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Derived Differential Geometry

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Contents

Abstract

Moduli spaces in differential geometry, such as those arising in symplectic topology and gauge theory, are constructed via intersection theory of nonlinear elliptic operators in infinite dimensions. These spaces are often not smooth manifolds due to transversality issues. The purpose of this work is to resolve these problems in the paradigm of derived geometry due to Lurie and Toën-Vezzosi. We characterize the ∞-category of derived manifolds via a universal property in the $(\infty, 2)$-category of finitely complete ∞-categories and show that it admits a description as the ∞-category of homotopically finitely presented simplicial C^∞-rings. We do the same thing for derived manifolds with corners, which we show are simplicial C^∞-rings equipped with positive logarithmic structures. We then show that these objects admit a good theory of higher derived stacks and investigate their deformation theory.

Résumé

Les espaces de modules en géométrie différentielle, tels que ceux qui apparaissent dans la topologie symplectique et la théorie de jauge, sont construits via la théorie d'intersection d'opérateurs elliptiques non linéaires en dimensions infinies. Ces espaces ne sont souvent pas des variétés lisses en raison de problèmes de transversalité. Le but de ce travail est de résoudre ces problèmes dans le paradigme de la géométrie dérivée dû à Lurie et Toën-Vezzosi. Nous caractérisons l'∞-category des variétés dérivées via une propriété universelle dans la $(\infty, 2)$-catégorie des ∞-categories finiment complètes et montrons qu'elle admet une description comme l'∞-category des anneaux C^∞ simpliciaux de présentation finie. On fait la même chose pour les variétés dérivées à bord, dont on montre qu'elles sont des anneaux C^∞ simpliciaux équipés de structures logarithmiques positives. Nous montrons ensuite que ces objets admettent une bonne théorie des champs dérivés supérieurs et étudions leur théorie de déformation.

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Chapter 1

Introduction

The purpose of this thesis is to lay the foundations of *derived geometry* in the differentiable, that is, C^∞-setting for applications in the theory of moduli spaces in differential geometry, symplectic geometry and mathematical physics, using the modern language and powerful tools of higher category theory, higher topos theory, and higher algebra. The corresponding theory of derived *algebraic* geometry has been well established for a number of years due to the seminal works of Lurie (DAG series, [Lur11b] through [Lur14], and [Lur]) and Toën-Vezzosi (Homotopical Algebraic Geometry, [TV04; TV06]). Derived geometry has been established in other contexts as well; there are derived versions of analytic geometry due to Lurie [Lur11a] and Porta-Yu [Por15; PY17]. In fact, a substantial literature on derived differential geometry already exists since the pioneering work of Spivak [Spi10], including a substantial work-in-progress of Joyce [Joy12b], the model categorical efforts of Carchedi-Roytenberg [CR12b; CR12a], the work of Borisov-Noël [BN11], recent work of Behrend-Liao-Xu [BLX20] and Amorim-Tu [AT20], and the thesis of Nuiten [Nui18] (and undoubtedly others that would deserve to be mentioned).
Derived geometry is a confluence of classical geometry, homological and homotopical algebra, intersection theory, deformation theory and higher sheaf theory, and the subject may be approached and appreciated from any of these avenues, and there are a number of excellent introductions available that do the subject justice; let us mention in particular the survey's of Toën and Anel [Toë14; Ane]. We motivate the theory we wish to develop in this work via an intersection problem, but one quite different from the well-known algebro-geometric story that passes from Serre's intersection formula to Koszul resolutions and derived pushouts of dg-algebras, as in the introduction of [Lur11b], for instance. We will be concerned with intersection theory in infinite dimensions.
From a sufficiently abstract vantage, in geometry influenced by Quantum Field Theory such as symplectic geometry and gauge theory, one studies the geometry of moduli spaces of solutions of nonlinear elliptic equations on manifolds -which are usually required to be compact(if not, the function spaces need to satisfy some decay estimates to admit well behaved moduli spaces)- up to the action of a (possibly infinite dimensional) Lie group of symmetries and perhaps suitably compactified. Dispensing with the issues of compactification and symmetries for the moment, we are interested in the following situation:

(1) M a compact smooth manifold.

(2) $V \to M$ a smooth fibre bundle over M.

(3) $F \to M$ a smooth vector bundle over M.

(4) $P : \Gamma(V) \to \Gamma(F)$ a nonlinear elliptic differential operator acting between smooth sections of V and F.

Let $\mathsf{Sol}(P) = P^{-1}(0)$. Let $x \in \mathsf{Sol}(P)$ and suppose that the linearization $dP_x : \Gamma(x^*TM) \to \Gamma(E)$, a 2-term Fredholm complex with finite dimensional homology, is surjective. Then $\mathsf{Sol}(P)$ admits the structure of a smooth manifold in a neighbourhood of x. If the linearized differential operator is not surjective, we still have the following important principle.

Fact 1.0.0.1 (Local finite dimensional reduction by Kuranishi models)**.** Locally, $\mathsf{Sol}(P)$ is given by the zero set of a smooth function $f : \mathbb{R}^n \to \mathbb{R}^k$ such that at each solution x of $f = 0$, the two-term complex determined by the linearization of f at x is *quasi-isomorphic* to the 2-term complex determined by the linearization of P at x.

This follows from an application of the inverse function theorem for Banach manifolds and elliptic bootstrapping methods, after replacing the spaces of smooth sections with Sobolev completions of sufficiently high regularity; we refer to the appendices of [MS12] for a textbook account in symplectic topology.
Depending on the geometric situation, it may or may not be possible to *perturb* the operator P and obtain a well defined cobordism class of smooth spaces of solutions. When this is not possible (when $\mathsf{Sol}(P)$ is the space of genus

0 pseudo-holomorphic curves on a non semipositive symplectic manifold, for instance), one is forced to make sense of $\mathsf{Sol}(P)$ using the zeroes of local finite dimensional reductions which are not transverse. We have the following two problems

(a) The local finite dimensional reductions are far from unique; only the homology complex induced by dP_x as x varies over $\mathsf{Sol}(P)$ is invariant.

(b) The space Sol is a gluing of the zero sets of local finite dimensional reductions, but as these spaces can have arbitrarily badly behaved topology (as subspaces of some Cartesian space) it is not clear how to perform this gluing and obtain some sort of geometric C^∞ structure on $\mathsf{Sol}(P)$.

Let us make an attempt at dealing with these issues.

Definition 1.0.0.2. An *affine Kuranishi model (without isotropy)* is a triple $(X, p : E \to X, s)$ where X is a smooth manifold, $p : E \to X$ is a vector bundle on E, and s is a section of p. We will usually just write E for the bundle $p : E \to X$. Given two affine Kuranishi models (X, E, s) and (Y, F, t), a morphism $f : (X, E, s) \to (Y, F, t)$ is a commuting diagram

$$\begin{tikzcd} E \arrow[r, "f_v"] \arrow[d, "p"] & F \arrow[d, "q"] \\ X \arrow[r, "f_b"] & Y \end{tikzcd}$$

where f_v is fibrewise linear such that $f_v \circ s = t \circ f_b$. Affine Kuranishi models and morphisms between them form a category, that we denote AffKur.

An *isomorphism* of affine Kuranishi models is far too strict a notion, largely irrelevant to the construction of geometric structure on moduli spaces.

Definition 1.0.0.3. Let $f : (X, E, s) \to (Y, F, t)$ be a morphism of affine Kuranishi models, and let $x \in Z(s)$ be a point. Then the diagram

$$\begin{tikzcd} T_x X \arrow[r, "T_x s"] \arrow[d, "T_x f_b"'] & T_{s(x)} E \arrow[d, "T_{s(x)} f_v"] \\ T_{f_b(x)} Y \arrow[r, "T_{f_b(x)} t"] & T_{s(x)} F \end{tikzcd}$$

commutes, and we say that f is a *weak equivalence at x* if the diagram is a quasi-isomorphism. We say that f is a *weak equivalence* if f induces a bijection $Z(s) \cong Z(t)$ and f is a weak equivalence at all points of $Z(s)$. Let $W \subset \mathrm{Fun}(\Delta^1, \mathsf{AffKur})$ be the full subcategory spanned by the weak equivalences. This full subcategory contains all identity maps and has the 2-out-of-6 property so the pair (AffKur, W) is a *homotopical category.*

Remark 1.0.0.4. It can be shown that if f is a weak equivalence at p, then f induces a homeomorphism from a neighbourhood of $p \in Z(s)$ onto a neighbourhood of $f(p) \in Z(t)$ (see for instance corollary 5.1.3.27); thus, a weak equivalence always induces a homeomorphism on zero sets.

We see that the ambiguity of local finite dimensional reduction is neatly resolved by the notion of a weak equivalence of affine Kuranishi models. Now we could apply the abstract principle of *localizing a category at a subcategory of weak equivalences* to obtain the 'correct' category of affine Kuranishi models. We define, up to essentially unique equivalence of categories, a new category $\mathsf{AffKur}[W^{-1}]$ equipped with a functor

$$L : \mathsf{AffKur} \longrightarrow h\mathsf{AffKur}^{\mathrm{c}}$$

by declaring that L has the following universal property in the 2-category of categories: for each category $\mathcal{C}$, the restriction functor along the functor $L : \mathsf{AffKur} \to \mathsf{AffKur}[W^{-1}]$ induces an equivalence of categories

$$\mathrm{Fun}(\mathsf{AffKur}[W^{-1}], \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}_W(\mathsf{AffKur}, \mathcal{C}),$$

where $\mathrm{Fun}_W(\mathsf{AffKur}, \mathcal{C}) \subset \mathrm{Fun}(\mathsf{AffKur}, \mathcal{C})$ is the full subcategory spanned by functors that carry weak equivalences in AffKur to isomorphisms in $\mathcal{C}$. By abstract nonsense, a localization at a subcategory of weak equivalences always exists, but it might be difficult to get a handle on the morphism sets. Very often, it is convenient to have a bit more structure. For instance, the classical homotopy category $h\mathcal{S}$ is obtained from the category CW of CW-complexes by inverting the weak homotopy equivalences, the maps $f : X \to Y$ of CW-complexes that induce isomorphisms $\pi_n(X) \cong \pi_n(Y)$ on all homotopy groups. It follows from Whitehead's theorem that a map is a weak homotopy equivalence if and only if it admits a *homotopy inverse.* This extra structure -the notion of a homotopy between maps- allows for a concrete description of the homotopy category: homotopy of maps is an equivalence relation, so simply take the category whose morphisms sets $\mathrm{Hom}_{h\mathcal{S}}(X, Y)$ are the homotopy equivalence classes of maps $X \to Y$ between CW-complexes.

Construction 1.0.0.5. An object (X,E,s) of AffKur determines a commutative $\mathbb{R}$-algebra $C^\infty(X)$ and a finitely generated projective module $\Gamma(E)$ of smooth sections of E over $C^\infty(X)$. We denote by $\Gamma(E^\vee)$ the sections of the dual vector bundle of E. Consider the object $\Gamma(E^\vee)[1]\oplus C^\infty(X)\in \mathrm{Mod}^{\mathrm{gr}}_{C^\infty(X)}$ in the category of graded $C^\infty(X)$-modules, which has the projective module $\Gamma(E^\vee)$ sitting in degree 1 and the object $C^\infty(X)$ sitting in degree 0. Contracting sections of $E^\vee$ with the section s furnishes a differential on $\Gamma(E^\vee)[1]\oplus C^\infty(X)$ making it a differentially graded $C^\infty(X)$-module. Taking the symmetric algebra over $C^\infty(X)$, we obtain the commutative differential graded exterior algebra $\mathrm{Sym}^\bullet_{C^\infty(X)}(\Gamma(E^\vee)[1])$ concentrated in nonnegative degrees, equipped with the differential ι_s contracting with s. Given affine Kuranishi models (X,E,s) and (Y,F,t) and a smooth map $f:X\to Y$, it is not hard to see that there is a canonical bijection

$$\left\{\begin{array}{c}\begin{tikzcd} E \arrow[r, "f_v"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ X \arrow[r, "f_b"] & Y, \end{tikzcd}\end{array} \in \mathrm{Fun}(\Delta^1,\mathsf{AffKur});\ f_b=f\right\}\longleftrightarrow\left\{\mathrm{Sym}^\bullet_{C^\infty(Y)}(\Gamma(F^\vee)[1])\to \mathrm{Sym}^\bullet_{C^\infty(X)}(\Gamma(E^\vee)[1])\right\}$$

Since maps $C^\infty(X)\to C^\infty(Y)$ of commutative algebras are in bijection with the smooth maps $X\to Y$ [KMS91], we have a fully faithful embedding

$$\mathsf{AffKur}^{op}\longhookrightarrow \mathbf{cdga}^{\geq 0}_{\mathbb{R}},\qquad (X,E,s)\longmapsto \mathrm{Sym}^\bullet_{C^\infty(X)}(\Gamma(E^\vee)[1]).$$

If $\mathbf{K}$ is an affine Kuranishi model, we will denote its associated cdga by $\mathcal{O}(\mathbf{K})$.

Remark 1.0.0.6. Since we have a fully faithful embedding $\mathsf{AffKur}^{op}\hookrightarrow \mathbf{cdga}^{\geq 0}_{\mathbb{R}}$, there seems to be another natural notion of weak equivalence on AffKur^{op}, namely the subcategory of $\mathrm{Fun}(\Delta^1,\mathsf{AffKur}^{op})$ spanned by the maps which induce quasi-isomorphisms on the exterior algebras, giving AffKur another structure of a homotopical category. It can be shown (see corollary 5.1.3.27 again) that the two notions of weak equivalence are the same, that is, a map $f:\mathbf{K}\to\mathbf{J}$ is a weak equivalence in the sense of definition 1.0.0.3 if and only if f induces a quasi-isomorphism $\mathcal{O}(\mathbf{J})\to\mathcal{O}(\mathbf{K})$.

There is a natural notion of homotopy among morphisms between commutative differentially graded algebras (over $\mathbb{Q}$) [BG76]: let $\Omega^\bullet_{\mathrm{poly}}(\Delta^1)$ be the (nonpositively graded) algebra of *polynomial differential forms* on the 1-simplex which admits the presentation

$$\Omega^\bullet_{\mathrm{poly}}(\Delta^1)=\mathbb{R}[t_0,t_1,dt_0,dt_1]/(t_0+t_1-1,dt_0+dt_1)$$

and comes with two evaluation maps $\mathrm{ev}_0,\mathrm{ev}_1:\Omega^\bullet_{\mathrm{poly}}(\Delta^1)\to\mathbb{R}$. We say that two morphisms $f,g:A\to B$ of differentially graded $\mathbb{R}$-algebras are *homotopic* if there is a morphism

$$H:A\longrightarrow B\otimes_{\mathbb{R}}\Omega^\bullet_{\mathrm{poly}}(\Delta^1)$$

in $\mathbf{cdga}_{\mathbb{R}}$ such that $(\mathrm{id}\otimes\mathrm{ev}_0)\circ H=f$ and $(\mathrm{id}\otimes\mathrm{ev}_1)\circ H=g$. While this immediately yields a notion of homotopy among morphisms between affine Kuranishi models, this does not accurately reflect the differential nature of Kuranishi models. For instance, if A is of the form $C^\infty(M)$ for a smooth manifold and B is of the form $\mathcal{O}(\mathbf{K})$ for $\mathbf{K}=(X,E,s)$ an affine Kuranishi model, a homotopy between two maps $f,g:C^\infty(M)\to\mathcal{O}(\mathbf{K})$ of differentially graded algebras consists of a diagram

$$\begin{tikzcd} C^\infty(M) \arrow[r, "H"] \arrow[dr, "0"'] & C^\infty(X)[t]\oplus\Gamma(E^\vee)[t] \arrow[d, "\frac{d}{dt}+\iota_s"] \\ & C^\infty(X)[t] \end{tikzcd}$$

where H is as a map of cdga's for the *trivial square zero extension* algebra structure on $C^\infty(X)[t]\oplus\Gamma(E^\vee)[t]$. The map H can be identified with a pair of maps (h_t,λ_t), where h_t is a polynomial family of maps $C^\infty(M)\to C^\infty(X)$ and λ_t is polynomial family of maps $\Omega^1_{C^\infty(M)}\to\Gamma(E^\vee)$ equivariant for the action of $C^\infty(M)$ on the left and $C^\infty(X)$ on the right. Here $\Omega^1_{C^\infty(M)}$ is the module of algebraic Kähler differentials of $C^\infty(M)$ [Eis95], which admits a universal derivation $d_{\mathrm{dR}}:C^\infty(M)\to\Omega^1_{C^\infty(M)}$. The map H is then given by

$$H(\_)=h_t(\_)+\lambda_t(d_{\mathrm{dR}}(\_))$$

and is subject to the constraint

$$\frac{dh_t}{dt}(\_)+\iota_s(\lambda_t(d_{\mathrm{dR}}(\_)))=0.$$

Our notion of homotopy between morphism is adequate, as long as we restrict to a subcategory of *cofibrant objects* (which is morally the same kind of operation as restricting to CW complexes).

Variant 1.0.0.7. Let $\mathsf{KurAff}^{\mathrm{c}} \subset \mathsf{KurAff}$ be the full subcategory spanned by affine Kuranishi models of the form $(\mathbb{R}^n, \mathbb{R}^{m+n}, s)$, where s is a section of the trivial bundle $\mathbb{R}^{n+m} \to \mathbb{R}^n$, that is, simply a function $\mathbb{R}^n \to \mathbb{R}^m$.

The following lemma is easy to prove using model category techniques.

Lemma 1.0.0.8. *Let* $\mathbf{K}$ *and* $\mathbf{J}$ *be a affine Kuranishi models.*

(1) *Suppose that* $\mathbf{J}$ *is cofibrant. Then on* $\mathrm{Hom}_{\mathsf{AffKur}}(\mathbf{K}, \mathbf{J})$*, the homotopy relation is transitive. Thus, the homotopy relation is an equivalence relation.*

(2) *Suppose that* $\mathbf{K}$ *and* $\mathbf{J}$ *are cofibrant. Let* $f : \mathbf{K} \to \mathbf{J}$ *be a weak equivalence (either definition 1.0.0.3 is satisfied or* $\mathcal{O}(\mathbf{J}) \to \mathcal{O}(\mathbf{K})$ *is a quasi-isomorphism) then* f *admits a homotopy inverse* g*.*

Notation 1.0.0.9. We write $h\mathsf{AffKur}^{\mathrm{c}}$ for the category obtained by taking as morphism sets the homotopy equivalence classes.

Remark 1.0.0.10. We cannot conclude that $h\mathsf{AffKur}^{\mathrm{c}}$ coincides with the localization of $\mathsf{AffKur}^{\mathrm{c}}$ at the set of homotopy equivalences or weak equivalences. This discrepancy is due to the problem that AffKur does not 'have enough path objects'.

Remark 1.0.0.11. There is an obvious forgetful functor $Z : \mathsf{AffKur} \to \mathsf{Top}$ sending an affine Kuranishi model $\mathbf{K}$ to the topological space $Z(\mathbf{K}) := Z(s)$. It follows from the definition of a weak equivalence that this forgetful functor factors through a functor $h\mathsf{AffKur}^{\mathrm{c}} \to \mathsf{Top}$ that we abusively also denote Z.

Now if we are given a nice topological space X, we would like to express the idea that X arises as the gluing of a collection of affine Kuranishi models, and this gluing should be compatible on the overlaps, that is, a cocycle condition should hold, all up to homotopy. To express gluing of two Kuranishi models (X, E, s) and (Y, F, t) along a common open subset in $Z(s)$ and $Z(t)$, we have to verify that restricting along open subsets of the zero locus is well defined in the homotopy category.

Definition 1.0.0.12. Let $\mathbf{K} = (X, E, s)$ be an affine Kuranishi model and let $U \subset Z(s)$ be an open subset, then we say that a morphism $\mathbf{H} \to \mathbf{K}$ in $h\mathsf{AffKur}^{\mathrm{c}}$ *exhibits* $\mathbf{H}$ *as a localization of* $\mathbf{K}$ *with respect to* U if for every affine Kuranishi model $\mathbf{J}$, restriction along the map $\mathbf{H} \to \mathbf{K}$ induces a bijection

$$\mathrm{Hom}_{h\mathsf{AffKur}^{\mathrm{c}}}(\mathbf{J}, \mathbf{H}) \longrightarrow \mathrm{Hom}^{U}_{h\mathsf{AffKur}^{\mathrm{c}}}(\mathbf{J}, \mathbf{K}),$$

where $\mathrm{Hom}^{U}_{h\mathsf{AffKur}^{\mathrm{c}}}(\mathbf{J}, \mathbf{K}) \subset \mathrm{Hom}_{h\mathsf{AffKur}^{\mathrm{c}}}(\mathbf{J}, \mathbf{K})$ is the subset of those maps $f : \mathbf{J} \to \mathbf{K}$ that satisfy the condition that $Z(f) : Z(\mathbf{J}) \to Z(\mathbf{K})$ factors through U.

It follows immediately from the definition that a localization of $\mathbf{K}$ with respect to $U \subset Z(s)$ is unique up to unique isomorphism, provided it exists; we will denote it $\mathbf{K}|_U \to \mathbf{K}$.

Lemma 1.0.0.13. *For every open* $U \subset Z(s)$ *and every open set* $V \subset X$ *such that* $V \cap Z(s) = U$*, the morphism of affine Kuranishi models* $(V, E|_V, s|_V) \to (X, E, s)$ *exhibits a localization with respect to* U*.*

It follows easily that sending a localization of $\mathbf{K}$ to the underlying open subset of $Z(\mathbf{K})$ induces an equivalence of categories between the full subcategory of the slice category $h\mathsf{AffKur}^{\mathrm{c}}_{/\mathbf{K}}$ spanned by localizations of $\mathbf{K}$ and the lattice of open subsets of $Z(\mathbf{K})$ (viewed as a category). If $f : \mathbf{K} \to \mathbf{H}$ is a morphism in $h\mathsf{AffKur}^{\mathrm{c}}$, then for each open $U \subset Z(\mathbf{K})$ we denote by $f|_U$ the composition $\mathbf{K}|_U \to \mathbf{K} \to \mathbf{H}$.
The following definition is the most naive approach one might be tempted to try.

Incorrect Definition 1.0.0.14. Let X be a paracompact Hausdorff topological space. A *naive Kuranishi atlas (without isotropy)* on X consists of the following data

(a) An open covering $\{U_i \to X\}_{i \in I}$ of X (not necessarily finite).

(b) A collection of affine Kuranishi models $\{\mathbf{K}_i\}_{i \in I}$ with zero loci $\{Z(\mathbf{K}_i)\}_{i \in I}$ called *charts*.

(c) A collection of homeomorphisms $\psi_i : Z(\mathbf{K}_i) \to U_i$ called *footprint maps* or *chart maps*.

(d) For every pair of indices $i, j \in I$ such that $U_{ij} := U_i \cap U_j$ is nonempty, an isomorphism $\phi_{ij} : \mathbf{K}_i|_{\psi_i^{-1}(U_{ij})} \to \mathbf{K}_j|_{\psi_j^{-1}(U_{ij})}$ in the homotopy category $h\mathsf{AffKur}^{\mathrm{c}}$. Moreover, we require that $\phi_{ii} = \mathrm{id}_{\mathbf{K}_i}$.

These data are required to satisfy the following conditions.

(1) The transition maps ϕ_{ij} are compatible with the footprint maps: for all pairs $i, j \in I$ such that U_{ij} is nonempty, the diagram

$$\begin{tikzcd} \psi_i^{-1}(U_{ij}) = Z(\mathbf{K}_i|_{\psi_i^{-1}(U_{ij})}) \arrow[rr, "Z(\phi_{ij})"] \arrow[dr, "\psi_i|_{U_{ij}}"'] & & Z(\mathbf{K}_j|_{\psi_j^{-1}(U_{ij})}) = \psi_j^{-1}(U_{ij}) \arrow[dl, "\psi_j|_{U_{ij}}"] \\ & U_{ij} & \end{tikzcd}$$

commutes.

(2) The cocycle condition holds: for every triple $i, j, k \in I$ such that $U_{ijk} \coloneqq U_i \cap U_j \cap U_k$ is nonempty, we see that (1) and the universal property of localization imply that the composition

$$\phi_{ij}|_{\psi^{-1}(U_{ijk})} : \mathbf{K}_i|_{\psi_i^{-1}(U_{ijk})} \longrightarrow \mathbf{K}_i|_{\psi_i^{-1}(U_{ij})} \xrightarrow{\phi_{ij}} \mathbf{K}_j|_{\psi_j^{-1}(U_{ij})}$$

where the first map is a localization, factors through $\mathbf{K}|_{\psi_j^{-1}(U_{ijk})}$. Then we can apply ϕ_{jk}, and we demand that the equality

$$\phi_{jk}|_{\psi_j^{-1}(U_{ijk})} \circ \phi_{ij}|_{\psi_i^{-1}(U_{ijk})} = \phi_{ik}|_{\psi_i^{-1}(U_{ijk})}$$

holds.

Note that this description is in almost complete analogy with the notion of an atlas on a manifold; indeed, suppose that for each i in the set I indexing the charts, the section s_i is transverse to the zero section, then a naive Kuranishi atlas in the sense above gives X the structure of a smooth manifold. To understand the sort of pathologies that this definition produces when the sections s_i are not transverse, we should contemplate what kind of objects we can extract from a naive Kuranishi atlas on a nice space X. If X is covered by a single affine Kuranishi model, then there exists a distinguished object (up to isomorphism) in the derived category of sheaves of $\mathbb{R}$-modules on X; indeed, for an affine Kuranishi model $\mathbf{K} = (X, E, s)$ we have the complex

$$\mathbb{T}_{\mathbf{K}} \coloneqq T_M \xrightarrow{ds} s^* T_E \in \mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(Z(\mathbf{K}))).$$

which we call the *tangent complex* or the *virtual tangent sheaf* of $\mathbf{K}$. If the topological space X is $\mathsf{Sol}(P)$ for some moduli problem defined by an elliptic equation, then the homology of linearization also determines a well defined element in $\mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(X)))$. Thus, we should at least demand that the local tangent complexes should glue nicely to produce a global object in the derived category of sheaves of $\mathbb{R}$-modules on X. To facilitate this gluing process, it's important to understand how the tangent complex is functorial in the transition maps of the Kuranishi atlas. To this end, it is convenient to recast incorrect definition 1.0.0.14 as follows.

Incorrect Definition 1.0.0.15. Consider the category $\mathsf{Top}^{\mathsf{open}}$ whose objects are paracompact Hausdorff spaces and whose morphisms are open topological embeddings of such spaces. Similarly, let $\mathsf{hKurAff}^{c,\mathrm{open}}$ be the subcategory of $\mathsf{hKurAff}^c$ on the morphisms $f : \mathbf{J} \to \mathbf{K}$ such that $Z(f)$ is an open topological embedding. We have a *Grothendieck fibration*

$$Z : \mathsf{hKurAff}^{c,\mathrm{open}} \longrightarrow \mathsf{Top}^{\mathsf{open}}$$

taking the underlying topological space of an affine Kuranishi space. Let $X \in \mathsf{Top}^{\mathsf{open}}$, then a *naive Kuranishi atlas* on X consists of the following data.

(a') A collection of maps $\{V_i \to X\}_{i \in I}$ in $\mathsf{Top}^{\mathrm{open}}$ of X with images being open sets $\{U_i \subset X\}$ that cover X. We can view this data as a functor $\mathfrak{U} : I \to \mathsf{Top}^{\mathsf{open}}$ from the set I viewed as a category with only identity morphisms. Consider the poset

$$P_I^{\leq 3} \coloneqq \{J \subset I;\ J \neq \varnothing,\ |J| \leq 3\}$$

of nonempty subsets of I of cardinality at most 3 ordered by reverse inclusion, then the functor $\mathfrak{U}$ induces a functor

$$f : P_I^{\leq 3} \longrightarrow \mathsf{Top}^{\mathsf{open}}$$

which sends J to the limit of the diagram $J \subset I \to \mathsf{Top}^{\mathrm{open}}_{/X}$.

(b') A dotted lift $\tilde{f}$ of f as follows

$$\begin{tikzcd} & h\mathsf{AffKur}^{c,\mathrm{open}} \arrow[d, "Z"] \\ P_I^{\leq 3} \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & \mathsf{Top}^{\mathsf{open}} \end{tikzcd}$$

that makes the diagram of categories (strictly) commute. Moreover, we require that $\tilde{f}$ carries every morphism in $P_I^{\leq 3}$ to a Cartesian morphism with respect to Z.

The two definitions of a naive Kuranishi atlas are equivalent: given a naive Kuranishi atlas in the sense of definition 1.0.0.15, the restriction of $\tilde{f}$ to I determines a collection of affine Kuranishi models $\{\mathbf{K}_i\}_{i\in I}$ with homeomorphisms $\psi_i : Z(\mathbf{K}_i) = V_i \cong U_i \subset X$. For every nonempty intersection $U_i \cap U_j$ with $i \neq j$, we have maps

$$\mathbf{K}_i|_{\psi_i^{-1}(U_i\cap U_j)} \longleftarrow \tilde{f}(\{i,j\}) \longrightarrow \mathbf{K}_j|_{\psi_j^{-1}(U_i\cap U_j)}$$

induced by the subset inclusions $\{i\} \subset \{i,j\} \supset \{j\}$. These maps are isomorphisms because $\tilde{f}$ carries all morphisms to Cartesian morphisms, giving us the isomorphisms $\{\phi_{ij}\}$. The compatibility conditions (1) and (2) are guaranteed by the fact that the diagram of (b') commutes and that $\tilde{f}$ is a functor. Conversely, from datum (a) of definition 1.0.0.14 we can construct a functor as in (a'), and given data (b) through (d) satisfying (1) and (2), it is possible to construct a lift $\tilde{f}$, and it can be shown that for a suitable choice of morphisms between naive Kuranishi atlases for both of the definitions we have given, this correspondence determines an equivalence of categories.

Construction 1.0.0.16. We have a pseudofunctor

$$(\mathsf{Top}^{\mathrm{open}})^{op} \longrightarrow \mathsf{Cat}, \quad X \longmapsto \mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(X))$$

that carries each space X to the derived category of sheaves of $\mathbb{R}$-modules on X. We can apply the Grothendieck construction to this pseudofunctor, obtaining a functor

$$\int_{\mathsf{Top}^{\mathrm{open}}} \mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(\_)) \longrightarrow \mathsf{Top}^{\mathrm{open}}.$$

Concretely, this category is given as follows.

(1) Objects are pairs $(U, \mathcal{F})$, where $U \in \mathsf{Top}^{\mathrm{open}}$ and $\mathcal{F}$ is a complex of sheaves on U.

(2) Morphisms $(U,\mathcal{F}) \to (V,\mathcal{G})$ are maps $i : U \subset V$ together with a map $i^*\mathcal{G} \to \mathcal{F}$.

The *tangent complex functor* $\mathbb{T}$ is the functor that carries an object $\mathbf{K}$ to the pair $(Z(\mathbf{K}), \mathbb{T}_{\mathbf{K}})$. Note that by definition of a weak equivalence, this assignment is well defined up to equivalence. The tangent complex functor fits into a commuting diagram

$$\begin{tikzcd} \mathsf{hKurAff}^{c,\mathrm{open}} \arrow[rr, "\mathbb{T}"] \arrow[dr, "Z"'] & & \int_{\mathsf{Top}^{\mathrm{open}}} \mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(\_)) \arrow[dl, "p"] \\ & \mathsf{Top}^{\mathrm{open}} & \end{tikzcd}$$

and carries Cartesian morphisms with respect to Z to Cartesian morphisms with respect to p.

Fact 1.0.0.17 (Descent)**.** Let X be a topological space with an open cover $\{U_i \subset X\}_{i\in I}$ determining the diagram

$$f : P_I^{\leq 3} \longrightarrow \mathsf{Top}^{\mathsf{open}}$$

as in definition 1.0.0.15, and let $\mathcal{A}$ be a Grothendieck abelian category, such as Mod_A for a A a unital commutative ring. There is a canonical equivalence between the category of $\mathrm{Ch}(\mathcal{A})$-valued sheaves on X, and the lifts

$$\begin{tikzcd} & \int_{\mathsf{Top}^{\mathrm{open}}} \mathsf{Shv}_{\mathrm{Ch}(\mathcal{A})}(\_) \arrow[d, "p"] \\ P_I^{\leq 3} \arrow[ur, dotted, "\mathcal{F}"] \arrow[r, "f"'] & \mathsf{Top}^{\mathsf{open}} \end{tikzcd}$$

such that $\mathcal{F}$ sends all morphisms to Cartesian morphisms with respect to p. This equivalence is implemented by the 'coCartesian pushforward', which applies to each $\mathcal{F}(J)$ the functor $i_!$ on sheaves of chain complexes induced by the map $i : f(J) \subset X$, resulting in a diagram $P_I^{\leq 3} \to \mathsf{Shv}_{\mathrm{Ch}(\mathcal{A})}(X)$, and takes the limit.

The equivalence between sheaves of vector spaces and descent data suggests a recipe for producing a global tangent complex. In our situation, however fact 1.0.0.17 does not apply since the tangent complex is only well defined in the derived category due to the ambiguity in the choice of affine Kuranishi model; the tangent complex determines an honest sheaf of $\mathbb{R}$-vector spaces on the zero locus of $(X, E \to X, s)$ if and only if s is transverse to the zero section. On the other hand, the similar statement for $\mathsf{Shv}_{\mathrm{Ch}(\mathcal{A})}(\_)$ replaced with $\mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathcal{A})}(\_))$ is false. While we can 'push the diagram forward', the derived category $\mathbf{D}(\mathsf{Shv}_{\mathrm{Ch}(\mathcal{A})}(\_))$ has very few limits, so we might not be able to preform the gluing construction. Even more seriously, if a limit exists, the resulting object $\mathbb{T}_X$ need not have the property that $\mathbb{T}_X|_{U_i}$ is isomorphic in the derived category to $\mathbb{T}_{\mathbf{K}_i}$, so $\mathbb{T}_X$ would not deserve to be called 'the gluing' of the complexes of sheaves $\mathbb{T}_{\mathbf{K}_i}$. The correct construction of limits in the derived category has to keep track of the various homotopies.

Definition 1.0.0.18. Fix again a topological space X with an open cover $\{U_i \subset X\}_{i \in I}$ determining a diagram

$$f : P_I \longrightarrow \mathsf{Top}^{\mathsf{open}}$$

where P_I is now the poset of all nonempty subsets.
A *homotopical descent datum* is a lift

$$\begin{tikzcd} & \int_{\mathsf{Top}^{\mathrm{open}}} \mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(\_) \arrow[d, "p"] \\ P_I \arrow[ur, dotted, "\mathcal{F}"] \arrow[r, "f"'] & \mathsf{Top}^{\mathsf{open}} \end{tikzcd}$$

such that $\mathcal{F}$ sends all morphisms to *homotopy* Cartesian morphisms with respect to p, which are those maps $(i, \alpha) : (U, \mathcal{F}) \to (V, \mathcal{G})$ for which $\alpha : i^*\mathcal{G} \to \mathcal{F}$ is a quasi-isomorphism.

Now a version descent holds: we can identify the derived category of $\mathsf{Shv}_{\mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})}(X)$ with the derived category of homotopical descent data, i.e. lifts as in the definition above. Now we clearly see the problem with definitions 1.0.0.14 and 1.0.0.15: the tangent complex of a naive Kuranishi atlas on X does *not* induce a homotopical descent datum for complexes of sheaves on X because the transition isomorphisms ϕ_{ij} only satisfy the cocycle condition in the homotopy category, while we should demand that they satisfy the cocycle condition up to *coherent* homotopy. At this point, one might be tempted to sidestep the issue altogether: it may seem as if an obvious improvement of incorrect definition 1.0.0.14 would be to require that the isomorphisms ϕ_{ij} are not extended zig-zags, but single maps, and that the cocycle condition holds on the nose.

Incorrect Definition 1.0.0.19. Let X be a paracompact Hausdorff topological space. A *strict Kuranishi atlas* on X consists of the following data.

(a) An open covering $\{U_i \to X\}_{i \in I}$ of X (not necessarily finite).

(b) A collection of affine Kuranishi models $\{K_i\}_{i \in I}$ with zero loci $\{Z_i\}_{i \in I}$ called *charts*.

(c) A collection of homeomorphisms $\psi_i : Z_i \to U_i$ called *footprints maps*.

(d) For every pair of indices $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, a weak equivalence $K_i|_{U_{ij}} \to K_j|_{U_{ji}}$.

This data is required to satisfy the following conditions.

(i) The cocycle condition holds: for every triple $i, j, k \in I$ such that $U_i \cap U_j \cap U_k$ is nonempty, we have the equality $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$.

(ii) The footprint maps are compatible with the transition maps: for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, we have the equality $\psi_j \circ \phi_{ij} = \psi_i$.

Now we do (trivially) have a homotopical descent datum, so a space X equipped with a Kuranishi atlas admits a well defined tangent complex in the derived category of sheaves of vector spaces on X, but this definition is too strong: the local finite dimensional reductions cannot be chosen to fit together in such a strict Kuranishi atlas. The problem of constructing a homotopical descent datum from the data of an elliptic equation is complicated as well. There is a vast literature on Kuranishi spaces, which have their inception in the work of Fukaya-Ono [FO99], which aims to find the correct notion of a space equipped with a Kuranishi atlas and prove that moduli spaces have such a structure, solving this coherence problem; see for instance [Par16; Joy15; MW12; Fuk+00] (we do not pretend to approximate these theories here or prove any sort of comparison with the notions of a Kuranishi atlas defined above, which merely serve a pedagogical purpose). In all these (slightly) different theories, the construction of the Kuranishi atlas is done essentially by hand, by induction on the size of the atlas, which requires one to solve a new elliptic moduli problem at each step.
In this work, we take our cue from derived algebraic geometry, and bring the homotopical and higher categorical machinery to bear on problems (a) and (b) above: to obtain the desired coherence for the tangent complex, we replace the homotopy category $\mathsf{hKurAff}^c$ with a suitable ∞*-category*. One way to do achieve this is to promote KurAff to a simplicial category by tensoring with $\Omega^{\bullet}_{\mathrm{poly}}(\Delta^n)$, but there is a more conceptually satisfying approach: since we are attempting to find an ∞-category of geometric objects that arise as non-transverse intersections of manifolds, let us consider the *universal* one.

Definition 1.0.0.20. The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}$ of *affine derived manifolds* (of finite presentation) is the smallest enlargement of the category of manifolds that has all finite limits (and idempotents).

The precise meaning of this statement will be given in chapter 2, and is based on the theory developed by Lurie [Lur11b]: if we let Man denote the category of smooth manifolds, then $\mathsf{d}C^\infty\mathsf{Aff}$ is an ∞-category that admits finite limits and idempotents together with a fully faithful functor $\mathsf{Man} \hookrightarrow \mathsf{d}C^\infty\mathsf{Aff}$ that preserves pullbacks along transverse maps and is universal with respect to this property. We give the following characterization of this ∞-category.

Theorem 1.0.0.21 ([CS19])**.** *Let* CartSp *be the category of Cartesian spaces* $\{\mathbb{R}^n\}$ *and smooth maps between them, and let* $\mathcal{S}$ *denote the* ∞*-category of spaces. Let* $sC^\infty\mathsf{ring}$ *be the* ∞*-category of* simplicial C^∞-rings*, the full subcategory of the functor* ∞*-category* $\mathrm{Fun}(\mathsf{CartSp},\mathcal{S})$ *spanned by those functors that preserve finite products. Then* $\mathsf{d}C^\infty\mathsf{Aff}$ *can be identified with the full subcategory of compact objects of* $sC^\infty\mathsf{ring}$.

One should think of simplicial C^∞-ring as a derived $\mathbb{R}$-algebra that is moreover equipped with a compatible (homotopy coherent) C^∞ functional calculus. Most of the approaches to derived C^∞-geometry (like the work of Spivak and Joyce) start with the notion of a C^∞-ring, and our result validates this choice.
Upon compactifying, moduli spaces such as those of pseudo-holomorphic polygons, acquire a boundary and corners. To adequately handle such cases, a version of derived C^∞-geometry with corners is desirable. Let Man_c denote the category of manifolds with *faces* and interior b-maps between them (see [Mel93], or chapter 3). Just as Man, this category has a universal 'derived geometry', denoted $\mathsf{d}C^\infty\mathsf{Aff}_c$.

Theorem 1.0.0.22. *The* ∞*-category* $\mathsf{d}C^\infty\mathsf{Aff}_c$ *can be identified as the full subcategory spanned by compact objects in a certain localization of the* ∞*-category of product preserving functors* $\mathsf{CartSp}_c \to \mathcal{S}$*, where* CartSp_c *is the category of Cartesian spaces with corners* $\{\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}\}$*. The latter category admits an alternative characterization as the* ∞*-category of pairs* (A,M) *where* A *is a simplicial* C^∞*-ring and* M *is a (derived)* logarithmic structure *on the (derived) monoid* $A_{\geq 0}$*, the positive elements of* A.

We cannot compare to other approaches as a result of this theorem since no work on derived C^∞-geometry with corners has yet been done.
The remainder of this work consists essentially of the verification that the familiar machinery of derived geometry applies to the contexts of C^∞-geometry mentioned above.

(1) For each simplicial C^∞-ring A, the ∞-category of A-modules, obtained by taking modules of the underlying $\mathbb{E}_\infty$-algebra, is equivalent to the ∞-category of spectrum objects in $\mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{/A})$ [Lur17a]. The *cotangent complex* of a simplicial C^∞-ring A may be defined as suspension spectrum

$$\Sigma^\infty_+\left(A \overset{\mathrm{id}}{\to} A\right) \in \mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{/A}) \simeq \mathsf{Mod}_A.$$

It can be computed via Kähler differentials, and on manifolds it coincides with the cotangent bundle.

(2) Consider the ∞-category $C^\infty\mathsf{RingTop}$ of *structured* ∞*-topoi* [Lur11b], informally given by pairs $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_{\mathcal{X}}$ a sheaf of local simplicial C^∞-rings on $\mathcal{X}$. The global sections functor

$$\Gamma : C^\infty\mathsf{RingTop} \longrightarrow sC^\infty\mathsf{ring}^{op}$$

admits a right adjoint *spectrum functor* **Spec** that is fully faithful on the subcategory of almost finitely presented simplicial C^∞-rings. The spectrum functor takes values in 0-localic ∞-topoi and the essential image consists of *derived affine* C^∞*-schemes*. Derived C^∞-schemes are defined in the obvious way.

(3) The ∞-topos $\mathsf{dSt} \coloneqq \mathsf{Shv}(\mathsf{dSmAff})$ of derived C^∞-stacks has for each $n \geq 0$ full subcategories of *derived* n*-Artin* C^∞*-stacks* and *derived* n*-Deligne-Mumford* C^∞*-stacks*, defined by inductively by gluing $(n-1)$-Artin/DM stacks along submersive/étale maps respectively [TV06; Sim96]. These stacks have a deformation theory (nilcomplete and inf-cohesive [Lur14; TV06])) and a cotangent complex. Furthermore, n-localic derived C^∞-schemes are equivalent to derived n-Deligne-Mumford stacks.

(4) The inverse function theorem holds: a map between derived manifold $\mathbf{Spec}\,A \to \mathbf{Spec}\,B$ is a local equivalence if and only if the *relative cotangent complex* $\mathbb{L}_{B/A}$ vanishes.

A similar list of properties is satisfied by derived manifolds with corners.
With a robust theory of derived geometry in the smooth setting available, we may now return to the elliptic moduli problem that prompted our discussion: we wish to understand $\mathsf{Sol}(P)$ as a derived geometric space. A valid perspective on the difficulties arising in moduli problems of geometric PDE's is that the problems arise, fundamentally, when one tries to work bottom-up, starting with a *set* of solutions, and then attempting to endow this set with some 'generalized smooth' *structure*. In algebraic geometry, it has long been recognized that moduli problems are best described as sheaves (in groupoids) on the category of affine schemes, which then fixes the geometric structure of

these generalized spaces. Let $V \to M$ be a fibre bundle, $F \to M$ a vector bundle, and $P : \Gamma(V) \to \Gamma(F)$ be a nonlinear elliptic differential equation. Then P defines a natural map

$$\widehat{P} : \mathsf{Map}_M(M, Y)_{\mathsf{dSt}} \longrightarrow \mathsf{Map}_M(M, V)_{\mathsf{dSt}}$$

between *derived stacks of sections*, which admits an entirely analysis-free definition as internal Homs in the Cartesian symmetric monoidal ∞-category $\mathsf{d}C^\infty\mathsf{St} = \mathsf{Shv}(\mathsf{d}C^\infty\mathsf{Aff})$. We define a derived stack Sol of solutions of P as the cone in the pullback diagram

$$\begin{tikzcd} \mathsf{Sol} \arrow[r] \arrow[d] & \mathsf{Map}_M(M, Y)_{\mathsf{dSt}} \arrow[d, "\widehat{P}"] \\ * \arrow[r, "0"] & \mathsf{Map}_M(M, V)_{\mathsf{dSt}} \end{tikzcd}$$

among derived stacks. We will prove the following result in upcoming work.

Theorem 1.0.0.23. *If M is compact and $\widehat{P}$ is elliptic, then Sol is* representable *by a (possibly non-affine) derived manifold. The perfect tangent complex $\mathbb{T}_{\mathsf{Sol}}$ is at each solution $s \in \mathsf{Sol}$ identified with the Fredholm map given by the linearization of the operator P at s.*

This result does not follow simply from the inverse function theorem for Banach manifolds. Instead, we have to resort to an inverse function theorem that produces local finite dimensional reductions at the Fréchet level, such as the Nash-Moser theorem [Ham82].

1.1 Sommaire

Le but de cette thèse est de jeter les bases de la *géométrie dérivée* dans le cadre différentiable, c'est-à-dire C^∞, pour des applications dans la théorie des espaces de modules en géométrie différentielle, en géométrie symplectique et en physique mathématique, en utilisant le langage moderne et les puissants outils de la théorie des catégories supérieures, de la théorie des topos supérieurs et de l'algèbre supérieure. La théorie correspondante de la géométrie *algébrique* dérivée est bien établie depuis un certain nombre d'années grâce aux travaux fondateurs de Lurie (série DAG, [Lur11b] à [Lur14], et [Lur]) et Toën-Vezzosi (Homotopical algebraic geometry [TV04; TV06]).

Contenu

Nous décrivons le contenu de cet ouvrage chapitre par chapitre.

Chapter 2: Recollections on ∞-Categories and ∞-Topoi

Ce chapitre est consacré à un rappel des notions et des résultats de base en théorie des ∞-catégories. Nos principales références sont [Lur17b] and [Lur17a]. Dans la deuxième partie de ce chapitre, nous approfondissons un peu la théorie des topoi supérieurs et prouvons plusieurs résultats qui seront utilisés plus tard dans le texte, comme une caractérisation des ∞-topoi n-localiques. Nous donnons également une construction tout à fait générale des groupes de jauge dans des ∞-topoi arbitraires.

Chapter 3: Pregeometries and Geometric Contexts

Ce chapitre introduit les notions fondamentales de *pregeometries* et de *geometries*, dues à Lurie. En particulier, nous expliquons le processus de passage d'une pregeometry à une geometry au moyen d'une construction universelle qui ajoute des limites à une pregeometry de manière minimale. Après avoir donné plusieurs exemples algébriques, nous traitons la théorie classique des C^∞*-rings* dans le paradigme de Lurie. La dernière partie du chapitre concerne les champs supérieurs de Simpson [Sim96] dans un cadre très général. La souplesse du formalisme développé ici nous aide quand nous avons plusieurs geometries et plusieurs catégories de schémas affines autour.

Chapter 4: Derived C^∞-geometry: foundational aspects

Dans ce long chapitre, plusieurs résultats principaux sont démontrés. Par exemple, nous prouvons la caractérisation suivante de *geometric envelope* des variétés lisses.

Theorem. *Soit* $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}^{op}$ *le foncteur évident portant une variété lisse vers son* C^∞*-anneau simplicial de fonctions lisses. Ensuite,* $C^\infty(\_)$ *factorise par la sous-catégorie plein* $sC^\infty\mathsf{ring}_{\mathrm{fp}} \subset sC^\infty\mathsf{ring}$ *des objets compacts, et le foncteur résultant se trouve dans* $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op})$ *et il existe une structure naturelle d'une geometry sur* $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ *telle que* $C^\infty(\_)$ *présente une* geometric envelope*, i.e. l'∞-catégorie* $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ *2-représente le foncteur* $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \_)$.

Nous fournissons ensuite un certain nombre de résultats sur l'algèbre homologique des fonctions lisses qui joueront un rôle crucial dans les autres contextes géométriques que nous développerons, comme la géométrie analytique réelle dérivée, et la géométrie dérivée différentiable à bord. En fait, pour les variétés dérivées avec bord, nous prouvons un théorème similaire à celui ci-dessus pour les variétés dérivées sans bord.
La dernière partie de ce chapitre traite des modules pour les C^∞-anneaux simpliciaux. Nous donnons deux caractérisations des modules : une en termes de stabilisation de la fibration du codomaine, et une utilisant l'algèbre sous-jacente

Chapter 5: The cotangent complex and differential calculus

Dans ce chapitre, nous établissons le complexe cotangent pour les anneaux simpliciaux (avec et sans bord) ainsi que les résultats usuels de fonctorialité. La principale nouveauté ici concerne les anneaux de fonctions de Whitney: nous prouvons que (pour certains ensembles fermés) les anneaux de fonctions de Whitney ont un complexe cotangent libre, que nous utilisons pour prouver que l'anneau de fonctions de l'intersection dérivée d'ensembles régulièrement situés coïncide avec l'anneau de fonctions de l'intersection des ensembles.

Notations

Here are the notations and conventions we use throughout the text.

- We handle the interplay between small and large categories via the usual device of Grothendieck universes, i.e. we assume Tarski-Grothendieck set theory. For any cardinal κ, we denote by $\mathcal{U}(\kappa)$ the collection of sets of rank $< \kappa$. We fix once and for all three strongly inaccessible cardinals $\kappa_s < \kappa_l < \kappa_{vl}$; then we call the sets in $\mathcal{U}(\kappa_s)$ *small*, those in $\mathcal{U}(\kappa_l)$ *large*, and those in $\mathcal{U}(\kappa_{vl})$ *very large*.

- The ordinary category of (small) sets is denoted as Set. The ordinary category of (small) simplicial sets is denoted as Set_Δ. When we speak of the model category of simplicial sets, we always mean its standard Quillen model structure.

- An ∞-category or $(\infty,1)$-category is a *weak Kan complex*, also known as a *quasi-category*. Our reference on the foundations of such higher categories is J. Lurie's book *Higher Topos Theory* [Lur17b].

- The homotopy category of an ∞-category $\mathcal{C}$ is denoted by $h\mathcal{C}$.

- For $\mathcal{C}$ an ∞-category, the Kan complex of morphisms between two objects X and Y is denoted by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

- For $C, D \in \mathcal{C}$ two morphisms in an ∞-category, a morphism f in the opposite ∞-category $\mathcal{C}^{op}$ from C to D is denoted $C \leftarrow D : f$.

- The nerve-realization adjunction defined by the cosimplicial simplicial(ly enriched) category $\mathfrak{C}(\Delta^\bullet)$ realizes a Quillen equivalence between $\mathbf{sSet}_{\mathrm{Joyal}}$ and the category of small simplicial categories endowed with Bergner's model structure. For a simplicial category M, we denote by $\mathbf{N}(M)$ the ∞-category obtained by taking the homotopy coherent nerve of M. For a simplicial model category $\mathcal{M}$, we denote by $\mathcal{M}^{fc}$ the simplicial category of fibrant-cofibrant objects in M. In the case of Set_Δ with the standard Kan-Quillen simplicial model structure, we write $\mathbf{N}(\mathsf{Set}_\Delta^{fc}) = \mathcal{S}$. The classical homotopy category $h\mathcal{S}$ is denoted $\mathcal{H}$.

- For a relative category $(\mathbf{A}, W)$, we denote by $A[W^{-1}]$ the ∞-category obtained by taking a fibrant replacement of the marked simplicial set $(\mathbf{N}(A), \mathbf{N}(W))$ in the model category Set_Δ^+ of marked simplicial sets.

- For $\mathcal{C}$ a simplicial set (usually an ∞-category) and $\mathcal{D}$ an ∞-category, the simplicial set of morphism from $\mathcal{C}$ to $\mathcal{D}$ is denoted as $\mathrm{Fun}(\mathcal{C},\mathcal{D})$. It is an ∞-category and it is called *the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$*. When $\mathcal{C} = \mathcal{S}$, the ∞-category of spaces, we write $\mathsf{PShv}(\mathcal{D})$ for $\mathrm{Fun}(\mathcal{D}^{op},\mathcal{S})$, and call it the *$\infty$-category of presheaves on $\mathcal{D}$*.

- An adjunction or ∞-adjunction $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$, with L the left adjoint and R the right adjoint is written as $(L \dashv R)$.

- The inclusion $\mathcal{S} \subset \mathsf{Cat}_\infty$ admits a left and a right adjoint. The right adjoint of the inclusion takes an ∞-category $\mathcal{C}$ to the *underlying ∞-groupoid* $\mathcal{C}^\simeq$, the wide subcategory on the invertible morphisms, that is, the largest Kan complex contained in $\mathcal{C}$, and comes with a counit map $\mathcal{C}^\simeq \hookrightarrow \mathcal{C}$. The left adjoint takes an ∞-category to the localization at all morphisms, which we denote $\|\mathcal{C}\|$, and comes with a unit map $\mathcal{C} \to \|\mathcal{C}\|$.

- Our grading conventions are *homological*, that is, the differential on a complex *lowers* the degree. Accordingly, a complex of R-modules $C \in \mathbf{Mod}_R$ for some commutative ring R is called *connective* if $H_n(C) = 0$ for all $n \leq -1$. A complex is called *eventually connective* if there exists some n such that $H_k(C) = 0$ for all $k < n$.

- A functor $f : \mathcal{C} \to \mathcal{D}$ of small ∞-categories is *left cofinal* if the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible for all objects $D \in \mathcal{D}$. By HTT, theorem 4.1.3.1, f is left cofinal if and only if composing with f identifies $\mathcal{D}$-indexed colimits with $\mathcal{C}$-indexed colimits in any ∞-category. A functor is *right cofinal* if $f^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is left cofinal.

- A *manifold* is a second countable, Hausdorff topological manifold without boundary whose topological dimension is globally bounded, equipped with a maximal C^∞-atlas. The category of manifolds is denoted Man. A manifold in our sense may have connected components of differing dimensions, as long as there is not a (countable) sequence of connected components whose dimensions grow to infinity. An *n-manifold* is a manifold each connected component of which has dimension n.

Chapter 2

Recollections on ∞-Categories and ∞-Topoi

In this work, we will freely make use of the theory and semantics of higher category theory incarnated through quasi-categories, as developed by Joyal [Joy], and, very extensively, by Lurie [Lur17b; Lur17a; Lur09]. We will occasionally employ, and the reader may benefit from, some other foundational texts on ∞-categories and homotopical algebra that offer different perspectives; see for instance Cisinski book [Cis18], or that of Riehl-Verity [RV18].
In the following we will very tersely go through the most basic of these notions -mainly to fix notations- while occasionally giving a result that will be used later in the text. Following this introduction, we take a little bit more time to review some aspects of the theory of ∞-topoi.

2.1 A ménagerie of ∞-categories of ∞-categories

We record the following ∞-categories of varieties of fibrations of ∞-categories:

- The ∞-category Cat_∞ ($\widehat{\mathsf{Cat}}_\infty$) of small (large) ∞-categories, obtained as the nerve $\mathbf{N}(\mathsf{Cat}_\infty^\Delta)$ of the fibrant simplicial subcategory of the simplicial category Set_Δ whose objects are quasi-categories and whose morphisms between two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is the largest Kan complex contained in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, that we denote $\mathrm{Fun}^\simeq(\mathcal{C},\mathcal{D})$. If $f:\mathcal{C}\to\mathcal{D}$ is a categorical fibration, then for any functor $\mathcal{C}'\to\mathcal{C}$, the induced map $\mathrm{Fun}^\simeq(\mathcal{C}',\mathcal{C})\to\mathrm{Fun}^\simeq(\mathcal{C}',\mathcal{D})$ is a Kan fibration, which implies that the full subcategory of the simplicial slice category $(\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}$ spanned by categorical fibrations over $\mathcal{C}$ is also a fibrant simplicial category.

- For any ∞-category $\mathcal{C}$, the subcategories $\mathsf{coCart}_\mathcal{C}$ and $\mathsf{Cart}_\mathcal{C}$ of $(\mathsf{Cat}_\infty)_{/\mathcal{C}}$ of *coCartesian* respectively *Cartesian* fibrations over $\mathcal{C}$, defined as follows. Consider the subcategory $\mathsf{coCart}_\mathcal{C}^\Delta$ of the simplicial category $(\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}$ spanned by inner fibrations $p:\mathcal{D}\to\mathcal{C}$ such that for each edge $e:\Delta^1\to\mathcal{C}$, there is an edge $\tilde{e}:\Delta^1\to\mathcal{D}$ such that $p(\tilde{0}) = e(0)$ and each diagram

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d] \arrow[dr, "\tilde{e}"] & \\ \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{C} \end{tikzcd}$$

admits a filler as indicated. Any edge satisfying this lifting property is a *p-coCartesian edge* and the edge $\tilde{e}$ is called a *p-coCartesian lift of e starting at* $\tilde{e}(0)$. The space of morphisms between coCartesian fibrations $\mathcal{D}\to\mathcal{C}$ and $\mathcal{D}'\to\mathcal{C}$ in $\mathsf{coCart}_\mathcal{C}^\Delta$ is the space of those connected components of the Kan complex $\mathrm{Hom}_{(\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}}(\mathcal{D},\mathcal{D}')$ that consists of functors over $\mathcal{C}$ preserving coCartesian edges. Then the ∞-category $\mathsf{coCart}_\mathcal{C}$ is the coherent nerve of the fibrant simplicial category $\mathsf{coCart}_\mathcal{C}^\Delta$. We have a (non-full) subcategory inclusion

$$\mathsf{coCart}_{/\mathcal{C}}^\Delta \longrightarrow (\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}^f,$$

of fibrant simplicial categories, where $(\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}^f$ is the full subcategory of $(\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}$ spanned by categorical fibrations $\mathcal{D}\to\mathcal{C}$. We have a functor of ∞-categories

$$\mathsf{coCart}_\mathcal{C} = \mathbf{N}(\mathsf{coCart}_\mathcal{C}^\Delta) \longrightarrow \mathbf{N}((\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}}^f) \longrightarrow \mathbf{N}(\mathsf{Cat}_\infty^\Delta)_{/\mathcal{C}} = (\mathsf{Cat}_\infty)_{/\mathcal{C}}.$$

Here, the second map can be shown to be an equivalence of ∞-categories (see for instance [Lur17b] lem. 6.1.3.13), realizing $\mathsf{coCart}_\mathcal{C}$ as a subcategory of $(\mathsf{Cat}_\infty)_{/\mathcal{C}}$. To define $\mathsf{Cart}_\mathcal{C}$, we repeat this definition, taking

opposites everywhere.
These ∞-categories are obtained as the nerves of the simplicial categories $(\mathsf{Set}_\Delta^{+,\mathrm{cocart}})_{/\mathcal{C}}^{fc}$, the fibrant-cofibrant objects in the simplicial model categories of marked simplicial sets with the coCartesian, respectively Cartesian model structure. In particular, we have $\mathbf{N}((\mathsf{Set}_\Delta^+)^{fc}) = \mathsf{Cat}_\infty$. The marked straightening-unstraightening construction of [Lur17b], section 3.2. provides Quillen equivalences

$$(\mathsf{Set}_\Delta^+)_{/\mathcal{C}} \underset{\mathrm{Un}^+}{\overset{\mathrm{St}^+}{\rightleftarrows}} \mathrm{Fun}(\mathfrak{C}(\mathcal{C}), \mathsf{Set}_\Delta^+), \qquad (\mathsf{Set}_\Delta^+)_{/\mathcal{C}} \underset{\mathrm{Un}^{+,\mathrm{co}}}{\overset{\mathrm{St}^{+,\mathrm{co}}}{\rightleftarrows}} \mathrm{Fun}(\mathfrak{C}(\mathcal{C})^{op}, \mathsf{Set}_\Delta^+)$$

of combinatorial model categories, where the slice category on the left is endowed with the coCartesian model structure and the slice category on the right with the Cartesian model structure, and both functor categories with the projective model structure. The marked unstraightening functors can be equipped with the structure of a simplicial functor, which then provide an equivalence of ∞-categories $\mathsf{coCart}_\mathcal{C} \simeq \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)$ and $\mathsf{Cart}_\mathcal{C} \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathsf{Cat}_\infty)$.

- For any ∞-category $\mathcal{C}$, the ∞-categories $\mathsf{LFib}_\mathcal{C}$ and $\mathsf{RFib}_\mathcal{C}$ of *left* respectively *right fibrations* over $\mathcal{C}$. The ∞-category $\mathsf{LFib}_\mathcal{C}$ is defined as the full subcategory of $\mathsf{coCart}_\mathcal{C}$ spanned by those coCartesian fibrations $p : \mathcal{D} \to \mathcal{C}$ for which *every* edge in $\mathcal{D}$ is p-coCartesian. Of course, the ∞-category $\mathsf{RFib}_\mathcal{C}$ is defined by taking opposites in the previous definition. The straightening-unstraightening equivalence for (co)Cartesian fibrations restricts to equivalences $\mathsf{LFib}_\mathcal{C} \simeq \mathsf{PShv}(\mathcal{C}^{op})$ and $\mathsf{RFib}_\mathcal{C} \simeq \mathsf{PShv}(\mathcal{C})$.

- For any ∞-category $\mathcal{C}$, the ∞-categories $\mathsf{locoCart}_\mathcal{C}$ and $\mathsf{loCart}_\mathcal{C}$ of *locally coCartesian fibrations* respectively *locally Cartesian fibrations* over $\mathcal{C}$. Consider an inner fibration $p : \mathcal{D} \to \mathcal{C}$, then an edge $e : \Delta^1 \to \mathcal{D}$ is said to be *locally p-coCartesian* if e is a p-coCartesian edge of the induced inner fibration $\mathcal{D} \times_\mathcal{C} \Delta^1 \to \Delta^1$. The inner fibration is locally coCartesian if every edge in $\mathcal{C}$ has a locally p-coCartesian lift with specified domain in $\mathcal{D}$. The Kan complex of maps between two locally coCartesian fibrations is the union of connected components of functors that preserve locally coCartesian edges. This defines a fibrant simplicial category $\mathsf{locoCart}_\mathcal{C}^\Delta$ whose coherent nerve is $\mathsf{locoCart}_\mathcal{C}$. The ∞-category $\mathsf{locoCart}_\mathcal{C}$ is defined similarly. The ∞-category $\mathsf{coCart}_\mathcal{C}$ sits inside $\mathsf{locoCart}_\mathcal{C}$ as a full subcategory. The ∞-categories of locally coCartesian and locally Cartesian fibrations are also obtained as the nerve of the fibrant-cofibrant objects in a simplicial model category of marked simplicial sets over $\mathcal{C}$ determined by the *categorical pattern* that marks all 1-simplices of $\mathcal{C}$ see [Lur17a], appendix B or [Lur09], section 3.2 on how to produce the such model structures.

Remark 2.1.0.1. The subcategory of coCartesian fibrations can also be defined without reference to simplicial categories as follows: say that an edge $e : \Delta^1 \to \mathcal{D}$ between objects $x = e(0)$ and $y = e(1)$ is *p-coCartesian* for a functor $p : \mathcal{D} \to \mathcal{C}$ if the diagram

$$\begin{tikzcd} \mathcal{D}_{y/} \arrow[r] \arrow[d] & \mathcal{D}_{x/} \arrow[d] \\ \mathcal{C}_{py/} \arrow[r] & \mathcal{D}_{px/} \end{tikzcd}$$

is a pullback in the ∞-category Cat_∞; that is, a homotopy pullback for the Joyal model structure. Passing to homotopy fibres in the diagram above, we see that this is equivalent to asking that for every object $z \in \mathcal{D}$, the diagram of Hom spaces

$$\begin{tikzcd} \mathrm{Hom}_\mathcal{D}(y,z) \arrow[r] \arrow[d] & \mathrm{Hom}_\mathcal{D}(x,z) \arrow[d] \\ \mathrm{Hom}_\mathcal{C}(p(y),p(z)) \arrow[r] & \mathrm{Hom}_\mathcal{C}(p(x),p(z)) \end{tikzcd}$$

is a homotopy pullback, where the horizontal functors compose with the edge e and $p(e)$. For the result that every p-coCartesian morphism as defined just now is uniquely up to equivalence represented by the earlier strict notion of a p-coCartesian morphism, we refer to [Maz15].

Remark 2.1.0.2. The main theorem of Gepner-Haugseng-Nikolaus [GHN15] states that for a functor $F : \mathcal{C} \to \mathsf{Cat}_\infty$, the application of the coCartesian unstraightening functor coincides with taking the *left lax colimit* of the functor F; that is, the total space of $\mathrm{Un}^{+,\mathrm{co}}(F)$ is given by the colimit of the diagram

$$\mathrm{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}_{-/} \times F} \mathsf{Cat}_\infty,$$

where $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$ is a right fibration representing the Yoneda embedding: the *twisted arrow ∞-category* of Joyal; see [Lur17a], section 5.2.1 and the work of Barwick [Bar13], for example.

CoCartesian and Cartesian fibrations, as well as left and right fibrations, and local versions are stable under a number of natural operations, such as composition of fibrations and the formation of over and under ∞-categories. The fact that compositions of (co)Cartesian fibrations are (co)Cartesian admits the following partial converse.

Proposition 2.1.0.3. *Consider a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}$$

of ∞-categories where p and q are Cartesian fibrations and f is an inner fibration that sends p-Cartesian edges to q-Cartesian edges. Suppose the following hold.

(1) *For each object $E \in \mathcal{E}$, the induced map on the fibres $f_E : \mathcal{C}_E \to \mathcal{D}_E$ is a Cartesian fibration.*

(2) *For every morphism $e : E \to E'$, the functor $e^* : \mathcal{C}_E \to \mathcal{C}_{E'}$ takes f_E-Cartesian edges to $f_{E'}$-Cartesian edges.*

Then f is a Cartesian fibration.

Proof. See, for instance, lemma 1.4.14 of [Lur09], or proposition 9.8 of [GHN15]. □

Remark 2.1.0.4. It is a fact that (co)Cartesian fibrations are *flat fibrations* in the terminology of Lurie ([Lur17a], section B.3) or *exponentiable fibrations* in the terminology of Ayala-Francis [AF20]. Flat categorical fibrations $p : \mathcal{D} \to \mathcal{C}$ are characterized by the property that pulling back along p preserves categorical equivalences. Since the functor $p^* : (\mathsf{Set}_\Delta)_{/\mathcal{C}} \to (\mathsf{Set}_\Delta)_{/\mathcal{D}}$ has a right adjoint and p^* preserves cofibrations, flatness amounts to the assertion that p_* is left Quillen for the Joyal model structure.

We further record the following ∞-categories whose objects are ∞-categories characterized by having certain limits or colimits, or being generated under certain limits or colimits.

- We write $\mathsf{Cat}_\infty^{\mathrm{Lex}}$ for the subcategory of Cat_∞ whose objects are small ∞-categories admitting finite limits and whose morphisms are functors preserving finite limits (i.e. left exact functors). Dually, we have an ∞-category $\mathsf{Cat}_\infty^{\mathrm{Rex}}$.

- An ∞-category $\mathcal{C}$ is *stable* if $\mathcal{C}$ has finite limits and colimits, a zero object 0 (an object that is both final and initial), and a composition $X \to Y \to Z$ is a fibre sequence, that is, we have a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

  if and only if $X \to Y \to Z$ is also a cofibre sequence, that is, the diagram above is also a pushout. A functor between stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$ is *exact* if f preserves finite limits and colimits. We have an inclusion $\mathsf{Cat}_\infty^{\mathrm{Ex}} \subset \mathsf{Cat}_\infty$ of the subcategory of stable ∞-categories and exact functors between them, and this inclusion preserves (small) limits and filtered colimits.

- A small ∞-category $\mathcal{C}$ is *κ-filtered* (*filtered* if $\kappa = \omega$) for some regular cardinal κ if the map $\mathcal{C} \to \Delta^0$ has the right lifting property against all inclusions $K \hookrightarrow K^{\triangleright}$ where K is a κ-small simplicial set. An object C ∞-category is *κ-compact* if the functor corepresented by C preserves κ-filtered colimits. Given any small ∞-category $\mathcal{C}$, we may construct an ∞-category $\mathrm{Ind}_\kappa(\mathcal{C})$ of *κ-Ind objects of $\mathcal{C}$*: $\mathrm{Ind}_k(\mathcal{C})$ is the smallest full subcategory of $\mathsf{PShv}(\mathcal{C})$ containing the image of the Yoneda embedding and is stable under filtered colimits. This construction has the following universal property: for every ∞-category $\mathcal{D}$ that admits κ-filtered colimits, composition with the Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathrm{Ind}_k(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^{\kappa-\mathrm{cont}}(\mathrm{Ind}_k(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

  where $\mathrm{Fun}^{\kappa-\mathrm{cont}}(\mathrm{Ind}_k(\mathcal{C}), \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the full subcategory spanned by *κ-continuous functors*, those functors that preserve κ-filtered colimits. The inverse is given by a functor taking left Kan extensions (see [Lur17b], section 4.3.2 or the next subsection). An ∞-category $\mathcal{C}$ is *κ-accessible* if $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$, which may be taken to be the ∞-category of κ-compact objects of $\mathcal{C}$. For each regular cardinal κ, we have a subcategory $\mathsf{Acc}_\kappa \subset \widehat{\mathsf{Cat}}_\infty$ of κ-accessible ∞-categories whose morphisms are functors preserving κ-filtered colimits.

