,n,\Gamma}$ along $\tilde{\gamma}$: this is an element in $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$, as $\nabla_{\tau,n,\Gamma}$ is defined on a principal $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$-bundle obtained as a quotient of the trivial one on $\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}$ (see §6.1.1), that we abusively denote $\mu_{\mathbf{z}_0,\tau,n,\Gamma}(\tilde{\gamma})$.
- Finally, $\mu_{\mathbf{z}_0,(\tau,\Gamma),[n]}(\gamma) = g\mu_{\mathbf{z}_0,\tau,n,\Gamma}(\tilde{\gamma})$.

Having such a morphism of exact sequences guaranties that it factors through a morphism

$$\begin{tikzcd} 1 \arrow[r] & \hat{\mathrm{PB}}_{1,n}^{\Gamma}(\mathbb{C}) \arrow[r] \arrow[d] & \hat{\mathrm{B}}_{1,n}(\varphi_n, \mathbb{C}) \arrow[r] \arrow[d] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] \arrow[d, equal] & 1\ , \\ 1 \arrow[r] & \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \arrow[r] & \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n) \arrow[r] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] & 1 \end{tikzcd}$$

where $\hat{\mathrm{B}}_{1,n}(\varphi_n, \mathbb{C})$ is is the relative prounipotent completion of the morphism $\mathrm{B}_{1,n} \longrightarrow \Gamma^n \rtimes \mathfrak{S}_n$, and $\hat{\mathrm{PB}}_{1,n}^{\Gamma}(\mathbb{C})$ is the prounipotent completion of $\mathrm{PB}_{1,n}^{\Gamma}$.

We will call the vertical maps the completed monodromy morphisms.

In the remainder of this Section we will prove that these completed monodromy morphisms are isomorphisms, which implies in particular the relative formality of $\mathrm{B}_{1,n} \longrightarrow \Gamma^n \rtimes \mathfrak{S}_n$.

Theorem 7.1.6. *The completed monodromy morphism*

$$\hat{\mathrm{B}}_{1,n}(\varphi_n, \mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$$

is an isomorphism. Equivalently, the completed monodromy morphism

$$\hat{\mathrm{PB}}_{1,n}^{\Gamma}(\mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$$

is an isomorphism.

Our aim now is to prove that Theorem 7.1.6, namely that the completed monodromy morphism

$$\hat{\mu}_{\mathbf{z}_0,\tau,n,\Gamma}(\mathbb{C}) \,:\, \hat{\mathrm{PB}}^{\Gamma}_{1,n}(\mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}^{\Gamma}_{1,n})$$

is an isomorphism. For this we will prove that the induced morphism on Malcev Lie algebras

$$Lie(\mu_{\mathbf{z}_0,\tau,n,\Gamma}) \,:\, \mathfrak{pb}^{\Gamma}_{1,n} \longrightarrow \hat{\mathfrak{t}}^{\Gamma}_{1,n}$$

is an isomorphism of filtered Lie algebras.

7.1.5 A morphism $\mathfrak{t}^{\Gamma}_{1,n} \longrightarrow \mathrm{gr}(\mathfrak{pb}^{\Gamma}_{1,n})$

Let us start with a few algebraic facts about $\mathrm{PB}_{1,n}$ and $\mathrm{PB}^{\Gamma}_{1,n}$.

The group $\mathrm{PB}_{1,n}$ is generated by the X_i's and Y_i's ($i = 1,\dots,n$), where X_i (resp. Y_i) is the class of the path given by $[0,1] \ni t \mapsto \mathbf{z}_0 - t\delta_i/M$ (resp. $[0,1] \ni t \mapsto \mathbf{z}_0 - t\tau\delta_i/N$). One sees very easily that X_i^M (resp. Y_i^N) is the class of the path given by $[0,1] \ni t \mapsto \mathbf{z}_0 - t\delta_i$ (resp. $[0,1] \ni t \mapsto \mathbf{z}_0 - t\tau\delta_i$), so that X_i^M and Y_i^N are elements of $\mathrm{PB}^{\Gamma}_{1,n}$. One has an obvious inclusion $\mathrm{PB}_n \hookrightarrow \mathrm{PB}^{\Gamma}_{1,n}$ coming from the identification of $\mathbb{C}$ with the fundamental domain

$$\{z = a + b\tau \in \mathbb{C} | 0 < a < \frac{1}{M}, 0 < b < \frac{1}{N}\}$$

of $E_{\tau,\Gamma}$.

Then one can check (by simply drawing) that the following relations are satisfied in $\mathrm{PB}_{1,n}$:

(T1) $(X_i, X_j) = 1 = (Y_i, Y_j)$ $(i < j)$,

(T2) $(X_j, Y_i^{-1}) = P_{ij} = (X_i, Y_j^{-1})$ $(i < j)$,

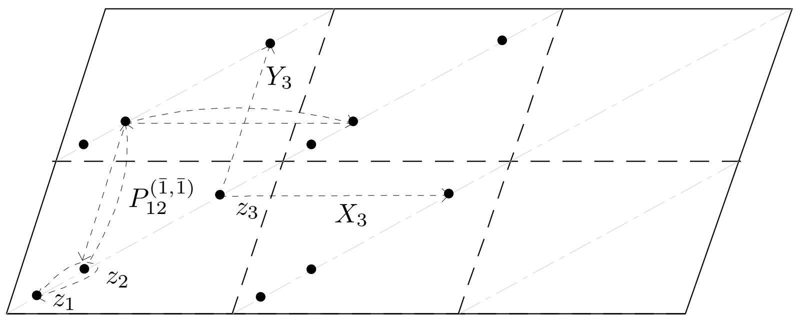
(T3) $(X_n, Y_n) = P_{n-1,n} \cdots P_{1n}$,

(T4) $(X_i, P_{jk}) = 1 = (Y_i, P_{jk})$ $(\forall i,\ j < k)$,

(T5) $(X_iX_j, P_{ij}) = 1 = (Y_iY_j, P_{ij})$ $(i < j)$.

In particular PB_n identifies with the subgroup of commutators in $\mathrm{PB}_{1,n}$. Moreover, one observes that $X_1 \cdots X_n$ and $Y_1 \cdots Y_n$ are central in $\mathrm{PB}_{1,n}$.

Now it follows from the geometric description of $\mathrm{PB}^{\Gamma}_{1,n}$ that it is generated by X_i^M, Y_i^N ($i = 1,\dots,n$) and $P_{ij}^{\alpha} := X_j^{-p}Y_j^{-q}P_{ij}Y_j^qX_j^p$ ($i < j$, $1 \le p \le M$, $1 \le q \le N$ and $\alpha = (\bar{p},\bar{q})$). One can for instance represent lifts of X_3, Y_3 and $P_{12}^{(\bar{1},\bar{1})}$ in $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ as follows



Observe that the standard descending filtration on $\hat{\mathfrak{t}}_{1,n}^{\Gamma}$ coincides with the descending filtration coming from the grading of $\mathfrak{t}_{1,n}^{\Gamma}$ defined in §4.3.3.

Proposition 7.1.7. *There is a surjective graded Lie algebra morphism* $p_n : \mathfrak{t}_{1,n}^{\Gamma} \longrightarrow \mathrm{gr}(\mathfrak{pb}_{1,n}^{\Gamma})$, *sending*

- $x_i \longmapsto \sigma\big(\log(X_i^M)\big)$ *for* $i = 1, \ldots, n$,
- $y_i \longmapsto \sigma\big(\log(Y_i^N)\big)$ *for* $i = 1, \ldots, n$,
- $t_{ij}^{\alpha} \longmapsto \sigma\big(\log(P_{ij}^{\alpha})\big)$ *for* $i < j$,
- $t_{ij}^{\alpha} \longmapsto \sigma\big(\log(P_{ji}^{-\alpha})\big)$ *for* $j < i$,

where σ *denotes the symbol map* $\mathfrak{pb}_{1,n}^{\Gamma} \longrightarrow \mathrm{gr}(\mathfrak{pb}_{1,n}^{\Gamma})$.

Proof. It is sufficient to check that the defining relations of $\mathfrak{t}_{1,n}^{\Gamma}$ are preserved by the above assignment.

The relation $[x_i, x_j] = 0 = [y_i, y_j]$ is obviously preserved. Now using (T2) and the relation

$$(X^M, Y^N) = \prod_{i=0}^{M-1} X^{M-i+1}(\prod_{j=0}^{N-1} Y^j (X, Y) Y^{-j}) X^{i-M-1}$$

(which is true in the free group F_2, and thus in any group) with $X = X_i$ and $Y = Y_j$ $(i \neq j)$, one obtains that $[x_i, y_j] = [x_j, y_i] = \sum_{\alpha} t_{ij}^{\alpha}$ is preserved. Using (T3) one also obtains that $[x_1, y_1] = -\sum_{\alpha} \sum_{j:1\neq j} t_{1j}^{\alpha}$ is preserved. Now it is obvious that the centrality of $\sum_i x_i$ and $\sum_i y_i$ is preserved, and thus it follows that $[x_i, y_i] = -\sum_{\alpha} \sum_{j:j\neq i} t_{ij}^{\alpha}$ is also preserved for any $i \in \{1, \ldots, n\}$. For any $\alpha = (\bar{p}, \bar{q})$ we compute

$$\begin{aligned}
(X_i^M, P_{jk}^{\alpha}) &= X_i^M X_k^{-p} Y_k^{-q} P_{jk} Y_k^q X_k^p X_i^{-M} X_k^{-p} Y_k^{-q} P_{jk}^{-1} Y_k^q X_k^p \\
&= X_k^{-p} (X_i^M, Y_k^{-q}) Y_k^{-q} X_i^M P_{jk} X_i^{-M} Y_k^q (X_i^M, Y_k^{-q})^{-1} Y_k^{-q} P_{jk}^{-1} Y_k^q X_k^p \\
&= X_k^{-p} (X_i^M, Y_k^{-q}) Y_k^{-q} P_{jk} Y_k^q (X_i^M, Y_k^{-q})^{-1} Y_k^{-q} P_{jk}^{-1} Y_k^q X_k^p .
\end{aligned}$$

One sees that the log of the l.h.s. lies in $(\mathfrak{pb}_{1,n}^{\Gamma})_3$ and its symbol is equal to $[\sigma(\log(X_i^M)), \sigma(\log(P_{jk}^{\alpha}))]$, and that the log of the r.h.s. lies in $(\mathfrak{pb}_{1,n}^{\Gamma})_4$. Hence one obtains that $[x_i, t_{jk}^{\alpha}] = 0$ is preserved. The proof that $[y_i, t_{jk}^{\alpha}] = 0$ is preserved is identical, and the proof that $[x_i + x_j, t_{ij}^{\alpha}] = 0 = [y_i + y_j, t_{ij}^{\alpha}]$, $[t_{ij}^{\alpha}, t_{kl}^{\beta}] = 0$ and $[t_{ij}^{\alpha}, t_{ik}^{\alpha+\beta} + t_{jk}^{\beta}] = 0$ are preserved is very similar. □

7.1.6 The formality of $\mathrm{PB}_{1,n}^{\Gamma}$ (end of the proof of Theorem 7.1.6)

To prove that $Lie(\mu_{\mathbf{z}_0,\tau,n,\Gamma})$ is an isomorphism, it is sufficient to prove that it is an isomorphism on associated graded. According to Proposition 7.1.7, we simply have to prove that $\phi := \mathrm{gr}Lie(\mu_{\mathbf{z}_0,\tau,n,\Gamma}) \circ p_n$ is an isomorphism of graded Lie algebras.

We will actually be more specific on prove the following:

Lemma 7.1.8. *We have* $\phi(x_i) = -y_i$, $\phi(y_i) = 2\pi \mathrm{i} x_i - \tau y_i$ *and* $\phi(t_{ij}^{\alpha}) = 2\pi \mathrm{i} t_{ij}^{\alpha}$. *In particular,* ϕ *is an automorphism.*

Proof. Recall that $\mu_{\mathbf{z}_0,\tau,n,\Gamma}$ can be computed as follows. Let $F_{\mathbf{z}_0} : U_\tau \longrightarrow \exp(\hat{\mathfrak{t}}^\Gamma_{1,n})$ be such that

$$\begin{cases} (\partial/\partial z_i)F_{\mathbf{z}_0}(\mathbf{z}) = K_i^\Gamma(\mathbf{z}|\tau)F_{\mathbf{z}_0}(\mathbf{z}) \,, \\ F_{\mathbf{z}_0}(\mathbf{z}_0) = 1 \,. \end{cases}$$

Then consider

$$H^\Gamma_{\tau,n} := \left\{ \mathbf{z} = (z_1, ..., z_n) | z_i = a_i + \tau b_i, 0 < a_n < ... < a_1 < \frac{1}{M} \right\}$$

and

$$V^\Gamma_{\tau,n} := \left\{ \mathbf{z} = (z_1, ..., z_n) | z_i = a_i + \tau b_i, 0 < b_n < ... < b_1 < \frac{1}{N} \right\} .$$

Let $F^{H^\Gamma}_{\mathbf{z}_0}$ (resp. $F^{V^\Gamma}_{\mathbf{z}_0}$) be the analytic prolongations of $F_{\mathbf{z}_0}$ to $H^\Gamma_{\tau,n}$ (resp. $V^\Gamma_{\tau,n}$). Then

$$F^{H^\Gamma_\tau}_{\mathbf{z}_0}(\mathbf{z} - \delta_i) = F^{H^\Gamma_\tau}_{\mathbf{z}_0}(\mathbf{z})\mu_{\mathbf{z}_0,\tau,n,\Gamma}(X_i^M) \quad \text{and} \quad e^{2\pi \mathrm{i} x_i} F^{V^\Gamma_\tau}_{\mathbf{z}_0}(\mathbf{z} - \tau\delta_i) = F^{V^\Gamma_\tau}_{\mathbf{z}_0}(\mathbf{z})\mu_{\mathbf{z}_0,\tau,n,\Gamma}(Y_i^N) \,.$$

Knowing that $\log F^{H^\Gamma_\tau}_{\mathbf{z}_0}(\mathbf{z}) = -\sum_i (z_i - z_i^0)y_i$ + terms of degree ≥ 2, we get

$$\log \mu_{\mathbf{z}_0,\tau,n,\Gamma}(X_i^M) = -y_i + \text{terms of degree } \geq 2$$

and

$$\log \mu_{\mathbf{z}_0,\tau,n,\Gamma}(Y_i^N) = 2\pi \mathrm{i} x_i - \tau y_i + \text{terms of degree } \geq 2 \,.$$

This gives us that $\phi(x_i) = -y_i$ and $\phi(y_i) = 2\pi \mathrm{i} x_i - \tau y_i$.

In order to compute $\log \mu_{\mathbf{z}_0,\tau,n,\Gamma}(P^\alpha_{ij})$, which is also equal to $\log \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P^\alpha_{ij})$, we will need to compute $\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i)$, $\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_i)$ and $\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P_{ij})$:

- As usual, we have

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P_{ij}) = \exp(2\pi \mathrm{i} t^{\mathbf{0}}_{ij} + \text{terms of degree } \geq 3) \,,$$

  where $\mathbf{0} = (\bar{0}, \bar{0})$.

- We also have

$$F^{H^\Gamma}_{\mathbf{z}_0}(\mathbf{z} + \frac{\delta_i}{M}) = (\bar{1}, \bar{0})_i F^{H^\Gamma}_{\mathbf{z}_0}(\mathbf{z})\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i) \,,$$

  which implies that

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i) \in (-\bar{1}, \bar{0})_i \exp(\mathfrak{t}^\Gamma_{1,n}) \,.$$

- We finally have

$$e^{2\pi \mathrm{i} \frac{x_i}{N}} F^{V^\Gamma}_{\mathbf{z}_0}(\mathbf{z} + \frac{\tau\delta_i}{N}) = (\bar{0}, \bar{1})_i F^{V^\Gamma}_{\mathbf{z}_0}(\mathbf{z})\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_i) \,,$$

  which implies that

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_i) \in (\bar{0}, \bar{-1})_i \exp(\mathfrak{t}^\Gamma_{1,n}) \,.$$

Hence, if $\alpha = (\bar{p}, \bar{q}) \in \Gamma$, then

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i^{-p} Y_j^{-q}) = g(\bar{p}, \bar{0})_i (\bar{0}, \bar{q})_j \,,$$

with $g \in \exp(\mathfrak{t}_{1,n}^{\Gamma})$, and

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_j^{q} X_i^{p}) = (\bar{0}, \bar{-q})_j (\bar{-p}, \bar{0})_i g^{-1} \,.$$

Therefore

$$\begin{aligned}
\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P_{ij}^{\alpha}) &= g(\bar{p}, \bar{0})_i (\bar{0}, \bar{q})_j \exp(t_{ij}^{\mathbf{0}}) (\bar{0}, \bar{-q})_j (\bar{-p}, \bar{0})_i g^{-1} \\
&= g \exp(t_{ij}^{\alpha} + \text{terms of degree } \geq 3) g^{-1} \,.
\end{aligned}$$

This shows that $\log \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P_{ij}^{\alpha}) = t_{ij}^{\alpha}$ + terms of degree ≥ 3, so that $\phi(t_{ij}^{\alpha}) = 2\pi \mathrm{i} t_{ij}^{\alpha}$. This ends the proof of the Lemma. □

7.2 The KZB ellipsitomic associator

First of all, recall that $\bar{\mathfrak{t}}_{1,2}^{\Gamma}$ is the Lie $\mathbb{C}$-algebra generated by $x := x_1$, $y := y_2$ and $t^{\alpha} := t_{12}^{\alpha}$, for $\alpha \in \Gamma$, such that $[x, y] = \sum_{\alpha \in \Gamma} t^{\alpha}$. We define the KZB ellipsitomic associator as the couple $e^{\Gamma}(\tau) := (A^{\Gamma}(\tau), B^{\Gamma}(\tau)) \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2}^{\Gamma}) \times \exp(\hat{\bar{\mathfrak{t}}}_{1,2}^{\Gamma})$ consisting in the renormalized holonomies from the straight paths from 0 to $1/M$ and from 0 to τ/N respectively of the differential equation

$$J'(z) = -\sum_{\alpha \in \Gamma} e^{-2\pi \mathrm{i} a x} \frac{\theta(z - \tilde{\alpha} + \mathrm{ad}(x)|\tau)}{\theta(z - \tilde{\alpha}|\tau)\theta(\mathrm{ad}(x)|\tau)} (t^{\alpha}) \cdot J(z), \tag{7.1}$$

with values in the group $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}^{\Gamma}) \rtimes \Gamma^n/\Gamma$. More precisely, for all $\alpha \in \Gamma$ and $\tilde{\alpha} = (a_0, a) \in \Lambda_{\tau,\Gamma}$ a lift of α, this equation has a unique solution $J_{\alpha}(z)$ defined over $\{\tilde{\alpha} + \frac{s_1}{M} + \frac{s_2}{N}\tau, \text{ for } s_1, s_2 \in]0, 1[\}$ such that we have

$$J_{\alpha}(z) \simeq (-2\pi \mathrm{i}(z - \tilde{\alpha}))^{e^{-2\pi \mathrm{i} a \,\mathrm{ad}(x)} t^{\alpha}}$$

at $z - \tilde{\alpha} \longrightarrow 0$. By denoting $J(z) := J_{\mathbf{0}}(z)$ we define

$$\underline{A}^{\Gamma}(\tau) := J(z)^{-1} (\bar{1}, \bar{0}) J(z + \frac{1}{M}) = J(z)^{-1} \theta(\bar{1}, \bar{0}) \cdot (J(z + \frac{1}{M}))(\bar{1}, \bar{0}) \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2}^{\Gamma}) \rtimes \Gamma^n/\Gamma.$$