Remark 2.1.0.5. More generally, if $\mathcal{K}$ is a collection of small simplicial sets and $\mathcal{C}$ is a small ∞-category, we may consider the smallest full subcategory $\mathsf{PShv}(\mathcal{C})_{\mathcal{K}} \subset \mathsf{PShv}(\mathcal{C})$ stable under colimits indexed by simplicial sets in $\mathcal{K}$, then for any ∞-category $\mathcal{D}$ that admits colimits indexed by simplicial sets in $\mathcal{K}$, composition with the Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathsf{PShv}(\mathcal{C})_{\mathcal{K}}$ induces an equivalence

$$\mathrm{Fun}_{\mathcal{K}}(\mathsf{PShv}(\mathcal{C})_{\mathcal{K}}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}_{\mathcal{K}}(\mathsf{PShv}(\mathcal{C})_{\mathcal{K}}, \mathcal{D}) \subset \mathrm{Fun}(\mathsf{PShv}(\mathcal{C})_{\mathcal{K}}, \mathcal{D})$ denotes the full subcategory spanned by functors preserving colimits indexed by simplicial sets in $\mathcal{K}$. When $\mathcal{K} = \mathrm{Idem}$, the simplicial set constructed in [Lur17b], section 4.4.4, this procedure constructs the idempotent completion of $\mathcal{C}$, freely adding retracts of idempotents. When $\mathcal{C}$ has finite coproducts and $\mathcal{K}$ is the collection of sifted simplicial sets (that is, the diagonal $K \to K \times K$ is left cofinal), freely adding sifted colimits yields the *algebraic theories* that we will study in chapter 3.

2.1.1 Colimits and Kan extensions

Recall that a diagram $\mathcal{J} : K^{\triangleright} \to \mathcal{C}$ is a *colimit diagram* if $\mathcal{J}(-\infty)$ is an initial object of $\mathcal{C}_{\mathcal{J}|_K/}$. We will have need of the version of this notion relative to an inner fibration $X \to S$, which can be thought of as interpolating between the usual theory of colimits (when $S = \Delta^0$) and the theory of coCartesian fibrations (when $K = \Delta^0$).

Definition 2.1.1.1. Let $p : X \to S$ be an inner fibration of ∞-categories, then a diagram $\mathcal{J} : K^{\triangleright} \to X$ is a *p-colimit* if the diagram

$$\begin{tikzcd} X_{\mathcal{J}/} \arrow[r] \arrow[d] & S_{\mathcal{J}/} \arrow[d] \\ X_{\mathcal{J}|_K/} \arrow[r] & S_{\mathcal{J}|_K/} \end{tikzcd}$$

is a homotopy pullback, which is equivalent to demanding that the map $X_{\mathcal{J}/} \to X_{\mathcal{J}|_K/} \times_{S_{\mathcal{J}|_K/}} S_{\mathcal{J}/}$ is a trivial Kan fibration; that is, for each $n > 0$, each diagram

$$\begin{tikzcd} K \star \partial\Delta^n \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ K \star \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

such that $f|_K = \mathcal{J}$ and $f|_{\partial\Delta^n}(\{0\}) = \mathcal{J}(-\infty)$ admits a diagonal lift as indicated.

For the most important results on relative colimits, such as that for $p : X \to S$ a coCartesian fibration of ∞-categories, the theory of p-colimits can be reduced to the theory of ordinary colimits in the fibres of p, we refer to [Lur17b], section 4.3.1.
It is notoriously difficult to construct by hand functors between higher categories that keep track of all possible coherences. In order to exhibit a functor between two ∞-categories, one often has to resort to enlarging the source of the desired functor until one is guaranteed that a functor must exist. Then, the original source has to somehow be found as lying inside the enlargement in a natural way. Among the most crucial tools for carrying out this strategy is the theory of *(relative) Kan extensions.*

Definition 2.1.1.2. Let $p : \mathcal{D} \to \mathcal{D}'$ be an inner fibration of ∞-categories and $i : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ an inclusion of a full subcategory, then a diagram

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "f"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] \arrow[ur, "F"] & \mathcal{D}' \end{tikzcd}$$

exhibits F as a p-left Kan extension of f along i if for each $C \in \mathcal{C}$, the induced diagram

$$\begin{tikzcd} \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C} \arrow[r, "F_C"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ (\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C})^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D}' \end{tikzcd}$$

determines a p-colimit diagram.

Undoubtedly the most important and useful technical result in [Lur17b] is the following very general existence and uniqueness theorem concerning Kan extensions along inclusions.

Proposition 2.1.1.3. *Let*

$$\mathcal{C} \longrightarrow \mathcal{D}' \xleftarrow{p} \mathcal{D}$$

be a diagram of ∞-categories, where p is a categorical fibration. Let $\mathcal{C}^0 \subset \mathcal{C}$ be a full subcategory. Let $\mathcal{K} \subset \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ that are p-left Kan extensions of $F|_{\mathcal{C}^0}$, and let $\mathcal{K}' \subset \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D})$ be the full subcategory spanned by functors $F_0 : \mathcal{C}^0 \to \mathcal{D}$ such that for every $C \in \mathcal{C}$, the induced functor $\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{D}$ admits a p-colimit. Then the restriction map $\mathcal{K} \to \mathcal{K}'$ is a trivial Kan fibration.

Proof. This is [Lur17b], prop. 4.3.2.15. □

When p is a coCartesian fibration, we have the following useful variant of this result, where the functor $\mathcal{C} \to \mathcal{D}'$ may also vary.

Theorem 2.1.1.4. *Let $p : \mathcal{D} \to \mathcal{D}'$ be a coCartesian fibration of ∞-categories, and let $\mathcal{C}^0 \subset \mathcal{C}$ be the inclusion of a full subcategory. Let $\mathcal{E} \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ that are p-left Kan extensions of $F|_{\mathcal{C}^0}$, and let $\mathcal{E}' \subset \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0, \mathcal{D}')} \mathrm{Fun}(\mathcal{C}, \mathcal{D}')$ be the full subcategory spanned by commuting diagrams*

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "f"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] & \mathcal{D}' \end{tikzcd}$$

such that for all $C \in \mathcal{C}$, the induced diagram $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{D}$ admits a p-colimit. Then the restriction map $\mathcal{E} \to \mathcal{E}'$ is a trivial Kan fibration.

Before we prove this, we need the following lemmata.

Lemma 2.1.1.5. *Let $\mathcal{C}^0 \subset \mathcal{C}$ be an inclusion of a full subcategory and consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}^0 \times \Delta^1 \arrow[r, "f"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \times \Delta^1 \arrow[r] \arrow[ur, "F"] & \mathcal{D}' \end{tikzcd}$$

where p is a categorical fibration. Suppose that $F|_{\mathcal{C}\times\{0\}}$ is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{0\}}$ and that for each object $C \in \mathcal{C}^0$, the edge $f|_{\{C\}\times\Delta^1}$ is a p-coCartesian lift of $pf|_{\{C\}\times\Delta^1}$. Then $F|_{\mathcal{C}\times\{1\}}$ is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{1\}}$ if and only if for each object $C \in \mathcal{C}$, the edge $F|_{\{C\}\times\Delta^1}$ is a p-coCartesian lift of $pF|_{\{C\}\times\Delta^1}$.

Proof. The equivalence of the conditions in the lemma follows from the following series of equivalent conditions, that we explain below.

(a) $F|_{\mathcal{C}\times\{1\}}$ is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{1\}}$.

(b) F is a p-left Kan extension of f.

(c) F is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{0\}}$.

(d) F is a p-left Kan extension of $F|_{\mathcal{C}\times\{0\}}$.

(e) For all $C \in \mathcal{C}$, $F|_{\{C\}\times\Delta^1}$ is a p-coCartesian lift of $pF|_{\{C\}\times\Delta^1}$.

We observe that $(a) \Leftrightarrow (b)$ is a consequence of [Lur17b], prop. 4.3.2.9, since $F|_{\mathcal{C}\times\{0\}}$ is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{0\}}$. The equivalence of (b) and (c) follows from [Lur17b], prop. 4.3.2.8, since we assume that $f|_{\{C\}\times\Delta^1}$ is a p-coCartesian lift of $pf|_{\{C\}\times\Delta^1}$ for all $C \in \mathcal{C}^0$, which, by [Lur17b], prop. 4.3.2.9 again, amounts to the assumption that f is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{0\}}$. Now $(c) \Leftrightarrow (d)$ follows from [Lur17b], prop. 4.3.2.9 again and the assumption that $F|_{\mathcal{C}\times\{0\}}$ is a p-left Kan extension of $F|_{\mathcal{C}^0\times\{0\}}$. One more application of [Lur17b], prop. 4.3.2.9. shows that (d) and (e) are equivalent. □

Lemma 2.1.1.6. *Let $\mathcal{C}^0 \subset \mathcal{C}$ be an inclusion of a full subcategory and consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}^0 \times \Delta^1 \arrow[r, "f"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \times \Delta^1 \arrow[r] & \mathcal{D}' \end{tikzcd}$$

where p is a coCartesian fibration. Suppose that for each object $C \in \mathcal{C}^0$, the edge $f|_{\{C\}\times\Delta^1}$ is a p-coCartesian lift of $pf|_{\{C\}\times\Delta^1}$. If $f|_{\mathcal{C}^0\times\{0\}}$ admits a p-left Kan extension along the inclusion $\mathcal{C}^0 \subset \mathcal{C}$, then $f|_{\mathcal{C}^0\times\{1\}}$ also admits a p-left Kan extension along the inclusion $\mathcal{C}^0 \subset \mathcal{C}$.

Proof. We are given a p-left Kan extension $F_0 : \mathcal{C} \times \{0\} \to \mathcal{D}$ that fits into a diagram

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "f|_{\mathcal{C}\times\{0\}}"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] \arrow[ur, "F_0"] & \mathcal{D}' \end{tikzcd}$$

Thus, combining F_0 with the maps already given, we have a diagram

$$\begin{tikzcd} ((\mathcal{C}^0)^\flat \times (\Delta^1)^\sharp) \coprod_{(\mathcal{C}^0)^\flat \times \{0\}^\sharp} (\mathcal{C}^\flat \times \{0\}^\sharp) \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C}^\flat \times (\Delta^1)^\sharp \arrow[r] \arrow[ur, dotted] & \mathcal{D}' \end{tikzcd}$$

Since the left vertical map is marked anodyne and the right vertical map is a coCartesian fibration, we can find a dotted lift F. Note that F satisfies the conditions of lemma 2.1.1.5, so $F|_{\mathcal{C}\times\{1\}}$ is a p-left Kan extension of $f|_{\mathcal{C}^0\times\{1\}}$. ☐

Proof of theorem 2.1.1.4. Since $\mathcal{D} \to \mathcal{D}'$ is a coCartesian fibration and $\mathcal{C}^0 \to \mathcal{C}$ is an inclusion of simplicial sets, the map $r' : \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D}') \times_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D}')} \mathrm{Fun}(\mathcal{C}^0,\mathcal{D})$ is a coCartesian fibration. We check that the map $r : \mathcal{E} \to \mathcal{E}'$ is a coCartesian fibration as well. We only have to show that r'-coCartesian lifts of edges in $\mathcal{E}'$ are also r-coCartesian lifts; the relevant horn fillings for higher dimensional horns are then automatically satisfied because r' is a coCartesian fibration (in particular, we see that r is an inner fibration). We need to verify that, given a map $H : h_1 \to h_1$ in $\mathcal{E}'$ and a coCartesian lift $F : f_1 \to f_2$ in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ of H such that f_1 is a p-left Kan extension of $f_1|_{\mathcal{C}^0}$, f_2 is a p-left Kan extension of $f_2|_{\mathcal{C}^0}$. Unwinding the definitions, we see that this is guaranteed by lemma 2.1.1.5. Thus, to prove that r is a trivial Kan fibration, it suffices to check that it is a categorical equivalence. We have a commuting diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "r"] \arrow[dr, "q_1"'] & & \mathcal{E}' \arrow[dl, "q_2"] \\ & \mathrm{Fun}(\mathcal{C},\mathcal{D}') & \end{tikzcd}$$

of simplicial sets; we check that q_2 is a coCartesian fibration. Since the map $q_2' : \mathrm{Fun}(\mathcal{C},\mathcal{D}') \times_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D}')} \mathrm{Fun}(\mathcal{C}^0,\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D}')$ is a coCartesian fibration, it suffices to check that for q_2'-coCartesian lifts of morphisms in $\mathrm{Fun}(\mathcal{C},\mathcal{D}')$ for which the domain admits a p-left Kan extension, the codomain also admits a p-left Kan extension. This follows from lemma 2.1.1.6. Now q_1 and q_2 are both coCartesian fibrations, and r takes q_1-coCartesian edges to q_2-coCartesian edges. Invoking [Lur17b], prop. 2.4.4.4, we can conclude that r is a categorical equivalence if we show that for each functor $f : \mathcal{C} \to \mathcal{D}$, the induced map on fibres $r_f : \mathcal{E}_f \to \mathcal{E}'_f$ is a categorical equivalence, but [Lur17b], prop. 4.3.2.15 asserts that r_f is a trivial Kan fibration. ☐

2.1.2 Adjunctions and adjointability

We formulate adjoints in terms of unit/counit transformations. We say that a right fibration $p : \mathcal{D} \to \mathcal{C}$ is *representable* if $\mathcal{D}$ has a final object; this implies that $\mathcal{D} \to \mathcal{C}$ is equivalent to $\mathcal{C}_{/p(D)}$, where D is a final object of $\mathcal{D}$. It's easy to see that the representable right fibrations are exactly the objects in the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathsf{RFib}_{\mathcal{C}}$.

Definition 2.1.2.1. (1) Let $f : \mathcal{C} \to \mathcal{D}$ be a functor, then we say that an object $\epsilon_D \in \mathcal{C}_{/D} := \mathcal{D}_{/D} \times_{\mathcal{D}} \mathcal{C}$ depicted as a pair $(C, f(C) \to D)$ is a *counit transformation at* D if ϵ_D is final; that is, if the right fibration $\mathcal{C}_{/D} \to \mathcal{C}$ is representable. We say that f is a *left adjoint* if there is a counit transformation for every $D \in \mathcal{D}$.

(2) Let $f : \mathcal{C} \to \mathcal{D}$ be a functor, then we say that an object $\eta_D \in \mathcal{C}_{D/} := \mathcal{D}_{D/} \times_{\mathcal{D}} \mathcal{C}$ depicted as a pair $(C, D \to f(C))$ is a *unit transformation at* D if η_D is initial. We say that f is a *right adjoint* if there is a unit transformation for every $D \in \mathcal{D}$.

Remark 2.1.2.2. Unpacking the definition, we see that $(C, f(C) \to D)$ is a counit transformation at D if and only if the composition

$$\mathrm{Hom}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Hom}_{\mathcal{D}}((f(C'), f(C)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(f(C'), D)$$

is an equivalence of spaces.

Remark 2.1.2.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor, then f is a left adjoint if and only if any associated coCartesian fibration $p : \mathcal{M} \to \Delta^1$ is also Cartesian. This is so because for every $D \in \mathcal{D} \simeq p^{-1}(1)$, the ∞-category $\mathcal{M}_{/D} \times_{\mathcal{M}} \mathcal{C}$, whose final objects are p-Cartesian lifts of $0 \to 1$ starting at D, is equivalent to $\mathcal{C}_{/D}$. It follows that if f is a left adjoint, we obtain a functor $g : \mathcal{D} \to \mathcal{C}$ that is a right adjoint by reversing the previous argument, and g then in turn determines f essentially uniquely. Thus, we have an *adjoint pair* $(f \dashv g)$, and the counit and unit transformations, locally defined at all objects, become natural transformations $\epsilon : f \circ g \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to g \circ f$.

In view of the previous remark, we may identify adjunctions $(f \dashv g) : \mathcal{C} \rightleftarrows \mathcal{D}$ with correspondences $\mathcal{M} \to \Delta^1$ that are both Cartesian and coCartesian associated to both g and f. A functor $f : \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if the inclusion into the fibre $\mathcal{C} \simeq p^{-1}(\{0\}) \hookrightarrow \mathcal{M}$ of the associated correspondence $p : \mathcal{M} \to \Delta^1$ is a left adjoint. This follows from the equivalence $\mathcal{M}_{/D} \simeq \mathcal{C}_{/D}$ for all $D \in \mathcal{D}$. It follows easily that a functor $q : \mathcal{C} \to \mathcal{D}$ is a locally coCartesian fibration if and only if for each $D \in \mathcal{D}$, the canonical map $\mathcal{C}_D \to \mathcal{C}_{/D}$ is a left adjoint. We also have the following result.

Proposition 2.1.2.4. *Let $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration, and consider for each object $C \in \mathcal{C}$ the induced coCartesian fibration $p' : \mathcal{C}_{/C} \to \mathcal{D}_{/p(C)}$ ([Lur17b], prop. 2.4.3.2). Then the inclusion of the fibre $(\mathcal{C}_{p(C)})_{/C} \hookrightarrow \mathcal{C}_{/C}$ has a left adjoint, and a map $\eta : C \to C''$ with $p(C'') = p(C)$ is a unit transformation if and only if η is a p-coCartesian lift starting at C.*

Proof. We should show that for each morphism $f : C' \to C$ of $\mathcal{C}_{/C}$, the left fibration

$$(\mathcal{C}_{p(C)})_{/C} \times_{\mathcal{C}_{/C}} \mathcal{C}_{C'//C} \to (\mathcal{C}_{p(C)})_{/C}$$

is corepresentable, that is, the ∞-category $(\mathcal{C}_{p(C)})_{/C} \times_{\mathcal{C}_{/C}} \mathcal{C}_{C'//C}$ has an initial object. We have a diagram of simplicial sets

$$\begin{tikzcd}
\{f\} \times_{\mathcal{D}_{p(C')//p(C)}} \mathcal{C}_{C'//C} \arrow[r] \arrow[d] & (\mathcal{C}_{p(C)})_{/C} \times_{\mathcal{C}_{/C}} \mathcal{C}_{C'//C} \arrow[r] \arrow[d] & \mathcal{C}_{C'//C} \arrow[d] \\
\{f\} \arrow[r, "i"] & \mathrm{Hom}^{\mathrm{L}}_{\mathcal{D}_{/p(C)}}(p(C'), p(C)) \arrow[r] & \mathcal{D}_{p(C')//p(C)}
\end{tikzcd}$$

where both squares are pullbacks and the indicated map i is a homotopy equivalence of Kan complexes and thus a categorical equivalence. As all objects in the diagram are fibrant and all vertical maps are categorical fibrations, the left upper horizontal map is also a categorical equivalence. But an initial object in the ∞-category $\{f\} \times_{\mathcal{D}_{p(C')//p(C)}} \mathcal{C}_{C'//C}$ is exactly a p-coCartesian lift of the map $p(C') \to p(C)$ starting at C'. □

We will also make use of the theory of *relative adjunctions*.

Definition 2.1.2.5. Given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "p"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "q"] \\
& \mathcal{E} &
\end{tikzcd}$$

where p and q are categorical fibrations, we say that the functor G is a *parametrized right adjoint* or a *right adjoint relative to* $\mathcal{E}$ if G admits a left adjoint F such that the unit transformation $\eta : \mathrm{id}_{\mathcal{D}} \to G \circ F$ maps to the identity transformation on $\mathcal{E}$ under q.

Under mild assumptions, relative adjunctions are guaranteed to exist once adjunctions on the fibres exist.

Proposition 2.1.2.6. *Given a diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "p"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "q"] \\
& \mathcal{E} &
\end{tikzcd}$$

where p and q are locally coCartesian categorical fibrations, G is a right adjoint relative to $\mathcal{E}$ if and only if G takes locally q-coCartesian edges to locally p-coCartesian edges.

Proof. This is [Lur17a]prop. 7.3.2.6. □

Definition 2.1.2.7. Consider a diagram $\sigma : \Delta^1 \times \Delta^1 \to \widehat{\mathsf{Cat}}_\infty$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "L"] \arrow[d, "F"] & \mathcal{D} \arrow[d, "F'"] \\
\mathcal{C}' \arrow[r, "L'"] & \mathcal{D}'
\end{tikzcd}$$

commuting up to a specified homotopy α. Let $D \in \mathcal{D}$ an object, then we say that this diagram is *L-right adjointable locally at D* (or *horizontally right adjointable*) if L and L' admit right adjoints U and U' respectively, and the *Beck-Chevalley transformation*

$$F \circ U \longrightarrow U' \circ L' \circ F \circ U \overset{\alpha}{\simeq} U' \circ F' \circ L \circ U \longrightarrow U' \circ F'$$

is an equivalence at D. A square diagram σ as above is *U-right adjointable* if it is U-right adjointable locally at every $D \in \mathcal{D}$. Using elementary manipulations of units and counits, it is easy to see that the diagram is L-right adjointable locally at D if L and L' admit right adjoints and the map F' takes counit transformations at D to counit transformations at $F'(D)$.

Remark 2.1.2.8. By unstraightening, a square diagram $\sigma : \Delta^1 \times \Delta^1 \to \widehat{\mathsf{Cat}}_\infty$ as in definition 2.1.2.7 determines a diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\widehat{F}"] \arrow[dr, "p"'] & & \mathcal{M}' \arrow[dl, "q"] \\ & \Delta^1 & \end{tikzcd}$$

where both p and q are coCartesian fibrations associated to U and U' respectively and $\widehat{F}$ carries p-coCartesian edges to q-coCartesian edges. The diagram σ is right adjointable locally at D if and only if p and q are also Cartesian fibrations, and $\widehat{F}$ carries p-Cartesian edges starting at D to q-Cartesian edges. We see in particular that a right adjointable square determines a left adjointable square up to contractible ambiguity and vice versa. More generally, we may for any ∞-category $\mathcal{C}$ consider the subcategory $\mathrm{Fun}^{\mathrm{RAd}}(\mathcal{C}, \mathsf{Cat}_\infty) \subset \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)$ whose objects are those functors that send all edges of $\mathcal{C}$ to functors admitting right adjoints, and whose morphisms are those natural transformations that determine a right adjointable square for each edge in $\mathcal{C}$. This subcategory corresponds under unstraightening to the simplicial subcategory $\mathsf{coCart}^{\Delta}_{\mathcal{C}} \cap \mathsf{Cart}^{\Delta}_{\mathcal{C}} \to (\mathsf{Cat}^{\Delta}_{\infty})_{/\mathcal{C}}$. This simplicial subcategory actually arises as the fibrant-cofibrant objects in the simplicial model category $(\mathsf{Set}^{++}_{\Delta})_{/\mathcal{C}}$ of *bimarked simplicial sets* over $\mathcal{C}$, of [Lur17a], section 4.7.4 and its nerve is therefore presentable. Since the inclusions $(\mathsf{Set}^{++}_{\Delta})_{/\mathcal{C}} \subset (\mathsf{Set}^{+,\mathrm{cocart}}_{\Delta})_{/\mathcal{C}}$ and $(\mathsf{Set}^{++}_{\Delta})_{/\mathcal{C}} \subset (\mathsf{Set}^{+,\mathrm{cocart}}_{\Delta})_{/\mathcal{C}}$ are both right Quillen, the subcategory inclusion $\mathrm{Fun}^{\mathrm{RAd}}(\mathcal{C}, \mathsf{Cat}_\infty) \subset \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)$ preserves limits.

We record the following ∞-categories.

- The subcategory $\mathsf{Pr}^{\mathrm{L}} \subset \widehat{\mathsf{Cat}}_\infty$, whose objects are *presentable ∞-categories*: accessible ∞-categories that admit small colimits. We say that a presentable ∞-category $\mathcal{C}$ is *κ-compactly generated* if $\mathcal{C}$ is κ-accessible. If $\kappa = \omega$, we say that $\mathcal{C}$ is *compactly generated*. By Simpson's theorem ([Lur17b], thm. 5.5.1.1) presentable ∞-categories are equivalently ∞-categories of the form $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ for $\mathcal{C}_0$ an ∞-category admitting κ-small colimits, or ∞-categories that are accessible localizations of presheaf ∞-categories. Morphisms are those functors $f : \mathcal{C} \to \mathcal{D}$ that preserve small colimits, or equivalently by the adjoint functor theorem ([Lur17b], prop. 5.5.2.9), functors that admit a right adjoint. We denote the full subcategory spanned by functors that admit a right adjoint by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ for any two ∞-categories $\mathcal{C}$ and $\mathcal{D}$. The inclusion $\mathsf{Pr}^{\mathrm{L}} \subset \widehat{\mathsf{Cat}}_\infty$ preserves small limits.
  A coCartesian fibration $p : \mathcal{D} \to \mathcal{C}$ of ∞-categories is *presentable* if the straightening of p factors through Pr^{L}, which is equivalent to demanding that the fibres of p are presentable ∞-categories and that for each edge $f : \Delta^1 \to \mathcal{C}$, the functor $f_!$ has a right adjoint (so that the fibration is also Cartesian).

- By taking opposites, Pr^{L} is equivalent to $(\mathsf{Pr}^{\mathrm{R}})^{op}$, where $\mathsf{Pr}^{\mathrm{R}} \subset \widehat{\mathsf{Cat}}_\infty$ is the subcategory that has the same objects as Pr^{L}, but morphisms are those functors that are accessible and preserve small limits, or equivalently, functors that admit a left adjoint. We denote the full subcategory spanned by functors that admit a left adjoint by $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ for any two ∞-categories $\mathcal{C}$ and $\mathcal{D}$. The inclusion $\mathsf{Pr}^{\mathrm{R}} \subset \widehat{\mathsf{Cat}}_\infty$ also preserves small limits.

2.1.3 Localization of ∞-categories

Definition 2.1.3.1. Let $(\mathcal{C}, W)$ be a pair of an ∞-category together with a collection edges of $\mathcal{C}$ that contains all degenerate ones. A functor $f : \mathcal{C} \to \mathcal{D}$ *exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W* if the following conditions are satisfied.

(1) f carries the edges of W into equivalences.

(2) For every ∞-category $\mathcal{E}$, composition with f induces an equivalence

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}_W(\mathcal{C}, \mathcal{E})$$

where $\mathrm{Fun}_W(\mathcal{C}, \mathcal{E})$ is the full subcategory spanned by functors sending W into equivalences.

Localizations are easily constructed: consider $(\mathcal{C}, W)$ as a marked simplicial set, then a fibrant replacement yields a localization of $(\mathcal{C}, W)$. We denote this localization by $\mathcal{C}[W^{-1}]$.

Remark 2.1.3.2. Localizations are to the (co)Cartesian marked model category of simplicial sets as cofinal maps are to the covariant/contravariant model categories. Since the covariant/contravariant model structures are obtained via (Bousfield) localization of the (co)Cartesian model structures, localizations are left and right cofinal.

Example 2.1.3.3. In [Lur17b], proposition 4.2.3.14 it is shown that for any simplicial set K, the map $\mathbf{N}(\mathbf{\Delta}_{/K})$ of simplices of K admits a left cofinal map $\mathbf{N}(\mathbf{\Delta}_{/K}) \to K$, the *last vertex map*. In fact, more is true. Let $\mathbf{\Delta}' \subset \mathbf{\Delta}$ be the subcategory containing all objects whose morphisms are maps $f : [m] \to [n]$ such that $f(m) = n$. Let $W = \mathbf{N}(\mathbf{\Delta}'/K) \subset \mathbf{N}(\mathbf{\Delta}_{/K})$, then the last vertex map sends every edge of W to a degenerate edge of K. If we denote by $K \to \mathbf{R}K$ a fibrant replacement of K in the Joyal model structure, the functor

$$\mathbf{N}(\mathbf{\Delta}_{/K}) \longrightarrow K \longrightarrow \mathbf{R}K$$

exhibits $\mathbf{R}K$ as a localization of $\mathbf{N}(\mathbf{\Delta}_{/K})$ with respect to W ([Cis18], proposition 7.3.15). In particular, every ∞-category is a localization of the nerve of a category.

Example 2.1.3.4. Let $\mathbf{A}$ be a model category and let $L^H(\mathbf{A})$ be the *hammock localization* of $\mathbf{A}$ [DD80], then we have a commuting square of Dwyer-Kan equivalences of simplicial categories

$$\begin{array}{ccc} \mathbf{N}(L^H(\mathbf{A}^{fc})) & \longrightarrow & \mathbf{N}(L^H(\mathbf{A}^{f})) \\ \downarrow & & \downarrow \\ \mathbf{N}(L^H(\mathbf{A})^{c}) & \longrightarrow & \mathbf{N}(L^H(\mathbf{A})) \end{array}$$

and upon taking a fibrant replacement of the coherent nerve, these simplicial categories model the localization $\mathbf{N}(\mathbf{A})[W^{-1}]$

Example 2.1.3.5. Let $\mathcal{A}$ be an abelian category with enough projectives, then the ∞-category $\mathcal{D}(\mathbf{A})^-$, the dg-nerve of the dg-category $\mathrm{Ch}^-(\mathcal{A}_{\mathrm{proj}})$ of left bounded chain complexes of projectives, then $\mathcal{D}(\mathbf{A})^-$ is a localization of the pair $(\mathbf{N}(\mathrm{Ch}^-(\mathcal{A})), W)$, where W is the collection of quasi-isomorphisms.

Definition 2.1.3.6. (1) A localization functor $f : \mathcal{C} \to \mathcal{D}$ is *reflective* if f has a right adjoint (in which case g must be fully faithful, see [Cis18], proposition 7.1.17). We say that a morphism $C \to C'$ in $\mathcal{C}$ *exhibits C' as an f-localization of C* if C' lies in the essential image of g and the induced morphism $f(C) \to f(C')$ is an equivalence; in other words, if $C \to C'$ is a unit transformation for the adjunction $(f \dashv g)$.

(2) Suppose that $\mathcal{C}$ is presentable, and let $f : \mathcal{C} \to \mathcal{D}$ be a reflective localization with right adjoint g, then this localization is *accessible* if $g \circ f$ is an accessible functor. In this case, $\mathcal{D}$ is also presentable, and we say that $\mathcal{D}$ is a *strongly reflective* subcategory of $\mathcal{C}$.

There is a one-to-one correspondence between equivalence classes of accessible localizations on a presentable ∞-category $\mathcal{C}$ and equivalence classes of strongly saturated collections of morphisms in $\mathcal{C}$ that are of small generation.

Example 2.1.3.7. For $n \geq -2$, any ∞-category $\mathcal{C}$ has a full subcategory of n-truncated objects. If $\mathcal{C}$ is presentable, then these full subcategories are strongly reflective. For $n \geq -2$, the localization functor is denoted $\tau_{\leq n}$, and $\tau_{\leq n}$-localizations are called *n-truncations*.

Remark 2.1.3.8. A tower diagram $C : \mathbf{N}(\mathbb{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{C}$ is a *Postnikov tower* if for each C_n, the map $C_\infty \to C_n$ exhibits an n-truncation, and a pretower diagram $C : \mathbf{N}(\mathbb{Z}_{\geq 0}^{op}) \to \mathcal{C}$ is a *Postnikov pretower* if for each C_n, the map $C_n \to C_{n-1}$ exhibits an $(n-1)$-truncation. We say that an ∞-category $\mathcal{C}$ *has truncations* if $\mathcal{C}$ has the property that for each $n \in \mathbb{Z}_{\geq 0}$, the n-truncated objects form a reflective subcategory of $\mathcal{C}$ (which is the case, for instance, if $\mathcal{C}$ is presentable). If this condition is satisfied, we may form the Postnikov completion of $\mathcal{C}$, denoted $\widehat{\mathcal{C}}^{\mathrm{Post}}$ by taking the limit of the tower

$$\longrightarrow \tau_{\leq n}\mathcal{C} \longrightarrow \tau_{\leq (n-2)}\mathcal{C} \longrightarrow \ldots \longrightarrow \tau_{\leq 0}\mathcal{C}$$

which is obtained as the ∞-category of coCartesian sections of the unstraightening of the functor $\mathbf{N}(\mathbb{Z}_{\geq 0}^{op}) \to \mathsf{Cat}_\infty$, $n \mapsto \tau_{\leq n}\mathcal{C}$. There is an obvious functor $\mathcal{C} \to \widehat{\mathcal{C}}^{\mathrm{Post}}$ that associates to each object $C \in \mathcal{C}$ an essentially unique Postnikov tower. An ∞-category that has all truncations is *Postnikov complete* (alternatively, *Postnikov towers are convergent in $\mathcal{C}$*) if the functor $\mathcal{C} \to \widehat{\mathcal{C}}^{\mathrm{Post}}$ is an equivalence. If this is the case, then every object in $\mathcal{C}$ is in particular a limit of its Postnikov tower, but Postnikov completeness is stronger in general: $\mathcal{C}$ is Postnikov complete if and only if a tower is a Postnikov tower precisely if it is a limit diagram and (its restriction to $\mathbf{N}(\mathbb{Z}_{\geq 0}^{op})$) a Postnikov pretower.
Let $\mathcal{C}$ be an ∞-category, then we denote by $\mathcal{C}^b$ the full subcategory spanned by objects that are k-truncated for some nonnegative integer $k < \infty$. We say that an ∞-category is *bounded* if $\mathcal{C}^b = \mathcal{C}$. If $\mathcal{C}$ has all truncations and is Postnikov

complete, we have an obvious equivalence $\mathcal{C} \simeq \widehat{\mathcal{C}^b}^{\text{Post}}$, and if $\mathcal{C}$ has all truncations and is bounded, we have an obvious equivalence $\mathcal{C}^b \simeq \left(\widehat{\mathcal{C}}^{\text{Post}}\right)^b$. It follows that the constructions $\mathcal{C} \mapsto \widehat{\mathcal{C}}^{\text{Post}}$ and $\mathcal{C} \mapsto \mathcal{C}^b$ furnish an equivalence between the theory of bounded ∞-categories that have all truncations and Postnikov complete ∞-categories that have all truncations.

We say a few words about derived functors.

Definition 2.1.3.9. Let $(\mathcal{C}, W)$ be an ∞-category with weak equivalences, and let $\mathcal{D}$ be any ∞-category. Choose a functor $f : \mathcal{C} \to \mathcal{C}[W^{-1}]$ which exhibits $\mathcal{C}[W^{-1}]$ as a localization with respect to W, and consider the pullback

$$\_ \circ f : \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \longrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D}).$$

Let $g \in \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ and suppose that we are given a natural transformation

$$\alpha : g \longrightarrow f \circ \mathbf{R}g,$$

then we say that α *exhibits* $\mathbf{R}g$ *as a right derived functor of* g if α is a unit transformation at g. Dually, a natural transformation $\beta : f \circ \mathbf{L}g \to g$ *exhibits* $\mathbf{L}g$ *as a left derived functor of* g if β is a counit transformation at g.

The following is theorem 7.5.30 of [Cis18].

Proposition 2.1.3.10. *Suppose that*

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$$

are adjoint functors, where f preserves weak equivalences between cofibrant objects, and g preserves weak equivalences between fibrant objects, then there is a canonical adjunction

$$\mathcal{C}[W_{\mathcal{C}}^{-1}] \underset{\mathbf{R}g}{\overset{\mathbf{L}f}{\rightleftarrows}} \mathcal{D}[W_{\mathcal{D}}^{-1}]$$

Corollary 2.1.3.11. *Let $\mathcal{C}$ be a fibrant simplicial category and let $\mathbf{A}$ be a category with weak equivalences. Then the derived functor of the colimit functor is equivalent to the colimit functor.*

Definition 2.1.3.12. Let $(\mathcal{C}, W)$ be an ∞-category with weak equivalences. A diagram $f : K^{\triangleright} \to \mathcal{C}$ is a *homotopy colimit diagram (with respect to W)* if the diagram

$$K^{\triangleright} \xrightarrow{f} \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$$

is a colimit diagram. There is an evident dual notion of a *homotopy limit diagram.*

Remark 2.1.3.13. Using these ideas it is not hard to prove that for any model category $\mathbf{A}$ (not necessarily combinatorial), the localization $\mathbf{N}(A)[W^{-1}]$ admits all limits and colimits. By self duality of the notion of a model category, it suffices to prove the case of colimits. Since every simplicial set S admits a left (and right) cofinal map $\mathbf{N}(\mathbf{\Delta}_{/S}) \to S$, it suffices to consider colimits indexed by nerves of categories of the form $\mathbf{\Delta}_{/S}$. These are Reedy categories, so a derived colimit functor exists for diagrams indexed by such categories, which are ∞-categorical colimits by [Cis18], remark 7.9.10.

2.1.4 Stability and Homological Algebra

Recall the ∞-category $\mathsf{Cat}_{\infty}^{\text{Ex}}$ whose objects are stable ∞-categories and whose morphisms are exact functors between them.

Notation 2.1.4.1. For a stable ∞-category $\mathcal{C}$, we will denote the suspension functor interchangeably by Σ and $[1]$, and the loop functor by Ω and $[-1]$. Sometimes, if multiple ∞-categories are in play, we write $\Omega_{\mathcal{C}}$ and $\Sigma_{\mathcal{C}}$ to emphasize the relevant ∞-category.

The homotopy category $h\mathcal{C}$ of a stable ∞-category $\mathcal{C}$ is triangulated, and t-structure on $\mathcal{C}$ is simply a t-structure on $h\mathcal{C}$. For future reference, we record that a t-structure on $\mathcal{C}$ consists of the following data.

($*$) A pair of full subcategories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of *coconnective* respectively *connective* objects, such that $\mathcal{C}^{\geq 0}$ is stable under the suspension functor $[1]$ and $\mathcal{C}^{\leq 0}$ is stable under the loop functor $[-1]$. For $n \in \mathbb{Z}$, we write $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq n}[n]$ and $\mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[n]$.

We require that $\mathrm{Hom}_{\mathcal{C}}(X, Y[-1]) = \varnothing$ if $X \in \mathcal{C}^{\geq 0}$ and $Y \in \mathcal{C}^{\leq 0}$ and that for each $X \in \mathcal{C}$, we have a fibre sequence $X' \to X \to X''$ with $X \in \mathcal{C}^{\geq 0}$ and $X'' \in \mathcal{C}^{\leq -1}$. The full subcategories $\mathcal{C}^{\leq n} \subset \mathcal{C}$ and $\mathcal{C}^{\geq n} \subset \mathcal{C}$ are a localization respectively a colocalization for each $n \in Z$, and we have a left adjoint $\tau_{\leq n}$ respectively a right adjoint $\tau_{\geq n}$ to these inclusions. Moreover, the Beck-Chevalley transformation

$$\tau_{\leq m} \circ \tau_{\geq n} \longrightarrow \tau_{\geq n} \circ \tau_{\leq m}$$

is an equivalence of functors $\mathcal{C} \to \mathcal{C}^{\leq m} \cap \mathcal{C}^{\geq n}$. The functor $\tau_{\leq 0} \circ \tau_{\geq 0}$ lands in $\mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq 0}$. This ∞-category is denoted $\mathcal{C}^{\heartsuit}$, the *heart* of the t-structure, and is the nerve of an abelian category. For every n, we denote by π_n the functor

$$\mathcal{C} \xrightarrow{[-n]} \mathcal{C} \xrightarrow{\tau_{\leq 0} \circ \tau_{\geq 0}} \mathcal{C}^{\heartsuit}$$

Definition 2.1.4.2. An exact functor $f : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories equipped with t-structures is *left* t-*exact* if f carries $\mathcal{C}^{\leq 0}$ into $\mathcal{D}^{\leq 0}$ and *right* t-*exact* if f carries $\mathcal{C}^{\geq 0}$ into $\mathcal{D}^{\geq 0}$. An exact functor f is t-*exact* if f is both left and right t-exact.

Remark 2.1.4.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a left t-exact functor between stable ∞-categories, then f is t-exact if and only if for all $C \in \mathcal{C}$, the map $f(C) \to f(\tau_{\leq 0} C)$ exhibits $f(\tau_{\leq 0} C)$ as a $\tau_{\leq 0}$-localization of $f(C)$. Combining this with the dual statement for right t-exact functors, we see that a t-exact functor commutes with the localization and colocalization functors $\tau_{\leq n}$ and $\tau_{\geq n}$ for all $n \in \mathbb{Z}$.

Definition 2.1.4.4. A t-structure on a stable ∞-category $\mathcal{C}$ is

(1) *left bounded* if $\mathcal{C} = \mathcal{C}^{+} := \bigcup_{n \in \mathbb{Z}} \mathcal{C}_{\leq n}$.

(2) *left complete* if $\mathcal{C}$ coincides with the limit $\widehat{\mathcal{C}}$ of the diagram

$$\dots \longrightarrow \tau_{\leq 3} \mathcal{C} \xrightarrow{\tau_{\leq 2}} \tau_{\leq 2} \mathcal{C} \xrightarrow{\tau_{\leq 1}} \tau_{\leq 1} \mathcal{C} \longrightarrow \dots$$

via the natural map $\mathcal{C} \to \widehat{\mathcal{C}}$.

There are evident notions of *right bounded* (and *bounded*) and *right complete* stable ∞-categories.

Remark 2.1.4.5. We will also say that an object $C \in \mathcal{C}$ in a stable ∞-category equipped with a t-structure is *left bounded* if $C \in \mathcal{C}^{+}$. Suppose that $\mathcal{C}$ admits countable products and that the inclusion $\mathcal{C}^{\leq 0} \subset \mathcal{C}$ preserves countable products, then the canonical functor $\mathcal{C} \to \widehat{\mathcal{C}}$ admits a fully faithful right adjoint whose essential image consists of those $X \in \mathcal{C}$ such that $X \to \lim_{n \in \mathbb{Z}} \tau_{\leq n} X$ is an equivalence. We say that an object of $\mathcal{C}$ (under the assumption involving countable products) is *left complete* if X lies in the image of the right adjoint $\widehat{\mathcal{C}} \hookrightarrow \mathcal{C}$.

Let us recall some of the foundational theory of stabilization of ∞-categories from Higher Algebra, section 1.4.

Definition 2.1.4.6. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories.

(1) Suppose $\mathcal{C}$ has a final object $*$. f is *reduced* if $f(*)$ is a final object of $\mathcal{D}$.

(2) Suppose $\mathcal{C}$ admits pushouts. f is *excisive* if f sends pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.

Whenever the notions of reduced and/or excisive functors make sense, the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by reduced functors is denoted $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by excisive functors is denoted $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$, and their intersection, the full subcategory of reduced and excisive functors is denoted $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$.

Definition 2.1.4.7. Let $\mathcal{C}$ be an ∞-category with finite limits. The *stabilization* of $\mathcal{C}$ (also called the ∞-category of *spectrum objects* of $\mathcal{C}$), denoted by $\mathcal{S}\mathrm{p}(\mathcal{C})$, is the stable ∞-category $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ of reduced excisive functors from the ∞-category of pointed finite spaces to $\mathcal{C}$.

Notation 2.1.4.8. For $n \geq 0$, the functor $\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ given by evaluating on the n-sphere is denoted $\Omega^{\infty - n}$. For $n < 0$, we define the functor $\Omega^{\infty - n}$ by the composition $\Omega^{\infty} \circ \Omega^{-n}$, where Ω^{-n} denotes the $(-n)$-fold composition of the loop functor $\Omega : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{S}\mathrm{p}(\mathcal{C})$

Remark 2.1.4.9. Here are a few properties of the stabilization.

(1) $\mathcal{S}\mathrm{p}(\mathcal{C})$ is an accessible localization of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$.

(2) An ∞-category $\mathcal{D}$ is stable if and only if the functor $\Omega^{\infty} : \mathcal{S}\mathrm{p}(\mathcal{D}) \to \mathcal{D}$ is an equivalence. If $\mathcal{D}$ is a stable ∞-category and $\mathcal{C}$ is an ∞-category that admits finite limits, composition with $\Omega^{\infty} : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{S}\mathrm{p}(\mathcal{C})) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}).$$

(3) For $\mathcal{C}$ an ∞-category that admits finite limits, $\mathcal{S}\mathrm{p}(\mathcal{C})$ can be identified with the limit of the tower

$$\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*,$$

where $\mathcal{C}_*$ denotes the ∞-category of pointed objects in $\mathcal{C}$.

(4) If $\mathcal{C}$ is a presentable ∞-category, then $\mathcal{S}\mathrm{p}(\mathcal{C})$ is presentable as $\mathcal{S}\mathrm{p}(\mathcal{C})$ is a limit in the ∞-category Pr^{R}. Let $\mathcal{S}\mathrm{p}(\mathcal{C})_{\le -1}$ denote the full subcategory of $\mathcal{S}\mathrm{p}(\mathcal{C})$ spanned by those objects X such that $\Omega^\infty X$ is a final object in $\mathcal{C}$. Then $\mathcal{S}\mathrm{p}(\mathcal{C})_{\le -1}$ determines an accessible t-structure on $\mathcal{S}\mathrm{p}(\mathcal{C})$, that is, a t-structure such that $\mathcal{S}\mathrm{p}(\mathcal{C})^{\ge 0}$ is presentable.

Taking $\mathcal{C} = \mathcal{S}$, we obtain the presentable stable ∞-category of *spectra* $\mathcal{S}\mathrm{p}$, equipped with its canonical t-structure. The heart of this t-structure is the nerve of the category Ab of abelian groups.

If $f : \mathcal{C} \to \mathcal{D}$ is a left exact functor between ∞-categories admitting finite limits, composition with f induces a functor $\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{S}\mathrm{p}(\mathcal{D})$. We will later on need to understand this construction as a functor on some suitable ∞-category of ∞-categories into $\mathsf{Cat}_\infty^{\mathrm{Ex}}$, but we will introduce this theory in due time.

2.1.5 Higher algebra

We will make light use of ∞-operadic methods in this work. This subsection is devoted to the recollection of the relevant notions. We record the following ∞-categories:

- Let $\mathbf{N}(\mathsf{Fin}_*)$ be the nerve of the category of pointed finite sets, written $\langle n\rangle = \{*, 1, 2, \ldots, n\}$, and basepoint preserving maps between them. This category admits a factorization system (S_L, S_R) given by *active* and *inert* maps: S_L consists of maps $f : \langle n\rangle \to \langle m\rangle$ such that $f^{-1}(*) = *$, and S_R consists of maps $g : \langle n\rangle \to \langle m\rangle$ such that $g^{-1}(i)$ consists of exactly one element, for $i \in \langle n\rangle^\circ := \langle n\rangle \smallsetminus \{*\}$. We let $\mathsf{Op}_\infty^\Delta$ be the simplicial subcategory of $(\mathsf{Cat}_\infty^\Delta)_{/\mathbf{N}(\mathsf{Fin}_*)}$ whose objects are ∞-*operads*: functors $p : \mathcal{O}^\otimes \to \mathbf{N}(\mathsf{Fin}_*)$ such that the following conditions are satisfied.

  (*i*) For every $\langle n\rangle$ and every object C in the fibre $\mathcal{O}^\otimes_{\langle n\rangle}$ over $\langle n\rangle$, every inert morphism $f : \langle n\rangle \to \langle m\rangle$ admits a p-coCartesian lift starting at C.

  (*ii*) For every $\langle n\rangle \in \mathbf{N}(\mathsf{Fin}_*)$, there are exactly n inert maps $\{\rho^i : \langle n\rangle \to \langle 1\rangle\}_{i \in \langle n\rangle^\circ}$ given by

  $$\rho^i(k) = \begin{cases} 1 & \text{if } k = i \\ * & \text{if } k \neq i \end{cases}$$

  Then the coCartesian transformations $\rho^i_! : \mathcal{O}^\otimes_{\langle n\rangle} \to \mathcal{O}^\otimes_{\langle 1\rangle}$ determine an equivalence $\mathcal{O}^\otimes_{\langle n\rangle} \simeq \prod_{i\in\langle n\rangle} \mathcal{O}^\otimes_{\langle 1\rangle}$.

  (*iii*) For every map $C \to C'$ in $\mathcal{O}^\otimes$ lying over some map $f : \langle n\rangle \to \langle m\rangle$, the maps $C' \to \rho_i^!(C)$ determined by the coCartesian lifts of the inert maps $\rho^i : \langle m\rangle \to \langle 1\rangle$ for $1 \le i \le m$ induce a homotopy equivalence

  $$\mathrm{Hom}^f_{\mathcal{O}^\otimes}(C, C') \longrightarrow \prod_{i\in\langle m\rangle^\circ} \mathrm{Hom}^{\rho_i\circ f}_{\mathcal{O}^\otimes}(C, \rho^i_!(C')),$$

  where $\mathrm{Hom}^f_{\mathcal{O}^\otimes}(C, C')$ denotes the union of connected components of $\mathrm{Hom}_{\mathcal{O}^\otimes}(C, C')$ of morphisms that lie over $\mathrm{Hom}_{\mathbf{N}(\mathsf{Fin}_*)}(\langle n\rangle, \langle m\rangle)$, and similarly for $\mathrm{Hom}^{\rho_i\circ f}_{\mathcal{O}^\otimes}(C, \rho^i_!(C'))$.
  Equivalently, the maps $C \to \rho^i_!(C')$ determine a p-product diagram

  $$\begin{tikzcd} \langle n\rangle^\circ \arrow[r] \arrow[d] & \mathcal{O}^\otimes \arrow[d, "p"] \\ (\langle n\rangle^\circ)^{\lhd} \arrow[r] \arrow[ur, dotted] & \mathbf{N}(\mathsf{Fin}_*) \end{tikzcd}$$

  It is easy to see that all ∞-operads $\mathcal{O}^\otimes \to \mathbf{N}(\mathsf{Fin}_*)$ are categorical fibrations, so that $\mathrm{Hom}_{(\mathsf{Cat}_\infty^\Delta)_{/\mathbf{N}(\mathsf{Fin}_*)}}(\mathcal{O}^\otimes, \mathcal{O}^{\otimes'})$ is a Kan complex. The space of morphisms between two ∞-operads $\mathcal{O}^\otimes$ and $\mathcal{O}^{\otimes'}$ is the union of those connected components of $\mathrm{Hom}_{(\mathsf{Cat}_\infty^\Delta)_{/\mathbf{N}(\mathsf{Fin}_*)}}(\mathcal{O}^\otimes, \mathcal{O}^{\otimes'})$ of functors over $\mathbf{N}(\mathsf{Fin}_*)$ that preserve coCartesian lifts of inert edges. The ∞-*category of* ∞-*operads* Op_∞ is the nerve of the fibrant simplicial category $\mathsf{Op}_\infty^\Delta$. Lurie's device of categorical patterns furnishes a simplicial model structure on the category $(\mathsf{Set}_\Delta^+)_{\mathbf{N}(\mathsf{Fin}_*)}$ of marked simplicial sets over $\mathbf{N}(\mathsf{Fin}_*)$ (where inert edges of $\mathbf{N}(\mathsf{Fin}_*)$ are marked), such that $\mathsf{Op}_\infty^\Delta$ arises as the fibrant objects of this simplicial model category.

- A *symmetric monoidal ∞-category* is a coCartesian fibration $\mathcal{C}^{\otimes} \to \mathbf{N}(\mathsf{Fin}_*)$ that is also an ∞-operad. The *∞-category of symmetric monoidal ∞-categories*, denoted $\mathbb{E}_\infty\mathsf{Alg}(\mathsf{Cat}_\infty)$ (for reasons we explain below) is the nerve of the fibrant simplicial subcategory of $(\mathsf{Cat}_\infty^\Delta)_{\mathbf{N}(\mathsf{Fin}_*)}$ whose objects are symmetric monoidal ∞-categories and whose space of morphisms is the union of those connected components of $\mathrm{Hom}_{(\mathsf{Cat}_\infty^\Delta)_{/\mathbf{N}(\mathsf{Fin}_*)}}(\mathcal{C}^{\otimes}, \mathcal{C}^{\otimes\prime})$ of functors over $\mathbf{N}(\mathsf{Fin}_*)$ that preserve coCartesian edges. We also have an ∞-category of symmetric monoidal ∞-category and *lax* monoidal functors between them, defined as the full subcategory $\mathbb{E}_\infty\mathsf{Alg}(\mathsf{Cat}_\infty)^{\mathrm{lax}} \subset \mathsf{Op}_\infty$ spanned by symmetric monoidal ∞-categories.

Remark 2.1.5.1. Let $p : \mathcal{O}^{\otimes} \to \mathbf{N}(\mathsf{Fin}_*)$ be an ∞-operad. Say that a morphism $f : C \to C'$ of $\mathcal{O}^{\otimes}$ is *inert* if $p(f)$ is inert and f is a coCartesian morphism. We say that f is active if $p(f)$ is active. Then the pair (S_L, S_R) where S_L consists of active morphisms and S_R consists of inert morphisms determines a factorization system on $\mathcal{O}^{\otimes}$. One should think of the inert maps as not participating in the essential structure of the ∞-operad $\mathcal{O}^{\otimes}$; indeed, such maps simply forget colours, whereas all the interesting structure encoded by $\mathcal{O}^{\otimes}$ can be extracted from the active morphisms.

Remark 2.1.5.2. The most basic examples of ∞-operads are obtained from simplicial coloured operads. Let $\mathbf{O}$ be a simplicial multicategory, that is, the data of a collection $\{X, Y, Z, \ldots\}$ of colours, together with a simplicial set $\mathrm{Mul}_{\mathbf{O}}(\{Y_i\}_{i\in I}, X)$ of multimorphisms for each finite set I. Additionally, for each map of finite sets $f : I \to J$, each $Z \in \mathrm{Col}(\mathbf{Op})$, each collection $\{Y_j\}_{j\in J}$ indexed by J and each family $\{\{X_{i_j}\}_{i_j \in f^{-1}(I)}\}_{j\in J}$ indexed by J consisting of collections indexed by the fibres of f, the sets of multimorphisms $\{\mathrm{Mul}_{\mathbf{O}}(\{X_i\}_{i\in f^{-1}(j)}, Y_j)\}_{j\in J}$ can be composed with the multimorphisms $\mathrm{Mul}_{\mathbf{O}}(\{Y_j\}, Z)$ in a manner that is unital and associative. We may associate to this data a *simplicial category of operators* as follows: Define a functor $\mathbf{O}^{\otimes} \to \mathsf{Fin}_*$ by declaring objects of $\mathbf{O}^{\otimes}$ to be pairs $(\langle n\rangle, (C_1, \ldots, C_n))$ of an object in Fin_* together with a tuple of colours of $\mathbf{Op}$. The simplicial set of morphisms between two pairs $(\langle n\rangle, (C_1, \ldots, C_n))$ and $(\langle m\rangle, (D_1, \ldots, D_m))$ is given by the formula

$$\coprod_{\alpha:\langle n\rangle m} \prod_{j\in\langle m\rangle} \mathrm{Mul}_{\mathbf{O}}(\{C_i\}_{i\in\alpha^{-1}(j), Y_j}),$$

then the unitality and associative of the composition follows immediately from the definition of a simplicial multicategory. The functor $\mathbf{O}^{\otimes} \to \mathsf{Fin}_*$ simply forgets colours. Taking the coherent nerve of this diagram yields an ∞-operad in the sense defined above.

Example 2.1.5.3. The following ∞-operads play a role in the sequel.

- The trivial ∞-operad $\mathrm{Triv}^{\otimes} \subset \mathbf{N}(\mathsf{Fin}_*)$ obtained as the subcategory spanned by inert maps.

- The commutative ∞-operad $\mathrm{Comm}^{\otimes} := \mathbf{N}(\mathsf{Fin}_*)$, obtained as the operadic nerve of the discrete simplicial commutative operad.

- The associative ∞-operad $\mathrm{Assoc}^{\otimes}$, obtained as the operadic nerve of the simplicial operad $\mathbf{Assoc}^{\otimes}$, whose objects are those of $\mathbf{N}(\mathsf{Fin}_*)$ and whose morphism are maps $f : \langle n\rangle \to \langle m\rangle$ together with a linear order on the fibre of each element in $\langle n\rangle^\circ$. The linear order on the fibre over $i \in \langle k\rangle^\circ$ of the composition $\langle n\rangle \xrightarrow{f} \langle m\rangle \xrightarrow{g} \langle k\rangle$ in $\mathbf{Assoc}^{\otimes}$ is given by concatenating the linear orders of the sets $f^{-1}(j)$ for $j \in g^{-1}(i)$ according to the linear order on $g^{-1}(i)$. There is a Dwyer-Kan equivalence ${}^t\mathbb{E}_1 \to \mathbf{Assoc}^{\otimes}$ of fibrant simplicial categories, where ${}^t\mathbb{E}_1$ is the simplicial (1-coloured) operad of little intervals, whose multimorphisms are given by the spaces of rectilinear embeddings of intervals into another, yielding an equivalence of ∞-operads $\mathbb{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$.

- The ∞-operad $\mathrm{MComm}^{\otimes}$ controlling pairs (A, M) of a commutative algebra and a module over it. $S = \{\mathfrak{a}, \mathfrak{m}\}$ denote the set of colours. Let $(x_i)_{i\in I} \in S^I$ be an I-tuple of colours, then the set of multimaps $\mathrm{Mul}_{\mathbf{MComm}}(\{x_i\}_I; \mathfrak{a})$ is the one element set if $x_i = \mathfrak{a}$ for all $i \in I$, and is empty otherwise. The set of multimaps $\mathrm{Mul}_{\mathbf{MComm}}(\{x_i\}_I; \mathfrak{m})$ is the one element set if there exists exactly one $j \in I$ such that $x_j = \mathfrak{m}$, and is empty otherwise. Taking categories of operators, we have a simplicial operad $\mathbf{MComm}^{\otimes} \to \mathsf{Fin}_*$. We denote by $\mathrm{MComm}^{\otimes}$ the operadic nerve of this operad. We have the following explicit description of $\mathbf{MComm}^{\otimes}$:

  (1) Objects of $\mathbf{MComm}^{\otimes}$ are pairs $(\langle n\rangle, T)$ where $T \subset \langle n\rangle^\circ$.

  (2) Morphisms between pairs $(\langle n\rangle, T)$ and $(\langle m\rangle, T')$ are maps $f : \langle n\rangle \to \langle m\rangle$ in $\mathbf{N}(\mathsf{Fin}_*)$ that satisfy the following conditions.

     (*i*) f carries $T \cup \{*\}$ into $T' \cup \{*\}$.

     (*ii*) For every $t' \in T'$, $f^{-1}(t')$ contains exactly one element of T.

  The set $T \subset \langle n\rangle$ indexes the elements of $\langle n\rangle^\circ$ corresponding to the $\mathfrak{m}$-coloured objects.

- The ∞-operads $\mathrm{LM}^{\otimes}$ and $\mathrm{RM}^{\otimes}$ controlling pairs (A, M) of an associative algebra with a left module respectively a right module over it. These ∞-operads are defined similarly to $\mathrm{MComm}^{\otimes}$: $\mathrm{LM}^{\otimes}$ is defined as the operadic nerve of the simplicial operad $\mathbf{LM}^{\otimes}$ admitting the following description.

  (1) Objects of $\mathbf{LM}^{\otimes}$ are pairs $(\langle n\rangle, T)$ where $T \subset \langle n\rangle^{\circ}$.

  (2) Morphisms between pairs $(\langle n\rangle, T)$ and $(\langle m\rangle, T')$ are maps $f : \langle n\rangle \to \langle m\rangle$ in $\mathbf{Assoc}^{\otimes}$ that satisfy the following conditions.

     (i) f carries $T \cup \{*\}$ into $T' \cup \{*\}$.

     (ii) For every $t' \in T'$, $f^{-1}(t')$ contains exactly one element of T, and this element is maximal with respect to the linear order of $f^{-1}(t')$.

  $\mathbf{RM}^{\otimes}$ is defined similarly, the only difference being that the element in the fibre of $f^{-1}(t')$ is minimal with respect to the linear order.

Example 2.1.5.4. We have the following symmetric monoidal ∞-categories:

(1) For any symmetric monoidal model category $\mathbf{M}$, the underlying ∞-category $\mathbf{N}(\mathbf{M}^{c})[W^{-1}]$ is canonically endowed with a symmetric monoidal structure.

(2) For any ∞-category $\mathcal{C}$ admitting finite products, the ∞-category $\mathcal{C}^{\times} \to \mathbf{N}(\mathsf{Fin}_*)$ exhibiting the *Cartesian* symmetric monoidal structure on $\mathcal{C}$ constructed in section 2.4.1 of HA. Similarly, for any ∞-category admitting finite coproducts, we have the ∞-category $\mathcal{C}^{\amalg} \to \mathbf{N}(\mathsf{Fin}_*)$ exhibiting the *coCartesian* symmetric monoidal structure on $\mathcal{C}$, constructed in section 2.4.3 of HA.

(3) The ∞-category of presentable ∞-category $(\mathsf{Pr}^{\mathrm{L}})^{\otimes}$ equipped with the *Lurie tensor product* on presentable ∞-categories . The tensor product of two presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is the presentable ∞-category $\mathcal{C} \otimes \mathcal{D}$ universal among presentable ∞-categories that admit functor from $\mathcal{C} \times \mathcal{D}$ that preserves colimits separately in each variable.

(4) The ∞-category of spectra $\mathcal{S}\mathrm{p}^{\otimes}$ with the smash product symmetric monoidal structure. This symmetric monoidal structure can be recovered in (at least) three ways: one can localize one of the symmetric monoidal model categories of spectra; for instance, one can take the symmetric monoidal model categories of symmetric or orthogonal spectra. Alternatively, one can take the Goodwillie derivative of the Cartesian symmetric monoidal structure on $\mathcal{S}$. In [Lur17a], section 4.8.2, the symmetric monoidal structure on $\mathcal{S}\mathrm{p}$ is recovered by observing that the underlying ∞-category of the unit of the Lurie tensor product on $\mathsf{Pr}^{\mathrm{Ex}}$ is $\mathcal{S}\mathrm{p}$. Thus, $\mathcal{S}\mathrm{p}$ is initial in $\mathbb{E}_{\infty}\mathsf{Alg}(\mathsf{Pr}^{\mathrm{St}})$ and therefore admits a symmetric monoidal structure that commutes with small colimits separately in each variable.

Definition 2.1.5.5. Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}^{\otimes'}$ be ∞-operads, then ∞-operad maps are functors lying over $\mathbf{N}(\mathsf{Fin}_*)$ preserving coCartesian lifts of inert maps. We denote by $\mathsf{Alg}_{\mathcal{O}}(\mathcal{O}')$ the full subcategory of $\mathrm{Fun}_{\mathbf{N}(\mathsf{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}^{\otimes'})$ spanned by ∞-operad maps. More generally, if $\mathcal{O}^{\otimes'} \to \mathcal{C}^{\otimes}$ is a categorical fibration and $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is a map of ∞-operads, we denote the full subcategory of $\mathrm{Fun}_{\mathcal{C}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{O}^{\otimes'})$ (which is an ∞-category by assumption that $\mathcal{O}^{\otimes'} \to \mathcal{C}^{\otimes}$ is a categorical fibration) spanned by ∞-operad maps by $\mathsf{Alg}_{\mathcal{O}/\mathcal{C}}(\mathcal{O}')$.

Example 2.1.5.6. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $\mathcal{C}^{\times}$ be a Cartesian symmetric monoidal ∞-category, then the ∞-category $\mathsf{Alg}_{\mathcal{O}}(\mathcal{C})$ admits a convenient description in terms of *$\mathcal{O}$-monoids.* Let $\mathcal{O}^{\otimes}_{\mathrm{in}}$ be the subcategory of $\mathcal{O}^{\otimes}$ spanned by inert morphisms, then we say that a functor $f : \mathcal{O}^{\otimes} \to \mathcal{C}$ is an $\mathcal{O}$-monoid if $f|_{\mathcal{O}^{\mathrm{in}}}$ is a right Kan extension of $f|_{\mathcal{O}^{\otimes}_{\langle 1\rangle}}$; that is, for each $X \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, the images under f of the inert morphisms $X \to X_i = \rho^{i}_{!}(X)$ exhibit $f(X)$ as a product of the objects $\{f(X_i)\}_{i \in \langle n\rangle^{\circ}}$. Let $\mathsf{Mon}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C})$ be the full subcategory spanned by $\mathcal{O}$-monoids, then it follows from [Lur17a], prop. 2.4.1.7 that composition with the projection $\pi : \mathcal{C}^{\times} \to \mathcal{C}$ induces a trivial fibration $\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Mon}_{\mathcal{O}}(\mathcal{C})$.
In the case $\mathcal{C}^{\times} = \mathsf{Cat}^{\times}_{\infty}$, the equivalence above together with unstraightening furnishes an equivalence between $\mathsf{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty})$ and the full subcategory of $\mathsf{coCart}_{/\mathcal{O}}$ spanned by coCartesian fibrations $\mathcal{D} \to \mathcal{O}^{\otimes}$ such that for each $X \in \mathcal{O}^{\otimes}$ lying over some $\langle n\rangle$, the inert maps $X \to \rho^{i}_{!}(X)$ induce an equivalence $\mathcal{D}_X \simeq \prod_{i \in \langle n\rangle} \mathcal{D}_{X_i}$. According to [Lur17a], prop. 2.1.2.12, this is precisely the condition that $\mathcal{D} \to \mathcal{C} \to \mathbf{N}(\mathsf{Fin}_*)$ is an ∞-operad. Such a coCartesian fibration is then called a *coCartesian fibration of ∞-operads.*

Example 2.1.5.7 (Commutative algebras)**.** Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, then we denote by $\mathbb{E}_{\infty}\mathsf{Alg}(\mathcal{C})$ the ∞-category $\mathsf{Alg}_{\mathrm{Comm}}(\mathcal{C})$, the ∞-category of Comm-algebra objects of $\mathcal{C}$. This terminology is justified as we have an equivalence $\mathrm{Comm}^{\otimes} \simeq \mathrm{colim}_{\,n}\mathbb{E}^{\otimes}_{n}$, where the ∞-operads $\mathbb{E}^{\otimes}_{n}$ are the little cubes operads. For $\mathcal{C}^{\otimes} = \mathcal{S}\mathrm{p}^{\otimes}$, we simply write $\mathbb{E}_{\infty}\mathsf{Alg}$ for $\mathbb{E}_{\infty}\mathsf{Alg}(\mathcal{S}\mathrm{p})$.

For $\mathcal{C}^{\otimes} = \mathsf{Cat}_{\infty}^{\times}$, the previous example identifies $\mathbb{E}_\infty$-algebras in $\mathcal{C}^{\otimes}$ with symmetric monoidal ∞-categories. Taking $\mathcal{C}^{\otimes} = (\mathsf{Pr}^{\mathrm{L}})^{\otimes}$, we obtain *presentably symmetric monoidal ∞-categories*: those symmetric monoidal ∞-categories whose underlying ∞-category is presentable such that the tensor products preserves small colimits in each variable.

Remark 2.1.5.8. Suppose that $\mathcal{C}^{\otimes} \to \mathbf{N}(\mathsf{Fin}_*)$ is a symmetric monoidal ∞-category such that $\mathcal{C}$ admits countable colimits and the tensor product functors preserve countable colimits in each variable separately, then the forgetful functor $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint denoted $\mathrm{Sym}^\bullet_{\mathcal{C}}$, the *free algebra functor* constructed via operadic left Kan extension in [Lur17a], 3.1.3. The unit of this map can be described as follows: let $X \in \mathcal{C}$, then the unit is the canonical inclusion

$$X \to \coprod_{n \geq 0} \mathrm{Sym}^n(X) = 1_{\mathcal{C}} \coprod X \coprod \mathrm{Sym}^2(X) \coprod \dots$$

where $1_{\mathcal{C}}$ is the tensor unit and $\mathrm{Sym}^n(X) \simeq (\bigotimes_n X)_{h\Sigma_n}$ are the homotopy coinvariants of the Σ_n-equivariant object $\bigotimes_n X$.

Example 2.1.5.9 (Modules over commutative algebras)**.** Let $\mathcal{C}^{\otimes} \to \mathbf{N}(\mathsf{Fin}_*)$ be a symmetric monoidal ∞-category, then there are several ways to think about module objects in $\mathcal{C}^{\otimes}$.

(1) As the ∞-operad $\mathrm{Comm}^{\otimes}$ is *coherent* in the sense of [Lur17a], defn. 3.3.1.9, we can define a categorical fibration $\mathsf{Mod}(\mathcal{C})^{\otimes} \to \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \times \mathbf{N}(\mathsf{Fin}_*)$ as in the construction of [Lur17a], section 3.3.3. Taking the fibre at A, we obtain an ∞-operad $\mathsf{Mod}_A^{\otimes}(\mathcal{C}) \to \mathbf{N}(\mathsf{Fin}_*)$, whose fibre over $\langle 1 \rangle$ gives an ∞-category of A-modules. The ∞-operad $\mathsf{Mod}_A^{\otimes}(\mathcal{C})$ is in fact a symmetric monoidal ∞-category.

(2) We can simply take ∞-category $\mathsf{Alg}_{\mathrm{MComm}^{\otimes}}(\mathcal{C})$, which by composition with $\mathrm{Comm}^{\otimes} \to \mathrm{MComm}^{\otimes}$ induces a categorical fibration $\mathsf{Alg}_{\mathrm{MComm}^{\otimes}}(\mathcal{C}) \to \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$. We will write $\mathsf{Mod}(\mathcal{C})$ for $\mathsf{Alg}_{\mathrm{MComm}^{\otimes}}(\mathcal{C})$.

It is a consequence of [Lur17a], thm. 4.4.1.28 that the left vertical map in the pullback diagram

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \times \mathbf{N}(\mathsf{Fin}_*)} \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \times \{\langle 1 \rangle\} \arrow[r] \arrow[d] & \mathsf{Mod}(\mathcal{C})^{\otimes} \arrow[d] \\
\mathbb{E}_\infty\mathsf{Alg} \times \{\langle 1 \rangle\} \arrow[r] & \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \times \mathbf{N}(\mathsf{Fin}_*)
\end{tikzcd}$$

is equivalent to $\mathsf{Alg}_{\mathrm{MComm}}(\mathcal{C}) \to \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$.
Now let k be a commutative ring. We would like to consider commutative algebra objects and modules over them inside the symmetric monoidal ∞-category of k-modules. To obtain a symmetric monoidal ∞-category of k-modules, we again have several options.

(i) Think of $k \in \mathbb{E}_\infty\mathsf{Alg}(\mathsf{Ab}) \subset \mathbb{E}_\infty\mathsf{Alg}$ as lying in the full subcategory of $\mathbb{E}_\infty$-algebras in spectra spanned by those $\mathbb{E}_\infty$-algebras whose underlying spectra lie in the heart, then we have a symmetric monoidal ∞-category $\mathsf{Mod}_k^{\otimes} := \mathsf{Mod}(\mathcal{S}\mathrm{p})^{\otimes} \times_{\mathbb{E}_\infty\mathsf{Alg} \times \mathbf{N}(\mathsf{Fin}_*)} \{k\} \times \mathbf{N}(\mathsf{Fin}_*)$ where the tensor product is given by the smash product of spectra.

(ii) Consider the left proper combinatorial model category $\mathbf{Mod}_k := \mathrm{Ch}(\mathrm{Mod}_k)$ of chain complexes of k-modules with the projective model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are taken levelwise (so that all objects are fibrant). This is a symmetric monoidal model category, so the ∞-category $\mathbf{N}(\mathbf{Mod}_k^{fc})[W^{-1}] = \mathbf{N}(\mathbf{Mod}_k^{c})[W^{-1}]$ is symmetric monoidal. Via the equivalences $\mathbf{N}(\mathbf{Mod}_k^{fc}[W^{-1}]) \simeq \mathbf{N}(\mathbf{Mod}_k)[W^{-1}] \simeq \mathcal{D}(\mathbf{Mod}_k)$, we obtain a symmetric monoidal structure on the dg-nerve of the dg category $\mathbf{Mod}_k$.

These two constructions yield equivalent ∞-categories of k-modules, essentially via a kind of monadic reconstruction (see [Lur17a], prop. 7.2.1.13). For any connective $\mathbb{E}_\infty$-ring A, the ∞-category Mod_A is presentably symmetric monoidal $\mathsf{Mod}_A \to \mathcal{S}\mathrm{p}$ is stable and comes equipped with a canonical t-structure determined by the forgetful functor $\theta : \mathsf{Mod}_A \to \mathcal{S}\mathrm{p}$, that is $(\mathsf{Mod}_A^{\leq 0} = \theta^{-1}(\mathcal{S}\mathrm{p}^{\leq 0})$ and $(\mathsf{Mod}_A^{\geq 0} = \theta^{-1}(\mathcal{S}\mathrm{p}^{\geq 0})$. This t-structure is compatible with the symmetric monoidal structure (in the sense that for the active map $\langle n \rangle \to \langle 1 \rangle$, the map $\prod_{i \in \langle n \rangle^\circ} \mathcal{C} \to \mathcal{C}$ carries $\prod_{i \in \langle n \rangle^\circ} \mathcal{C}^{\geq 0}$ into $\mathcal{C}^{\geq 0}$), and compatible with filtered colimits (in the sense that $\mathcal{C}^{\leq 0} \subset \mathcal{C}$ is stable under filtered colimits). If $A = k$ a commutative ring, the t-structure just described coincides with the one obtained via the derived ∞-category of $\mathbf{Mod}_k$.
We can take once again different points of view on (commutative) algebra objects in Mod_k.

(i') Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and let $A \in \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ be an $\mathbb{E}_\infty$-algebra in $\mathcal{C}$. Then there is a canonical equivalence of ∞-categories

$$\mathbb{E}_\infty\mathsf{Alg}(\mathsf{Mod}_A) \longrightarrow \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})^{A/}$$

of $\mathbb{E}_\infty$-algebra objects in Mod_A and commutative algebra objects that come equipped with a map from A.

(ii') The model category $\mathbf{Mod}_k$ satisfies the monoid axiom of Schwede and Shipley [SS03], which implies that $\mathbb{E}_1\mathsf{Alg}(\mathbf{Mod}_k)$, the (ordinary) category of associative algebras in $\mathbf{Mod}_k$, admits a model structure right transferred along the adjunction

$$\mathbf{Mod}_k \rightleftarrows \mathbb{E}_1\mathsf{Alg}(\mathbf{Mod}_k)$$

such that the canonical map

$$\mathbb{E}_1\mathsf{Alg}(\mathbf{Mod}_k)^{fc}[W^{-1}] \longrightarrow \mathbb{E}_1\mathsf{Alg}(\mathsf{Mod}_k)$$

is an equivalence. If we assume that $\mathbb{Q} \subset k$, then the same statements hold with $\mathbb{E}_1$ replaced by $\mathbb{E}_\infty$. We will also use the notation $\mathbf{cdga}_k$ for the model category $\mathbb{E}_\infty\mathsf{Alg}(\mathbf{Mod}_k)$; its objects are *commutative differentially graded algebras*, or *cdga's* over k. The left proper combinatorial model category $\mathbf{Mod}_k^{\geq 0}$ also satisfies the monoid axiom and is freely powered, so we obtain an equivalence

$$\mathbb{E}_\infty\mathsf{Alg}(\mathbf{Mod}_k^{\geq 0})^{fc}[W^{-1}] \longrightarrow \mathbb{E}_\infty\mathsf{Alg}(\mathsf{Mod}_k^{\geq 0})$$

since $\mathbf{Mod}_k^{\geq 0, fc}[W^{-1}] \to \mathsf{Mod}_k^{\geq 0}$ is an equivalence. We will write $\mathbf{cdga}_k^{\geq 0}$ for the model category $\mathbb{E}_\infty\mathsf{Alg}(\mathbf{Mod}_k^{\geq 0})$; its objects are *connective cdga's* over k.

All these perspectives will play a role in this work.

Remark 2.1.5.10. Suppose that $\mathcal{C}^\otimes \to \mathbf{N}(\mathsf{Fin}_*)$ is presentably symmetric monoidal, then the categorical fibration $\mathsf{Mod}(\mathcal{C}) \to \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ is a presentable fibration. In fact, the map $\mathsf{Mod}(\mathcal{C})^\otimes \to \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \times \mathbf{N}(\mathsf{Fin}_*)$ is a coCartesian fibration, so by straightening, we have a functor $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \to \mathbb{E}_\infty\mathsf{Alg}(\mathsf{Pr}^{\mathrm{L}})$. Concretely, this amounts to the assertion that for any map $f : A \to B$, the functor $f_!$ given by $\_ \otimes_A B$ is symmetric monoidal.

Remark 2.1.5.11 (Two-sided Bar construction)**.** Let $\mathcal{C}^\otimes \to \mathbf{N}(\mathsf{Fin}_*)$ be a symmetric monoidal ∞-category, and suppose that the symmetric monoidal structure is compatible with geometric realizations of simplicial objects in the sense that $\mathcal{C}$ admits geometric realizations of simplicial objects and all the tensor product functors preserve geometric realizations of simplicial objects (automatically separately in each variable as $\mathbf{N}(\mathbf{\Delta}^{op})$ is sifted). Then the relative tensor product functor

$$\mathsf{Mod}_A \times \mathsf{Mod}_A \xrightarrow{\text{-}\otimes_A\text{-}} \mathsf{Mod}_A \longrightarrow \mathcal{C}$$

admits an explicit description. Let $\mathbf{M}^2\mathbf{Comm}^\otimes$ be the category defined as follows.

(1) Objects are ordered triples $(\langle n \rangle, T, S)$, where $T, S \subset \langle n \rangle^\circ$ and $T \cap S = \varnothing$.

(2) Morphisms between ordered triples $(\langle n \rangle, T, S)$ and $(\langle m \rangle, T', S')$ are maps $f : \langle n \rangle \to \langle m \rangle$ in $\mathbf{N}(\mathsf{Fin}_*)$ such that $f(T) \subset T' \cup \{*\}$ and for each $t' \in T'$, $f^{-1}(t')$ contains exactly one element of T, and similarly for S and S'.

This describes the category of operators for the ∞-operad controlling ordered triples (A, M, N) where A is an $\mathbb{E}_\infty$-algebra and M and N are A-modules. Note that the corresponding ∞-operad $\mathrm{M}^2\mathrm{Comm}^\otimes$ fits into a pushout diagram

$$\begin{tikzcd}
\mathrm{Comm}^\otimes \arrow[r] \arrow[d] & \mathrm{MComm}^\otimes \arrow[d] \\
\mathrm{MComm}^\otimes \arrow[r] & \mathrm{M}^2\mathrm{Comm}^\otimes
\end{tikzcd}$$

among ∞-operads. Let $\mathrm{M}^2\mathrm{Comm}^{\otimes'} \subset \mathrm{M}^2\mathrm{Comm}^\otimes$ be the full subcategory spanned by objects of the form $(\langle n \rangle, \{1\}, \{n\})$, then $\mathrm{M}^2\mathrm{Comm}^{\otimes'}$ is equivalent to $\mathrm{Comm}^\otimes$ via the functor

$$(\langle n \rangle, \{1\}, \{n\}) \mapsto \langle n \rangle \smallsetminus \{1, n\}.$$

Let ϕ denote an inverse to this functor, then we can consider the composition

$$U : \mathbf{N}(\mathbf{\Delta}^{op}) \xrightarrow{\mathrm{Cut}} \mathrm{Comm}^\otimes \xrightarrow{\phi} \mathrm{M}^2\mathrm{Comm}^{\otimes'} \subset \mathrm{M}^2\mathrm{Comm}^\otimes$$

where the functor Cut of [Lur17a], construction 4.1.2.9 is the nerve of the functor $\mathbf{N}(\mathbf{\Delta}^{op}) \to \mathbf{Comm}^\otimes$ sending $[n]$ to $\langle n \rangle$ and $\alpha : [n] \leftarrow [m]$ to the map $\mathrm{Cut}(\alpha) : \langle n \rangle \to \langle m \rangle$ defined by

$$\mathrm{Cut}(\alpha)(i) = \begin{cases} j & \text{if there is a } j \in \langle n \rangle \text{ such that } \alpha(j-1) < i \leq \alpha(j) \\ * & \text{otherwise} \end{cases}$$

We have an equivalence $\mathsf{Mod}_A \times \mathsf{Mod}_A \simeq \mathsf{Alg}_{\mathrm{M}^2\mathrm{Comm}}(\mathcal{C}) \times_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})} \{A\}$, so we may identify a pair (M, N) of A-modules with a functor $F : \mathrm{M}^2\mathrm{Comm}^\otimes \to \mathcal{C}^\otimes$ over $\mathbf{N}(\mathsf{Fin}_*)$ such that $F((\langle 1 \rangle, \varnothing, \varnothing)) = A$. Then we denote by $\mathsf{Bar}_A(M, N)_\bullet$ the simplicial object $F \circ U$ in $\mathcal{C}$, and we have a canonical equivalence between $M \otimes_A N$ and the geometric realization $|\mathsf{Bar}_A(M, N)_\bullet|$.

Remark 2.1.5.12. Let $\mathcal{C}^{\otimes} \in \mathbb{E}_\infty \mathsf{Alg}(\mathsf{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal ∞-category, and let $A \in \mathcal{C}^{\otimes}$ be an $\mathbb{E}_\infty$-algebra in $\mathcal{C}$. Then the forgetful functor

$$\rho_A : \mathbb{E}_\infty \mathsf{Alg}(\mathsf{Mod}_A) \longrightarrow \mathsf{Mod}_A$$

admits a left adjoint, the free A-algebra functor, given by $\mathrm{Sym}^{\bullet}_A = \coprod_{n \geq 0} \mathrm{Sym}^n_A$, which induces a functor

$$\mathsf{Mod}_A \simeq (\mathsf{Mod}_A)^{/0_{\mathcal{C}}} \longrightarrow \mathbb{E}_\infty \mathsf{Alg}(\mathcal{C})^{A//A}.$$

This functor has a right adjoint, denoted by I_A, which takes an A-augmented A-algebra $B \to A$ to the pullback along the unit map $0_{\mathcal{C}} \to A$ of A-modules ([Lur17b], prop 5.2.5.1), so that we have a fibre sequence

$$I_A(B) \longrightarrow B \longrightarrow A$$

in Mod_A. We call the functor I_A the *A-augmentation ideal functor*. The unit map of the adjunction $(\mathrm{Sym}^{\bullet}_A \dashv \rho_A)$ is given by the map $\mathrm{id} = \mathrm{Sym}^1_A \to \coprod_{n \geq 0} \mathrm{Sym}^n_A$, from which we easily deduce that the unit of the adjunction $(\mathrm{Sym}^{\bullet}_A \dashv I_A)$ is given by the map $\mathrm{id} = \mathrm{Sym}^1_A \to \coprod_{n \geq 1} \mathrm{Sym}^n_A$.
Suppose that we take $\mathcal{C}^{\otimes} = \mathsf{Mod}_k$, the symmetric monoidal ∞-category of k-modules for k a commutative ring, and suppose that A is connective, then the free A-algebra functor $\mathrm{Sym}^{\bullet}_A$ preserves connective objects, so the adjunction $(\mathrm{Sym}^{\bullet}_A \dashv \rho_A)$ restricts to an adjunction between connective objects. If $A \to B \to A$ is an A-augmented A-algebra such that B is connective, then $I_A(B)$ is also connective, as the long exact sequence shows. It follows that the adjunction $(\mathrm{Sym}^{\bullet}_A \dashv I_A)$ also restricts to an adjunction $\mathsf{Mod}^{\mathrm{cn}}_A \rightleftarrows (\mathbb{E}_\infty \mathsf{Alg}(\mathsf{Mod}_k)^{A//A})^{\mathrm{cn}}$. This construction will be important in chapter 4.

We discuss some examples of module objects in $\mathsf{Cat}^{\times}_{\infty}$.

Example 2.1.5.13 (Tensored, cotensored and enriched ∞-categories)**.** Let $\mathcal{C}^{\otimes} \to \mathbf{N}(\mathsf{Fin}_*)$ be an ∞-operad and $\mathcal{M}$ be ∞-categories, then following [Lur17a], section 4.2.1, we say that a fibration of ∞-operads $p : \mathcal{O}^{\otimes} \to \mathrm{MComm}^{\otimes}$ *exhibits $\mathcal{M}$ as weakly enriched over $\mathcal{C}$* if there are isomorphisms $\mathcal{C}^{\otimes} \cong \mathcal{O}^{\otimes} \times_{\mathrm{MComm}^{\otimes}} \mathbf{N}(\mathsf{Fin}_*)$ and $\mathcal{M} \cong \mathcal{O}^{\otimes}_{(\langle 1 \rangle, \{1\})}$. Suppose that the fibration p is coCartesian, so that we can identify p with the data of an MComm-module object $(\mathcal{C}^{\otimes}, \mathcal{M})$ in Cat_∞, then we say that $\mathcal{M}$ is *tensored* over the symmetric monoidal ∞-category $\mathcal{C}$.
Suppose that $p : \mathcal{O}^{\otimes} \to \mathrm{MComm}^{\otimes}$ exhibits $\mathcal{M}$ as tensored over $\mathcal{C}^{\otimes}$ so that there is an action map

$$\mathcal{C} \times \mathcal{M} \xrightarrow{-\otimes-} \mathcal{M},$$

and let M, N be two objects of $\mathcal{M}$, then a *morphism object* of M and N is an object $\mathsf{Mor}_{\mathcal{C}}(M, N) \in \mathcal{C}$ together with a map $\mathsf{Mor}_{\mathcal{C}}(M, N) \otimes M \to N$ that is a unit transformation at N of the functor

$$\mathcal{C} \xrightarrow{-\otimes M} \mathcal{M}.$$

If there is a morphism object for every pair of object $M, N \in \mathcal{M}$ then we say that $\mathcal{M}$ is *enriched over* $\mathcal{C}^{\otimes}$.
Let $N \in \mathcal{M}$ and $C \in \mathcal{C}$, then an *exponential object* of N and C is an object ${}^{C}N \in \mathcal{M}$ together with a map $N \to C \otimes {}^{C}N$ that is a counit transformation at C of the functor

$$\mathcal{M} \xrightarrow{C\otimes -} \mathcal{M}.$$

If there is an exponential object for every pair of objects $N \in \mathcal{M}$ and $C \in \mathcal{C}$, then we say that $\mathcal{M}$ is *cotensored over* $\mathcal{C}$.

Remark 2.1.5.14. The previous example gives notions of ∞-categories weakly enriched over (symmetric) monoidal ∞-categories. Gepner and Haugseng in [GH15] have given a detailed treatment of weak enrichment over general monoidal ∞-categories. We will not review this theory here, but we note that if $q : \mathcal{O}^{\otimes} \to \mathrm{MComm}^{\otimes}$ exhibits $\mathcal{M}$ as tensored and enriched over $\mathcal{C}$, then, as explained in section 7 of [GH15], a $\mathcal{C}$-enriched ∞-category in the sense of Gepner-Haugseng can be extracted from the coCartesian fibration q such that the morphism object in $\mathcal{C}$ between any two $M, N \in \mathcal{M}$ is the morphism object $\mathsf{Mor}_{\mathcal{C}}(M, N)$ satisfying the universal property of the unit transformation as defined above. In the setup of [GH15], it is quite straightforward to prove that $\mathcal{V}$-enriched ∞-categories with a single objects are associative monoids in $\mathcal{V}$ (see [GH15], remark 6.3.5), so we conclude that for each object $M \in \mathcal{V}$ the object $\mathsf{End}(M) \coloneqq \mathsf{Mor}_{\mathcal{C}}(M, M)$ lifts to an associative algebra of $\mathcal{C}$, the *endomorphism algebra* of M
Replacing $\mathrm{MComm}^{\otimes}$ with $\mathrm{LM}^{\otimes}$ or $\mathrm{RM}^{\otimes}$ yields evident notions of left/right tensored ∞-categories and ∞-categories enriched in merely monoidal ∞-categories.

Example 2.1.5.15 (Lurie's Barr-Beck theorem)**.** We will find many uses for Lurie's version of the Barr-Beck monadicity theorem, which is indispensable for constructing equivalences of ∞-categories that would otherwise be combinatorially intractable, by exhibiting ∞-categories as algebras over monads defined on ∞-categories that are easier to handle. For $\mathcal{D}, \mathcal{C}$ two ∞-categories, the pair $(\mathrm{Fun}(\mathcal{D},\mathcal{D}), \mathrm{Fun}(\mathcal{C},\mathcal{D}))$ determines a strict simplicial associative monoid object together with a left module over it, which we can identify with a coCartesian fibration $\mathcal{O}^{\otimes} \to \mathrm{LM}^{\otimes}$. Consider an adjunction $(F \dashv G) : \mathcal{D} \rightleftarrows \mathcal{C}$, then the counit map $F \circ G \to \mathrm{id}$ provides a map $G \circ F \circ G \to G$ which endows $T = G \circ F$ with the structure of an endomorphism algebra of G such that G canonically lifts to a T-module. The functor $\mathcal{C} \times \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathcal{D}$ lifts to a functor $\mathcal{C} \times \mathsf{LMod}_T(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \to \mathsf{LMod}_T(\mathcal{D})$, so given an adjunction $(F \dashv G)$ with its endomorphism monad T as above, we obtain a functor $\overline{G} : \mathcal{C} \to \mathsf{LMod}_T(\mathcal{D})$ such that the composition with the forgetful functor $\mathsf{LMod}_T(\mathcal{D}) \to \mathcal{C}$ is given by G. Then the adjunction $(F \dashv G)$ is said to be *monadic* if the functor $\overline{G}$ is an equivalence. According to the Barr-Beck theorem ([Lur17a], thm 4.7.3.5), the following are equivalent.

(1) The adjunction $(F \dashv G)$ is monadic.

(2) G is conservative and G admits and preserves colimits of G-split simplicial objects.

It follows that if these conditions are satisfied, every object of $X \in \mathcal{C}$ is the colimit of the monadic Bar construction $\mathsf{Bar}_T(T, X)$.

2.1.6 Groupoid objects

In ordinary category theory, an epimorphism $f : x \to y$ is an *effective epimorphism* if y is the coequalizer of the equivalence relation on x determined by y. We obtain a wealth of examples of effective epimorphism in ordinary 1-topos theory since 1-topoi are obtained, essentially, by turning a class of coverings defined by a Grothendieck topology on a category into effective epimorphisms. For example, if M is a manifold and we have a collection of submersions $\{U_\alpha \to M\}$ that cover M in the usual sense, then the induced morphism

$$\varphi : \coprod_\alpha j(U_\alpha) \to j(M),$$

where $j : \mathsf{Man} \to \mathrm{Fun}(\mathsf{Man}^{op}, \mathsf{Set})$ is the Yoneda embedding, is an effective epimorphism in the topos of sheaves on the site of smooth manifolds for the Grothendieck topology generated by submersions. The *kernel pair* of φ is the pullback $\coprod_{\alpha,\beta} j(U_\alpha \times_M U_\beta)$, and the diagram

$$\textstyle\coprod_{\alpha,\beta} j(U_\alpha \times_M U_\beta) \rightrightarrows \coprod_\alpha j(U_\alpha) \longrightarrow j(M)$$

is a coequalizer diagram. The diagram above is a (Lie) groupoid, known as the *Čech groupoid of M*. In higher category theory and higher topos theory, effective epimorphism play a very prominent role as well.

Definition 2.1.6.1. (1) Let $\mathcal{C}$ be an ∞-category that admits finite limits and let $f : X \to Y$ be a morphism in $\mathcal{C}$. View f as a functor $f : \mathbf{N}(\Delta_+^{\leq 0})^{op} \to \mathcal{C}$, then an augmented simplicial diagram $U_\bullet : \mathbf{N}(\mathbf{\Delta}_+)^{op} \to \mathcal{C}$ is a *Čech nerve* of f if $U_\bullet$ is a right Kan extension of $U_\bullet|_{\mathbf{N}(\Delta_+^{\leq 0})^{op}} = f$. Note that Čech nerves are defined up to contractible ambiguity, so we will speak of *the* Čech nerve of a morphism f and denote the augmented simplicial object by $\check{\mathrm{C}}(f)_\bullet$.

(2) A morphism $f : X \to Y$ in $\mathcal{C}$ is an *effective epimorphism* if $\check{\mathrm{C}}(f)_\bullet : \mathbf{N}(\mathbf{\Delta}_+)^{op} = \mathbf{N}(\mathbf{\Delta}^{op})^{\triangleright} \to \mathcal{C}$ is a colimit diagram.

Remark 2.1.6.2. It follows from the definition of the right Kan extension that for all $[n] \in \Delta$, we have an equivalence

$$\check{\mathrm{C}}(f)_n \simeq \underbrace{X \times_Y X \times_Y \ldots \times_Y X}_{(n+1)\text{-fold product}}.$$

The face maps $\check{\mathrm{C}}(f)_n \to \check{\mathrm{C}}(f)_{n-1}$ correspond to the obvious projections, and the degeneracies correspond to the obvious maps in the diagram determining the relative product.

Remark 2.1.6.3. In the ∞-topos $\mathcal{S}$ of spaces, effective epimorphisms are precisely those maps that induce surjections on connected components.

As Čech nerves replace coequalizer diagrams of kernel pairs in ordinary category theory, we expect that Čech nerves are groupoid objects. This is indeed the case.

Definition 2.1.6.4. Let $\mathcal{C}$ be an ∞-category. A simplicial object $U_\bullet : \mathbf{N}(\mathbf{\Delta}^{op}) \to \mathcal{C}$ is a *groupoid object* if for all $n \geq 0$ and every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element s, the diagram

$$\begin{tikzcd} U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\ U(S') \arrow[r] & U(\{s\}) \end{tikzcd}$$

is a pullback.

Definition 2.1.6.5. Let $\mathcal{C}$ be an ∞-category that admits finite products, then a *group object* in $\mathcal{C}$ is a groupoid object $G_\bullet$ such that G_0 is a final object.

Remark 2.1.6.6. Let $\mathcal{C}$ be an ∞-category that admits finite products, then we have equivalences

$$\mathbb{E}_1\mathsf{Alg}(\mathcal{C}) \simeq \mathsf{Mon}_{\mathrm{Assoc}}(\mathcal{C}) \simeq \mathsf{Mon}(\mathcal{C})$$

relating $\mathbb{E}_1$-algebras in $\mathcal{C}^\times$ to associative monoids in $\mathcal{C}$. An $\mathbb{E}_1$-algebra G is *grouplike* if the associated monoid object $\tilde{G}$ is a group object in $\mathcal{C}$. The inclusion $\mathsf{Mon}_{\mathrm{gp}}(\mathcal{C}) \subset \mathsf{Mon}(\mathcal{C})$ of grouplike objects into all monoids has a right adjoint, that we denote $G_\bullet \mapsto G_\bullet^\simeq$. For $\mathcal{C} = \mathcal{S}$, this operation takes $G_\bullet$ to the simplicial space $G_\bullet^\simeq$ that has as $G_n^\simeq$ the union of connected components of n-fold compositions of elements in G_1 that are invertible in the underlying monoid in sets. For arbitrary $\mathcal{C}$ with finite products, we can always reduce to this situation via the Yoneda embedding.

2.2 ∞-Topoi

This section is devoted to the recollection of the basic features of the theory of ∞-topoi.

Definition 2.2.0.1. Let $\mathcal{X}$ be an ∞-category. $\mathcal{X}$ is an *∞-topos* if $\mathcal{X}$ arises as a left exact accessible localization of an ∞-category of presheaves on a small ∞-category; that is, there is a fully faithful inclusion $\mathcal{X} \hookrightarrow \mathsf{PShv}(\mathcal{C})$ which admits a left exact left adjoint, for some small ∞-category $\mathcal{C}$.

An ∞-category $\mathcal{X}$ is an ∞-topos if and only if $\mathcal{X}$ satisfies the ∞-categorical Giraud axioms.

Theorem 2.2.0.2 ([Lur17b], thm. 6.1.0.6)**.** *Let $\mathcal{X}$ be an ∞-category, then $\mathcal{X}$ is an ∞-topos if and only if the following conditions are satisfied.*

(1) *$\mathcal{X}$ is presentable.*

(2) *Coproducts are disjoint in $\mathcal{X}$.*

(3) *Colimits in $\mathcal{X}$ are universal.*

(4) *Every groupoid in $\mathcal{X}$ is effective.*

Remark 2.2.0.3. Condition (2) simply means that the limit of the colimit diagram $X \to X \coprod Y \leftarrow Y$ in $\mathcal{X}$ is initial in $\mathcal{X}$. For condition (3), we note that for any ∞-category $\mathcal{C}$ that admits pullbacks, the codomain projection $\mathrm{ev}_{\{1\}} : \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a Cartesian fibration, and $\mathrm{ev}_{\{1\}}$-Cartesian edges are precisely pullback diagrams, so that we have a functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ for every map $f : X \to Y$. Let $\mathcal{K}$ be a collection of small simplicial set, then we say that *$\mathcal{K}$-indexed colimits are universal* if these pullback functors f^* preserve $\mathcal{K}$-indexed colimits.

Remark 2.2.0.4. An *n-topos* for $n \in \mathbb{Z}_{\geq -1}$ is an ∞-category $\mathcal{X}$ that arises as the left exact localization (automatically accessible in this case) of an ∞-category of $(n-1)$-truncated presheaves on a small ∞-category, which we may assume to be an n-category. If $\mathcal{X}$ is an ∞-topos, then the truncation $\tau_{\leq (n-1)}\mathcal{X}$ is an n-topos. In particular, $\tau_{\leq 0}\mathcal{X}$ is a 1-topos that we also denote by $\mathrm{Disc}(\mathcal{X})$, the underlying discrete topos of $\mathcal{X}$, and $\tau_{\leq -1}\mathcal{X}$ is a classical locale.

Remark 2.2.0.5. In any ∞-category $\mathcal{C}$, a map $i : U \to X$ is a *monomorphism* if i is (-1)-truncated in $\mathcal{C}_{/X}$. In an ∞-topos (or more generally any ∞-pretopos (see [Lur], appendix A.6)) the pair (S_L, S_R) where S_L consists of monomorphisms and S_R of effective epimorphisms constitutes an orthogonal factorization system in the sense of [Lur17b], defn. 5.2.8.8.

We will use below the following alternative characterizations of ∞-topoi: one by *descent* and the other in terms of Cartesian transformations.

Proposition 2.2.0.6. *Let $\mathcal{X}$ be a presentable ∞-category. Then $\mathcal{X}$ is an ∞-topos if and only if $\mathcal{X}$ satisfies either of the following equivalent conditions.*

(1) *The functor $\mathcal{X}^{op} \to \mathsf{Pr}^{\mathrm{L}}$ classified by the Cartesian fibration $\mathrm{ev}_{\{1\}} : \mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$ preserves small limits.*

(2) *For each small simplicial set K and each natural transformation $\overline{\alpha} : p \to q$ between functors $p, q : K^{\triangleright} \to \mathcal{X}$ the following holds: if q is a colimit diagram and $\overline{\alpha}|_K$ is a Cartesian transformation, then p is a colimit diagram if and only if $\overline{\alpha}$ is a Cartesian transformation.*

Remark 2.2.0.7. In this proposition, a *Cartesian transformation* $\alpha : f \to g$ between functors $f, g \in \mathrm{Fun}(K, \mathcal{D})$ is a natural transformation $\mathrm{Fun}(\Delta^1 \times K, \mathcal{D})$ such that for every edge $e : \Delta^1 \to K$, the induced diagram $\Delta^1 \times \Delta^1 \to \mathcal{D}$ is a pullback.

We record the following ∞-categories:

- The subcategory ${}^{\mathrm{L}}\mathsf{Top} \subset \widehat{\mathsf{Cat}}_\infty$ whose objects are ∞-topoi, and whose morphisms are functors that are left exact and admit a right adjoint. Such morphisms between ∞-topoi will be called *algebraic morphisms.* For $\mathcal{X}, \mathcal{Y} \in {}^{\mathrm{L}}\mathsf{Top}$, the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by algebraic morphisms is denoted $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$.
- The subcategory ${}^{\mathrm{R}}\mathsf{Top} \subset \widehat{\mathsf{Cat}}_\infty$ whose objects are ∞-topoi, and whose morphisms are functors that admit a left exact left adjoint. Morphisms in ${}^{\mathrm{R}}\mathsf{Top}$ will be called *geometric morphisms.* For $\mathcal{X}, \mathcal{Y} \in {}^{\mathrm{R}}\mathsf{Top}_n$, the full subcategory of $\mathrm{Fun}(\mathcal{Y}, \mathcal{X})$ spanned by geometric morphisms is denoted $\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X})$. The ∞-categories ${}^{\mathrm{R}}\mathsf{Top}$ and ${}^{\mathrm{L}}\mathsf{Top}$ are canonically antiequivalent, as are the ∞-categories $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ and $\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X})$.

Algebraic and geometric morphisms are usually denoted as in the adjoint pair $(f^*, f_*) : \mathcal{X} \rightleftarrows \mathcal{Y}$. We will also need to work with ∞-topoi relative over a given base.

Definition 2.2.0.8. Let $\mathcal{C}$ be an ∞-category. A *topos fibration* over $\mathcal{C}$ is a presentable fibration $p : \widetilde{\mathcal{X}} \to \mathcal{C}$ such that for each $C \in \mathcal{C}$, the fibre $\widetilde{\mathcal{X}}_C$ is an ∞-topos and for each edge $f : \Delta^1 \to \mathcal{C}$, the coCartesian transformation $f_! : \widetilde{\mathcal{X}}_{f(\{0\})} \to \widetilde{\mathcal{X}}_{f(\{1\})}$ is an algebraic morphism. This is equivalent to demanding that the functor $\mathrm{St}^{+,\mathrm{co}}(p) : \mathcal{C} \to \widehat{\mathsf{Cat}}_\infty$ factors through ${}^{\mathrm{L}}\mathsf{Top}$. We let ${}^{\mathrm{L}}\mathsf{Top}_{\mathcal{C}} \subset \mathsf{coCart}_{\mathcal{C}}$ denote the subcategory whose objects are topos fibrations and whose morphisms are commutative diagrams

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{Y}} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

such that the horizontal map preserves coCartesian edges and for each $C \in \mathcal{C}$, the induced map on the fibres is an algebraic morphism of ∞-topoi.

Example 2.2.0.9. Let $q : \overline{{}^{\mathrm{L}}\mathsf{Top}} \to {}^{\mathrm{L}}\mathsf{Top}$ be the coCartesian fibration associated to the subcategory inclusion ${}^{\mathrm{L}}\mathsf{Top} \hookrightarrow \widehat{\mathsf{Cat}}_\infty$. Then q is a topos fibration. In fact q is a *universal* topos fibration, uniquely (up to equivalence) determined by the property that pulling back along q induces, for any ∞-category $\mathcal{C}$, a canonical bijection between equivalence classes of topos fibrations over $\mathcal{C}$ and functors $\mathcal{C} \to {}^{\mathrm{L}}\mathsf{Top}$.

The characterization of ∞-topoi by descent shows that equivalences of ∞-topoi are locally determined, in the following sense.

Lemma 2.2.0.10. *Let $f^* : \mathcal{Y} \to \mathcal{X}$ be an algebraic morphism of ∞-topoi and suppose that there is an effective epimorphism $\coprod_\alpha V_\alpha \to 1_{\mathcal{Y}}$ such that for each α the induced algebraic morphism $\mathcal{Y}_{/V_\alpha} \to \mathcal{X}_{/f^*(V_\alpha)}$ is an equivalence. Then f^* is an equivalence.*

Proof. Consider a covering $\coprod_\alpha V_\alpha \to 1_{\mathcal{Y}}$ such that $\mathcal{Y}_{/V_\alpha} \simeq \mathcal{X}_{/f^*(V_\alpha)}$, and note that for each object of the form $V_{\alpha_{i_n}} \times \ldots \times V_{\alpha_{i_n}}$, the algebraic morphism

$$\mathcal{Y}_{/V_{\alpha_{i_n}} \times \ldots \times V_{\alpha_{i_n}}} \longrightarrow \mathcal{X}_{/f^*(V_{\alpha_{i_n}} \times \ldots \times V_{\alpha_{i_n}})} \simeq \mathcal{X}_{/f^*(V_{\alpha_{i_n}}) \times \ldots \times f^*(V_{\alpha_{i_n}})}$$

is an equivalence since $\mathcal{Y}_{/X \times Y} \simeq (\mathcal{Y}_{/X})_{/X \times Y}$ for every pair of objects $X, Y \in \mathcal{Y}$. Since $f^* : \mathcal{Y} \to \mathcal{X}$ preserves finite limits, the functor $\mathrm{Fun}(\Delta^1, \mathcal{Y}) \to \mathrm{Fun}(\Delta^1, \mathcal{X}) \times_{\mathcal{X}} \mathcal{Y}$ over $\mathcal{Y}$ carries $\mathrm{ev}_{\{1\}}$-Cartesian edges into $\mathrm{ev}_{\{1\}}$-Cartesian edges. The induced functors on the fibres are algebraic morphisms, so we have a morphism of topos fibrations over $\mathcal{Y}$ and therefore a natural transformation $\mathcal{O}_{\mathcal{Y}} \to f^* \mathcal{O}_{\mathcal{X}}$ of functors $\mathcal{Y}^{op} \to {}^{\mathrm{L}}\mathsf{Top}$. Composing this transformation with the Čech nerve of the map

$$h : \coprod_\alpha V_\alpha \longrightarrow 1_{\mathcal{X}}$$

induces a coaugmented cosimplicial object

$$F^\bullet : \mathbf{N}(\mathbf{\Delta})^{\triangleleft} \longrightarrow \mathrm{Fun}(\Delta^1, \mathsf{Pr}^{\mathrm{L}})$$

that carries the cone point to the functor f^* and each object $[n]$ to the algebraic morphism

$$\prod_{\alpha_{i_1}, \ldots, \alpha_{i_n}} \mathcal{Y}_{/V_{\alpha_{i_n}} \times \ldots \times V_{\alpha_{i_n}}} \longrightarrow \prod_{\alpha_{i_1}, \ldots, \alpha_{i_n}} \mathcal{X}_{/f^*(V_{\alpha_{i_n}} \times \ldots \times V_{\alpha_{i_n}})},$$

since by descent we have $\prod_{\alpha_{i_1},\dots,\alpha_{i_n}} \mathcal{Y}_{/V_{\alpha_{i_n}}\times\dots\times V_{\alpha_{i_n}}} \simeq \mathcal{Y}_{/\coprod_{\alpha_{i_1},\dots,\alpha_{i_n}} V_{\alpha_{i_n}}\times\dots\times V_{\alpha_{i_n}}}$ and similarly for $\mathcal{X}$. This algebraic morphism is an equivalence, as we have just verified. Since $\mathcal{X}$ and $\mathcal{Y}$ are ∞-topoi and the maps h and $f^*(h)$ are effective epimorphisms covering the respective unit objects, the functor $F^\bullet$ is a limit diagram, which implies that f^* is an equivalence. □

The effectiveness of groupoids shows that the functor $\mathsf{Gpd}(\mathcal{X}) \to \mathrm{Fun}(\Delta^1, \mathcal{X})$ carrying $G_\bullet$ to the morphism $G_0 \to |G_\bullet|$ determines an equivalence onto the full subcategory spanned by effective epimorphisms, so that we can pass back and forth between groupoids and their deloopings. In particular, if $G_0 = *$, we obtain a version of May's recognition principle as an equivalence between groups and pointed 1-connective objects in $\mathcal{X}$, and using Dunn-Lurie additivity ([Lur17a], thm 5.1.2.2), this equivalence immediately extends to an equivalence between grouplike $\mathbb{E}_n$-algebras and pointed n-connective objects in $\mathcal{X}$. In the next subsection, we will deduce some consequences of the delooping principle for groupoid actions.
In the remainder of this subsection we will give a sample application of the universality of colimits. When a presentable ∞-category $\mathcal{C}$ has universal colimits, $\mathcal{C}$ is in particular a closed Cartesian symmetric monoidal ∞-category. More generally, for each object X in an ∞-topos $\mathcal{X}$, the ∞-topos $\mathcal{X}_{/X}$ is tensored, cotensored and enriched over $\mathcal{X}$. To formalize this, we first take a more general point of view and show that a product preserving functor $g : \mathcal{D} \to \mathcal{C}$ between ∞-categories such that $\mathcal{D}$ admits finite products and all products with objects in the image of g exist in $\mathcal{C}$ can be extended to the data of a tensoring of $\mathcal{C}$ over $\mathcal{D}$.

Construction 2.2.0.11. Let $g : \mathcal{D} \to \mathcal{C}$ be a functor and let $p : \mathcal{M} \to \Delta^1$ be a Cartesian fibration associated to g such that we have equivalences $g^{-1}(\{0\}) \simeq \mathcal{C}$ and $g^{-1}(\{1\}) \simeq \mathcal{D}$. Let $\Gamma^\times_M$ be the category defined as follows.

(1) Objects of $\Gamma^\times_M$ are triples $((\langle n\rangle, T), S)$ where $T \subset \langle n\rangle^\circ$ is a subset and $S \subset \langle n\rangle^\circ$ is a subset containing at most one element of T. For a pair $(\langle n\rangle, T)$, we let $P(n,T)$ denote the poset of such subsets S of $\langle n\rangle^\circ$, ordered by reverse inclusion.

(2) Morphisms between triples $((\langle n\rangle, T), S)$ and $((\langle m\rangle, T'), S')$ are maps $\alpha : \langle n\rangle \to \langle m\rangle$ such that the following conditions are satisfied.

   (i) α carries $T \cup \{*\}$ into $T' \cup \{*\}$.
   (ii) For every $t' \in T'$, the set $\alpha^{-1}(t')$ contains exactly one element of T.
   (iii) $\alpha^{-1}(S') \subset S$.

There is an obvious forgetful functor $\mathbf{N}(\Gamma^\times_M) \to \mathrm{MComm}^\otimes$. Conditions ($i$) and ($ii$) imply that α^{-1} carries $P(m,T')$ into $P(n,T)$; then condition (iii) guarantees that the forgetful functor is a Cartesian fibration. Define a simplicial set $\widetilde{\mathcal{O}_g^\otimes}$ over $\mathrm{MComm}^\otimes$ by the universal property that for any map of simplicial sets $K \to \mathrm{MComm}^\otimes$, there is a canonical bijection

$$\mathrm{Hom}_{(\mathsf{Set}_\Delta)_{/\mathrm{MComm}^\otimes}}(K, \widetilde{\mathcal{O}_g^\otimes}) \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(K \times_{\mathrm{MComm}^\otimes} \Gamma^\times_M, \mathcal{M}).$$

It follows immediately from [Lur17b], cor. 3.2.2.12 that $\widetilde{\mathcal{O}_g^\otimes} \to \mathrm{MComm}^\otimes$ is a coCartesian fibration (note that we apply the result to the map $\mathcal{M} \to *$, *not* to the fibration $\mathcal{M} \to \Delta^1$; we do not assume that g has a left adjoint). We may identify the fibre $\widetilde{\mathcal{O}_g^\otimes}_{(\langle n\rangle,T)}$ with the ∞-category of functors $\mathrm{Fun}(\mathbf{N}(P(n,T), \mathcal{M})$, so that an edge $\alpha : f \to f'$ over $(\langle n\rangle, T) \to (\langle m\rangle, T')$ is coCartesian if and only if $f(\alpha^{-1}(S')) \to f'(S')$ is an equivalence for all $S' \in P(m,T')$. We let $\mathcal{O}_g^\otimes \subset \widetilde{\mathcal{O}_g^\otimes}$ be the full subcategory spanned by functors $f : \mathbf{N}(P(n,T)) \to \mathcal{M}$ such that the following conditions are satisfied.

(a) If $S \cap T \neq \varnothing$, then $f(S) \in p^{-1}(\{0\})$ and if $S \cap T = \varnothing$, then $f(S) \in p^{-1}(\{1\})$.

(b) For all $S \in P(n,T)$, the maps $f(S) \to f(\{i\})$ exhibit $f(S)$ as a p-product of the objects $\{f(\{i\})\}_{i\in S}$.

We observe that (b) is equivalent to the following condition on a functor $f : \mathbf{N}(P(n,T)) \to \mathcal{M}$.

(b') For all $S \in P(n,T)$ the following holds: for each object $X \in \mathcal{M}$ and each map $\varphi : p(X) \to p(f(S))$ in Δ^1, the maps $\beta_i : f(S) \to f(\{i\})$ for $i \in S$ determine an equivalence

$$\mathrm{Hom}^\varphi_{\mathcal{M}}(X, f(S)) \longrightarrow \prod_{i\in S} \mathrm{Hom}^{p(\beta_i)\circ\varphi}_{\mathcal{M}}(X, f(\{i\}))$$

where $\mathrm{Hom}^\varphi_{\mathcal{M}}(X, f(S)) \subset \mathrm{Hom}_{\mathcal{M}}(X, f(S))$ is the union of those connected components that lie over φ, and $\mathrm{Hom}^{p(\alpha_i)\circ\varphi}_{\mathcal{M}}(X, f(\{i\}))$ is defined similarly.

It is then easy to see that if $\alpha : f \to f'$ is a coCartesian edge of $\widetilde{\mathcal{O}_g^\otimes}$ and $f \in \mathcal{O}_g^\otimes$, then $f' \in \mathcal{O}_g^\otimes$ so that $\mathcal{O}_g^\otimes \to \mathrm{MComm}^\otimes$ is again a coCartesian fibration.

Proposition 2.2.0.12. *Let $g : \mathcal{D} \to \mathcal{C}$ be a functor that preserves finite products, and let $q : \mathcal{O}_g^{\otimes} \to \mathrm{MComm}^{\otimes}$ be the coCartesian fibration of construction 2.2.0.11. Then q exhibits the ∞-category $\mathcal{C}$ as tensored over the Cartesian symmetric monoidal ∞-category $\mathcal{D}^{\times}$ (that is, q is a coCartesian fibration of ∞-operads, and we have identifications $\mathcal{D}^{\times} \simeq \mathcal{O}_g^{\otimes} \times_{\mathrm{MComm}^{\otimes}} \mathrm{Comm}^{\otimes}$ and $\mathcal{C} \simeq (\mathcal{O}_g^{\otimes})_{(\langle 1\rangle,\{1\})}$) if and only if $\mathcal{D}$ admits finite products and for every pair $(D, C) \in \mathcal{D} \times \mathcal{C}$, the pair $(g(D), C)$ admits a product in $\mathcal{C}$.*

Proof. First we note that conditions (a) and (b) imply that $(\mathcal{O}_g^{\otimes})_{\langle 0\rangle}$ consists of p-final objects of $\mathcal{M}$ that lie in $p^{-1}(\{1\})$. Using that g preserves finite products and invoking [Lur17b], prop. 4.3.1.10, such p-final objects are precisely final objects in $\mathcal{D}$, so we may assume that $\mathcal{D}$ has a final object. In this case, the ∞-category $(\mathcal{O}_g^{\otimes})_{(\langle 1\rangle,\{\varnothing\})}$ is identified with the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{D})$ spanned by edges $X \to Y$ where Y is final, so that $(\mathcal{O}_g^{\otimes})_{(\langle 1\rangle,\{\varnothing\})} \simeq \mathcal{D}$. The ∞-category $(\mathcal{O}_g^{\otimes})_{(\langle 1\rangle,\{1\})}$ is identified with the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{M})$ spanned by edges $X' \to Y$ where $X' \in p^{-1}(\{0\})$ and $Y \in p^{-1}(\{1\})$ is p-final. Since g preserves final objects, we may identify $(\mathcal{O}_g^{\otimes})_{(\langle 1\rangle,\{1\})}$ with $\mathcal{C} \simeq p^{-1}(\{0\})$. Now the map q is a coCartesian fibration of ∞-operads precisely if for every $(\langle n\rangle, T) \in \mathrm{MComm}^{\otimes}$, the inert maps $\rho^i : (\langle n\rangle, T) \to (\langle 1\rangle, \{i\} \cap T)$ determine an equivalence $\phi : (\mathcal{O}_g^{\otimes})_{(\langle n\rangle,T)} \to \prod_{i\in\langle n\rangle^\circ} (\mathcal{O}_g^{\otimes})_{(\langle 1\rangle,\{i\}\cap T)}$. Condition (a) of of construction 2.2.0.11 defines a functor $\nu : \mathbf{N}(P(n, T)) \to \Delta^1$ and we can identify the ∞-category $(\mathcal{O}_g^{\otimes})_{(\langle n\rangle,T)}$ with the full subcategory of $\mathrm{Fun}_{\Delta^1}(\mathbf{N}(P(n, T)), \mathcal{M})$ spanned by functors satisfying condition (b), where the fibre is taken over ν. Let $P_0(n, T) \subset P(n, T)$ be the full subcategory spanned by subsets $\{j\} \in P(n, T)$ on a single element, then a functor $f : \mathbf{N}(P(n, T)) \to \mathcal{M}$ satisfies condition (b) precisely if f is a p-right Kan extension of $f|_{\mathbf{N}(P_0))}$. Using [Lur17b], prop. 4.3.2.15, we see that $(\mathcal{O}_g^{\otimes})_{(\langle n\rangle,T)}$ is equivalent to the full subcategory $\mathcal{E} \subset \mathrm{Fun}_{\Delta^1}(\mathbf{N}(P_0(n, T)), \mathcal{M})$ spanned by functors that admit finite p-products, and under this equivalence, the functor ϕ is identified with the inclusion $i : \mathcal{E} \subset \mathrm{Fun}_{\Delta^1}(\mathbf{N}(P_0(n, T)), \mathcal{M})$. The inclusion i is an equivalence for all pairs $(\langle n\rangle, T)$ if and only if every finite collection of objects $(M_1, \ldots, M_k)$ in $\mathcal{M}$ that contains at most one object in $p^{-1}(\{0\})$ admits a p-product, and this p-product lies in $p^{-1}(\{0\})$ if the collection $(M_1, \ldots, M_k)$ contains an object in $p^{-1}(\{0\})$ and lies in $p^{-1}(\{1\})$ otherwise. Using that g preserves finite products and invoking [Lur17b], prop. 4.3.1.10 and cor. 4.3.1.11, this is equivalent to demanding that the fibre $p^{-1}(\{1\}) \simeq \mathcal{D}$ admits finite products and that for a finite collection $(D_1, \ldots, D_k, C) \in \mathcal{D} \times \ldots \times \mathcal{D} \times \mathcal{C}$, the collection $(g(D_1), \ldots, g(D_k), C)$ admits a product in $\mathcal{C}$. As g preserves finite products, the collection $(g(D_1), \ldots, g(D_k), C)$ admits a product if and only if the pair $(g(D_1 \times \ldots \times D_k), C)$ admits a product, so we may assume $k = 1$.
The proof also shows that if $T = \varnothing$, functors $f : \mathbf{N}(P(n, T)) \to \mathcal{M}$ satisfying (a) and (b) are equivalent to functors $f : \mathbf{N}(P(n, \varnothing)) \to \mathcal{D}$ that are right Kan extensions of $\mathbf{N}(P_0(n, \varnothing)) \to \mathcal{D}$. Parsing the construction of [Lur17a], section 2.4.1, we obtain an identification $\mathcal{D}^{\times} \simeq \mathcal{O}_g^{\otimes} \times_{\mathrm{MComm}^{\otimes}} \mathrm{Comm}^{\otimes}$. □

Corollary 2.2.0.13. *Let $\mathcal{C}$ be a presentable ∞-category. If colimits in $\mathcal{C}$ are universal, then for each object $C \in \mathcal{C}$ the coCartesian fibration $q : \mathcal{O}_g \to \mathrm{MComm}^{\otimes}$ associated to the functor $g : \mathcal{C} \to \mathcal{C}_{/C}$ right adjoint to the right fibration $\mathcal{C}_{/C} \to \mathcal{C}$ exhibits $\mathcal{C}_{/C}$ as tensored, cotensored and enriched over $\mathcal{C}$.*

Proof. Since g is a right adjoint, it follows from proposition 2.2.0.12 that q exhibits $\mathcal{C}_{/C}$ as tensored over $\mathcal{C}$. Unwinding the definitions, we observe that the tensoring is given by the functor $\_ \otimes \_ : \mathcal{C} \times \mathcal{C}_{/C} \to \mathcal{C}_{/C}$ which takes $(X, D \to C) \mapsto X \times D \to C$. Under the assumption that colimits are universal, the functor $\_ \otimes (D \to C)$ preserves colimits, which implies that for every $D' \to C$ the presheaf $\mathcal{C}^{op} \to \mathcal{S}$, $C' \mapsto \mathrm{Hom}_{\mathcal{C}_{/C}}(C' \times D, D')$ is representable by a *mapping object* $\mathsf{Map}_C(D, D')_{\mathcal{C}}$ (a Weil restriction), so that q exhibits $\mathcal{C}_{/C}$ as enriched over $\mathcal{C}$, with morphism object $\mathsf{Mor}_{\mathcal{C}_{/C}}(D, D') = \mathsf{Map}_C(D, D')_{\mathcal{C}}$. For every $C' \in \mathcal{C}$, the functor $\mathcal{C}_{/C} \to \mathcal{C}_{/C}$ given by taking products with C' also preserves colimits, so that $\mathcal{C}_{/C}$ is also cotensored over $\mathcal{C}$, and for a pair $(C', D \to C)$, the exponential object is given by the internal mapping object $\mathsf{Map}_{\mathcal{C}_{/C}}(C' \times C, D)$ in $\mathcal{C}_{/C}$. □

Remark 2.2.0.14. It follows from the proof that for every morphism $D \to C$ of $\mathcal{C}$, the object $\mathsf{Map}_C(D, D)_{\mathcal{C}}$ is the object in $\mathcal{C}$ of morphisms of the full $\mathcal{C}$-enriched subcategory of $\mathcal{C}_{/C}$ on the single object $D \to C$, which lifts canonically to an associative algebra object of $\mathcal{C}$.

Remark 2.2.0.15. One could consider general ∞-categories enriched in $\mathcal{X}$, or more generally, (∞, n)-categories enriched in $\mathcal{X}$, in the sense of [GH15]. For certain ∞-topoi, this theory can be realized using category objects within $\mathcal{X}$. It is easy to see that an ∞-topos $\mathcal{X}$ is an *absolute distributor* in the sense of [Lur09] if an only if $\mathcal{X}$ is locally of constant shape and the shape of $\mathcal{X}$ is trivial, in the sense of [Lur17a], appendix A.1. Under these conditions, the results of [Hau15], section 7, show that the ∞-category $\mathsf{CSS}(\mathcal{X})$ of complete Segal space objects $X_\bullet \in \mathrm{Fun}(\mathbf{N}(\boldsymbol{\Delta}^{op}), \mathcal{X})$ such that X_0 lies in the image of the functor $\pi^* : \mathcal{S} \to \mathcal{X}$ is a model for the ∞-category of $\mathcal{X}$-enriched ∞-categories. As absolute distributors are stable under taking complete Segal space objects, iterating this construction yields ∞-categories of $\mathcal{X}$-enriched (∞, n)-categories. Thus, $\mathcal{X}$-enriched (∞, n)-categories may be viewed as stacks of (∞, n)-categories on $\mathcal{X}$ whose underlying sheaf of objects is constant. The ∞-topos SmSt of smooth stacks is an example of an ∞-topos for which this procedure can be performed.

2.2.1 Groupoid actions in ∞-topoi

In this work, we will at various points switch perspectives between viewing the objects of study as ∞-topoi *themselves*, equipped with some geometric structure encoded as sheaf of algebras, and as objects internal to some specific ∞-topoi we construct (namely, the ∞-topoi that arise as sheaves on some variety of derived manifolds). In the latter case, our focus will be on groupoid objects in ∞-topoi satisfying some conditions analogous to those satisfied by Lie groupoids. Since groupoids are effective, they are determined by the quotient map $G_0 \to G_{-1}$, and it will prove to be advantageous to have an understanding of the functor that the quotient object G_{-1} represents.

Definition 2.2.1.1. Let $\mathcal{C}$ be an ∞-category, and let $G_\bullet$ be a simplicial object in $\mathcal{C}$. A morphism $G'_\bullet \to G_\bullet$ of simplicial objects in $\mathcal{C}$ *exhibits $G'_\bullet$ as a $G_\bullet$-torsor* if the following conditions are satisfied.

(*i*) $G_\bullet$ is a groupoid object of $\mathcal{C}$.

(*ii*) For each finite ordinal $[n]$ and each $k \in [n]$, the diagram

$$\begin{array}{ccc} G'_n = G'_\bullet([n]) & \longrightarrow & G'_\bullet(\{k\}) = G'_0 \\ \downarrow & & \downarrow \\ G_n = G_\bullet([n]) & \longrightarrow & G_\bullet(\{k\}) = G_0 \end{array}$$

is a pullback square.

We denote the full subcategory of $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{C})_{/G_\bullet}$ spanned by $G_\bullet$-torsors by $G_\bullet\mathsf{Tor}(\mathcal{C})$, and the full subcategory of $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}) \times \Delta^1, \mathcal{C})$ spanned by maps $G'_\bullet \to G_\bullet$ that exhibit $G'_\bullet$ as a $G_\bullet$-torsor by $\mathsf{Tor}_{\mathcal{C}}$.

Remark 2.2.1.2. In an arbitrary ∞-category $\mathcal{C}$, there is the notion of a left/right action object of an associative monoid in $\mathcal{C}$ as definition 4.2.2.2 of [Lur17a], and the ∞-category of left action objects is naturally equivalent to the ∞-category of algebras for the ∞-operad controlling pairs of an associative algebra A and a left A-module. In case the underlying monoid is grouplike, the definition above generalizes action objects to groupoids.

Remark 2.2.1.3. If $G'_\bullet \to G_\bullet$ is a $G_\bullet$-torsor in $\mathcal{C}$, it is obvious that for all maps $[n] \to [m]$ of finite ordinals, the associated diagram

$$\begin{array}{ccc} G'_\bullet([n]) & \longrightarrow & G'_\bullet([m]) \\ \downarrow & & \downarrow \\ G_\bullet([n]) & \longrightarrow & G_\bullet([m]) \end{array}$$

is a pullback square, that is, a $G_\bullet$-torsor $G'_\bullet \to G_\bullet$ is a Cartesian transformation of simplicial objects. Using characterization ($4'''$) of [Lur17b], prop. 6.1.2.6, it is easy to see that if $G'_\bullet \to G_\bullet$ is a $G_\bullet$-torsor, then $G'_\bullet$ is also a groupoid object, so we have a full subcategory inclusion $G_\bullet\mathsf{Tor} \subset \mathsf{Gpd}(\mathcal{C})_{/G_\bullet}$.

Definition 2.2.1.4. Let $G_\bullet$ be a groupoid object in $\mathcal{C}$.

(1) The ∞-category of *$G_\bullet$-torsor structures on X*, denoted $G_\bullet\mathsf{Tor}_X$ is the fibre at $X \in \mathcal{C}$ of the functor

$$G_\bullet\mathsf{Tor} \subset \mathsf{Gpd}(\mathcal{C})_{/G_\bullet} \longrightarrow \mathsf{Gpd}(\mathcal{C}) \overset{\mathrm{ev}_{\{[0]\}}}{\longrightarrow} \mathcal{C}.$$

(2) Suppose $\mathcal{C}$ admits geometric realizations of simplicial objects, then the ∞-category of *$G_\bullet$-torsors with base X*, denoted $G_\bullet\mathsf{Tor}(X)$ and defined up to a contractible space of choices, is the fibre at $X \in \mathcal{C}$ of the functor

$$G_\bullet\mathsf{Tor} \subset \mathsf{Gpd}(\mathcal{C})_{/G_\bullet} \longrightarrow \mathsf{Gpd}(\mathcal{C}) \overset{\mathrm{colim}}{\longrightarrow} \mathcal{C}.$$

Remark 2.2.1.5. Suppose that $\mathcal{C}$ is presentable and has universal colimits. Let $G_\bullet$ be a group object in $\mathcal{C}$ viewed as an associative monoid, then we find that $G_\bullet\mathsf{Tor}_X$ is equivalent to the pullback $\{X\} \times_{\mathcal{C}} \mathsf{LMod}(\mathcal{C}) \times_{\mathsf{Mon}(\mathcal{X})} \{G_\bullet\}$. As $\mathcal{C}$ is enriched over itself, one can apply [Lur17a], cor. 4.7.1.41 and 4.7.1.42 to see that the projection $\mathsf{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{X\} \to \mathsf{Mon}(\mathcal{C})$ is a right fibration, representable by the endomorphism algebra $\mathsf{Mor}_{\mathcal{C}}(X,X)$; it follows that we have a canonical equivalence $G_\bullet\mathsf{Tor}_X \simeq \mathrm{Hom}_{\mathsf{Mon}(\mathcal{C})}(G_\bullet, \mathsf{Mor}_{\mathcal{C}}(X,X))$. Since $G_\bullet$ is grouplike and the grouplike monoids form a coreflective subcategory of the ∞-category of monoids, we have a canonical equivalence of ∞-categories

$$G_\bullet\mathsf{Tor}_X \simeq \mathrm{Hom}_{\mathsf{Grp}(\mathcal{C})}(G_\bullet, \mathsf{Aut}(X)),$$

where $\mathsf{Aut}(X) \coloneqq \mathsf{Map}_{\mathcal{C}}(X,X)^{\simeq}$, the automorphism group object of X.

Remark 2.2.1.6. A map $f : X_\bullet \to Y_\bullet$ of simplicial objects in an ∞-topos $\mathcal{X}$ is a *realization fibration* [Rez14] if for any map of simplicial objects $Z_\bullet \to Y_\bullet$ the canonical map

$$|X_\bullet \times_{Y_\bullet} Z_\bullet| \longrightarrow |X_\bullet| \times_{|Y_\bullet|} |Z_\bullet|$$

is an equivalence. It is easy to see that a Cartesian transformation of simplicial objects is a realization fibration. In particular, a map $G_\bullet'' \to G_\bullet'$ of $G_\bullet$-torsors for some groupoid object $G_\bullet$ is a realization fibration. Now suppose that G is a group object in an ∞-topos $\mathcal{X}$ and suppose that X and Y are objects carrying a G-action; that is, we are given maps $G \to \mathsf{Aut}(X)$ and $G \to \mathsf{Aut}(Y)$ of group objects in $\mathcal{X}$ (see remark 2.2.1.5). Let $f : X \to Y$ be a map, then the space of extensions of f to a G-equivariant map $X_\bullet \to Y_\bullet$, where $X_\bullet$ and $Y_\bullet$ are the associated action groupoids, is naturally equivalent to the space of extensions of f that exhibit $Y_\bullet$ as an $X_\bullet$-torsor. Thus, we conclude that for any map of simplicial objects $Z_\bullet \to Y_\bullet$, the canonical map $|X_\bullet \times_{Y_\bullet} Z_\bullet| \to |X_\bullet| \times_{|Y_\bullet|} |Z_\bullet|$ is an equivalence. In particular, if Z carries a G-action and $Z \to Y$ is a G-equivariant map, then $X \times_Y Z$ is a G-torsor and the canonical map

$$[X \times_Y Z/G] \longrightarrow [X/G] \times_{[Y/G]} [Z/G]$$

is an equivalence. If Z carries the trivial G-action, this reduces to an equivalence $[X \times_Y Z/G] \simeq [X/G] \times_{[Y/G]} Z \times \mathrm{B}G$. If $Z = *$, a final object, then a map $y : * \to Y$ determines a group object $(G_y)_\bullet = * \times_Y Y_\bullet$, the *isotropy group of Y at y*, so that the pullback $X_\bullet \times_{Y_\bullet} *$ carries the structure of a G_y-torsor and we have an equivalence $[X \times_Y */G_y] \simeq [X/G] \times_{[Y/G]} \mathrm{B}G_y$ (which coincides with the previous equivalence if $y : * \to Y$ is G-invariant).

Remark 2.2.1.7. Aside from group objects, the notion of a $G_\bullet$-torsor structure on $X \in \mathcal{C}$ subsumes a variety of geometric structures. For instance, if we let $\mathcal{C} = \mathsf{d}C^\infty\mathsf{St}$, the ∞-topos of derived C^∞-stacks, then the fundamental theorem of (parametrized) derived deformation theory [Lur11e; Nui19; CG18] shows that we can construct for each derived Lie algebroid A on a quasi-smooth derived manifold X (such as the tangent complex of X) a formal thickening $X \to X_A$ which exhibits the groupoid X_A as the *formal integration of A* (in case $A = \mathbb{T}_X$, the tangent complex, this is the de Rham stack of Simpson. Let $E \in \mathsf{Perf}(X)$ be a perfect complex on X and let $\mathbb{V}(E) \to X$ denote its relative spectrum, then we can identify X_A-torsor structures on a map $f : \mathbb{V}(E) \to X$ with flat A-connections on the complex E.

Using pasting of pullback squares, it is easy to see that all morphism in $G_\bullet\mathsf{Tor}_X$ are equivalences. If $\mathcal{C}$ is an ∞-topos, this is also true for $G_\bullet\mathsf{Tor}(X)$.

Proposition 2.2.1.8. *Let $\mathcal{X}$ be an ∞-topos, then for all $X \in \mathcal{X}$, there is a canonical equivalence*

$$\mathrm{Hom}_{\mathcal{X}}(X, G_{-1}) \simeq G_\bullet\mathsf{Tor}(X),$$

of ∞-categories, where $G_{-1} = \mathrm{colim}_{\mathbf{N}(\boldsymbol{\Delta}^{op})} G_\bullet$; in particular, $G_\bullet\mathsf{Tor}(X)$ is an ∞-groupoid.

Remark 2.2.1.9. See [TV06] for a proof in the setting of model topoi, and [NSS15] for a treatment in the special case that $G_\bullet$ is a group object.

Remark 2.2.1.10. It follows from the proposition that if G_{-1} is an n-truncated object of $\mathcal{X}$, then $G_\bullet\mathsf{Tor}(X) \simeq \mathrm{Hom}_{\mathcal{X}}(X, G_{-1})$ is an n-groupoid for any $X \in \mathcal{X}$. In particular, if A is a discrete abelian group object in $\mathcal{X}$, that is, an abelian group object in $\mathrm{Disc}(\mathcal{X})$, then $B^n A$, the n'th delooping of A which we can also write as the Eilenberg-MacLane object $K(A, n)$, is n-truncated. If $n \geq 2$, we can then identify the n-groupoid of $B^n A$-torsors with base X with the space of *n-gerbes on X banded by A*.

To prove the proposition, we pass to a larger model for $G_\bullet$-torsors.

Definition 2.2.1.11. Let $\mathcal{X}$ be an ∞-topos and let

$$\mathsf{Gpd}^+(\mathcal{X}) \subset \mathrm{Fun}(\mathbf{N}(\boldsymbol{\Delta}_+)^{op}, \mathcal{X})$$

be the full subcategory spanned by groupoid resolutions, that is, simplicial resolutions $G_\bullet^+$ such that the simplicial object $G_\bullet := G_\bullet^+|_{\mathbf{N}(\boldsymbol{\Delta}^{op})}$ is a groupoid. Then the full subcategory $G_\bullet^+\mathsf{Action} \subset \mathsf{Gpd}^+(\mathcal{X})_{/G_\bullet}$ spanned by those maps of augmented simplicial objects $G_\bullet'^+ \to G_\bullet^+$ such that the diagram

$$\begin{tikzcd} G_0'^+ \arrow[r] \arrow[d] & G_{-1}'^+ \arrow[d] \\ G_0^+ \arrow[r] & G_{-1}^+ \end{tikzcd}$$

is a pullback square, is the *∞-category of $G_\bullet^+$-action objects*. We denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathsf{Gpd}^+(\mathcal{X}))$ spanned by action objects by $\mathsf{Action}_{\mathcal{X}}$

Proposition 2.2.1.12. *The restriction map* $\mathsf{Gpd}^+(\mathcal{X}) \to \mathsf{Gpd}(\mathcal{X})$ *induces a trivial Kan fibration* $G_\bullet^+\mathsf{Action} \to G_\bullet\mathsf{Tor}$.

Proof. First, we show that for $\overline{\alpha} : G_\bullet^{'+} \to G_\bullet^+$ a $G_\bullet^+$-action object, the restriction $G_\bullet' \to G_\bullet$ exhibits a $G_\bullet$ torsor. An augmented simplicial object $U_\bullet^+$ is a groupoid resolution if and only if $U_\bullet^+$ is a right Kan extension of $U_\bullet^+|_{\Delta^{\{0,-1\}}}$. Viewing $\overline{\alpha}$ as a functor $\Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op}) \to \mathcal{X}$, we have a diagram

$$\begin{tikzcd} \Delta^1 \times \Delta^{\{0,-1\}} \arrow[r, "\overline{\alpha}|_{\Delta^{\{0,-1\}}}"] \arrow[d, hook] & \mathcal{X} \\ \Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op}) \arrow[ur, "\overline{\alpha}"'] & \end{tikzcd}$$

Restricted to both $\{0\}$ and $\{1\}$ in Δ^1, the diagram above is a right Kan extension, so the diagram itself is a right Kan extension, and by assumption, the horizontal functor $\overline{\alpha}|_{\Delta^{\{0,-1\}}}$ is a pullback, so $\overline{\alpha}$ is a right Kan extension of $\overline{\alpha}|_{\Lambda_2^2}$, where $\Lambda_2^2 \subset \Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})$ is the full subcategory determining the cospan $G_0^+ \to G_{-1}^+ \leftarrow G_{-1}^{'+}$ ([Lur17b], prop. 4.3.2.8 and 4.3.2.9). Let K denote the full subcategory $\{1\} \times \mathbf{N}(\mathbf{\Delta}_+^{op}) \coprod_{\{1\}\times\{[-1]\}} \Delta^1 \times \{[-1]\} \subset \Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})$ which contains the full subcategory $\Lambda_2^2 \subset \Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})$, then it follows from the arguments above that $\overline{\alpha}$ is a right Kan extension of $\overline{\alpha}|_K$; now let $J := K \times_{\Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})} \Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})_{(0,[n])/}$, then J is the full subcategory of $\Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})_{(0,[n])/}$ spanned by the maps

(*a*) $(0,[n]) \to (0,[-1])$,

(*b*) $(0,[n]) \to (1,[-1])$,

(*c*) all the maps $(0,[n]) \to (1,[m])$ for $[m] \in \mathbf{N}(\mathbf{\Delta}^{op})$.