Then the A-associator A^{Γ} is

$$A^{\Gamma}(\tau) := J(z)^{-1} \theta(\bar{1}, \bar{0}) \cdot J(z + \frac{1}{M}) \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2}^{\Gamma}).$$

In the same way, we define

$$\underline{B}^{\Gamma}(\tau) := J(z)^{-1} (\bar{0}, \bar{1}) e^{\frac{2\pi \mathrm{i}}{N} x} J(z + \frac{\tau}{N}) = J(z)^{-1} \theta((\bar{0}, \bar{1})) \cdot (e^{\frac{2\pi \mathrm{i}}{N} x} J(z + \frac{\tau}{N}))(\bar{0}, \bar{1}),$$

and the B-associator is then

$$B^{\Gamma}(\tau) := J(z)^{-1} \theta((\bar{0}, \bar{1})) \cdot (e^{\frac{2\pi \mathrm{i}}{N} x} J(z + \frac{\tau}{N})) \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2}^{\Gamma}).$$

We have $\underline{A}^p \in \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma})(\bar{p},\bar{0})$. Indeed, one checks for example that

$$\begin{aligned}\underline{A}^3 &= \underline{A}\cdot\underline{A}\cdot\underline{A}\cdot \\ &= A(\bar{1},\bar{0})A(\bar{1},\bar{0})A(\bar{1},\bar{0}) \\ &= A(\theta((\bar{1},\bar{0}))\cdot A)(\theta((\bar{2},\bar{0}))\cdot A(\bar{3},\bar{0})) \\ &= A(\theta((\bar{1},\bar{0}))\cdot A)(\theta((\bar{2},\bar{0}))\cdot A)(\bar{3},\bar{0}).\end{aligned}$$

Now, let $p,q\geq 1$. Define $A^{(p)}$ and $B^{(q)}$ such that $\underline{A}^p = \underline{A}^{(p)}(\bar{p},\bar{0})$ and $\underline{B}^q = \underline{A}^{(q)}(\bar{0},\bar{q})$. These are elements of $\exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma})$ and we have

$$A^{(p)} = \overrightarrow{\prod_{k=0,\dots,p-1}}(\theta((\bar{k},\bar{0}))\cdot A) = A(\theta((\bar{1},\bar{0}))\cdot A)(\theta((\bar{2},\bar{0}))\cdot A)\cdots(\theta((\overline{p-1},\bar{0}))\cdot A)$$

and

$$B^{(q)} = \overrightarrow{\prod_{k=0,\dots,q-1}}(\theta((\bar{0},\bar{k}))\cdot B) = B(\theta((\bar{0},\bar{1}))\cdot B)(\theta((\bar{0},\bar{2}))\cdot B)\cdots(\theta((\bar{0},\overline{q-1}))\cdot B).$$

Recall from Theorem 4.3.10 that the set of ellipsitomic associators $\mathrm{Ell}^{\Gamma}(\mathbf{k})$ can be regarded either as the set of Γ-equivariant $\widehat{\mathbf{PaB}}(\mathbf{k})$-module isomorphisms $\widehat{\mathbf{PaB}}^{\Gamma}_{e\ell\ell}(\mathbf{k}) \longrightarrow G\mathbf{PaCD}^{\Gamma}_{e\ell\ell}(\mathbf{k})$ which are the identity on objects or either as tuples $(\lambda,\Phi,A^{\Gamma},B^{\Gamma})$, where $(\lambda,\Phi)\in\mathrm{Ass}(\mathbf{k})$ and $A^{\Gamma},B^{\Gamma}\in\exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}(\mathbf{k}))$, satisfying relations (tN1), (tN2) and (tE). We are ready to show that the set $\mathrm{Ell}^{\Gamma}(\mathbb{C})$ is not empty. Write $\mathrm{Ell}^{\Gamma}_{\mathrm{KZB}} := \mathrm{Ell}^{\Gamma}(\mathbb{C})\times_{\mathrm{Ass}(\mathbb{C})}\{2\pi\mathrm{i},\Phi_{\mathrm{KZ}}\}$.

Theorem 7.2.1. *There is an analytic map*

$$\begin{aligned}\mathfrak{h} &\longrightarrow \mathrm{Ell}^{\Gamma}_{\mathrm{KZB}} \\ \tau &\longmapsto e^{\Gamma}(\tau) = (A^{\Gamma}(\tau),B^{\Gamma}(\tau)).\end{aligned}$$

In particular, for each $\tau\in\mathfrak{h}$, the element $(2\pi\mathrm{i},\Phi_{\mathrm{KZ}},A^{\Gamma}(\tau),B^{\Gamma}(\tau))$ is an ellipsitomic $\mathbb{C}$-associator (i.e. it belongs to $\mathrm{Ell}^{\Gamma}(\mathbb{C})$).

The rest of this section is devoted to the proof of the above theorem.

7.2.1 The solution $F^{(n)}_{\Gamma}(\mathbf{z}|\tau)$

The ellipsitomic KZB system is

$$(\partial/\partial z_i)F^{\Gamma}(\mathbf{z}|\tau) = \bar{K}^{\Gamma}_i(\mathbf{z}|\tau)F^{\Gamma}(\mathbf{z}|\tau), \quad (\partial/\partial\tau)F^{\Gamma}(\mathbf{z}|\tau) = \bar{\Delta}^{\Gamma}(\mathbf{z}|\tau)F^{\Gamma}(\mathbf{z}|\tau),$$

where $F^{\Gamma}(\mathbf{z}|\tau)$ is a function $(\mathbb{C}^n\times\mathfrak{H})-\Delta_{n,\Gamma}\supset U\longrightarrow \mathbf{G}_n\rtimes\mathfrak{S}_n$ invariant under translation by $\mathbb{C}(\sum_i\delta_i)$. Let

$$D^{\Gamma}_n := \left\{(\mathbf{z},\tau)\in\mathbb{C}^n\times\mathfrak{H}|z_i=a_i+b_i\tau, a_i,b_i\in\mathbb{R}, a_1<a_2<\ldots<a_n<a_1+\frac{1}{M}, b_1<b_2<\ldots<b_n<b_1+\frac{1}{N}\right\}.$$

Then $D_n^\Gamma \subset (\mathbb{C}^n \times \mathfrak{H}) - \Delta_{n,\Gamma}$ is simply connected and invariant under $\mathbb{C}(\sum_i \delta_i)$. A solution of the ellipsitomic KZB system on this domain is then unique, up to right multiplication by a constant. We now determine a particular solution $F_\Gamma^{(n)}(\mathbf{z}|\tau)$ of the ellipsitomic KZB system.

Let us denote $z_{ij} = z_i^{\mathbf{0}} - z_j^{\mathbf{0}}$ and let us compute the expansions of $\bar{K}_i(\mathbf{z}|\tau)$ and $\bar{\Delta}(\mathbf{z}|\tau)$ in the region $z_{ij} \ll 1$, $\tau \to i\infty$. We have

$$\begin{aligned}
\bar{K}_i(\mathbf{z}|\tau) &= -\bar{y}_i + \sum_{j;j\neq i}\sum_{\alpha\in\Gamma}\left(e^{-2\pi i a\,\mathrm{ad}(\bar{x}_i)}\frac{\theta(z_i - z_j - \tilde{\alpha} + \mathrm{ad}(\bar{x}_i);\tau)}{\theta(z_i - z_j - \tilde{\alpha};\tau)\theta(\mathrm{ad}(\bar{x}_i);\tau)} - \frac{1}{\mathrm{ad}(\bar{x}_i)}\right)(\bar{t}_{ij}^{\alpha}) \\
&= \sum_{j;j\neq i}\sum_{\alpha\in\Gamma}\left(\frac{1}{\mathrm{ad}(\bar{x}_i)} + \frac{\bar{t}_{ij}^{\alpha}}{z_i - z_j - \tilde{\alpha}} - \frac{1}{\mathrm{ad}(\bar{x}_i)}\right)(\bar{t}_{ij}^{\alpha}) + O(1) \\
&= \sum_{j;j\neq i}\sum_{\alpha\in\Gamma}\frac{\bar{t}_{ij}^{\alpha}}{z_i - z_j - \tilde{\alpha}} + O(1) = \sum_{j;j\neq i}\sum_{\alpha\in\Gamma}\frac{\bar{t}_{ij}^{\alpha}}{z_i - z_j - \frac{a_0}{M}} + O(1)
\end{aligned}$$

Notice the resemblance with the function which defines the universal cyclotomic KZ connection defined in [33, Section 1.4].

For the expansion of $\bar{\Delta}$, recall that if $\gamma \in \Gamma$ and $\tilde{\gamma} = (c_0, c) \in \Lambda_{\tau,\Gamma}$ is any lift of γ, we have $g_\gamma(z, x|\tau) := \partial_x k_\gamma(z, x|\tau)$ and

$$g_{-\gamma}(0, x|\tau) = \sum_{s\geq 0} A_{s,\gamma}(\tau)x^s.$$

We then have

$$\bar{\Delta}^\Gamma(\mathbf{z}|\tau) = \frac{-1}{2i\pi}\left(\Delta_0 + \frac{1}{2}\sum_{s\geqslant 0}\sum_{\gamma\in\Gamma}A_{s,\gamma}(\tau)\left(\delta_{s,\gamma} + 2\sum_{i,j:i<j}\mathrm{ad}(\bar{x}_i)^s(\bar{t}_{ij}^{-\gamma})\right)\right) + o(1),$$

for $z_{ij} \ll 1$ and any $\tau \in \mathfrak{H}$.

In section 13 we will relate $A_{s,\gamma}(\tau)$ to Eisenstein-Hurwitz series which have a q_N-expansion and we define the normalized version $\tilde{A}_{s,\gamma}(\tau)$ of the twisted Eisenstein series $A_{s,\gamma}(\tau)$ such that

$$A_{s,\gamma}(\tau) = a_{s,\gamma}\tilde{A}_{s,\gamma}(\tau),$$

and such that we have an expansion $\tilde{A}_{s,\gamma}(\tau) = 1 + \sum_{l>0} a_{kl,\gamma}e^{2\pi i l\tau/N}$ as $\tau \longrightarrow i\infty$. Then, by applying Proposition 3 in Appendix A of [24] with $u_n = z_{n1}$, $u_{n-1} = z_{n-1,1}/z_{n1}$,..., $u_2 = z_{21}/z_{31}$, $u_1 = q(\tau) = e^{2\pi i\tau/N}$, we obtain a unique solution $F_\Gamma^{(n)}(\mathbf{z}|\tau)$ with the expansion

$$\begin{aligned}
F_\Gamma^{(n)}(\mathbf{z}|\tau) \simeq{}& z_{21}^{\bar{t}_{12}^{\mathbf{0}}} z_{31}^{\bar{t}_{13}^{\mathbf{0}}+\bar{t}_{23}^{\mathbf{0}}} \ldots z_{n1}^{\bar{t}_{1n}^{\mathbf{0}}+\ldots+\bar{t}_{n-1,n}^{\mathbf{0}}} \\
&\exp\left(-\frac{\tau}{2\pi i}\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} a_{s,\gamma}\left(\delta_{s,\gamma} - 2\sum_{i<j}\mathrm{ad}^s(\bar{x}_i)(\bar{t}_{ij}^{-\gamma})\right)\right)\right)
\end{aligned}$$

in the region $z_{21} \ll z_{31} \ll \ldots \ll z_{n1} \ll 1$, $\tau \longrightarrow i\infty$, $(\mathbf{z}, \tau) \in D_n^\Gamma$. The sign $\simeq$ means here that any of the ratios of both sides is of the form

$$1 + \sum_{k>0}\sum_{i,a_1,\ldots,a_n} r_k^{i,a_1,\ldots,a_n}(u_1, \ldots, u_n),$$

where the second sum is finite with $a_i \geq 0$, $i \in \{1, ..., n\}$, $r_k^{i,a_1,\dots,a_n}(u_1, ..., u_n)$ has degree k, and is $O(u_i(\log u_1)^{a_1}...(\log u_n)^{a_n})$. We denote $F_\Gamma^{([n])}$ the solution with values on $\mathbf{G}_n \rtimes (\Gamma^n \mathfrak{S}_n)$ induced by $F_\Gamma^{(n)}$

7.2.2 A presentation of $\bar{\mathrm{B}}_{1,n}^\Gamma$

We use the same presentation of $\bar{\mathrm{B}}_{1,n}$ coming from [24] that we used in the proof of Theorem 4.3.2. Let us define $\mathrm{B}_{1,n}^\Gamma := \pi_1(\mathrm{Conf}(E_{\tau,\Gamma}, [n], \Gamma), [\mathbf{z}_0])$ and recall that $\mathrm{B}_{1,n} = \pi_1(\mathrm{Conf}(E_{\tau,\Gamma}, [n]), [\mathbf{z}_0])$. Now, since the canonical surjective map $\mathrm{Conf}(E_{\tau,\Gamma}, [n], \Gamma) \twoheadrightarrow \mathrm{Conf}(E_{\tau,\Gamma}, [n])$ defines a Γ-covering, then $\mathrm{B}_{1,n}^\Gamma = \ker(\rho)$, where $\rho : \mathrm{B}_{1,n} \longrightarrow \Gamma$ sends σ_i to $\mathbf{0} = (\bar{0}, \bar{0})$, A_i to $(\bar{1}, \bar{0})$ and B_i to $(\bar{0}, \bar{1})$. If A_i^M (resp. B_i^N) is the class of the path given by $[0, 1] \ni t \mapsto \mathbf{z}_0 + t\sum_{j=i}^n \delta_i$ (resp. $[0, 1] \ni t \mapsto \mathbf{z}_0 + t\tau\sum_{j=i}^n \delta_i$), then it follows from the geometric description of $\mathrm{B}_{1,n}^\Gamma$ that A_i^M, B_i^N $(i = 1, \dots, n)$ and

$$R_{ij}^\alpha := X_j^{-p} Y_j^{-q} C_{ij} Y_j^q X_j^p$$

(for $i < j$, $1 \leq p \leq M$, $1 \leq q \leq N$ and $\alpha = (\bar{p}, \bar{q})$) are generators of $\mathrm{B}_{1,n}^\Gamma$.

We denote again A_i^M and B_i^N $(i = 1, ..., n)$ for the projections of these elements to $\bar{\mathrm{B}}_{1,n}^\Gamma$.

7.2.3 The monodromy morphism $\gamma_n : \mathrm{B}_{1,n} \longrightarrow \mathbf{G}_n^\Gamma \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$

The monodromy of the flat $\mathbf{G}_n^\Gamma \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$-bundle $(P_{\Gamma,[n]}, \nabla_{\Gamma,[n]})$ on $\mathcal{M}_{1,[n]}$ provides us with a group morphism

$$\mu_{\mathbf{z}_0,\Gamma,[n]} : \pi_1(\mathcal{M}_{1,n}^\Gamma/(\Gamma^n \rtimes \mathfrak{S}_n)) \longrightarrow \mathbf{G}_n^\Gamma \rtimes (\Gamma^n \rtimes \mathfrak{S}_n),$$

where $\pi_1(\mathcal{M}_{1,n}^\Gamma/(\Gamma^n \rtimes \mathfrak{S}_n))$ is the mapping class group (i.e. the orbifold fundamental group) associated to $\mathcal{M}_{1,n}^\Gamma/(\Gamma^n \rtimes \mathfrak{S}_n)$. This actually fits into a morphism of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{PB}_{1,n}^\Gamma \arrow[r] \arrow[d] & \mathrm{B}_{1,n} \arrow[r] \arrow[d] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] \arrow[d, equal] & 1 \ , \\
1 \arrow[r] & \mathrm{MCG}_{1,n}^\Gamma \arrow[r] \arrow[d] & \pi_1(\mathcal{M}_{1,n}^\Gamma/(\Gamma^n \rtimes \mathfrak{S}_n)) \arrow[r] \arrow[d] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{G}_n^\Gamma \arrow[r] & \mathbf{G}_n^\Gamma \rtimes (\Gamma^n \rtimes \mathfrak{S}_n) \arrow[r] & \Gamma^n \rtimes \mathfrak{S}_n \arrow[r] & 1
\end{tikzcd}$$

where $\mathrm{MCG}_{1,n}^\Gamma := \pi_1(\mathcal{M}_{1,n}^\Gamma)$ is the mapping class group associated to $\mathcal{M}_{1,n}^\Gamma$, the top vertical arrows are injections and the bottom first vertical morphism is the monodromy morphism

$$\mu_{\mathbf{z}_0,n,\Gamma} : \mathrm{MCG}_{1,n}^\Gamma \longrightarrow \mathbf{G}_n^\Gamma$$

of associated with the flat $\mathbf{G}_n^\Gamma$-bundle $(P_{n,\Gamma}, \nabla_{n,\Gamma})$ on $\mathcal{M}_{1,n}^\Gamma$.

Indeed, this comes from the fact that $\nabla_{\Gamma,[n]}$ is obtained by descent, from $\nabla_{n,\Gamma}$ and using its equivariance properties of Proposition 6.3.14. We denote

$$\tilde{\gamma}_n^\Gamma : \mathrm{PB}_{1,n}^\Gamma \longrightarrow \mathbf{G}_n^\Gamma$$

and

$$\gamma_n^\Gamma : \mathrm{B}_{1,n} \longrightarrow \mathbf{G}_n^\Gamma \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$$

the corresponding vertical composites.

Let $F_\Gamma(\mathbf{z}|\tau)$ be a solution of the ellipsitomic KZB system defined on D_n^Γ with values in $\mathbf{G}_n^\Gamma \rtimes \Gamma^n$. Let us consider the domains

$$H_n^\Gamma := \left\{(\mathbf{z},\tau) \in \mathbb{C}^n \times \mathfrak{H} | z_i = a_i + b_i\tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < ... < a_n < a_1 + \frac{1}{M}\right\}$$

and

$$V_n^\Gamma := \left\{(\mathbf{z},\tau) \in \mathbb{C}^n \times \mathfrak{H} | z_i = a_i + b_i\tau, a_i, b_1 < b_2 < ... < b_n < b_1 + \frac{1}{N}\right\}.$$

Both of these domains are simply connected and invariant. We denote $F_\Gamma^H(\mathbf{z}|\tau)$ and $F_\Gamma^V(\mathbf{z}|\tau)$ the prolongations of $F_\Gamma(\mathbf{z}|\tau)$ to these domains.