Let $J' \subset J$ be the full subcategory spanned by the morphisms (a) and (b) and the map $(0,[n]) \to (1,[n])$ corresponding to the identity on $[n]$. Note that there is an isomorphism $J' \cong \Lambda_2^2$. Consider for each of the maps $(0,[n]) \to (i,[m])$, $i \in \{0,1\}$ in J, the full subcategory $J'' \subset \Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op})_{(0,[n])//(i,[m])}$ spanned by compositions $(0,[n]) \to (j,[k]) \to (i,[m])$ where the first map is an object of J' and the composition is an object of J. For $(i,[m]) = (0,[-1])$, we note that J'' is the trivial category, and for each $(1,[m])$, the category J'' has an initial object given by the composition $(0,[n]) \to (1,[n]) \to (1,[m])$ where the first map induces the identity on $[n]$. Using [Lur17b], thm. 4.1.3.1, we deduce that the inclusion $\Lambda_2^2 \cong J' \subset J$ is right cofinal. By definition of right Kan extension we have $G_\bullet^{'+}([n]) \simeq \lim_{(i\times[m])\in J} \overline{\alpha}(i \times [m])$, so we conclude that the diagram

$$\begin{tikzcd} G_\bullet^{'+}([n]) \arrow[r] \arrow[d] & G_\bullet^{'+}([-1]) \arrow[d] \\ G_\bullet^+([n]) \arrow[r] & G_\bullet^+([-1]) \end{tikzcd}$$

is a pullback. Using pasting of pullback squares, one sees that $G_\bullet^{'+} \to G_\bullet^+$ is a Cartesian transformation. It follows that restriction to $\mathbf{N}(\mathbf{\Delta}^{op})$ induces a functor $G_\bullet^+\mathsf{Action} \to G_\bullet\mathsf{Tor}$. This functor is obviously a categorical fibration, so we need to show it is an equivalence of ∞-categories. For this, it suffices to show that the projection $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}_+^{op}) \times \Delta^1, \mathcal{X}) \to \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}) \times \Delta^1, \mathcal{X})$ induces a trivial fibration $\mathsf{Action}_\mathcal{X} \to \mathsf{Tor}_\mathcal{X}$. Since all groupoids are effective in the arrow ∞-topos $\mathrm{Fun}(\Delta^1, \mathcal{X})$, we have a trivial fibration $\mathrm{Fun}(\Delta^1, \mathsf{Gpd}_+(\mathcal{X})) \to \mathrm{Fun}(\Delta^1, \mathsf{Gpd}(\mathcal{X}))$ with inverse given by a functor taking colimits. As we have just verified, a map $G_\bullet^{'+} \to G_\bullet^+$ of groupoid resolutions that is a $G_\bullet^+$-action on $G_\bullet^{'+}$ restricts to a Cartesian transformation of simplicial objects. Conversely, if $\overline{\alpha} : G_\bullet^{+'} \to G_\bullet^+$ is a natural transformation of colimit diagrams and $\overline{\alpha}|_{\mathbf{N}(\mathbf{\Delta}^{op})}$ is a Cartesian transformation, then $\overline{\alpha}$ is a Cartesian transformation since $\mathcal{X}$ is an ∞-topos, that is, the diagram

$$\begin{tikzcd} G_0^{+'} \arrow[r] \arrow[d] & G_{-1}^{+'} \arrow[d] \\ G_0^+ \arrow[r] & G_{-1}^+ \end{tikzcd}$$

is a pullback square. □

Proof of proposition 2.2.1.8. Let $G_\bullet \in \mathsf{Gpd}(\mathcal{X})$ and let $G_\bullet^+$ be a corresponding groupoid resolution, unique up to contractible ambiguity. We have a diagram

$$\begin{tikzcd} & G_\bullet^+\mathsf{Action} \arrow[dl, "r"'] \arrow[dr, "p"] & \\ \mathcal{X}_{/G_{-1}^+} & & G_\bullet\mathsf{Tor} \end{tikzcd}$$

where r restricts along $\Delta^{\{0,-1\}} \hookrightarrow \mathbf{N}(\boldsymbol{\Delta}^{op})_+$ and p is the trivial fibration of proposition 2.2.1.12. We now show that the functor r is also a trivial fibration. As effective groupoids are right Kan extensions along $\Delta^{\{0,-1\}}$, the restriction functor $\mathsf{Gpd}^+(\mathcal{X}) \to \mathrm{Eff}(\mathcal{X}) \subset \mathrm{Fun}(\Delta^1, \mathcal{X})$ taking values in the full subcategory spanned by effective epimorphisms is a trivial fibration. Since effective epimorphisms are stable under pullbacks, the functor $\mathrm{ev}_{\{1\}} : \mathrm{Eff}(\mathcal{X}) \to \mathcal{X}$ is a Cartesian fibration. Let u denote the map $G_0^+ \to G_{-1}^+$, then the restriction functor $G_\bullet^+\mathsf{Action} \to \mathrm{Eff}(\mathcal{X})_{/u}$ is a trivial fibration onto the full subcategory $\mathrm{Eff}'(\mathcal{X})_{/u} \subset \mathrm{Eff}(\mathcal{X})_{/u}$ spanned by pullback squares. Notice that the induced Cartesian fibration $\mathrm{Eff}'(\mathcal{X})_{/u} \to \mathcal{X}_{/G_{-1}^+}$ is in fact a right fibration. Indeed, a morphism in $\mathrm{Eff}'(\mathcal{X})_{/u}$ depicted as a diagram $\Delta^2 \times \Delta^1 \to \mathcal{X}$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & G_0^+ \arrow[d] \\ Y' \arrow[r] & Y \arrow[r] & G_{-1}^+ \end{tikzcd}$$

is $\mathrm{ev}_{\{1\}}$-Cartesian if and only if the left square is a pullback, but by assumption the right square and large rectangle are pullbacks. It follows that $G_\bullet^+\mathsf{Action} \to \mathcal{X}$ is a right fibration, which is representable because it has a final object, the tautological $G_\bullet^+$-action on $G_\bullet$ itself. This implies that the functor $r : G_\bullet^+\mathsf{Action}(\mathcal{X}) \to \mathcal{X}_{/G_{-1}^+}$ is a trivial fibration. Consequently, we have for each object $X \in \mathcal{X}$ a canonical equivalence

$$\mathrm{Hom}_{\mathcal{X}}(X, G_{-1}) \simeq G_\bullet\mathsf{Tor}(X)$$

of ∞-categories. $\square$

The arguments above also yield the following useful result, that can be used to construct gauge groups for arbitrary groupoid actions in general ∞-topoi.

Corollary 2.2.1.13. *Let $\mathcal{X}$ be an ∞-topos and let $G_\bullet$ be a groupoid object in $\mathcal{X}$. Then the ∞-category $G_\bullet\mathsf{Tor}$ is canonically tensored, cotensored and enriched over $\mathcal{X}$. Moreover, for two $G_\bullet$-torsors $P_\bullet$ and $P'_\bullet$, the morphism object in $\mathcal{X}$ is given up to equivalence by $\mathsf{Map}_{G_{-1}}(P_{-1}, P'_{-1})_{\mathcal{X}}$, where G_{-1} is a colimit of the simplicial object $G_\bullet$, and similarly for P_{-1} and P'_{-1}.*

Proof. Let $g : \mathcal{X} \to \mathcal{X}_{/G_{-1}}$ be a functor taking products with G_{-1} right adjoint to the right fibration $\mathcal{X}_{/G_{-1}} \to \mathcal{X}$. Choose a section s of the trivial fibration $r : G_\bullet^+\mathsf{Action} \to \mathcal{X}_{/G_{-1}}$ and apply proposition 2.2.0.12 to the functor $p \circ s \circ g$, where $p : G_\bullet^+\mathsf{Action} \to G_\bullet\mathsf{Tor}$ is the trivial fibration of proposition 2.2.1.12. The resulting coCartesian fibration $\mathcal{O}^\otimes_{p\circ s\circ g} \to \mathrm{MComm}^\otimes$ is equivalent to $\mathcal{O}_g^\otimes \to \mathrm{MComm}^\otimes$. $\square$

Remark 2.2.1.14. Let $G_\bullet$ be a groupoid object in an ∞-topos $\mathcal{X}$, then for any $P_\bullet \in G_\bullet\mathsf{Tor}$, the object $\mathsf{Mor}_{\mathcal{X}}(P_\bullet, P_\bullet) \simeq \mathsf{Map}_{G_{-1}}(P_{-1}, P_{-1})$ is a monoid in $\mathcal{X}$. The group object $\mathsf{Aut}_{G_\bullet}(P)$ obtained by discarding noninvertible morphisms in $\mathsf{Mor}_{\mathcal{X}}(P_\bullet, P_\bullet)_{\mathcal{X}}$ is familiar when $G_\bullet$ is a group object: it is the gauge group of $P_\bullet$. If $\mathcal{X} = \mathsf{d}C^\infty\mathsf{St}$, the ∞-topos of derived C^∞-stacks with which this work is concerned and G is a compact smooth Lie group, then $P_\bullet$ is represented by an infinite dimensional manifold modelled on nuclear Fréchet spaces. If G is a noncompact Lie group, then $P_\bullet$ is no longer represented by an infinite dimensional manifold in any reasonable sense, yet $P_\bullet$ is still *smooth* in the sense that the counit of the adjunction $\mathsf{Shv}(\mathsf{Mfd}) \leftrightarrows \mathsf{d}C^\infty\mathsf{St}$ applied to this object is an equivalence.

Let us give one final application of the constructions in this section. There are standard notions of a *vector bundle over an orbifold*, and more generally, of a *vector bundle groupoid* (VB-groupoid) [MM03; Mac05]. If $G = G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$ is a Lie groupoid, then a vector bundle over G consists of a vector bundle $\pi : E \to G_0$ together with an equivalence

$$\alpha : s^*(E) \xrightarrow{\simeq} t^*(E)$$

of vector bundles over G_1. We can think of α as a smooth section of the bundle of homomorphisms $\mathrm{Hom}(s^*E, t^*E) \to G_1$, that is, as a family of linear isomorphisms

$$\alpha(e) : E_{s(e)} \longrightarrow E_{t(e)},$$

depending smoothly on the morphisms in G, and we require α to satisfy that

$$\alpha(x \xrightarrow{\mathrm{id}} x) : E_x \longrightarrow E_x$$

is the identity and if $e_1, e_2 \in G_1$ satisfy $s(e_1) = t(e_2)$, then

$$\alpha(e_2) \circ \alpha(e_1) : E_{s(e_1)} \longrightarrow E_{t(e_1)=s(e_2)} \longrightarrow E_{t(e_2)}$$

is the map $\alpha(m(e_1,e_2))$ where $m : G_1 \times_{G_0} G_1 \to G_1$ is the groupoid multiplication. If such an α is given, then the diagram

$$\begin{tikzcd} s^*E \arrow[r, shift left, "\pi^* s"] \arrow[r, shift right, "\pi^* t\circ\alpha"'] \arrow[d] & E \arrow[d, "\pi"] \\ G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$

commutes and $s^*E \rightrightarrows E$ is a again a Lie groupoid. We can formulate this in general ∞-topoi for torsors with base being an arbitrary diagram in $\mathcal{X}$.

Definition 2.2.1.15. Let $G_\bullet$ be a groupoid object in an ∞-topos $\mathcal{X}$, let K be a small simplicial set and let $U : K \to \mathcal{X}$ be a diagram in $\mathcal{X}$. The ∞-category of *$G_\bullet$-torsors with base U*, denoted $G_\bullet\mathsf{Tor}_U^K$ and defined up to a contractible space of choices, is the fibre at U of the functor

$$\mathrm{Fun}(K, G_\bullet\mathsf{Tor}) \xrightarrow{\_\circ\mathrm{colim}} \mathrm{Fun}(K, \mathcal{X}).$$

Proposition 2.2.1.16. *Let $G_\bullet^+$ be a groupoid resolution associated to $G_\bullet$. Then there are canonical equivalences of ∞-categories*

$$G_\bullet\mathsf{Tor}_U^K \simeq \mathrm{Fun}(K, \mathcal{X}_{/G_{-1}^+}) \times_{\mathrm{Fun}(K,\mathcal{X})} \{U\} \simeq \mathrm{Hom}_{\mathcal{X}}(\mathrm{colim}_{k\in K} U(k), G_{-1}^+) \simeq \lim_{k\in K^{op}} G_\bullet\mathsf{Tor}_{U(k)}.$$

Proof. The proof of proposition 2.2.1.8 shows that $G_\bullet\mathsf{Tor}_U^K$ is canonically equivalent to the ∞-category $\mathrm{Fun}(K, \mathcal{X}_{/G_{-1}^+}) \times_{\mathrm{Fun}(K,\mathcal{X})} \{U\}$, which in turn is isomorphic to the space $S = \mathrm{Fun}_K(K, K \times_{\mathcal{X}} \mathcal{X}_{/G_{-1}^+})$ of sections of the right fibration $K \times_{\mathcal{X}} \mathcal{X}_{/G_{-1}^+} \to K$. Let $\overline{U} : K^{\triangleright} \to \mathcal{X}$ be a colimit diagram extending K, then the space of sections T of the right fibration $K^{\triangleright} \times_{\mathcal{X}} \mathcal{X}_{/G_{-1}^+} \to K^{\triangleright}$ is equivalent to S: indeed, the restriction map $T \to S$ is a right fibration whose fibre over a map $K \to K \times_{\mathcal{X}} \mathcal{X}_{/G_{-1}^+}$ can be identified with the space of lifts

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathcal{X}_{/G_{-1}^+} \arrow[d, "p"] \\ K^{\triangleright} \arrow[r, "\overline{U}"'] \arrow[ur, dotted] & \mathcal{X} \end{tikzcd}$$

As $\overline{U}$ is a colimit diagram and p is a representable right fibration, every such lift is a p-left Kan extension, so the space of such lifts is contractible. As the inclusion of the cone point $\{*\} \hookrightarrow K^{\triangleright}$ is right anodyne, the space T is equivalent to the space $\mathrm{Fun}_{K^{\triangleright}}(\{*\}, K^{\triangleright} \times_{\mathcal{X}} \mathcal{X}_{/G_{-1}^+}) \simeq \mathrm{Hom}_{\mathcal{X}}(\overline{U}(*), G_{-1}^+)$. For the last equivalence, we note that proposition 2.2.1.8 shows that $G_\bullet\mathsf{Tor}_U^K$ is canonically equivalent to the ∞-category $\mathrm{Fun}(K, G_\bullet^+\mathsf{Action}) \times_{\mathrm{Fun}(K,\mathcal{X})} \{U\}$, which in turn is isomorphic to the space of sections of the right fibration $K \times_{\mathcal{X}} G_\bullet^+\mathsf{Action} \to K$. This space is identified with the limit of the functor

$$K^{op} \longrightarrow \mathcal{X}^{op} \xrightarrow{G_\bullet\mathsf{Tor}_-} \mathcal{S}$$

via [Lur17b], corollary 3.3.3.2 □

Lemma 2.2.1.17. *Let $G_\bullet$ be a groupoid object in an ∞-topos $\mathcal{X}$, let K be a small simplicial set and let $V : K \to G_\bullet\mathsf{Tor}$ be a $G_\bullet$-torsor with base $U : K \to \mathcal{X}$. Consider the induced functor $V : K \to G_\bullet^+\mathsf{Action}$, which may be viewed as a diagram*

$$K^{\triangleright} \times \mathbf{N}(\mathbf{\Delta}_+^{op}) \longrightarrow \mathcal{X}.$$

For each morphism $e : \Delta^1 \to \mathbf{N}(\mathbf{\Delta}_+^{op})$, the induced functor $K^{\triangleright} \times \Delta^1 \to \mathcal{X}$ is a Cartesian transformation.

Proof. We need to show that for each morphism $e' : \Delta^1 \to K^{\triangleright}$, the square $\Delta^1 \times \Delta^1 \to \mathcal{X}$ induced by V is Cartesian. This is obvious from fact that $\Delta^1 \times \mathbf{N}(\mathbf{\Delta}_+^{op}) \to \mathcal{X}$ is morphism of $G_\bullet^+$-action objects. □

Corollary 2.2.1.18. *Let $U_\bullet$ and $G_\bullet$ be groupoid objects in an ∞-topos $\mathcal{X}$ and let $V : \mathbf{N}(\mathbf{\Delta}^{op}) \to G_\bullet\mathsf{Tor}$ be a G-torsor with base $U_\bullet$. Then for each $[n] \in \mathbf{N}(\mathbf{\Delta}^{op})$, the induced functor*

$$V : \mathbf{N}(\mathbf{\Delta}^{op}) \longrightarrow G_\bullet\mathsf{Tor} \xrightarrow{\mathrm{ev}_{\{[n]\}}} \mathcal{X}$$

is a $U_\bullet$-torsor.

2.2.2 Grothendieck topologies

We discuss some generalities on Grothendieck topologies and sheaves, taken from [TV04] and [Lur17b].

Definition 2.2.2.1. Let $\mathcal{C}$ be an ∞-category, then we say that *sieve* on $C \in \mathcal{C}$ is a subobject of $j(C) \in \mathsf{PShv}(\mathcal{C})$, where $j : \mathcal{C} \hookrightarrow \mathsf{PShv}(\mathcal{C})$ denotes the Yoneda embedding.
A Grothendieck topology on a small ∞-category consists of a collection of sieves $\{U \hookrightarrow j(C)\}$ for each object $C \in \mathcal{C}$, called *covering sieves*, such that

(1) $j(C) \to j(C)$ is covering.

(2) If $U \to j(C)$ is covering and $D \to C$ is any map, then $U \times_{j(C)} j(D)$ is covering on D.

(3) If $U \to j(C)$ is a sieve and $V \to j(C)$ is a covering sieve, then if for each $j(D) \to j(C)$ that factors through V, $j(D) \times_{j(C)} U$ is a covering sieve on D, then U is a covering sieve on D.

Let τ be a Grothendieck topology on $\mathcal{C}$, then the full subcategory $\mathsf{Shv}(\mathcal{C}) \subset \mathsf{PShv}(\mathcal{C})$ spanned by objects that are S-local for S the class of covering sieves $U \to j(C)$ are *sheaves*. If $\mathcal{C}$ is small so that $\mathsf{PShv}(\mathcal{C})$ is presentable, localizing at the collection of monomorphisms that are covering sieves induces a sheafification functor $L : \mathsf{PShv}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C})$, which is left exact, so that $\mathsf{Shv}(\mathcal{C})$ is an ∞-topos. Conversely, if $\mathcal{X} \subset \mathsf{PShv}(\mathcal{C})$ is a localization obtained by inverting a strongly saturated class S of morphisms that is stable under pullbacks and generated by a small set of monomorphisms (so that the class S is *topological* in the sense of [Lur17b], defn. 6.1.2.4), then $\mathcal{X}$ coincides with the ∞-topos $\mathsf{Shv}(\mathcal{C})$ for the Grothendieck topology given by those sieves $i : U \hookrightarrow j(C)$ such that Li is an equivalence. It is easy to characterize the τ-coverings in $\mathsf{PShv}(\mathcal{C})$; that is, those maps $X \to Y$ of presheaves that become effective epimorphisms after sheafifying.

Proposition 2.2.2.2. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology, and let $f : X \to Y$ be a map in $\mathsf{PShv}(\mathcal{C})$. Let L denote a sheafification functor, then the following are equivalent.*

(1) *The map Lf is an effective epimorphism in $\mathsf{Shv}(\mathcal{C})$.*

(2) *For each map $j(C) \to Y$ in $\mathsf{PShv}(\mathcal{C})$, there exists a collection of morphism $\{C_i \to C\}$ which generate a covering sieve and a commuting diagram*

$$\begin{tikzcd} \coprod_i j(C_i) \arrow[r] \arrow[d] & j(C) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. [Lur17b] lem. 6.2.4.5 gives (2) ⇒ (1) when X and Y are sheaves, but the proof also holds for presheaves. For the converse, factor $X \to \check{C}(f) \to Y$ as an effective epimorphism followed by a monomorphism and form a pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & j(C) \arrow[d] \\ \check{C}(f) \arrow[r] & Y \end{tikzcd}$$

then $U \to j(C)$ is a subobject, that is, a sieve on C. After sheafifying, $\check{C}(f) \to Y$ becomes an equivalence, so we deduce that U is covering as sheafification is left exact. Choose a collection of objects $\{C_i \to C\}$ that generates this sieve (i.e. each $C_i \to C$ factors through U and the map $\coprod_i j(C_i) \to U \to j(C)$ exhibits an epi-mono factorization in $\mathsf{PShv}(\mathcal{C})$), then we should show that the associated map $\coprod_i j(C_i) \to U \to \check{C}(f)$ factors through X. But the functors evaluating on objects of $\mathcal{C}$ on presheaves preserve effective epimorphisms as limits and colimits are computed objectwise, so the map of spaces $X(C_i) \to \check{C}(f)(C_i)$ is a surjection on connected components. □

Usually, we will specify a topology on an ∞-category $\mathcal{C}$ by giving a collection of morphisms $\{U_\alpha \to j(C)\}$ on each object $C \in \mathcal{C}$ that does not necessarily form a subobject of $j(C)$. We can always turn a collection into a sieve by taking the epi-mono factorization $\coprod_\alpha U_\alpha \to U \to j(C)$. The following definition gives conditions for when this procedure produces a Grothendieck topology.

Definition 2.2.2.3. Let $\mathcal{C}$ be an ∞-category. A *Grothendieck pretopology* $\mathcal{B}$ on $\mathcal{C}$ is the following data.

- For each object $C \in \mathcal{C}$, a collection $\mathcal{B}(C)$ of families $\{U_\alpha \to C\}$ of morphisms. Such distinguished families will be called *coverings*.

These collections are required to satisfy the following conditions.

(*i*) For each $C \in \mathcal{C}$, the family $\{\mathrm{id} : C \to C\}$ is a covering.

(*ii*) For each map $f : C' \to C$, and each covering $\{U_\alpha \to C\}$ of C, the pullbacks $U_\alpha \times_C C'$ exist for all α and the family $\{U_\alpha \times_C C' \to C'\}$ is a covering of C'.

(*iii*) Let $\{U_\alpha \to C\}$ be a covering, and suppose we are given a covering $\{W_{\beta_\alpha} \to U_\alpha\}$ for each α. Then the induced family $\{W_{\beta_\alpha} \to C\}$ is a covering.

Proposition 2.2.2.4. *Let $\mathcal{B}$ be a Grothendieck pretopology on an ∞-category $\mathcal{C}$. Consider, for each $C \in \mathcal{C}$, the collection of those sieves $U \hookrightarrow j(C)$ that contain a sieve generated by some covering in $\mathcal{B}(C)$. Then this collection of sieves specifies a Grothendieck topology on $\mathcal{C}$.*

Proof. We show that the three conditions on covering sieves defining a Grothendieck topology hold. Since for each $C \in \mathcal{C}$, the collection $\mathcal{B}(C)$ is nonempty, the maximal sieve $j(C) \to j(C)$ on C is a covering sieve.
Let $U \to j(C)$ be a sieve on C, and let $\{V_\alpha \to C\}$ be a covering generating a sieve $V \hookrightarrow U \hookrightarrow j(C)$. Let $f : C' \to C$ be any morphism. As colimits are universal and taking pullbacks preserves subobjects, we have an epi-mono factorization $\coprod_\alpha (j(V_\alpha \times_C C') \to V \times_{j(C)} j(C') \to j(C)$, so $V \times_{j(C)} j(C') \to j(C)$ is a covering sieve by (*ii*). It follows that we have an inclusion of covering sieves $V \times_{j(C)} j(C') \hookrightarrow U \times_{j(C)} j(C') \to j(C)$.
Now suppose that we have a covering sieve V on C and sieve U on C, and that for each $(f : D \to C)$ that factors through V, the sieve $U \times_{j(C)} j(D)$ is a covering sieve on D. Choose a covering $\{V_\alpha \to C\}$ on C that generates a sieve contained in V and choose for each V_α a covering family $\{W_{\beta_\alpha} \to V_\alpha\}$ that generates a sieve contained in $U \times_{j(C)} j(V_\alpha)$. Now every morphism in the family $\{W_{\beta_\alpha} \to C\}$ factors through $U \to j(C)$, so this sieve is a covering sieve, by (*iii*). □

Definition 2.2.2.5. Let $\mathcal{C}$ be an ∞-category equipped with a Grothendieck topology τ. Let $\mathcal{B}$ be a Grothendieck pretopology, then $\mathcal{B}$ induces a Grothendieck topology described by the previous proposition. If this topology is τ, we say that $\mathcal{B}$ is a *basis for* τ.

The following construction gives another way to express that a Grothendieck topology is determined by a basis. It asserts that the sheafification procedure only involves covering sieves generated by covering families.

Construction 2.2.2.6. Let $\mathcal{C}$ be an ∞-category equipped with a Grothendieck topology and let $\mathcal{B}$ be a basis for this topology. Let $\mathsf{Cov}(\mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\{1\}, \mathcal{C}) \times_{\mathrm{Fun}(\{1\}, \mathsf{PShv}(\mathcal{C}))} \mathrm{Fun}(\{\Delta^1\}, \mathsf{PShv}(\mathcal{C}))$ spanned by pairs $(C, U \to j(C))$ where $U \to j(C)$ is a covering sieve. The functor $\rho : \mathsf{Cov}(\mathcal{C}) \to \mathcal{C}$ is a Cartesian fibration by (2) of definition 2.2.2.1. For each $C \in \mathcal{C}$, the set $\mathcal{B}(C)$ is partially ordered by refinement, and the assignment $C \mapsto \mathcal{B}(C)$ determines a functor $\mathcal{C}^{op} \to \mathsf{Cat}_\infty$ (which factors through $h\mathcal{C}$). We let $\mathsf{CovFam}(\mathcal{C}) \to \mathcal{C}$ denote the associated Cartesian fibration, whose objects are pairs $(C, \{U_\alpha \to C\})$ where C is an object in $\mathcal{C}$ and $\{U_\alpha \to C\}$ a covering family of C. By sending covering families to the covering sieves they generate, we obtain a fully faithful functor

$$\begin{tikzcd} \mathsf{CovFam}(\mathcal{C}) \arrow[rr] \arrow[dr, "\varphi"'] & & \mathsf{Cov}(\mathcal{C}) \arrow[dl, "\rho"] \\ & \mathcal{C} & \end{tikzcd} \qquad (C, \{U_\alpha \to C\}) \mapsto (C, \tau_{\leq -1}(\textstyle\coprod j(U_\alpha) \to j(C)))$$

preserving Cartesian edges.

Proposition 2.2.2.7. *Let $\mathcal{C}$ be an ∞-category equipped with a basis for a Grothendieck topology. Then a presheaf $F \in \mathsf{PShv}(\mathcal{C})$ is a sheaf for the induced topology if and only if F is local for covering sieves generated by covering families.*

Proof. It follows from proposition 2.2.2.4 that $\mathcal{B}$ induces a Grothendieck topology τ. Let S denote the class of covering sieves for τ and let $S' \subset S$ denote the collection of sieves generated by covering families, and let $\overline{S}$ and $\overline{S'}$ denote their respective strong saturations. We clearly have $\overline{S'} \subset \overline{S}$. For the other inclusion, we need to show that $S \subset \overline{S'}$. We claim that it suffices to show that for any map $X \to j(C)$ in $\mathsf{PShv}(\mathcal{C})$ and any covering sieve $V \to j(C)$ generated by a covering family, the pullback $X \times_{j(C)} V \to X$ lies in $\overline{S'}$. Suppose this is the case, then we note that for any map of subobjects $V \hookrightarrow U \to j(C)$ the pullback diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & j(C) \end{tikzcd}$$

guarantees that $V \to U$ lies in $\overline{S'}$ if V is generated by a covering family. Since $\overline{S'}$ has the 2-out-of-3 property, $U \to j(C)$ also lies in $\overline{S}$. To prove the claim, we use that colimits are universal in $\mathsf{PShv}(\mathcal{C})$ and that any $X \in \mathsf{PShv}(\mathcal{C})$

is generated under colimits by a representables, so that the map $X \times_{j(C)} V \to X$ is a colimit of maps of the form $j(C') \times_{j(C)} V \to j(C')$. Any such map is a sieve generated by a covering family, by (ii) of definition 2.2.2.3. As $\overline{S'}$ is stable under colimits of arrows, we conclude. □

Definition 2.2.2.8. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small ∞-categories equipped with Grothendieck pretopologies. We say that this functor is *covering-preserving* if

(1) For every covering family $\{U_i \to C\}$, f preserves pullbacks along each $U_i \to C$.

(2) Each covering family $\{U_i \to C\}$ in $\mathcal{C}$, the family $\{f(U_i) \to f(C)\}$ is a covering.

Remark 2.2.2.9. It's easy to see that if f is covering-preserving, then the pullback $f^* : \mathsf{PShv}(\mathcal{D}) \to \mathsf{PShv}(\mathcal{C})$ carries sheaves to sheaves, since in this case f carries (-1)-truncations of covering families to (-1)-truncations of covering families (we refrain from calling such functors *continuous*; this terminology is reserved for functors commuting with filtered colimits).

Example 2.2.2.10. Given any topology τ on an ∞-category $\mathcal{C}$, there is a maximal basis for τ, whose covering families are those families of morphisms that generate a covering sieve.

Example 2.2.2.11. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{B}$ be a pretopology on $\mathcal{C}$. Then we say that $\mathcal{B}$ is *finitary* if each covering family has a finite refinement. Given any pretopology $\mathcal{B}$, there is an associated finitary pretopology $\mathcal{B}$ such that the identity functor $\mathcal{C} \to \mathcal{C}$, where the first copy of $\mathcal{C}$ is endowed with $\mathcal{B}'$ and the second copy with $\mathcal{B}$, is covering-preserving: say that a family $\{f_i : U_i \to C\}_{i \in I}$ lies in $\mathcal{B}'(C)$ if it lies in $\mathcal{B}(C)$ and I is finite. Conditions (i) through (iii) are obvious.

Remark 2.2.2.12. If $\mathcal{B}$ is finitary, the associated ∞-topos is *locally coherent* in the sense of [Lur11c], definitions 3.1 and 3.12; that is, for each sheaf $X \in \mathsf{Shv}(\mathcal{C})$ there is an effective epimorphism $\coprod_i F_i \to X$, such that the sheaves F_i belong to the collection of sheaves $F \in \mathsf{Shv}(\mathcal{C})$ satisfying the following.

(0) F is *quasi-compact*: for effective epimorphism of the form $\coprod_{i \in I} G_i \to F$ there is a finite subset $I' \subset I$ such that $\coprod_{i \in I'} G_i \to F$ is still an effective epimorphism.

(1) For each morphism $F' \to F$, the object F' admits a cover $\coprod_{j \in J} V_j \to F'$ by quasi-compact objects V_j, and the collection of all quasi-compact objects in $\mathsf{Shv}(\mathcal{C})_{/F}$ is stable under products in $\mathsf{Shv}(\mathcal{C})_{/F}$.

(2) For each morphism $F'' \to F$, the object F'' admits a cover $\coprod_{j \in J'} V'_j \to F''$, by object V'_j that satisfy (0) and (1) with F replaced by V'_j and the collection of such objects satisfying (0) and (1) is stable under products in $\mathsf{Shv}(\mathcal{C})_{/F}$.

(3) ...

(i) For each morphism $F''' \to F$, the object F''' admits a cover $\coprod_{j \in J''} V''_j \to F'''$, by object V''_j that satisfy (0) through $(i-1)$ with F replaced by V''_j and the collection of such objects satisfying (0) through $(i-1)$ is stable under products in $\mathsf{Shv}(\mathcal{C})_{/F}$.

$(> i)$ etc.

Sheaves F satisfying (0) through (n) above are said to be *n-coherent*.

We will often use the following elementary yet useful principle.

Proposition 2.2.2.13. *Let $f : \mathcal{C} \to \mathcal{C}'$ be a covering-preserving functor between ∞-categories equipped with Grothendieck topologies, and denote by $f^* : \mathsf{PShv}(\mathcal{C}') \to \mathsf{PShv}(\mathcal{C})$ the functor induced by composing with f, which descends to a functor $f^* : \mathsf{Shv}(\mathcal{C}') \to \mathsf{Shv}(\mathcal{C})$*

(1) *If for each $\alpha : F \to F'$ that exhibits F' as a sheafification of F in $\mathsf{PShv}(\mathcal{C}')$, the map $f^*(\alpha)$ exhibits a sheafification, then f^* preserves colimits and admits a right adjoint.*

(2) *Suppose that f is fully faithful and that both the topology on $\mathcal{C}$ and $\mathcal{C}$ are subcanonical. If the condition in (1) is satisfied, then the left adjoint $f_!$ and the right adjoint f_* to f^* are fully faithful.*

Proof. To prove (1), it suffices to show that f^* preserves colimits, in view of the adjoint functor theorem. Let $q : K^{\triangleright} \to \mathsf{Shv}(\mathcal{C}')$ be a colimit diagram, then q is of the form $L' \circ q'$ for $q' : K^{\triangleright} \to \mathsf{PShv}(\mathcal{C}')$ a colimit diagram, where $q'|_K = q|_K$ and $L' : \mathsf{PShv}(\mathcal{C}') \to \mathsf{Shv}(\mathcal{C}')$ is a sheafification functor. We have a natural equivalence $f^* \circ L' \circ q' \simeq L \circ f^* \circ q'$, where L is a sheafification functor on $\mathsf{PShv}(\mathcal{C})$. Since both $f^* : \mathsf{PShv}(\mathcal{C}') \to \mathsf{PShv}(\mathcal{C})$ and $L : \mathsf{PShv}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C})$ preserve colimits, we are done.
For (2), we first show that the unit $\mathrm{id} \to f^* f_!$ is an equivalence. Point (1) grants that $f^* f_!$ preserves colimits, so

using [Lur17b], prop. 4.3.2.15 we deduce that the full subcategory $\mathcal{D} \subset \mathsf{Shv}(\mathcal{C})$ spanned by those sheaves F such that $F \to f^* f_! F$ is an equivalence, is stable under small colimits. Now we conclude by observing that under the hypothesis that the pretopologies are subcanonical, the image of the Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathsf{Shv}(\mathcal{C})$ lies in $\mathcal{D}$. We have a pair of adjunctions $(f_! \dashv f^* \dashv f_*)$ and an induced adjunction $(f^* f_! \dashv f^* f_*)$. Since $f^* f_!$ is homotopic to the identity via the unit, $f^* f_*$ is homotopic to the identity via the counit, so it follows that f_* is also fully faithful. □

Remark 2.2.2.14. Let Site be the category whose objects are pairs $(\mathcal{C}, \mathcal{B})$ of a small idempotent complete ∞-category $\mathcal{C}$ together with a Grothendieck pretopology on $\mathcal{C}$, and whose morphisms are equivalence classes of covering-preserving functors, then we have an obvious forgetful functor Site $\to h\mathsf{Cat}_\infty^\vee$, where $\mathsf{Cat}_\infty^\vee$ is the full subcategory of Cat_∞ spanned by idempotent complete ∞-categories. The *∞-category of sites* denoted Site is the pullback $\mathsf{Cat}_\infty^\vee \times_{h\mathsf{Cat}_\infty^\vee}$ Site. The forgetful functor $\mathsf{Site} \to \mathsf{Cat}_\infty^\vee$ is a Cartesian fibration and the fibre over each small idempotent complete ∞-category $\mathcal{C}$ can be identified with the partially ordered set of pretopologies on $\mathcal{C}$. Let $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{ccont}}$ be the subcategory containing all objects whose morphisms are *completely continuous* functors, that is, those functors that preserve small colimits and carry completely compact objects to completely compact objects. The construction $\mathcal{C} \mapsto \mathsf{PShv}(\mathcal{C})$ determines a functor $\mathsf{Cat}_\infty^\vee \to \mathsf{Pr}^{\mathrm{L}}$ that factors fully faithfully through the subcategory $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{ccont}}$, so we can identify $\mathsf{Cat}_\infty^\vee$ with a certain (non full) subcategory $\chi \subset \mathsf{Pr}^{\mathrm{L}}$ and we have a Cartesian fibration $\mathsf{Site} \to \chi$. Unwinding the definitions, we see that Site is equivalent to the nerve of the fibrant simplicial category whose objects are pairs $(\mathsf{PShv}(\mathcal{C}), \mathcal{B})$ where $\mathcal{B}$ is a pretopology on the idempotent complete ∞-category $\mathcal{C}$. The Kan complex of morphisms

$$\mathrm{Hom}_{\mathsf{Site}}((\mathsf{PShv}(\mathcal{C}), \mathcal{B}), (\mathsf{PShv}(\mathcal{D}), \mathcal{E}))$$

is the union of those connected components of $\mathrm{Hom}_{\widehat{\mathsf{Cat}}_\infty^\Delta}(\mathsf{PShv}(\mathcal{C}), \mathsf{PShv}(\mathcal{D}))$ spanned by functors $f_! : \mathsf{PShv}(\mathcal{C}) \to \mathsf{PShv}(\mathcal{D})$ that are left Kan extensions of functors of the form $\mathcal{C} \xrightarrow{f} \mathcal{D} \hookrightarrow \mathsf{PShv}(\mathcal{D})$ such that f is covering-preserving. It follows that the spaces of morphisms of ∞-category Site^{op} is the union of connected components of $\mathrm{Hom}_{\widehat{\mathsf{Cat}}_\infty^\Delta}(\mathsf{PShv}(\mathcal{C}), \mathsf{PShv}(\mathcal{D}))$ spanned by functors $f^* : \mathsf{PShv}(\mathcal{D}) \to \mathsf{PShv}(\mathcal{C})$ obtained as the pullback of some covering-preserving functor $f : \mathcal{C} \to \mathcal{D}$. Consider the functor $\mathrm{Sys} : h\widehat{\mathsf{Cat}}_\infty^{op} \to \mathsf{Poset}$ carrying an ∞-category $\mathcal{D}$ to the partially ordered set of *systems* (lluf subcategories of $h\mathcal{D}$) on $\mathcal{D}$ (see [Lur17a], section 4.1.8) which is classified by a Cartesian fibration $\mathcal{W}\widehat{\mathsf{Cat}}_\infty \to \widehat{\mathsf{Cat}}_\infty$. We can identify The assignment $\mathcal{B} \mapsto S$ carrying a pretopology on $\mathcal{C}$ to the strong saturation of the class of covering sieves in $\mathsf{PShv}(\mathcal{C})$ is a natural transformation $h\mathsf{Pretop} \to \mathrm{Sys}$ that corresponds via unstraightening to a diagram

$$\begin{tikzcd} \mathsf{Site} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{W}\chi \arrow[dl, "q"] \\ & \chi & \end{tikzcd}$$

where the horizontal functor takes p-Cartesian edges to q-Cartesian edges. The functor q admits a section sending an ∞-category $\mathcal{D}$ to the system containing only the equivalences of $\mathcal{D}$, and this section has a left adjoint that sends a pair $(\mathcal{D}, W)$ to the localization $\mathcal{D}[W^{-1}]$. By construction of the functor $\mathsf{Site}^{op} \to \mathcal{W}\widehat{\mathsf{Cat}}_\infty$, the composition $\mathsf{Site}^{op} \to \mathcal{W}\widehat{\mathsf{Cat}}_\infty \to \widehat{\mathsf{Cat}}_\infty$ factors through Pr^{R}, so we obtain a functor

$$\mathsf{Shv}(\_) \coloneqq \mathsf{Site} \longrightarrow \mathsf{Pr}^{\mathrm{L}}$$

informally given by the formula $(\mathcal{C}, \mathcal{B}) \mapsto \mathsf{Shv}_{\mathcal{B}}(\mathcal{C})$. Beware that $\mathsf{Shv}(\_)$ need not send covering-preserving functors to algebraic morphisms.

2.2.3 n-Topoi and localic ∞-topoi

The class of n-topoi is extrinsically defined as containing the ∞-categories that come about as left exact localizations of $(n-1)$-truncated presheaves on small ∞-categories. There are more intrinsic characterization as well in the form of Giraud axioms and descent conditions for n-topoi, summarized as [Lur17b], theorem 6.4.1.5. We record the following ∞-categories.

- The subcategory ${}^{\mathrm{L}}\mathsf{Top}_n \subset \widehat{\mathsf{Cat}}_\infty$ whose objects are n-topoi, and whose morphisms are functors that are left exact and admit a right adjoint. Such morphisms between ∞-topoi will be called *algebraic morphisms*. For $\mathcal{X}, \mathcal{Y} \in {}^{\mathrm{L}}\mathsf{Top}_n$, the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by algebraic morphisms is denoted $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$.

- The subcategory ${}^{\mathrm{R}}\mathsf{Top}_n \subset \widehat{\mathsf{Cat}}_\infty$ whose objects are n-topoi, and whose morphisms are functors that admit a left exact left adjoint. Morphisms in ${}^{\mathrm{R}}\mathsf{Top}$ will be called *geometric morphisms*. For $\mathcal{X}, \mathcal{Y} \in {}^{\mathrm{L}}\mathsf{Top}_n$, the full subcategory of $\mathrm{Fun}(\mathcal{Y}, \mathcal{X})$ spanned by geometric morphisms is denoted $\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X})$. The ∞-categories ${}^{\mathrm{R}}\mathsf{Top}_n$ and ${}^{\mathrm{L}}\mathsf{Top}_n$ are canonically antiequivalent, and the ∞-categories $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ and $\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X})$ are canonically equivalent.

[Lur17b], theorem 6.4.1.5 in particular implies that taking full subcategories of $(n-1)$-truncated objects induces a functor

$$ {}^{\mathrm{R}}\mathsf{Top} \xrightarrow{\tau_{\leq(n-1)}} {}^{\mathrm{R}}\mathsf{Top}_n. $$

This functor admits a fully faithful right adjoint that embeds the ∞-category of n-topoi into the ∞-category of ∞-topoi. The essential image of this embedding is characterized as follows.

Definition 2.2.3.1. Let $n \in \mathbb{Z}_{\geq 0}$. An ∞-topos $\mathcal{X}$ is *n-localic* if for every ∞-topos $\mathcal{Y}$, the canonical map

$$ \mathrm{Fun}_*(\mathcal{Y},\mathcal{X}) \longrightarrow \mathrm{Fun}_*(\tau_{\leq(n-1)}\mathcal{Y}, \tau_{\leq(n-1)}\mathcal{X}) $$

is an equivalence of ∞-categories.

For any n-topos $\mathcal{X}$, there exists an n-localic ∞-topos $\mathcal{Y}$ together with an equivalence $g_* : \tau_{\leq n}\mathcal{Y} \to \mathcal{X}$ such that for each ∞-topos $\mathcal{Z}$, taking $(n-1)$-truncated objects and composing with g_* induces an equivalence

$$ \mathrm{Fun}_*(\mathcal{Z},\mathcal{Y}) \longrightarrow \mathrm{Fun}_*(\tau_{\leq(n-1)}\mathcal{Z}, \mathcal{X}). $$

Using [Lur17b], theorem 6.4.1.5, we may assume that there is a small n-category $\mathcal{C}$ that admits finite limits equipped with a Grothendieck topology such that $\mathcal{X} \simeq \mathsf{Shv}_{(n-1)}(\mathcal{C})$, the ∞-category of $(n-1)$-truncated sheaves on $\mathcal{C}$. Then the associated n-localic ∞-topos is simply $\mathsf{Shv}(\mathcal{C})$ and the equivalence g_* is the identity. This construction determines a collection of counit transformations (which are equivalences) yielding the fully faithful right adjoint to the functor $\tau_{\leq n}$. We denote this right adjoint by υ_n:

$$ {}^{\mathrm{R}}\mathsf{Top} \underset{\upsilon_n}{\overset{\tau_{\leq n}}{\rightleftarrows}} {}^{\mathrm{R}}\mathsf{Top}_n. $$

Recall (from [MM92] for instance) that taking set-valued sheaves on topological spaces furnishes an equivalence of categories between sober topological spaces and spatial locales. Let SobSp be the category of sober topological spaces, then by composing υ_n we have in particular a fully faithful inclusion

$$ \mathbf{N}(\mathsf{SobSp}) \hookrightarrow {}^{\mathrm{R}}\mathsf{Top}, $$

which coincides with the functor $\mathsf{Shv}(\_)$ of the previous subsection, restricted to locales. We have the following important stability result for n-localic ∞-topoi.

Proposition 2.2.3.2. *Let $\mathcal{X}$ be an n-localic ∞-topos and let $U \in \mathcal{X}$ be an object. Then the following are equivalent.*

(1) *$\mathcal{X}_{/U}$ is n-localic.*

(2) *U is n-truncated.*

Proof. This is lemma 2.3.16 of [Lur11b]. □

Being n-localic is not a local property of ∞-topoi, but we nevertheless have the following useful characterization.

Proposition 2.2.3.3. *Let $\mathcal{X}$ be an ∞-topos, then the following are equivalent for all $n \in \mathbb{Z}_{\geq 0}$.*

(1) *$\mathcal{X}$ is n-localic.*

(2) *$\mathcal{X}$ is equivalent to the ∞-category of sheaves on an n-category that admits finite limits equipped with a Grothendieck topology.*

(3) *There exists a collection of $(n-1)$-truncated objects $U_\alpha \in \mathcal{X}$ determining an effective epimorphism $\coprod_\alpha U_\alpha \to 1_{\mathcal{X}}$ such that $\mathcal{X}_{/U_\alpha}$ is n-localic for all α.*

Proof. If $\mathcal{X}$ is n-localic, then the unit map $\pi_* : \mathcal{X} \to \mathcal{Y}$ of the adjunction constructed above is an equivalence (this map is the *n-localic reflection*). The construction shows that $\mathcal{Y}$ is of the form $\mathsf{Shv}(\mathcal{C})$ for $\mathcal{C}$ an n-category that admits finite limits. The converse follows from [Lur17b], lem. 6.4.5.6. The implication (1) ⇒ (3) is immediate. For the reverse implication, we will show that the canonical geometric morphism

$$ \pi_* : \mathcal{X} \longrightarrow \mathcal{Y} = \upsilon_n(\tau_{\leq(n-1)}\mathcal{X}) $$

to the n-localic reflection is an equivalence. Choose a collection of $(n-1)$-truncated objects U_α such that $\mathcal{X}_{/U_\alpha}$ is n-localic for all n. Since the adjoint π^* induces an equivalence of n-topoi $\tau_{\leq(n-1)}\mathcal{Y} \to \tau_{\leq(n-1)}\mathcal{X}$, the objects U_α are of the form $\pi^* W_\alpha$ for a collection of objects $W_\alpha \in \mathcal{Y}$. Moreover, the map $h : \coprod_\alpha W_\alpha \to 1_{\mathcal{Y}}$ is an effective epimorphism. To see this, we note that if we factorize h as $\coprod_\alpha W_\alpha \to \tilde{W} \to 1_{\mathcal{Y}}$, an effective epimorphism followed by a monomorphism in $\mathcal{Y}$, then we also have such a factorization $\coprod_\alpha U_\alpha \to \pi^*(\tilde{W}) \to 1_{\mathcal{X}}$ so that $\pi^*(\tilde{W}) \simeq 1_{\mathcal{X}}$. Since $\tilde{W}$ and $\pi^*(\tilde{W})$ are

(−1)-truncated, and π^* is an equivalence on $(n-1)$-truncated objects, we also have $\tilde{W} \simeq 1_{\mathcal{Y}}$. We now show that for each α, the algebraic morphism

$$\pi^*|_{W_\alpha} : \mathcal{Y}_{/W_\alpha} \longrightarrow \mathcal{X}_{/U_\alpha}$$

is an equivalence. By assumption, $\mathcal{X}_{/U_\alpha}$ is n-localic and $\mathcal{Y}_{/W_\alpha}$ is n-localic by proposition 2.2.3.2; thus it will suffice to show that $\pi^*|_{W_\alpha}$ induces an equivalence on $(n-1)$-truncated objects. As W_α and U_α are $(n-1)$-truncated and $\mathcal{Y}$ is an n-localic reflection, we have equivalences

$$\tau_{\leq(n-1)}(\mathcal{Y}_{/W_\alpha}) \simeq \tau_{\leq(n-1)}\mathcal{Y}_{/W_\alpha} \simeq \tau_{\leq(n-1)}\mathcal{X}_{/U_\alpha} \simeq \tau_{\leq(n-1)}(\mathcal{X}_{/U_\alpha}),$$

the composition being induced by $\pi^*|_{W_\alpha}$. This finishes the proof as the conditions of lemma 2.2.0.10 for the functor π^* are satisfied. □

Warning 2.2.3.4. In classical topos theory, a 1-topos $\mathcal{X}$ is by definition *localic* if $\mathcal{X}$ is generated under colimits by subobjects of the unit object [MM92]. The obvious generalization of this definition to n-topoi is adequate for $n < \infty$ but breaks down for ∞-topoi. On the one hand, an n-localic ∞-topos is clearly generated under colimits by its $(n-1)$-truncated objects, but the converse is *false* in general; indeed, consider the hypercompletion of the ∞-category of sheaves on the Hilbert cube H (see [Lur17b], sections 6.5.3 and 6.5.4). This ∞-topos is generated under colimits by its (-1)-truncated objects, but the hypercompletion $L : \mathsf{Shv}(H) \to \widehat{\mathsf{Shv}}(H)$ is the 0-localic reflection.

2.2.4 Simplicial homotopy theory in ∞-topoi

To any object X in an ∞-topos $\mathcal{X}$, one associates homotopy sheaves in the underlying discrete topos $\mathrm{Disc}(\mathcal{X})$ by the 0'th truncation of the morphism $X^{S_n} \to X$, using the cotensoring of $\mathcal{X}$ over $\mathcal{S}$. These homotopy sheaves have the same properties as do homotopy groups of spaces; in particular, we can define homotopy sheaves for maps $f : X \to Y$ of objects in $\mathcal{X}$; we say that a map is *n-connective* if f is an effective epimorphism and $\pi_k(f)$ is final for $0 \leq k < n$. For any $n \geq 0$ the classes of n-connective and n-truncated objects form a factorization system. In the case $n = \infty$, the class right orthogonal to ∞-connective morphisms does *not* necessarily consist only of equivalences. For any ∞-topos $\mathcal{X}$, the *hypercompletion* is the accessible left localization $L^\wedge : \mathcal{X} \to \mathcal{X}^\wedge$ by the set $S^\wedge$ of ∞-connective maps ([Lur17b], 6.5.2.8). The $S^\wedge$-local objects (those objects that have the property that for maps between them, Whitehead's theorem holds), are intimately related to objects in $\mathcal{X}$ that satisfy a stronger descent property.

Definition 2.2.4.1. Let $\mathcal{X}$ be an ∞-topos, then we say that an augmented simplicial object $C_\bullet$ in $\mathcal{X}$ with augmentation map $C_0 \to C_{-1} = X$ is a *hypercover* if the unit map $C_n \to \mathrm{cosk}_{n-1}C_\bullet$ is an effective epimorphism for all $n \geq 0$.
Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology τ, then we say that an augmented simplicial object $C_\bullet$ in $\mathsf{PShv}(\mathcal{C})$ with augmentation map $C_0 \to C_{-1} = j(C)$ is a *semi-representable hypercover* if each C_n is a small coproduct of representables and if the unit map $C_n \to \mathrm{cosk}_{n-1}C_\bullet$ is a τ-covering (i.e. a map that becomes an effective epimorphism after localization). An augmented simplicial object $C_\bullet$ in $\mathsf{Shv}_{\mathcal{C}}$ is a *semi-representable hypercover* if it is a hypercover and C_n is a small coproduct of sheafified representables.
A presheaf F on $\mathcal{X}$ is a *hypersheaf* or *satisfies hyperdescent* if F satisfies descent with respect to hypercovers.

To explain what these stricter sheaf conditions have to do with ∞-connectiveness, we make the following observation.

Lemma 2.2.4.2. *A map $f : X \to Y$ in an ∞-topos $\mathcal{X}$ is ∞-connective if and only if the matching map*

$$f_n : X \longrightarrow X^{S_n} \times_{Y^{S_n}} Y$$

is an effective epimorphism for all $n \geq 0$.

Proof. The truncation of the map $X^{S_n} \times_{Y^{S_n}} Y \to X$ in $\mathcal{X}_{/X}$ yields the object $\pi_n(f)$. The maps f_n are effective epimorphisms if and only if their 0'th truncations are effective epimorphisms. We can view f_n as a map in $\mathcal{X}_{/X}$, then after taking the 0'th truncation, we have a map $1_{\mathrm{Disc}(\mathcal{X}_{/X})} \to \pi_n(f)$ which is an equivalence of discrete sheaves over X for all n if and only if f is ∞-connective. □

Let $X \to Y$ be map of sheaves in $\mathsf{Shv}(\mathcal{C})$. This map is an equivalence if and only if the map $X(C) \to \left(X^{S_n} \times_{Y_{s_n}} Y\right)(C)$ of spaces is an effective epimorphism for all objects $C \in \mathcal{C}$. However, if f is an ∞-connective map between sheaves, this condition is *not necessarily* satisfied, since we cannot lift maps $j(C) \to X^{S_n} \times_{Y_{s_n}} Y$ to X; we can only find a covering of C that lifts. Crucially though, by iterating (in a suitable sense) the procedure of passing to coverings, we *can* construct a (semi-representable) hypercover of C that lifts to X. Thus, if X sees all such hypercoverings as effective, the map $X \to Y$ is an equivalence after all. This result is due to Dugger-Hollander-Isaksen and Toën-Vezzosi:

Proposition 2.2.4.3 ([DI04; TV04]). *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology. Then the hypercompletion of $\mathsf{Shv}(\mathcal{C})$ can be identified with the full subcategory of $\mathsf{PShv}(\mathcal{C})$ of S-local objects for S any of the following collections of morphisms.*

(1) *The collection of morphisms $f : X \to Y$ that induces an equivalence on all homotopy sheaves (equivalently, the morphisms that become ∞-connective after sheafifying with respect to the topology)*

(2) *The collection of morphisms $f : X \to Y$ of the form $|C_\bullet| \to Y$ for $C_\bullet$ a hypercover of Y.*

(3) *The collection of morphisms $f : X \to Y$ of the form $|C_\bullet| \to j(C)$ for $C_\bullet$ a semi-representable hypercover of $j(C)$.*

(4) *The collection of morphisms $f : X \to Y$ of the form $|C_\bullet| \to j(C)$ for $C_\bullet$ a semi-representable hypercover of $j(C)$ where each C_n is a τ-small coproduct of representables for τ a sufficiently large regular cardinal.*

Proof. Choose, using [Lur17b], prop. 5.4.7.4, an uncountable regular cardinal τ such that $\mathcal{C}$ is τ-small and the full subcategory $\mathsf{PShv}_\tau(\mathcal{C}) \subset \mathsf{PShv}(\mathcal{C})$ spanned by τ-compact objects is stable under finite limits. According to [Lur17b], prop. 5.3.4.7, every τ-compact object Z of $\mathsf{PShv}(\mathcal{C})$ is a retract of a τ-small colimit of representables, so using that every representable object of $\mathsf{PShv}(\mathcal{C})$ is completely compact, we deduce that $Z(C)$ is a τ-small space for each object $C \in \mathcal{C}$. The full subcategory $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op})^{\triangleright}, \mathsf{PShv}_\tau(\mathcal{C})) \subset \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op})^{\triangleright}, \mathsf{PShv}(\mathcal{C}))$ spanned by augmented simplicial objects with τ-compact simplices is essentially small. Consider the full subcategory $\mathcal{D} \subset \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op})^{\triangleright}, \mathsf{PShv}_\kappa(\mathcal{C}))$ spanned by augmented simplicial sets that are semi-representable hypercovers $C_\bullet$ of representable objects, and choose for each homotopy class of objects $[C_\bullet]$ in $h\mathcal{D}$ a colimit $|C_\bullet| \to j(C')$. Denote by S the collection of all such maps, which is a small set, then it follows from [Lur17b], prop. 5.5.4.15 that the subcategory inclusion $S^{-1}\mathsf{PShv}(\mathcal{C}) \subset \mathsf{PShv}(\mathcal{C})$ is an accessible localization. Since every hypercover determines an ∞-connective morphism, all hypercomplete objects in $\mathsf{PShv}(\mathcal{C})$ satisfy hyperdescent, so we have an inclusion $\mathsf{Shv}^\wedge(\mathcal{C}) \subset S^{-1}\mathsf{PShv}(\mathcal{C})$, corresponding to the localization at the first and fourth collections of morphisms described in the proposition. The second and third collection lie in between these two, so in order to prove the proposition, it suffices to prove the reverse inclusion $S^{-1}\mathsf{PShv}(\mathcal{C}) \subset \mathsf{Shv}^\wedge(\mathcal{C})$. We show that the unit $X \to LX$ of the hypercompletion is an equivalence whenever X satisfies hyperdescent. Note that since ∞-connective morphisms form a strongly saturated collection, the map $X \to LX$ is ∞-connective. We will show more generally that every ∞-connective morphism between S-local objects is an equivalence. Let $f : X \to Y$ be such a morphism, then we are required to show that for every $C \in \mathcal{C}$ and all $n \in \mathbb{Z}_{\geq 0}$, the map $X(C) \to X^{S_n} \times_{Y^{S^n}} Y(C)$ is an effective epimorphism of spaces. If $f : X \to Y$ is ∞-connective, then so are all the maps $X \to X^{S_n} \times_{Y^{S^n}} Y$, so we may replace $X^{S_n} \times_{Y^{S^n}} Y$ by Y, and it is sufficient to show that $X(C) \to Y(C)$ is surjection on connected components. Now we build a hypercover in $\mathcal{D}$ of the object $j(C)$ compatible with the map $f : X \to Y$ using Reedy methods (also known as an *Artin-Mazur* argument in this case [AM69]). We inductively define a sequence of functors $g_n : \mathbf{N}(\mathbf{\Delta}^{\leq n})^{op} \to \mathsf{PShv}(\mathcal{C})_{/j(C)}$ for all $n \geq 0$, together with a sequence of natural transformations $\alpha_n : g_n \to \underline{X}_n$, where $\underline{X}_n : \mathbf{N}(\mathbf{\Delta}^{\leq n})^{op} \to \mathsf{PShv}(\mathcal{C})_{/Y}$ is the constant n-truncated simplicial object on X. We require that g_n satisfies the following conditions.

(1) For $n \geq 0$, let $L_n(g)$ denote then n'th latching object, given by the colimit of composite functor

$$\mathbf{N}(\mathbf{\Delta}^{\leq n-1})^{op} \times_{\mathbf{N}(\mathbf{\Delta})^{op}} \mathbf{N}(\mathbf{\Delta}_{[n]/})^{op} \longrightarrow \mathbf{N}(\mathbf{\Delta}^{\leq n-1})^{op} \xrightarrow{g_{n-1}} \mathsf{PShv}(\mathcal{C})_{/j(C)},$$

which has a canonical map $L_n(g) \to g_n([n])$. We require that there exists an object $V_n \in \mathsf{PShv}(\mathcal{C})$ which is a τ-small coproduct of representables and a map $V_n \to g_n([n])$ such that the induced map $V_n \coprod L_n(g) \to g_n([n])$ is an equivalence.

(2) For $n \geq 0$ let $M_n(g)$ denote the n'th matching object, given by the limit of the composite functor

$$\mathbf{N}(\mathbf{\Delta}^{\leq n-1})^{op} \times_{\mathbf{N}(\mathbf{\Delta})^{op}} \mathbf{N}(\mathbf{\Delta}_{/[n]})^{op} \longrightarrow \mathbf{N}(\mathbf{\Delta}^{\leq n-1})^{op} \xrightarrow{g_{n-1}} \mathsf{PShv}(\mathcal{C})_{/j(C)},$$

which has a canonical map $g_n([n]) \to M_n(g)$ in $\mathsf{PShv}(\mathcal{C})_{/j(C)}$, and we require that this map is an effective epimorphism.

We construct this sequence by induction: for $n = 0$, we take a covering sieve of C, necessarily generated by τ-small set of maps $\{C_i \to C\}$, such that each map $C_i \to Y$ factors through X, and set $g_0([0]) = \coprod_i j(C_i)$. We have a commuting square

$$\begin{array}{ccc} \coprod_i j(C_i) & \longrightarrow & j(C) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in $\mathsf{PShv}(\mathcal{C})$ which determines a natural transformation $g_0 \to \underline{X}_0$ of functors $\Delta^0 \cong \mathbf{N}(\mathbf{\Delta}^{\leq 0})^{op} \to \mathsf{PShv}(\mathcal{C})_{/Y}$. Now let $n \geq 1$ and suppose that g_m and $\alpha_m : g_m \to \underline{X}_m$ have been constructed for $m < n$. To construct g_n and α_n, we use [Lur17b], prop. A.2.9.14 to conclude that it suffices to provide the following data:

(*i*) An object V_n which is a τ-small coproduct of representables.

(*ii*) An effective epimorphism $g : V_n \to M_n(g)$ in $\mathsf{PShv}(\mathcal{C})$.

(*iii*) Let $M_n(\underline{X})$ denote the n'th matching object of $\underline{X}$, given by the limit

$$\mathbf{N}(\mathbf{\Delta}^{\le n-1}_{/[n]})^{op} \longrightarrow \mathbf{N}(\mathbf{\Delta}^{\le n-1})^{op} \xrightarrow{\underline{X}_{n-1}} \mathsf{PShv}(\mathcal{C})_{/Y},$$

and note that the canonical map $X \to M_n(\underline{X})$ may be identified with the map $X \to X^{S^n} \times_{Y^{S^n}} Y$. The natural transformation α_{n-1} determines a map $M'_n(g) \to M_n(\underline{X})$, where $M'_n(g)$ is the matching object of g_{n-1} taken in $\mathsf{PShv}(\mathcal{C})_{/Y}$, and the right fibration $\mathsf{PShv}(\mathcal{C})_{/j(C)} \to \mathsf{PShv}(\mathcal{C})_{/Y}$ determines a map $M_n(g) \to M'_n(g)$. The effective epimorphism from (*ii*) may be composed with these maps to produce a map $V_n \to M_n(\underline{X})$. Then we require the existence of a lift $V_n \to X$ fitting into a 2-simplex

$$\begin{tikzcd} & X \arrow[dr] & \\ V_n \arrow[ur] \arrow[rr] & & M_n(\underline{X}). \end{tikzcd}$$

in $\mathsf{PShv}(\mathcal{C})$ (such a simplex will automatically extend to a 2-simplex in $\mathsf{PShv}(\mathcal{C})_{/Y}$ because $\mathsf{PShv}(\mathcal{C})$ is an ∞-category).

By induction, the matching object $M_n(g)$ is a finite limit τ-compact objects and therefore also τ-compact. It follows that the domain of the effective epimorphism

$$\coprod_{C' \in \mathcal{C}} \coprod_{\pi_0(M_n(g)(C'))} j(C') \longrightarrow M_n(g)$$

is a coproduct of representables indexed by a τ-small set. Using that $X \to M_n(\underline{X})$ is an effective epimorphism, we can choose for each summand $j(C') \to M_n$ a τ-small collection $\{C'_i \to C'\}$ generating a covering sieve such that each composition $j(C'_i) \to M_n(g) \to M_n(\underline{X})$ factors through X. Taking the (τ-small) coproduct over all $j(C'_i)$ for all $C' \in \mathcal{C}$, we obtain an object V_n which satisfies the required conditions. This concludes the construction of the hypercover. By construction, we have a commuting diagram

$$\begin{tikzcd} \operatorname*{colim}_{[n] \in \mathbf{N}(\mathbf{\Delta}^{op})} g_n([n]) \arrow[r] \arrow[d] & j(C) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where the upper horizontal map is equivalent to one in S. Since X and Y are S-local, we conclude. □

Remark 2.2.4.4. An ∞-topos *has enough points* if all the functors $\mathcal{X} \to \mathcal{S}$ in ${}^{\mathrm{L}}\mathsf{Top}$ are jointly conservative. By (strictly) decreasing strength, we have the following conditions on an ∞-topos.

(1) $\mathcal{X}$ has enough points.

(2) Postnikov towers converge in $\mathcal{X}$.

(3) $\mathcal{X}$ is hypercomplete.

If $\mathcal{X}$ is locally coherent (think locally compact space) and hypercomplete, then $\mathcal{X}$ has enough points, by the Lurie-Deligne completeness theorem. If $\mathcal{X}$ is locally of homotopy dimension $\le n$, then Postnikov towers converge in $\mathcal{X}$. If $\mathcal{X} = \mathsf{Shv}(X)$, the ∞-topos of sheaves on a space, then $\mathcal{X}$ has enough points if $\mathcal{X}$ is locally of homotopy dimension $\le n$,

Remark 2.2.4.5 (Godement resolution)**.** Let $\mathcal{X}$ be an ∞-topos and $X \in \mathcal{X}$ an object. If there exists a space $K \in \mathcal{S}$, a geometric morphism $p_* : \mathcal{S} \to \mathcal{X}$ (i.e. a point), and an equivalence $X \simeq p_* K$, then we say that X is a *skyscraper object (at p)*. Let $\mathsf{Sky}_{\prod}(\mathcal{X}) \subset \mathcal{X}$ be the smallest full subcategory stable under finite products that contains all skyscraper objects. The functor $\mathsf{Sky}^{op}_{\prod} \hookrightarrow \mathcal{X}^{op}$ extends to a colimit preserving functor $\Psi : \mathrm{Fun}^{\pi}(\mathsf{Sky}_{\prod}, \mathcal{S}) \to \mathcal{X}^{op}$. Then it can be shown that the functor Ψ is essentially surjective if and only if $\mathcal{X}$ has enough points.

We have looked in some detail at 1-groupoid objects and their spaces of torsors in ∞-topoi. In the remainder of this subsection, we will study *n-groupoids internal in some ∞-topos.*

Notation 2.2.4.6. Let $\mathcal{C}$ be an arbitrary ∞-category and let $X_\bullet : \mathbf{N}(\mathbf{\Delta}^{op}) \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$, then we denote by $X_\bullet^K$ the *matching object*

$$X_\bullet^K := \lim_{(\Delta^n \to K) \in \mathbf{N}(\mathbf{\Delta}^{op})_{K/}} X_n,$$

provided this limit exists. Consider the faithful inclusion $i : \mathbf{N}(\mathbf{\Delta})^{\mathrm{nd}}_{/K} \hookrightarrow \mathbf{N}(\mathbf{\Delta})_{/K}$ on the nondegenerate simplices in K and assume that every face of every nondegenerate simplex is nondegenerate, then the Eilenberg-Zilber lemma implies that i has a left adjoint and is therefore left cofinal. In this case, it follows, for instance, that if $\mathcal{C}$ has finite limits, the object $X_\bullet^K$ exists if K is a finite simplicial set.
Suppose that for some fixed simplicial set K and simplicial object $X_\bullet$ in $\mathcal{C}$, all the matching objects $X_\bullet^{K \times \Delta^n}$ exist in $\mathcal{C}$. Then we denote by $[K, X_\bullet]_\bullet \in \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{C})$ the simplicial object whose n-simplices are given by the object $X_\bullet^{K \times \Delta^n}$ and whose face and degeneracy operators are the ones induced from the canonical cosimplicial simplicial set $\mathbf{\Delta} \hookrightarrow \mathsf{Set}_\Delta$.

Remark 2.2.4.7. If $\mathcal{C}$ has all small limits, it is not very difficult to enhance the assignment $(K, X_\bullet) \mapsto [K, X_\bullet]_\bullet$ to the data of a tensoring of $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{C})^{op}$ over the Cartesian monoidal category Set_Δ.

It is proven in [Lur17b], prop. 6.1.2.6 that a simplicial object $U_\bullet$ is a groupoid if and only if for every $n \geq 2$ and $0 \leq i \leq n$, the map $U_n \to U_\bullet^{\Lambda_i^n}$ is an equivalence. The following definition is essentially due to Duskin and Glenn.

Notation 2.2.4.8. Recall that a *semitopos* is a presentable ∞-category $\mathcal{X}$ such that colimits are universal in $\mathcal{X}$, and the Čech nerve of any morphism in $\mathcal{X}$ determines an effective groupoid. In a semitopos, effective epimorphisms and n-truncated morphisms are stable under the formation of pullbacks. Moreover, equivalences and effective epimorphisms are reflected by pullback functors along effective epimorphisms.

Definition 2.2.4.9. Let $\mathcal{X}$ be a semitopos.

(1) Let $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial objects in $\mathcal{X}$ is an *n-fibration* if f satisfies the following conditions.

- For all $m \geq 1$ and all $0 \leq i \leq m$, the natural map $X_m \to X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m$ is an effective epimorphism.
- For $k > n$, the natural map $X_k \to X_\bullet^{\Lambda_i^k} \times_{Y_\bullet^{\Lambda_i^k}} Y_k$ is an equivalence. Note that for $n = \infty$, this condition does not apply.

We call an ∞-fibration simply a *fibration*. A simplicial object $X_\bullet$ is an *n-hypergroupoid* if the map $X_\bullet \to *$ to a final object is an n-fibration. We denote by $\mathsf{Gpd}_n(\mathcal{X})$ the full subcategory of $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}_+^{op}), \mathcal{X})$ spanned by n-hypergroupoids.

(2) Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial objects is a *trivial n-fibration* if f satisfies the following conditions.

- For all $m \geq 0$, the natural map $X_m \to X_\bullet^{\partial\Delta^m} \times_{Y_\bullet^{\partial\Delta^m}} Y_m$ is an effective epimorphism.
- For $k \geq n$, the natural map $X_k \to X_\bullet^{\partial\Delta^k} \times_{Y_\bullet^{\partial\Delta^k}} Y_k$ is an equivalence. Note that for $n = \infty$, this condition does not apply.

We call a trivial ∞-fibration simply a *trivial fibration*, or a *hypercover*. A simplicial object $X_\bullet$ is a *trivial n-hypergroupoid* if the map $X_\bullet \to *$ to a final object is a trivial n-fibration.

Example 2.2.4.10. For $n = 1$ and $\mathcal{X}$ a semitopos, an n-hypergroupoid is simply a groupoid object in $\mathcal{X}$. If $G_\bullet$ is a groupoid object in an ∞-topos $\mathcal{X}$ and $G'_\bullet$ is a $G_\bullet$-torsor, then the map $G'_\bullet \to G_\bullet$ is a 1-fibration. For $m > 1$, we clearly have that

$$G'_m \longrightarrow G'^{\Lambda_i^m}_\bullet \times_{G_\bullet^{\Lambda_i^m}} G_m$$

is an equivalence since both $G_m \to G_\bullet^{\Lambda_i^m}$ and $G'_m \to G'^{\Lambda_i^m}_\bullet$ are equivalences. For $m = 1$, the maps $G'_1 \to G'_0 \times_{G_0} G_1$ are equivalences, so in particular effective epimorphisms.

We study n-hypergroupoids and n-fibrations in ∞-topoi for some *finite* n. The following easy lemma summarizes the elementary consequences of the definition of (trivial) n-fibrations in semitopoi.

Lemma 2.2.4.11. *Let $\mathcal{X}$ be a semitopos.*

(1) *If $f : X_\bullet \to Y_\bullet$ is a trivial n-fibration, then f is an n-fibration.*

(2) *If $f : X_\bullet \to Y_\bullet$ is a trivial fibration and an n-fibration and X is an ∞-hypergroupoid, then f is a trivial n-fibration.*

(3) *For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the classes of n-fibrations and trivial n-fibrations are stable under compositions in* $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{X})$.

(4) *Let $\Delta^2 \to \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{X})$ be a diagram depicted as*

$$\begin{tikzcd} & Y_\bullet \arrow[dr, "g"] & \\ X_\bullet \arrow[ur, "f"] \arrow[rr, "h"] & & Z_\bullet \end{tikzcd}$$

and suppose all maps are fibrations. Then if g is an n-fibrations, f is an n-fibration if and only if h is an n-fibration. The same result holds for trivial fibrations.

(5) *For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the classes of n-fibrations and trivial n-fibrations are stable under the formation of pullbacks in* $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{X})$.

Proof. (1) We should show that the map

$$X_m \longrightarrow X_\bullet^{\Lambda_i^m} \times_{X_\bullet^{\Lambda_i^m}} Y_m$$

is an effective epimorphism for all m and an equivalence for $m > n$. The pushout $\partial\Delta^m = \Lambda_i^m \coprod_{\partial\Delta^{m-1}} \Delta^{m-1}$ yields a pullback diagram

$$\begin{tikzcd} X_\bullet^{\partial\Delta^m} \times_{Y_\bullet^{\partial\Delta^m}} Y_m \arrow[r] \arrow[d] & X_{m-1} \arrow[d] \\ X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m \arrow[r] & X_\bullet^{\partial\Delta^{m-1}} \times_{Y_\bullet^{\partial\Delta^{m-1}}} Y_{m-1} \end{tikzcd}$$

By hypothesis, the right vertical map is an effective epimorphism and for $m > n$ an equivalence. Since we have a factorization

$$X_m \longrightarrow X_\bullet^{\partial\Delta^m} \times_{Y_\bullet^{\partial\Delta^m}} Y_m \longrightarrow X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m$$

and the first map is an effective epimorphism and for $m \geq n$ an equivalence, we conclude.