Then

$$(\mathbf{z},\tau) \longmapsto F_\Gamma^H\left(\mathbf{z} + \sum_{j=i}^n \frac{\delta_i}{M}|\tau\right)$$
$$(\mathbf{z},\tau) \longmapsto e^{2\pi\mathrm{i}\frac{(\bar{x}_i+\ldots+\bar{x}_n)}{N}} F_\Gamma^V\left(\mathbf{z} + \tau(\sum_{j=i}^n \frac{\delta_i}{N})|\tau\right)$$

are solutions of the ellipsitomic KZB system on H_n^Γ and V_n^Γ respectively. Let us define $\underline{A}_i^F, \underline{B}_i^F \in \mathbf{G}_n^\Gamma \rtimes \Gamma^n$ by

$$F_\Gamma^H\left(\mathbf{z} + \sum_{j=i}^n \frac{\delta_i}{M}|\tau\right) = \prod_{j=i}^n (\bar{1},\bar{0})_j F_\Gamma^H(\mathbf{z}|\tau)\underline{A}_i^F,$$

$$e^{2\pi\mathrm{i}\frac{(\bar{x}_i+\ldots+\bar{x}_n)}{N}} F_\Gamma^V\left(\mathbf{z} + \tau(\sum_{j=i}^n \frac{\delta_i}{N})|\tau\right) = \prod_{j=i}^n (\bar{0},\bar{1})_j F_\Gamma^V(\mathbf{z}|\tau)\underline{B}_i^F.$$

We also define $\sigma_i^F \in \mathfrak{S}_n$ by means of

$$\sigma_i F_\Gamma(\sigma_i^{-1}\mathbf{z}|\tau) = F_\Gamma(\mathbf{z}|\tau)\sigma_i^F,$$

where, on the left hand side, F_Γ is extended to the universal cover of $(\mathbb{C}^n \times \mathfrak{h}) - \mathrm{Diag}_{n,\Gamma}$. Notice that σ_i exchanges $z_i^{\mathbf{0}}$ and $z_{i+1}^{\mathbf{0}}$, $z_{i+1}^{\mathbf{0}}$ passing to the right of $z_i^{\mathbf{0}}$. Its monodromy is given by $e^{\pi\mathrm{i}t^{\mathbf{0}}_{i(i+1)}}$.

Let us denote $\underline{X}_i^p := \underline{A}_i^p(\underline{A}_{i+1}^p)^{-1}$ and $\underline{Y}_i^q := \underline{B}_i^q(\underline{B}_{i+1}^q)^{-1}$, $X_i^{(p)} := A_i^{(p)}(A_{i+1}^{(p)})^{-1}$ and $Y_i^{(q)} := B_i^{(q)}(B_{i+1}^{(q)})^{-1}$ and recall that $\theta((-\alpha)_j) \cdot \bar{t}^0_{ij} = \bar{t}^\alpha_{ij}$.

Lemma 7.2.2. *The morphism $\tilde{\gamma}_n : \mathrm{PB}_{1,n}^\Gamma \longrightarrow \mathbf{G}_n^\Gamma$ induced by the solution F^Γ takes A_i^M to $(A_i^F)^M$, B_i^N to $(B_i^F)^N$. Let us denote $R_{ij}^\alpha := X_j^{-p}Y_j^{-q}C_{ij}Y_j^qX_j^p$ for all $\alpha = (\bar{p},\bar{q}) \in \Gamma$ and denote $\underline{\tilde{X}}_i^p := \tilde{\gamma}_n(X_i^p)$ and $\underline{\tilde{Y}}_i^q := \tilde{\gamma}_n(X_i^q)$. Then R_{ij}^α is sent via $\tilde{\gamma}_n$ to*

$$\underline{\tilde{R}}_{ij}^\alpha = g_1(\bar{p},\bar{0})_j(\bar{0},\bar{q})_j e^{2\pi\mathrm{i}t^{\mathbf{0}}_{ij}}(\bar{0},\bar{-q})_j(\bar{-p},\bar{0})_j g_1^{-1}$$

and σ_i^α *is sent via* $\tilde\gamma_n$ *to*

$$\underline{\tilde{C}}_i^\alpha \quad = \quad g_2(\bar p,\bar 0)_{i+1}(\bar 0,\bar q)_{i+1}e^{\pi \mathrm{i} t^{\mathbf{0}}_{i,i+1}}(\bar 0,\bar{-q})_i(\bar{-p},\bar 0)_i g_2^{-1}$$

Proof. This follows from the geometric description of the generators of $\mathrm{B}_{1,[n]}$: if $(\mathbf{z}_0,\tau_0)\in D_n^\Gamma$, then A_i is the class of the projection of the path $[0,1]\ni t\mapsto(\mathbf{z}_0+t\sum_{j=i}^n(\delta_j/M),\tau_0)$ and B_i is the class of the projection of $[0,1]\ni t\mapsto(\mathbf{z}_0+t\tau\sum_{j=i}^n(\delta_j/N),\tau_0)$. Finally, as paths in H_n^Γ, A^M and $A^{(M)}$ are homotopic. Likewise, as paths in V_n^Γ, B^N and $B^{(N)}$ are homotopic. □

Thus, following the same conventions as before, we set the following elements in $\mathbf{G}_n^\Gamma$

$$\tilde{R}_{ij}^\alpha := g_1 = \prod_{l=0}^{p-1}(\theta((\overline{p-l},\bar q))\cdot X_j^{-1})\prod_{l=0}^{q-1}(\theta((\bar 0,\overline{q-l}))\cdot Y_j^{-1})e^{2\pi\mathrm{i}t^{\mathbf{0}}_{ij}}\prod_{l=0}^{q-1}(\theta((\bar 0,\bar l))\cdot Y_j)\prod_{l=0}^{p-1}(\theta((\bar l,\bar q))\cdot X_j),$$

and

$$\tilde{C}_i^\alpha := g_2 = \prod_{l=0}^{p-1}(\theta((\overline{p-l},\bar q))\cdot X_{i+1}^{-1})\prod_{l=0}^{q-1}(\theta((\bar 0,\overline{q-l}))\cdot Y_{i+1}^{-1})e^{\pi\mathrm{i}t^{\mathbf{0}}_{i,i+1}}\prod_{l=0}^{q-1}(\theta((\bar 0,\bar l))\cdot Y_i)\prod_{l=0}^{p-1}(\theta((\bar l,\bar q))\cdot X_i).$$

We will denote by $\tilde\gamma_n:\mathrm{PB}_{1,n}^\Gamma\longrightarrow\mathbf{G}_n^\Gamma$ the morphism induced by the solution $F_\Gamma^{(n)}(\mathbf{z}|\tau)$ and $\gamma_n:\mathrm{B}_{1,n}\longrightarrow\mathbf{G}_n^\Gamma\rtimes(\Gamma^n\rtimes\mathfrak{S}_n)$ the one induced by $F_\Gamma^{([n])}$.

7.2.4 Expression of $\gamma_n:\mathrm{B}_{1,n}\longrightarrow\mathbf{G}_n^\Gamma\rtimes(\Gamma^n\rtimes\mathfrak{S}_n)$ using γ_1 and γ_2

Lemma 7.2.3. $\tilde\gamma_2(A_2^M)$ *and* $\tilde\gamma_2(B_2^N)$ *belong to* $\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)\subset\mathbf{G}_2^\Gamma$.

Proof. If $F_\Gamma(\mathbf{z}|\tau):H_2^\Gamma\longrightarrow\mathbf{G}_2^\Gamma$ is a solution of the ellipsitomic KZB equation for $n=2$, then $A_2^F=F_\Gamma^H(\mathbf{z}-\delta_2|\tau)F_\Gamma^H(\mathbf{z}|\tau)^{-1}$ is the iterated integral, from $\mathbf{z}_0\in D_n^\Gamma$ to $\mathbf{z}_0-\delta_2$, of $K_2(\mathbf{z}|\tau)\in\hat{\mathfrak{t}}_{1,2}^\Gamma$. Thus, $A_2^F\in\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)$. Then, as $\gamma_2(A_2^M)$ is a conjugate of $(A_2^F)^M$, it belongs to $\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)$ as $\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)\subset\mathbf{G}_2^\Gamma\rtimes\mathfrak{S}_2$ is normal. One proves in the same way that $\gamma_2(B_2^N)\in\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)$. □

We let the reader check that the above lemma remains true in the reduced case.

7.2.5 Algebraic relations for the ellipsitomic KZB associator

Let us set

$$\Phi_i:=\Phi^{1\ldots i-1,i,i+1\ldots n}\ldots\Phi^{1\ldots n-2,n-1,n}\in\exp(\hat{\mathfrak{t}}_n),$$

and denote by $x\mapsto\{x\}$ the morphism $\exp(\hat{\mathfrak{t}}_n)\longrightarrow\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$ induced by $t_{ij}\mapsto t_{ij}^{\mathbf{0}}$.

Proposition 7.2.4. *If* $n\geq 3$*, then*

$$\gamma_n(A_i)=\{\Phi_i\}\gamma_2(A_2)^{1\ldots i-1,i\ldots n}(\bar 1,\bar 0)_i\{\Phi_i\}^{-1},$$

$$\gamma_n(B_i)=\{\Phi_i\}\gamma_2(B_2)^{1\ldots i-1,i\ldots n}(\bar 0,\bar 1)_i\{\Phi_i\}^{-1},\ (i=1,\ldots,n),$$

and

$$\gamma_n(\sigma_i^\alpha) \quad = \quad g_2(\bar{p},\bar{0})_{i+1}(\bar{0},\bar{q})_{i+1}e^{\pi \mathrm{i} t^{\mathbf{0}}_{i,i+1}}(\bar{0},\bar{-q})_i(\bar{-p},\bar{0})_i g_2^{-1}$$

where $i=1,...,n-1,$ *and* $\alpha=(\bar{p},\bar{q})$.

Proof. Let $G_i^\Gamma(\mathbf{z}|\tau)$ be the solution of the elliptic Γ-KZB system, such that

$$\begin{aligned} G_i^\Gamma(\mathbf{z}|\tau) =& z_{21}^{t^{\mathbf{0}}_{12}}...z_{i-1,1}^{t^{\mathbf{0}}_{12}+...+t^{\mathbf{0}}_{1,i-1}} z_{n,i}^{t^{\mathbf{0}}_{i,n}+...+t^{\mathbf{0}}_{n-1,n}}...z_{n,n-1}^{t^{\mathbf{0}}_{n-1,n}} \\ & \times \exp\left(-\frac{\tau}{2\pi\mathrm{i}}\left(\Delta_0+\frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} a_{s,\gamma}\left(\delta_{s,\gamma}-2\sum_{i<j}\mathrm{ad}^s(x_i)(t_{ij}^{-\gamma})\right)\right)\right), \end{aligned}$$

when $z_{21}\ll ...\ll z_{i-1,1}\ll 1$, $z_{n,n-1}\ll ...\ll z_{n,i}\ll 1$, $\tau\longrightarrow \mathrm{i}\infty$ and $(\mathbf{z},\tau)\in D_n^\Gamma$ (we set $z_{ij}=z_i^{\mathbf{0}}-z_j^{\mathbf{0}}$ as before). Then

$$G_i^\Gamma(\mathbf{z}+\sum_{j=i}^n\delta_i|\tau)=\prod_{j=i}^n(\bar{1},\bar{0})_j G_i^\Gamma(\mathbf{z}|\tau)\gamma_2(A_2^M)^{1...i-1,i...n},$$

because in the domain considered $K_i(\mathbf{z}|\tau)$ is close to $K_2(z_1,z_n|\tau)^{1...i-1,i...n}$ (where $K_2(...)$ corresponds to the 2-point system); on the other hand, $F^\Gamma(\mathbf{z}|\tau)=G_i^\Gamma(\mathbf{z}|\tau)\{\Phi_i\}$, which implies the formula for $\gamma_n(A_i)$. The formula for $\gamma_n(B_i)$ is proved in the same way. Finally, the behavior of $F_\Gamma^{(n)}(\mathbf{z}|\tau)$ for $z_{21}^{\mathbf{0}}\ll ...\ll z_{n1}^{\mathbf{0}}\ll 1$ is similar to that of a solution of the KZ equations and we know that the twisted elliptic KZB connection is Γ-equivariant. This implies the formula for $\gamma_n(\sigma_i^\alpha)$. □

Let us now finish the proof of Theorem 7.2.1. We set, for $\alpha\in\Gamma$,

$$\theta((\alpha)_i)\cdot A^{1_0,2_0,...,i_0,...,n_0}=A^{1_0,2_0,...,i_\alpha,...,n_0}.$$

`changer a partir d'ici`

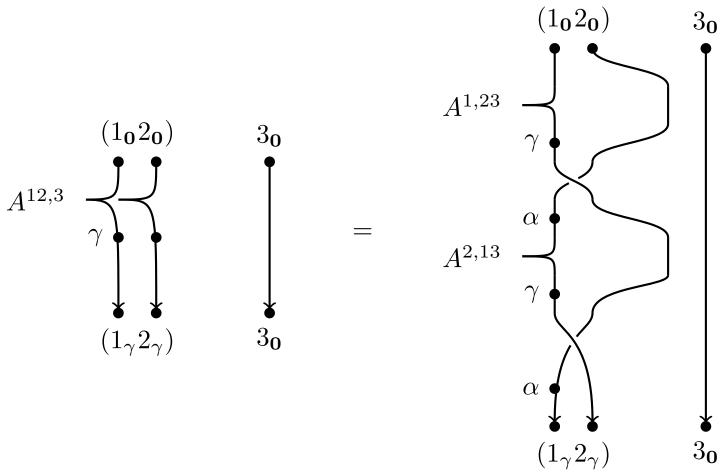
Set $\tilde{A}:=\gamma_2(A_2)$, $\tilde{B}:=\gamma_2(B_2)$. The image of the relation

$$A_2(\theta((\bar{1},\bar{0})_1)\cdot A_3^{-1})=(\sigma_1^\alpha)^{-1}\theta((-\alpha)_1)\cdot(A_2^{-1}(\sigma_1^\alpha)^{-1})$$

by γ_3 yields

$$\begin{aligned} \tilde{A}^{12,3} \quad = \quad & \{\Phi\}^{1,2,3}\tilde{A}^{1,23}\theta((\bar{1},\bar{0})_1)\cdot \\ & \left((\{\Phi\}^{1,2,3})^{-1}\tilde{C}_1^\alpha\theta((\alpha)_2)\left(\{\Phi\}^{2,1,3}\tilde{A}^{2,13}\theta\left((\bar{1},\bar{0})_2\right)\cdot((\{\Phi\}^{2,1,3})^{-1}\tilde{C}_1^\alpha)\right)\right). \end{aligned}$$

This relation can be depicted as follows:



(TNAbis)

where $\gamma = (\bar{1}, \bar{0})$ and $\alpha \in \Gamma$. In the same way, we obtain

$$\begin{aligned}\tilde{B}^{12,3} \quad = \quad & \{\Phi\}^{1,2,3}\tilde{B}^{1,23}\theta((\bar{0},\bar{1})_1)\cdot \\ & \left((\{\Phi\}^{1,2,3})^{-1}(\tilde{C}_1^{\alpha})^{-1}\theta((\alpha)_2)\left(\{\Phi\}^{2,1,3}\tilde{B}^{2,13}\theta\left((\bar{0},\bar{1})_2\right)\cdot((\{\Phi\}^{2,1,3})^{-1}(\tilde{C}_1^{\alpha})^{-1})\right)\right)\end{aligned}$$

Accordingly, the image by γ_3 of the lift of the relation $(B_3, A_3A_2^{-1}) = (B_3B_2^{-1}, A_3) = C_{23}$ to $\bar{\mathrm{B}}_{1,n}^{\Gamma}$ then gives

$$\tilde{B}^{12,3}\theta((\bar{0},\bar{1})_{1,2})\left(\Phi(\tilde{A}^{1,23})^{-1}\theta((\overline{-1},\bar{0})_1)\left(\Phi^{-1}\tilde{A}^{12,3}\theta((\bar{1},\bar{0})_{1,2})\left((\tilde{B}^{12,3})^{-1}\theta\left((\bar{0},\overline{-1})_{1,2}\right)\cdot X\right)\right)\right)$$

$$= \Phi(\tilde{B}^{1,23})^{-1}\theta((\bar{0},\overline{-1})_1)\left(\Phi^{-1}\tilde{B}^{12,3}\theta((\bar{0},\bar{1})_{1,2})\left(\tilde{A}^{12,3}\theta((\bar{1},\bar{0})_{1,2})\left((\tilde{B}^{12,3})^{-1}\theta\left((\bar{0},\overline{-1})_{1,2}\right)\cdot Y\right)\right)\right)$$

$$= \Phi^{-1}e^{2\pi \mathrm{i}\bar{t}_{23}^{\mathbf{1}}}\Phi,$$

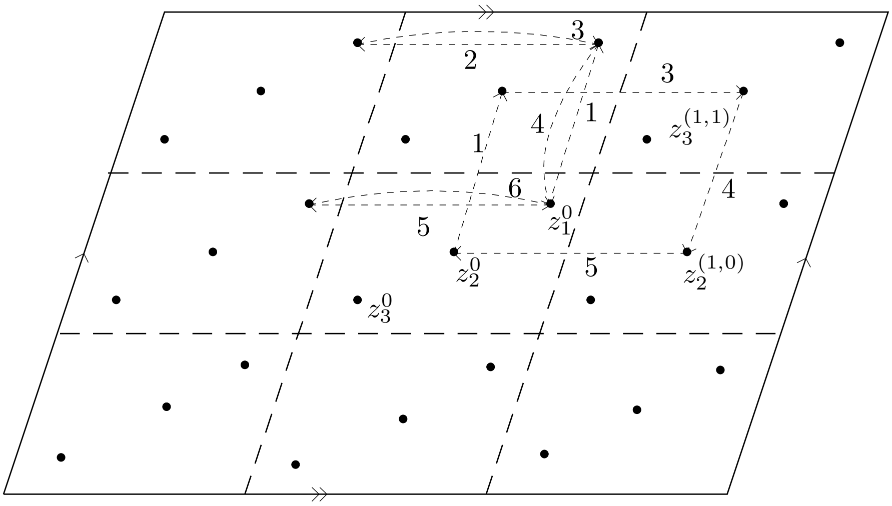
where $\mathbf{1} = (\bar{1}, \bar{1})$,

$$X = ((\tilde{A}^{12,3})^{-1}\Phi^{-1}\theta((\overline{-1},\bar{0})_{1,2})(\Phi\tilde{A}^{1,23}\theta((\bar{1},\bar{0})_1)\Phi^{-1})),$$

and

$$Y = (\Phi\tilde{B}^{1,23}\theta((\bar{0},\bar{1})_1)((\Phi)^{-1}(\tilde{A}^{12,3})^{-1})).$$

One can simply draw the l.h.s. of these double equation as follows: we simplify the paths by just neglecting the associators and we suppose that the central portion of the torus corresponds to the $(\bar{0},\bar{0})$-labelled region with respect to the sublattice $\Lambda_{\tau,\Gamma}$. Then we enumerate the different movements (read from left to right in the l.h.s of the equation) of the marked points in the twisted configuration space:



We can see that the $z_2^{\mathbf{0}}$ is only braided with $z_3^{\mathbf{0}}$ since $z_1^{\mathbf{0}}$ moved to $z_1^{(\bar{0},\bar{1})}$ in the first movement. By applying $x \mapsto x^{\emptyset,1,2}$, this identity implies

$$\tilde{A}(\theta((\bar{1},\bar{0})) \cdot \tilde{B})(\theta((\bar{1},\bar{1})) \cdot \tilde{A}^{-1})(\theta((\bar{0},\bar{1})) \cdot \tilde{B}^{-1}) = e^{-2\pi \mathrm{i} \bar{t}_{12}^{\mathbf{0}}}.$$

Since the universal twited elliptic KZB connection is Γ-equivariant, then this equations are also Γ-equivariant. Now, let us denote

$$S = \tilde{B}^{12,3}\theta((\bar{0},\bar{1})_{1,2}) \left(\Phi(\tilde{A}^{1,23})^{-1}\theta((\overline{-1},\bar{0})_1) \left(\Phi^{-1}\tilde{A}^{12,3}\theta((\bar{1},\bar{0})_{1,2}) \left((\tilde{B}^{12,3})^{-1}\theta\left((\bar{0},\overline{-1})_{1,2}\right) \cdot X\right)\right)\right).$$

We then have

$$e^{-2\pi \mathrm{i}\sum_{i=0}^{M-1} \bar{t}_{12}^{(\bar{i},\bar{0})}} =$$

$$SA_1\theta((\bar{1},\bar{0})_{1,2,3}) \cdot (SA_1^{-1})A_1^{(2)}\theta((\bar{2},\bar{0})_{1,2,3}) \cdot (SA_1^{-2}) \cdots A_1^{(M-1)}\theta((\overline{M-1},\bar{0})_{1,2,3}) \cdot (SA_1^{-(M-1)}).$$

Now denoting by T the r.h.s of this equation we get

$$e^{-2\pi \mathrm{i}\sum_{\alpha\in\Gamma} \bar{t}_{12}^{\alpha}} =$$

$$TB_1\theta((\bar{0},\bar{1})_{1,2,3}) \cdot (TB_1^{-1})B_1^{(2)}\theta((\bar{0},\bar{2})_{1,2,3}) \cdot (TB_1^{-2}) \cdots B_1^{(N-1)}\theta((\bar{0},\overline{N-1})_{1,2,3}) \cdot (TB_1^{-(N-1)}).$$

By taking the log of this last equation we retrieve relation $[\overline{x}_1, \overline{y}_2] = \sum_{\alpha\in\Gamma} \bar{t}_{12}^{\alpha}$. In the same way, one can show that $A^{(M)}$ satisfy the elliptic first nonagon equation. The same will be satisfied by $B^{(N)}$. The elliptic mixed equation for $n = 2$ will be then written as

$$(\tilde{A}^{(M)}, \tilde{B}^{(N)}) = e^{-2\pi \mathrm{i} \sum\limits_{\beta\in\Gamma} \bar{t}_{12}^{\beta}}.$$

Finally, one can see that if we take Γ to be trivial, we retrieve equations (22), (23), (24), (25) and (26) in [24].

In order to finish the proof of Theorem 7.2.1, one has to take different boundary conditions for our KZB solutions. The couple $e^{\Gamma}(\tau) := (A^{\Gamma}(\tau), B^{\Gamma}(\tau)) \in \exp(\widehat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2}) \times \exp(\widehat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})$ is defined by

$$A^{\Gamma}(\tau) := J(z)^{-1}\theta(\bar{1},\bar{0})J(z+\frac{1}{M}), \quad B^{\Gamma}(\tau) := J(z)^{-1}\theta(\bar{0},\bar{1})e^{\frac{2\pi\mathrm{i}}{N}x}J(z+\frac{\tau}{N}),$$

where $J(z)$ is the unique solution defined over $\{\frac{a}{M}+\frac{b}{N}\tau, \text{for } a,b \in]0,1[\}$ such that we have $J(z) \simeq (-2\pi\mathrm{i}z)^{t^{\mathbf{o}}}$ at $z \longrightarrow 0$. The couple $(\tilde{A}^{\Gamma}, \tilde{B}^{\Gamma}) \in \exp(\widehat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2}) \times \exp(\widehat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})$ is defined by

$$\tilde{A}^{\Gamma} := \tilde{J}(z)^{-1}\theta(\bar{1},\bar{0})\tilde{J}(z+\frac{1}{M}), \quad \tilde{B}^{\Gamma} := \tilde{J}(z)^{-1}\theta(\bar{0},\bar{1})e^{\frac{2\pi\mathrm{i}}{N}x}\tilde{J}(z+\frac{\tau}{N}),$$

where $\tilde{J}(z)$ is the unique solution defined over $\{\frac{a}{M}+\frac{b}{N}\tau, \text{for } a,b \in]0,1[\}$ such that we have $\tilde{J}(z) \simeq z^{t^{\mathbf{o}}}\varphi(\tau)$ at $z \longrightarrow 0$, where

$$\varphi(\tau) := \exp\left(-\frac{\tau}{2\pi\mathrm{i}}\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} a_{s,\gamma}\bar{\xi}^{(2)}_{s,\gamma}\right)\right).$$

Thus, we have $J(z) = (-2\pi\mathrm{i})^{t^{\mathbf{o}}}\tilde{J}(z)\varphi(\tau)^{-1}$ and $\tilde{J}(z) = (-2\pi\mathrm{i})^{-t^{\mathbf{o}}}J(z)\varphi(\tau)$. We compute

$$\begin{aligned}(\bar{1},\bar{0})J(z+\frac{1}{M})J(z)^{-1} &= (-2\pi\mathrm{i})^{t^{\mathbf{o}}}(\bar{1},\bar{0})\tilde{J}(z+\frac{1}{M})\varphi(\tau)^{-1}\varphi(\tau)\tilde{J}(z)^{-1}(-2\pi\mathrm{i})^{-t^{\mathbf{o}}}\\ (\bar{1},\bar{0})J(z+\frac{1}{M})J(z)^{-1} &= \mathrm{Ad}((-2\pi\mathrm{i})^{t^{\mathbf{o}}})((\bar{1},\bar{0})\tilde{J}(z+\frac{1}{M})\tilde{J}(z)^{-1})\end{aligned}$$

This means that $\underline{A}^{\Gamma}(\tau) = \mathrm{Ad}((-2\pi\mathrm{i})^{t^{\mathbf{o}}})((\bar{1},\bar{0})\underline{\tilde{A}}^{\Gamma})$. The same argument for $\underline{B}^{\Gamma}(\tau)$ and $\underline{\tilde{B}}^{\Gamma}$ shows that $\underline{B}^{\Gamma}(\tau) = \mathrm{Ad}((-2\pi\mathrm{i})^{t^{\mathbf{o}}})((\bar{0},\bar{1})\underline{\tilde{B}}^{\Gamma})$. We conclude that $e^{\Gamma}(\tau) = (A^{\Gamma}_{+}(\tau), A^{\Gamma}_{-}(\tau))$ satisfy (tN1) and (tN2). Next, (tE) is obtained in the same way as in the untwisted case (see [34] Proposition 3.8) and this concludes the proof of Theorem 7.2.1.

Remark 7.2.5. *The modularity relations of* $e^{\Gamma}(\tau)$*, depending on the chosen congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$*, will be investigated in forthcoming works by the second author.*

7.3 The Eisenstein-Hurwitz series

For any $\gamma \in \Gamma$, recall that $g_{\gamma}(z,x|\tau) := \partial_x k_{\gamma}(z,x|\tau)$. Until now, the terms $A_{s,\gamma}(\tau)$ were determined as the coefficients of the expansion

$$g_{-\gamma}(0,x|\tau) = \sum_{s\geq 0} A_{s,\gamma}(\tau)x^s.$$

In this section we give an explicit definition of these functions, show that they are modular forms for the group $\mathrm{SL}^{\Gamma}_2(\mathbb{Z})$ and relate them to cyclotomic zeta values. We also determine their normalized variant $\tilde{A}_{s,\gamma}(\tau)$ with constant term 1 on their q_N-expansion that we used to apply [24, Proposition A.3] at the end of Section 11.1.

Recall that the Weierstrass function is the function $\wp : \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right).$$

It is even, periodic with respect to the latice $(\mathbb{Z} \oplus \tau\mathbb{Z})$ and meromorphic with poles of order exactly 2 in $(\mathbb{Z} \oplus \tau\mathbb{Z})$.

We have the following identities for $z \in \mathbb{C} - (\mathbb{Z} \oplus \tau\mathbb{Z})$:

$$\wp(z,\tau) = -\left(\frac{\theta'}{\theta}\right)'(z,\tau) + c = -\partial_z^2(\log(\theta(z,\tau))) + c,$$

for a constant $c \in \mathbb{C}$. Next, for z in a suitable punctured neighborhood of $z_0 = 0$ (i.e. in the maximal punctured open disk centered at 0 which does not contain any non-zero lattice point), we have a Laurent expansion

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{k=0}^{\infty} b_{2k} z^{2k} = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2}(\tau) z^{2k},$$

where $b_{2n} = \frac{f^{(2n)}(0)}{(2n)!}$ with $f(z) = \wp(z) - \frac{1}{z^2}$. Here $G_k(\tau)$ are the Eisenstein series defined for all $k \geq 1$, by

$$G_k(\tau) := \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{m=-\infty \\ m \neq 0 \text{ if } n=0}}^{\infty} \frac{1}{(m+n\tau)^k} \right) = 2\zeta(k) + \frac{2 \cdot (2\pi \mathrm{i})^k}{(k-1)!} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m,$$

where $\sigma_\alpha(k) = \sum_{d|k} d^\alpha$. We have $G_k(\tau) = 0$ if k is odd. We will also use the normalized Eisenstein series $E_k(\tau)$, defined for $k \geq 4$ even, by $E_k := \frac{G_k(\tau)}{2\zeta(k)}$ so that, for $n \geq 1$, we have

$$(2n+1) G_{2n+2}(\tau) = \tilde{a}_{2n} E_{2n+2}(\tau)$$

where

$$\tilde{a}_{2n} = -(2n+1) \mathcal{B}_{2n+2} (2\mathrm{i}\pi)^{2n+2} / (2n+2)!,$$

where $\mathcal{B}_n$ are the Bernoulli numbers given by $x/(e^x - 1) = \sum_{r \geq 0} (\mathcal{B}_r / r!) x^r$. In particular, the constant term in the q-expansion of the series E_{2n} is equal to 1.

Finally, also recall the expansion $\theta(x,\tau) = x + 2\pi \mathrm{i} \partial_\tau \log \eta(\tau) x^3 + O(x^5)$.

7.3.1 Twisted Eisenstein series

First of all, set $\gamma = \mathbf{0}$. We get, as in [24, Section 4.1],

$$g_{\mathbf{0}}(0, x|\tau) = (\theta'/\theta)'(x) + 1/x^2 = -\sum_{k \geq 0} \tilde{a}_{2k} E_{2k+2}(\tau) x^{2k},$$

where $\tilde{a}_0 = \pi^2/3$, $E_2(\tau) = \frac{24}{2\pi \mathrm{i}} \partial_\tau \log \eta(\tau)$, and for $n \geq 1$, .

We now concentrate to the case where $\gamma \in \Gamma - \mathbf{0}$. Let $\gamma \in \Gamma - \{0\}$ and let $\tilde{\gamma} = (c_0, c) \in \Lambda_{\tau,\Gamma} - \Lambda_\tau$ be any lift of γ.

By using the identity $\partial_x f(x) = \partial_x(\log(f(x))) \times f(x)$, we get

$$\begin{aligned} g_\gamma(z,x|\tau) &= \partial_x k_\gamma(z,x|\tau) \\ &= \left(2\pi \mathrm{i} c + \left(\frac{\theta'}{\theta}\right)(z+x-\tilde{\gamma}) - \left(\frac{\theta'}{\theta}\right)(x)\right) e^{2\pi \mathrm{i} cx} \frac{\theta(z-\tilde{\gamma}+x)}{\theta(z-\tilde{\gamma})\theta(x)} + \frac{1}{x^2} \\ &= \sum_{n=0}^{\infty} \frac{g_\gamma^{(n)}(0,x|\tau)}{n!} z^n. \end{aligned}$$

Let us determine $g_{-\gamma}(0,x|\tau) = \sum_{s\geq 0} A_{s,\gamma}(\tau)x^s = \sum_{s\geq 0} \frac{g_{-\gamma}^{(s)}(0,0|\tau)}{s!} x^s$. We have

$$\begin{aligned} g_{-\gamma}(0,x|\tau) &= \lim_{z\to 0}\left(\left(2\pi \mathrm{i} c + \left(\frac{\theta'}{\theta}\right)(z+x+\tilde{\gamma}) - \left(\frac{\theta'}{\theta}\right)(x)\right) e^{2\pi \mathrm{i} cx} \frac{\theta(z+\tilde{\gamma}+x)}{\theta(z+\tilde{\gamma})\theta(x)}\right) + \frac{1}{x^2} \\ &= \left(2\pi \mathrm{i} c + \left(\frac{\theta'}{\theta}\right)(x+\tilde{\gamma}) - \left(\frac{\theta'}{\theta}\right)(x)\right) e^{2\pi \mathrm{i} cx} \frac{\theta(\tilde{\gamma}+x)}{\theta(\tilde{\gamma})\theta(x)} + \frac{1}{x^2} \\ &= (2\pi \mathrm{i} c + \left(\frac{\theta'}{\theta}\right)(\tilde{\gamma}) - \frac{1}{x})(1+2\pi \mathrm{i} cx + 2\pi \mathrm{i} cx^2)(\frac{1}{x} + \left(\frac{\theta'}{\theta}\right)(\tilde{\gamma})) + \frac{1}{x^2} + o(x) \\ &= (2\pi \mathrm{i} c)^2 - \left(\frac{\theta'}{\theta}\right)^2(\tilde{\gamma}) - 2\pi \mathrm{i} c\left(\frac{\theta'}{\theta}\right)(\tilde{\gamma}) - \pi \mathrm{i} c + o(x). \end{aligned}$$

Set $F_\gamma(x) := e^{2\pi \mathrm{i} cx} \frac{\theta(\tilde{\gamma}+x)}{\theta(\tilde{\gamma})\theta(x)}$ so that

$$\log(F_\gamma(x)) = \log(\theta(\tilde{\gamma}+x)) - \log(\theta(x)) + 2\pi \mathrm{i} cx - \log(\theta(\tilde{\gamma})). \tag{7.2}$$

We have

$$\begin{aligned} \partial_x^2(\log(F_\gamma(x))) &= \partial_x^2(\log(\theta(\tilde{\gamma}+x))) - \partial_x^2(\log(\theta(x))) \\ &= \wp(x) - \wp(\tilde{\gamma}+x) \\ &= \frac{1}{x^2} - \frac{1}{(x+\tilde{\gamma})^2} + \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \left(\frac{1}{(x+m+n\tau)^2} - \frac{1}{(x+\tilde{\gamma}+m+n\tau)^2}\right). \end{aligned}$$

Now let $s \geqslant 0$. Recall the expansion

$$\frac{1}{(x+y)^2} = \sum_{s\geqslant 0} a_s \frac{x^s}{y^{s+2}},$$

where a_s is the generalized binomial coefficient

$$\begin{pmatrix} -2 \\ s \end{pmatrix} = (-1)^s \begin{pmatrix} 2+s-1 \\ s \end{pmatrix} = (-1)^s(s+1).$$

On the one hand, for $y = m + n\tau$, we have

$$\begin{aligned} H(x,\tau) &:= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \left(\frac{1}{(x+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right) \\ &= \sum_{s\geqslant 1} (2s+1) G_{2s+2}(\tau) x^{2s}. \end{aligned}$$

On the other hand, for $y = m + n\tau + \tilde{\gamma}$, we obtain

$$
\begin{aligned}
H_\gamma(x,\tau) &:= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \left(\frac{1}{(x+\tilde{\gamma}+m+n\tau)^2} - \frac{1}{(\tilde{\gamma}+m+n\tau)^2} \right) \\
&= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \sum_{s\geqslant 1} \frac{1}{(m+n\tau+\tilde{\gamma})^2} a_s \frac{x^s}{(\tilde{\gamma}+m+n\tau)^s} \\
&= \sum_{s\geqslant 1} \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} a_s \frac{x^s}{(\tilde{\gamma}+m+n\tau)^{s+2}} \\
&= \sum_{s\geqslant 1} (-1)^s (s+1) G_{s+2,\gamma}(\tau) x^s,
\end{aligned}
$$

where, for $s \geq 3$, we define

$$
G_{s,\gamma}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau+\tilde{\gamma})^s}.
$$

Then, for $s \geq 3$, we write $B_{s,\gamma}(\tau) = G_s(\tau) - G_{s,\gamma}(\tau)$ and we have

$$
H(x,\tau) - H_\gamma(x,\tau) = \sum_{s\geqslant 1} (-1)^s (s+1) B_{s+2,\gamma}(\tau) x^s.
$$

and we write $\bar{A}_{s,\gamma}(\tau) = G_s(\tau) + G_{s,\gamma}(\tau) = -B_{s,\gamma}(\tau)$, as $H(x,\tau)$ and $G_s(\tau)$ are even. If Γ is the trivial group, $\bar{A}_{s,\gamma}(\tau)$ reduces to twice the classical Eisenstein series $G_s(\tau)$.

Notice that $G_{s,\gamma}(\tau)$ is not pair for the variable x but is pair for the variable $x+\gamma$ i.e. it is invariant under the transformation $x+\tilde{\gamma} \mapsto -x-\tilde{\gamma}$. Thus, we obtain $G_{s,\gamma} = (-1)^s G_{s,-\gamma}$, which implies that $\bar{A}_{s,\gamma} = (-1)^s \bar{A}_{s,-\gamma}$.

In conclusion, we obtain

$$
\partial_x^2(\log(F_\gamma(x))) = \frac{1}{x^2} - \frac{1}{(x+\tilde{\gamma})^2} + \sum_{s\geqslant 1} (-1)^{s+1} (s+1) \bar{A}_{s+2,\gamma}(\tau) x^s,
$$

which gives

$$
\log(F_\gamma(x)) = \log(x) - \log(x+\tilde{\gamma}) + \sum_{s\geqslant 1} (-1)^{s+1} \frac{\bar{A}_{s+2,\gamma}(\tau)}{s+2} x^{s+2} + lx + m,
$$

Thus,

$$
\begin{aligned}
F_\gamma(x) &= -x(x+\tilde{\gamma}) \exp\left(\sum_{s\geqslant 1} (-1)^{s+1} \frac{\bar{A}_{s+2,\gamma}(\tau)}{s+2} x^{s+2} + 2\pi i c x - \log(\theta(\tilde{\gamma})) \right) \\
&= \frac{-x(x+\tilde{\gamma})}{\theta(\tilde{\gamma})} e^{2\pi i c x} \exp\left(\sum_{s\geqslant 1} (-1)^{s+1} \frac{\bar{A}_{s+2,\gamma}(\tau)}{s+2} x^{s+2} \right)
\end{aligned}
$$

where the term $+2\pi i c x - \log(\theta(\tilde{\gamma}))$ comes from the identification of the above formula with equation (7.2).