(2) We should show that the map

$$X_m \longrightarrow X_\bullet^{\partial\Delta^m} \times_{Y_\bullet^{\partial\Delta^m}} Y_m$$

is an equivalence for $m \geq n$. Consider again the composition

$$X_m \xrightarrow{\alpha} X_\bullet^{\partial\Delta^m} \times_{Y_\bullet^{\partial\Delta^m}} Y_m \xrightarrow{\beta} X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m.$$

For $m \geq n+1$, this composition is an equivalence. Since the map α is an effective epimorphism, both maps are equivalences. For the remaining case $m = n$ we consider again the pullback diagram

$$\begin{tikzcd} X_\bullet^{\partial\Delta^{n+1}} \times_{Y_\bullet^{\partial\Delta^{n+1}}} Y_{n+1} \arrow[r, "\gamma"] \arrow[d, "\beta"] & X_n \arrow[d, "\zeta"] \\ X_\bullet^{\Lambda_i^{n+1}} \times_{Y_\bullet^{\Lambda_i^{n+1}}} Y_{n+1} \arrow[r, "\theta"] & X_\bullet^{\partial\Delta^n} \times_{Y_\bullet^{\partial\Delta^n}} Y_n \end{tikzcd}$$

The left vertical map β is an equivalence as was just proven, and the right vertical map ζ is an effective epimorphism because f is a hypercover. We note that the composition

$$X_{n+1} \xrightarrow{\alpha} X_\bullet^{\partial\Delta^{n+1}} \times_{Y_\bullet^{\partial\Delta^{n+1}}} Y_{n+1} \xrightarrow{\gamma} X_n$$

is an effective epimorphism because the face map $\Delta^n \to \Delta^{n+1}$ is anodyne, so it follows that γ is an effective epimorphism as well. Thus, $\zeta \circ \gamma$ is an effective epimorphism, so the lower horizontal map θ is one as well. Since β is an equivalence, the right vertical map ζ is also an equivalence, by [Lur17b], lem. 6.2.3.16.

(3) Let $\Delta^2 \to \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{X})$ be a diagram of simplicial objects depicted as

$$\begin{tikzcd} & Y_\bullet \arrow[dr, "g"] & \\ X_\bullet \arrow[ur, "f"] \arrow[rr, "h"] & & Z_\bullet \end{tikzcd}$$

and suppose that g and f are n-fibrations. The map

$$X_m \longrightarrow X_\bullet^{\Lambda_i^m} \times_{Z_\bullet^{\Lambda_i^m}} Z_m$$

factorizes as

$$X_m \longrightarrow X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m \longrightarrow X_\bullet^{\Lambda_i^m} \times_{Z_\bullet^{\Lambda_i^m}} Z_m.$$

The first map is an effective epimorphism and for $m > n$ an equivalence. The second map fits into a pullback diagram

$$\begin{tikzcd} X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m \arrow[r] \arrow[d] & Y_m \arrow[d] \\ X_\bullet^{\Lambda_i^m} \times_{Z_\bullet^{\Lambda_i^m}} Z_m \arrow[r] & Y_\bullet^{\Lambda_i^m} \times_{Z_\bullet^{\Lambda_i^m}} Z_m \end{tikzcd}$$

where the right vertical map is an effective epimorphism and for $m > n$ an equivalence. The same argument applies for trivial n-fibrations.

(4) We need to show that for $k > n$, the first map in the diagram

$$X_k \longrightarrow X_\bullet^{\Lambda_i^k} \times_{Y_\bullet^{\Lambda_i^k}} Y_k \longrightarrow X_\bullet^{\Lambda_i^k} \times_{Z_\bullet^{\Lambda_i^k}} Z_k$$

is an equivalence if and only the composition is an equivalence, but the second map is a pullback of an equivalence by assumption. The same argument applies for trivial n-fibrations.

(5) Let $Z_\bullet \to Y_\bullet$ be an arbitrary map between simplicial objects in $\mathcal{X}$, and let $X_\bullet \to Y_\bullet$ be an n-fibration. We are required to show that the map

$$X_m \times_{Y_m} Z_m \longrightarrow X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Z_\bullet^{\Lambda_i^m} \times_{Z_\bullet^{\Lambda_i^m}} Z_m \simeq X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Z_m$$

is an effective epimorphism and for $m > n$ an equivalence. Consider the following diagram

$$\begin{tikzcd} X_m \times_{Y_m} Z_m \arrow[r] \arrow[d] & X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Z_m \arrow[r] \arrow[d] & Z_m \arrow[d] \\ X_m \arrow[r, "\alpha"] & X_\bullet^{\Lambda_i^m} \times_{Y_\bullet^{\Lambda_i^m}} Y_m \arrow[r] \arrow[d] & Y_m \arrow[d] \\ & X_\bullet^{\Lambda_i^m} \arrow[r] & Y_\bullet^{\Lambda_i^m} \end{tikzcd}$$

in which all squares are pullbacks. Since the indicated map α is an effective epimorphism and for $m > n$ an equivalence, we conclude. The same argument shows that trivial n-fibrations are stable under the formation of pullbacks.

□

Now we will give some results showing that fibrations and trivial fibrations behave well with respect to the geometric realization functor. The common idea in the proofs below is to use Boolean localization [Jar15] to reduce to the case $\mathcal{X} = \mathcal{S}$, where the results are amenable to elementary bisimplicial homotopy theory. To facilitate this strategy, we recall that the diagonal of a bisimplicial set $X_{\bullet\bullet}$ is weakly equivalent to the homotopy colimit of the simplicial diagram $[n] \mapsto X_{n\bullet}$ (or of the diagram $[n] \to X_{\bullet n}$). We will need a functorial version of this fact. The diagonal functor $\Delta^* : \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \longrightarrow \mathsf{Set}_\Delta^{\mathcal{C}^{op}}$ is obtained by pulling back along the functor

$$\mathbf{\Delta}^{op} \times \mathcal{C}^{op} \xrightarrow{\Delta \times \mathrm{id}} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \times \mathcal{C}^{op}.$$

The functor Δ^* admits a right adjoint Δ_* that takes a simplicial presheaf $\mathcal{F}: \mathcal{C}^{op} \to \mathsf{Set}_\Delta$ to the composition

$$\mathcal{C}^{op} \xrightarrow{\mathcal{F}} \mathsf{Set}_\Delta \xrightarrow{\mathrm{Fun}(\mathbf{\Delta}^\bullet, -)} \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta)$$

where the second functor sends a simplicial set S to the bisimplicial set whose (n,k)-bisimplices are given by $\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n \times \Delta^k, S)$. The functor Δ^* preserves weak equivalences and sends Reedy cofibrations to cofibrations for both the projective and injective model structures on $\mathsf{Set}_\Delta^{\mathcal{C}^{op}}$, so we have a Quillen adjunction

$$\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \underset{\Delta_*}{\overset{\Delta^*}{\rightleftarrows}} \mathsf{Set}_\Delta^{\mathcal{C}^{op}}$$

of combinatorial simplicial model categories. This is a *simplicial* Quillen adjunction: let $\mathcal{F}_\bullet$ be a bisimplicial presheaf, then for $S \in \mathsf{Set}_\Delta$, the tensoring $\mathcal{F}_\bullet \otimes S$ is given by the product $\mathcal{F}_\bullet \times S$, where we now view S as the simplicial object constant on the simplicial presheaf taking the constant value S. Since the functor Δ^* preserves limits, we have $\Delta^*(\mathcal{F}_\bullet \otimes S) \cong \Delta^*(\mathcal{F}_\bullet) \otimes S$ functorially in $\mathcal{F}_\bullet$ and S. To see that Δ_* is a simplicial adjoint to Δ^*, we note that the fact that Δ^* is simplicial induces via the adjunction a map

$$\Delta_*(\mathcal{F})^S \longrightarrow \Delta_*(\mathcal{F}^S)$$

between cotensorings for $\mathcal{F}$ a simplicial presheaf and S a simplicial set. It is immediate from the definition of Δ_* that this map is an isomorphism, which implies that $(\Delta^* \dashv \Delta_*)$ is a simplicial adjunction.

Lemma 2.2.4.12. *Let $\mathcal{C}$ be a small fibrant simplicial category, and let $\mathsf{Set}_\Delta^{\mathcal{C}^{op}}$ be the category of simplicial presheaves on $\mathcal{C}^{op}$, equipped with the injective model structure making it a combinatorial simplicial model category. Then for the Reedy model structure on $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}})$, the left derived functor of the simplicial left Quillen diagonal functor*

$$\Delta^* : \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \longrightarrow \mathsf{Set}_\Delta^{\mathcal{C}^{op}}$$

is equivalent to the colimit functor

$$\mathrm{colim} : \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathsf{PShv}(\mathbf{N}(\mathcal{C})) \longrightarrow \mathsf{PShv}(\mathbf{N}(\mathcal{C}))$$

in the ∞-categorical sense (e.g. produced by [Lur17b], prop. 4.3.2.15) defined up to a contractible space of choices.

Proof. It follows from theorem 7.5.30 of [Cis18] that we have an adjunction between derived functors. To see that Δ^* is a colimit functor, it suffices to show that the right derived functor of Δ_* is equivalent to the constant diagram functor

$$\mathsf{PShv}(\mathbf{N}(\mathcal{C})) \longrightarrow \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathsf{PShv}(\mathbf{N}(\mathcal{C})).$$

It follows from theorem 7.9.8 of [Cis18] that the constant diagram functor is the right derived functor of the constant diagram functor

$$\mathrm{cst} : \mathsf{Set}_\Delta^{\mathcal{C}^{op}} \longrightarrow \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}}).$$

We can view this functor as the one that takes a simplicial presheaf $\mathcal{F}: \mathcal{C}^{op} \to \mathsf{Set}_\Delta$ to the composition

$$\mathcal{C}^{op} \xrightarrow{\mathcal{F}} \mathsf{Set}_\Delta \xrightarrow{\mathrm{Fun}(*, -)} \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta)$$

where we view $*$ as the constant cosimplicial simplicial set on the final object. The map of cosimplicial simplicial sets $\Delta^\bullet \to *$ induces a (simplicial) natural transformation $\alpha : \mathrm{cst} \to \Delta_*$. On the category $(\mathsf{Set}_\Delta^{\mathcal{C}^{op}})^{\mathrm{f}}$, the natural transformation α is a weak equivalence since for each pair $(X, [n]) \in \mathsf{Set}_\Delta \times \mathcal{C}^{op}$ we have a retraction

$$\mathcal{F}(X) \longrightarrow \mathrm{Fun}(\Delta^n, \mathcal{F}(X)) \xrightarrow{\mathrm{ev}_{\{0\}}} \mathcal{F}(X)$$

where the second map is a trivial Kan fibration because $\mathcal{F}(X)$ is a Kan complex. $\square$

Corollary 2.2.4.13. *The following diagram among ∞-categories*

$$\begin{array}{ccc}
\mathbf{N}(\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}})) & \xrightarrow{\Delta^*} & \mathbf{N}(\mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathsf{PShv}(\mathbf{N}(\mathcal{C})) & \xrightarrow{\mathrm{colim}} & \mathsf{PShv}(\mathbf{N}(\mathcal{C})
\end{array}$$

canonically commutes, where the vertical functors implement localizations. More precisely, the left vertical functor is obtained by adjunction from the functor

$$\mathbf{N}(\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \times \mathbf{\Delta}^{op}) \xrightarrow{\mathrm{ev}} \mathbf{N}(\mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \xrightarrow{\gamma} \mathsf{PShv}(\mathbf{N}(\mathcal{C})$$

Proof. In the category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}})$, viewed as an ∞-category with weak equivalences and cofibrations, every object is cofibrant, so the functor $\gamma \circ \Delta^*$ carries weak equivalences to equivalences in $\mathsf{PShv}(\mathbf{N}(\mathcal{C})$. It follows that the left derived functor fits into a commuting diagram

$$\begin{tikzcd}
\mathbf{N}(\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}_\Delta^{\mathcal{C}^{op}})) \arrow[r, "\Delta^*"] \arrow[d] & \mathbf{N}(\mathsf{Set}_\Delta^{\mathcal{C}^{op}}) \arrow[d, "\gamma"] \\
\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathsf{PShv}(\mathbf{N}(\mathcal{C})) \arrow[r, "\mathbf{L}\Delta^*"] & \mathsf{PShv}(\mathbf{N}(\mathcal{C})
\end{tikzcd}$$

but lemma 2.2.4.12 shows that the lower horizontal map is equivalent to colim. $\square$

Homotopy colimits of diagrams in model categories are computed by taking colimits of projectively cofibrant replacements. Using the previous result, we can show that for simplicial diagrams in simplicial sets, there is a way to extract a homotopy colimit from an *injectively fibrant* replacement.

Corollary 2.2.4.14. *Let $X_\bullet$ be a simplicial space and let $X_\bullet \to X'_\bullet$ be a Reedy fibrant replacement of $X_\bullet$. Then for any $n \geq 0$, there is an isomorphism*

$$X_n'^{\perp} \cong \operatorname*{hocolim}_{\mathbf{\Delta}^{op}} X_\bullet$$

in the homotopy category $\mathcal{H}$.

Proof. Since weakly equivalent diagrams have weakly equivalent homotopy colimits, we may assume that $X_\bullet = X'_\bullet$, that is, $X_\bullet$ is Reedy fibrant. Let $S \hookrightarrow T$ be a trivial cofibration of simplicial sets. By formal nonsense of two variable adjunctions [JT07], there is a bijection of lifting problems

$$\left\{ \begin{tikzcd} S \arrow[r] \arrow[d, hook] & X_m \arrow[d] \\ T \arrow[r] \arrow[ur, dotted] & X_\bullet^{\partial\Delta^m} \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & X_\bullet^{\perp T} \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dotted] & X_\bullet^{\perp S} \end{tikzcd} \right\}$$

so we deduce that the map $X_\bullet^{\perp T} \to X_\bullet^{\perp S}$ is a trivial Kan fibration. In particular, for every injective map of ordinals $[n] \to [m]$, the face map $X_n^\perp \to X_m^\perp$ is a trivial fibration. By 2-out-of-3, all degeneracy maps of $X_\bullet^\perp$ are trivial cofibrations, so it follows that the diagram $X_\bullet^\perp$ is essentially constant, and its homotopy colimit is therefore equivalent to any of the simplicial sets $X_n^\perp$. Since we have weak equivalences

$$\operatorname*{hocolim}_{\mathbf{\Delta}^{op}} X_\bullet^\perp \simeq \Delta^*(X_\bullet^\perp) = \Delta^*(X_\bullet) \simeq \operatorname*{hocolim}_{\mathbf{\Delta}^{op}} X_\bullet$$

by lemma 2.2.4.12, we conclude. $\square$

The following proposition is a version of the Bousfield-Friedlander theorem for bisimplicial sets ([GJ99], chapter IV, thm. 4.9) in the setting of ∞-topoi.

Proposition 2.2.4.15. *Let $\mathcal{X}$ be an ∞-topos, and let $X_\bullet \to Y_\bullet$ be a fibration in $\mathcal{X}$, then for any simplicial object $Z_\bullet$, the canonical map $|X_\bullet \times_{Y_\bullet} Z_\bullet| \to |X_\bullet| \times_{|Y_\bullet|} |Z_\bullet|$ is ∞-connective.*

Proof. We have a functor $Q : \Lambda_2^2 \times \mathbf{N}(\mathbf{\Delta}^{op}) \to \mathcal{X}$ determining the diagram

$$\begin{tikzcd}
& X_\bullet \arrow[d, "f"] \\
Z_\bullet \arrow[r] & Y_\bullet
\end{tikzcd}$$

We claim that the functor sending an ∞-topos $\mathcal{X}$ to the ∞-category of diagrams of shape $\Lambda_2^2 \times \mathbf{N}(\mathbf{\Delta}^{op})$ such that the vertical map is an ∞-fibration in $\mathcal{X}$ admits a locally coherent classifying ∞-topos. More precisely, we assert the following.

($*$) There is a locally coherent ∞-topos $\mathcal{Y}$ that comes equipped with a functor $P : \Lambda_2^2 \times \mathbf{N}(\mathbf{\Delta}^{op}) \to \mathcal{Y}$ determining a diagram

$$\begin{tikzcd}
& R_\bullet \arrow[d] \\
T_\bullet \arrow[r] & S_\bullet
\end{tikzcd}$$

such that for each ∞-topos $\mathcal{X}$, restriction along P determines an equivalence

$$\mathrm{Fun}^*(\mathcal{Y},\mathcal{X}) \xrightarrow{\simeq} \mathrm{Fun}'(\Lambda^2_2 \times \mathbf{N}(\mathbf{\Delta}^{op}),\mathcal{X})$$

where $\mathrm{Fun}'(\Lambda^2_2 \times \mathbf{N}(\mathbf{\Delta}^{op}),\mathcal{X})$ denotes the full subcategory spanned by pullback diagrams where $X_\bullet \to Y_\bullet$ is a fibration between simplicial objects.

Assuming this for a moment, we may find some algebraic morphism $g^* : \mathcal{Y} \to \mathcal{X}$ such that $Q \simeq g^* \circ P$. As g^* preserves finite limits and small colimits, it suffices to show that the canonical map $h : |R_\bullet \times_{S_\bullet} T_\bullet| \to |R_\bullet| \times_{|S_\bullet|} |T_\bullet|$ is ∞-connective in $\mathcal{Y}$. Since $\mathcal{Y}$ is locally coherent, this will follow once we show that for every point $p^* : \mathcal{S} \to \mathcal{Y}$, the induced map $p^*(h)$ is an equivalence. Using again that $\mathcal{Y}$ is a classifying ∞-topos for pullbacks along fibrations, we are reduced to proving the proposition for $\mathcal{X} = \mathcal{S}$. We can identify Q with a diagram $\Lambda^2_2 \to \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}),\mathcal{S})$, so using [Lur17b], prop. 4.2.4.4, we may suppose that Q with is an injectively fibrant diagram $\Lambda^2_2 \to \mathrm{Fun}(\mathbf{\Delta}^{op},\mathsf{Set}_\Delta)$ where the category of bisimplicial sets is equipped with the injective model structure; that is, we may suppose that the maps $X_\bullet \to Y_\bullet$ and $Z_\bullet \to Y_\bullet$ are injective fibrations between injectively fibrant diagrams of simplicial sets. As $X_\bullet \to Y_\bullet$ is a Reedy fibration, the relative matching maps

$$X_n \longrightarrow X_\bullet^{\Lambda^n_i} \times_{Y_\bullet^{\Lambda^n_i}} Y_n$$

are Kan fibrations. By assumption, the map

$$\pi_0(X_n) \longrightarrow \pi_0(X_\bullet^{\Lambda^n_i} \times_{Y_\bullet^{\Lambda^n_i}} Y_n)$$

is a surjection, so the map $X_n \to X_\bullet^{\Lambda^n_i} \times_{Y_\bullet^{\Lambda^n_i}} Y_n$ is a levelwise surjection for all n and all $0 \le i \le n$. Denoting by $V_\bullet^\perp$ the simplicial object $\mathrm{Fun}(\mathbf{\Delta}^{op},\mathsf{Set}_\Delta)$ obtained by adjunction from a simplicial object $V_\bullet$ by interchanging the two opposite categories of ordinals, we deduce that $X_\bullet^\perp \to Y_\bullet^\perp$ is a levelwise fibration. This implies that the map $\Delta^*(X_\bullet) \to \Delta^*(Y_\bullet)$ is a Kan fibration, which by right propereness of the Kan-Quillen model structure on Set_Δ guarantees that the pullback diagram

$$\begin{tikzcd}
\Delta^*(X_\bullet \times_{Y_\bullet} Z_\bullet) \arrow[r] \arrow[d] & \Delta^*(X_\bullet) \arrow[d] \\
\Delta^*(Z_\bullet) \arrow[r] & \Delta^*(Y_\bullet)
\end{tikzcd}$$

is a homotopy pullback diagram. It follows from lemma 2.2.4.12 that after a fibrant replacement, this diagram is equivalent to the square

$$\begin{tikzcd}
|X_\bullet \times_{Y_\bullet} Z_\bullet| \arrow[r] \arrow[d] & |X_\bullet| \arrow[d] \\
|Z_\bullet| \arrow[r] & |Y_\bullet|.
\end{tikzcd}$$

We are left to prove the assertion $(*)$. Let $\mathcal{J} : \Lambda^2_2 \times \mathbf{N}(\mathbf{\Delta}^{op}) \to \mathcal{C}$ be the ∞-category obtained from $\Lambda^2_2 \times \mathbf{N}(\mathbf{\Delta}^{op})$ by freely adding finite limits according the procedure described in [Lur17b], section 5.3.6 and remark 2.1.0.5; that is, we have for each ∞-category admitting finite limits $\mathcal{D}$ an equivalence $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\Lambda^2_2 \times \mathbf{N}(\mathbf{\Delta}^{op}),\mathcal{D})$. Equip $\mathcal{C}$ with the coarsest Grothendieck pretopology such that the single map

(i) $\mathcal{J}(1,n) \longrightarrow \mathcal{J}(1,\bullet)^{\Lambda^n_i} \times_{\mathcal{J}(2,\bullet)^{\Lambda^n_i}} \mathcal{J}(2,n)$

constitutes a covering family for all n and all $0 \le i \le n$. This Grothendieck topology is finitary: if it were not, we would obtain via example 2.2.2.11 a coarser finitary pretopology containing the covering family above. Thus, $\mathsf{Shv}(\mathcal{C})$ is a locally coherent ∞-topos. Now it follows easily from [Lur17b], prop. 6.2.3.20 that the composition

$$\Lambda^2_2 \times \mathbf{N}(\mathbf{\Delta}^{op}) \xrightarrow{\mathcal{J}} \mathcal{C} \xrightarrow{j} \mathsf{PShv}(\mathcal{C}) \xrightarrow{L} \mathsf{Shv}(\mathcal{C})$$

satisfies the conditions of assertion $(*)$. □

The following proposition appears in [Lur11c], and its proof is slightly easier version of that of proposition 2.2.4.15.

Proposition 2.2.4.16. *Let $\mathcal{X}$ be an ∞-topos, and let $f : X_\bullet \to Y_\bullet$ be a hypercover (i.e, a trivial ∞-fibration) in $\mathcal{X}$, then the canonical map $|X_\bullet| \to |Y_\bullet|$ is ∞-connective.*

Proof. We can reduce this proposition to the case of spaces using the same stratagem as in the proof of proposition 2.2.4.15: we construct a locally coherent classifying ∞-topos for hypercovers between simplicial objects in $\mathcal{X}$ to reduce to the case $\mathcal{X} = \mathcal{S}$. Using [Lur17b], prop. 4.2.4.4 we may suppose that the hypercover f is given by an injectively fibrant diagram

$$Q : \Delta^1 \longrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathsf{Set}_\Delta)$$

where $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathsf{Set}_\Delta)$ is endowed with the injective model structure. As $X_\bullet \to Y_\bullet$ is a Reedy fibration, the relative matching maps

$$X_n \longrightarrow X_\bullet^{\partial\Delta^n} \times_{Y_\bullet^{\partial\Delta^n}} Y_n$$

are Kan fibrations. By assumption, the maps

$$\pi_0(X_n) \longrightarrow \pi_0(X_\bullet^{\partial\Delta^n} \times_{Y_\bullet^{\partial\Delta^n}} Y_n)$$

are surjections, so using that the relative matching maps above are Kan fibrations, we find that the maps

$$X_n \longrightarrow X_\bullet^{\partial\Delta^n} \times_{Y_\bullet^{\partial\Delta^n}} Y_n$$

are levelwise surjections. Denoting by $V_\bullet^\perp$ the simplicial object $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathsf{Set}_\Delta)$ obtained by adjunction from a simplicial object $V_\bullet$ by interchanging the two opposite categories of ordinals, we deduce that $f^\perp : X_\bullet^\perp \to Y_\bullet^\perp$ is a levelwise trivial fibration. The diagonal functor preserves weak equivalences and takes injective (Reedy) fibrations that are also horizontal fibrations to Kan fibrations, so we deduce that the map

$$\Delta^*(X_\bullet) \longrightarrow \Delta^*(Y_\bullet)$$

is a trivial fibration. But this map is isomorphic to the map $|X_\bullet| \to |Y_\bullet|$ in $h\mathcal{S}$, by lemma 2.2.4.12. □

We now give a converse to proposition 2.2.4.16.

Proposition 2.2.4.17. *Let $\mathcal{X}$ be an ∞-topos and suppose that $f : X_\bullet \to Y_\bullet$ be a fibration between ∞-hypergroupoids in $\mathcal{X}$ and suppose that $|X_\bullet| \to |Y_\bullet|$ is ∞-connective. Then f is a trivial fibration.*

Proof. By Boolean localization, there exists a surjective algebraic morphism $\mathcal{X} \to \mathsf{Shv}(B)$ to the ∞-topos of sheaves on a complete Boolean algebra, so we may suppose that $\mathcal{X} = \mathsf{Shv}(B)$. The ∞-topos of sheaves on any complete Boolean algebra has homotopy dimension 0, so using that $\mathsf{Shv}(B)_{/j(U)} \simeq \mathsf{Shv}(B_{/U})$ and that $B_{/U}$ is again a complete Boolean algebra for any $U \in B$, we see that a morphism $X \to Y$ is an effective epimorphism in $\mathsf{Shv}(B)$ if and only if $X(U) \to Y(U)$ is an effective epimorphism in $\mathcal{S}$ for each $U \in B$. We claim that this property of $\mathsf{Shv}(B)$ implies the following:

(∗) Let $X_\bullet$ be an ∞-hypergroupoid in $\mathsf{Shv}(B)$, then the canonical map $|X_\bullet(U)| \to |X_\bullet|(U)$ of spaces is an equivalence for every $U \in B$.

Assuming this for the moment, it follows easily that we are reduced to proving the proposition for $\mathcal{X} = \mathcal{S}$. We may assume that $X_\bullet \to Y_\bullet$ is an injective fibration between injectively fibrant diagrams, so that the relative matching maps $X_k \to X_\bullet^{\partial\Delta^k} \times_{Y_\bullet^{\partial\Delta^k}} Y_k$ are Kan fibrations. We wish to show that

(•) The map $X_k \to X_\bullet^{\partial\Delta^k} \times_{Y_\bullet^{\partial\Delta^k}} Y_k$ is a surjection on connected components for all k. As this map is a Kan fibration, this is equivalent to the map being a surjection in simplicial degree 0. In turn, this means that $f_{\bullet 0} : X_{\bullet 0} \to Y_{\bullet 0}$ is a trivial Kan fibration.

To prove this, it suffices to show that the map $f_{\bullet 0} : X_{\bullet 0} \to Y_{\bullet 0}$ has contractible fibres, since it is a Kan fibration between Kan complexes. The fibre of $f_{\bullet 0}$ at any element $\{p\}$ of $Y_{0,0}$ is given by the Kan complex $(X_{\bullet\bullet} \times_{Y_{\bullet\bullet}} \{p\})_{\bullet 0}$, the horizontal simplicial set in degree 0 of the pullback of bisimplicial sets. We note that the bisimplicial set

$$F_{\bullet\bullet} := X_{\bullet\bullet} \times_{Y_{\bullet\bullet}} \{p\}$$

has the following properties:

(1) $F_{\bullet\bullet}$ is injectively fibrant with respect to the vertical model structure.

(2) The maps $F_{k\bullet} \to F_{\bullet\bullet}^{\Lambda_i^k}$ are levelwise surjections for all k and all $0 \le i \le k$; that is, the simplicial sets $F_{\bullet l}$ are Kan fibrant.

(3) The diagonal $\Delta^*(F_{\bullet\bullet})$ is a weakly contractible Kan complex.

Now we show that for bisimplicial sets that satisfy the properties (1) through (3) above, the horizontal simplicial set $F_{\bullet 0}$ is weakly contractible Kan complex (note that this implies that $F_{\bullet l}$ is then a weakly contractible for all $l \geq 0$ by (1)). Consider the flipped simplicial space $F_\bullet^\perp$. By corollary 2.2.4.14, this is a essentially constant simplicial space such that each level $F_n^\perp$ is weakly equivalent to the homotopy colimit of $F_\bullet$, so by (3), $F_n^\perp$ is weakly contractible for all n. Since $F_n^\perp$ is a Kan complex, we conclude.

It remains to prove assertion $(*)$. The map $|X_\bullet(U)| \to |X_\bullet|(U)$ is given by applying the evaluation functor ev_U to the map $\mathrm{colim}_{\mathbf{N}(\mathbf{\Delta}^{op})} X_\bullet \to L\mathrm{colim}_{\mathbf{N}(\mathbf{\Delta}^{op})} X_\bullet$ in $\mathsf{PShv}(B)$ exhibiting $L\mathrm{colim}_{\mathbf{N}(\mathbf{\Delta}^{op})} X_\bullet$ as a sheafification of $\mathrm{colim}_{\mathbf{N}(\mathbf{\Delta}^{op})} X_\bullet$ (taken in presheaves), so it suffices to show that this latter presheaf is already a sheaf. Since the diagonal is a homotopy colimit functor, it suffices to show that the map

$$h : \Delta^*(X_{\bullet\bullet}) \longrightarrow \mathbf{R}\Delta^*(X_{\bullet\bullet})$$

is an objectwise weak equivalence.

Let $\mathcal{C}$ be a category equipped with a Grothendieck topology, then recall that a morphism $Z_\bullet \to Z'_\bullet$ of simplicial objects in $\mathsf{Shv}_{\mathsf{Set}}(\mathcal{C}) \simeq \mathrm{Disc}(\mathsf{Shv}(\mathcal{C}))$ is a *local (trivial) fibration* if the map $Z_n \to Z^{\Lambda^n_i} \times_{Z'^{\Lambda^n_i}} Z'_n$ (the map $Z_n \to Z^{\partial\Delta^n} \times_{Z'^{\partial\Delta^n}} Z'_n$) is an effective epimorphism. Just as in the category Set_Δ, it is not hard to prove that a map that is both a local weak equivalence and a local fibration is a local trivial fibration (see for instance [Jar15], lemma 4.18). In $\mathsf{Shv}_{\mathsf{Set}}(B)$ for B a complete Boolean algebra, all effective epimorphism are objectwise epimorphisms (i.e. the axiom of choice holds in this topos), so local (trivial) fibrations are simply projective (trivial) fibrations. As $X_\bullet$ is an ∞-hypergroupoid, $\Delta^*(X_{\bullet\bullet})$ is locally fibrant and thus also projectively fibrant, and because $\mathbf{R}\Delta^*(X_{\bullet\bullet})$ is injectively fibrant for the local model structure, it is automatically projectively fibrant. It follows that we have a factorization

$$\begin{tikzcd} \Delta^*(X_{\bullet\bullet}) \arrow[r] \arrow[dr, "h"'] & \Delta^*(X_{\bullet\bullet}) \times_{\mathbf{R}\Delta^*(X_{\bullet\bullet})} \mathbf{R}\Delta^*(X_{\bullet\bullet})^{\Delta^1} \arrow[d] \\ & \mathbf{R}\Delta^*(X_{\bullet\bullet}) \end{tikzcd}$$

where the horizontal map is an objectwise weak equivalence (as it is a section of a projective trivial fibration) and the vertical map is a projective fibration and a local weak equivalence, and therefore a local trivial fibration. Since local trivial fibrations are projective trivial fibrations in $\mathsf{Shv}_{\mathsf{Set}}(B)$, we deduce that h is an objectwise weak equivalence. $\square$

Corollary 2.2.4.18. *Let $\mathcal{X}$ be a hypercomplete ∞-topos and suppose that $f : X_\bullet \to Y_\bullet$ is a fibration between ∞-hypergroupoids in $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{X})$, then f is a trivial fibration if and only if f induces an equivalence $|X_\bullet| \to |Y_\bullet|$.*

Remark 2.2.4.19. Let us remark that in an arbitrary (hypercomplete) ∞-topos $\mathcal{X}$, we *cannot* conclude that an n-fibration between n-hypergroupoids which induces an equivalence after geometric realization is a trivial n-fibration, because it is not necessarily the case that an n-fibration between n-hypergroupoids that is also a hypercover is a trivial n-fibration. However, this becomes if we slightly modify the notion of fibrations and hypergroupoids, when we work with ∞-topoi of sheaves on affine scheme-like objects in a quite general sense that come equipped with notions of submersive and local diffeomorphisms, for instance. In such a context, it is natural to demand that the matching maps of fibrations are not only epimorphisms of sheaves, but also a submersion or local diffeomorphism.

Remark 2.2.4.20. It is possible to remove the hypercompleteness assumption in the results above, but we do not bother as all our ∞-topoi will be hypercomplete.

2.2.5 $\mathcal{C}$-valued sheaves

One of the advantages of applying (higher) topos theory to geometry is the fact that it treats two kinds of mathematical objects on the same footing: ∞-topoi serve as generalized spaces underlying the geometric objects of interest, while the arena in which this geometry takes place forms itself an ∞-topos. In the first instance, it does not suffice to study bare ∞-topoi: we will have need of *structured* spaces, that is, we will need to consider notions of *sheaves of algebras and modules* on an ∞-topos. This subsection is meant as an introduction to this theory, containing the basic results that we will have need of.

Definition 2.2.5.1. Let $\mathcal{C}$ be an ∞-category and $\mathcal{X}$ an ∞-topos, then we denote

$$\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$$

for the ∞-category of *$\mathcal{C}$-valued sheaves*, the full subcategory spanned by those functors that preserve small limits.

Remark 2.2.5.2. Clearly we have an isomorphism $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \cong \mathrm{Fun}(\mathcal{X}, \mathcal{C}^{op})^{op}$, identifying $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ with the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{C}^{op})^{op}$ spanned by functors preserving small colimits. Suppose that $\mathcal{C}$ is locally small, then by [Lur17b], prop. 5.5.2.9 and cor. 5.5.2.10, this full subcategory coincides with $\mathrm{Fun}^{\mathrm{L}}(\mathcal{X}, \mathcal{C}^{op})^{op}$, so that $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ is identified with $\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{op}, \mathcal{C})$.

Definition 2.2.5.3. Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck pretopology $\mathcal{B}$ and let $\mathcal{D}$ be an ∞-category that admits small limits, then a functor $f : \mathcal{C}^{op} \to \mathcal{D}$ is a *$\mathcal{D}$-valued sheaf* if for every $C \in \mathcal{C}$ and every covering family $\{C_i \to C\}_{i \in I} \in \mathcal{B}(C)$, the Čech nerve $\check{\mathrm{C}}(h)_\bullet : \mathbf{N}(\mathbf{\Delta}^{op})^{\triangleright} \to \mathsf{PShv}(\mathcal{C})$ of the map

$$h : \coprod_i j(C_i) \longrightarrow j(C)$$

determines a colimit diagram $F \circ \check{\mathrm{C}}(h)_\bullet : \mathbf{N}(\mathbf{\Delta}^{op})^{\triangleright} \to \mathcal{D}^{op}$, where F is a left Kan extension of f^{op} along $j : \mathcal{C} \to \mathsf{PShv}(\mathcal{C})$. The ∞-category of $\mathcal{D}$-valued sheaves on $\mathcal{C}$ is denoted $\mathsf{Shv}_{\mathcal{D}}(\mathcal{C})$. If $\mathcal{C} = \mathbf{N}(\mathsf{Open}(X))$ for some topological space X, we write $\mathsf{Shv}_{\mathcal{D}}(X)$ for $\mathsf{Shv}_{\mathcal{D}}(\mathcal{C})$.
The functor f is a *hypersheaf* if the left Kan extension F of f along j sends every augmented semi-representable hypercover $C_\bullet : \mathbf{N}(\mathbf{\Delta}^{op}) \to \mathsf{PShv}(\mathcal{C})$ to a colimit diagram in $\mathcal{D}^{op}$. The ∞-category of $\mathcal{D}$-valued hypersheaves on $\mathcal{C}$ is denoted $\mathsf{Shv}^{\wedge}_{\mathcal{D}}(\mathcal{C})$.

Remark 2.2.5.4. If $\mathcal{D}$ is an arbitrary ∞-category, we say that a functor $j \circ f : \mathcal{C}^{op} \to \mathcal{D}$ is a sheaf if the composition $f : \mathcal{C} \to \mathsf{PShv}(\mathcal{D})$ is a sheaf.

Remark 2.2.5.5. If $\mathcal{D}$ is an ∞-category that admits small limits, then unwinding definition 2.2.5.3 above, we see that a functor $f : \mathcal{C}^{op} \to \mathcal{D}$ is a sheaf if the cosimplicial diagram

$$f(C) \longrightarrow \textstyle\prod_i f(C_i) \rightrightarrows \prod_{i,j} f(C_i \times_C C_j) \;\substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow}\; \cdots$$

is a limit diagram for all covering families $\{C_i \to C\}$. A similar explicit description holds for hypersheaves.

Remark 2.2.5.6 ([Lur11b], prop. 1.1.12)**.** The notation of definitions 2.2.5.1 and 2.2.5.3 is consistent in the following sense: let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck pretopology and let $\mathcal{D}$ be an ∞-category that admits small limits, then [Lur17b], prop. 5.5.4.20 implies that the functor

$$\mathsf{Shv}_{\mathcal{D}}(\mathsf{Shv}(\mathcal{C})) \xrightarrow{L\circ} \mathrm{Fun}'(\mathsf{PShv}(\mathcal{C})^{op}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$$

is fully faithful, where $\mathrm{Fun}'(\mathsf{PShv}(\mathcal{C})^{op}, \mathcal{D})$ is the full subcategory spanned by functors preserving small limits. The essential image consists of those colimit preserving functors $F : \mathsf{PShv}(\mathcal{C}) \to \mathcal{D}$ that take the class of maps that become an equivalence after sheafifying to equivalences in $\mathcal{D}$. This class is the strongly saturated collection $\overline{S}$ generated by the class S of Čech nerves of covering families. Because F is presumed to preserve colimits, the collection of maps in $\mathsf{PShv}(\mathcal{C})$ that become an equivalence after applying F is strongly saturated, so it contains $\overline{S}$ if and only if it contains S, which is the case precisely if F is a sheaf in accordance with definition 2.2.5.3. Thus we have an equivalence $\mathsf{Shv}_{\mathcal{D}}(\mathsf{Shv}(\mathcal{C})) \simeq \mathsf{Shv}_{\mathcal{D}}(\mathcal{C})$. We can repeat this argument with the class $\overline{S^{\wedge}}$ of morphisms in $\mathsf{PShv}(\mathcal{C})$ that become an equivalence after sheafifying *and* passing to the hypercompletion. This strongly saturated class is generated by the class $S^{\wedge}$ of maps $|C_\bullet| \to j(C)$ for $C_\bullet$ a semi-representable hypercover of C, so we see that the equivalence $\mathsf{Shv}_{\mathcal{D}}(\mathsf{Shv}(\mathcal{C})) \simeq \mathsf{Shv}_{\mathcal{D}}(\mathcal{C})$ restricts to an equivalence $\mathsf{Shv}_{\mathcal{D}}(\mathsf{Shv}^{\wedge}(\mathcal{C})) \simeq \mathsf{Shv}^{\wedge}_{\mathcal{D}}(\mathcal{C})$.

Lemma 2.2.5.7. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be an ∞-category that admits small limits, then the functor*

$$\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \longrightarrow \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$$

induced by $\Omega^\infty_{\mathcal{C}}$ induces an equivalence $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \simeq \mathcal{S}\mathrm{p}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}))$.

Proof. If $\mathcal{C}$ is presentable, this can be viewed as a consequence of the associativity (up to coherent homotopy) of the tensor product on presentable ∞-categories, which gives equivalences

$$(\mathcal{X} \otimes \mathcal{C}) \otimes \mathcal{S}\mathrm{p} \simeq \mathcal{X} \otimes (\mathcal{C} \otimes \mathcal{S}\mathrm{p}).$$

In general, we have the following argument. We have isomorphisms of simplicial sets

$$\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})) \cong \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_* \times \mathcal{X}^{op}, \mathcal{C}) \cong \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}))$$

Since the full subcategory $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$ is stable under limits, a functor $f : \mathcal{S}^{\mathrm{fin}}_* \to \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ is reduced excisive if and only if it is reduced excisive as a functor into $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$, which is the case if and only the corresponding functor $\mathcal{S}^{\mathrm{fin}}_* \times \mathcal{X}^{op} \to \mathcal{C}$ is reduced excisive in the first argument and limit-preserving in the second. Similarly, because $\mathcal{S}\mathrm{p}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ is stable under limits, a functor $g : \mathcal{X}^{op} \to \mathcal{S}\mathrm{p}(\mathcal{C})$ preserves limits if and only if the corresponding functor $\mathcal{S}^{\mathrm{fin}}_* \times \mathcal{X}^{op} \to \mathcal{C}$ is reduced excisive in the first argument and limit preserving in the second. We conclude that the isomorphisms above restrict to an isomorphism $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \cong \mathcal{S}\mathrm{p}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}))$ which intertwines the functor evaluating at S^0. $\square$

If $\mathcal{C}$ is presentable, then the proposition guarantees the existence of a canonical t-structure on $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$, but in general, it seems we cannot say much about this t-structure unless we put some extra conditions on $\mathcal{C}$. In the following, we will assume that $\mathcal{C}$ is *compactly generated*.

Lemma 2.2.5.8. *Let $\mathcal{C}$ be a compactly generated ∞-category, then for any ∞-topos $\mathcal{X}$, $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ is an accessible left exact localization of an ∞-category of $\mathcal{C}$-valued presheaves.*

Proof. We have a natural equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_0)$ for $\mathcal{C}_0$ the full subcategory of compact objects, then using [Lur17b], prop. 5.5.3.3, prop 5.3.5.10 and prop. 5.5.1.9 we have canonical equivalences

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{op}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{X}).$$

Realize $\mathcal{X}$ as a left exact accessible localization $L : \mathsf{PShv}(\mathcal{D}) \to \mathcal{X}$ for some small ∞-category $\mathcal{D}$, then the adjunction $\mathrm{Fun}(\mathcal{C}_0^{op}, \mathcal{X}) \leftrightarrows \mathrm{Fun}(\mathcal{C}_0^{op}, \mathsf{PShv}(\mathcal{D}))$ induced by the reflection L restricts to an adjunction $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{X}) \leftrightarrows \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathsf{PShv}(\mathcal{D}))$ because both L and the fully faithful inclusion $\mathcal{X} \hookrightarrow \mathsf{PShv}(\mathcal{D})$ are left exact. The counit of the adjunction $(L \dashv \iota)$ is an equivalence, so the counit of the induced adjunction on functor ∞-categories is one as well. Thus, the functor $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathsf{PShv}(\mathcal{D}))$ is fully faithful, so this functor is right adjoint to an accessible ([Lur17b], prop. 5.5.1.2) localization. This localization is left exact because L is left exact and limits in the ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{X})$ and $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathsf{PShv}(\mathcal{D}))$ are computed objectwise. □

Corollary 2.2.5.9. *Let $\mathcal{C}$ be a compactly generated ∞-category, then for any ∞-topos $\mathcal{X}$, filtered colimits are left exact in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ (see [Lur17b], defn. 7.3.4.2).*

Proof. First we claim that filtered colimits are left exact in $\mathcal{C}$ because $\mathcal{C}$ is compactly generated: we may choose a small ∞-category $\mathcal{C}'$ and an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}')$, so the assertion follows from the fact that the full subcategory $\mathrm{Ind}(\mathcal{C}') \hookrightarrow \mathsf{PShv}(\mathcal{C}')$ is stable under filtered colimits and finite limits, and filtered colimits are left exact in $\mathsf{PShv}(\mathcal{C}')$. It follows that for any simplicial set K, filtered colimits are left exact in $\mathrm{Fun}(K, \mathcal{C})$. By lemma 2.2.5.8, the ∞-category $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ is an accessible left exact localization of such an ∞-category of $\mathcal{C}$-valued presheaves, which implies the result. □

Remark 2.2.5.10. Let $L : \mathsf{PShv}(\mathcal{E}) \to \mathcal{X}$ be an ∞-topos arising as a left exact localization of the ∞-category of presheaves on a small ∞-category $\mathcal{E}$, and let $\mathcal{C}$ compactly generated presentable ∞-category. Then we have a commuting diagram of fully faithful inclusions

$$\begin{tikzcd}
\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \arrow[r, hook, "f"] \arrow[d, hook, "i"] & \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \arrow[d, hook, "i'"] \\
\mathsf{Shv}_{\mathcal{C}}(\mathsf{PShv}(\mathcal{E})) \simeq \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{C}) \arrow[r, hook, "g"] & \mathrm{Fun}(\mathsf{PShv}(\mathcal{E})^{op}, \mathcal{C})
\end{tikzcd}$$

The functor g is given by right Kan extension along the opposite of the Yoneda embedding, and thus admits a left adjoint, and the functor i has a left adjoint $L_{\mathcal{D}}$ given by composition with L as in lemma 2.2.5.8. It follows that the functor f also has a left adjoint, that we denote with by $F_{\mathcal{C}}$.
Suppose that $\mathcal{C}_0 \subset \mathcal{C}$ is a full subcategory stable under small colimits that is generated under small colimits by compact objects, then it is easy to see that $\mathcal{C}_0$ is also compactly generated so that the inclusion $\mathcal{C}_0 \subset \mathcal{C}$ has a right adjoint, that we denote G. Composing with G induces a functor $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{C}_0}(\mathcal{X})$ that admits a fully faithful left adjoint given by the composition

$$\mathsf{Shv}_{\mathcal{C}_0}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}_0) \longrightarrow \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \xrightarrow{F_{\mathcal{C}}} \mathsf{Shv}_{\mathcal{C}}(\mathcal{X}).$$

These facts are easy to show; see proposition 1.21 of [Lur11c] for instance.

Definition 2.2.5.11. Let $\mathcal{C}$ be a presentable ∞-category, then an object $C \in \mathcal{C}$ is *n-connective* for $n \geq -1$ if $\tau_{\leq (n-1)} C$ is a final object.

If $\mathcal{C}$ is stable and admits a t-structure, there is a clash of terminology with the connective objects defined by the t-structure, but context should allow one to avoid confusion.

Definition 2.2.5.12. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be a compactly generated ∞-category. Consider the following full subcategories of $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$:

(*a*) $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\leq 0}$ consists of those objects $\mathcal{F}$ such that $\Omega^{\infty}\mathcal{F}$ is a discrete object in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$.

(*b*) $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\geq 0}$ consists of those objects $\mathcal{F}$ such that $\Omega^{\infty - n}\mathcal{F}$ is n-connective in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ for all $n \geq 0$.

Proposition 2.2.5.13. *Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{C}$ be a compactly generated ∞-category.*

(1) *The full subcategories* ($\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\le 0}$,$\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\ge 0}$) *determine an accessible* t*-structure on* $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$.

(2) *The* t*-structure on* $\mathcal{S}\mathrm{p}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})) \simeq \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ *is compatible with filtered colimits.*

(3) *The* t*-structure on* $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ *is right complete.*

Proof. It follows from [Lur17a], prop. 1.4.3.4 and prop. 1.4.4.11 that $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ admits an accessible t-structure such that $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\le 0}$ consists of those spectrum objects $\mathcal{F}$ such that $\Omega^{\infty+1}\mathcal{F}$ is a final object and $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\ge 0}$ is the smallest full subcategory containing the essential image of the suspension functor $\Sigma^\infty_+ : \mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ that is stable under extensions and small colimits. $\Omega^{\infty+1}\mathcal{F} \simeq \Omega_{\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})}\Omega^\infty\mathcal{F}$ is final if and only if $\Omega^\infty\mathcal{F}$ is 0-truncated, which shows that coconnective part of the t-structure coincides with the full subcategory described in (a). A spectrum object $\mathcal{G}$ is connective if and only if the map unit map $\mathcal{G} \to \mathcal{G}_{\le -1}$ is equivalent to $\mathcal{G} \to 0$. The following useful criterion, which is easy enough to prove and left to the reader, shows that this is equivalent to demanding that $\Omega^{\infty-n}\mathcal{G}$ is n-connective for all $n \ge 0$.

($*$) For all $k \in \mathbb{Z}$, a map $\mathcal{F} \to \mathcal{F}'$ exhibits $\mathcal{F}'$ as $\tau_{\le k}$-localization in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ if and only if for all $n \ge 0$, the map $\Omega^{\infty-n}\mathcal{F} \to \Omega^{\infty-n}\mathcal{F}'$ exhibits an $(n+k)$-truncation, where use the convention that a map exhibiting an m-truncation is the canonical map to a final object if $m \le -2$.

Since filtered colimits are left exact in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ by corollary 2.2.5.9, the loop functor commutes with filtered colimits, which implies that the functor $\Omega^{\infty-n} : \mathcal{S}\mathrm{p}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})) \to \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ preserves filtered colimits for all $n \in \mathbb{Z}$. This implies in turn that the fibre of $\Omega^{\infty+1}$ over the final object is stable under filtered colimits, which proves (2).
To prove (3), we note that in view of [Lur17a], prop. 1.2.1.19, it suffices to show that $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\le 0}$ is stable under countable coproducts and that if $\Omega^{\infty-n}\mathcal{F}$ is discrete for all $n \in \mathbb{Z}$, then $\mathcal{F}$ is a zero object in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$. The first assertion is true because $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\le 0}$ is stable under filtered colimits and the second assertion is obvious. $\square$

Remark 2.2.5.14. For each ∞-topos $\mathcal{X}$ and each stable ∞-category $\mathcal{D}$, there is an objectwise t-structure on $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D})$. Letting $\mathcal{D} = \mathcal{S}\mathrm{p}(\mathcal{C})$ for $\mathcal{C}$ compactly generated, we have

Lemma 2.2.5.15. *Let $f^* : \mathcal{X} \to \mathcal{Y}$ be an algebraic morphism of ∞-topoi, then the functor $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{Y})$ induced by composing reduced excisive functors with the left exact functor*

$$f^* \circ \_ : \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{X}) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{Y})$$

is t*-exact, where $\mathcal{C}_0 \subset \mathcal{C}$ is the full subcategory spanned by compact objects, and the adjoint functor $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{Y}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ induced by composition with f_* right adjoint to f^* is left* t*-exact.*

Proof. By formal nonsense, it suffices to show that the functor induced by f^* is t-exact. Denote by $\partial(f^* \circ \_)$ the functor obtained by composing reduced excisive functors with $f^* \circ \_$. We have a commuting diagram

$$\begin{tikzcd}
\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \arrow[r, "\partial(f^* \circ _)"] \arrow[d, "\Omega^\infty"] & \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{D})}(\mathcal{Y}) \arrow[d, "\Omega^\infty"] \\
\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \arrow[r, "f^* \circ _"] & \mathsf{Shv}_{\mathcal{C}}(\mathcal{Y})
\end{tikzcd}$$

where $\partial(f^* \circ \_)$ is exact. Using the description of the t-structures of definition 2.2.5.12, it suffices to show that the functor $f^* \circ \_$ preserves truncatedness and connectivity. Since $f^* \circ \_$ is a left exact left adjoint, this follows from [Lur17b], prop. 5.5.6.28. $\square$

Remark 2.2.5.16. It follows from the previous lemma that if we realize the ∞-topos $\mathcal{X}$ as a left exact accessible localization $L : \mathsf{PShv}(\mathcal{D}) \to \mathcal{X}$, then the functor $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathsf{PShv}(\mathcal{D})) \simeq \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S}\mathrm{p}(\mathcal{C}))$ is left t-exact. Similarly, for any ∞-topos, the global sections functor $\Gamma : \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \to \mathcal{S}\mathrm{p}(\mathcal{C})$ induced by the geometric morphism $\mathcal{X} \to \mathcal{S}$ to the final ∞-topos in ${}^{\mathrm{R}}\mathsf{Top}$ is left t-exact. Also, if $i_* : \mathcal{S} \to \mathcal{X}$ is a point, then the functor $\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ is left t-exact.

In the remainder of this section, we will assume that the ∞-category $\mathcal{C}$ is *projectively generated*, that is, $\mathcal{C}$ is generated under small colimits by a set of compact projective object. Recall that $C \in \mathcal{C}$ is *compact projective* if the $\mathcal{S}$-valued functor $\mathrm{Hom}_{\mathcal{C}}(C, \_)$ corepresented by C preserves sifted colimits. This implies that $\mathcal{C} \simeq \mathrm{Fun}^\pi(\mathcal{C}_0^{op}, \mathcal{S})$, where $\mathcal{C}_0$ is the smallest full subcategory of $\mathcal{C}$ that is stable under finite coproducts and contains a set of compact projective generators.

Remark 2.2.5.17. If $\mathcal{C}$ is a presentable ∞-category, then $\mathcal{C}$ is projectively generated if and only if there exists a small collection of functors $\{g_\alpha : \mathcal{C} \to \mathcal{S}\}_{\alpha \in A}$ that is jointly conservative and preserves limits and sifted colimits, by the Barr-Beck theorem ([Lur17a], prop 4.7.3.18). For our purposes, we will only need ∞-categories with a single projective generator. In fact, since we intend to stabilize anyway, the reader may assume that $\mathcal{C}_0$ is additive, that is, $\mathcal{C}$ is a *Grothendieck prestable ∞-category* in the sense of [Lur], appendix C. In this case, the ∞-categorical version of the Gabriel-Popescu theorem asserts that $\mathcal{C}$ is an accessible left exact localization of the connective objects in an ∞-category of right modules for some $\mathbb{E}_1$-ring.

Remark 2.2.5.18. Let $\mathcal{C}$ be presentable and projectively generated and let $\mathcal{C}_0$ be the smallest full subcategory stable under finite coproducts containing the compact projective objects, then restriction along $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ yields an equivalence $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{Fun}^{\pi}(\mathcal{C}_0^{op}, \mathcal{X})$ ([Lur17b], prop. 5.5.8.15). For any sifted diagram $\mathcal{J} : K \to \mathcal{X}$, the colimit functor preserves finite products because colimits are universal in $\mathcal{X}$, which shows that evaluation at any object $C \in \mathcal{C}_0$ preserves small limits and small sifted colimits. Moreover, an object $F : \mathcal{C}_0^{op} \to \mathcal{X}$ is n-truncated in this ∞-category if and only if it takes n-truncated values in $\mathcal{X}$ and the n-truncation functor $\tau_{\leq n}^{\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})}$ is equivalent to the composition with $\tau_{\leq n}^{\mathcal{X}}$; this follows easily from the fact that the truncation functor $\tau_{\leq n} : \mathcal{X} \to \mathcal{X}$ preserves finite products.

For the following proposition, we will use that the inclusion $\mathcal{X}^{\wedge} \hookrightarrow \mathcal{X}$ induces a fully faithful left t-exact functor $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}^{\wedge}) \hookrightarrow \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ whose essential image is the full subcategory spanned by $\mathcal{S}\mathrm{p}(\mathcal{C})$-values sheaves $\mathcal{F}$ such that $\Omega^{\infty-n}\mathcal{F}$ is a hypercomplete $\mathcal{C}$-valued sheaf for all $n \geq 0$, and this functor is moreover a right adjoint.

Proposition 2.2.5.19 (Left completion is hypercompletion)**.** *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be a projectively generated presentable ∞-category, then a sheaf $\mathcal{F}$ valued in $\mathcal{C}$-spectrum objects is left complete if and only if $\mathcal{F}$ lies in the essential image of the inclusion $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}^{\wedge}) \hookrightarrow \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$.*

Proof. If $\mathcal{F}$ is left complete, then $\mathcal{F} \simeq \lim_n \tau_{\leq n}\mathcal{F}$, but using criterion $(*)$ of proposition 2.2.5.13, the object $\Omega^{\infty-m}\tau_{\leq n}\mathcal{F}$ is $(n+m)$-truncated in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ for all $m \geq 0$. By remark 2.2.5.18, this means that the for each $C \in \mathcal{C}_0$, the object $\Omega^{\infty-m}\tau_{\leq n}\mathcal{F}(C)$ in $\mathcal{X}$ is $(n+m)$-truncated, which implies that $\Omega^{\infty-m}\tau_{\leq n}\mathcal{F}$ is hypercomplete for all $m \geq 0$. It follows that $\tau_{\leq n}\mathcal{F}$ lies in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}^{\wedge})$ for all $n \in \mathbb{Z}$, so we conclude as the inclusion $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}^{\wedge}) \hookrightarrow \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ preserves limits.
Conversely, suppose that $\Omega^{\infty-m}\mathcal{F}$ is hypercomplete for all $m \geq 0$, then we should show that the fibre $\mathcal{G}$ of the map $\mathcal{F} \to \lim_n \tau_{\leq n}\mathcal{F}$ (which also has the property that $\Omega^{\infty-m}\mathcal{G}$ is hypercomplete for all $m \geq 0$) vanishes. This fibre is identified with $\lim_n \tau_{\geq (n+1)}\mathcal{F}$, which lies in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\geq\infty}$. This means that for any $m \geq 0$, the sheaf $\Omega^{\infty-m}\mathcal{G}$ is ∞-connective in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$. By remark 2.2.5.18, this means that for all $C \in \mathcal{C}_0$, the object $\Omega^{\infty-m}\mathcal{G}(C)$ is ∞-connective in $\mathcal{X}$. Since $\Omega^{\infty-m}\mathcal{G}$ is hypercomplete, this object is final. This holds for all $m \geq 0$, implying that $\mathcal{G}$ is the zero object. $\square$

Remark 2.2.5.20. The construction of homotopy sheaves for objects in an ∞-topos via the canonical cotensoring over spaces carries over without change to $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ (if $\mathcal{C}$ is presentable and projectively generated), so for each $\mathcal{C}$-valued sheaf Y on $\mathcal{X}$, we have for each $k \geq 0$ an object $\pi_k(Y)$ which lives in $\mathsf{Shv}_{\mathcal{C}}(\mathrm{Disc}(\mathcal{X})) \simeq \mathsf{Shv}_{\tau_{\leq 0}\mathcal{C}}(\mathrm{Disc}(\mathcal{X}))$. Moreover, the homotopy sheaves detect connectiveness; indeed, for an object $X \in \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ the following are equivalent.

(1) X is n-connective, i.e. $\tau_{\leq(n-1)}X$ is a final object.

(2) View X as a product preserving functor $X : \mathcal{C}_0^{op} \to \mathcal{X}$, then for all $C \in \mathcal{C}_0$, $X(C)$ is n-connective in $\mathcal{X}$.

(3) For all $C \in \mathcal{C}_0$, $\pi_k(X(C))$ is a final object in $\mathcal{X}$ for $k < n$ and the canonical map $X(C) \to 1_{\mathcal{X}}$ is an effective epimorphism.

(4) The object $\pi_k(X)$ is final in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ for all $k < n$ and the canonical map $X \to 1_{\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})}$ is an effective epimorphism.

The equivalence (1) $\Leftrightarrow$ (2) follows because final objects are detected objectwise in $\mathrm{Fun}^{\pi}(\mathcal{C}_0^{op}, \mathcal{X})$ and the truncation functor commutes with all evaluations functors by the previous remark. [Lur17b], prop. 6.5.1.12 gives (2) $\Leftrightarrow$ (3), and (3) $\Leftrightarrow$ (4) follows because the construction of homotopy sheaves commutes with all evaluation maps (because the homotopy sheaves are constructed using only limits) and effective epimorphisms are detected objectwise.

Proposition 2.2.5.21. *Let $\mathcal{C}$ be projectively generated presentable ∞-category, then for $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$, points* (1) *through* (3) *hold and in addition we have*

(4) *The functor $\pi_0 : \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^{\heartsuit}$ identifies the latter ∞-category with $\mathsf{Shv}_{\mathsf{Ab}(\tau_{\leq 0}\mathcal{C})}(\mathrm{Disc}(\mathcal{X}))$, the nerve of the category of sheaves of abelian group objects in the category $\tau_{\leq 0}\mathcal{C}$ on the* $\mathrm{Disc}(\mathcal{X})$.

Proof. The heart of $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ is identified with the full subcategory of the limit of the tower

$$\ldots \xrightarrow{\Omega} \mathsf{EM}_n(\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})) \xrightarrow{\Omega} \mathsf{EM}_{n-1}(\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})) \xrightarrow{\Omega} \ldots \xrightarrow{\Omega} \mathsf{EM}_1(\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})) \xrightarrow{\Omega} \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$$

where $\mathsf{EM}_n(\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X}))$ is the ∞-category of *Eilenberg-MacLane objects*, the full subcategory spanned by n-connective n-truncated objects. This tower stabilizes at $n = 2$ and is then given by the nerve of the category of abelian group objects in $\tau_{\leq 0}\mathsf{Shv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathsf{Shv}_{\tau_{\leq 0}\mathcal{C}}(\mathrm{Disc}(\mathcal{X}))$, which is canonically equivalent to the nerve of the category of $\mathsf{Ab}(\tau_{\leq 0}\mathcal{C})$-valued sheaves on $\mathrm{Disc}(\mathcal{X})$. □

Now suppose that $\mathcal{C}^{\otimes}$ is presentably symmetric monoidal, then $\mathcal{S}\mathrm{p}(\mathcal{C})$ has a canonical presentably symmetric monoidal structure, since the underlying presentable ∞-category of the coproduct $\mathcal{S}\mathrm{p}^{\otimes} \coprod \mathcal{C}^{\otimes}$ in $\mathbb{E}_\infty\mathsf{Alg}(\mathsf{Pr}^{\mathrm{L}})$ is given by the tensor product $\mathcal{S}\mathrm{p} \otimes \mathcal{C}$, and this symmetric monoidal structure is easily seen to be compatible with the t-structure (so that $\mathcal{S}\mathrm{p}(\mathcal{C})^{\geq 0}$ is stable under the tensor product and contains the tensor unit). We let $\mathcal{D}^{\otimes}$ denote this presentably symmetric monoidal ∞-category. For any simplicial set K, we get an objectwise symmetric monoidal structure on $\mathrm{Fun}(K, \mathcal{D})$, that is, the ∞-operad $\mathrm{Fun}(K, \mathcal{D}^{\otimes}) \times_{\mathrm{Fun}(K, \mathbf{N}(\mathsf{Fin}_*))} \mathbf{N}(\mathsf{Fin}_*) =: \mathrm{Fun}(K, \mathcal{D})^{\otimes}$ is a symmetric monoidal ∞-category. Let $L : \mathsf{PShv}(\mathcal{E}) \to \mathcal{X}$ be an ∞-topos arising as a localization of the ∞-category of presheaves on a small ∞-category $\mathcal{E}$, and let $\mathcal{C}$ compactly generated presentable ∞-category. In remark 2.2.5.10 we describe the commuting diagram of fully faithful inclusions

$$\begin{tikzcd}
\mathsf{Shv}_{\mathcal{D}}(\mathcal{X}) \arrow[r, hook, "f"] \arrow[d, hook, "i"] & \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D}) \arrow[d, hook, "i'"] \\
\mathsf{Shv}_{\mathcal{D}}(\mathsf{PShv}(\mathcal{E})) \simeq \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{C}) \arrow[r, hook, "g"] & \mathrm{Fun}(\mathsf{PShv}(\mathcal{E})^{op}, \mathcal{D})
\end{tikzcd}$$

where f has a left adjoint $F_{\mathcal{D}}$ given by the restriction to $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D})$ of the left adjoint to $g \circ i$. In the diagram above, the functor ∞-categories admit objectwise symmetric monoidal structures, and the functors g and i' exhibit $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D})$ respectively $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D})$ as symmetric monoidal subcategories. The objectwise symmetric monoidal structure on $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D}) \simeq \mathsf{Shv}_{\mathcal{D}}(\mathsf{PShv}(\mathcal{E}))$ localizes to $\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$. Indeed, according to [Lur17a], prop. 2.2.1.9 it suffices to show that for an $L_{\mathcal{D}}$-equivalence $\mathcal{F} \to \mathcal{F}'$ in $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D})$ and any $\mathcal{G} \in \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D})$, the morphism $\mathcal{F} \otimes \mathcal{G} \to \mathcal{F}' \otimes \mathcal{G}$ is an $L_{\mathcal{D}}$-equivalence. This follows because the functor $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D} \times \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D})$ given by composition with the tensor product takes $L_{\mathcal{D} \times \mathcal{D}}$-equivalences to $L_{\mathcal{D}}$-equivalences. It follows easily that the symmetric monoidal structure on $\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$ is also the localization of the one on $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D})$; that is, given an object $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n \in \mathsf{Shv}_{\mathcal{D}}(\mathcal{X})_{\langle n \rangle}$ where each $F_i : \mathcal{X}^{op} \to \mathcal{D}$ preserves limits, then the coCartesian lift of the unique active map $\langle n \rangle \to \langle 1 \rangle$ starting at this object is given by

$$\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_n \longrightarrow \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \longrightarrow F_{\mathcal{D}}(\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$$

where the first map is the objectwise tensor product.

Remark 2.2.5.22. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be an algebraic morphism of ∞-topoi. Then the preceding discussion implies that the map $\mathsf{Shv}_{\mathcal{D}}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{D}}(\mathcal{Y})$ is symmetric monoidal.

The symmetric monoidal structure on $\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$ allows for the consideration of algebra objects for ∞-operads. In particular, the map of ∞-operads $\mathrm{Comm}^{\otimes} \hookrightarrow \mathrm{MComm}^{\otimes}$ induces a map $\mathsf{Mod}(\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})) \to \mathbb{E}_\infty\mathsf{Alg}(\mathsf{Shv}_{\mathcal{D}}(\mathcal{X}))$, but as the map $\mathsf{Mod}(\mathcal{D}) \to \mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})$ preserves small limits, there is also a map $\mathsf{Shv}_{\mathsf{Mod}(\mathcal{D})}(\mathcal{X}) \to \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})}(\mathcal{X})$.

Lemma 2.2.5.23. *For an ∞-topos $\mathcal{X}$ and any presentable symmetric monoidal ∞-category $\mathcal{C}$, there is a canonical isomorphism of maps of simplicial sets between*

$$\mathsf{Mod}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})) \longrightarrow \mathbb{E}_\infty\mathsf{Alg}(\mathsf{Shv}_{\mathcal{D}}(\mathcal{X}))$$

and

$$\mathsf{Shv}_{\mathsf{Mod}(\mathcal{C})}(\mathcal{X}) \longrightarrow \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})}(\mathcal{X}).$$

The lemma is obvious enough when the definitions are unwinded, which we do below.

Construction 2.2.5.24. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{C}^{\otimes} \to \mathbf{N}(\mathsf{Fin}_*)$ be an ∞-category with a symmetric monoidal structure. Define a simplicial set $S(\mathcal{O}^{\otimes})$ by the universal property that for each simplicial set K, there is a canonical bijection

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(K, S(\mathcal{O}^{\otimes})) \cong \mathrm{Hom}_{(\mathsf{Set}_\Delta)_{/\mathbf{N}(\mathsf{Fin}_*)}}(K \times \mathcal{O}^{\otimes} \times \mathcal{X}^{op}, \mathcal{C}))$$

Let $\mathsf{Alg}_{\mathcal{X},\mathcal{C}}(\mathcal{O}^{\otimes}) \subset S(\mathcal{O}^{\otimes})$ denote the full simplicial subset spanned by those maps

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{X}^{op} \arrow[rr, "F"] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\
& \mathbf{N}(\mathsf{Fin}_*) &
\end{tikzcd}$$

such that

(1) for all $C \in \mathcal{O}$, the functor $F|_{\{C\}\times\mathcal{X}^{op}} : \mathcal{X}^{op} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} \cong \mathcal{C}$ preserves small limits.

(2) for all $X \in \mathcal{X}$, the functor $F_{\mathcal{O}^{\otimes}\times\{X\}} : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is a map of ∞-operads.

Clearly, the construction of the simplicial set $\mathsf{Alg}_{\mathcal{X},\mathcal{C}}(\mathcal{O}^{\otimes})$ is contravariantly functorial in the sense that a map $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ of ∞-operads induces a map of simplicial sets $\mathsf{Alg}_{\mathcal{X},\mathcal{C}}(\mathcal{O}'^{\otimes}) \to \mathsf{Alg}_{\mathcal{X},\mathcal{C}}(\mathcal{O}^{\otimes})$. When (2) is satisfied, (1) is equivalent to the apparently stronger condition

(1′) For all $n \geq 1$ and all $C = C_1 \oplus \ldots \oplus C_n \in \mathcal{O}$, the functor $F|_{\{C\}\times\mathcal{X}^{op}} : \mathcal{X}^{op} \to \mathcal{C}^{\otimes}_{\langle n\rangle}$ preserves small limits.

Indeed, using that F preserves coCartesian lifts of each map $\rho^i : \langle n\rangle \to \langle 1\rangle$, there is a commuting diagram

$$\begin{tikzcd}
\mathcal{O}^{\otimes}_{\langle n\rangle} \times \mathcal{X}^{op} \arrow[r, "F|_{\langle n\rangle}"] \arrow[d, "\rho^i_!"] & \mathcal{C}^{\otimes}_{\langle n\rangle} \arrow[d, "\rho^i_!"] \\
\mathcal{O}^{\otimes}_{\langle 1\rangle} \times \mathcal{X}^{op} \arrow[r, "F|_{\langle 1\rangle}"] & \mathcal{C}
\end{tikzcd}$$

so we see that the composition $\rho^i_! \circ F|_{\{C\}\times\mathcal{X}^{op}}$ is equivalent to $F_{\{C_i\}\times\mathcal{X}^{op}}$. The functors $\rho^i_!$ induce an equivalence $\mathcal{C}^{\otimes}_{\langle n\rangle} \simeq \prod_{i=1}^n \mathcal{C}$, so we see that the functor $F|_{\{C\}\times\mathcal{X}^{op}}$ preserves small limits if each of the functors $F|_{\{C_i\}\times\mathcal{X}^{op}}$ preserves small limits.

Proof of lemma 2.2.5.23. For each ∞-operad $\mathcal{O}^{\otimes}$, the ∞-categories $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes})$ and $\mathsf{Shv}_{\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}^{\otimes})}(\mathcal{X})$ are both full simplicial subsets of the simplicial set $S(\mathcal{O}^{\otimes})$ of construction 2.2.5.24. Under these identifications, both functors

$$\mathsf{Mod}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes}) \longrightarrow \mathbb{E}_\infty\mathsf{Alg}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes})$$

and

$$\mathsf{Shv}_{\mathsf{Mod}(\mathcal{C}^{\otimes})}(\mathcal{X}) \longrightarrow \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}^{\otimes})}(\mathcal{X})$$

are the one induced by the obvious map of ∞-operads $\mathrm{Comm}^{\otimes} \hookrightarrow \mathrm{MComm}^{\otimes}$. Thus, it suffices to check that the two simplicial subsets of $S(\mathrm{Comm}^{\otimes})$ and $S(\mathrm{MComm}^{\otimes})$ are the same. We only treat the case of $\mathrm{MComm}^{\otimes}$, the other one is similar (and easier).
We show that $\mathsf{Mod}(\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes})$ and $\mathsf{Shv}_{\mathsf{Mod}(\mathcal{C}^{\otimes})}(\mathcal{X})$ correspond by adjunction to the simplicial subset $\mathsf{Alg}_{\mathcal{X},\mathcal{C}}(\mathcal{O}^{\otimes}) \subset S(\mathrm{Comm}^{\otimes})$. A map $G : \mathrm{MComm}^{\otimes} \to \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes}$ over $\mathbf{N}(\mathsf{Fin}_*)$ is a map of ∞-operads if and only if the composition $\mathrm{MComm}^{\otimes} \to \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes} \subset \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})^{\otimes}$ is a map of ∞-operads, and this condition is satisfied if and only if the adjoint map F to G in $S(\mathrm{MComm}^{\otimes})$ satisfies condition (2). Using the fact that $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes} \subset \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})^{\otimes}$ consists of those pairs (n, F) of an integer $n \geq 1$ and a functor $F : \mathcal{X}^{op} \to \mathcal{C}^{\otimes}_{\langle n\rangle}$ such that for each $i \in \langle n\rangle^{\circ}$ the composition $\mathcal{X}^{op} \to \mathcal{C}^{\otimes}_{\langle n\rangle} \xrightarrow{\rho^i_!} \mathcal{C}$ preserves small limits, we see that G takes values in $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes}$ if and only if the adjoint F satisfies (1′) and (2).
Conversely, we immediately see that an object $F' \in S(\mathrm{MComm}^{\otimes})$ satisfies (2) if and only if the adjoint $G' : \mathcal{X}^{op} \to \mathrm{Fun}_{\mathbf{N}(\mathsf{Fin}_*)}(\mathrm{MComm}^{\otimes}, \mathsf{Shv}_{\mathcal{C}}(\mathcal{X})^{\otimes})$ lands in the full subcategory of ∞-operad maps. We are left to show that the map G' preserves small limits if and only if (1) is satisfied. For this, we note that the functor $\mathsf{Mod}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ taking a pair (A, M) of an algebra and module over it to the pair of underlying spectrum objects is conservative and preserves small limits. But the composition $\mathcal{X}^{op} \to \mathsf{Mod}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is precisely given the pair of functors described in (1) for the two objects $\mathfrak{a}$ and $\mathfrak{m}$ of MComm. □

Remark 2.2.5.25. The full subcategory $\mathcal{D}^{\geq 0} \subset \mathcal{D}$ of connective objects is stable under colimits and is generated by the compact objects of $\mathcal{D}$ that are connective, so remark 2.2.5.10 yields a fully faithful functor $g : \mathsf{Shv}_{\mathcal{D}^{\geq 0}}(\mathcal{X}) \hookrightarrow \mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$. An object $\mathcal{F}$ lies in the essential image of this functor if and only if the counit map $\widetilde{\tau_{\geq 0}\mathcal{F}} \to \mathcal{F}$ is an $F_{\mathcal{D}}$-equivalence. As $F_{\mathcal{D}}$ is t-exact, this is the case precisely if $\mathcal{F} \in \mathsf{Shv}_{\mathcal{D}}(\mathcal{X})^{\geq 0}$, so that the functor g induces an equivalence $\mathsf{Shv}_{\mathcal{D}^{\geq 0}}(\mathcal{X}) \hookrightarrow \mathsf{Shv}_{\mathcal{D}}(\mathcal{X})^{\geq 0}$
Similarly, the full subcategory $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D}))^{\mathrm{cn}} \subset \mathbb{E}_\infty\mathsf{Alg}(\mathcal{D}))$ spanned by connective $\mathbb{E}_\infty$-algebras is stable under colimits and generated by compact objects of $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D}))$ whose underlying object in $\mathcal{D}$ is connective. We obtain a fully faithful functor $\overline{g} : \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})^{\mathrm{cn}}}(\mathcal{X}) \hookrightarrow \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})}(\mathcal{X})$. Let $\mathcal{O}_{\mathcal{X}} \in \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})}(\mathcal{X})$, then $\mathcal{O}_{\mathcal{X}}$ lies in the image of $\overline{g}$ if and only if the counit $\widetilde{\tau_{\geq 0}}\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is an $F_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})}$-equivalence, where $\tau_{\geq 0}$ is the functor taking the connective cover of $\mathbb{E}_\infty$-algebras. The forgetful functor $\mathbb{E}_\infty\mathsf{Alg}(\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})) \to \mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$ is conservative and commutes with the connective cover functor, so, as we have an equivalence $\mathsf{Shv}_{\mathcal{D}^{\geq 0}}(\mathcal{X}) \simeq \mathsf{Shv}_{\mathcal{D}}(\mathcal{X})^{\geq 0}$, we deduce that the functor $\widetilde{g}$ induces an equivalence $\mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})^{\mathrm{cn}}}(\mathcal{X}) \simeq \mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}(\mathcal{D})}(\mathcal{X})^{\mathrm{cn}}$ onto the full subcategory of $\mathbb{E}_\infty$-algebras in $\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$ whose underlying $\mathbb{E}_\infty$-algebra is connective in the t-structure on $\mathsf{Shv}_{\mathcal{D}}(\mathcal{X})$.

Now that these subtleties are dealt with, we have, for any sheaf of (connective) $\mathbb{E}_\infty$-algebras $\mathcal{O}_\mathcal{X}$ in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$, a presentably symmetric monoidal ∞-category $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ via the procedures of example 2.1.5.9. This symmetric monoidal structure is moreover stable by [Lur17a], prop. 7.1.1.4.

Proposition 2.2.5.26. *Suppose that $\mathcal{O}_\mathcal{X}$ is a connective $\mathbb{E}_\infty$-algebra in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$.*

(1) *The full subcategories $(\theta^{-1}(\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})(\mathcal{X})^{\leq 0}}), \theta^{-1}(\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})(\mathcal{X})^{\geq 0}}))$ determine an accessible* t*-structure on $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$.*

(2) *The* t*-structure on $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ is compatible with filtered colimits.*

(3) *The* t*-structure on $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ is compatible with with the symmetric monoidal structure.*

(4) *The* t*-structure on $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ is right complete.*

(5) *If $\mathcal{X}$ is hypercomplete, then the* t*-structure on $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ is left complete.*

Proof. Once (1) is proven, (2), and (4) are obvious consequences of the fact that θ is limit and colimit preserving and conservative, using that the t-structure on $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ satisfies the analogous conditions, and (5) is proven similarly using proposition 2.2.5.19. To prove (3), we need to show that given $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ and $\mathcal{G}$ such that the underlying objects of $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ are connective, the tensor product $\theta(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G})$ is also connective in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$, but this last object can be identified with the colimit of the Bar construction $\mathsf{Bar}_\mathcal{O}(\mathcal{F}, \mathcal{G})$ whose entries are of the form

$$\mathcal{F} \otimes \mathcal{O} \otimes \ldots \otimes \mathcal{O} \otimes \mathcal{G}.$$

We conclude that $\theta(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G})$ is connective since $\mathcal{O}$ is connective and the t-structure on $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ is compatible with the symmetric monoidal structure. Part (1) follows as in proposition 2.1.3 of [Lur11d]. ☐

The last part of this subsection is concerned with abelian sheaf cohomology; that is, we study the following problem.

- Let X be a topological space and $\mathcal{O}_X$ a sheaf of commutative algebras on X. $\mathcal{F}$ be a (differentially graded or simplicial) sheaf of $\mathcal{O}_X$-modules on X. How does one compute the homology/homotopy groups of $\Gamma(\mathcal{O}_X)$ in terms of the homology/homotopy groups of sheaves of $\mathcal{O}_X$?

The global sections functor is in general only left t-exact, so we merely have a functor

$$\Gamma : \mathsf{Shv}_{\mathsf{Mod}_k}(\mathcal{X})^\heartsuit \longrightarrow \mathsf{Mod}_k^{\leq 0}.$$

Definition 2.2.5.27. Let $\mathcal{C}$ be a presentable projectively generated ∞-category, and consider for each $n \geq 0$ the functor

$$H^n(\_, \mathcal{X}) : \mathsf{Shv}_{\mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))}(\mathrm{Disc}(\mathcal{X})) \simeq \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^\heartsuit \xrightarrow{\Gamma} \mathcal{S}\mathrm{p}(\mathcal{C})^{\leq 0} \xrightarrow{\pi_{-n}} \mathcal{S}\mathrm{p}(\mathcal{C})^\heartsuit \simeq \mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C})).$$

Let F be a sheaf of abelian group objects in $\tau_{\leq 0}\mathcal{C}$ on $\mathrm{Disc}(\mathcal{X})$, then we call the abelian group object $H^n(F, \mathcal{X})$ the *n'th sheaf cohomology group of F*. The functor $H^0(\_, \mathcal{X})$ coincides with the global sections functor of 1-categories

$$\mathsf{Shv}_{\mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))}(\mathrm{Disc}(\mathcal{X})) \longrightarrow \mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C})).$$

To a sheaf $\mathcal{F}$ of k-modules on space X, we may assign the homotopy sheaves $\pi_n(\mathcal{F})$ in the abelian category of discrete sheaves of k-modules. Then we may ask how the discrete graded k-modules $H^*(\pi_n(\mathcal{F}), X)$ and $\pi_*(\Gamma(\mathcal{F}))$ are related. We will show that if $\mathcal{F}$ is a left bounded object in $\mathcal{X} \otimes \mathcal{S}\mathrm{p} \otimes \mathcal{C}$ for $\mathcal{C}$ an arbitrary projectively generated presentable ∞-category, there is a hypercohomology spectral sequence relating these two graded objects of $\mathsf{Ab}(\tau_{\leq 0}\mathcal{C})$. First, we treat the case when the relation is very simple.

Proposition 2.2.5.28. *Let F be an injective object in $\mathsf{Shv}_{\mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))}(\mathcal{X})$, then $H^n(F, X) \cong 0$ for $n > 0$.*

Proof. View F as a functor $\mathcal{E}^{op} \to \mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))$, then under the equivalence $\mathsf{Shv}_{\mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))}(\mathrm{Disc}(\mathcal{X})) \simeq \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})^\heartsuit$, F is sent to $L_{\mathcal{S}\mathrm{p}(\mathcal{C})}(F)$. Since the global sections of the presheaf F is acyclic, it suffices to show that under the assumption that F is injective, the sheafification map $F \to L_{\mathcal{S}\mathrm{p}(\mathcal{C})}(F)$ is an equivalence; that is, F is already a $\mathcal{S}\mathrm{p}(\mathcal{C})$-valued sheaf on $\mathcal{E}$. Let $\{C_i \to C\}_i$ be a covering family in $\mathcal{E}$, then we should show that the coaugmented cosimplicial diagram

$$F(C) \longrightarrow \textstyle\prod_i F(C_i) \rightrightarrows \prod_{i,j} F(C_i \times_C C_j) \rightrightarrows\!\!\!\!\!\rightarrow \ldots$$

is a limit diagram in $\mathcal{S}\mathrm{p}(\mathcal{C})$. This follows as in the proof of lemma 2.1.10 of [Lur11d]. ☐

Definition 2.2.5.29. Let $\mathcal{D}$ be a stable ∞-category equipped with a t-structure, then we say that an object X in $\mathcal{D}$ is *quasi-injective* if there exists an integer k such that X is of the form $\prod_{n \in \mathbb{Z}_{\leq k}} I_n[n]$ for I_n an injective object of $\mathcal{D}^\heartsuit$.

Warning 2.2.5.30. The terminology introduced in the previous definition is not standard, and bears no relation to the notion of a quasi-injective object in an abelian category (i.e. every submodule inclusion extends to an endomorphism).

Lemma 2.2.5.31. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ a projectively generated presentable ∞-category, then for each quasi-injective object $\mathcal{F}$ of $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$, the canonical map*

$$H^0(\pi_*(\mathcal{F}),\mathcal{X}) \longrightarrow \pi_*(\Gamma(\mathcal{F}))$$

is an equivalence of $\mathbb{Z}$-graded objects in $\mathsf{Ab}(\tau_{\le 0}\mathcal{C})$.

Proof. By lemma 2.2.5.28, we have for each injective I of $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ an equivalence $H^0(\pi_k(I[n]),\mathcal{X}) \simeq \pi_k(\Gamma(I[n]))$ for all k and n (both sides are only nonzero for $k=n$). Thus, it suffices to show that for each $k \in \mathbb{Z}$, both the functors $H^0(\pi_k(\mathcal{F}),\_)$ and $\pi_k(\Gamma(\_))$ commute with taking products indexed by $\mathbb{Z}_{\ge l}$ for some l such that for each $n \in \mathbb{Z}_{\le l}$, the n'th factor is an injective object sitting in degree n. To see this, it suffices to note that Γ and $H^0(\_,\mathcal{X})$ commute with limits, and that the functor $\pi_k = \tau_{\le 0} \circ \tau_{\ge 0} \circ [-k]$ commutes with $\mathbb{Z}_{\le l}$-indexed products of the form described above because the connective cover functor $\tau_{\ge 0}$ preserves limits and applying $\tau_{\ge 0}$ to a quasi-injective object returns a finite product, which commutes with $\tau_{\le 0}$. □

Lemma 2.2.5.32. *Let $\mathcal{A}$ be a Grothendieck abelian category (i.e. $\mathcal{A}$ is presentable and the collection of monomorphisms is stable under filtered colimits in $\mathrm{Fun}(\Delta^1,\mathcal{A})$), and let $D^+(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ be the left bounded derived ∞-category of $\mathcal{A}$. Suppose that $\pi_n(\mathcal{F})$ is an injective object of $\mathcal{A}$ for all $n \in \mathbb{Z}$. Then $\mathcal{F}$ is quasi-injective.*

Proof. By the characterization of fibrant(-cofibrant) objects in $D^+(\mathcal{A})$ and the assumption that $\mathcal{F} \in \mathcal{D}^{\le k}(\mathcal{A})$ for some $k \in \mathbb{Z}$, the chain complex $\pi_\bullet(\mathcal{F})$ given by

$$\dots \xrightarrow{0} \pi_n(\mathcal{F}) \xrightarrow{0} \pi_{n-1}(\mathcal{F}) \xrightarrow{0} \pi_{n-2}(\mathcal{F}) \xrightarrow{0} \dots$$

with zero differential is fibrant in the model structure on $\mathcal{A}$ which models the object $\prod_{n\le k}\pi_n(\mathcal{F})[n]$ in the ∞-category $\mathcal{D}(\mathcal{A})$. Let $F_\bullet$ be a left bounded chain complex of injectives that models $\mathcal{F}$, then using injectivity of $\pi_n(\mathcal{F})$, we can find a dotted lift in the diagram

$$\begin{tikzcd} \ker(\partial_n) \arrow[r] \arrow[d, hook] & \pi_n(\mathcal{F}) \\ F_n \arrow[ur, dotted] & \end{tikzcd}$$

for every $n \le k$. These maps determine a map $F_\bullet \to \pi_\bullet(\mathcal{F})$ of chain complexes, which is clearly a quasi-isomorphism. □

Lemma 2.2.5.33. *Let $\mathcal{F}$ be a left bounded object of $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$, then there exists a coaugmented cosimplicial object $\mathscr{I}^\bullet$ such that $\mathscr{I}^{-1} \simeq \mathcal{F}$ with the following properties.*

(1) *The diagram $\mathscr{I}^\bullet$ is a limit diagram.*

(2) *For each $n \in \mathbf{N}(\mathbf{\Delta})$, the object $\mathscr{I}^n$ is quasi-injective.*

(3) *For each $q \in \mathbb{Z}$, the map $\pi_q(\mathcal{F}) \to \pi_q(\mathscr{I}^0)$ exhibits the unnormalized cochain complex*

$$\pi_q(\mathscr{I}^0) \longrightarrow \pi_q(\mathscr{I}^1) \longrightarrow \pi_q(\mathscr{I}^2) \longrightarrow \dots$$

as an injective resolution of $\pi_q(\mathcal{F})$.

Proof. It is a classical fact (see [Bry07] for instance) that every left bounded chain complex $K_\bullet$ admits an injective resolution $K_\bullet \hookrightarrow I_{\bullet\bullet}$, where $I_{\bullet\bullet}$ is a $\mathbb{Z}\times\mathbb{Z}_{\le 0}$ graded double complex consisting of injective objects that has the following properties

(1) For each $m \ge 0$, the chain complex $I_{\bullet m}$ has injective homology (and is thus quasi-injective in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$).

(2) For each $n \in Z$, the chain complex $H_n^V(I_{\bullet\bullet})$ of the vertical homology of $I_{\bullet\bullet}$ is an injective resolution of $H_n(K_\bullet)$.

The stable Dold-Kan correspondence now provides the desired object. □

Since $\mathcal{A} = \mathsf{Shv}_{\mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))}(\mathcal{X})$ is a Grothendieck abelian category with enough injective objects, the inclusion $A_{\text{inj}} \subset \mathsf{Shv}_{\mathbf{N}(\mathsf{Ab}(\tau_{\leq 0}\mathcal{C}))}(\mathcal{X}) \to \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ induces an equivalence $\mathcal{D}^+(\mathcal{A}) \simeq \mathsf{Shv}^+_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$. We deduce that each left bounded object $\mathcal{F}$ has a cosimplicial resolution $\mathscr{I}^\bullet$ satisfying (2) and (3) of lemma 2.2.5.33. Let $\mathcal{F} \in \mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$ be coconnective and let $\mathscr{I}^\bullet$ be a cosimplicial resolution, then we have an equivalence

$$\Gamma(\mathcal{F}) \xrightarrow{\simeq} \mathrm{Tot}(\Gamma(\mathscr{I}^\bullet))$$

as Γ preserves limits. The stable Dold-Kan correspondence provides a *co*filtered object $\mathbf{N}(Z_{\geq 0})^{op} \to \mathcal{S}\mathrm{p}(\mathcal{C})$ which induces a cohomological spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq 1}$ that converges

$$E_1^{p,q} \Rightarrow \pi_{q-p}(\mathrm{Tot}(\Gamma(\mathscr{I}^\bullet))) \cong \pi_{q-p}(\Gamma(\mathcal{F})).$$

The complex $\{E_1^{*,q}, d_1\}$ is the normalized cochain complex associated to the cosimplicial object $\pi_q(\Gamma(\mathscr{I}^\bullet)) \cong \Gamma(\pi_q(\mathscr{I}^\bullet))$. But the normalization of $\pi_q(\mathscr{I}^\bullet)$ is an injective resolution of the object $\pi_q(\mathcal{F})$. Thus, we have proven the following.

Proposition 2.2.5.34 (Hypercohomology spectral sequence)**.** *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{C}$ a projectively generated presentable ∞-category and $\mathcal{F}$ a left bounded object in $\mathsf{Shv}_{\mathcal{S}\mathrm{p}(\mathcal{C})}(\mathcal{X})$. Then there is a convergent spectral sequence*

$$E_2^{p,q} = H^p(\pi_q(\mathcal{F}), X) \Rightarrow \pi_{q-p}(\Gamma(\mathcal{F}))$$

Remark 2.2.5.35. Note that lemma 2.2.5.31 ceases to hold if we were to work with quasi-injective indexed by $\mathbb{Z}$. This has the consequence that in the absence of a hypercompleteness condition on $\mathcal{X}$, the construction of the hypercohomology spectral sequence does not work for unbounded objects.

The underlying topological spaces of derived manifolds are closed subspaces of $\mathbb{R}^n$, which are a paracompact Hausdorff of finite covering dimension. Moreover, all sheaves of algebras and modules we will consider have partitions of unity, which has rather strong consequences for the interaction between the (stable) homotopy theory and the sheaf theory on such spaces, some of which we will establish in what follows.

Lemma 2.2.5.36. *Let k be a commutative ring and let $\mathcal{F}$ be a left bounded sheaf of k-modules on a topological space X such that each homotopy sheaf of $\mathcal{F}$ is acyclic, then there is an equivalence*

$$H^0(\pi_n(\mathcal{F})) \simeq \pi_n(\Gamma(\mathbb{R})).$$

In particular, if $\pi_n(\mathcal{F}) = 0$ as a sheaf, then $\pi_n(\Gamma(\mathcal{F})) = 0$.

Proof. As $\mathcal{F}$ is left bounded, this follows at once from the collapse at the E_2-page of the hypercohomology spectral sequence associated to $\mathcal{F}$. □

For the next proposition, recall that an *F_σ-subset* of a space X is a countable union of closed sets. An F_σ subset of a paracompact space is paracompact, and the collection of open F_σ-sets of a paracompact space forms a basis for the topology that is closed under finite intersections.

Proposition 2.2.5.37. *Let X be a paracompact Hausdorff space, and let $\mathcal{O}_X$ be sheaf of connective $\mathbb{E}_\infty$-algebras over a commutative ring k on $\mathcal{X} = \mathsf{Shv}(X)$ (which we can view as a connective $\mathbb{E}_\infty$-algebra object in $\mathsf{Shv}_{\mathsf{Mod}_k}(\mathcal{X})$ or as a $\mathbb{E}_\infty\mathsf{Alg}_k^{\mathrm{cn}}$-valued sheaf on X). Suppose that $\pi_0(\mathcal{O}_{\mathcal{X}})$ is a fine sheaf on X. Then*

(1) *For each left complete sheaf $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules, the map $\Gamma(\mathcal{F}) \to \Gamma(\tau_{\leq n}\mathcal{F})$ induced by the unit of the truncation functor $\mathcal{F} \to \tau_{\leq n}\mathcal{F}$ exhibits $\Gamma(\tau_{\leq n}\mathcal{F})$ as a $\tau_{\leq n}$-localization of $\Gamma(\mathcal{F})$.*

(2) *Let $\mathcal{B}$ be the basis of open F_σ-sets of X, so that restriction induces an equivalence $\mathsf{Shv}(X) \simeq \mathsf{Shv}(\mathbf{N}(\mathcal{B}))$. Then for each left complete sheaf $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}$, the presheaf $\widetilde{\tau_{\leq n}\mathcal{F}} \in \mathsf{PShv}_{\mathsf{Mod}_k}(\mathbf{N}(\mathcal{B}))$ given by applying the functor $\tau_{\leq n}$ objectwise is already a sheaf.*

(3) *For each left complete sheaf $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}$, the presheaf $\widetilde{\pi_n(\mathcal{F})}$ on $\mathcal{B}$ given by applying the n'th homotopy group functor objectwise is already a sheaf.*

Proof. (1) First, let $\mathcal{F}$ be a left bounded sheaf of $\mathcal{O}_X$-modules. Because $\mathcal{F}$ is left bounded, all the objects in the fibre sequence

$$\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \tau_{\leq n}\mathcal{F}$$

are left bounded. Since $\tau_{\leq n}\mathcal{F}$ is n-truncated, $\Gamma(\tau_{\leq n}\mathcal{F})$ is also n-truncated. For any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, the homotopy sheaves of $\mathcal{G}$ are sheaves of $\pi_0(\mathcal{O}_X)$-modules, which are fine sheaves on a paracompact Hausdorff space and therefore acyclic, as $\pi_0(\mathcal{O}_X)$ is fine. Applying this to $\mathcal{F}'$, we see that because this sheaf of modules is $(n+1)$-connective and left bounded, $\Gamma(\mathcal{F}')$ is also $(n+1)$-connective by lemma 2.2.5.36. Since Γ preserves fibre sequences, the result follows.

If $\mathcal{F}$ is left complete, $\mathcal{F}$ is the limit of its left bounded truncations, that is, $\Gamma(\mathcal{F}) \simeq \lim_n \Gamma(\tau_{\leq n}\mathcal{F})$. To show that the fibre of $\theta_n : \Gamma(F) \to \Gamma(\tau_{\leq n}\mathcal{F})$ is $(n+1)$-connective, consider the factorization

$$\Gamma(\mathcal{F}) \xrightarrow{\theta_{n+1}} \Gamma(\tau_{\leq(n+1)}\mathcal{F}) \xrightarrow{\theta_{n+1,n}} \Gamma(\tau_{\leq n}\mathcal{F})$$

then the octahedral axiom provides a fibre sequence

$$\mathrm{fib}(\theta_{n+1}) \longrightarrow \mathrm{fib}(\theta_n) \longrightarrow \mathrm{fib}(\theta_{n+1,n})$$

whose long exact sequence implies that it suffices to show that $\mathrm{fib}(\theta_{n+1})$ and $\mathrm{fib}(\theta_{n+1,n})$ are $(n+1)$-connective. Using the first part of the proof, we see that $\theta_{n+1,n}$ exhibits an n-truncation so that $\mathrm{fib}(\theta_{n+1,n})$ is indeed $(n+1)$-connective. As we have the equivalence $\Gamma(\mathcal{F}) \simeq \lim_{k\geq n+1} \Gamma(\tau_{\leq(n+1)}\mathcal{F})$, we also have an equivalence $\mathrm{fib}(\theta_{n+1}) \simeq \lim_{k\geq n+2} \mathrm{fib}(\theta_{k,n+1})$, but by the first part of the proof, the map $\theta_{k,n+1}$ exhibits an $(n+1)$-truncation so $\mathrm{fib}(\theta_{k,n+1})$ is $(n+2)$-connective for all $k \geq n+2$. Since the limit of a tower of $(n+2)$-connective objects in Mod_k is $(n+1)$-connective, we conclude.

(2) We have to show that the sheafification map $\widetilde{\tau_{\leq n}\mathcal{F}} \to \tau_{\leq n}\mathcal{F}$ is an equivalence, but as truncation is preserved by passing to slice topoi, the map $\widetilde{\tau_{\leq n}\mathcal{F}}(U) \to \tau_{\leq n}\mathcal{F}(U)$ is identified with the global sections of the map $\widetilde{\tau_{\leq n}\mathcal{F}|_U} \to \tau_{\leq n}(\mathcal{F}|_U)$ for each open set $U \subset X$. Then letting U range over the basis $\mathcal{B}$, we see that (1) applies because each $U \in \mathcal{B}$ is paracompact Hausdorff and left completeness is preserved by passing to slice topoi, which implies that $\Gamma(\widetilde{\tau_{\leq n}\mathcal{F}|_U}) \to \Gamma(\tau_{\leq n}(\mathcal{F}|_U))$ is an equivalence.

(3) The homotopy groups of sheaves are given by a composition of Ω^n for some integer n, $\tau_{\leq 0}$ and $\tau_{\geq 0}$. The functor Ω^n is clearly defined objectwise because the functor $\mathsf{Shv}_{\mathsf{Mod}_k}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathbf{N}(\mathcal{B})^{op}, \mathsf{Mod}_k)$ is exact, and by (2), the functor $\tau_{\leq 0}$ is defined objectwise on left complete sheaves. We wish to show that $\tau_{\geq 0}$ is also defined objectwise on a left complete sheaf $\mathcal{F}$ of $\mathsf{Mod}_{\mathcal{O}_X}$. We have a morphism of fibre sequences

$$\begin{tikzcd}
\widetilde{\tau_{\geq 0}\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & \widetilde{\tau_{\leq -1}\mathcal{F}} \arrow[d] \\
\tau_{\geq 0}\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \tau_{\leq -1}\mathcal{F}.
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{N}(\mathcal{B})^{op}, \mathsf{Mod}_k)$. Since the right vertical map is an equivalence by (2), the left vertical map is one as well. □

Remark 2.2.5.38. In view of proposition 2.2.5.19, we can remove the left completeness assumption if the space X of proposition 2.2.5.37 has finite covering dimension. In this case, proposition 2.2.5.37 can be equivalently expressed by stating that the functors $\mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_{\Gamma(\mathcal{O}_X)}$ and $\mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_{\widetilde{\mathcal{O}_X}}$ are t-exact, where $\widetilde{\mathcal{O}_X}$ is the object $\mathcal{O}_X$ viewed as a connective $\mathbb{E}_\infty$-algebra in $\mathrm{Fun}(\mathbf{N}(\mathcal{B}), \mathsf{Mod}_k)$ (note that $\widetilde{\mathcal{O}_X}$ is indeed connective in the objectwise t-structure as its homotopy groups are fine sheaves).

Chapter 3

(Pre)geometries and Geometric Contexts

In this chapter, we will add additional layers of categorical structure permitting discussions on *geometry*. Let us outline the general procedure we have in mind, following [Lur11b] and [TV04; TV06]:

(1) We start with a suitably 'geometric' ∞-category $\mathcal{T}$ with a Grothendieck topology, consisting of the objects we would like to think of as 'affine'. We may consider 'scheme-like' objects that locally look like objects of $\mathcal{T}$, by taking the cocompletion with respect to étale maps in the ∞-category of *$\mathcal{T}$-structured spaces*.

(2) We notice that our ∞-category of affine objects does not have all finite limits and/or not all limits that exist are of the correct geometric nature. However, we can identify a natural subcategory for which limits do exist and are correct (e.g. transverse pullbacks). This gives $\mathcal{T}$ the structure of what we shall call a *pregeometry*. We wish to 'derive' this pregeometry, that is, consider the ∞-category $\mathcal{G}$ such that $\mathcal{G}$ comes with a map $\mathcal{T} \hookrightarrow \mathcal{G}$ which respects the limits we have deemed correct, and is otherwise freely generated by finite limits. $\mathcal{G}$ is known as the *geometric envelope* of $\mathcal{T}$, and is an example of a *geometry*.

(3) An essentially unique geometric envelope always exists for a pregeometry and it is characterized by a universal property. We show that in our case of interest, we can explicitly realize a geometric envelope for $\mathcal{T}$ as a natural subcategory of the ∞-category of $\mathcal{T}$-structured spaces.

(4) The geometry $\mathcal{G}$ gives us an ∞-site of *derived affine $\mathcal{T}$-spaces*, which has finite limits. To get all colimits as well, we take the localization of $\mathsf{PShv}(\mathcal{G})$ with respect to covers and the resulting ∞-topos contains the objects that we call *derived $\mathcal{T}$-stacks*.

(5) General derived $\mathcal{T}$-stacks are just homotopy sheaves on $\mathcal{G}$ and one can not expect to make sense of deformation theory for such objects. Our next goal is to identify a subcategory of stacks having a good infinitesimal theory. To this end, we introduce a subcategory $\mathcal{P}$ in $\mathcal{G}$ that is local for the topology, and the pair $(\mathcal{G},\mathcal{P})$ becomes a *geometric context*. We can then inductively define n-geometric derived $\mathcal{T}$-stacks as those derived $\mathcal{T}$-stacks that have an atlas by $(n-1)$-geometric stacks, where (-1)-geometric stacks are the derived $\mathcal{T}$-spaces, the stacks in the essential image of the Yoneda embedding.

For $\mathcal{T} = \mathsf{Man}$ the étale site of smooth manifolds, we obtain the enveloping geometry of affine derived manifolds dSmAff and the ∞-topos dSmSt of derived C^∞-stacks. Letting $\mathcal{P}$ be the subcategory spanned by étale or submersive morphisms yields the ∞-category dSmDM or dSmAr of derived Deligne-Mumford or derived Artin geometric C^∞-stacks respectively.

3.1 Pregeometries and Geometries

In this section, we introduce the notion of a *pregeometry*, a structure on an ∞-category $\mathcal{T}$ that will ensure the existence of a *scheme theory for* $\mathcal{T}$. The structure we are looking for should somehow blend two pieces of data:

(1) A collection of well behaved pullbacks in $\mathcal{T}$, generating the finite limits that are *good*, such as transverse intersections.

(2) A Grothendieck topology on $\mathcal{T}$, specifying which maps are *local*.

We expect these data to be suitably compatible. The desired structure is encapsulated by J. Lurie's elegant notion of an *admissibility structure*.

Definition 3.1.0.1 (J. Lurie [Lur11b]). Let $\mathcal{T}$ be an ∞-category. An *admissibility structure* on $\mathcal{T}$ is the data of

(1) A subcategory $\mathcal{T}^{ad} \subseteq \mathcal{T}$ that contains all objects of $\mathcal{T}$. Morphisms in $\mathcal{T}^{ad}$ will be called *admissible.*

(2) A Grothendieck topology on $\mathcal{T}$ such that for each covering sieve $\mathcal{T}^{S}_{/X} \subseteq \mathcal{T}_{/X}$ of an object X in $\mathcal{T}$, there is covering sieve in $\mathcal{T}^{S}_{/X}$ generated by a collection $\{U_\alpha \to X\}$ of admissible morphisms. In other words, there is a basis for the topology on $\mathcal{T}$ whose covering families contain only admissible morphisms. If a collection $\{U_\alpha \to X\}$ generates a covering sieve, it is called an *admissible covering.*

This data is required to satisfy the following conditions:

(1) For every admissible map $f : U \to X$ and any map $g : Y \to X$, there is a pullback

$$\begin{tikzcd} V \arrow[r, "g'"] \arrow[d, "f'"'] & U \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

with f' an admissible map.

(2) For a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

with f and g admissible, h is also admissible.

(3) A retract of an admissible map is admissible.

Definition 3.1.0.2. (1) A *pregeometry* is a pair $(\mathcal{T}, \mathcal{T}^{ad})$ of an essentially small ∞-category $\mathcal{T}$ with finite products, together with an admissibility structure $\mathcal{T}^{ad}$ on $\mathcal{T}$.

(2) A *geometry* is a pair $(\mathcal{G}, \mathcal{G}^{ad})$ of an essentially small idempotent complete ∞-category $\mathcal{G}$ with finite limits together with an admissibility structure $\mathcal{G}^{ad}$ on $\mathcal{G}$

Definition 3.1.0.3. (1) Let $\mathcal{T}$ and $\mathcal{T}'$ be pregeometries. A *transformation of pregeometries* is a functor $f \in \mathrm{Fun}(\mathcal{T}, \mathcal{T}')$ that preserves products and pullbacks along admissibles such that $f(\mathcal{T}^{\mathrm{ad}}) \subset (\mathcal{T}')^{\mathrm{ad}}$, and f takes admissible coverings to admissible coverings.

(2) Let $\mathcal{G}$ and $\mathcal{G}'$ be geometries. A *transformation of geometries* is a functor $f \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{G}')$ such that $f(\mathcal{G}^{\mathrm{ad}}) \subset (\mathcal{G}')^{\mathrm{ad}}$, and f takes admissible coverings to admissible coverings.

We will usually just write $\mathcal{T}$ (or $\mathcal{G}$) for a pregeometry $(\mathcal{T}, \mathcal{T}^{ad})$ (or a geometry $(\mathcal{G}, \mathcal{G}^{ad})$). Now we will define what it means to be a $\mathcal{T}$- or $\mathcal{G}$-structure on an ∞-topos $\mathcal{X}$, which, having the theory of Lawvere theories and algebraic theories in mind, one should think of as being a kind of 'algebra object' in $\mathcal{X}$ with possibly very intricate multiplication rules determined by the (pre)geometry.

Definition 3.1.0.4. Let $\mathcal{T}$ be a pregeometry and let $\mathcal{C}$ be an ∞-category. We denote by $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}, \mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathcal{T}, \mathcal{C})$ spanned by those functors $\mathcal{O} : \mathcal{T} \to \mathcal{C}$ such that

(1) $\mathcal{O}$ preserves finite products.

(2) $\mathcal{O}$ preserves pullbacks along admissible maps.

Definition 3.1.0.5. Let $\mathcal{X}$ be an ∞-topos.

(1) For $\mathcal{T}$ a pregeometry, the ∞*-category* $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ *of* $\mathcal{T}$*-structures on* $\mathcal{X}$ is the ∞-category $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}, \mathcal{X})$.

(2) For $\mathcal{G}$ a geometry, ∞*-category* $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ *of* $\mathcal{G}$*-structures on* $\mathcal{X}$ is the ∞-category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{X})$

Warning 3.1.0.6. Note that while a geometry $\mathcal{G}$ can also be viewed as a pregeometry, a $\mathcal{G}$-structure on an ∞-topos $\mathcal{X}$ *with* $\mathcal{G}$ *viewed as a geometry* is *not* the same thing as a $\mathcal{G}$-structure on $\mathcal{X}$ *with* $\mathcal{G}$ *viewed as a pregeometry.* To prevent ambiguity, we will always use the symbol $\mathcal{G}$ to mean a geometry, and $\mathcal{T}$ to mean a pregeometry.

Remark 3.1.0.7. Let $\mathcal{C}$ be an ∞-category with finite limits, and let $\mathcal{X}$ be an ∞-topos. By [Lur17b], prop. 5.5.1.9 and 5.3.5.10, the Yoneda embedding $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathrm{Pro}(\mathcal{C}), \mathcal{X}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{X}),$$

where the left hand side is the ∞-category of functors that admit a left adjoint. We have a natural equivalence $\mathrm{Fun}^{\mathrm{R}}(\mathrm{Pro}(\mathcal{C}), \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{op}, \mathrm{Ind}(\mathcal{C}^{op}))$, and because a functor from $\mathcal{X}^{op}$ to $\mathrm{Ind}(\mathcal{C}^{op})$ admits a left adjoint if and only if it preserves small limits (apply the adjoint functor theorem [Lur17b], prop 5.5.2.9 to the functor of opposite categories), we get $\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}^{op}, \mathrm{Ind}(\mathcal{C}^{op})) = \mathsf{Shv}_{\mathrm{Ind}(\mathcal{C}^{op})}(\mathcal{X})$. We conclude that for a geometry $\mathcal{G}$, a $\mathcal{G}$-structure on $\mathcal{X}$ can be equivalently viewed (perhaps more geometrically intuitively) as an $\mathrm{Ind}(\mathcal{G}^{op})$-valued sheaf on $\mathcal{X}$. To cement this intuition, we encourage the reader to have for $\mathcal{G}^{op}$ the category $(\mathsf{CAlg}_k)_{\mathrm{fp}}$ of k-algebras of the form $k[x_1, \dots, x_n]/I$ in mind, where k is a commutative ring and I a finitely generated ideal, whose opposite category is the category of finitely presented affine k-schemes. By taking the ind-completion we obtain $\mathrm{Ind}((\mathsf{CAlg}_k)_{\mathrm{fp}}) \simeq \mathsf{CAlg}_k$, the category of all commutative k algebras. Thus, a $(\mathsf{CAlg}_k)_{\mathrm{fp}}^{op}$-structure on an ∞-topos $\mathsf{Shv}(X)$ for X a topological space can be canonically identified with a sheaf of commutative k-algebras on the space X. We will make $(\mathsf{CAlg}_k)_{\mathrm{fp}}$ into an actual geometry in examples 3.1.0.15 and 3.1.0.16.
A similar geometric intuition applies to $\mathcal{T}$-structures for some *pre*geometry $\mathcal{T}$, but only after we have introduced the crucial notion of a 'geometric envelope'.

Using the Grothendieck topology on a (pre)geometry arising from the admissibility structure, we can ask that $\mathcal{T}$- or $\mathcal{G}$-structures and maps between them on an ∞-topos $\mathcal{X}$ can be recovered from local data on $\mathcal{T}$ or $\mathcal{G}$.

Definition 3.1.0.8. Let $\mathcal{T}$ be a pregeometry and let $\mathcal{O}$ be a $\mathcal{T}$-structure on an ∞-topos $\mathcal{X}$. $\mathcal{O}$ is a *local* $\mathcal{T}$-structure if for each collection of admissible maps $\{U_\alpha \to X\}$ that generates a covering sieve, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \to \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$.
For $\mathcal{O}$, $\mathcal{O}'$ local $\mathcal{T}$-structures on $\mathcal{X}$, a *morphism of local $\mathcal{T}$-structures* is a natural transformation $\alpha : \mathcal{O} \to \mathcal{O}'$ such that for all admissible maps $X \to Y$, the commuting diagram of induced maps

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r] \arrow[d] & \mathcal{O}'(X) \arrow[d] \\ \mathcal{O}(Y) \arrow[r] & \mathcal{O}'(Y) \end{tikzcd}$$

is a pullback square. We denote by $\mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X})$ the subcategory of $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ spanned by local $\mathcal{T}$-structures on $\mathcal{X}$ and morphisms of local $\mathcal{T}$-structures.
Replacing the pregeometry $\mathcal{T}$ with a geometry $\mathcal{G}$ in this definition, we obtain the ∞-category of local $\mathcal{G}$-structures on $\mathcal{X}$.

Some examples of geometries and pregeometries are in order.

Example 3.1.0.9 (Discrete (pre)geometries)**.** Let $\mathcal{T}$ be an ∞-category with finite products. We make $\mathcal{T}$ into a pregeometry by declaring that only equivalences are admissible, which generates the trivial Grothendieck topology. Pregeometries for which the subcategory of admissible maps is the subcategory spanned by equivalences are called *discrete pregeometries*. For a discrete pregeometry $\mathcal{T}$, all $\mathcal{T}$-structures are local. Discrete pregeometries are the same thing as *finite limit theories*, a subclass of which will be studied in more detail in section 4.1. A basic example of a discrete pregeometry is the following: let k be a commutative ring, and let $\mathcal{T}_k^{\mathrm{disc}} L := \mathbf{N}(\mathrm{Poly}_k)$, where Poly_k is the (ordinary) category whose objects are the affine k-spaces $\mathbb{A}^n_k$ for $n \geq 0$, and whose morphisms are polynomial maps.
Of course, starting from an ∞-category $\mathcal{G}$ that has finite limits and is idempotent complete, we have an associated *discrete geometry* with underlying ∞-category $\mathcal{G}$. By remark 3.1.0.7, there are equivalences $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \simeq \mathsf{Shv}_{\mathrm{Ind}(\mathcal{G}^{op})}(\mathcal{X})$ for any ∞-topos $\mathcal{X}$.

Example 3.1.0.10 (Pregeometry of Smooth Manifolds)**.** The motivating example in this work is the following: let $\mathcal{T}_{\mathrm{Diff}}$ be the pregeometry whose underlying ∞-category is $\mathbf{N}(\mathsf{Man})$, endowed with the étale topology. A morphism $f : U \to M$ is admissible if it is an injective local diffeomorphism, that is, an open embedding. Similarly, we let $\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}}$ be the pregeometry whose underlying ∞-category is the nerve of the category of open submanifolds of $\mathbb{R}^n$, for some $n \geq 0$, endowed with the étale topology. Admissible morphisms are again open embeddings. There is an obvious transformation of pregeometries $\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}} \to \mathcal{T}_{\mathrm{Diff}}$.

Example 3.1.0.11 (Pregeometry of Smooth Manifolds with Corners)**.** Let Man_c be the category whose objects are manifolds with corners, and whose morphisms are the *b-maps* of Melrose [Mel93; Mela; JF19]. A map $f : M \to N$ between manifolds with corners is locally of the form $\mathbb{R}^n \times \mathbb{R}^k_{\geq 0} \to \mathbb{R}^m \times \mathbb{R}^l_{\geq 0}$ so we may replace M and N by these Cartesian spaces with corners. f is a *b-map* if f is smooth (i.e. there is an extension $\tilde{f}$ of f to some neighbourhood $U \subset \mathbb{R}^n \times \mathbb{R}^k$ such that $\tilde{f}$ is smooth) and either of the following two conditions hold.

(1) f maps $\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$ into $\{0\} \times \mathbb{R}^l_{\geq 0}$

(2) Write $f = (f_1, \dots, f_m, f_{m+1,\dots,f_{n+l}})$, then each f_{n+i} decomposes uniquely as $g_{n+k} \prod h_1^{\alpha_1} \dots h_k^{\alpha_k}$ where $g_{n+k} > 0$, the α_j are nonnegative integers and the $\{h_j\}$ form a complete set of boundary defining functions.

If a b-map f does *not* satisfy (1), we say that f is an *interior b-map.* We define a pregeometry $\mathcal{T}_{\mathrm{Diffc}}$ as the nerve of the category of manifolds with corners and interior b-maps among them. The pregeometry structure is generated by open inclusions.

Example 3.1.0.12 (Pregeometry of Complex Manifolds)**.** The starting point of *derived analytic geometry* as developed in the final sections of [Lur11a] and in [Por15; PY17] is the *complex analytic pregeometry* $\mathcal{T}_{\mathrm{An}_{\mathbb{C}}}$. Here, the underlying ∞-category is the nerve of the category of open submanifolds of $\mathbb{C}^n$ for some $n \in \mathbb{N}$, and a morphism is admissible if it is an injective local biholomorphism.

The following pregeometries are among the main players in the passage from classical algebraic geometry to derived algebraic geometry. We will explain later on how these pregeometries 'generate' the geometries wherein derived algebraic geometry takes place.

Example 3.1.0.13 (Pregeometry of Zariski open subschemes of affine k-space)**.** Let k be a commutative ring, and let $\mathcal{T}_{\mathrm{Zar}}(k) \coloneqq \mathbf{N}(\mathsf{CAlg}_k^{\mathrm{Zar}})^{op}$, where $\mathsf{CAlg}_k^{\mathrm{Zar}}$ is the (ordinary) category of k-algebras of the form $k[x_1, \dots, x_n, (f(x_1, \dots, x_n))^{-1}]$, where f is a polynomial function on affine n-dimensional k-space $\mathbb{A}^n_k$. Given an object $A \in \mathsf{CAlg}_k^{\mathrm{Zar}}$, we denote the corresponding object in $\mathcal{T}_{\mathrm{Zar}}(k)$ by $\operatorname{Spec} A$ (for the moment, this is just abstract notation, not meant to indicate that $\operatorname{Spec} A$ is a locally ringed space). Recall that, given a commutative k-algebra and any element $b \in B$, the *localization* of B by b is the universal object (defined up to isomorphism) $f : B \to B[1/b]$ such that $f(b)$ is invertible, and should be thought of as the algebra of functions on the open set where b is nonzero. With these preliminaries out of the way, we can make the ∞-category $\mathcal{T}_{\mathrm{Zar}}(k)$ into a pregeometry by endowing it with the following admissibility structure:

(1) A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if and only if there exists some element $b \in B$ such that the map $B \to A$ induces an isomorphism $B[1/b] \cong A$.

(2) A collection of admissible morphism $\{\operatorname{Spec} B_i[1/b_i] \to \operatorname{Spec} B\}_i$ is an admissible covering if and only if the elements $\{b_i\}$ generate the unit ideal in B.

Example 3.1.0.14 (Pregeometry of étale open subschemes of affine k-space)**.** Let k be a commutative ring, and let $\mathcal{T}_{\text{ét}}(k) \coloneqq \mathbf{N}(\mathsf{CAlg}_k^{sm})^{op}$, where CAlg_k^{sm} is the (ordinary) category of k-algebras A that admit an étale map $f : k[x_1, \dots, x_n] \to A$ (that is, f is finitely presented, flat, and the module of relative Kähler differential Ω_f vanishes). We can make the ∞-category $\mathcal{T}_{\text{ét}}(k)$ into a pregeometry by endowing it with the following admissibility structure:

(1) A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if and only if the map $B \to A$ is étale.

(2) A collection of admissible morphism $\{\operatorname{Spec} B_i \to \operatorname{Spec} B\}_i$ is an admissible covering if and only if there exists a finite set of indices $\{i_j\}_{1 \leq j \leq n}$ such that the induced map $g : B \to \prod_{1 \leq j \leq n} B_{i_j}$ is faithfully flat (that is, the base change functor along g preserves and reflects exact sequences of B-modules).

The following two examples of *geometries* describe the arena of *classical* algebraic geometry.

Example 3.1.0.15 (Geometry of affine k-schemes (Zariski))**.** Let k be a commutative ring, and let $\mathcal{G}_{\mathrm{Zar}}(k) \coloneqq \mathbf{N}((\mathsf{CAlg}_k)_{\mathrm{fp}})$, where $(\mathsf{CAlg}_k)_{\mathrm{fp}}$ is the (ordinary) category of finitely presented k-algebras; that is k-algebras of the form $k[x_1, \dots, x_n]/I$ for some finitely generated ideal I. We make $\mathcal{G}_{\mathrm{Zar}}(k)$ into a geometry by endowing it with the following admissibility structure, which is the obvious extension of the admissibility structure on $\mathcal{T}_{\mathrm{Zar}}(k) \subset \mathcal{G}_{\mathrm{Zar}}(k)$, using the same notations:

(1) A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if and only if there exists some element $b \in B$ such that the map $B \to A$ induces an isomorphism $B[1/b] \cong A$.

(2) A collection of admissible morphism $\{\operatorname{Spec} B_i[1/b_i] \to \operatorname{Spec} B\}_i$ is an admissible covering if and only if the elements $\{b_i\}$ generate the unit ideal in B.

Example 3.1.0.16 (Geometry of affine k-schemes (étale))**.** Continuing the notation of the previous examples, we let $\mathcal{G}_{\text{ét}}(k)$ be the geometry that has the same underlying ∞-category as $\mathcal{G}_{\mathrm{Zar}}(k)$, and whose admissibility structure is as follows:

(1) A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if and only if the map $B \to A$ is an étale map of commutative k-algebras.

(2) A collection of admissible morphism $\{\operatorname{Spec} B_i \to \operatorname{Spec} B\}_i$ is an admissible covering if and only if there exists a finite set of indices $\{i_j\}_{1 \leq j \leq n}$ such that the induced map $g : B \to \prod_{1 \leq j \leq n} B_{i_j}$ is faithfully flat.

We will introduce some geometries describing various levels of derived differential geometry in later subsections. Recall from chapter 1 and [Lur17b], section 6.3 the two antiequivalent ∞-categories ${}^{\mathrm{L}}\mathsf{Top}$ and ${}^{\mathrm{R}}\mathsf{Top}$ of ∞-topoi and algebraic respectively geometric morphisms between them. We have a non-full subcategory inclusion ${}^{\mathrm{L}}\mathsf{Top} \hookrightarrow \widehat{\mathsf{Cat}}_\infty$ which classifies a coCartesian fibration over ${}^{\mathrm{L}}\mathsf{Top}$, the *universal topos fibration*, which was denoted as $p : \overline{{}^{\mathrm{L}}\mathsf{Top}} \to {}^{\mathrm{L}}\mathsf{Top}$.

Definition 3.1.0.17. Let $\mathcal{T}$ be a pregeometry. Let ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ denote the subcategory of $\mathrm{Fun}(\mathcal{T}, \overline{{}^{\mathrm{L}}\mathsf{Top}}) \times_{\mathrm{Fun}(\mathcal{T}, {}^{\mathrm{L}}\mathsf{Top})} {}^{\mathrm{L}}\mathsf{Top}$ defined as follows:

(1) An object V of $\mathrm{Fun}(\mathcal{T}, \overline{{}^{\mathrm{L}}\mathsf{Top}}) \times_{\mathrm{Fun}(\mathcal{T}, {}^{\mathrm{L}}\mathsf{Top})} {}^{\mathrm{L}}\mathsf{Top}$, which we can identify with a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, for some ∞-topos $\mathcal{X}$ and a functor $\mathcal{O}_{\mathcal{X}} : \mathcal{T} \to \mathcal{X}$, lies in ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ precisely if $\mathcal{O}_{\mathcal{X}}$ is a local $\mathcal{T}$-structure on $\mathcal{X}$.

(2) For a morphism $\alpha : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}'_{\mathcal{Y}})$ in $\mathrm{Fun}(\mathcal{T}, \overline{{}^{\mathrm{L}}\mathsf{Top}}) \times_{\mathrm{Fun}(\mathcal{T}, {}^{\mathrm{L}}\mathsf{Top})} {}^{\mathrm{L}}\mathsf{Top}$, let $f^* : \mathcal{X} \to \mathcal{Y}$ be the underlying algebraic morphism. We declare α to be a morphism in ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ if the induced morphism $f^* \circ \mathcal{O}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{Y}}$ is a morphism of local $\mathcal{T}$-structures on $\mathcal{Y}$.

Replacing the pregeometry $\mathcal{T}$ with a geometry $\mathcal{G}$ in this definition, we obtain an ∞-category which we denote by ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G})$.

Remark 3.1.0.18. The canonical projection $p : {}^{\mathrm{L}}\mathsf{Top}(\mathcal{T}) \to {}^{\mathrm{L}}\mathsf{Top}$ is a coCartesian fibration. To see this, recall that the class of coCartesian fibrations is stable under formation of functor categories and pullbacks, showing that the map $p' : \mathrm{Fun}(\mathcal{T}, \overline{{}^{\mathrm{L}}\mathsf{Top}}) \times_{\mathrm{Fun}(\mathcal{T}, {}^{\mathrm{L}}\mathsf{Top})} {}^{\mathrm{L}}\mathsf{Top} \to {}^{\mathrm{L}}\mathsf{Top}$ is a coCartesian fibration; for an algebraic morphism $f^* : \mathcal{X} \to \mathcal{Y}$ and a $\mathcal{T}$-structure $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$, the (unique up to contractible ambiguity) p'-coCartesian lift of f^* with domain $\mathcal{O}_{\mathcal{X}}$ is the morphism $\mathcal{O}_{\mathcal{X}} \to f^* \circ \mathcal{O}_{\mathcal{X}}$. Because f^* preserves small colimits and finite limits, $f^* \circ \mathcal{O}_{\mathcal{X}}$ is als a local $\mathcal{T}$-structure on $\mathcal{Y}$, we also have a p-coCartesian lift of f^*. The fibre of p over an ∞-topos $\mathcal{X}$ can be identified with $\mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X}')$ with $\mathcal{X}'$ an ∞-topos canonically equivalent to $\mathcal{X}$ (not isomorphic, because the fibre over $\mathcal{X}$ of the universal topos fibration is in general only canonically categorically equivalent to $\mathcal{X}$).

Definition 3.1.0.19. We call the opposite category of ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ the ∞-category of *$\mathcal{T}$-structured spaces*, and denote it ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T})$. Similarly, we have the category ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ of $\mathcal{G}$-structured spaces.

Remark 3.1.0.20. The definition of $\mathcal{T}$- or $\mathcal{G}$-structured spaces allows for structured spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which the underling ∞-topos is not 0-localic, that is, not the ∞-category of sheaves on a topological space. The underlying n-localic ∞-topos of a structured space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ will (for specific (pre)geometries) have to be interpreted as the ∞-category of sheaves on the small ∞-site of a *higher orbifold*, or *n-Deligne-Mumford stack*.

The existence of certain limits and colimits in the ∞-category of $\mathcal{T}$- or $\mathcal{G}$-structures spaces is of great import to us. Remark 3.1.0.18 and the theory of relative colimits suggests that limits in the ∞-category of structured spaces are controlled by limits in the ∞-categories $\mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X})$ and $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X})$.

Remark 3.1.0.21. It's easy to see that the ∞-category $\mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ is presentable for $\mathcal{G}$ a geometry; this follows from the equivalence $\mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \simeq \mathsf{Shv}_{\mathrm{Ind}(\mathcal{G}^{op})}(\mathcal{X})$ of remark 3.1.0.7. Furthermore, the ∞-category $\mathsf{Shv}_{\mathrm{Ind}(\mathcal{G}^{op})}(\mathcal{X})$ can be identified with the ∞-category $\mathsf{Shv}(\mathcal{X}) \otimes \mathrm{Ind}(\mathcal{G}^{op})$, where $\otimes$ is the Lurie tensor product of presentable ∞-categories. The same is true for a pregeometry $\mathcal{T}$, since there exists a geometry $\mathcal{G}'$ such that $\mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \simeq \mathrm{Str}_{\mathcal{G}'}(\mathcal{X})$ (this is the geometric envelope of $\mathcal{T}$, see definition 4.1.4.6)

Proposition 3.1.0.22. *Let $\mathcal{X}$ be an ∞-topos.*

(1) Let $\mathcal{T}$ be a pregeometry. The ∞-category $\mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X})$ has sifted colimits and the inclusion $\mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{T}, \mathcal{X})$ preserves sifted colimits.

(2) Let $\mathcal{G}$ be a pregeometry. The ∞-category $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X})$ has filtered colimits and the inclusion $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{G}, \mathcal{X})$ preserves filtered colimits.

Proof. (1) is proposition 3.3.1 of [Lur11b] and (2) is proposition 1.5.1 of [Lur11b]. □

Corollary 3.1.0.23. *(1) Let $\mathcal{T}$ be a pregeometry. The ∞-category ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ has sifted colimits and the projection $p : {}^{\mathrm{L}}\mathsf{Top}(\mathcal{T}) \to {}^{\mathrm{L}}\mathsf{Top}$ preserves sifted colimits.*

(2) Let $\mathcal{G}$ be a pregeometry. The ∞-category ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G})$ has filtered colimits and the projection $p : {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}) \to {}^{\mathrm{L}}\mathsf{Top}$ preserves filtered colimits.

Proof. The proof is the same in both cases, using proposition 3.1.0.22; we write it for a pregeometry $\mathcal{T}$ only. Given a sifted diagram $q : K \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$, we have a sifted diagram $p \circ q$ in ${}^{\mathrm{L}}\mathsf{Top}$ which has a colimit, by [Lur17b], cor. 6.3.4.7. By proposition 3.1.0.22 and [Lur17b], cor. 4.3.1.11, we can lift this colimit to a p-colimit in ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$, which is also a colimit in ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ by [Lur17b], prop. 4.3.1.5. □

The existence of limits in ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ is a little bit more subtle, but for the moment, we will have need only of geometric realizations in ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{T})$ by *étale* maps.

Definition 3.1.0.24. Let $\mathcal{T}$ be a pregeometry. A map $(f^*,\alpha):(\mathcal{X},\mathcal{O}_\mathcal{X})\to(\mathcal{Y},\mathcal{O}_\mathcal{Y})$ of $\mathcal{T}$-structured spaces is *étale* if the underlying algebraic morphism f^* is étale ([Lur17b], section 6.3.5) and the morphism $\alpha: f^*\circ\mathcal{O}_\mathcal{X}\to\mathcal{O}_\mathcal{Y}$ is an equivalence of local $\mathcal{T}$-structures on $\mathcal{Y}$. Replacing $\mathcal{T}$ with $\mathcal{G}$ in this definition, we obtain the notion of an étale morphism of $\mathcal{G}$-structured spaces.

The collection of étale maps defines a Grothendieck topology on the ∞-category of $\mathcal{T}$-structured spaces; the following proposition shows that this topology is subcanonical.

Proposition 3.1.0.25. *Let $(\mathcal{X},\mathcal{O}_\mathcal{X})$ be a $\mathcal{T}$-structured space and let $\coprod U_\alpha\to\mathbf{1}$ be an effective epimorphism in $\mathcal{X}$ with codomain the final object in $\mathcal{X}$. Then $(\mathcal{X},\mathcal{O}_\mathcal{X})$ is a colimit of the Čech nerve of the étale covering $\{(\mathcal{X}_{/\mathcal{U}_\alpha},\mathcal{O}|_{\mathcal{U}_\alpha})\to(\mathcal{X},\mathcal{O}_\mathcal{X})\}$ in ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T})$.*

Proof. The Čech nerve is an augmented simplicial diagram $\overline{q}:(\mathbf{N}(\Delta)^{op})^{\triangleright}\to{}^{\mathrm{R}}\mathsf{Top}(\mathcal{T})$ such that $p\circ\overline{q}$ is a colimit diagram in ${}^{\mathrm{R}}\mathsf{Top}$. Moreover, by definition of étale morphisms, every edge in $(\mathbf{N}(\Delta)^{op})^{\triangleright}$ is sent to a p-Cartesian edge of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T})$. The result now follows by proposition 1.5.6 of [Lur11b]. ☐

Definition 3.1.0.26. Let $\mathcal{T}$ be a pregeometry. We say that a morphism $\alpha:\mathcal{O}\to\mathcal{O}'$ of $\mathcal{T}$-structures on an ∞-topos $\mathcal{X}$ is an *effective epimorphism* if α is objectwise an effective epimorphism. A map $(f^*,\alpha):(\mathcal{X},\mathcal{O}_\mathcal{X})\to(\mathcal{Y},\mathcal{O}_\mathcal{Y})$ of $\mathcal{T}$-structured ∞-topoi is a *closed immersion* if the algebraic morphism f^* is a closed immersion of ∞-topoi ([Lur17b], definition 7.3.2.7) and $\alpha: f^*\circ\mathcal{O}_\mathcal{X}\to\mathcal{O}_\mathcal{Y}$ is an effective epimorphism.

Definition 3.1.0.27. Let $\mathcal{T}$ be a pregeometry. We say that a morphism $f:U\to X$ in $\mathsf{PShv}(\mathcal{T})$ is *admissible* if f fits into a pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"] & Lj(U') \arrow[d] \\ X \arrow[r] & Lj(X') \end{tikzcd}$$

where L is a left exact left adjoint to the inclusion $\mathsf{Shv}(\mathcal{T})\hookrightarrow\mathsf{PShv}(\mathcal{T})$ and $U'\to X'$ is admissible in $\mathcal{T}$.

Warning 3.1.0.28. Admissible morphisms in $\mathsf{PShv}(\mathcal{T})$ are *not* in general stable under composition.

The following proposition is a variation on [Lur17b], prop. 6.2.3.20, characterizing sheaves on pregeometries (instead of ∞-sites with finite limits) via a universal property, replacing 'finite limits' with 'finite products and admissible pullbacks' in the aforementioned proposition in [Lur17b].

Proposition 3.1.0.29. *Let $\mathcal{T}$ be a pregeometry, and let $\mathcal{X}$ be an ∞-topos. Let $\mathrm{Fun}^{*\mathrm{ad}}(\mathsf{Shv}(\mathcal{T}),\mathcal{X})$ and $\mathrm{Fun}^{*\mathrm{ad}}(\mathsf{Shv}(\mathcal{T}),\mathcal{X})$ denote the full subcategories spanned by those functors that preserve all small colimits, finite products and admissible pullbacks. Then the composition*

$$J:\mathrm{Fun}^{*\mathrm{ad}}(\mathsf{Shv}(\mathcal{T}),\mathcal{X})\xrightarrow{L\circ}\mathrm{Fun}^{*\mathrm{ad}}(\mathsf{PShv}(\mathcal{T}),\mathcal{X})\xrightarrow{j\circ}\mathrm{Fun}(\mathcal{T},\mathcal{X})$$

is fully faithful, and if a functor $\mathcal{O}:\mathcal{T}\to\mathcal{X}$ lies in the essential image of J, then $\mathcal{O}$ is a local $\mathcal{T}$-structure. If the Grothendieck topology on $\mathsf{Shv}(\mathcal{T})$ is subcanonical, then $\mathcal{O}$ is a local $\mathcal{T}$-structure if and only if $\mathcal{O}$ lies in the essential image of J.

Proof. It follows from [Lur17b], prop. 5.5.4.20 that the functor $L\circ$ is fully faithful, and the functor $j\circ$ is fully faithful by the universal property of presheaf ∞-categories.
Let $\mathcal{O}:\mathcal{T}\to\mathcal{X}$ be a functor in the essential image of J, then $\mathcal{O}$ is equivalent to a functor of the form $F\circ Lj$, with $F\in\mathrm{Fun}^{*\mathrm{ad}}(\mathsf{Shv}(\mathcal{T}),\mathcal{X})$. Because L and j preserve finite limits, $F\circ Lj$ is a $\mathcal{T}$-structure. To show that $F\circ Lj$ is local, we note that the functor $Lj:\mathcal{T}\to\mathsf{Shv}(\mathcal{T})$ sends admissible coverings to effective epimorphisms, so it suffices to show that F preserves effective epimorphisms of the form

$$\coprod_i Lj(U_i)\longrightarrow Lj(X)$$

where each $U_i\to X$ is admissible. This follows because F preserves admissible pullbacks and small colimits.
Now assume that the Grothendieck topology on $\mathsf{Shv}(\mathcal{T})$ is subcanonical. Suppose that $\mathcal{O}:\mathcal{T}\to\mathcal{X}$ is a local $\mathcal{T}$-structure, then $\mathcal{O}$ is the restriction of a left Kan extension $j_!\mathcal{O}:\mathsf{PShv}(\mathcal{T})\to\mathcal{X}$. We note that $j_!\mathcal{O}$ preserves final

objects, since final objects are representable and $\mathcal{O}$ preserves final objects. To show that $j_!\mathcal{O}$ preserves finite products, consider a pullback square

$$\begin{array}{ccc} X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

where $*$ is a final object, then we should show that the square

$$\begin{array}{ccc} j_!\mathcal{O}(X \times Y) & \longrightarrow & j_!\mathcal{O}(Y) \\ \downarrow & & \downarrow \\ j_!\mathcal{O}(X) & \longrightarrow & * \simeq j_!\mathcal{O}(*) \end{array}$$

is a pullback. Writing Y as a colimit of representables, and using the fact that colimits are universal in $\mathsf{PShv}(\mathcal{T})$ and $\mathcal{X}$, and that $j_!\mathcal{O}$ preserves small colimits, we may assume that Y is itself representable. The same holds for X, so the claim reduces to the assertion that $\mathcal{O}$ preserves binary products. To show that $j_!\mathcal{O}$ preserves admissible pullbacks, we argue similarly. Since we assumed the topology to be subcanonical, we need not apply L in the definition of admissible pullbacks. By the pasting property of pullback squares, it clearly suffices to prove that pullback diagrams of the form

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ j(U') & \xrightarrow{f} & j(X') \end{array}$$

are preserved by $j_!\mathcal{O}$, where $f : U' \to X'$ is admissible in $\mathcal{T}$. Again, writing X as a colimit of representables, and using universality of colimits in $\mathsf{PShv}(\mathcal{T})$ and $\mathcal{X}$, and that $j_!\mathcal{O}$ preserves small colimits, we may assume that X is itself representable, so that the claim follows from the assumption that $\mathcal{O}$ is a $\mathcal{T}$-structure. Now the assertion that $j_!\mathcal{O}$ lies in the image of the functor

$$\circ L : \mathrm{Fun}^L(\mathsf{Shv}(\mathcal{T}), \mathcal{X}) \longrightarrow \mathrm{Fun}^L(\mathsf{PShv}(\mathcal{T}), \mathcal{X})$$

is equivalent to the assertion that the adjoint $\mathcal{O}^* : \mathcal{X} \to \mathsf{PShv}(\mathcal{T})$ factors through $\mathsf{Shv}(\mathcal{T})$. Unwinding definitions, this means that for any admissible covering $\{U_i \to X\}$ with associated morphism $h : \coprod j(U_i) \to j(X)$, the functor $j_!\mathcal{O}$ takes the canonical monomorphism

$$|\check{C}(h)_\bullet| \xrightarrow{g} j(X)$$

to an equivalence in $\mathcal{X}$. We have just proven that $j_!\mathcal{O}$ preserves finite products and pullbacks along admissible maps in $\mathsf{PShv}(\mathcal{T})$ which implies that the canonical map of simplicial objects $j_!\mathcal{O}(\check{C}(h)_\bullet) \to \check{C}(j_!\mathcal{O}(h))_\bullet$ is an equivalence, so it follows that the composition

$$\coprod_i j_!\mathcal{O}(j(U_i)) \longrightarrow j_!\mathcal{O}(|\check{C}(h)_\bullet|) \xrightarrow{j_!\mathcal{O}(g)} j_!\mathcal{O}(j(X))$$

is the unique -up to contractible ambiguity- factorization of the map $j_!\mathcal{O}(h) : \coprod_i j_!\mathcal{O}(j(U_i)) \to j_!\mathcal{O}(j(X))$ into an effective epimorphism followed by a monomorphism. But $j_!\mathcal{O}(h)$ is already an effective epimorphism because $\mathcal{O}$ is a local $\mathcal{T}$-structure, so the map $j_!\mathcal{O}(g)$ is indeed an equivalence. We have an equivalence of functors $j_!\mathcal{O} \simeq j_!\mathcal{O}|_{\mathsf{Shv}(\mathcal{T})} \circ L$, so we immediately conclude that $j_!\mathcal{O}|_{\mathsf{Shv}(\mathcal{T})}$ preserves finite products. Since we assumed the topology to be subcanonical, $j_!\mathcal{O}|_{\mathsf{Shv}(\mathcal{T})}$ also preserves pullbacks along admissible maps in $\mathsf{Shv}(\mathcal{T})$. □

Definition 3.1.0.30. Let $\mathcal{T}$ be a pregeometry. Let $\coprod_i X_i \to X$ be a morphism in $\mathsf{Shv}(\mathcal{T})$, and denote for each $\bar{i} = (i_1, \ldots, i_n) \in I^n$ the object $X_{i_1} \times_X \ldots \times_X X_{i_n}$ by $X_{\bar{i}}$, so that the n'th level of the Čech nerve is given by $\coprod_{\bar{i} \in I^n} X_{\bar{i}}$. We say that a map $f : X \to Y$ in $\mathsf{Shv}(\mathcal{T})$ is

(1) *strongly étale* if there is a small collection of admissible maps $\{X_i \to X\}$ that determines an effective epimorphism $\coprod_{i \in I} X_i \to X$ such that for each $\bar{i} \in I^n$ the maps $X_{\bar{i}} \to X$ and $X_{\bar{i}} \to X \to Y$ are admissible.

(2) *strongly submersive* if there is a collection of admissible maps $\{X_i \to X\}$ that determines an effective epimorphism $\coprod_{i \in I} X_i \to X$ such that for each $\bar{i} \in I^n$, the map $X_{\bar{i}} \to X$ is admissible, and there exists an admissible map $X_{\bar{i}} \to V_{\bar{i}} \times Y$ for some object $V_{\bar{i}} \in \mathsf{Shv}(\mathcal{T})$ that fits into a commuting diagram

$$\begin{array}{ccc} X_{\bar{i}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_{\bar{i}} \times Y & \longrightarrow & Y \end{array}$$

where the lower horizontal map is the projection.