We conclude that

$$\begin{aligned}
g_{-\gamma}(0,x|\tau) &= \partial_x(F_\gamma(x)) + \frac{1}{x^2} \\
&= \frac{1}{x^2} + \frac{-2x-\tilde{\gamma}}{\theta(\tilde{\gamma})} e^{2\pi i c x} \exp\left(\sum_{s\geqslant 1}(-1)^{s+1}\frac{\bar{A}_{s+2,\gamma}(\tau)}{s+2}x^{s+2}\right) \\
&\quad -2\pi i c \frac{x(x+\tilde{\gamma})}{\theta(\tilde{\gamma})} e^{2\pi i c x} \exp\left(\sum_{s\geqslant 1}(-1)^{s+1}\frac{\bar{A}_{s+2,\gamma}(\tau)}{s+2}x^{s+2}\right) \\
&\quad -\frac{-x(x+\tilde{\gamma})}{\theta(\tilde{\gamma})} e^{2\pi i c x}\left(\sum_{s\geqslant 1}(-1)^{s+1}\bar{A}_{s+2,\gamma}(\tau)x^{s+1}\right) \exp\left(\sum_{s\geqslant 1}(-1)^{s+1}\frac{\bar{A}_{s+2,\gamma}(\tau)}{s+2}x^{s+2}\right).
\end{aligned}$$

Now, we define $G_{2,\gamma}(\tau)$ by

$$G_{2,\gamma}(\tau) = \sum_{n=-\infty}^{\infty}\left(\sum_{\substack{m=-\infty \\ m\neq 0 \text{ if } n=0}}^{\infty} \frac{1}{(\tilde{\gamma}+m+n\tau)^2}\right).$$

and $\bar{A}_{2,\gamma}(\tau) := G_2(\tau) + G_{2,\gamma}(\tau)$. We also define

$$A_{2,\gamma}(\tau) := (2\pi i c)^2 - \left(\frac{\theta'}{\theta}\right)^2(\tilde{\gamma}) - 2\pi i c\left(\frac{\theta'}{\theta}\right)(\tilde{\gamma}) - \pi i c.$$

7.3.2 Modularity of the Eisenstein-Hurwitz series $A_{s,\gamma}$

Consider for $s \geq 2$, the function

$$G_{s,\gamma}(\tau) := \sum_{n=-\infty}^{\infty}\left(\sum_{\substack{m=-\infty \\ m\neq 0 \text{ if } n=0}}^{\infty} \frac{1}{(m+n\tau+\tilde{\gamma})^k}\right).$$

and denote as above $\bar{A}_{s,\gamma}(\tau) = G_s(\tau) + G_{s,\gamma}(\tau)$.

Proposition 7.3.1. *Let $s \geq 3$. The function $\bar{A}_{s,\gamma}$ is a modular form of weight s for $\mathrm{SL}_2^\Gamma(\mathbb{Z})$.*

Proof. We will proceed as follows. First, we will show the modular quasi-invariance. Then we will show holomorphy at the cusps by characterising holomorphy in terms of a q_N-expansion, where $q_N = e^{2\pi i\tau/N}$ (see [30, Definition 1.2.3]). For $s \geq 3$, the series $\bar{A}_{s,\gamma}(\tau)$ converge normally. Let us first show that, if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2^\Gamma(\mathbb{Z})$, then $\bar{A}_{s,\gamma}(\alpha\cdot\tau) = (c\tau+d)^s\bar{A}_{s,\gamma}(\tau)$. We already know that the Eisenstein series $G_s(\tau)$ are modular forms of weight s, for $s \geq 4$ and $G_3(\tau) = 0$. We have

$$G_{s,\gamma}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m+\frac{u}{M}+(n+\frac{v}{N})\tau)^s}.$$

Thus,

$$G_{s,\gamma}(\alpha \cdot \tau) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m+\frac{u}{M}+(n+\frac{v}{N})\frac{a\tau+b}{c\tau+d})^s}$$

$$= (c\tau+d)^s \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(md+nb+(mc+na)\tau+\frac{u}{M}d+\frac{v}{N}b+(\frac{u}{M}c+\frac{v}{N}a)\tau)^s}$$

$$= (c\tau+d)^s \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau+\tilde{\gamma}')^s}$$

for some lift $\tilde{\gamma}'$ of $\gamma \in \Gamma$. The last line holds by the fact that, since $a \equiv 1 \bmod M$, $d \equiv 1 \bmod N$, $b \equiv 0 \bmod N$ and $c \equiv 0 \bmod M$, we have $\frac{u}{M}d \in \frac{\mathbb{Z}}{M}$, $\frac{v}{N}b \in \mathbb{Z}$, $\frac{u}{M}c \in \mathbb{Z}$ and $\frac{v}{N}a \in \frac{\mathbb{Z}}{N}$. Then we can rewrite the term $md+nb+(mc+na)\tau$ as $m+n\tau$ by applying

$$\begin{pmatrix} n & m \end{pmatrix} \longmapsto \begin{pmatrix} n & m \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and we can rewrite the term $\frac{u}{M}d+\frac{v}{N}b+(\frac{u}{M}c+\frac{v}{N}a)\tau$ as $m+n\tau+\tilde{\gamma}'$ by applying

$$\begin{pmatrix} \frac{v}{N} & \frac{u}{M} \end{pmatrix} \longmapsto \begin{pmatrix} \frac{v}{N} & \frac{u}{M} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Finally, as we already know that $\bar{A}_{s,\gamma}$ does not depend on the choice of the lift $\tilde{\gamma}$ of γ, we obtain $G_{s,\gamma}(\alpha \cdot \tau) = (c\tau+d)^s G_{s,\gamma}(\tau)$. The function $\bar{A}_{s,\gamma}$ being holomorphic on $\mathfrak{h}$, it remains to show that it is also holomorphic at all cusps of the compactified modular curve $X(\Gamma)$.

Recall that the Hurwitz zeta function is defined by

$$\zeta(s,z) := \sum_{m\geq 0} \frac{1}{(m+z)^s},$$

where $s, q \in \mathbb{C}$ are such that $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(q) > 0$.

Lemma 7.3.2. *The function $G_{s,\gamma}(\tau)$ admits a q_N-expansion, where $q_N = e^{2\pi \mathrm{i}\tau/N}$.*

Proof. We have

$$\begin{aligned}
G_{s,\gamma}(\tau) &= \sum_{m\in\mathbb{Z}} \frac{1}{(m+\tilde{\gamma})^s} + \sum_{n\in\mathbb{Z}-0}\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau+\tilde{\gamma})^s} \\
&= \sum_{m\in\mathbb{Z}} \frac{1}{(m+\tilde{\gamma})^s} + \sum_{n\in\mathbb{Z}-0}\sum_{m\in\mathbb{Z}} \frac{1}{(m+\frac{u}{M}+(n+\frac{v}{N})\tau)^s} \\
&= \sum_{m\in\mathbb{Z}} \frac{1}{(m+\tilde{\gamma})^s} + \frac{(-2i\pi)^s}{(s-1)!} \sum_{n-\in\mathbb{Z}-0}\sum_{r\geq 1} r^{s-1} e^{2\pi \mathrm{i} r(\frac{u}{M}+\tau(n+\frac{v}{N}))} \\
&= \sum_{m\in\mathbb{Z}} \frac{1}{(m+\tilde{\gamma})^s} + \frac{(-2\pi \mathrm{i})^s}{(s-1)!} \sum_{n\in\mathbb{Z}-0}\sum_{r\geq 1} r^{s-1} e^{\frac{2\pi \mathrm{i} r u}{M}} q_N^{(Nn+v)r} \\
&= \sum_{m\geq 0} \frac{1}{(m+\tilde{\gamma})^s} + (-1)^s \sum_{m\geq 1} \frac{1}{(m-\tilde{\gamma})^s} + \frac{(-2\pi \mathrm{i})^s}{(s-1)!} \sum_{n\geq 1}\sum_{r\geq 1} r^{s-1} e^{\frac{2\pi \mathrm{i} r u}{M}} q_N^{(Nn+v)r}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{(2\pi \mathrm{i})^s}{(s-1)!}\sum_{n\geq 1}\sum_{r\geq 1} r^{s-1}e^{\frac{-2\pi iru}{M}}q_N^{(Nn-v)r} \\
= \quad & -\frac{1}{\tilde{\gamma}^s}+\zeta(s,\gamma)+(-1)^s\zeta(s,-\gamma) \\
& +\frac{(-2\pi \mathrm{i})^s}{(s-1)!}\sum_{n\geq 1}\sum_{r\geq 1} r^{s-1}e^{\frac{2\pi iru}{M}}q_N^{(Nn+v)r}+\frac{(2\pi \mathrm{i})^s}{(s-1)!}\sum_{n\geq 1}\sum_{r\geq 1} r^{s-1}e^{\frac{-2\pi iru}{M}}q_N^{(Nn-v)r},
\end{aligned}
$$

where $\zeta(s,\gamma)$ is the Hurwitz zeta function evaluated at (s,γ). $\square$

This shows that $G_{s,\gamma}(\tau)$ is N-periodic and is holomorphic at $\mathrm{i}\infty$ and we define, for $\gamma = u/M$,

$$a_{s,\gamma} = -\frac{1}{\tilde{\gamma}^s}+\zeta(s,\gamma)+(-1)^s\zeta(s,-\gamma)$$

to be the constant term in this expansion (it also depends on τ but logarithmically). In other words, $G_{s,\gamma}(\tau)$ has constant term equal to $a_{s,\gamma}$ if $\gamma = u/M$ and 0 else.

The term $a_{s,\gamma}$ tends to 0 when $\tau \longrightarrow \mathrm{i}\infty$.

We now show that this function is also holomorphic at the remaining cusps of the modular curve $X(\Gamma)$.

Lemma 7.3.3. *For all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the function*

$$\tau \mapsto (c\tau+d)^{-s}G_{s,\gamma}(\alpha\cdot\tau)$$

has a q_N-expansion.

Proof. We have

$$
\begin{aligned}
(c\tau+d)^{-s}G_{s,\gamma}(\alpha\cdot\tau) &= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{(md+nb+(mc+na)\tau+\frac{u}{M}d+\frac{v}{N}b+(\frac{u}{M}c+\frac{v}{N}a)\tau)^s} \\
&= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{((m+\frac{u}{M})d+(\frac{v}{N}+n)b+(mc+na+\frac{u}{M}c+\frac{v}{N}a)\tau)^s} \\
&= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{(md+\frac{u}{M}d+(\frac{v}{N}+n)b+(mc+na+\frac{u}{M}c+\frac{v}{N}a)\tau)^s} \\
&= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{\left(d\left(m+\frac{1}{d}\left(\frac{u}{M}d+(\frac{v}{N}+n)b+(mc+na+\frac{u}{M}c+\frac{v}{N}a)\tau\right)\right)\right)^s} \\
&= \frac{1}{d^s}\sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{\left(m+\frac{1}{d}\left(\frac{u}{M}d+(\frac{v}{N}+n)b+(mc+na+\frac{u}{M}c+\frac{v}{N}a)\tau\right)\right)^s},
\end{aligned}
$$

By denoting $z = \frac{1}{d}\left(\frac{u}{M}d + (\frac{v}{N}+n)b + (mc+na+\frac{u}{M}c+\frac{v}{N}a)\tau\right)$, we have

$$\begin{aligned}
(c\tau+d)^{-s}G_{s,\gamma}(\alpha\cdot\tau) &= \frac{1}{d^s}\sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{(m+z)^s} \\
&= \frac{1}{d^s}\frac{(-2\pi\mathrm{i})^s}{(s-1)!}\sum_{n\in\mathbb{Z}}\sum_{r\geq 1} r^{s-1}e^{2\pi\mathrm{i}rz} \\
&= \frac{1}{d^s}\frac{(-2\pi\mathrm{i})^s}{(s-1)!}\sum_{n\in\mathbb{Z}}\sum_{r\geq 1} r^{s-1}e^{2\pi\mathrm{i}r\frac{1}{d}\left(\frac{u}{M}d+(\frac{v}{N}+n)b+(mc+na+\frac{u}{M}c+\frac{v}{N}a)\tau\right)} \\
&= \frac{1}{d^s}\frac{(-2\pi\mathrm{i})^s}{(s-1)!}\sum_{n\in\mathbb{Z}}\sum_{r\geq 1} r^{s-1}e^{2\pi\mathrm{i}r(m+\frac{nb}{d}+\frac{u}{M}+\frac{vb}{Nd})}e^{2\pi\mathrm{i}r\tau(\frac{mc+na}{d}+\frac{uc}{Md}+\frac{va}{Nd})} \\
&= \frac{1}{d^s}\frac{(-2\pi\mathrm{i})^s}{(s-1)!}\sum_{n\in\mathbb{Z}}\sum_{r\geq 1} r^{s-1}e^{2\pi\mathrm{i}r(m+\frac{nb}{d}+\frac{u}{M}+\frac{vb}{Nd})}e^{\frac{2\pi\mathrm{i}r\tau}{N}(N(\frac{mc+na}{d}+\frac{uc}{Md})+\frac{va}{d})} \\
&= \frac{1}{d^s}\frac{(-2\pi\mathrm{i})^s}{(s-1)!}\sum_{n\in\mathbb{Z}}\sum_{r\geq 1} r^{s-1}e^{2\pi\mathrm{i}r(m+\frac{nb}{d}+\frac{u}{M}+\frac{vb}{Nd})}q_N^{(N(\frac{mc+na}{d}+\frac{uc}{Md})+\frac{va}{d})r},
\end{aligned}$$

which concludes the proof. □

We conclude that, for all $\alpha\in\mathrm{SL}_2^\Gamma(\mathbb{Z})$, the function

$$\tau\mapsto(c\tau+d)^{-s}\bar{A}_{s,\gamma}(\alpha\cdot\tau)$$

is holomorphic at $\mathrm{i}\infty$, which concludes the proof. □

Remark 7.3.4. *From the expression of the function $(c\tau+d)^{-s}\bar{A}_{s,\gamma}(\alpha\cdot\tau)$, we can notice that our functions $\bar{A}_{s,\gamma}$ will degenerate at all cusps of $X(\Gamma)$ to functions closely related to cyclotomic zeta values. More precisely, the function $\sum_{\gamma\in\Gamma-\{\mathbf{0}\}}\bar{A}_{s,\gamma}(\tau)$ has a q_N-expansion whose constant term (in the sense that if $\tau\longrightarrow\mathrm{i}\infty$, its remaining non zero component) is*

$$\sum_{1\leq u\leq M-1}\left(-(\frac{M}{u})^s+\zeta(s,\frac{u}{M})+(-1)^s\zeta(s,-\frac{u}{M})\right).$$

7.4 Representations of Cherednik algebras

7.4.1 The Cherednik algebra of a wreath product

In this paragraph Γ is any finite group such that $\Gamma\subset\mathrm{Aut}(\mathbb{C})$, $\underline{k}=(k_\alpha)_\alpha\in\mathbb{C}^\Gamma$ is such that $k_\alpha=k_{-\alpha}$ and $G:=\Gamma\wr\mathfrak{S}_n$. We define the Cherednik algebra $H_n^\Gamma(\underline{k})$ as the quotient of the algebra $\mathbb{C}\langle x_1,\ldots,x_n,y_1,\ldots,y_n\rangle\rtimes\mathbb{C}[G]$ by the relations

- $\sum_i \mathrm{x}_i=\sum_i \mathrm{y}_i=0$
- $[\mathrm{x}_i,\mathrm{x}_j]=0=[\mathrm{y}_i,\mathrm{y}_j]$,
- $[\mathrm{x}_i,\mathrm{y}_j]=\frac{1}{n}-\sum_{\alpha\in\Gamma}k_\alpha s_{ij}^\alpha\ (i\neq j)$,

where $s_{ij}^\alpha=(\alpha_i-\alpha_j)s_{ij}$, and s_{ij} is the permutation of i and j.

Remark 7.4.1. *As $\Gamma \subset \mathrm{Aut}(\mathbb{C})$, $H_n^\Gamma(\underline{k})$ admits a geometric construction. Define $X := \{\mathbf{z} \in \mathbb{C}^n | \sum_i z_i = 0\}$ and consider the following action of G on it: $\mathfrak{S}_n$ acts in an obvious way and*

$$\alpha_i(\mathbf{z}) = (\alpha^{(i)} - \frac{1}{n}\sum_j \alpha^{(j)})(\mathbf{z}),$$

where $\alpha^{(k)}$ is the action of $\alpha \in \Gamma$ on the k-th factor of $\mathbb{C}^n$. Following [41] one can construct a Cherednik algebra $H_{1,\underline{k},0}(X,G)$ on X/G. It can be defined as the subalgebra of $\mathrm{Diff}(X) \rtimes \mathbb{C}[G]$ generated by the function algebra $\mathcal{O}_X$, the group G and the Dunkl-Opdam operators $D_i - D_j$, where

$$D_i = \partial_{z_i} + \sum_{\substack{j:j\neq i\\ \alpha\in\Gamma}} k_\alpha \frac{1 - s_{ij}^\alpha}{(-\alpha)(z_i) - \alpha(z_j)} .$$

One can then prove that there is a unique isomorphism of algebras $H_n^\Gamma(\underline{k}) \longrightarrow H_{1,\underline{k},0}(X,G)$ defined by

$$\begin{aligned} \mathrm{x}_i &\longmapsto z_i, \\ \mathrm{y}_i &\longmapsto D_i - \frac{1}{n}\sum_j D_j, \\ G \ni g &\longmapsto g. \end{aligned}$$

7.4.2 Morphisms from $\bar{\mathfrak{t}}_{1,n}^\Gamma$ to the Cherednik algebra

Proposition 7.4.2. *For any $a, b \in \mathbb{C}$ there is a morphism of Lie algebras $\phi_{a,b} : \bar{\mathfrak{t}}_{1,n}^\Gamma \longrightarrow H_n^\Gamma(\underline{k})$ defined by*

$$\begin{aligned} \bar{x}_i &\longmapsto a\,\mathrm{x}_i \\ \bar{y}_i &\longmapsto b\,\mathrm{y}_i\,, \\ \bar{t}_{ij}^\alpha &\longmapsto ab\left(\frac{1}{n} - k_\alpha s_{ij}^\alpha\right) . \end{aligned}$$

Proof. Straightforward from the alternative presentation of $\bar{\mathfrak{t}}_{1,n}^\Gamma$ in Lemma 4.3.5. □

Hence any representation V of $H_n^\Gamma(\underline{k})$ yields a family of flat connections $\nabla_{a,b}^{(V)}$ over the configuration space $\mathrm{C}(E,[n],\Gamma)$.

7.4.3 Monodromy representations of Hecke algebras

Let E be an elliptic curve and $\tilde{E} \longrightarrow E$ the Γ-covering as in §6.1.1. Define $X = \tilde{E}^n/\tilde{E}$ and $G = (\Gamma \wr \mathfrak{S}_n)/\Gamma^{\mathrm{diag}}$. Then the set $X' \subset X$ of points with trivial stabilizer is such that $X'/G = \mathrm{C}(E,[n],\Gamma)$.