Remark 3.1.0.31. Since admissible maps in $\mathsf{Shv}(\mathcal{T})$ are not stable under compositions, the notions of strongly étale and submersive maps would not be well behaved if we only demanded that the maps $X_i \to X \to Y$ were admissible.

Proposition 3.1.0.32. *Let $\mathcal{T}$ be a pregeometry*

(1) Strongly étale and strongly submersive maps are stable under pullbacks.

(2) For any ∞-topos $\mathcal{X}$, if $F \in \mathrm{Fun}^{*\mathrm{ad}}(\mathsf{Shv}(\mathcal{T}), \mathcal{X})$, then F preserves pullbacks along strongly étale maps.

(3) For any ∞-topos $\mathcal{X}$, if $F \in \mathrm{Fun}^{*\mathrm{ad}}(\mathsf{Shv}(\mathcal{T}), \mathcal{X})$, then F preserves pullbacks along strongly submersive maps.

Proof. (1) is immediate since admissible maps and effective epimorphisms are stable under pullbacks and colimits are universal in $\mathsf{Shv}(\mathcal{T})$. Strongly étale maps are strongly submersive, so it suffices to prove (3). Consider a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where f is strongly étale. Since f is strongly étale, there is an effective epimorphism $h : \coprod_{i \in I} X_i \to X$ where each map $h_i : X_i \to X$ is admissible. Because the map F preserves pullbacks along admissible maps, F preserves the Čech nerve of the map h; that is, the map of simplicial objects $F(\check{C}(h)_\bullet) \to \check{C}(F(h))_\bullet$ is an equivalence. Because colimits are universal in $\mathsf{Shv}(\mathcal{T})$, we also have an effective epimorphism $h' : \coprod_{i \in I} X_i \times_Y Y' \to X'$ and an equivalence $F(\check{\mathrm{C}}(h'))_\bullet \to \check{\mathrm{C}}(F(h'))_\bullet$. Because F preserves colimit diagrams, we obtain a diagram

$$\begin{tikzcd} |\check{C}(F(h'))_\bullet| \arrow[r, "\simeq"] \arrow[d] & F(X') \arrow[r] \arrow[d] & F(Y') \arrow[d] \\ |\check{C}(F(h))_\bullet| \arrow[r, "\simeq"] & F(X) \arrow[r] & F(Y) \end{tikzcd}$$

We will be done once we show that the outer rectangle is a pullback. Let $\bar{i} = (i_1, \ldots, i_n) \in I^n$, and denote $X_{\bar{i}} = X_{i_1} \times_X \ldots \times_X X_{i_n}$ and $X'_{\bar{i}} = (X_{i_1} \times_Y Y') \times_{X'} \ldots \times_{X'} (X_{i_n} \times_Y Y') \simeq X_{\bar{i}} \times_X X'$. Because colimits are universal in $\mathcal{X}$, it suffices to show that for each $n \geq 1$ and each $\bar{i} \in I^n$, the diagram

$$\begin{tikzcd} F(X'_{\bar{i}}) \arrow[r] \arrow[d] & F(Y') \arrow[d] \\ F(X_{\bar{i}}) \arrow[r] & F(Y) \end{tikzcd}$$

is a pullback. Considering the diagram

$$\begin{tikzcd} F(X'_{\bar{i}}) \arrow[r] \arrow[d] & F(V_{\bar{i}}) \times F(Y') \arrow[r] \arrow[d] & F(Y') \arrow[d] \\ F(X_{\bar{i}}) \arrow[r] & F(V_{\bar{i}}) \times F(Y) \arrow[r] & F(Y) \end{tikzcd}$$

we note that the right square is a pullback because F preserves finite products, so it suffices to show that the left square is a pullback. The $\Delta^2 \times \Delta^1$-shaped diagram above is the image under F of the diagram

$$\begin{tikzcd} X'_{\bar{i}} \arrow[r] \arrow[d] & V_{\bar{i}} \times Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X_{\bar{i}} \arrow[r] & V_{\bar{i}} \times Y \arrow[r] & Y \end{tikzcd}$$

In this diagram, the outer rectangle and right square are pullbacks by construction, so the left square is also a pullback. Since the left lower horizontal map is admissible by assumption, we conclude, since F preserves pullbacks along admissible maps. ☐

Remark 3.1.0.33. We refrain from calling the class of morphisms just defined *étale* since this phrase has already been standardized ('étale map of geometric stacks'), and we will use that terminology in the same sense in later sections. In contrast, we will encounter strongly étale morphisms mainly when we deal with infinite dimensional manifolds that have enough smooth bump functions. Viewed as higher stacks on the site of manifolds, infinite dimensional manifolds (in derived differential geometry) are not geometric in the same sense that algebraic stacks (in derived algebraic geometry) are geometric.

3.1.1 Spectra and schemes

We have seen that for the geometry $\mathcal{G}_{\mathrm{Zar}}(k)$ and a topological space X, there is an equivalence $\mathrm{Str}_{\mathcal{G}_{\mathrm{Zar}}(k)}(\mathsf{Shv}(X)) \simeq \mathsf{Shv}_{\mathsf{CAlg}_k}(X)$ and with a little more work, it can be shown that $\mathrm{Str}^{loc}_{\mathcal{G}_{\mathrm{Zar}}(k)}(\mathsf{Shv}(X))$ may actually be identified with the subcategory of $\mathsf{Shv}_{\mathsf{CAlg}_k}(X)$ of sheaves for which the stalk at each point of X is a *local* (in the ordinary sense of commutative algebra) k-algebra, and local morphisms between them. It follows that the ∞-category ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}_{\mathrm{Zar}}(k))$ contains the category $\mathsf{RingSpace}^{loc}_k$ of locally k-ringed spaces as a full, discrete subcategory. Taking the Zariski spectrum $\operatorname{Spec} A$ of a commutative k-algebra A yields a functor $\operatorname{Spec} : \mathsf{CAlg}^{op}_k \to \mathsf{RingSpace}^{loc}_k$ right adjoint to the global sections functor $\Gamma : \mathsf{RingSpace}^{loc}_k \to \mathsf{CAlg}^{op}_k$. The goal of this subsection is to describe, for any geometry $\mathcal{G}$, a spectrum functor $\mathbf{Spec}^{\mathcal{G}} : \mathrm{Pro}(\mathcal{G}) \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ right adjoint to the functor taking global sections $\Gamma^{\mathcal{G}}$, so that we have a weak equivalence of Kan complexes

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{G})}(\Gamma^{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})), A) \simeq \mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), \mathbf{Spec}^{\mathcal{G}} A)$$

Taking suitable geometries, we will see that this spectrum functor encompasses many classical constructions, such as the prime spectrum of commutative k-algebras just discussed (for $\mathcal{G} = \mathcal{G}_{\mathrm{Zar}}(k)$), but also the 'real spectrum' of a C^∞-ring as reviewed in [Joy12a] (for $\mathcal{G}$ the geometry defined in subsection 3.1.3). Armed with $\mathbf{Spec}^{\mathcal{G}}$, we immediately have a notion of *affine $\mathcal{G}$-schemes*, which are just the $\mathcal{G}$-structured spaces in the essential image of the spectrum functor. Arbitrary $\mathcal{G}$-schemes are then constructed as étale gluings of affine ones.

We start by defining global sections. For each pair $(\mathcal{X}, X) \in \overline{{}^{\mathrm{L}}\mathsf{Top}}$ where X is an object of $\mathcal{X}$, the global sections $\Gamma(X) \in \mathcal{S}$ are given by $\mathrm{Hom}_{\mathcal{X}}(\mathbf{1}_{\mathcal{X}}, X)$ where $\mathbf{1}_{\mathcal{X}}$ is a final object of $\mathcal{X}$. Because $\mathcal{S}$ is initial in ${}^{\mathrm{L}}\mathsf{Top}$, we have a weak equivalence $\mathrm{Hom}_{\overline{{}^{\mathrm{L}}\mathsf{Top}}}((\mathcal{S}, \mathbf{1}_{\mathcal{S}}), (\mathcal{X}, X) \simeq \mathrm{Hom}_{\mathcal{X}}(\mathbf{1}_{\mathcal{X}}, X))$. Accordingly, we take the global sections functor

$$\Gamma : \overline{{}^{\mathrm{L}}\mathsf{Top}} \longrightarrow \mathcal{S}$$

to be the functor corepresented by $(\mathcal{S}, \mathbf{1}_{\mathcal{S}})$. For a geometry $\mathcal{G}$, we define $\Gamma^{\mathcal{G}}$ as the composition

$$\Gamma^{\mathcal{G}} : {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}) \longrightarrow \mathrm{Fun}(\mathcal{G}, \overline{{}^{\mathrm{L}}\mathsf{Top}}) \longrightarrow \mathrm{Fun}(\mathcal{G}, \mathcal{S}).$$

Concretely, $\Gamma^{\mathcal{G}}$ is given on objects by

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (A \mapsto \mathrm{Hom}_{\mathcal{X}}(p^* \mathbf{1}_{\mathcal{S}}, \mathcal{O}_{\mathcal{X}}(A))), \quad A \in \mathcal{G},$$

so $\Gamma^{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a left exact functor, that is, $\Gamma^{\mathcal{G}}$ factors through $\mathrm{Ind}(\mathcal{G}^{op})$.

Construction 3.1.1.1 ($\mathcal{G}$-spectrum)**.** Let $\mathcal{G}$ be a geometry. We say that a morphism $f : U \to X$ in $\mathrm{Pro}(\mathcal{G})$ is *admissible* if f fits into a pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"] & j(U') \arrow[d, "j(f')"] \\ X \arrow[r] & j(X') \end{tikzcd}$$

where $f' : U' \to X'$ is an admissible morphism in $\mathcal{G}$ and $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ is the Yoneda embedding. Lemma 2.2.4 of [Lur11b] tells us that

(1) Every equivalence in $\mathrm{Pro}(\mathcal{G})$ is admissible.

(2) The collection of admissible morphism is closed under taking pullbacks along any morphism in $\mathrm{Pro}(\mathcal{G})$.

(3) For a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

with g admissible, f is admissible if and only if h is admissible.

Let $\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X}$ be the full subcategory of $\mathrm{Pro}(\mathcal{G})_{/X}$ spanned by admissible morphisms to X. This ∞-category is essentially small, as all admissible maps are pullbacks of admissible maps in $\mathcal{G}$. We make $\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X}$ into an ∞-site by endowing it with the Grothendieck topology generated by the covering families of the form $\{j'(V_\alpha) \times_{j(U')} U \to U\}_{\alpha \in I}$ for $U \to j(U')$ some morphism and $\{V_\alpha \to U'\}_{\alpha \in I}$ an admissible covering of U' in $\mathcal{G}$.

Let $\operatorname{Spec} X$ denote the ∞-topos $\mathsf{Shv}(\mathrm{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X})$. We define a $\mathcal{G}$-structure on $\operatorname{Spec} X$, given by sending an object $Y \in \mathcal{G}$

to the sheafification of the presheaf sending an admissible map $U \to X$ to $\operatorname{colim}_i \operatorname{Hom}_{\mathcal{G}}(U_i, Y)$, where U_i is a filtered diagram with colimit U in $\mathcal{G}^{op}$. More formally, we have a functor

$$\mathcal{G} \times \operatorname{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X} \longrightarrow \mathcal{G} \times \operatorname{Pro}(\mathcal{G}) = \mathcal{G} \times \operatorname{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{S}) \xrightarrow{\mathrm{ev}} \mathcal{S}$$

where the last functor is the evaluation pairing. By adjunction we get a functor $\rho : \mathcal{G} \to \mathsf{PShv}(\operatorname{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X})$; now we let $\mathcal{O}_{\operatorname{Spec} X}$ be the composition

$$\mathcal{O}_{\operatorname{Spec} X} : \mathcal{G} \xrightarrow{\rho} \mathsf{PShv}(\operatorname{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X}) \xrightarrow{L} \mathsf{Shv}(\operatorname{Pro}(\mathcal{G})^{\mathrm{ad}}_{/X}),$$

where L is a sheafification functor. Notice that $\mathcal{O}_{\operatorname{Spec} X}$ is indeed a local $\mathcal{G}$-structure on $\operatorname{Spec} X$: ρ is manifestly left exact and L is a left exact localization, so $\mathcal{O}_{\operatorname{Spec} X}$ is also left exact. It's easy to see that $\mathcal{O}_{\operatorname{Spec} X}$ is local, that is, for an admissible covering $\{V_\beta \to Y\}_{\beta \in J}$ the map $\coprod_\beta \mathcal{O}_{\operatorname{Spec} X}(V_\beta) \to \mathcal{O}_{\operatorname{Spec} X}(Y)$ is an effective epimorphism.

Proposition 3.1.1.2. *Let $\mathcal{G}$ be a geometry. The global sections functor $\Gamma^{\mathcal{G}} : {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}) \to \operatorname{Ind}(\mathcal{G}^{op})$ has a left adjoint $\mathbf{Spec}^{\mathcal{G}}$ which on objects coincides with the $\mathcal{G}$-structured ∞-topoi of construction 3.1.1.1.*

Proof. This is [Lur11b], theorem 2.2.12. □

The spectrum functor for a *pre*geometry is defined similarly.

Construction 3.1.1.3 ($\mathcal{T}$-spectrum)**.** Let $\mathcal{T}$ be a pregeometry. For $X \in \mathcal{T}$, let $\mathcal{T}^{\mathrm{ad}}_{/X}$ be the ∞-category of admissible maps in $\mathcal{T}$ over X, endowed with its Grothendieck topology generated by admissible morphisms. The spectrum $\mathbf{Spec}^{\mathcal{T}} X$ is the pair $(\mathsf{Shv}(\mathcal{T}^{\mathrm{ad}}_{/X}), \mathcal{O}_X$, where $\mathcal{O}_X$ denotes the composition $\mathcal{T} \to \mathsf{PShv}(\mathcal{T}^{\mathrm{ad}}_{/X}) \to \mathsf{Shv}(\mathcal{T}^{\mathrm{ad}}_{/X})$, where the last map is a sheafification functor. One easily verifies that $\mathcal{O}_X$ is a $\mathcal{T}$-structure.

Example 3.1.1.4. Consider the pregeometry $\mathcal{T}_{\mathrm{Diff}}$. Clearly, $(\mathcal{T}_{\mathrm{Diff}})^{\mathrm{ad}}_{/M} \simeq \operatorname{Open}(M)$, so the ∞-topos $\operatorname{Spec} M$ is simply $\mathsf{Shv}(M)$. The spectrum functor produces a local $\mathcal{T}_{\mathrm{Diff}}$-structure on $\mathsf{Shv}(M)$ that coincides with the functor

$$\mathcal{T}_{\mathrm{Diff}} \to \mathsf{Shv}(M), \quad N \mapsto (U \mapsto \operatorname{Hom}_{\mathcal{T}_{\mathrm{Diff}}}(U, N)).$$

Notice that this functor lands in sheaves because the topology on $\mathcal{T}_{\mathrm{Diff}}$ is subcanonical.

Remark 3.1.1.5. The flexibility of the spectrum functor constructed here is not just philosophically satisfying. Construction 3.1.1.1 will reappear in section 4 when we deal with *geometries of modules*.

Now that we have set up the necessary theory, we can give a first definition of derived manifolds.

Definition 3.1.1.6. A $\mathcal{T}_{\mathrm{Diff}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an *affine derived manifold of finite presentation* if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a retract of a finite limit of $\mathcal{T}_{\mathrm{Diff}}$-structured ∞-topoi of the form $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diff}}} M$, for M a smooth manifold. Since the functor ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diff}}) \to \mathsf{Top}$ preserves finite limits and 0-localic Hausdorff ∞-topoi are stable under limits and retracts, we deduce that affine derived manifolds of finite presentation have 0-localic underlying ∞-topoi. A 0-localic Hausdorff $\mathcal{T}_{\mathrm{Diff}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an *derived manifold of finite presentation* if there is a countable collection of objects U_α of $\mathcal{X}$ and an effective epimorphism $\coprod_\alpha U_\alpha \to \mathbf{1}$ such that for each α, $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha})$ is an affine derived manifold of finite presentation.

Remark 3.1.1.7. The definition above is basically the one given by Spivak [Spi10] and Wallbrige [Wal17] (but note that Spivak only takes pullbacks of manifolds, not arbitrary finite limits). While this definition is conceptually appealing, it is far from practical. In the coming sections our main concern will be with finding more suitable models for the ∞-category spanned by affine derived manifolds of finite presentation. In fact, we will give an explicit description of a geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ such that the affine $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-schemes are precisely the affine derived manifolds of finite presentation.

3.1.2 Geometric envelopes and truncations

Let $\mathcal{T}$ be a pregeometry. While sometimes $\mathcal{T}$ has all finite limits, it usually does not; the point of introducing admissibility structures is to specify a collection of extant, well-behaved finite limits. When passing from ordinary geometry (smooth, analytic, algebraic,...) to derived geometry, we start with the pregeometry $\mathcal{T}$ of 'smooth affines' which we complete with respect to finite limits, in such a way that the admissible pullbacks are preserved, that is, we produce a canonical *geometry* $\mathcal{G}$ out of $\mathcal{T}$. We will see that producing such a *geometric envelope* is always possible. This canonical derived geometry is related to the classical theory of non-smooth affine spaces (C^∞-schemes, analytic spaces, schemes,...) by *truncation*: we can choose to form the finite limit-completion of $\mathcal{T}$ in the ∞-category of n-categories for each $n \geq 1$. The resulting *n-truncated geometric envelopes* $\mathcal{G}_{\leq n}$ are approximations to the derived geometry $\mathcal{G}$, and it can be shown that $\mathcal{G}_{\leq n}$ is obtained as the compact objects of $\tau_{\leq n}\operatorname{Ind}(\mathcal{G})$. In many cases, the 0-truncated geometric envelope $\mathcal{G}_{\leq 0}$ actually coincides with the usual theory of $\mathcal{T}$-schemes of finite type. For example, for the pregeometry $\mathcal{T}_{\mathrm{Zar}}(k)$ of example 3.1.0.13, there is a geometric envelope of derived affine k-schemes $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$

describing derived algebraic geometry for the Zariski topology, and the 0-truncated geometric envelope is the geometry $\mathcal{G}_{\mathrm{Zar}}(k)$ of affine k-schemes describing *ordinary* algebraic geometry.
We define the geometric envelope of a pregeometry by a universal property.

Definition 3.1.2.1. Let $\mathcal{T}$ be a pregeometry. A functor $\varphi : \mathcal{T} \to \mathcal{G}$ *exhibits $\mathcal{G}$ as geometric envelope of $\mathcal{T}$* if $\mathcal{G}$ is an essentially small idempotent complete ∞-category that has finite limits, $\varphi \in \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T},\mathcal{G})$, and for each idempotent complete ∞-category $\mathcal{C}$ that has finite limits, composition with φ yields an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G},\mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T},\mathcal{C}).$$

If $\varphi : \mathcal{T} \to \mathcal{G}$ exhibits $\mathcal{G}$ as geometric envelope of $\mathcal{T}$, we endow $\mathcal{G}$ with the coarsest admissibility structure such that φ is a transformation of pregeometries.

Remark 3.1.2.2. The uniqueness of the geometric envelope may be explained as follows: let $\mathsf{Cat}_\infty^{\mathrm{lex},\mathrm{Idem}}$ denote the subcategory of Cat_∞ spanned by idempotent complete ∞-categories that admit finite limits, and left exact functors between them. The assignment $\mathcal{C} \mapsto \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T},\mathcal{C})$ defines a functor $\mathsf{Cat}_\infty^{\mathrm{lex},\mathrm{Idem}} \to \mathsf{Cat}_\infty$ and thus a coCartesian fibration $\mathfrak{T} \to \mathsf{Cat}_\infty^{\mathrm{lex},\mathrm{Idem}}$. Now a geometric envelope of $\mathcal{T}$ is precisely an initial object in $\mathfrak{T}$; in other words, a geometric envelope $\mathcal{G}$ *2-represents* the functor $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T},\_)$, in the sense of [GHN15], for instance.

The existence of geometric envelopes is guaranteed by the procedure of adding (co)limits to ∞-categories as exposed in [Lur17b], section 5.3.6.4: we consider the collection $\mathcal{K}$ of simplicial sets indexing finite colimits and idempotents. First, we consider the ∞-category $S^{-1}\mathsf{PShv}(\mathcal{T}^{op})$ obtained by localizing $\mathsf{PShv}(\mathcal{T}^{op})$ at the set S of maps $X \to j(Y)$, where X is a colimit of an admissible pushout diagram in $\mathsf{PShv}(\mathcal{T}^{op})$ and $j(Y)$ is the image of the Yoneda embedding of a colimit of the same pushout diagram taken in $\mathcal{T}^{op}$ itself. Then we let $\mathcal{G}^{op}$ be the smallest full subcategory of $S^{-1}\mathsf{PShv}(\mathcal{T}^{op})$ containing the essential image of $\mathcal{T}^{op} \to \mathsf{PShv}(\mathcal{T}^{op}) \to S^{-1}\mathsf{PShv}(\mathcal{T}^{op})$ that is stable under colimits of diagrams indexed by simplicial sets in the collection $\mathcal{K}$. The ∞-category $\mathcal{G}$ so obtained satisfies the desired properties by [Lur17b], prop. 5.3.6.2.
Passing to the geometric envelope $\mathcal{G}$ of a pregeometry $\mathcal{T}$ yields the same theory of structured spaces:

Proposition 3.1.2.3. *Let $\mathcal{T}$ be a pregeometry and let $f : \mathcal{T} \to \mathcal{G}$ exhibit $\mathcal{G}$ as a geometric envelope of $\mathcal{T}$. Then composition with f induces equivalences $\mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ and $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X})$ for any ∞-topos $\mathcal{X}$. Moreover, the functors $\mathbf{Spec}^{\mathcal{T}}$ and $\mathbf{Spec}^{\mathcal{G}} \circ j \circ f$ are canonically equivalent, where $j : \mathcal{G} \to \mathrm{Pro}(\mathcal{G})$ is the Yoneda embedding.*

Proof. This is [Lur11b], prop. 3.4.7 and prop. 3.5.7. □

Remark 3.1.2.4. Let $f : \mathcal{T} \hookrightarrow \mathcal{G}$ exhibit $\mathcal{G}$ as a geometric envelope of $\mathcal{T}$. Composing f with the functor $L \circ j : \mathcal{G} \to \mathsf{Shv}(\mathcal{G})$ induces a functor $f' : \mathcal{T} \to \mathsf{Shv}(\mathcal{G})$ which is a local $\mathcal{T}$-structure because f is a transformation of pregeometries. Suppose that the topology on $\mathcal{T}$ is subcanonical, then by proposition 3.1.0.29, f' comes from a functor

$$F : \mathsf{Shv}(\mathcal{T}) \longrightarrow \mathsf{Shv}(\mathcal{G})$$

which preserves small colimits, finite products and admissible pullbacks. We claim that F is a left adjoint to the functor $f^* : \mathsf{Shv}(\mathcal{G}) \to \mathsf{Shv}(\mathcal{T})$ given by restriction of sheaves. In particular, F can be identified with the functor

$$\mathsf{Shv}(\mathcal{T}) \longrightarrow \mathsf{PShv}(\mathcal{T}) \xrightarrow{f_!} \mathsf{PShv}(\mathcal{G}) \xrightarrow{L} \mathsf{Shv}(\mathcal{G}).$$

To see this, note that F is the restriction to sheaves of a left Kan extension $j_! f'$ of f' along the Yoneda embedding into presheaves. Now both $j_! f'$ and $L \circ f_!$ are colimit preserving functors on $\mathsf{PShv}(\mathcal{T})$, so they are equivalent if and only if their restrictions to $\mathcal{T}$ are equivalent, which is obviously the case.

If $\mathcal{O}_{\mathcal{X}}$ is a local $\mathcal{T}$-structure on $\mathcal{X}$, it might not be the case that the truncation

$$\tau_{\leq n}\mathcal{O}_{\mathcal{X}} \coloneqq \tau_{\leq n} \circ \mathcal{O}_{\mathcal{X}} : \mathcal{T} \to \mathcal{X} \overset{\tau_{\leq n}}{\to} \mathcal{X}$$

is a local $\mathcal{T}$-structure. Being able to take truncations of the structure sheaves of objects in ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T})$ is a highly desirable feature for a pregeometry $\mathcal{T}$ to have, as it allows for arguments by induction up the Postnikov tower, which can be handled by obstruction theory in specific cases. If $\mathcal{T}$ satisfies the following definition, then truncated local $\mathcal{T}$-structures are still local $\mathcal{T}$-structures.

Definition 3.1.2.5. A pregeometry $\mathcal{T}$ is *compatible with n-truncations* if for each ∞-topos $\mathcal{X}$, each $\mathcal{T}$-structure $\mathcal{O}$ on $\mathcal{X}$ and each admissible map $U \to X$, the diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \tau_{\leq n}\mathcal{O}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \tau_{\leq n}\mathcal{O}(X) \end{array}$$

is a pullback.

Remark 3.1.2.6. All the pregeometries we have seen so far are discrete. Proposition 3.3.5 of [Lur11b] tells us that such pregeometries are compatible with n-truncations for $n \geq -1$.

For each truncation $n \geq -1$, we can make sense of n-truncated geometric envelopes, which should be thought of as interpolating between the 'classical' enveloping geometry and the 'fully derived' geometry.

Definition 3.1.2.7. Let $\mathcal{T}$ be a pregeometry. A functor $\varphi : \mathcal{T} \to \mathcal{G}$ *exhibits $\mathcal{G}$ as an n-truncated geometric envelope of $\mathcal{T}$* if $\mathcal{G}$ is an essentially small n-category that has finite limits[1], $\varphi \in \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}, \mathcal{G})$, and for each idempotent complete ∞-category $\mathcal{C}$ that has finite limits, composition with φ yields an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}, \mathcal{C}).$$

If $\varphi : \mathcal{T} \to \mathcal{G}$ exhibits $\mathcal{G}$ as an n-truncated geometric envelope of $\mathcal{T}$, we endow $\mathcal{G}$ with the coarsest admissibility structure such that φ is a transformation of pregeometries.

Remark 3.1.2.8. Given the existence of geometric envelopes, the existence of n-truncated geometric envelopes can be established as follows: take a functor $f : \mathcal{T} \to \mathcal{G}$ exhibiting $\mathcal{G}$ as a geometric envelope of $\mathcal{T}$, then the canonical functor $\mathcal{T} \to \mathcal{G} \to \mathcal{G}'_n$, where $\mathcal{G}'_n$ is the opposite category of the n-category of compact objects in $\tau_{\leq n}\mathrm{Ind}(\mathcal{G}^{op})$, exhibits $\mathcal{G}'_n$ as an n-truncated geometric envelope of $\mathcal{T}$.

Here we give some examples of geometric envelopes for pregeometries mentioned.

Example 3.1.2.9 (Geometric envelopes of $\mathcal{T}_{\mathrm{Zar}}(k)$)**.** For k a commutative ring, we may consider the ∞-category of *simplicial commutative k-algebras*, denoted sCAlg_k, which is defined as the ∞-category of algebras for the finite limit theory whose objects are all the affine k-spaces $\mathbb{A}^n_k$ and whose morphisms are polynomial maps (see section 4.1 for finite limit theories or section 4 of [Lur11b] where this example is discussed in great detail). When $\mathrm{char}(k) = 0$, sCAlg_k can be shown to be equivalent to the ∞-category of $\mathbb{E}_\infty\mathsf{Alg}^{cn}_k$ of connective $\mathbb{E}_\infty$-algebra objects in the ∞-category of k-modules, via the Barr-Beck theorem. The ∞-category $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$ is defined as the opposite of the ∞-category of finitely presented objects in the presentable ∞-category sCAlg_k. We have an equivalence $\tau_{\leq 0}\mathsf{sCAlg}_k \simeq \mathbf{N}(\mathsf{CAlg}_k)$ which remains an equivalence after restricting to finitely presented objects. Again, we may define for each simplicial k-algebra B and each $b \in \tau_{\leq 0}B$, a localization $B \to B[1/b]$ defined up to equivalence. We endow $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$ with the following admissibility structure:

(1) A morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is admissible if and only if there exists some element $b \in \tau_{\leq 0}B$ such that the map $B \to A$ induces an isomorphism $B[1/b] \cong A$.

(2) A collection of admissible morphism $\{\mathrm{Spec}\, B_i[1/b_i] \to \mathrm{Spec}\, B\}_i$ is an admissible covering if and only if the elements $\{b_i\}$ generate the unit ideal in B.

There is an obvious functor $\mathcal{T}_{\mathrm{Zar}}(k) \hookrightarrow \mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$ which, according to [Lur11b], prop. 4.2.3, exhibits $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$ as a geometric envelope of $\mathcal{T}_{\mathrm{Zar}}(k)$. It follows that for each $0 \leq n < \infty$, the map $\mathcal{G}^{\mathrm{der},\leq n}_{\mathrm{Zar}}(k)$ defined as the opposite of the full subcategory of $\tau_{\leq n}\mathsf{sCAlg}_k$ spanned by finitely presented objects, is an n-truncated geometric envelope of $\mathcal{T}_{\mathrm{Zar}}(k)$. In particular, the geometry $\mathcal{G}_{\mathrm{Zar}}(k) \simeq \tau_{\leq 0}\mathsf{sCAlg}_k$ of example 3.1.0.15 is a 0-truncated geometric envelope of $\mathcal{T}_{\mathrm{Zar}}(k)$.

The construction of the geometric envelopes of the *étale* pregeometry follows along similar lines, but we do not give the details here since we do not yet have the means to talk about étale maps of simplicial commutative rings. One may wonder what the geometric envelopes of $\mathcal{T}_{\mathrm{Diff}}$ look like. A partial answer is given in the following subsection.

3.1.3 The geometry of finitely presented C^∞-rings

In this section, we take some time to study C^∞-rings and C^∞-schemes within the framework of geometries and structured spaces. While not strictly necessary, some familiarity with ordinary C^∞-rings and synthetic differential geometry, as exposed, for instance, in the textbook [MR91] or the more recent [Joy12a], will be advantageous to the reader.

Definition 3.1.3.1. Let $\mathsf{CartSp} \subset \mathcal{T}_{\mathrm{Diff}}$ be the full subcategory spanned by objects of the form $\mathbb{R}^n$ for $n \geq 0$. A *C^∞-ring* is an algebra for the Lawvere theory CartSp, i.e., a finite product preserving functor $\mathsf{CartSp} \to \mathsf{Set}$. The full subcategory of $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})$ spanned by C^∞-rings is denoted $C^\infty\mathsf{ring}$. This is a strongly reflective, and thus presentable, subcategory of $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})$.

We will discuss Lawvere theories and their ∞-categories of space-valued algebras in more detail in section 4.1.1. Unwinding the definition, a C^∞-ring A consists of a set $A(\mathbb{R})$ equipped with a functional calculus for all smooth functions; that is, we have functorial operations

$$f_* : A(\mathbb{R})^n \longrightarrow A(\mathbb{R})^m$$

[1] Idempotent completeness is automatic for n-categories that have finite limits

for each smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$. We will often abuse notation and write 'an element $a \in A$' for $a \in A(\mathbb{R})$. Let $\mathcal{T}_{\mathbb{R}}^{disc}$ be the category that has the same objects as CartSp, but only polynomial maps; this is a Lawvere theory, and its algebras are precisely commutative $\mathbb{R}$-algebras. The transformation of Lawvere theories $\mathcal{T}_{\mathbb{R}}^{disc} \to \mathsf{CartSp}$ induces an 'underlying commutative $\mathbb{R}$-algebra' functor $(\_)^{\mathrm{alg}} : C^\infty\mathsf{ring} \to \mathsf{CAlg}_{\mathbb{R}}$. Many C^∞-rings of interest are subsumed by the following examples.

(1) The forgetful functor $(\_)^{\mathrm{alg}} : C^\infty\mathsf{ring} \to \mathsf{CAlg}_{\mathbb{R}}$ preserves limits and sifted colimits, and thus admits a left adjoint, the *free C^∞-ring functor F*, which takes the polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$ to $C^\infty(\mathbb{R}^n)$.

(2) Let A be a C^∞-ring and let $I \subset A$ be an ideal of the underlying $\mathbb{R}$-algebra. Then A/I is a C^∞-ring and the map $A \to A/I$ is regular epimorphism, i.e. it is the coequalizer in $C^\infty\mathsf{ring}$ of the equivalence relation determined by I.

(3) For a subset $X \subset \mathbb{R}^n$, the algebra of smooth functions

$$C^\infty(X) := \{f : X \to \mathbb{R};\ \forall x \in X \text{ there exists } x \in U \subset \mathbb{R}^n \text{ open and } \tilde{f} \in C^\infty(U) \text{ such that } \tilde{f}|_{X \cap U} = f|_U\}$$

is a C^∞-ring by composition. If X is a closed subset, then an application of the Tietze extension theorem shows that the natural map $C^\infty(\mathbb{R}^n) \to C^\infty(X)$ induces an isomorphism $C^\infty(\mathbb{R}^n)/\mathfrak{m}_X^0 \to C^\infty(X)$, where $\mathfrak{m}_X^0$ is the ideal of functions that vanish on X. In particular (by the Whitney embedding theorem), the algebra of smooth functions on a manifold M is a C^∞-ring of the form $C^\infty(\mathbb{R}^n)/\mathfrak{m}_M^0$.

(4) For $x \in \mathbb{R}^n$, the local algebra of germs of smooth functions at x is a C^∞-ring, given by $C^\infty(\mathbb{R}^n)_x := C^\infty(\mathbb{R}^n)/\mathfrak{m}_x$ with $\mathfrak{m}_x^g$ the ideal of smooth functions vanishing in some neighbourhood of x.

(5) Every local Artin $\mathbb{R}$-algebra $W = \mathbb{R} \oplus \mathfrak{m}$ is a C^∞-ring, whose C^∞-ring structure is uniquely determined by the underlying algebra. Such C^∞-rings are also called *Weil algebras*.

(6) Let $\mathfrak{m}$ be the maximal ideal of the C^∞-ring of germs $C^\infty(\mathbb{R}^n)_0$. The C^∞-ring $J_n^k := C^\infty(\mathbb{R}^n)_0/\mathfrak{m}^k$ of *k'th order jets at* 0 is a Weil algebra.

(7) Let R be a complete local Noetherian $\mathbb{R}$-algebra with residue class field $\mathbb{R}$, then R is of the form $R \cong \mathbb{R}[[x_1, \dots, x_n]]/I$ for some ideal I, by Cohen's structure theorem. By Borel's lemma on formal power series, there is an equivalence $C^\infty(\mathbb{R}^n)/\mathfrak{m}_0^\infty \cong \mathbb{R}[[x_1, \dots, x_n]]$, where $\mathfrak{m}_0^\infty$ is the ideal of functions that are flat at 0 (all partial derivatives vanish at 0). Thus, R can be written as a quotient by $\mathfrak{m}_0^\infty$ of $C^\infty(\mathbb{R}^n)/\tilde{I}$, where $\tilde{I}$ is a finitely generated (because $\mathbb{R}[[x_1, \dots, x_n]]$ is Noetherian) ideal lifting I, so we conclude that R is a C^∞-ring. It's easy to see that all algebra morphism between complete local Noetherian $\mathbb{R}$-algebras are morphisms of C^∞-rings, so the C^∞-ring structure of R is also uniquely determined by the underlying algebra, as in the case of Weil algebras (which this example subsumes).

Remark 3.1.3.2. The essential image of the free C^∞-ring functor $F : \mathsf{CAlg}_{\mathbb{R}} \to C^\infty\mathsf{ring}$ already contains many interesting objects. For instance, $F(\mathsf{CAlg}_{\mathbb{R}})$ contains all C^∞-rings of smooth functions on *compact* manifolds. This is an immediate consequence of the Nash-Tognoli theorem [Nas52; Tog73], which extends an older result of Whitney that all compact submanifolds of Euclidean space are diffeomorphic to zero loci of systems of real analytic equations.

Remark 3.1.3.3. Clearly, functions on manifolds that have less regularity also form C^∞-rings: let M be a manifold, then there are C^∞-rings $C^k(M)$ of k-times differentiable functions and $\mathrm{Lip}^k(M)$ of k-times differentiable functions with locally Lipschitz derivatives. Let M be an n-dimensional manifold and let $k \in \mathbb{Q}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$ such that $kp > n$, then the space $W^{k,p}_{\mathrm{loc}}(M)$ of Sobolev functions of class (k,p) is also a C^∞-ring by an extension of the Sobolev multiplication theorems, which can be deduced from the Gagliardo-Nirenberg interpolation estimates (see, for instance, [Melb]).

Notation 3.1.3.4. The functor $(\_)^{\mathrm{alg}}$ does not preserve pushouts nor coproducts in general. We reserve the symbol $\otimes^\infty$ for the pushout of C^∞-rings.

Definition 3.1.3.5. A C^∞-ring A is *finitely generated* if $A \simeq C^\infty(\mathbb{R}^n)/I$ for some $n < \infty$. A is *finitely presented* if $A \simeq C^\infty(\mathbb{R}^n)/I$ for some $n < \infty$ and I a finitely generated ideal.

Remark 3.1.3.6. A C^∞-ring A is finitely presented if and only if the functor corepresented by A (on the category of C^∞-rings) preserves filtered colimits. A is finitely generated if and only if the functor corepresented by A preserves filtered colimits of diagrams consisting only of monomorphisms. See [AR94], chapter 3 for proofs of these facts. As the category of C^∞-rings is presentable, we see that the full subcategories spanned by finitely generated and finitely presented C^∞-rings have finite colimits.

Remark 3.1.3.7. Let $f : N \to M$ and $g : P \to M$ be smooth maps of manifolds. We say that the pullback $N \times_M P$ is *transverse* if for each $x_1 \in N$, $x_2 \in P$ such that $f(x_1) = x = p(x_2)$, the induced map $T_{x_1} f \oplus T_{x_2} g : T_{x_1} N \oplus T_{x_2} P \to T_x M$ is a surjection. An elementary but crucial result in synthetic differential geometry is the following: the functor $C^\infty : \mathcal{T}_{\mathrm{Diff}} \to C^\infty\mathsf{ring}^{op}$ is fully faithful, takes values in finitely presented objects, and preserves finite products and transverse pullbacks. For a proof, see [MR91], chapter 1, theorem 2.8. The next chapter shall be concerned with proving a derived version of this result.

Remark 3.1.3.8. For $f : M \to \mathbb{R}^n$ a function on a manifold, we call the set $\mathrm{Carr}(f) := f^{-1}(\mathbb{R}^n \setminus \{0\})$ the *carrier* of f, and we call the set $\mathrm{Supp}(f) := \overline{\mathrm{Carr}(f)}$ the *support* of f. We will use frequently that any open set $U \to M$ in a manifold has a *characteristic function* $\chi_U : M \to \mathbb{R}$, a function on M such that $\mathrm{Carr}(\chi_U) = U$. We will also use that any function $f \in C^\infty(U)$ defined on an open set $U \subset M$ of a manifold M is divisible by some function $g|_U$ where g is defined on all of M and nonzero on U.

For any $n > 0$, the C^∞-rings $C^\infty(\mathbb{R}^n)$ do not satisfy the the conclusion of the Nullstellensatz for arbitrary ideals; instead, we single out three classes of ideals for which the weak version of the Nullstellensatz does hold. Let M be a smooth manifold of dimension $n > 0$ and let I be an ideal of the commutative algebra $C^\infty(M)$. For $x \in M$, we have the ideals

(1) $\mathfrak{m}_x^0$ of functions that vanish at x, and the quotient map $C^\infty(M) \to C^\infty(M)/\mathfrak{m}_x^0 \cong \mathbb{R}$ is the map ev_x evaluating at x.

(2) $\mathfrak{m}_x^\infty$ of functions that are flat at x, and choosing coordinates centered at x, the quotient map $C^\infty(M) \to C^\infty(M)/\mathfrak{m}_x^\infty \cong \mathbb{R}[[x_1, \dots, x_n]]$ is the map j_x^∞ taking the infinite jet at x.

(3) $\mathfrak{m}_x^g$ of functions that vanish in a neighbourhood of x, and choosing coordinates centered at x, the quotient map $C^\infty(M) \to C^\infty(M)/\mathfrak{m}_x^g \cong C^\infty(\mathbb{R}^n)_0$ is the map taking the germ at x.

Since surjections of ring maps carry ideals to ideals, it makes sense to ask whether a function $f \in C^\infty(M)$ is pointwise, formally, or locally contained in an ideal I.

Definition 3.1.3.9. Let M be a smooth manifold of dimension $n > 0$, and let $I \subset C^\infty(M)$ be an ideal. Write $Z(I)$ for the common zero locus of the functions in I.

(1) I is *point determined* iff for all $f \in C^\infty(\mathbb{R}^n)$, $f \in I$ iff $f(x) = 0$ for all $x \in Z(I)$.

(2) I is *jet determined* or *closed* iff for all $f \in C^\infty(\mathbb{R}^n)$, $f \in I$ iff $j_x^\infty(f) \in j_x^\infty(I)$ for all $x \in Z(I)$, where $j_x^\infty : C^\infty(\mathbb{R}^n) \to \mathbb{R}[[x_1, \dots, x_n]]$ carries a function to its formal power series at x.

(3) I is *locally determined* or *germ determined* iff for all $f \in C^\infty(\mathbb{R}^n)$, $f \in I$ iff $f_x \in I_x$ for all $x \in Z(I)$, where f_x and I_x are the germ of f at x and the ideal of $C^\infty(M)_x$ of germs at x of functions in I.

Remark 3.1.3.10. Here are some properties of the classes of ideals just defined.

(1) Point determined implies jet determined implies germ determined. None of these implications can be reversed in general. For instance, the ideal $I \subset C^\infty(\mathbb{R})$ of functions whose jet at 0 vanishes is jet determined but not point determined. An ideal of $C^\infty(\mathbb{R})$ generated by a compactly supported function is germ determined but not jet determined. Finally, for an ideal that satisfies none of the conditions above -and for which the Nullstellensatz fails completely- consider the ideal of compactly supported functions in $C^\infty(\mathbb{R})$.

(2) A collection of functions $\{f_1, \dots, f_m\}$ on M generates a point determined ideal if the functions $\{f_1, \dots, f_m\}$ are *independent*, that is, the zero locus of $(f_1, \dots, f_m) : M \to \mathbb{R}^m$ consists of regular points.

(3) Recall that given a collection of functions $\{f_\alpha\} \subset C^\infty(M)$ such that their carriers furnish a locally finite collection of opens on M, the pointwise sum $\sum_\alpha f_\alpha$ exists in $C^\infty(M)$ and is called a locally finite sum. An ideal $I \subset C^\infty(M)$ is germ determined if and only if I is closed under taking locally finite sums. It follows easily that finitely generated ideals are germ determined.

(4) Let $I \subset \mathbb{R}[[x_1, \dots, x_n]]$ be an ideal, then in order to conclude that $h \in I$, it suffices to show that for all $k \in \mathbb{Z}_{>0}$, $h \in I + \mathfrak{m}^k$, where $\mathfrak{m} = (x_1, \dots, x_n)$, the unique maximal ideal. Indeed, it suffices to show that $I = \cap_{k \geq 1}(I + \mathfrak{m}^k)$. The inclusion $I \subset \cap_{k \geq 1}(I + \mathfrak{m}^k)$ is obvious. For the other inclusion, it suffices to show that $p(\cap_{k \geq 1}(I + \mathfrak{m}^k)) = 0$, where $p : \mathbb{R}[[x_1, \dots, x_n]] \to \mathbb{R}[[x_1, \dots, x_n]]/I$ is the projection, but we clearly have

$$p(\bigcap_{k>1}(I + \mathfrak{m}^k)) \subset \bigcap_{k \geq 1} p(I + \mathfrak{m}^k) = \bigcap_{k \geq 1} p(\mathfrak{m})^k.$$

Since p is a local morphism, $p(\mathfrak{m})$ is the maximal ideal of $\mathbb{R}[[x_1, \dots, x_n]]/I$ so Krull's intersection theorem yields $\cap_{k \geq 1} p(\mathfrak{m})^k = 0$ as $\mathbb{R}[[x_1, \dots, x_n]]/I$ is local and Noetherian.

(5) We say that a finitely generated C^∞-ring $A = C^\infty(\mathbb{R}^n)/I$ is *point determined* (*closed*, *germ determined*) if I is a point determined (closed, germ determined) ideal. This does *not* depend on the presentation of A. Thus, if $C^\infty(\mathbb{R}^n)/I \cong C^\infty(\mathbb{R}^m)/J$ and I is point determined (closed, germ determined), then J is point determined (closed, germ determined) as well. As an application, let M be a manifold and note that as M lies in some $\mathbb{R}^n$ as a closed submanifold, $C^\infty(M)$ is a point determined C^∞-ring. As $C^\infty(M)$ is finitely presented, this shows that we have a presentation $C^\infty(M) \cong C^\infty(\mathbb{R}^n)/I$ where I is a finitely generated and point determined ideal.

Remark 3.1.3.11. Let $X \subset \mathbb{R}^n$ be a subset. We define the following ideals of $C^\infty(\mathbb{R}^n)$ associated to X:

$$\begin{aligned}
\mathfrak{m}_X^0 &:= \{f \in C^\infty(\mathbb{R}^n) | \, f(p) = 0 \, \forall p \in X\},\\
\mathfrak{m}_X^\infty &:= \{f \in C^\infty(\mathbb{R}^n); \, D_\alpha f(p) = 0 \, \forall p \in X\},\\
\mathfrak{m}_X^g &:= \{f \in C^\infty(\mathbb{R}^n); \, \exists U \supset X \text{ open}, \, f|_U = 0\}.
\end{aligned}$$

In the second line, D_α denotes the differential operator $\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ for α a multi-index $(\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$. $\mathfrak{m}_X^0$ is point determined, $\mathfrak{m}_X^\infty$ is closed and $\mathfrak{m}_X^g$ is germ determined. If $X \subset \overline{X^\circ}$, then $\mathfrak{m}_X^0 = \mathfrak{m}_X^\infty$.

As we have seen, the Tietze extension theorem shows that for $X \subset \mathbb{R}^n$ closed subset, we have $C^\infty(\mathbb{R}^n)/\mathfrak{m}_X^0 \cong C^\infty(X)$. There is a similar characterization of C^∞-rings of the form $C^\infty(\mathbb{R}^n)/\mathfrak{m}_X^\infty$ that uses Whitney's extension theorem.

Definition 3.1.3.12. Let $X \subset U$ be a closed subset of an open subset in $\mathbb{R}^n$, and let $F = (f^k)_{k \in \mathbb{Z}^n_{\geq 0}}$ be a collection of continuous functions for each multi-index k. F is a *Whitney function* if for each $m \geq 0$, we have for $\mathbf{x}, \mathbf{y} \in X$

$$f^k(\mathbf{x}) = \sum_{|l| = m - |k|} \frac{f^{l+k}(\mathbf{y})}{l!} (\mathbf{x} - \mathbf{y})^l + R^k(\mathbf{x}, \mathbf{y}),$$

where $R^k(\mathbf{x}, \mathbf{y})$ is a term that goes to 0 as $|\mathbf{x} - \mathbf{y}| \to 0$ faster than $|\mathbf{x} - \mathbf{y}|^{m - |k|}$.

The following easy lemma shows that if $X \subset \mathbb{R}^n$ is a closed quadrant, then the Whitney functions on X coincide with the functions that have infinitely many derivatives up to the boundary.

Lemma 3.1.3.13. *Let $X \subset \mathbb{R}^n$ be a closed convex subset with nonempty interior. Then restriction to X° induces an equivalence between $C^\infty(X; \mathbb{R}^n)$ and the space*

$$\{f \in C^\infty(X^\circ); \, D_\alpha f \text{ is bounded on } X^\circ\}.$$

Proposition 3.1.3.14 (Whitney Extension Theorem [Whi34])**.** *Let $X \subset U$ be a closed subset of an open subset in $\mathbb{R}^n$, then taking the infinite jet prolongation and restricting to X yields an isomorphism $C^\infty(U)/\mathfrak{m}_X^\infty \cong C^\infty(X; U)$.*

A proof can be found in [Mal66]. We record the following pleasant property of flat ideals, i.e. ideals of the form $\mathfrak{m}_X^\infty$ for $X \subset \mathbb{R}^n$ closed.

Theorem 3.1.3.15 (Reyes-van Quê [QR82])**.** *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed, then as ideals of $C^\infty(\mathbb{R}^{n+m})$ we have the equality $(\mathfrak{m}_X^\infty, \mathfrak{m}_Y^\infty) = \mathfrak{m}_{X \times Y}^\infty$.*

Corollary 3.1.3.16. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed subsets, then the canonical map*

$$C^\infty(X; \mathbb{R}^n) \otimes^\infty C^\infty(Y; \mathbb{R}^m) \longrightarrow C^\infty(X \times Y; \mathbb{R}^{n+m})$$

is an equivalence.

Remark 3.1.3.17. We will also prove a derived version of the result above, which shows in particular that the local models for manifolds with corners behave well under the derived tensor product of C^∞-rings, which is the starting point for derived C^∞-geometry with corners.

We now define the admissible maps for a geometry with underlying category $C^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$. These maps will correspond to open inclusions of *C^∞-schemes*.

Definition 3.1.3.18. Let A be a C^∞-ring and let $a \in A$. A map $f : A \to B$ such that $f(a)$ is invertible *exhibits B as a localization of A* if for each C^∞-ring C, composition with f induces a bijection

$$\mathrm{Hom}_{C^\infty\mathsf{ring}}(B, C) \xrightarrow{\simeq} \mathrm{Hom}^0_{C^\infty\mathsf{ring}}(A, C)$$

where $\mathrm{Hom}^0_{C^\infty\mathsf{ring}}(A, C)$ is the subset of maps that send a to an invertible element.

Remark 3.1.3.19. A localization of an element $a \in A$ is clearly unique up to unique isomorphism, and we denote it $A \to A[a^{-1}]$. The localization always exists and can be constructed as the pushout $C^\infty(\mathbb{R}^\infty \setminus \{0\}) \otimes^\infty_{C^\infty(\mathbb{R})} A$ where the map $C^\infty(\mathbb{R}) \to A$ corresponds by the Yoneda bijection $\mathrm{Hom}_{\mathsf{ring}}(C^\infty(\mathbb{R}), A) \simeq A(\mathbb{R})$ to a. To see this, we write A as a filtered colimit of its finitely generated subalgebras that contain a, reducing to the case $A = C^\infty(\mathbb{R}^n)/I$. It is easy to see that the localization of A is the pushout of $C^\infty(\mathbb{R}^n)[\tilde{a}^{-1}] \otimes^\infty_{C^\infty(\mathbb{R}^n)} A$, where $\tilde{a}$ is some lift of a to $C^\infty(\mathbb{R}^n)$, so we reduce to the case of free C^∞-rings (for a more detailed version of this argument, see the proof of proposition 4.1.3.13). For this case, the localization of $\tilde{a} \in C^\infty(\mathbb{R}^n)$ coincides with $C^\infty(\mathbb{R}^{n+1})/(y \cdot \tilde{a} - 1)$ where y is the $(n+1)$'st coordinate, for algebraic reasons. Since the ideal $(y \cdot \tilde{a} - 1)$ is point determined, it is also the ideal of functions vanishing on the zero locus of $y \cdot \tilde{a} - 1$, which is diffeomorphic to $\tilde{a}^{-1}(\mathbb{R} \setminus \{0\})$, whose C^∞-ring of smooth functions is in turn given by $C^\infty(\mathbb{R}^n) \otimes^\infty_{C^\infty(\mathbb{R})} C^\infty(\mathbb{R} \setminus \{0\})$.

Remark 3.1.3.20. The analysis of the previous remark shows that in many cases, the C^∞-ring localization is very different from the $\mathbb{R}$-algebraic localization. Indeed, inverting the identity in $C^\infty(\mathbb{R})$ yields only those smooth functions $f(x)$ on $\mathbb{R} \setminus \{0\}$ that approach infinity at most polynomially fast as $x \to 0$.

Notation 3.1.3.21. We will denote $\mathcal{G}_{\mathrm{Diff}}$ for the opposite category of the category of finitely presented C^∞-rings. To notationally distinguish a finitely presented C^∞-ring A from A as an object of $\mathcal{G}_{\mathrm{Diff}}$, we use the notation $\mathrm{Spec}\, A$ in the latter case. We also say that an ideal J of a finitely presented object $C^\infty(\mathbb{R}^n)/I$ is *germ determined* if the pullback of J along the quotient map $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/I$ is germ determined (equivalently, the C^∞-ring $(C^\infty(\mathbb{R}^n)/I)/J$ is germ determined).

We endow $\mathcal{G}_{\mathrm{Diff}}$ with the structure of a geometry according to the following prescription:

(1) A map $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ in $\mathcal{G}_{\mathrm{Diff}}$ is admissible if and only if there exists an element $b \in B$ such that the image of b under f is invertible in A and the induced map $B[1/b] \to A$ is an equivalence.

(2) A collection $\{\mathrm{Spec}\, B[1/b_\alpha] \to \mathrm{Spec}\, B\}_{\alpha \in J}$ generates a covering sieve if and only if the germ determined ideal generated by the elements b_α in B contains the unit.

Remark 3.1.3.22. To see that this is a geometry, we only have to check that the admissible maps are stable under retracts and that, if g is admissible and h another map with codomain being the domain of g, then h is admissible if and only if $g \circ h$ is admissible. The stability under pullbacks follows at once from remark 3.1.3.19. For stability under retracts, consider a localization $f : A \to A[1/a]$ and a retraction diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[r, "h"] \arrow[d, "f"] & A' \arrow[d] \\
B \arrow[r] & A[1/a] \arrow[r] & B
\end{tikzcd}$$

Now B is the localization $A'[1/h(a)]$. To see this, note that for a map $A' \to C$ that inverts $h(a)$, we get a unique map $q : A[1/a] \to C$ as in the commuting diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[r, "h"] \arrow[d, "f"] & A' \arrow[d] \arrow[dr, bend left] & \\
B \arrow[r, "g"] & A[1/a] \arrow[r] \arrow[rr, bend right, "q"] & B & C
\end{tikzcd}$$

so we have map $B \to C$ as $q \circ g$. Note that this map is unique: suppose we have p and p' as in the commuting diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[r, "h"] \arrow[d, "f"] & A' \arrow[d] \arrow[dr, bend left] & \\
B \arrow[r, "g"] & A[1/a] \arrow[r, "k"] & B \arrow[r, shift left, "p"] \arrow[r, shift right, "p'"'] & C
\end{tikzcd}$$

then by uniqueness k equalizes p and p', and because the diagram is a retraction we have $p = p \circ k \circ g = p' \circ k \circ g = p'$. We have the claims about compositions of admissibles left to check. It is clear that the localization $A[1/a][1/b_a]$ is equivalent to $A[1/d_{ab}]$ for some $d_{ab} \in A$ as this is obvious for localizations of $C^\infty(\mathbb{R}^n)$, and all localizations of finitely presented objects are pushouts of these. It is also easy to verify that for a diagram $A \xrightarrow{f} A[1/a] \to A[1/b]$ where f and the composition are localizations, we have $A[1/b] \simeq A[1/a][1/f(b)]$.

Remark 3.1.3.23. The relation between the pregeometry $\mathcal{T}_{\mathrm{Diff}}$ and the geometry $\mathcal{G}_{\mathrm{Diff}}$ is as follows: by corollary 4.1.4.7, $\mathcal{G}_{\mathrm{Diff}}$ is a 0-truncated geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$.

Remark 3.1.3.24. We observe that by remark 3.1.3.19, every admissible map into $A = \operatorname{Spec} C^\infty(\mathbb{R}^n)/I \in \mathcal{G}_{\text{Diff}}$ is pulled back from an admissible map into $\operatorname{Spec} C^\infty(\mathbb{R}^n)$. Given an admissible covering $\{\operatorname{Spec} A[1/a_\alpha] \to \operatorname{Spec} A\}_{\alpha\in J}$, we may invoke the axiom of choice and find $\tilde{a}_\alpha \in C^\infty(\mathbb{R}^n)$ such that $A[1/a_\alpha]$ is a pushout of $C^\infty(\mathbb{R}^n)[\tilde{a}_\alpha]$ along $C^\infty(\mathbb{R}^n) \to A$. I is finitely generated, so $I = (g_1,\dots,g_n)$ for some smooth functions $g_1,\dots,g_n$; because the germ determined ideal generated by the collection $\{a_\alpha\}_{\alpha\in J}$ contains the unit in A, the germ determined ideal collection $\{a_\alpha\}_{\alpha\in J} \bigcup \{g_i\}_{1\le i\le n}$ contains the unit in $C^\infty(\mathbb{R}^n)$, so we have an admissible covering

$$\{\operatorname{Spec} C^\infty(\mathbb{R}^n)[1/a_\alpha] \to C^\infty(\mathbb{R}^n)\}_{\alpha\in J} \cup \{\operatorname{Spec} C^\infty(\mathbb{R}^n)[1/g_i] \to C^\infty(\mathbb{R}^n)\}_{1\le i\le n}.$$

By remark 3.1.3.19, each map in this covering corresponds to an open inclusion into $\mathbb{R}^n$. Clearly, if the germ determined ideal of a collection $\{f_\alpha\}$ of functions in $C^\infty(\mathbb{R}^n)$ generates the unit ideal, then for each maximal ideal I with residue field $\mathbb{R}$ in $C^\infty(\mathbb{R}^n)$, there is some f_α not contained in I, so the open collection $\{f_\alpha^{-1}(\mathbb{R}\smallsetminus\{0\})\}$ covers $\mathbb{R}^n$. Conversely, given an open cover $\{U_\alpha\}$ of $\mathbb{R}^n$, the germ determined ideal generated by the collection of characteristic functions $\{\chi_{U_\alpha}\}$ contains a partition of unity subordinate to a locally finite refinement of the cover $\{U_\alpha\}$, by point 3 of remark 3.1.3.10. This shows that the condition on a collection of admissibles to be a covering of $C^\infty(\mathbb{R}^n)$ corresponds precisely to the condition that the corresponding collection of open inclusions is a covering of $\mathbb{R}^n$ in the usual sense. As a result, the Grothendieck topology on $\mathcal{G}_{\text{Diff}}$ is generated by the open cover topology on CartSp, in the sense that every covering family in $\mathcal{G}_{\text{Diff}}$ is pulled back from a covering family in CartSp.

Let $\mathcal{O} : \mathcal{G}_{\text{Diff}} \to \mathcal{S}$ be a $\mathcal{G}_{\text{Diff}}$-structure in spaces, which can be identified with a C^∞-ring by the equivalences $\operatorname{Str}_{\mathcal{G}_{\text{Diff}}}(\mathcal{S}) \simeq \operatorname{Ind}(\mathcal{G}_{\text{Diff}}^{op}) \simeq C^\infty\mathsf{ring}$; the corresponding C^∞-ring $A_{\mathcal{O}}$ is up to unique isomorphism determined by $\operatorname{Hom}_{C^\infty\mathsf{ring}}(B, A_{\mathcal{O}}) = \mathcal{O}(B)$ for B a finitely presented C^∞-ring. We'd like to give a characterization of what it means to be local as a $\mathcal{G}_{\text{Diff}}$-structure on $\mathcal{S}$ in terms of the corresponding C^∞-ring. We need the following lemma, due to Bunge, Dubuc and Joyal [BD87].

Lemma 3.1.3.25. *Any open covering on $\mathbb{R}^n$ is generated under pullbacks, composition and refinement by coverings on $\mathbb{R}$.*

Proof. Fix an open covering $\{U_\alpha \to \mathbb{R}^n\}_{\alpha\in J}$. This covering is a composition of the coverings $W_I = \coprod_{i\in I} U_{\alpha_i}$ for the finite subsets $I \subset J$. To see that such finite coverings are pulled back from coverings on $\mathbb{R}$, we first reduce any finite covering to a covering $U_1 \coprod U_2 = W$ by induction. Choosing characteristic functions χ_{U_1} and χ_{U_2}, we may replace them by $\frac{\chi_{U_1}^2}{\chi_{U_1}^2+\chi_{U_1}^2}$ and $\frac{\chi_{U_2}^2}{\chi_{U_1}^2+\chi_{U_1}^2}$, so that $\chi_{U_1} + \chi_{U_2} = 1$. Now $U_1 = \chi_1(\mathbb{R}\smallsetminus\{0\})$ and $U_2 = \chi_1(\mathbb{R}\smallsetminus\{1\})$.
Now we show that the covering $\{W_I \to \mathbb{R}^n\}_{I\subset J, |I|\le\infty}$ is refined by a covering pulled back from $\mathbb{R}$. Choose some covering $\{Y_k \to \mathbb{R}\}$ by bounded open sets, and a proper smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ (for instance, the square length function $(x_1,\dots,x_n) \to x_1^2 + \ldots + x_n^2$). The opens $\varphi^{-1}(Y_k)$ cover $\mathbb{R}^n$ and are bounded, so each such open is covered by a finite collection $U_{\alpha_{j_1}},\dots,U_{\alpha_{j_n}}$ and thus $\varphi^{-1}(Y_k) \subset W_S$ for some finite index set S. Consequently, there is a refinement map $\coprod_k \varphi^{-1}(Y_k) \to \coprod_{I\subset J,|I|<\infty} W_I$. ☐

Proposition 3.1.3.26. *$\mathcal{O}$ is a local $\mathcal{G}_{\text{Diff}}$-structure on $\mathcal{S}$ if and only if $A_{\mathcal{O}}$ is local as a commutative ring and the residue field is isomorphic to $\mathbb{R}$.*

Proof. We should check that $A_{\mathcal{O}}$ is a local ring with residue field $\mathbb{R}$ if and only if for each finitely presented C^∞-ring B and each admissible covering $\{B \to B[1/b_\alpha]\}_{\alpha\in J}$, the map

$$\coprod_\alpha \operatorname{Hom}_{C^\infty\mathsf{ring}}(B[1/b_\alpha], A_{\mathcal{O}}) \longrightarrow \operatorname{Hom}_{C^\infty\mathsf{ring}}(B, A_{\mathcal{O}})$$

is an epimorphism. By remark 3.1.3.24, any admissible covering on B is pulled back from a covering on a free C^∞-ring, so, because epimorphisms are stable under pullbacks in Set, we note that it is enough to prove the claim for the collection of free C^∞-rings. By the existence of characteristic functions and remark 3.1.3.19, an admissible covering of $\operatorname{Spec} C^\infty(\mathbb{R}^n)$ is the same thing as an open covering of $\mathbb{R}^n$. Thus, we should check that $A_{\mathcal{O}}$ is local with residue field $\mathbb{R}$ if only if for each open cover $\{U_\alpha \to \mathbb{R}^n\}_{\alpha\in J}$, the map $\coprod_\alpha \mathcal{O}(U_\alpha) \to \mathcal{O}(\mathbb{R}^n)$ is an epimorphism. Because epimorphisms are stable under pullback, composition and refinement in Set, we reduce further to having to check the statement only for covering families of $\mathbb{R}$. In one direction, consider the open covering $\mathbb{R}\smallsetminus\{0\} \cap \mathbb{R}\smallsetminus\{1\} \to \mathbb{R}$, and note that we have transverse pullback diagrams

$$\begin{tikzcd} \mathbb{R}\smallsetminus\{0\} \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[d, "(a,b)\mapsto ab"] \\ * \arrow[r, "1"] & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}\smallsetminus\{1\} \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[d, "(a,b)\mapsto(a-1)b"] \\ * \arrow[r, "1"] & \mathbb{R} \end{tikzcd}$$

showing that $\mathcal{O}(\mathbb{R}\smallsetminus\{0\}) = A_{\mathcal{O}}^2 \times_{A_{\mathcal{O}}} \{1\}$ is the set $A_{\mathcal{O}}^*$ of invertible elements of $A_{\mathcal{O}}$, and similarly $\mathcal{O}(\mathbb{R}\smallsetminus\{1\})$ is the set $1 - A_{\mathcal{O}}^*$ of elements $a \in A_{\mathcal{O}}$ such that $1-a$ is invertible. Clearly, $A_{\mathcal{O}}^* \coprod (1 - A_{\mathcal{O}}^*) \to A$ is an epimorphism if and only if $A_{\mathcal{O}}$ is local as a commutative ring. Now we show that there is a map $p : A_{\mathcal{O}} \to \mathbb{R}$ of C^∞-rings which is nonzero if $A_{\mathcal{O}}$ is nonzero: $\mathcal{O}$ gives a functor

$$\mathrm{Open}(\mathbb{R}) \longrightarrow \mathrm{Sub}(A_{\mathcal{O}}(\mathbb{R})), \quad U \mapsto \mathcal{O}(U),$$

(note that since $\mathcal{O}$ preserves all pullbacks in $\mathrm{Open}(\mathbb{R})$, $\mathcal{O}(U)$ is a subobject of $A_{\mathcal{O}}(\mathbb{R})$), which is a map of locales as it is left exact and sends coverings to epimorphism. Since the underlying topological spaces of these locales are sober, we get a map $p : A_{\mathcal{O}}(\mathbb{R}) \to \mathbb{R}$ of sets. To see that this map is in fact a morphism of C^∞-rings, it suffices to show that for a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, the diagram of locales

$$\begin{tikzcd}
\mathrm{Open}(\mathbb{R}^m) \arrow[r, "\mathcal{O}"] \arrow[d, "f^{-1}"] & \mathrm{Sub}(A_{\mathcal{O}}(\mathbb{R})^m) \arrow[d, "f^*"] \\
\mathrm{Open}(\mathbb{R}^n) \arrow[r, "\mathcal{O}"] & \mathrm{Sub}(A_{\mathcal{O}}(\mathbb{R})^n)
\end{tikzcd}$$

commutes, where the right vertical map f^* sends a subobject X of $A_{\mathcal{O}}(\mathbb{R})^m$ to the pullback $X \times_{A_{\mathcal{O}}(\mathbb{R})^m} A_{\mathcal{O}}(\mathbb{R})^n$ along the map $f_* : A_{\mathcal{O}}(\mathbb{R})^n \to A_{\mathcal{O}}(\mathbb{R})^m$. Concretely, for any smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ and any open in U, we ask that $\mathcal{O}(f^{-1}(U)) \simeq \mathcal{O}(U) \times_{\mathcal{O}(\mathbb{R}^m)} \mathcal{O}(\mathbb{R}^n)$. This is clearly the case since $\mathcal{O}$ preserves pullbacks along open inclusions. The kernel of the map $p : A_{\mathcal{O}} \to \mathbb{R}$ just constructed is a maximal ideal, so by locality of $A_{\mathcal{O}}$, p must be the projection onto the residue field.
For the converse, suppose that $A_{\mathcal{O}}$ is local with residue field $\mathbb{R}$. We want to show that for any open covering $\coprod_\alpha U_\alpha \to \mathbb{R}$, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \to A_{\mathcal{O}}(\mathbb{R})$ is an epimorphism. Points in $A_{\mathcal{O}}$ corresponds by Yoneda to maps $q : C^\infty(\mathbb{R}) \to A_{\mathcal{O}}$, so it suffices to show that each such map factors as $C^\infty(\mathbb{R}) \to C^\infty(U_\alpha) \to A_{\mathcal{O}}$ for some index α. The Yoneda embedding $\mathsf{CartSp}^{op} \to C^\infty\mathsf{ring}$ is fully faithful, so the composition

$$C^\infty(\mathbb{R}) \longrightarrow A_{\mathcal{O}} \longrightarrow \mathbb{R}$$

is given by evaluation ev_x at some $x \in U_\alpha \subset \mathbb{R}$ for some index α; let χ_{U_α} be a characteristic function for U_α, then $\mathrm{ev}_x(\chi_{U_\alpha}) \neq 0$, implying that the image of χ_{U_α} under q is not in the maximal ideal $\ker(q)$ of $A_{\mathcal{O}}$, so $q(\chi_{U_\alpha})$ is invertible in $A_{\mathcal{O}}$ by locality. Now q factors through the localization $C^\infty(\mathbb{R}) \to C^\infty(U_\alpha)$ of χ_{U_α} so we are done. ☐

Proposition 3.1.3.27. *Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be morphism of local $\mathcal{G}_{\mathrm{Diff}}$-structure on $\mathcal{S}$. Then α is a local morphism if and only if the corresponding morphism $f_\alpha : A_{\mathcal{O}} \to A_{\mathcal{O}'}$ is local as a map of commutative rings.*

Proof. The map f_α is local as a map of commutative rings if and only if f_α reflects invertibility, which is true if and only if $A^*_{\mathcal{O}} \simeq A_{\mathcal{O}} \times_{A^*_{\mathcal{O}'}} A^*_{\mathcal{O}'}$. If α is local, this obviously holds. In the other direction, we want to show that for each localization $B \to B[1/b]$ of finitely presented C^∞-rings, the naturality square induced by α gives an equivalence $\mathcal{O}(B[1/b]) \simeq \mathcal{O}(B) \times_{\mathcal{O}'(B)} \mathcal{O}'(B[1/b])$. Because B is finitely presented, we have a pushout $B[1/b] \simeq C^\infty(\mathbb{R} \setminus \{0\}) \otimes^\infty_{C^\infty(\mathbb{R})} B$ so we get a commuting cube

$$\begin{tikzcd}
& A^*_{\mathcal{O}} \arrow[rr] \arrow[dd] & & A^*_{\mathcal{O}'} \arrow[dd] \\
\mathcal{O}(B[1/b]) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{O}'(B[1/b]) \arrow[ur] \arrow[dd] & \\
& A_{\mathcal{O}} \arrow[rr] & & A_{\mathcal{O}'} \\
\mathcal{O}(B) \arrow[ur] \arrow[rr] & & \mathcal{O}'(B) \arrow[ur] &
\end{tikzcd}$$

Because the side faces are pullbacks and the back face is a pullback by assumption, the front face is a pullback as well. ☐

Corollary 3.1.3.28. *Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a morphism of local $\mathcal{G}_{\mathrm{Diff}}$-structures. Then α is a local morphism.*

Proof. As the residue field of both $A_{\mathcal{O}}$ and $A_{\mathcal{O}'}$ is $\mathbb{R}$, any morphism of rings between them is local. ☐

In summary, the correct notion of a local C^∞-ring (with respect to the geometry $\mathcal{G}_{\mathrm{Diff}}$), is that of a *local Archimedean* C^∞-ring, i.e. a C^∞-ring A such that the underlying commutative $\mathbb{R}$-algebra of A is a local ring, and the residue field of A is $\mathbb{R}$. Whenever we talk about local C^∞-rings in the sequel, we mean this notion.