Let us recall from [41] the construction of the Hecke algebra $\mathcal{H}_n^\Gamma(q,\underline{t})$ of X/G. It is the quotient of the group algebra of the orbifold fundamental group $\bar{B}_{1,n}^\Gamma$ of $\mathrm{C}(E,[n],\Gamma)$ by the additional relations $(T_\alpha - q^{-1}t_\alpha)(T_\alpha + q^{-1}t_\alpha^{-1}) = 0$, where T_α is an element of $\bar{B}_{1,n}^\Gamma$ homotopic as a free loop to a small loop around the divisor $Y_\alpha := \cup_{i\neq j}\{z_i = \alpha \cdot z_j\}$ in X/G, in the counterclokwise direction.[1]

Let us consider the flat connection $\nabla_{a,b}^{(V)}$ and set

$$q = e^{-2\pi \mathrm{i} ab/n}\,, \quad t_\alpha = e^{-2\pi \mathrm{i} k_\alpha ab}\,.$$

Then the monodromy representation $\bar{B}_{1,n}^\Gamma \longrightarrow GL(V)$ of $\nabla_{a,b}^{(V)}$ obviously gives a representation of $\mathcal{H}_n^\Gamma(q,\underline{t})$ either if V is finite dimensional or if a, b are formal parameters. In particular, taking $a = b$ a formal parameter and $V = H_n^\Gamma(\underline{k})$, one obtains an algebra morphism

$$\mathcal{H}_n^\Gamma(q,\underline{t}) \longrightarrow H_n^\Gamma(\underline{k})[[a]]\,.$$

We do not know if this morphism is an isomorphism upon inverting a.

7.4.4 The modular extension of $\phi_{a,b}$.

Now assume that $a, b \neq 0$.

Proposition 7.4.3. *The Lie algebra morphism $\phi_{a,b}$ can be extended to the algebra $U(\bar{\mathfrak{t}}_{1,n}^\Gamma \rtimes \mathfrak{d}^\Gamma) \rtimes G$ by the formulas*

$$\phi_{a,b}(s_{ij}^\alpha) = s_{ij}^\alpha,$$

$$\phi_{a,b}(d) = \frac{1}{2}\sum_i (\mathrm{x}_i\mathrm{y}_i + \mathrm{y}_i\mathrm{x}_i), \quad \phi_{a,b}(X) = -\frac{1}{2}ab^{-1}\sum_i \mathrm{x}_i^2,$$

$$\phi_{a,b}(\Delta_0) = \frac{1}{2}ba^{-1}\sum_i \mathrm{y}_i^2, \quad \phi_{a,b}(\xi_{s,\gamma}) = -a^{s-1}b^{-1}\sum_{i<j}(\gamma\cdot(\mathrm{x}_i - \mathrm{x}_j))^s.$$

Thus, the flat connections $\nabla_{a,b}^\Gamma$ extend to flat connections on $\mathcal{M}_{1,[n]}^\Gamma$.

[1] Here the sugroup of G acting trivially on Y_α is the order 2 cyclic subgroup generated by s_{ij}^α.

Chapter 8

Elliptic multiple zeta values at torsion points

We propose in this chapter a twisted version $\mathfrak{u}^\Gamma$ of Pollack's special derivation algebra contructed in [90] and [100] by relating it to the twisted derivation algebra $\mathfrak{d}^\Gamma$ constructed in subsection 6.2. Next, we state and prove a differential equation in the variable τ for the ellipsitomic KZB associator and use the iterated integral machinery developped in [35] to give a well-defined notion of elliptic multiple zeta values at torsion points, closely related to that which appeared in the physics paper [19].

8.1 The Lie algebra $\mathfrak{u}^\Gamma$ of special twisted derivations

We give a definition of the twisted version of Pollack's Lie algebra $\mathfrak{u}$ of special derivations.

8.1.1 The case of the twisted configuration space $\mathrm{Conf}(E, n, \Gamma)$

Proposition 8.1.1. *There is a unique bigraded Lie algebra morphism*

$$\begin{array}{rcl} \rho : \mathfrak{d}^\Gamma & \longrightarrow & \mathfrak{t}^\Gamma_{1,n} \rtimes \mathfrak{d}^\Gamma \\ \tilde{e}, \tilde{f}, \tilde{h} & \longmapsto & \tilde{e}, \tilde{f}, \tilde{h} \\ \delta_{s,\gamma} & \longmapsto & \delta^{(n)}_{s,\gamma} := \delta_{s,\gamma} + \displaystyle\sum_{1\leqslant i<j\leqslant n} (\mathrm{ad}\, x_i)^s t^{-\gamma}_{ij} + (-\,\mathrm{ad}\, x_i)^s t^{\gamma}_{ij}. \end{array}$$

This induces a group morphism $\mathbf{G}^\Gamma_1 \longrightarrow \mathbf{G}^\Gamma_n$ that will be denoted $h \mapsto \tilde{h}$.

Proof. Let us first show that the relation $\delta_{s,\gamma} = (-1)^s \delta_{s,-\gamma}$ is preserved by ρ:

$$\begin{aligned}
(-1)^s \delta^{(n)}_{s,-\gamma} &= (-1)^s \delta_{s,-\gamma} + \sum_{1\leqslant i<j\leqslant n} (-1)^s ((\operatorname{ad} x_i)^s t^{\gamma}_{ij} + (-\operatorname{ad} x_i)^s t^{-\gamma}_{ij}) \\
&= \delta_{s,\gamma} + \sum_{1\leqslant i<j\leqslant n} (-1)^s (\operatorname{ad} x_i)^s t^{\gamma}_{ij} + (-1)^s (-1)^s (\operatorname{ad} x_i)^s t^{-\gamma}_{ij} \\
&= \delta_{s,\gamma} + \sum_{1\leqslant i<j\leqslant n} (\operatorname{ad} x_i)^s t^{-\gamma}_{ij} + (-\operatorname{ad} x_i)^s t^{\gamma}_{ij} \\
&= \delta^{(n)}_{s,\gamma}.
\end{aligned}$$

Next, we show that the highest weight relations are preserved for $\delta^{(n)}_{s,\gamma}$ i.e. that we have relations $[\tilde{e}, \delta^{(n)}_{s,\gamma}] = 0$, $[\tilde{h}, \delta^{(n)}_{s,\gamma}] = s\delta^{(n)}_{s,\gamma}$ and $\operatorname{ad}^{s+1}(\tilde{f})(\delta^{(n)}_{s,\gamma}) = 0$. The relation $[\tilde{e}, \delta^{(n)}_{s,\gamma}] = 0$ is obviously satisfied. Next, we have

$$\begin{aligned}
[\tilde{h}, \delta^{(n)}_{s,\gamma}] &= s\delta_{s,-\gamma} + [\tilde{h}, \sum_{1\leqslant i<j\leqslant n} (\operatorname{ad} x_i)^s t^{-\gamma}_{ij} + (-\operatorname{ad} x_i)^s t^{\gamma}_{ij}] \\
&= s\delta_{s,-\gamma} + \sum_{1\leqslant i<j\leqslant n} [\tilde{h}, (\operatorname{ad} x_i)^s t^{-\gamma}_{ij}] + [\tilde{h}, (-\operatorname{ad} x_i)^s t^{\gamma}_{ij}] \\
&= s\delta_{s,-\gamma} + \sum_{1\leqslant i<j\leqslant n} s(\operatorname{ad} x_i)^s t^{-\gamma}_{ij} + s(-\operatorname{ad} x_i)^s t^{\gamma}_{ij} \\
&= s\delta^{(n)}_{s,\gamma},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{ad}^{s+1}(\tilde{f})(\delta^{(n)}_{s,\gamma}) &= 0 + \operatorname{ad}^{s+1}(\tilde{f}) \left(\sum_{1\leqslant i<j\leqslant n} (\operatorname{ad} x_i)^s t^{-\gamma}_{ij} + (-\operatorname{ad} x_i)^s t^{\gamma}_{ij} \right) \\
&= \sum_{1\leqslant i<j\leqslant n} (\tilde{f}) \underset{n \texttt{ times}}{\cdots} (\tilde{f}) \big(\ (\operatorname{ad} x_i)^s t^{-\gamma}_{ij} + (-\operatorname{ad} x_i)^s t^{\gamma}_{ij} \big) \\
&= \sum_{1\leqslant i<j\leqslant n} (\tilde{f}) \underset{n-1 \texttt{ times}}{\cdots} (\tilde{f}) \big(\ \operatorname{ad}(y_i)(\operatorname{ad} x_i)^{s-1} t^{-\gamma}_{ij} + \operatorname{ad}(y_i)(-\operatorname{ad} x_i)^{s-1} t^{\gamma}_{ij} \big) \\
&= \sum_{1\leqslant i<j\leqslant n} (\tilde{f}) \big(\ (\operatorname{ad} y_i)^s t^{-\gamma}_{ij} + (-\operatorname{ad} y_i)^s t^{\gamma}_{ij} \big) \\
&= 0.
\end{aligned}$$

This finishes the proof. $\square$

Remark 8.1.2. *Since*

$$\sum_{\gamma\in\Gamma} A_{s,\gamma} (\operatorname{ad} x_i)^s (t^{-\gamma}_{ij}) = \sum_{\gamma\in\Gamma} (-1)^s A_{s,-\gamma} (\operatorname{ad} x_i)^s (t^{-\gamma}_{ij}) = \sum_{\gamma\in\Gamma} A_{s,-\gamma} (-\operatorname{ad} x_i)^s t^{\gamma}_{ij},$$

we obtain

$$\frac{1}{2} \sum_{\gamma\in\Gamma} A_{s,\gamma} ((\operatorname{ad} x_i)^s (t^{-\gamma}_{ij}) + (-\operatorname{ad} x_i)^s (t^{\gamma}_{ij})) = \sum_{\gamma\in\Gamma} A_{s,\gamma} (\operatorname{ad} x_i)^s (t^{-\gamma}_{ij}).$$

Recall that there is a bigraded Lie algebra morphism

$$\begin{aligned}
\mathfrak{d}^{\Gamma} &\longrightarrow \operatorname{Der}(\mathfrak{t}^{\Gamma}_{1,n}) \\
e, f, h &\longmapsto \xi^{(n)}_e, \xi^{(n)}_f, \xi^{(n)}_h \\
\delta_{s,\gamma} &\longmapsto \xi^{(n)}_{s,\gamma},
\end{aligned}$$

where $\xi_e^{(n)} := \xi_e, \xi_f^{(n)} := \xi_f, \xi_h^{(n)} := \xi_h$ are the usual derivations given by the $\mathfrak{sl}_2$-basis $\{e, f, h\}$ and

- $\xi_{s,\gamma}^{(n)}(x_i) = 0$,
- $\xi_{s,\gamma}^{(n)}(y_i) = \sum_{j:j\neq i} \sum_{p+q=s-1} \sum_{\beta\in\Gamma} [(\mathrm{ad}\, x_i)^p t_{ij}^{\beta-\gamma}, (-\mathrm{ad}\, x_i)^q t_{ij}^{\beta}]$,
- $\xi_{s,\gamma}^{(n)}(t_{ij}^\alpha) = [t_{ij}^\alpha, (\mathrm{ad}\, x_i)^s t_{ij}^{\alpha-\gamma} + (-\mathrm{ad}\, x_i)^s t_{ij}^{\alpha+\gamma}]$.

This morphism induces a morphism $\mathfrak{d}^\Gamma \longrightarrow \mathrm{Der}(\bar{\mathfrak{t}}_{1,n}^\Gamma)$ and we denote $\bar{\xi}_e^{(n)}, \bar{\xi}_f^{(n)}, \bar{\xi}_h^{(n)}$ and $\bar{\xi}_{s,\gamma}^{(n)}$ the images of e, f, h and $\delta_{s,\gamma}$ by this map.

Proposition 8.1.3. *The derivation* $\tilde{\xi}_{s,\gamma}^{(n)} := \xi_{s,\gamma}^{(n)} + \sum_{1\leqslant i<j\leqslant n} [(\mathrm{ad}\, x_i)^s t_{ij}^{-\gamma} + (-\mathrm{ad}\, x_i)^s t_{ij}^{\gamma}, -]$ *of* $\mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$ *is given on generators by*

$$\tilde{\xi}_{s,\gamma}^{(n)}(x_i) = \sum_{j;i\neq j} -(\mathrm{ad}\, x_i)^{s+1}(t_{ij}^{-\gamma}) + (-\mathrm{ad}\, x_i)^{s+1}(t_{ij}^{\gamma})$$

and

$$\tilde{\xi}_{s,\gamma}^{(n)}(t_{ij}^\alpha) = \sum_{k\neq j} [-((\mathrm{ad}\, x_k)^s t_{kj}^{\alpha-\gamma} + (-\mathrm{ad}\, x_k)^s t_{kj}^{\alpha+\gamma}) + (\mathrm{ad}\, x_k)^s t_{kj}^{-\gamma} + (-\mathrm{ad}\, x_k)^s t_{kj}^{\gamma}, t_{ij}^\alpha].$$

Proof. We have

$$\begin{aligned}
\tilde{\xi}_{s,\gamma}^{(n)}(x_i) &= \sum_{j<k} [(\mathrm{ad}\, x_j)^s t_{jk}^{-\gamma} + (-\mathrm{ad}\, x_j)^s t_{jk}^{\gamma}, x_i] \\
&= \sum_{j<k} [(\mathrm{ad}\, x_j)^s t_{jk}^{-\gamma}, x_i] + [(-\mathrm{ad}\, x_j)^s t_{jk}^{\gamma}, x_i] \\
&= \sum_{i<k} [(\mathrm{ad}\, x_i)^s t_{ik}^{-\gamma}, x_i] + [(-\mathrm{ad}\, x_i)^s t_{ik}^{\gamma}, x_i] + \sum_{j<i} [(\mathrm{ad}\, x_j)^s t_{ji}^{-\gamma}, x_i] + [(-\mathrm{ad}\, x_j)^s t_{ji}^{\gamma}, x_i] \\
&= \sum_{i<j} [(\mathrm{ad}\, x_i)^s t_{ij}^{-\gamma}, x_i] + [(-\mathrm{ad}\, x_i)^s t_{ij}^{\gamma}, x_i] + \sum_{j<i} [(-\mathrm{ad}\, x_i)^s t_{ij}^{\gamma}, x_i] + [(\mathrm{ad}\, x_i)^s t_{ij}^{-\gamma}, x_i] \\
&= \sum_{j;i\neq j} [(\mathrm{ad}\, x_i)^s (t_{ij}^{-\gamma}), x_i] + [(-\mathrm{ad}\, x_i)^s (t_{ij}^{\gamma}), x_i] \\
&= \sum_{j\neq i} -(\mathrm{ad}\, x_i)^{s+1}(t_{ij}^{-\gamma}) + (-\mathrm{ad}\, x_i)^{s+1}(t_{ij}^{\gamma}).
\end{aligned}$$

Next,

$$\begin{aligned}
\tilde{\xi}_{s,\gamma}^{(n)}(t_{ij}^\alpha) &= [t_{ij}^\alpha, (\mathrm{ad}\, x_i)^s t_{ij}^{\alpha-\gamma} + (-\mathrm{ad}\, x_i)^s t_{ij}^{\alpha+\gamma}] + \sum_{k<l} [(\mathrm{ad}\, x_k)^s t_{kl}^{-\gamma} + (-\mathrm{ad}\, x_k)^s t_{kl}^{\gamma}, t_{ij}^\alpha] \\
&= [t_{ij}^\alpha, (\mathrm{ad}\, x_i)^s t_{ij}^{\alpha-\gamma} + (-\mathrm{ad}\, x_i)^s t_{ij}^{\alpha+\gamma}] + \sum_{i<j} [(\mathrm{ad}\, x_i)^s t_{ij}^{-\gamma} + (-\mathrm{ad}\, x_i)^s t_{ij}^{\gamma}, t_{ij}^\alpha] \\
&\quad + \sum_{k<j,k\neq i} [(\mathrm{ad}\, x_k)^s t_{kj}^{-\gamma}, t_{ij}^\alpha] + \sum_{j<l} [(\mathrm{ad}\, x_j)^s t_{jl}^{-\gamma}, t_{ij}^\alpha] + \sum_{i<l,l\neq j} [(\mathrm{ad}\, x_j)^s t_{il}^{-\gamma}, t_{ij}^\alpha] \\
&\quad + \sum_{k<j,k\neq i} [(-\mathrm{ad}\, x_k)^s t_{kj}^{\gamma}, t_{ij}^\alpha] + \sum_{j<l} [(-\mathrm{ad}\, x_j)^s t_{jl}^{\gamma}, t_{ij}^\alpha] + \sum_{i<l,l\neq j} [(-\mathrm{ad}\, x_j)^s t_{il}^{\gamma}, t_{ij}^\alpha]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k<i}[(\operatorname{ad}x_k)^s t_{ki}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{k<i}[(-\operatorname{ad}x_k)^s t_{ki}^{\gamma}, t_{ij}^{\alpha}] \\
= \;& [t_{ij}^{\alpha}, (\operatorname{ad}x_i)^s t_{ij}^{\alpha-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\alpha+\gamma}] + \sum_{i<j}[(\operatorname{ad}x_i)^s t_{ij}^{-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}[(\operatorname{ad}x_k)^s t_{kj}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{j<l}[(-\operatorname{ad}x_l)^s t_{jl}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{i<l,l\neq j}[(-\operatorname{ad}x_l)^s t_{il}^{-\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}[(-\operatorname{ad}x_k)^s t_{kj}^{\gamma}, t_{ij}^{\alpha}] + \sum_{j<l}[(\operatorname{ad}x_l)^s t_{jl}^{\gamma}, t_{ij}^{\alpha}] + \sum_{i<l,l\neq j}[(\operatorname{ad}x_l)^s t_{il}^{\gamma}, t_{ij}^{\alpha}] \\
&+\sum_{k<i}[(\operatorname{ad}x_k)^s t_{ki}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{k<i}[(-\operatorname{ad}x_k)^s t_{ki}^{\gamma}, t_{ij}^{\alpha}] \\
= \;& [t_{ij}^{\alpha}, (\operatorname{ad}x_i)^s t_{ij}^{\alpha-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\alpha+\gamma}] + \sum_{i<j}[(\operatorname{ad}x_i)^s t_{ij}^{-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}[(\operatorname{ad}x_k)^s t_{kj}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}[(-\operatorname{ad}x_k)^s t_{jk}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{i<k,k\neq j}[(-\operatorname{ad}x_k)^s t_{ik}^{-\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}[(-\operatorname{ad}x_k)^s t_{kj}^{\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}[(\operatorname{ad}x_k)^s t_{jk}^{\gamma}, t_{ij}^{\alpha}] + \sum_{i<k,k\neq j}[(\operatorname{ad}x_k)^s t_{ik}^{\gamma}, t_{ij}^{\alpha}] \\
&+\sum_{k<i}[(\operatorname{ad}x_k)^s t_{ki}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{k<i}[(-\operatorname{ad}x_k)^s t_{ki}^{\gamma}, t_{ij}^{\alpha}] \\
= \;& [t_{ij}^{\alpha}, (\operatorname{ad}x_i)^s t_{ij}^{\alpha-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\alpha+\gamma}] + \sum_{i<j}[(\operatorname{ad}x_i)^s t_{ij}^{-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}(\operatorname{ad}x_k)^s[t_{kj}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}(-\operatorname{ad}x_k)^s[t_{jk}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{i<k,k\neq j}(-\operatorname{ad}x_k)^s[t_{ik}^{-\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}(-\operatorname{ad}x_k)^s[t_{kj}^{\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}(\operatorname{ad}x_k)^s[t_{jk}^{\gamma}, t_{ij}^{\alpha}] + \sum_{i<k,k\neq j}(\operatorname{ad}x_k)^s[t_{ik}^{\gamma}, t_{ij}^{\alpha}] \\
&+\sum_{k<i}(\operatorname{ad}x_k)^s[t_{ki}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{k<i}(-\operatorname{ad}x_k)^s[t_{ki}^{\gamma}, t_{ij}^{\alpha}] \\
= \;& [t_{ij}^{\alpha}, (\operatorname{ad}x_i)^s t_{ij}^{\alpha-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\alpha+\gamma}] + \sum_{i<j}[(\operatorname{ad}x_i)^s t_{ij}^{-\gamma} + (-\operatorname{ad}x_i)^s t_{ij}^{\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}(\operatorname{ad}x_k)^s[t_{kj}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}(-\operatorname{ad}x_k)^s[t_{jk}^{-\gamma}, t_{ij}^{\alpha}] - \sum_{i<k,k\neq j}(-\operatorname{ad}x_k)^s[t_{kj}^{\alpha+\gamma}, t_{ij}^{\alpha}] \\
&+\sum_{j<k}(\operatorname{ad}x_k)^s[t_{jk}^{\gamma}, t_{ij}^{\alpha}] - \sum_{i<k,k\neq j}(\operatorname{ad}x_k)^s[t_{kj}^{\alpha-\gamma}, t_{ij}^{\alpha}] - \sum_{k<i}(-\operatorname{ad}x_k)^s[t_{kj}^{\alpha+\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}(-\operatorname{ad}x_k)^s[t_{kj}^{\gamma}, t_{ij}^{\alpha}] - \sum_{k<i}(\operatorname{ad}x_k)^s[t_{kj}^{\alpha-\gamma}, t_{ij}^{\alpha}] \\
= \;& -[(\operatorname{ad}x_i)^s t_{ij}^{\alpha-\gamma}, t_{ij}^{\alpha}] - [(-\operatorname{ad}x_i)^s t_{ij}^{\alpha+\gamma}, t_{ij}^{\alpha}] + \sum_{i<j}[(\operatorname{ad}x_i)^s t_{ij}^{-\gamma}, t_{ij}^{\alpha}] + [(-\operatorname{ad}x_i)^s t_{ij}^{\gamma}, t_{ij}^{\alpha}] \\
&- \sum_{i<k,k\neq j}(\operatorname{ad}x_k)^s[t_{kj}^{\alpha-\gamma}, t_{ij}^{\alpha}] + \sum_{k<j,k\neq i}(\operatorname{ad}x_k)^s[t_{kj}^{-\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}(\operatorname{ad}x_k)^s[t_{kj}^{-\gamma}, t_{ij}^{\alpha}] \\
&+ \sum_{k<j,k\neq i}(-\operatorname{ad}x_k)^s[t_{kj}^{\gamma}, t_{ij}^{\alpha}] + \sum_{j<k}(-\operatorname{ad}x_k)^s[t_{kj}^{\gamma}, t_{ij}^{\alpha}] - \sum_{k<i}(-\operatorname{ad}x_k)^s[t_{kj}^{\alpha+\gamma}, t_{ij}^{\alpha}] \\
&- \sum_{i<k,k\neq j}(-\operatorname{ad}x_k)^s[t_{kj}^{\alpha+\gamma}, t_{ij}^{\alpha}] - \sum_{k<i}(\operatorname{ad}x_k)^s[t_{kj}^{\alpha-\gamma}, t_{ij}^{\alpha}]
\end{aligned}
$$

$$\begin{aligned}
&= -[(\operatorname{ad} x_i)^s t_{ij}^{\alpha-\gamma}, t_{ij}^\alpha] - [(-\operatorname{ad} x_i)^s t_{ij}^{\alpha+\gamma}, t_{ij}^\alpha] + \sum_{i<j}[(\operatorname{ad} x_i)^s t_{ij}^{-\gamma}, t_{ij}^\alpha] + [(-\operatorname{ad} x_i)^s t_{ij}^{\gamma}, t_{ij}^\alpha] \\
&\quad - \sum_{k\neq i,j}(\operatorname{ad} x_k)^s [t_{kj}^{\alpha-\gamma}, t_{ij}^\alpha] + \sum_{k\neq i,j}(\operatorname{ad} x_k)^s [t_{kj}^{-\gamma}, t_{ij}^\alpha] - \sum_{k\neq i,j}(-\operatorname{ad} x_k)^s [t_{kj}^{\alpha+\gamma}, t_{ij}^\alpha] \\
&\quad + \sum_{k\neq i,j}(-\operatorname{ad} x_k)^s [t_{kj}^{\gamma}, t_{ij}^\alpha] \\
&= -[(\operatorname{ad} x_i)^s t_{ij}^{\alpha-\gamma}, t_{ij}^\alpha] - [(-\operatorname{ad} x_i)^s t_{ij}^{\alpha+\gamma}, t_{ij}^\alpha] + \sum_{i<j}[(\operatorname{ad} x_i)^s t_{ij}^{-\gamma}, t_{ij}^\alpha] + [(-\operatorname{ad} x_i)^s t_{ij}^{\gamma}, t_{ij}^\alpha] \\
&\quad - \sum_{k\neq i,j}(\operatorname{ad} x_k)^s [t_{kj}^{\alpha-\gamma}, t_{ij}^\alpha] - \sum_{k\neq i,j}(-\operatorname{ad} x_k)^s [t_{kj}^{\alpha+\gamma}, t_{ij}^\alpha] + \sum_{k\neq i,j}(\operatorname{ad} x_k)^s [t_{kj}^{-\gamma}, t_{ij}^\alpha] \\
&\quad + \sum_{k\neq i,j}(-\operatorname{ad} x_k)^s [t_{kj}^{\gamma}, t_{ij}^\alpha] \\
&= \sum_{k\neq j}\Big([-(\operatorname{ad} x_k)^s t_{kj}^{\alpha-\gamma}, t_{ij}^\alpha] - [(-\operatorname{ad} x_k)^s t_{kj}^{\alpha+\gamma}, t_{ij}^\alpha] + [(\operatorname{ad} x_k)^s t_{kj}^{-\gamma}, t_{ij}^\alpha] + [(-\operatorname{ad} x_k)^s t_{kj}^{\gamma}, t_{ij}^\alpha]\Big) \\
&= \sum_{k\neq j}[-((\operatorname{ad} x_k)^s t_{kj}^{\alpha-\gamma} + (-\operatorname{ad} x_k)^s t_{kj}^{\alpha+\gamma}) + (\operatorname{ad} x_k)^s t_{kj}^{-\gamma} + (-\operatorname{ad} x_k)^s t_{kj}^{\gamma}, t_{ij}^\alpha].
\end{aligned}$$

This finishes the proof.

□

Remark 8.1.4. *In particular, there is a Lie algebra morphism*

$$\begin{aligned}
\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}^\Gamma &\longrightarrow \operatorname{Der}(\mathfrak{t}_{1,n}^\Gamma), \\
e, f, h &\longmapsto \xi_e^{(n)}, \xi_f^{(n)}, \xi_h^{(n)} \\
\delta_{s,\gamma}^{(n)} &\longmapsto \tilde{\xi}_{s,\gamma}^{(n)}
\end{aligned}$$

and the equality

$$\tilde{\xi}_{s,\gamma}^{(n)}\left(\sum_{\alpha\in\Gamma} t_{ij}\right) = [\tilde{\xi}_{s,\gamma}^{(n)}(x_i), y_j] + [x_i, \tilde{\xi}_{s,\gamma}^{(n)}(y_j)] \tag{8.1}$$

implies that it is sufficient to determine the image of the x_i's and all the t_{ij}^α's to fully determine $\tilde{\xi}_{s,\gamma}^{(n)}$.

8.1.2 The Lie algebra of twisted special derivations

Recall that the fibers at τ of the Γ-punctured universal curve over $\bar{\mathcal{M}}_{1,1}^\Gamma$ are the spaces $E_{\tau,\Gamma}^\times$ consisting of an elliptic curve minus torsion points indexed by a finite group $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ is defined as the space $\left(\mathbb{C} - \left\{\left(\frac{1}{M}\right)\mathbb{Z} + \left(\frac{\tau}{N}\right)\mathbb{Z}\right\}\right)/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$.

Lemma 8.1.5. *The de-Rham fundamental Lie algebra $\mathfrak{p}(E_{\tau,\Gamma}^\times)$ of $E_{\tau,\Gamma}^\times$ is the $\mathbb{C}$-Lie algebra generated by symbols x, y and t^α, for $\alpha \in \Gamma$, such that $[x, y] = \sum_{\alpha\in\Gamma} t^\alpha$.*

Proof. The space $E_{\tau,\Gamma}^\times$ can be identified with the reduced twisted configuration space $\mathrm{C}(E_{\tau,\Gamma}, 2, \Gamma)$ whose de-Rham fundamental Lie algebra is $\bar{\mathfrak{t}}_{1,2}^\Gamma$, which is nothing but the $\mathbb{C}$-Lie algebra generated by symbols $x := \bar{x}_1$, $y := \bar{y}_2$ and $t^\alpha := \bar{t}_{12}^\alpha$, for $\alpha \in \Gamma$, such that $[x, y] = \sum_{\alpha\in\Gamma} t^\alpha$. □

For any $s \in \mathbb{N}$ and $\gamma \in \Gamma$ we set

$$D_{s,\gamma} := \sum_{p+q=s-1} \sum_{\beta \in \Gamma} [(\operatorname{ad} x)^p t^{\beta-\gamma}, (-\operatorname{ad} x)^q t^{\beta}],$$

and $(C_{s,\gamma})_\alpha := (\operatorname{ad} x)^s t^{\alpha-\gamma} + (-\operatorname{ad} x)^s t^{\alpha+\gamma}$. Observe that $(D_{s,\gamma}, C_{s,\gamma}) = (-1)^s (D_{s,-\gamma}, C_{s,-\gamma})$. One has shown (e.g. Proposition 6.2.4 and the fact that $D_{s,\gamma}(x, t^\beta) = D_{s,\gamma}(-x, t^{-\beta})$) that the bidegree of $(D_{s,\gamma}, C_{s,\gamma})$ is $(s+1, 1)$. The derivation $\bar{\xi}^{(2)}_{s,\gamma}$ is then given by

- $\bar{\xi}^{(2)}_{s,\gamma}(x) = 0$,
- $\bar{\xi}^{(2)}_{s,\gamma}(y) = D_{s,\gamma}(x, t^\beta)$,
- $\bar{\xi}^{(2)}_{s,\gamma}(t^\alpha) = [t^\alpha, C^\alpha_{s,\gamma}(x, t^\beta)]$.

The image of $\tilde{\delta}_{s,\gamma} := \delta_{s,\gamma} + (\operatorname{ad} x)^s t^{-\gamma} + (-\operatorname{ad} x)^s t^{\gamma}$ under the Lie algebra morphism $\mathfrak{d}^\Gamma \rtimes \operatorname{Der}(\hat{\mathfrak{t}}^\Gamma_{1,2}) \to \operatorname{Der}(\mathfrak{p}(E^\times_{\tau,\Gamma}))$ yields the derivation $\tilde{\bar{\xi}}^{(2)}_{s,\gamma}$ given by

- $\tilde{\bar{\xi}}^{(2)}_{s,\gamma}(x) = -(\operatorname{ad} x)^{s+1}(t^{-\gamma}) + (-\operatorname{ad} x)^{s+1}(t^{\gamma})$,
- $\tilde{\bar{\xi}}^{(2)}_{s,\gamma}(t^\alpha) = [-((\operatorname{ad} x)^s t^{\alpha-\gamma} + (-\operatorname{ad} x)^s t^{\alpha+\gamma}) + (\operatorname{ad} x)^s t^{-\gamma} + (-\operatorname{ad} x)^s t^{\gamma}, t^\alpha]$.

Remark 8.1.6. *We have*

$$\tilde{\xi}^{(2)}_{s,\gamma}(y) \quad = \quad -\xi^{(2)}_{s,\gamma}(y) + [(\operatorname{ad} x)^s t^{-\gamma} + (-\operatorname{ad} x)^s t^{\gamma}, y].$$

Let $\mathfrak{u}^\Gamma$ be the Lie subalgebra of $\operatorname{Der}(\mathfrak{p}(E^\times_{\tau,\Gamma}))$ generated by the derivations $\varepsilon_{s,\gamma}$ for $s \geqslant 1$ and $\gamma \in \Gamma$, defined by

- $\varepsilon_{s,\gamma}(x) = (\operatorname{ad} x)^s(t^{-\gamma}) + (-\operatorname{ad} x)^s(t^{\gamma})$,
- $\varepsilon_{s,\gamma}(t^\alpha) = [-((\operatorname{ad} x)^s t^{\alpha-\gamma} + (-\operatorname{ad} x)^s t^{\alpha+\gamma}) + (\operatorname{ad} x)^s t^{-\gamma} + (-\operatorname{ad} x)^s t^{\gamma}, t^\alpha]$.

Let $\mathfrak{u}$ be the Pollack's Lie subalgebra of $\operatorname{Der}^0(\mathfrak{f}_2(a,b))$ generated by the $\varepsilon_s \in \operatorname{Der}(\mathfrak{f}_2(x,y))$, for $s \geqslant 1$, given by

- $\varepsilon_{2s}(x) := \operatorname{ad}^{2s}(x)(y)$,
- $\varepsilon_{2s}(y) := \sum_{0 \leq j \leq s} (-1)^j [\operatorname{ad}^j(x)(y), \operatorname{ad}^{2s-1-j}(x)(y)]$.
- $\varepsilon_{2s+1}(x) = \varepsilon_{2s+1}(y) = 0$.

Proposition 8.1.7. *There is a surjective Lie algebra morphism*

$$\begin{array}{ccc} \mathfrak{u}^\Gamma & \longrightarrow & \mathfrak{u} \\ \varepsilon_{s,\gamma} & \longmapsto & \varepsilon_s. \end{array}$$

Proof. This is consequence of the definition of the commutativity of the comparison morphism diagram

$$\begin{tikzcd} \tilde{\mathfrak{d}}^{\Gamma_1} \ltimes \mathfrak{t}^{\Gamma_1}_{1,n} \arrow[r] \arrow[d] & \mathfrak{t}^{\Gamma_1}_{1,n} \arrow[d] \\ \tilde{\mathfrak{d}}^{\Gamma_2} \ltimes \mathfrak{t}^{\Gamma_2}_{1,n} \arrow[r] & \mathfrak{t}^{\Gamma_2}_{1,n} \end{tikzcd}$$

applied to the case where Γ_2 is trivial and of the definition of $\varepsilon_{s,\gamma}$, as $\varepsilon_{2s,\mathbf{0}}(x) = \varepsilon_{2s}(x)$ and $\varepsilon_{s,\mathbf{0}}(t^{\mathbf{0}}) = 0$. $\square$

8.2 Differential equations in τ

In this section we prove a differential equation in τ for the ellipsitomic KZB associator. Namely we have :

Theorem 8.2.1. *We have*

$$2\pi\mathrm{i}\frac{\partial}{\partial\tau}A^{\Gamma}(\tau) = \left(-\Delta_0 - \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{s\geqslant 0}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma}\right)A^{\Gamma}(\tau),$$

$$2\pi\mathrm{i}\frac{\partial}{\partial\tau}B^{\Gamma}(\tau) = \left(-\Delta_0 - \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{s\geqslant 0}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma}\right)B^{\Gamma}(\tau).$$

Proof. Recall that $z = z_{21}$ $x = \bar{x}_1$, $t^{\alpha} = \bar{t}^{\alpha}_{12}$. In Remark 8.1.2 we established

$$\frac{1}{2}\sum_{\gamma\in\Gamma}A_{s,\gamma}((\operatorname{ad}x)^s(t^{-\gamma}) + (-\operatorname{ad}x)^s(t^{\gamma})) = \sum_{\gamma\in\Gamma}A_{s,\gamma}(\operatorname{ad}x)^s(t^{-\gamma}).$$

Now, seen in $\mathrm{Der}(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})$, the (reduced) ellipsitomic KZB system for $n = 2$ is

$$\frac{\partial}{\partial z}F^{\Gamma}(z;\tau) = \left(-\sum_{\alpha\in\Gamma}e^{-2\pi\mathrm{i}a\operatorname{ad}(x)}\frac{\theta(z-\tilde{\alpha}+\operatorname{ad}(x)|\tau)}{\theta(z-\tilde{\alpha}|\tau)\theta(\operatorname{ad}(x)|\tau)}(t^{\alpha})\right)F^{\Gamma}(z;\tau)$$

$$\begin{aligned}2\mathrm{i}\pi\frac{\partial}{\partial\tau}F^{\Gamma}(z;\tau) &= -\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma}A_{s,\gamma}(\tau)\bar{\bar{\xi}}_{s,\gamma} - \sum_{\alpha\in\Gamma}g_{\alpha}(\operatorname{ad}(x),z|\tau)(t^{\alpha})\right)F^{\Gamma}(z;\tau)\\ &= -\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma} - \sum_{\alpha\in\Gamma}g_{\alpha}(z|\tau)(t^{\alpha})\right)F^{\Gamma}(z;\tau),\end{aligned}$$

where $g_{\alpha}(z|\tau) := g_{\alpha}(z,\operatorname{ad}x|\tau)(t^{\alpha}) - g_{\alpha}(0,\operatorname{ad}x|\tau)(t^{\alpha})$ and where $F^{\Gamma}(z;\tau)$ is defined on $\{(z,\tau)\in\mathbb{C}\times\mathfrak{H}|z = a+b\tau,(a,b)\in]0,1/M[\times\mathbb{R}\cup\mathbb{R}\times]0,1/N[\}$, valued in $\exp(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})\rtimes\Gamma\rtimes\mathrm{Aut}(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})\rtimes\Gamma$ and is determined by the behaviour

$$F^{\Gamma}(z;\tau)\simeq z^{t^{0}}\exp\left(-\frac{\tau}{2\pi\mathrm{i}}\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma}a_{s,\gamma}\bar{\bar{\xi}}^{(2)}_{s,\gamma}\right)\right)$$

when $z\longrightarrow 0^{+},\tau\longrightarrow\mathrm{i}\infty$. We have

$$\begin{aligned}F^{H}_{\Gamma}(z+\frac{1}{M}|\tau) &= (\bar{1},\bar{0})F^{H}_{\Gamma}(z|\tau)\underline{\tilde{A}}(\tau),\\ e^{2\pi\mathrm{i}\frac{x}{N}}F^{V}_{\Gamma}(z+\frac{\tau}{N}|\tau) &= (\bar{0},\bar{1})F^{V}_{\Gamma}(z|\tau)\tilde{B}(\tau).\end{aligned}$$

These conditions imply that the image of $F^{\Gamma}(z|\tau)$ in $\mathrm{Aut}(\hat{\bar{\mathfrak{t}}}^{\Gamma}_{1,2})$ is independent of z. Now let us write

$$x^{\Gamma}(\tau) := \Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma}.$$

We define

$$(A^\Gamma)_{z_0}^{z_1}(\tau) := F^\Gamma(z_1|\tau)F^\Gamma(z_0|\tau)^{-1} \in \exp(\hat{\mathfrak{t}}^\Gamma_{1,2}),$$

which satisfies

$$2\pi\,\mathrm{i}\,\frac{\partial}{\partial\tau}(A^\Gamma)_{z_0}^{z_1}(\tau) = -x^\Gamma(\tau)((A^\Gamma)_{z_0}^{z_1}(\tau)) + \sum_{\gamma\in\Gamma} g_\gamma(z_1|\tau)\cdot(A^\Gamma)_{z_0}^{z_1}(\tau) - (A^\Gamma)_{z_0}^{z_1}(\tau)\cdot g_\gamma(z_0|\tau).$$

The function $J(z|\tau)$ appearing in the definition of $A^\Gamma(\tau), B^\Gamma(\tau)$, is related to the function $F^\Gamma(z|\tau)$ by $F^\Gamma(z|\tau) = F(z|\tau)\varphi(\tau)$, where

$$\varphi(\tau) := (-2\pi\mathrm{i})^{t^{\mathbf{0}}}\exp\left(-\frac{\tau}{2\pi\mathrm{i}}\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} a_{s,\gamma}\bar{\bar{\xi}}^{(2)}_{s,\gamma}\right)\right)$$

takes values in $\exp(\hat{\mathfrak{t}}^\Gamma_{1,2}) \rtimes \mathrm{Aut}(\hat{\mathfrak{t}}^\Gamma_{1,2})$, because both them satisfy the same differential equation in z. It follows that

$$(A^\Gamma)_{z_0}^{z_1}(\tau) = J(z_1|\tau)J(z_0|\tau)^{-1}.$$

We conclude that $\underline{A}^\Gamma(\tau) = J(z|\tau)^{-1}(\bar{1},\bar{0})(A^\Gamma)_z^{z+\frac{1}{M}}(\tau)J(z|\tau)$. Now, taking $z \longrightarrow 0$, this implies

$$\underline{A}^\Gamma(\tau) = \lim_{z\longrightarrow 0}(-2\pi\,\mathrm{i}\,z)^{-\,\mathrm{ad}(t^{\mathbf{0}})}\big((\bar{1},\bar{0})(A^\Gamma)_z^{z+\frac{1}{M}}(\tau)\big).$$

As z is fixed, $(-2\pi\,\mathrm{i}\,z)^{-\,\mathrm{ad}(t^{\mathbf{0}})}\big((\bar{1},\bar{0})(A^\Gamma)_z^{z+\frac{1}{M}}(\tau)\big)$ satisfies the same differential equation in τ as $(A^\Gamma)_{z_0}^{z_1}(\tau)$, with $g(z_0|\tau)$ replaced by $(-2\pi\,\mathrm{i}\,z)^{-\,\mathrm{ad}(t^{\mathbf{0}})}(g(z|\tau))$ and $g(z_1|\tau)$ replaced by

$$(-2\pi\,\mathrm{i}\,z)^{-\,\mathrm{ad}(t^{\mathbf{0}})}((\bar{1},\bar{0})g(z+\frac{1}{M}|\tau)),$$

which both tend to 0 when $z \longrightarrow 0$. It follows that these terms disappear from the differential equation satisfied by $\underline{A}^\Gamma(\tau)$, so

$$2\pi\,\mathrm{i}\,\frac{\partial}{\partial\tau}A^\Gamma(\tau) = -(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma})A^\Gamma(\tau).$$

Let us now show the differential equation for $B^\Gamma(\tau)$.

We have, $\underline{B}^\Gamma(\tau) = F(z|\tau)^{-1}(\bar{0},\bar{1})e^{\frac{2\pi\,\mathrm{i}\,x}{N}}(A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau)F(z|\tau)$, thus

$$\underline{B}^\Gamma(\tau) = \lim_{z\longrightarrow 0}(-2\pi\,\mathrm{i}\,z)^{-t^{\mathbf{0}}}(\bar{0},\bar{1})e^{\frac{2\pi\,\mathrm{i}\,x}{N}}(A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau)(-2\pi\,\mathrm{i}\,z)^{t^{\mathbf{0}}}.$$

One computes, for $\tilde{\alpha} = \frac{a_0}{M} + \tau\frac{a}{N}$ any lift of $\alpha\in\Gamma$,

$$\begin{aligned}\frac{\partial}{\partial\tau}(A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau) = \quad & \frac{-1}{2\pi\,\mathrm{i}}x^\Gamma(\tau)((A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau)) \\ & +\Big(\frac{1}{2\pi\,\mathrm{i}}g(z+\frac{\tau}{N}|\tau) - \sum_{\alpha\in\Gamma} e^{-2\pi\mathrm{i}ax}\frac{\theta(z+\frac{\tau}{N}-\tilde{\alpha}+\mathrm{ad}\,x|\tau)}{\theta(z+\frac{\tau}{N}-\tilde{\alpha}|\tau)\theta(\mathrm{ad}\,x|\tau)}(t^\alpha)\Big)(A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau) \\ & -(A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau)\frac{1}{2\pi\,\mathrm{i}}g(z|\tau).\end{aligned}$$

Set $X_z(\tau) := (-2\pi\,\mathrm{i}\,z)^{-t^{\mathbf{0}}}(\bar{0},\bar{1})e^{\frac{2\pi\,\mathrm{i}\,x}{N}}(A^\Gamma)_z^{z+\frac{\tau}{N}}(\tau)(-2\pi\,\mathrm{i}\,z)^{t^{\mathbf{0}}}$. If we fix z, we get

$$\begin{aligned}
2\pi\,\mathrm{i}\,\frac{\partial}{\partial\tau}X_z(\tau) \quad = \quad & -x^\Gamma(\tau)(X_z(\tau)) \\
& -X_z(\tau)\cdot\big((-2\pi\,\mathrm{i}\,z)^{-t^{\mathbf{0}}}g(z|\tau)(-2\pi\,\mathrm{i}\,z)^{t^{\mathbf{0}}}\big) \\
& +\Big(\mathrm{Ad}((-2\pi\,\mathrm{i}\,z)^{-t^{\mathbf{0}}}(\bar{0},\bar{1})e^{\frac{2\pi\,\mathrm{i}\,x}{N}}\big(g(z+\frac{\tau}{N}|\tau) \\
& -2\pi\,\mathrm{i}\sum_{\alpha\in\Gamma}e^{-2\pi\mathrm{i}ax}\frac{\theta(z+\frac{\tau}{N}-\tilde{\alpha}+\mathrm{ad}\,x|\tau)}{\theta(z+\frac{\tau}{N}-\tilde{\alpha}|\tau)\theta(\mathrm{ad}\,x|\tau)}(t^\alpha)\big) \\
& -(-2\pi\,\mathrm{i}\,z)^{-t^{\mathbf{0}}}e^{\frac{2\pi\,\mathrm{i}\,x}{N}}((\bar{0},\bar{1})x^\Gamma(\tau)e^{-\frac{2\pi\,\mathrm{i}\,x}{N}}(-2\pi\,\mathrm{i}\,z)^{t^{\mathbf{0}}}\Big)\cdot X_z(\tau).
\end{aligned}$$

Then, as we showed that

$$\Delta(\mathbf{z}+\frac{\tau\delta_j}{N}|\tau) = e^{\frac{-2\pi\mathrm{i}\mathrm{ad}(x_j)}{N}}\theta((\bar{0},\bar{1})_j)\cdot(\Delta(\mathbf{z}|\tau)-K_j(\mathbf{z}|\tau)).$$

then the parenthesis in the last three lines is equal to

$$\mathrm{Ad}((-2\pi\,\mathrm{i}\,z)^{-t^{\mathbf{0}}})(g(z|\tau)).$$

We conclude that, in the limit $z\longrightarrow 0$,

$$2\pi\,\mathrm{i}\,\frac{\partial}{\partial\tau}B^\Gamma(\tau) = -(\Delta_0+\frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma})B^\Gamma(\tau).$$

□

Remark 8.2.2. *If we suppose that the group* $\mathrm{GRT}^\Gamma_{ell}(\mathbb{C})$ *has a semi-direct product decomposition into some group* $\mathrm{R}^\Gamma_{ell}(\mathbb{C})$ *and* $\mathrm{GRT}(\mathbb{C})$, *there is an action of* $\mathrm{R}^\Gamma_{ell}(\mathbb{C})$ *on* $\mathrm{Ell}^\Gamma_{KZB}$. *In this case, the above theorem can be rewritten in a more compact way by*

$$2\pi\mathrm{i}\frac{\partial}{\partial\tau}e^\Gamma(\tau) = e^\Gamma(\tau)*\left(-\Delta_0-\frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{s\geqslant 0}A_{s,\gamma}(\tau)\bar{\bar{\xi}}^{(2)}_{s,\gamma}\right).$$

where $*$ *is here a Lie algebra action.*

Let us fix $\tau\in\mathfrak{H}$, $\gamma\in\Gamma$ and $x\in\mathbb{C}$. Define

$$\sigma^\tau_{x,\gamma}(z) := \frac{\theta(z+\tilde{\gamma}+x)}{\theta(z+\tilde{\gamma})\theta(x)}.$$

Consider x as a formal variable close to 0 and $\sigma^\tau_{x,\gamma}$ as an element of $x^{-1}\mathcal{M}(\mathbb{C})[[x]]$, where

$$\mathcal{M}(\mathbb{C}) = \{\text{meromorphic functions defined over } \mathbb{C}\}.$$

Proposition 8.2.3. $\sigma^\tau_{x,\gamma}$ *has an expansion*

$$\sigma^\tau_{x,\gamma}(z) = \frac{1}{x}+\sum_{n\geq 0}k^\tau_{\gamma,n}(z)x^n,$$

where $k^\tau_{\gamma,0}(z) = (\theta'_\tau/\theta_\tau)(z+\tilde{\gamma})$ *and* $k^\tau_{\gamma,n}$ *is regular at* 0 *and* 1 *if* $n>0$.

Proof. In light of [35, Proposition 2.5], the only left thing to prove is that, for $\gamma\neq\mathbf{0}$, we have $k^\tau_{\gamma,0}(z) = (\theta'_\tau/\theta_\tau)(z+\tilde{\gamma})$ which is true by the very same computation (using that θ is an odd function) and the fact that for $\gamma\neq\mathbf{0}$, the term $k^\tau_{\gamma,0}(z)$ is regular when $z,x\longrightarrow 0$. □

8.3 Elliptic multiple zeta values at torsion points

The twisted elliptic KZB associator $e^{\Gamma}(\tau) := (A^{\Gamma}(\tau), B^{\Gamma}(\tau))$ has an expression in terms of iterated integrals. Let us denote

$$K^{\Gamma}(z) := -\sum_{\alpha\in\Gamma} e^{-2\pi i a x}\frac{\theta(z-\tilde{\alpha}+\mathrm{ad}(x)|\tau)}{\theta(z-\tilde{\alpha}|\tau)\theta(\mathrm{ad}(x)|\tau)}(t^{\alpha}).$$

By Picard iteration and well-known properties of iterated integrals, we have

$$I^{\Gamma}(\tau) = \left(\lim_{t\to 0} z^{-t^{\mathbf{0}}}\left((\bar{1},\bar{0})\exp\left[\int_{\alpha_t\left(\frac{1-t}{M}\right)} K^{\Gamma}(z)dz\right]\right) z^{t^{\mathbf{0}}}\right)^{\mathrm{op}}$$

and

$$2i\pi J^{\Gamma}(\tau) = \left(\lim_{t\to 0} z^{-t^{\mathbf{0}}}\left((\bar{0},\bar{1})\exp\left[\int_{\beta_t\left(\frac{\tau-t}{N}\right)} K^{\Gamma}(z)dz\right]\right) z^{t^{\mathbf{0}}}\right)^{\mathrm{op}}$$

where the superscript op denotes the opposite multiplication on the algebra $\mathbb{C}\langle\langle x, t^{\alpha}; \alpha\in\Gamma\rangle\rangle$, defined by $(f\cdot g)^{\mathrm{op}} = g\cdot f$. Here we choose the principal branch of the logarithm so that $log(\pm i) = \pm\pi i/2$.

Definition 8.3.1. *Let $n_1,\ldots,n_r \geqslant 0$ and $\alpha_1,\alpha_2,\ldots,\alpha_r\in\Gamma$. The twisted elliptic multizeta values*

$$I_A^{\Gamma}\begin{pmatrix} n_1 & n_2 & ,\ldots, & n_r \\ \alpha_1 & \alpha_2 & ,\ldots, & \alpha_r \end{pmatrix};\tau\Bigg) \text{ and } I_B^{\Gamma}\begin{pmatrix} n_1 & n_2 & ,\ldots, & n_r \\ \alpha_1 & \alpha_2 & ,\ldots, & \alpha_r \end{pmatrix};\tau\Bigg)$$

are defined equivalently

1. *as the coefficients of $\mathrm{ad}^{n_1}(x)(t^{\alpha_1})\ldots\mathrm{ad}^{n_r}(x)(t^{\alpha_r})$ in the renormalized generating series of regularized iterated integrals*

$$\lim_{t\to 0} z^{-t^{(\bar{1},\bar{0})}}\exp\left[\int_{\alpha_t\left(\frac{1-t}{M}\right)} F^{\Gamma}(z)dz\right] z^{t^{\mathbf{0}}} \text{ and } \lim_{t\to 0} z^{-t^{(\bar{0},\bar{1})}}\exp\left[\int_{\beta_t\left(\frac{\tau-t}{N}\right)} F^{\Gamma}(z)dz\right] z^{t^{\mathbf{0}}}$$

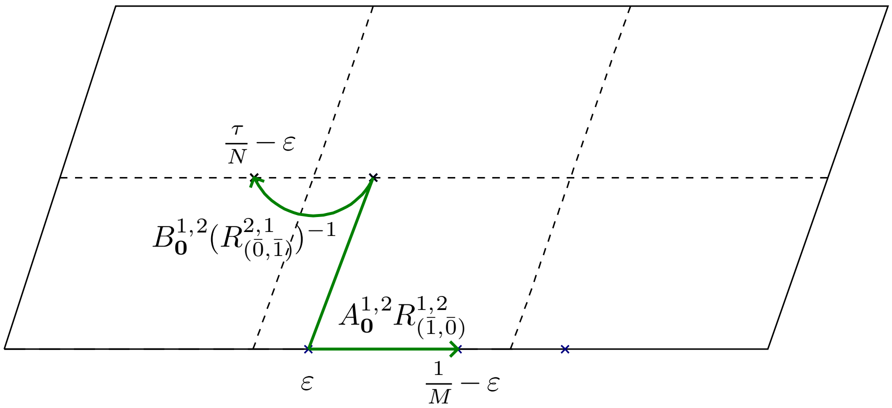
2. *by means of two functions $\mathcal{A}^{\Gamma}(\tau)$ and $\mathcal{B}^{\Gamma}(\tau)$, closely related to $A(\tau)$ and $B(\tau)$, of the form*

$$\mathcal{A}^{\Gamma}(\tau) = \sum_{n\geqslant 0}(-1)^n \sum_{n_1,\ldots,n_r\geqslant 0}\sum_{\alpha_1,\ldots,\alpha_r\in\Gamma} I_A^{\Gamma}\left(\begin{matrix} n_1 & n_2 & ,\ldots, & n_r \\ \alpha_1 & \alpha_2 & ,\ldots, & \alpha_r \end{matrix};\tau\right)\mathrm{ad}^{n_1}(x)(t^{\alpha_1})\ldots\mathrm{ad}^{n_r}(x)(t^{\alpha_r})$$

and

$$\mathcal{B}^{\Gamma}(\tau) = \sum_{n\geqslant 0}(-1)^n \sum_{n_1,\ldots,n_r\geqslant 0}\sum_{\alpha_1,\ldots,\alpha_r\in\Gamma} I_B^{\Gamma}\left(\begin{matrix} n_1 & n_2 & ,\ldots, & n_r \\ \alpha_1 & \alpha_2 & ,\ldots, & \alpha_r \end{matrix};\tau\right)\mathrm{ad}^{n_1}(x)(t^{\alpha_1})\ldots\mathrm{ad}^{n_r}(x)(t^{\alpha_r})$$

One can picturally see the relation between $(\mathcal{A}^{\Gamma}(\tau, \mathcal{B}^{\Gamma}(\tau))$ and $(A(\tau), B(\tau))$ by means of the following picture



Our approach to multiple zeta values at torsion points is somewhat different to that in the recent work of Broedel–Matthes–Richter–Schlotterer [19], and generalizes to the case of any surjective morphism $\mathbb{Z}^2 \longrightarrow \Gamma$ sending the generators of $\mathbb{Z}^2$ to their class modulo M and N, respectively. More general surjective morphisms could be considered. The relation between the twisted elliptic multiple zeta values obtained in this paper and that in [19] will be investigated by the second author and N. Matthes in a forthcoming collaboration.

Now, multiple Hurwitz values are defined, for $n_2, ..., n_{r-1} \geq 1$, $n_r \geq 2$, as the real numbers

$$\zeta(n_1, \ldots, n_r, a_1, \ldots, a_r) = \sum_{0 \leq k_1 < \cdots < k_r; m_i \in \mathbb{Z}} \frac{1}{(k_1 - a_1)^{n_1}(k_2 - a_2)^{n_2} \cdots (k_r - a_r)^{n_r}}$$

where $a_1, \ldots, a_r$ are rational numbers with $a_1 > 0$ and such that $\zeta(n_1, \ldots, n_r, 1, \ldots, 1) = \zeta(n_1, \ldots, n_r)$.

Then, the differential equation of Theorem 8.2.1 combined with the fact that, for real values of $\gamma \in \Lambda_{\tau,\Gamma}$, the Eisenstein-Hurwitz series have Hurwitz zeta values as constant coefficients in their q_N-expansion, permits us to expect the following:

- elliptic multiple zeta values at torsion points should have a q_N-expansion whose coefficients are special linear combinations of multiple Hurwitz values,
- elliptic multiple zeta values at (real) torsion points should degenerate to multiple Hurwitz values at the cusps of $Y(\Gamma)$.
- elliptic multiple zeta values at torsion points should be linear combinations of iterated integrals of Eisenstein-Hurwitz series whose coefficients are controlled by the Lie algebra $\mathfrak{u}^{\Gamma}$.

This gives hope of finding new periods of $\mathbb{P}^1 - \{0, \mu_M, \infty\}$ besides cyclotomic multiple-zeta values for special values of M.

List of notation

Operads

Groups

Spaces

Lie and associative algebras

Torsor sets

Bundles

Series

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