Remark 3.1.3.29. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}_{\text{Diff}}$-structured ∞-topos that has enough points. Then $\mathcal{O}_{\mathcal{X}}$ is local if and only if for each stalk $p^* : \mathcal{X} \to \mathcal{S}$ the induced C^∞-ring is local, and every morphism $\alpha : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$ of local $\mathcal{G}_{\text{Diff}}$-structures is local.

Remark 3.1.3.30. Local C^∞-rings are strictly Henselian (in fact, they are separably real closed). The admissibility structure on $\mathcal{G}_{\text{Diff}}$ is the C^∞-analog of the *Zariski* admissibility structure; however, in the *étale* geometry of example 3.1.0.16 describing algebraic geometry over a commutative ring k, the local objects are precisely the strictly Henselian local rings, so the fact that local C^∞-rings are strictly Henselian local rings is explained by the fact that the Zariski and étale topologies coincide on $C^\infty\mathsf{ring}$.

Recall that a 0-localic ∞-topos $\mathcal{X}$ such that the underlying locale has enough points arises as the ∞-category of sheaves on a sober topological space; in fact, taking ∞-categories of sheaves yields an equivalence of ∞-categories between $\mathbf{N}(\mathsf{Top}_{\text{sob}})$, the ∞-category of sober topological spaces and continuous maps, and ${}^{\text{R}}\mathsf{Top}'$, the full subcategory of ${}^{\text{R}}\mathsf{Top}$ spanned by 0-localic ∞-topoi for which the underlying locales have enough points[2]. Taking these facts together with propositions 3.1.3.26, 3.1.3.27 and remark 3.1.3.29, we have the following proposition (see [Lur11b], proposition 2.5.15 for the algebro-geometric situation[3])

Proposition 3.1.3.31. *Let* $\mathsf{RingSpace}_{C^\infty}$ *be the category of sober topological spaces equipped with sheaves of* C^∞*-rings, and let* ${}^{\text{R}}\mathsf{Top}'(C^\infty)$ *be the* ∞*-category of pairs* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *where* $\mathcal{X}$ *is a* 0*-localic* ∞*-topos for which the underlying locale has enough points and* $\mathcal{O}_{\mathcal{X}}$ *is a (possibly non-local)* $\mathcal{G}_{\text{Diff}}$*-structure. There is a canonical equivalence of* ∞*-categories*

$$\zeta : \mathbf{N}(\mathsf{RingSpace}_{C^\infty}) \xrightarrow{\simeq} {}^{\text{R}}\mathsf{Top}'(C^\infty).$$

Moreover, if we let $\mathsf{RingSpace}^{loc}_{C^\infty}$ *be the category of sober topological spaces equipped with sheaves of* local C^∞*-rings, then* ζ *restricts to an equivalence*

$$\zeta : \mathbf{N}(\mathsf{RingSpace}^{loc}_{C^\infty}) \xrightarrow{\simeq} {}^{\text{R}}\mathsf{Top}'(\mathcal{G}_{\text{Diff}}),$$

where ${}^{\text{R}}\mathsf{Top}'(\mathcal{G}_{\text{Diff}})$ *is the* ∞*-category of 0-localic* $\mathcal{G}_{\text{Diff}}$*-structured* ∞*-topoi for which the underlying locales have enough points.*

Now we'd like to describe the $\mathcal{G}_{\text{Diff}}$-spectrum in terms of a more classical differential geometry construction.

Definition 3.1.3.32. Let A be a C^∞-ring. The *real spectrum* $\text{Spec}_{\mathbb{R}} A$ of A is the topological space constructed as follows. For the underlying set, we take $\text{Hom}_{C^\infty\mathsf{ring}}(A, \mathbb{R})$. The topology is generated by the basis open sets

$$\{U_a\}_{a \in A}, \quad U_a := \text{ev}_a^{-1}(\mathbb{R} \smallsetminus \{0\})$$

where

$$\text{ev}_a : \text{Hom}_{C^\infty\mathsf{ring}}(A, \mathbb{R}) \to \mathbb{R}, \quad \text{ev}_a(f) = f(a).$$

The real spectrum of A has a canonical sheaf $\mathcal{O}_{\text{Spec}_{\mathbb{R}} A}$ of C^∞-rings whose stalks are local C^∞-rings, given by the sheafification of the presheaf sending U_a to $A[a^{-1}]$. We will usually abuse notation and write $\text{Spec}_{\mathbb{R}} A$ for the local C^∞-ringed space $(\text{Spec}_{\mathbb{R}} A, \mathcal{O}_{\text{Spec}_{\mathbb{R}} A})$.
A local C^∞-ringed space $(X, \mathcal{O}_X)$ is a C^∞*-scheme* if there is a covering $\{U_i \to X\}$ such that $(U, \mathcal{O}_X|_{U_i})$ is equivalent to the real spectrum of some C^∞-ring. We denote the full subcategory spanned by C^∞-schemes by Sch_{C^∞}.

Remark 3.1.3.33. Affine C^∞-schemes are regular topological spaces. If A is finitely generated, say $A \cong C^\infty(\mathbb{R}^n)/J$, then $\text{Spec}_{\mathbb{R}} A \cong Z(J) \overset{\iota}{\hookrightarrow} \mathbb{R}^n$, topologized as a subspace. It's straightforward to check that the sheaf $\mathcal{O}_{\text{Spec}_{\mathbb{R}} A}$ is obtained as $\iota^*(\mathcal{O}_{\mathbb{R}^n}/\mathcal{J})$ with $\mathcal{J}$ the sheaf of ideals obtained by sheafifying the presheaf $U_a \mapsto J \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^n)[a^{-1}]$. In particular, when J is finitely generated, there is a closed immersion $\text{Spec}_{\mathbb{R}} A \to \mathbb{R}^n$. A closed subspace of a space of covering dimension $\leq n$ also has covering dimension $\leq n$, so because $\text{Spec}_{\mathbb{R}} A$ is paracompact, the ∞-topos $\mathsf{Shv}(\text{Spec}_{\mathbb{R}} A)$ is locally of homotopy dimension $\leq n$. It follows that $\mathsf{Shv}(\text{Spec}_{\mathbb{R}} A)$ is hypercomplete and has enough points. Moreover, Postnikov towers converge in $\mathsf{Shv}(\text{Spec}_{\mathbb{R}} A)$.

Proposition 3.1.3.34. *The equivalence* ζ *of proposition 3.1.3.31 restricts to a fully faithful functor*

$$\mathbf{N}(\text{Sch}_{C^\infty}) \to \text{Sch}(\mathcal{G}_{\text{Diff}}).$$

The essential image of this functor consists of those $\mathcal{G}_{\text{Diff}}$*-schemes* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *such that* $\mathcal{X}$ *is* 0*-localic.*

[2] Beware: not every 0-localic ∞-topos $\mathcal{X}$ for which the underlying locale has enough points has itself enough points as an ∞-topos. Indeed, this would imply that the ∞-category of sheaves on a sober topological space is always hypercomplete. However, there are coherent (thus sober) topological spaces whose ∞-categories of (space-valued) sheaves are *not* hypercomplete (see [Lur17b], counterexamples 6.5.4.2, 6.5.4.5)

[3] But note that in that proposition, '0-localic ∞-topos with enough points' is written where '0-localic ∞-topos such that the underlying locale has enough points' is meant

Lemma 3.1.3.35. *Let A be a C^∞-ring. The category $(C^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/A}$ is equivalent to the poset of subsets of* $\mathrm{Hom}_{C^\infty\mathsf{ring}}(A,\mathbb{R})$ *of the form* $\mathrm{ev}_a^{-1}(\mathbb{R}\smallsetminus\{0\})$*, where* $\mathrm{ev}_a : \mathrm{Hom}_{C^\infty\mathsf{ring}}(A,\mathbb{R}) \to \mathbb{R}$ *is the evaluation map for some element $a \in A$.*

Proof. It is obvious from the description of localizations of C^∞-rings as pushouts along $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}\smallsetminus\{0\})$ that a map $f : A \to B$ in $C^\infty\mathsf{ring}$ is admissible as a morphism of $\mathrm{Pro}(\mathcal{G}_{\mathrm{Diff}})^{op}$ if and only if f exhibits B as a localization of A. Now the assignment $(A \to A[a^{-1}]) \mapsto \mathrm{ev}_a^{-1}(\mathbb{R}\smallsetminus\{0\})$ yields the desired equivalence. □

Proof of proposition 3.1.3.34. Fully faithfulness of ζ is contained in proposition 3.1.3.31. We check that ζ sends C^∞-schemes to $\mathcal{G}_{\mathrm{Diff}}$-schemes. It suffices to check this for affine objects. For A a C^∞-ring, the ∞-topos $\mathsf{Shv}(\mathrm{Pro}(\mathcal{G}_{\mathrm{Diff}})^{\mathrm{ad}}_{/A})$ is the ∞-topos $\mathsf{Shv}(\mathcal{B})$, where $\mathcal{B}$ is a lattice of basis open sets of $\mathrm{Spec}_{\mathbb{R}}A$, so restriction induces an equivalence $\mathsf{Shv}(\mathrm{Pro}(\mathcal{G}_{\mathrm{Diff}})^{\mathrm{ad}}_{/A}) \simeq \mathsf{Shv}(\mathrm{Spec}_{\mathbb{R}}A)$. We are left to show that the structure sheaves coincide as well. This follows because in both cases, the structure sheaf is the sheafification of the presheaf defined by

$$\mathcal{B}^{op} \to C^\infty\mathsf{ring}, \quad (A \to A[a^{-1}]) \mapsto A[a^{-1}].$$

If $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is a 0-localic $\mathcal{G}_{\mathrm{Diff}}$-scheme, $\mathcal{X}$ is locally the ∞-category of sheaves on a Hausdorff space, so by [Lur11b], lemma 2.5.21, $\mathcal{X} \simeq \mathsf{Shv}(X)$ for some topological space X. Since $(\mathsf{Shv}(X),\mathcal{O}_{\mathcal{X}})$ is locally an affine $\mathcal{G}_{\mathrm{Diff}}$-scheme, it is also locally a C^∞-scheme, as we have just identified affine C^∞-schemes with affine $\mathcal{G}_{\mathrm{Diff}}$-schemes under ζ. □

Remark 3.1.3.36. Every admissible map $U \to X$ in $\mathrm{Pro}(\mathcal{G}_{\mathrm{Diff}}) = C^\infty\mathsf{ring}^{op}$ corresponds to an open embedding of affine C^∞-schemes, but the converse is *not* true. A counterexample is example 4.31 of [Joy12a]: for I an infinite set, consider $C^\infty(\mathbb{R}^I) = \mathrm{colim}_{S\subset I, |S|<\infty} C^\infty(\mathbb{R}^{|S|})$, the free C^∞-ring generated by I. The inclusion $\mathbb{R}^I \smallsetminus\{0\} \to \mathbb{R}^I$ is an open embedding of affine C^∞-schemes, but the corresponding map $C^\infty(\mathbb{R}^I) \to C^\infty(\mathbb{R}^I\smallsetminus\{0\})$ is not a localization, since every element of $C^\infty(\mathbb{R}^I)$ is a function on only *finitely* many variables, and for a function a to exhibit V as the locus where it is nonzero would require a to depend on infinitely many variables. If X is finitely generated, then every open embedding does arise as an admissible map: this follows from the existence of characteristic functions for finitely generated C^∞-rings.

Remark 3.1.3.37. If A is finitely generated, then $\mathrm{Spec}_{\mathbb{R}} A$ admits a closed immersion into $\mathbb{R}^n$ and is therefore metrizable, so it follows that all open sets of $\mathrm{Spec}_{\mathbb{R}} A$ are F_σ-subsets. As $\mathrm{Spec}_{\mathbb{R}} A$ has finite covering dimension and the sheaf $\mathcal{O}_{\mathrm{Spec}\,A}$ is fine, proposition 2.2.5.37 implies that each sheaf of $\mathcal{O}^{\mathrm{alg}}_{\mathrm{Spec}\,A}$-modules $\mathcal{F}$ has the property that the openwise presheaf of n'th homotopy groups $\widetilde{\pi_n(\mathcal{F})}$ is already a sheaf.

The following is the content of [Joy12a], chapter 5.

Definition-Proposition 3.1.3.38. Let A be a C^∞-ring, and consider the category $\mathrm{Mod}_{A^{\mathrm{alg}}}$ of modules over the underlying commutative $\mathbb{R}$-algebra. Consider also the category $\mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}_{\mathrm{Spec}_{\mathbb{R}} A}}$ of sheaves of $\mathcal{O}^{\mathrm{alg}}_{\mathrm{Spec}_{\mathbb{R}} A}$-modules on $\mathrm{Spec}_{\mathbb{R}} A$. There is a module spectrum functor $M\mathbf{Spec}_A : \mathrm{Mod}_{A^{\mathrm{alg}}} \to \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}_{\mathrm{Spec}_{\mathbb{R}} A}}$ which sends a module M to the sheafification of the presheaf defined by

$$U_a \mapsto M \otimes_A A[a^{-1}].$$

This spectrum functor is left adjoint to the obvious global sections functor.

Remark 3.1.3.39. Just as the real spectrum of a C^∞-ring is obtained as the $\mathcal{G}_{\mathrm{Diff}}$-spectrum, so does the adjunction mentioned in the previous proposition come from construction 3.1.1.1 for a certain geometry. This geometry has as underlying ∞-category the opposite of the 1-category Perf of pairs (A,M) with A a C^∞-ring of finite presentation and M a perfect A^{alg}-module. The admissibility structure is given as follows: consider the Cartesian fibration $q : \mathrm{Perf}^{op} \to C^\infty\mathsf{ring}^{op}_{\mathrm{fp}} \simeq \mathcal{G}_{\mathrm{Diff}}$, then a morphism $f \in \mathrm{Perf}^{op}$ is admissible if and only if it is q-Cartesian and $q(f)$ is admissible in $\mathcal{G}_{\mathrm{Diff}}$, and a collection $\{f_\alpha : (\mathrm{Spec}\,A_\alpha, M_\alpha) \to (\mathrm{Spec}\,B, N)\}$ of admissibles generates a covering sieve if and only if $\{f_\alpha : \mathrm{Spec}\,A_\alpha \to \mathrm{Spec}\,B\}$ generates a covering sieve in $\mathcal{G}_{\mathrm{Diff}}$. We will come back to this point of view when we deal with modules of simplicial C^∞-rings.

Here are several properties of the spectrum-global sections adjunction for C^∞-schemes and modules that we will have need of in the sequel.

Proposition 3.1.3.40. *(1) If A is of the form $C^\infty(\mathbb{R}^n)/I$ with I with I germ determined, then the global sections of the sheafification of the presheaf*

$$U_a \mapsto A[a^{-1}]$$

coincides with A. Consequently, the counit of the adjunction $\Gamma \circ \mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}} A \to A$ is an equivalence.

(2) If A is of the form $C^\infty(\mathbb{R}^n)/I$ with I germ determined and M is a finitely presented A^{alg}-module (in the 1-category $\mathrm{Mod}_{A^{\mathrm{alg}}}$*), then the presheaf*

$$U_a \mapsto M \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^n)[a^{-1}]$$

is already a sheaf.

(3) *The unit* $\mathrm{id} \to \mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}} \circ \Gamma$ *of the adjunction is an equivalence on the essential image of* $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}}$.

(4) *For* A *a* C^∞*-ring, the counit* $M\mathbf{Spec}_A \circ \Gamma \to \mathrm{id}$ *of the adjunction, a natural transformation between endofunctors on* $\mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}_{\mathrm{Spec}_{\mathbb{R}} A}}$*, is an equivalence.*

Proof. (1) is originally due to Dubuc, and can be found in [Joy12a] as theorem 4.22. It is instructive to take a C^∞-ring $C^\infty(\mathbb{R}^n)/J$, carry out the sheafification of the ideal sheaf explicitly and observe that by taking global sections of the étalé space projection, one obtains precisely the smallest germ determined ideal containing J. The same argument applies to (2); it can be found in [Joy12a], example 5.28. (3) and (4) are propositions 4.34 and 5.20 respectively in [Joy12a]. □

Remark 3.1.3.41. In algebraic geometry, the Zariski spectrum furnishes a fully faithful embedding of the category of k-algebras into the category of k-locally ringed spaces. Since the global sections functor is a left adjoint, the category of affine k-schemes is a reflective subcategory of $\mathsf{RingSpace}_k^{loc}$, with localization functor $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}} \circ \Gamma$. For the geometry of C^∞-rings, this is *not* true; instead, the functor $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Diff}}} \circ \Gamma$ is an autoequivalence on the essential image of $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Diff}}}$. Using this fact, it is easy to see that the functor $L_{cplt} : \Gamma \circ \mathrm{Spec}^{\mathcal{G}_{\mathrm{Diff}}}$ is a localization functor on $C^\infty\mathsf{ring}$ with fully faithful right adjoint, so the situation is somewhat reversed with respect to algebraic geometry. The reflective full subcategory $L_{cplt}(C^\infty\mathsf{ring})$ contains the objects that are usually called *complete* C^∞*-rings*. Proposition 3.1.3.40 shows that finitely generated and germ determined C^∞-rings are complete. We will call such C^∞-rings *fair*, following Joyce [Joy12a].
Similarly, (4) of proposition 3.1.3.40 shows that the functor $R^M_{comp} : \Gamma \circ M\mathbf{Spec}_A$ is a reflective localization, and the objects of $R^M_{comp}(\mathrm{Mod}_{A^{\mathrm{alg}}})$ are called *complete modules*.

For later use, we record the following useful fact about complete modules.

Proposition 3.1.3.42. *Let* $f : A \to B$ *be a surjective map of finitely generated* C^∞*-rings, and let* M *be a* B*-module. If* M *is complete as an* A*-module (via* f*), then* M *is complete as a* B*-module.*

Proof. Considering M as an A-module, $M\mathbf{Spec}_A M$ is the sheaf $\mathcal{F}_M$ associated to the presheaf

$$U_a \mapsto M \otimes_A A[a^{-1}] \cong M \otimes_B B[f(a)^{-1}].$$

Meanwhile, $M\mathrm{Spec}_B M$ is the sheaf $\mathcal{F}'_M$ associated to the presheaf

$$U_b \mapsto M \otimes_B B[b^{-1}].$$

Using that f is surjective, it follows easily that for each $\mathbb{R}$-point of $\mathrm{Spec}\, B$, that is, for each $\phi : B \to \mathbb{R}$, the map of filtered posets

$$\{a \in A;\, \phi(f(a)) \neq 0\} \longrightarrow \{b \in A;\, \phi(b) \neq 0\}$$

is left cofinal. Using this fact, it follows by checking on stalks that $\mathcal{F}_M$ is simply the direct image sheaf of $\mathcal{F}'_M$ along the map $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}} f : \mathrm{Spec}_{\mathbb{R}} B \to \mathrm{Spec}_{\mathbb{R}} A$, so the global sections of $\mathcal{F}_M$ and $\mathcal{F}'_M$ coincide. Thus, if M is complete as an A-module, then M is complete as a B-module. □

Warning 3.1.3.43. We say that a property P on modules of C^∞-rings is *local* if the following holds for every pair (A, M) where A is a C^∞-ring and M is an A-module.

(1) If $A \to B$ is admissible, then $M \otimes_A B$ has the property P.

(2) Suppose there exists an admissible covering $\{A \to A[1/a_i]\}_{i \in I}$ such that for each $i \in I$, the module $M \otimes_A A[1/a_i] \in \mathrm{Mod}_{A[1/a_i]}$ has the property P, then M has the property P.

In algebraic geometry, many natural properties, such as being a finitely presented, finitely generated, or being a finite rank vector bundle, are local (for the Zarsiki/étale/fppf topology). In contrast, these same properties are *not* local in C^∞-geometry. For instance, let $U \coloneqq \coprod_{i=0}^\infty B_i \subset \mathbb{R}^n$ be a countable disjoint union of open balls in $B_i \subset \mathbb{R}^n$, then it is easy to construct $C^\infty(U)$-modules that are locally finitely generated, but not globally, such as a vector bundle whose rank is i on each open ball B_i. Similarly, it easy to construct fair C^∞-rings which are locally finitely presented, but not globally.

3.2 Geometric Contexts: the Language of Higher Stacks

Until now, we have investigated categories with admissibility structures, a notion that allows one to investigate the behaviour and existence of certain limits compatible with a given Grothendieck topology. The theory of pregeometries and geometric envelopes is designed to handle adding finite limits to a category in a controlled way, taking objects which function as affine spaces for some notion of geometry, and producing 'derived' affine spaces. Treating these notions of affine spaces on the same footing, we may then ask how to add finite colimits in the form of higher groupoid quotients. This is achieved by Simpson's device of *higher geometric stacks* [Sim96], which generalizes algebraic Artin and Deligne-Mumford stacks, as well as orbifolds and (higher) Lie groupoids in differential geometry to ∞-sites that come equipped with a collection of maps that are well-behaved in the sense that they enjoy the same formal properties as smooth maps in algebraic geometry.
Our approach is similar to that of Toën-Vezzosi [TV06], except for a crucial difference: for the applications we have in mind, we may *not* assume that our Grothendieck topology is *quasi-compact*, that is, for any covering family $\{U_i \to X\}_{i\in I}$, there exists a finite $I_0 \subset I$ such that $\{U_i \to X\}_{i\in I_0}$ is still a covering family. This complicates our theory of higher stacks slightly, since it is technically inconvenient to introduce finiteness or countability restrictions already in the definition of higher geometric stacks (these conditions will generally be satisfied only by virtue of specific features of the moduli spaces under consideration, e.g. Gromov compactness [Gro85]). Since we cannot assume that coverings are finite, and the ∞-category of affine derived manifolds does not admit arbitrary small coproducts, we are forced to consider a rather larger class of objects that we should consider as *(-1)-geometric*, i.e. the class of objects at which the inductive definition of higher geometric stacks begins. Since we will have multiple geometries and various subcategories of structured spaces around, we wish to encompass a class of examples as large as possible, so we start only with the following data: a pair $(\mathcal{G},\mathcal{L})$ of a geometry and a full subcategory $\mathcal{L} \subset {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ consisting of objects whose underlying ∞-topos is n-localic, such that a saturation condition with respect to open inclusions is satisfied.

3.2.1 Localic scheme theories

The notion of a *geometry equipped with a scheme theory* we develop below is based on sections 2.3 and 2.4 of [Lur11b], and extends the theory developed there for geometries whose spectrum functors take values in locales; most of the proofs here adapted from this reference and the detailed work Carchedi [Car16]. A less terse treatment (with less unimaginative terminology) of some of the material that follows can be found in this latter reference.

Definition 3.2.1.1. Let $\mathcal{G}$ be a geometry. We will also say that a morphism $f:(\mathcal{X},\mathcal{O}_{\mathcal{X}})\to(\mathcal{X},\mathcal{O}_{\mathcal{Y}})$ of $\mathcal{G}$-structured ∞-topoi is $(n-1)$-*étale* if there exists an $(n-1)$-truncated object $V\in(\mathcal{X},\mathcal{O}_{\mathcal{Y}})$ such that f is equivalent to the canonical morphism $(\mathcal{Y}_{/V},\mathcal{O}_{\mathcal{Y}}|_V)\to(\mathcal{X},\mathcal{O}_{\mathcal{Y}})$.

The following definition may appear somewhat baroque, but turns out to be very versatile and useful.

Definition 3.2.1.2. Let $\mathcal{G}$ be a geometry. We say that a full subcategory $\mathcal{L}\subset{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ is an *n-localic $\mathcal{G}$-scheme theory* if the following conditions are satisfied.

$\mathcal{L}1$. For every $\mathcal{G}$-structured ∞-topos $(\mathcal{X},\mathcal{O}_{\mathcal{X}})\in\mathcal{L}$, the underlying ∞-topos is n-localic.

$\mathcal{L}2$. Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}})\in\mathcal{L}$ let $U\in\mathcal{X}$ be an $(n-1)$-truncated object, and consider the inclusion

$$\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\hookrightarrow {}^{\mathrm{R}}\mathsf{Top}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}.$$

Then we can identify $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ with the full subcategory of $\tau_{\le(n-1)}\mathcal{X}$ spanned by those $(n-1)$-truncated objects U such that the object $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)\in\mathcal{L}$ (note that $\mathcal{X}_{/U}$ is n-localic by proposition 2.2.3.2). We demand that $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ contains a full subcategory $\mathcal{C}_{\mathcal{X}}$ such that the following conditions are satisfied.

(1) $\mathcal{C}_{\mathcal{X}}$ is essentially small.

(2) $\mathcal{C}_{\mathcal{X}}$ admits finite limits and the fully faithful functor $\mathcal{C}_{\mathcal{X}}\hookrightarrow\tau_{\le(n-1)}\mathcal{X}$ is left exact.

(3) There exists a regular cardinal κ such that the essential image $\mathcal{C}_{\mathcal{X}}\hookrightarrow\mathcal{X}$ consists of κ-compact objects and generates $\mathcal{X}$ under κ-filtered colimits.

$\mathcal{L}3$. $\mathcal{L}$ is stable under pullbacks by $(n-1)$-étale morphisms. That is, given a pullback diagram

$$\begin{tikzcd}
(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}) \arrow[r] \arrow[d] & (\mathcal{Y},\mathcal{O}_{\mathcal{Y}}) \arrow[d, "f"] \\
(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U) \arrow[r] & (\mathcal{X},\mathcal{O}_{\mathcal{X}})
\end{tikzcd}$$

among $\mathcal{G}$-structured ∞-topoi, where $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$, $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{X}, \mathcal{O}_{\mathcal{Y}})$ lie in $\mathcal{L}$ and U is $(n-1)$-truncated, the $\mathcal{G}$-structured space $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, which may be identified with $(\mathcal{Y}_{/f^*(U)}, \mathcal{O}_{\mathcal{Y}}|_{f^*(U)})$ where f^* is the algebraic morphism underlying f, lies in $\mathcal{L}$.

$\mathcal{L}4$. $\mathcal{L}$ is locally small.

For $n = \infty$, condition (1) is vacuous and we say that the subcategory $\mathcal{L} \subset {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ that merely satisfies (2), (3) and (4) is a *$\mathcal{G}$-scheme theory.*
We say that an n-localic $\mathcal{G}$-scheme theory is *saturated* if instead of $\mathcal{L}2$ and $\mathcal{L}3$, the following condition is satisfied:

$\mathcal{L}2'$ For each $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$ and each $(n-1)$-truncated object $U \in \mathcal{X}$, the object $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ lies in $\mathcal{L}$. In other words, the inclusion $\mathcal{L} \subset {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ induces an isomorphism

$$\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \cong {}^{\mathrm{R}}\mathsf{Top}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \hookrightarrow {}^{\mathrm{R}}\mathsf{Top}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}.$$

This clearly implies $\mathcal{L}3$. $\mathcal{L}2'$ also implies $\mathcal{L}2$: we have an equivalence $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \simeq \tau_{\le(n-1)}\mathcal{X}$, so when $\mathcal{X}$ is n-localic, we may choose, according to (the proof of) [Lur17b], prop. 6.4.3.6, an essentially small ∞-category $\mathcal{C}_X \subset \tau_{\le(n-1)}$ that generates $\mathcal{X}$ under κ-filtered colimits for some sufficiently large regular cardinal κ. If $n = \infty$, the same conclusion holds by the accessibility of $\mathcal{X}$ and [Lur17b], prop. 5.4.7.4.

Notation 3.2.1.3. In what follows, we take $n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$, and an ∞-localic ∞-topos is simply an ∞-topos.

Remark 3.2.1.4. Let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory for some geometry $\mathcal{G}$, then there is a smallest n-localic $\mathcal{G}$-scheme theory that contains $\mathcal{L}$, the *saturation* of $\mathcal{L}$, denoted $\overline{\mathcal{L}}$. It contains those $(\mathcal{X}, \mathcal{O}_{\mathcal{Y}})$ of the form $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ for U any $(n-1)$-truncated object in $\mathcal{X}$.

Remark 3.2.1.5. The basic datum for constructing a scheme theory is a geometry. Evidently we may also start from a pregeometry $\mathcal{T}$, but this does not constitute a generalization since we may always choose a geometric envelope $\mathcal{T} \hookrightarrow \mathcal{G}$ resulting in the same structured spaces.

Definition 3.2.1.6. If $\mathcal{G}$ is a geometry and $\mathcal{L}$ is a n-localic $\mathcal{G}$-scheme theory, we call the $\mathcal{G}$-structured spaces in $\mathcal{L}$ *affine $\mathcal{L}$-schemes.* Let $\mathrm{Sch}(\mathcal{G};\mathcal{L}) \subset {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ denote the full subcategory of spanned by object $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that the following condition is satisfied.

($*$) There exists an effective epimorphism $\coprod_\alpha U_\alpha \to 1_{\mathcal{X}}$ such that for each α, the object $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha})$ is equivalent to an object in $\mathcal{L}$.

The objects of this ∞-category will be called *$\mathcal{L}$-schemes.* Denote by j_{Sch} the composition

$$\mathrm{Sch}(\mathcal{G},\mathcal{L}) \hookrightarrow {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}) \xrightarrow{j} \widehat{\mathsf{PShv}}\left({}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})\right) \longrightarrow \widehat{\mathsf{PShv}}(\mathcal{L})$$

of the full subcategory inclusion with the restricted Yoneda functor.

Remark 3.2.1.7. We can endow $\mathcal{L}$ with a Grothendieck pretopology as follows: a collection of morphisms $\{(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})\}$ is a covering family if each $(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i})$ is of the form $(\mathcal{Y}_{/U_i}, \mathcal{O}_{\mathcal{Y}}|_{U_i})$ for U a $(n-1)$-truncated object of $\mathcal{Y}$ and the objects U_i define an effective epimorphism $\coprod_i U_i \to 1_{\mathcal{Y}}$. Using property $\mathcal{L}2$, it is easy to see that these covering families define a pretopology. We call the associated Grothendieck topology the $(n-1)$-*étale topology.*

Our first order of business is to establish some closure and generation properties of the full subcategory of $\mathcal{G}$-schemes of type $\mathcal{L}$.

Lemma 3.2.1.8. *Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory.*

(1) *If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an $\mathcal{L}$-scheme, then for any $U \in \mathcal{X}$, the object $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U)$ is an $\mathcal{L}$-scheme.*

(2) *If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an $\mathcal{L}$-scheme and $\coprod_i U_i \to 1_{\mathcal{X}}$ is an effective epimorphism such that for each i, $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i})$ is an $\mathcal{L}$-scheme, then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an $\mathcal{L}$-scheme.*

(3) *Let $\mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}$ be the subcategory of $\mathrm{Sch}(\mathcal{G};\mathcal{L})$ on the étale morphisms. Then $\mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}$ is stable under colimits in ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$.*

(4) *The ∞-category $\mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}$ is generated under small colimits by the full subcategory $\mathcal{L}^{\text{ét}} \subset \mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}$ spanned by affine $\mathcal{L}$-schemes.*

Proof. This is proven as lemmas 2.3.10 and 2.3.11 of [Lur11b]. Point (2) is obvious. To prove (1), we take an $\mathcal{L}$-scheme $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ and some object $U \in \mathcal{X}$. Choose an effective epimorphism $\coprod_i V_i \to 1_{\mathcal{X}}$ such that $(\mathcal{X}_{/V_i},\mathcal{O}_{\mathcal{X}}|_{V_i}) \in \mathcal{L}$. We have an effective epimorphism $\coprod_i V_i \times U \to U$, so by (2), it suffices to show that $(\mathcal{X}_{V_i\times U},\mathcal{O}_{\mathcal{X}}|_{V_i\times U}) \in \mathcal{L}$. Thus, we may replace $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ with $(\mathcal{X}_{V_i},\mathcal{O}_{\mathcal{X}}|_{V_i})$ and assume that $(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$. By $\mathcal{L}2$, there exists a small diagram $\theta : K \to \mathcal{C}_{\mathcal{X}}$ in the subcategory $\mathcal{C}_{\mathcal{X}} \subset \mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \hookrightarrow \tau_{\le(n-1)}\mathcal{X} \hookrightarrow \mathcal{X}$ with colimit U. It follows that the map $\coprod_{k\in K}\theta(k) \to U$ is an effective epimorphism in $\mathcal{X}$, and therefore also in $\mathcal{X}_{/U}$. By definition, we have $(\mathcal{X}_{/\theta(k)},\mathcal{O}_{\mathcal{X}}|_{\theta(k)}) \in \mathcal{L}$, so we conclude that $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)$ is an $\mathcal{L}$-scheme.
For (3), we note that a small diagram $K \to \mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}} \subset {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})^{\text{ét}}$ with colimit $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ determines a diagram $\xi : K \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})^{\text{ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \simeq \mathcal{X}$ an effective epimorphism $\coprod_{k\in K}\xi(k) \to 1_{\mathcal{X}}$ such that $(\mathcal{X}_{/\xi(k)},\mathcal{O}_{\mathcal{X}}|_{\xi(k)})$ is an $\mathcal{L}$-scheme, so by (2), $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is an $\mathcal{L}$-scheme as well.
We prove (4). Using again general yoga of ∞-topoi, it suffices to show that for any $(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$ and any $U \in \mathcal{X}$, the $\mathcal{L}$-scheme $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)$ (which is indeed an $\mathcal{L}$-scheme by (1)) is generated under small colimits by affine $\mathcal{L}$-schemes. But we may choose a small diagram $K \to \mathcal{C}_{\mathcal{X}} \hookrightarrow \tau_{\le(n-1)\mathcal{X}}$ with colimit U. This diagram determines a diagram $K \to \mathcal{C}_{\mathcal{X}} \subset \mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \to \mathcal{L}^{\text{ét}}$ with colimit $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)$. □

Lemma 3.2.1.9. *Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory. Then the ∞-category $\mathrm{Sch}(\mathcal{G};\mathcal{L})$ is locally small.*

Proof. This is proven as in proposition 2.3.13 of [Lur11b]. Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ and $(\mathcal{X},\mathcal{O}_{\mathcal{Y}})$ be $\mathcal{L}$-schemes. If both $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ and $(\mathcal{X},\mathcal{O}_{\mathcal{Y}})$ lie in $\mathcal{L}$, then the hom-space $\mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}}((\mathcal{X},\mathcal{O}_{\mathcal{X}}),(\mathcal{X},\mathcal{O}_{\mathcal{Y}}))$ is essentially small by $\mathcal{L}4$, so we aim to reduce the problem to this case. Consider the functor

$$\zeta : \mathcal{X}^{op}\times\mathcal{Y} \simeq \left({}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})^{\text{ét}}_{/\mathcal{X}}\right)^{op} \times {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})^{\text{ét}}_{/\mathcal{Y}} \longrightarrow {}^{\mathrm{R}}\mathsf{Top}^{op}\times{}^{\mathrm{R}}\mathsf{Top} \longrightarrow \widehat{\mathcal{S}}$$

given on objects by the formula

$$(U,V) \longmapsto \mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}}((\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U),(\mathcal{Y}_{/V},\mathcal{O}_{\mathcal{Y}}|_V)).$$

Since ${}^{\mathrm{R}}\mathsf{Top}^{\text{ét}} \subset {}^{\mathrm{R}}\mathsf{Top}$ is stable under small colimits, the functor ζ preserves small limits in its first variable, and the functor

$$\zeta^{\vee} : \mathcal{Y} \longrightarrow \mathrm{Fun}(\mathcal{X}^{op},\widehat{\mathcal{S}})$$

obtained from ζ by adjunction therefore factors through the full subcategory $\mathsf{Shv}_{\widehat{\mathcal{S}}}(\mathcal{X})$ (recall that for an ∞-category $\mathcal{C}$ admitting small limits, $\mathsf{Shv}_{\mathcal{C}}(\mathcal{X})$ is the full subcategory of functors $\mathcal{X}^{op}\to\mathcal{C}$ that preserve small limits). It suffices to show that $\zeta^{\vee}(1_{\mathcal{Y}})$ is equivalent to an object in the full subcategory $\mathcal{X} \simeq \mathsf{Shv}_{\mathcal{S}}(\mathcal{X}) \subset \mathsf{Shv}_{\widehat{\mathcal{S}}}(\mathcal{X})$, that is, $\zeta^{\vee}(1_{\mathcal{Y}})$ takes essentially small values. First, we show that for $V \in \mathcal{Y}$ such that $(\mathcal{X}_{/V},\mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$, the sheaf $\zeta^{\vee}(V)$ has essentially small values. Indeed, for each $U \in \mathcal{X}$, the $\mathcal{L}$-scheme $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)$ is generated under small colimits objects in $\mathcal{L}$ étale over $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)$, so the space

$$\zeta^{\vee}(V)(U) = \mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}((\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U),(\mathcal{Y}_{/V},\mathcal{O}_{\mathcal{Y}}|_V))$$

is small limit in $\widehat{\mathcal{S}}$ over hom spaces in $\mathcal{L}$, which are small by $\mathcal{L}4$. Since $\mathcal{S}\subset\widehat{\mathcal{S}}$ preserves small limits by [Lur17b], lem. 5.4.7.6, this limit is essentially small. Since $1_{\mathcal{Y}}$ is obtained as a small colimit of objects V such that $\zeta^{\vee}(V)$ takes essentially small values and the inclusion $\mathsf{Shv}_{\mathcal{S}}(\mathcal{X}) \subset \mathsf{Shv}_{\widehat{\mathcal{S}}}(\mathcal{X})$ is stable under small colimits by [Lur17b], rmk. 6.3.5.17, it suffices to show that the functor $\zeta^{\vee}$ preserves colimits. This is proven as in lemma 2.3.11 of [Lur11b]. □

Since we obviously have $\overline{\mathcal{L}} \subset \mathrm{Sch}(\mathcal{G};\mathcal{L})$, we have the following corollary.

Corollary 3.2.1.10. *Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory, then the associated saturated n-localic $\mathcal{G}$-scheme theory $\overline{\mathcal{L}}$ is locally small.*

Corollary 3.2.1.11. *Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory. For $n\le m\le\infty$, let $\mathrm{Sch}_m(\mathcal{G};\mathcal{L}) \subset \mathrm{Sch}(\mathcal{G};\mathcal{L})$ be the full subcategory spanned by m-localic $\mathcal{L}$-schemes. Then $\mathrm{Sch}_m(\mathcal{G};\mathcal{L})$ is a saturated m-localic $\mathcal{G}$-scheme theory. Moreover, if $n\le m\le k\le\infty$, then $\mathrm{Sch}_k(\mathcal{G};\mathrm{Sch}_m(\mathcal{G};\mathcal{L})) = \mathrm{Sch}_k(\mathcal{G};\mathcal{L})$.*

Proof. $\mathcal{L}1$ is clear, the saturation condition follows immediately from (1) of lemma 3.2.1.8 and proposition 2.2.3.2, and $\mathcal{L}4$ is the content of lemma 3.2.1.9. The last statement is obvious. □

Here are some examples of scheme theories; we'll give a few more later on.

Example 3.2.1.12. Let $\mathcal{G}$ be a geometry, then the ∞-category of affine $\mathcal{G}$-schemes is a $\mathcal{G}$-scheme theory. The ∞-category of affine $\mathcal{G}$-schemes of finite presentation is also a scheme theory. These scheme theories are not in general saturated.

Example 3.2.1.13. Choose a geometric envelope $\mathcal{G}^{\mathrm{der}}_{\mathrm{An}_{\mathbb{C}}}$ for the complex analytic pregeometry $\mathcal{T}_{\mathrm{An}_{\mathbb{C}}}$, then the ∞-category of derived complex analytic spaces of [Lur11a] is a 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{An}_{\mathbb{C}}}$-scheme theory. In fact, it is saturated.

Example 3.2.1.14. We can also describe derived algebraic and spectral geometry in the étale topology. For the algebraic étale topology, we should take for $\mathcal{L}$ the subcategory of affine schematic Deligne-Mumford stacks, which are 1-localic.

Lemma 3.2.1.15. *Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory. Denote the fully faithful functor $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \hookrightarrow \tau_{\leq(n-1)}\mathcal{X}$ by i. Then the restricted Yoneda functor*

$$j^{\mathcal{X}}_{(n-1)} : \mathcal{X} \xrightarrow{j} \mathsf{PShv}(\mathcal{X}) \xrightarrow{i^*} \mathsf{PShv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\right)$$

is fully faithful and admits a left exact left adjoint.

Proof. By $\mathcal{L}2$, we may choose a small category $\mathcal{C}_{\mathcal{X}}$ that admits finite limits and a fully faithful and finite limit preserving functor $h : \mathcal{C}_{\mathcal{X}} \hookrightarrow \mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \hookrightarrow \mathcal{X}$. By the proof of [Lur17b], 6.1.5.3, we see that the induced functor $L : \mathsf{PShv}(\mathcal{C}) \to \mathcal{X}$ is a left exact accessible localization such that $h = L \circ j$. Denote by $f : \mathcal{C}_{\mathcal{X}} \subset \mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ the inclusion, then composing with f yields a limit and colimit preserving functor $f^* : \mathsf{PShv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\right) \to \mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$. Set $L^{\mathcal{X}}_{(n-1)} \coloneqq L \circ f^*$. This functor is clearly left exact, and we should show it is a left adjoint. Recall that we have chosen a regular cardinal κ such that the essential image of f consists of κ-compact objects that generate $\mathcal{X}$ under small κ-filtered colimits. Under these assumptions, the composition

$$\mathcal{X} \xrightarrow{j^{\mathcal{X}}_{(n-1)}} \mathsf{PShv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\right) \xrightarrow{f^*} \mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$$

is a κ-accessible functor. Because $\mathcal{C}_{\mathcal{X}}$ is small, the functor f^* admits a left adjoint $f_!$ given by left Kan extension along f, and the unit of the adjunction $(f_! \dashv f^*)$ is the identity. This implies that the composition

$$\mathcal{C}_{\mathcal{X}} \xhookrightarrow{h} \mathcal{X} \xrightarrow{j^{\mathcal{X}}_{(n-1)}} \mathsf{PShv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\right) \xrightarrow{f^*} \mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$$

is equivalent to the Yoneda embedding. Now both the canonical inclusion $\mathcal{X} \subset \mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$ and $f^* \circ j^{\mathcal{X}}_{(n-1)}$ are κ-accessible functors that restrict to the Yoneda embedding on $\mathcal{C}_{\mathcal{X}}$, so we have an equivalence $\mathrm{id}_{\mathcal{X}} \simeq L \circ f^* \circ j^{\mathcal{X}}_{(n-1)} = L^{\mathcal{X}}_{(n-1)} \circ j^{\mathcal{X}}_{(n-1)}$. Note that the functor f^* also has a right adjoint f_* given by right Kan extension, and the counit of the adjunction (f^*, f_*) is the identity. Let $g : \mathcal{X} \subset \mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$ denote the canonical inclusion, then we have a composition of adjunctions $(f^* \circ L, g \circ f_*)$, the counit of which is the identity. We now have equivalences of functors

$$j^{\mathcal{X}}_{(n-1)} \simeq f_* \circ g \circ L \circ f^* \circ j^{\mathcal{X}}_{(n-1)} \simeq f_* \circ g.$$

This functor has the left adjoint $L \circ f^*$. ☐

Definition 3.2.1.16. Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory.

(1) Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$ then the ∞-category $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ can be identified with a full subcategory of $\tau_{\leq(n-1)}\mathcal{X}$. We say that a presheaf F on $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \subset \tau_{\leq(n-1)}\mathcal{X}$ is a *sheaf* if F lies in the essential image of the restricted Yoneda functor $j^{X}_{(n-1)}$.

(2) For each $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$, denote by $\phi_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ the functor $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \to \mathcal{L}$, which induces a pullback functor

$$\varphi(\mathcal{X},\mathcal{O}_{\mathcal{X}})^* : \mathsf{PShv}(\mathcal{L}) \longrightarrow \mathsf{PShv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\right).$$

Then a presheaf on F is a *sheaf* if for all $(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$, the presheaf $\phi^*_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}(F)$ is a sheaf on $\tau_{\leq(n-1)}\mathcal{X}$.

Remark 3.2.1.17. Let $\mathcal{L}$ be a saturated scheme theory, then a presheaf $F \in \mathsf{PShv}(\mathcal{L})$ is sheaf if and only if for each $(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$, the presheaf $\phi^*_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}(F)$ preserves limits, that is, if and only if it is representable.

The definition of sheaves on an n-scheme theory above admits a more familiar intepretation if n is finite.

Definition 3.2.1.18. Let $n < \infty$ and let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with an n-localic $\mathcal{G}$-scheme theory. Let S be the class of morphisms in $\mathsf{PShv}(\mathcal{L})$ obtained as follows: consider the class of morphisms

$$h : \coprod_i j(\mathcal{X}/_{U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \longrightarrow j(\mathcal{X},\mathcal{O}_{\mathcal{X}})$$

where $\{U_i\}$ is a small collection of $(n-1)$-truncated objects in $\mathcal{X}$, the the induced maps $|\check{C}(h)_\bullet| \to j(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ make up S. A presheaf $F : \mathcal{L} \to \mathcal{S}$ is a *sheaf* if F is S-local. In other words, if F is a sheaf for the $(n-1)$-étale topology on $\mathcal{L}$.

We have double-booked the terminology for sheaves on $\mathcal{L}$. We now resolve this point of tension.

Lemma 3.2.1.19. *Let $n < \infty$ and let $(\mathcal{G}, \mathcal{L})$ be a geometry equipped with a saturated n-localic $\mathcal{G}$-scheme theory. Then F is a sheaf in the sense of definition 3.2.1.18 if and only if for each $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$, the presheaf $\phi^*_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}(F)$ is a sheaf on $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})} \simeq \tau_{\le(n-1)}\mathcal{X}$ in the sense of definition 3.2.1.16.*

Proof. Consider for each $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$ the following collection of covering families on the n-topos $\tau_{\le(n-1)}\mathcal{X}$: a small collection of morphisms $\{U_i \to X\}$ generates a covering sieve if the map $\coprod_i U_i \to X$ is an effective epimorphism in $\mathcal{X}$. This collection of covering sieves determines a Grothendieck pretopology on $\tau_{\le(n-1)}\mathcal{X}$, whose associated topology is called the *canonical topology*, that we denote by τ (see for instance [Lur17b], section 6.2.4). We note that the statement of the lemma may be reformulated as follows.

($*$) A presheaf F on $\mathcal{L}$ is a sheaf in the sense of definition 3.2.1.16 if and only if for each $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$, the presheaf $\phi^*_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}(F)$ is a sheaf for the canonical topology.

To prove this, we will show that a presheaf F on $\tau_{\le(n-1)}\mathcal{X}$ lies in the essential image of the restricted Yoneda functor $j^{\mathcal{X}}_{(n-1)}$ if and only if F is a sheaf for the canonical topology. Because $\mathcal{X}$ is n-localic, we may identify $\mathcal{X}$ with $\mathsf{Shv}(\mathcal{C}_{\mathcal{X}})$, where we endow $\mathcal{C}_{\mathcal{X}}$ with the canonical topology relative to the inclusion $\mathcal{C}_{\mathcal{X}} \subset \tau_{(n-1)}\mathcal{X} \simeq \mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})$. This inclusion is thus tautologically covering-preserving, so we have an induced map $\mathsf{Shv}_\tau(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})) \to \mathsf{Shv}(\mathcal{C}_{\mathcal{X}}) \simeq \mathcal{X}$. By the constructions in the proof of lemma 3.2.1.15, the composition $\mathsf{Shv}_\tau(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})) \subset \mathsf{PShv}(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})) \to \mathsf{Shv}(\mathcal{C}_{\mathcal{X}})$ coincides with the functor $L^{\mathcal{X}}_{(n-1)}|_{\mathsf{Shv}_\tau(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}}))}$. It follows that if we can show that the functor $j^{\mathcal{X}}_{(n-1)}$ takes values in sheaves (for the canonical topology), then the adjunction

$$\mathsf{PShv}(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})) \underset{j^{\mathcal{X}}_{(n-1)}}{\overset{L^{\mathcal{X}}_{(n-1)}}{\rightleftarrows}} \mathsf{Shv}(\mathcal{C}_{\mathcal{X}})$$

restricts to an adjunction

$$\mathsf{Shv}_\tau(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})) \underset{j^{\mathcal{X}}_{(n-1)}}{\overset{L^{\mathcal{X}}_{(n-1)}}{\rightleftarrows}} \mathsf{Shv}(\mathcal{C}_{\mathcal{X}}).$$

To see this, we endow $\mathcal{X}$ also with its canonical topology, then the subcategory inclusion $\tau_{\le(n-1)}\mathcal{X} \subset \mathcal{X}$ is covering-preserving and the induced functor $\mathsf{PShv}(\mathcal{X}) \to \mathsf{PShv}(\tau_{\le(n-1)}\mathcal{X})$ carries sheaves to sheaves (for the canonical topologies). As the Yoneda embedding $j : \mathcal{X} \hookrightarrow \mathsf{PShv}(\mathcal{X})$ clearly takes values in sheaves, we conclude that the adjunction $(L^{\mathcal{X}}_{(n-1)} \dashv j^{\mathcal{X}}_{(n-1)})$ indeed restricts. We already know that the counit map is an equivalence. To see that the unit is an equivalence as well, we note that the proof of lemma 3.2.1.15 guarantees that $j^{\mathcal{X}}_{(n-1)}$ may be identified with a right Kan extension along the Yoneda embedding $\mathcal{C}_X \hookrightarrow \mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})$. We deduce that given a sheaf $F \in \mathsf{Shv}_\tau(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}}))$, the unit map $F \to j^{\mathcal{X}}_{(n-1)} L^{\mathcal{X}}_{(n-1)} F$ is an equivalence when restricted to the essential image of the fully faithful embedding $\mathcal{C}_{\mathcal{X}} \hookrightarrow \mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})$. We finish the proof by showing that the unit map is an equivalence on any object $Z \in \mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}})$. Choose an uncountable regular cardinal κ such that $\mathcal{C}_{\mathcal{X}}$ is κ-small, Z is κ-compact and the full subcategory of $\mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$ spanned by κ-compact objects is stable under finite limits, then using an Artin-Mazur argument as in the proof of proposition 2.2.4.3, we can construct an $(n+1)$-truncated hypercover $C_\bullet$ of Z in $\mathsf{PShv}(\mathcal{C}_{\mathcal{X}})$ such that each level C_n is a κ-small coproduct of representables. We may repeat the construction of this simplicial object in the ∞-category $\mathsf{PShv}(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}}))$ to produce an $(n+1)$-truncated simplicial object $\widetilde{C}_\bullet$ of Z, where now each level of $\widetilde{C}_\bullet$ is a κ-small coproduct of objects in the essential image of the 'double Yoneda embedding' $\mathcal{C}_{\mathcal{X}} \hookrightarrow \mathsf{Shv}_{(n-1)}(\mathcal{X}) \hookrightarrow \mathsf{PShv}(\mathsf{Shv}_{(n-1)}(\mathcal{C}_{\mathcal{X}}))$. By construction, $\widetilde{C}_\bullet$ is a semi-representable hypercover of Z for the canonical topology. As both F and $j^{\mathcal{X}}_{(n-1)} L^{\mathcal{X}}_{(n-1)} F$ are sheaves, they satisfy descent with respect to truncated semi-representable hypercovers, so we deduce that $F(Z) \to j^{\mathcal{X}}_{(n-1)} L^{\mathcal{X}}_{(n-1)} F(Z)$ is an equivalence. □

Remark 3.2.1.20. From the arguments in the proof above we can extract the following result: if $\mathcal{X}$ is an n-topos for n finite, then the associated $(n+1)$-localic ∞-topos is the ∞-category of sheaves on $\mathcal{X}$ for the canonical topology. The argument also applies if $\mathcal{X}$ is a hypercomplete ∞-topos and we consider hypersheaves for the canonical topology.

Suppose that $\mathcal{L}$ is a saturated n-localic $\mathcal{G}$-scheme theory that is also essentially small as an ∞-category (notice that this forces n to be 0 since there are no essentially small presentable ∞-categories containing an object which is not (-1)-truncated). Then using the class S in definition 3.2.1.18, we see that $\mathsf{Shv}(\mathcal{L})$ is a left exact (accessible) localization of $\mathsf{PShv}(\mathcal{L})$, so that $\mathsf{Shv}(\mathcal{L})$ is an ∞-topos. Even if the ∞-category $\mathcal{L}$ is not essentially small, we would still like to construct a sheafification functor. Unless we are willing to consider presheaves and sheaves valued in large spaces, this sheafification cannot be an accessible localization, but this is not an insurmountable issue. To overcome the problem, we will realize the ∞-category $\mathsf{PShv}(\mathcal{L})$ as an ∞-category consisting of a compatible collection of pairs $\big((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), F_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\big)$, where $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{L}$ and $F_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$ is a presheaf on $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$, the full subcategory of affine

$\mathcal{L}$-schemes on $(\mathcal{X},\mathcal{O}_\mathcal{X})$ which are induced by a $(n-1)$-truncated object in $\mathcal{X}$. Using lemma 3.2.1.15, we can find a sheafification functor, and assembling these sheafifications yields the desired localization. This strategy has the added benefit that it allows for a convenient description of colimits in the ∞-category $\mathsf{Shv}(\mathcal{L})$. The precise state of affairs is summarized in the following proposition.

Proposition 3.2.1.21. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with an n-localic $\mathcal{G}$-scheme theory.*

(1) The full subcategory inclusion $\mathsf{Shv}(\mathcal{L})\subset\mathsf{PShv}(\mathcal{L})$ admits a left exact left adjoint L.

(2) The ∞-category $\mathsf{Shv}(\mathcal{L})$ admits small limits and colimits.

(3) Let K be a (small) simplicial set. A diagram $K^{\triangleright}\to\mathsf{Shv}(\mathcal{L})$ is a colimit diagram if and only if for each $(\mathcal{X},\mathcal{O}_\mathcal{X})$, the composition

$$K^{\triangleright}\longrightarrow\mathsf{Shv}(\mathcal{L})\longrightarrow\mathsf{Shv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_\mathcal{X})}\right)\simeq\mathcal{X}$$

is a colimit diagram.

(4) Let $\widehat{\mathsf{PShv}}(\mathcal{L})$ respectively $\widehat{\mathsf{Shv}}(\mathcal{L})$)denote the very large ∞-topos of presheaves respectively sheaves on $\mathcal{L}$, and denote by $\widehat{L}:\widehat{\mathsf{PShv}}(\mathcal{L})\to\widehat{\mathsf{Shv}}(\mathcal{L})$ a sheafification functor. Then the diagram

$$\begin{tikzcd}
\mathsf{PShv}(\mathcal{L}) \arrow[r] \arrow[d, "L"] & \widehat{\mathsf{PShv}}(\mathcal{L}) \arrow[d, "\widehat{L}"] \\
\mathsf{Shv}(\mathcal{L}) \arrow[r] & \widehat{\mathsf{Shv}}(\mathcal{L})
\end{tikzcd}$$

commutes up to homotopy, where the left vertical map is the left adjoint L of point (1).

(5) The inclusion $\mathsf{Shv}(\mathcal{L})\subset\widehat{\mathsf{Shv}}(\mathcal{L})$ preserves small limits and colimits.

(6) If $\mathcal{L}$ is small, then the localization $\mathsf{Shv}(\mathcal{L})\subset\mathsf{PShv}(\mathcal{L})$ is accessible and $\mathsf{Shv}(\mathcal{L})$ is an ∞-topos.

Proof. As proposition 2.4.4 of [Lur11b] or proposition 5.2.10 of [Car16]. □

Remark 3.2.1.22. Parsing the proof in the references above gives an explicit sheafification procedure: let $\alpha:F\to F'$ be a morphism of presheaves on $\mathcal{L}$. Then α exhibits F' as a sheafification of F if and only if for each $(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}$, the map $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}(\alpha)$ exhibits a sheafification in $\mathsf{PShv}(\tau_{\le n-1}\mathcal{Y})$.

Since the inclusion $\mathsf{Shv}(\mathcal{L})\hookrightarrow\widehat{\mathsf{Shv}}(\mathcal{L})$ preserves small limits and colimits, we have

Corollary 3.2.1.23. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with a $\mathcal{G}$-scheme theory, then the following hold in $\mathsf{Shv}(\mathcal{L})$.*

(1) Groupoids are effective.

(2) Small colimits are universal.

(3) Small coproducts are disjoint.

Now that we have good control over the ∞-category of sheaves on a scheme theory, we continue our study of the restricted Yoneda functor j_{Sch}.

Proposition 3.2.1.24. *The functor*

$$j_{\mathrm{Sch}}:{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})\xrightarrow{j}\widehat{\mathsf{PShv}}\left({}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})\right)\longrightarrow\widehat{\mathsf{PShv}}(\mathcal{L})$$

takes values in the full subcategory of sheaves.

Proof. This is just a consequence of the fact that the topology on ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$ is subcanonical. Indeed, we are asked to show that for any (n-1)-étale covering $h:\coprod_i(\mathcal{X}_{/U_i},\mathcal{O}|_{U_i})\to(\mathcal{X},\mathcal{O}_\mathcal{X})$ in $\mathcal{L}$, we have an equivalence

$$\mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}((\mathcal{X},\mathcal{O}_\mathcal{X}),(\mathcal{Y},\mathcal{O}_\mathcal{Y}))\simeq\lim_{\mathbf{N}(\Delta)}\mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}(\check{C}(h)_\bullet,(\mathcal{Y},\mathcal{O}_\mathcal{Y})).$$

By proposition 3.1.0.25, we have an equivalence $\operatorname{colim}_{\mathbf{N}(\Delta)^{op}}\check{C}(h)_\bullet\simeq(\mathcal{X},\mathcal{O}_\mathcal{X})$ in ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})$, so we get weak equivalences

$$\begin{aligned}\lim_{\mathbf{N}(\Delta)}\mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}(\check{C}(h)_\bullet,(\mathcal{Y},\mathcal{O}_\mathcal{Y}))&\simeq\mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}(\operatorname{colim}_{\mathbf{N}(\Delta)^{op}}\check{C}(h)_\bullet,(\mathcal{Y},\mathcal{O}_\mathcal{Y}))\\&\simeq\mathrm{Hom}_{{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})}((\mathcal{X},\mathcal{O}_\mathcal{X}),(\mathcal{Y},\mathcal{O}_\mathcal{Y})),\end{aligned}$$

and we are done. □

Remark 3.2.1.25. Evidently, the proof above also shows that the $(n-1)$-étale topology on $\mathcal{L}$ is subcanonical.

Proposition 3.2.1.26. *The functor*

$$\mathrm{Sch}(\mathcal{G},\mathcal{L}) \subset {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})_{\acute{e}t} \xrightarrow{j} \widehat{\mathsf{PShv}}\left({}^{\mathrm{R}}\mathsf{Top}(\mathcal{G})\right) \longrightarrow \widehat{\mathsf{Shv}}(\mathcal{L})$$

preserves small colimits.

Proof. As proposition 5.2.11 of [Car16]. □

Theorem 3.2.1.27. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with an n-localic $\mathcal{G}$-scheme theory. Then the restricted Yoneda functor j_{Sch} is fully faithful, and takes values in $\mathsf{Shv}(\mathcal{L})$.*

Proof. It follows from corollary **??** that j_{Sch} takes values in small presheaves, and it follows from proposition 3.2.1.24 that j_{Sch} takes values in $\mathsf{Shv}(\mathcal{L})$.
We should check that for any pair $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$, $(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$ of $\mathcal{L}$-schemes, the map

$$\phi : \mathrm{Hom}_{\mathrm{Sch}(\mathcal{G};\mathcal{L})}((\mathcal{X},\mathcal{O}_{\mathcal{X}}),(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})) \longrightarrow \mathrm{Hom}_{\mathsf{PSh}(\mathcal{L})}(j_{\mathrm{Sch}}(\mathcal{X},\mathcal{O}_{\mathcal{X}}), j_{\mathrm{Sch}}(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}))$$

is a homotopy equivalence of Kan complexes. Since j_{Sch} preserves small colimits and $\mathrm{Sch}(\mathcal{G};\mathcal{L})$ is generated under small colimits by $\mathcal{L}$, it suffices to check that ϕ is fully faithful when $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is an affine $\mathcal{L}$-scheme, but this is obvious. □

Given a scheme theory $\mathcal{L}$ for a geometry $\mathcal{G}$, we may also consider sheaves on the ∞-category $\mathrm{Sch}(\mathcal{G};\mathcal{L})$, but the following result shows that taking sheaves on arbitrary $\mathcal{L}$-schemes does not constitute an enlargement.

Proposition 3.2.1.28. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with a $\mathcal{G}$-scheme theory, then the functor*

$$i^* : \mathsf{PShv}(\mathrm{Sch}(\mathcal{G};\mathcal{L})) \longrightarrow \mathsf{PShv}(\mathcal{L})$$

induces an equivalence $\mathsf{Shv}(\mathrm{Sch}(\mathcal{G};\mathcal{L})) \simeq \mathsf{Shv}(\mathcal{L})$, and the functor

$$i^* : \mathsf{PShv}(\mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}) \longrightarrow \mathsf{PShv}(\mathcal{L}^{\text{ét}})$$

induces an equivalence $\mathsf{Shv}(\mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}) \simeq \mathsf{Shv}(\mathcal{L}^{\text{ét}})$.

Proof. Since i is covering-preserving, the functor i^* takes sheaves to sheaves. We prove that the left adjoint $i_!$ to i^* is fully faithful. For this, it suffices to show that if a morphism $\alpha : F \to F'$ in $\mathsf{PShv}(\mathrm{Sch}(\mathcal{G};\mathcal{L}))$ exhibits F as a sheafification of F', then $i^*(\alpha)$ also exhibits a sheafification. Let S respectively S' denote the classes of covering sieves in $\mathcal{L}$ and $\mathrm{Sch}(\mathcal{G};\mathcal{L})$ respectively, and let $\overline{S}$ respectively $\overline{S}'$ be their strong saturations. Since i^* preserves sheaves, it suffices to show that $i^*(\overline{S}') \subset \overline{S}$, or equivalently $\overline{S}' \subset (i^*)^{-1}\overline{S}$. Since $(i^*)^{-1}\overline{S}$ is strongly saturated, it suffices to show that $S' \subset (i^*)^{-1}\overline{S}$. Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}$-scheme of type $\mathcal{L}$, let $\coprod U_i \to 1_{\mathcal{X}}$ be an effective epimorphism and consider the Čech nerve of the map

$$h : \coprod_i j(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \longrightarrow j(\mathcal{X},\mathcal{O}_{\mathcal{X}}),$$

then we should show that the augmented simplicial diagram $i^*(\check{C}(h)_\bullet)$ becomes a colimit diagram after applying the sheafification functor L. Each level of $Li^*(\check{C}(h)_\bullet)$ is given by an object of the form

$$\coprod_{i_1,\dots,i_n} Lj_{\mathrm{Sch}}(\mathcal{X}_{/U_{i_1}\times\dots\times U_{i_n}}, \mathcal{O}_{\mathcal{X}}|_{U_{i_1}\times\dots\times U_{i_n}}) \simeq \coprod_{i_1,\dots,i_n} j_{\mathrm{Sch}}(\mathcal{X}_{/U_{i_1}\times\dots\times U_{i_n}}, \mathcal{O}_{\mathcal{X}}|_{U_{i_1}\times\dots\times U_{i_n}})$$

where the equivalence is due to the fact that the essential image of j_{Sch} consists of sheaves. Since j_{Sch} commutes with coproducts it follows that the augmented simplicial object $Li^*(\check{C}(h)_\bullet)$ is equivalent to $j_{\mathrm{Sch}}(\check{C}(h'))$, where h' is the map $\coprod_i(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \to (\mathcal{X},\mathcal{O}_{\mathcal{X}})$. Because the diagram $\check{C}(h')_\bullet$ is a colimit diagram in $\mathrm{Sch}(\mathcal{G};\mathcal{L})$ and j_{Sch} preserves colimits, we conclude.
We prove that $i_!$ is essentially surjective. Let $\mathcal{C} \subset \mathsf{Shv}(\mathrm{Sch}(\mathcal{G};\mathcal{L}))$ be the smallest full subcategory stable under colimits containing the essential image of $i_!$. It suffices to show that $j(\mathrm{Sch}(\mathcal{G};\mathcal{L}))$ is contained in $\mathcal{C}$. This follows because $\mathrm{Sch}(\mathcal{G};\mathcal{L})$ is generated under small colimits of diagrams in $\mathcal{L}^{\text{ét}}$ and the Yoneda embedding $j : \mathrm{Sch}(\mathcal{G};\mathcal{L}) \to \mathsf{Shv}(\mathrm{Sch}(\mathcal{G};\mathcal{L}))$ preserves small colimits of diagrams in $\mathrm{Sch}(\mathcal{G};\mathcal{L})^{\text{ét}}$.

□

The ideas introduced in this chapter become particularly useful when we compare different scheme theories.

Proposition 3.2.1.29. *Fix a geometry $\mathcal{G}$ and let $\mathcal{L} \subset \mathcal{L}'$ be two saturated $\mathcal{G}$-scheme theories. Denote by $i : \mathcal{L} \subset \mathcal{L}'$ the inclusion and by $i^* : \mathsf{PShv}(\mathcal{L}') \to \mathsf{PShv}(\mathcal{L})$ the induced functor on presheaves, then i^* preserves colimits and the left adjoint*

$$i_! : \mathsf{Shv}(\mathcal{L}) \longrightarrow \mathsf{Shv}(\mathcal{L}')$$

to i^* is fully faithful. Thus, if $\mathcal{L}'$ is (essentially) small, then i^* exhibits $\mathsf{Shv}(\mathcal{L}')$ as a local $\mathsf{Shv}(\mathcal{L})$-topos.

Proof. Using proposition 2.2.2.13, it suffices to show that if $\alpha : F \to F'$ exhibits a sheafification, then $i^*(\alpha)$ exhibits a sheafification. Using remark 3.2.1.22, we see that $\varphi^*_{(\mathcal{X},\mathcal{O}_\mathcal{X})}(\alpha)$ is a sheafification for each $(\mathcal{X},\mathcal{O}_\mathcal{X}) \in \mathcal{L}'$. If $(\mathcal{X},\mathcal{O}_\mathcal{X}) \in \mathcal{L}$, then we have an isomorphism of ∞-categories $\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_\mathcal{X})} \cong \mathcal{L}'^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_\mathcal{X})}$ and a commuting diagram

$$\begin{tikzcd}
\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_\mathcal{X})} \arrow[rr, "\varphi_{(\mathcal{X},\mathcal{O}_\mathcal{X})}"] \arrow[dr, "\phi_{(\mathcal{X},\mathcal{O}_\mathcal{X})}"'] & & \mathcal{L}' \arrow[dl, "i"] \\
& \mathcal{L} &
\end{tikzcd}$$

so we have a commuting diagram

$$\begin{tikzcd}
\mathsf{PShv}(\mathcal{L}') \arrow[rr, "i^*"] \arrow[dr, "\varphi^*_{(\mathcal{X},\mathcal{O}_\mathcal{X})}"'] & & \mathsf{PShv}(\mathcal{L}) \arrow[dl, "\phi^*_{(\mathcal{X},\mathcal{O}_\mathcal{X})}"] \\
& \mathsf{PShv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{X},\mathcal{O}_\mathcal{X})}\right) &
\end{tikzcd}$$

so that $\varphi^*_{(\mathcal{X},\mathcal{O}_\mathcal{X})}(i^*(\alpha))$ is a sheafification for each $(\mathcal{X},\mathcal{O}_\mathcal{X}) \in \mathcal{L}$. Using remark 3.2.1.22 again, we conclude that $i^*(\alpha)$ exhibits a sheafification. □

We conclude this subsection with some observations that transfer the properties of the ∞-topoi that make up a scheme theory to $\mathsf{Shv}(\mathcal{L})$:

Proposition 3.2.1.30. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with a* small *$\mathcal{G}$-scheme theory. Then $\mathsf{Shv}(\mathcal{L})$ is hypercomplete if and only if for every $(\mathcal{Y},\mathcal{O}_\mathcal{Y}) \in \mathcal{L}$, the ∞-topos $\mathcal{Y}$ is hypercomplete.*

Proof. For the 'if' direction, we note that the collection of functors $\{\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\}_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}}$ is jointly conservative, so it suffices to prove that for each $(\mathcal{Y},\mathcal{O}_\mathcal{Y}) \in \mathcal{L}$, the functor $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}$ preserves k-connective morphisms for all $k \geq 0$. This follows from the fact that the functor $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}$ preserves limits and colimits. For the converse, we note that $\mathcal{Y}$ is a local subtopos of the slice topos $\mathsf{Shv}(\mathcal{L})_{/(\mathcal{Y},\mathcal{O}_\mathcal{Y})}$. □

The following result is based on a mathoverflow answer of Marc Hoyois.

Proposition 3.2.1.31. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with a* small *$\mathcal{G}$-scheme theory. Then Postnikov towers converge in $\mathsf{Shv}(\mathcal{L})$ if and only if for every $(\mathcal{Y},\mathcal{O}_\mathcal{Y}) \in \mathcal{L}$, Postnikov towers converge in the ∞-topos $\mathcal{Y}$.*

Proof. We prove the 'if' direction. Let $F_\bullet : \mathbf{N}(\mathbb{Z}^{\triangleright})^{op} \to \mathsf{Shv}(\mathcal{L})$ be a tower, then we need to check that $F_\bullet$ is a Postnikov tower if and only if $F_\bullet|_{\mathbf{N}(\mathbb{Z})^{op}}$ is a Postnikov pretower and $F_\bullet$ is a limit diagram. For every $(\mathcal{X},\mathcal{O}_\mathcal{X}) \in \mathcal{L}$, denote by $(F_\bullet)_{(\mathcal{X},\mathcal{O}_\mathcal{X})}$ the composition

$$(F_\bullet)_{(\mathcal{X},\mathcal{O}_\mathcal{X})} : \mathbf{N}(\mathbb{Z}^{\triangleright})^{op} \longrightarrow \mathsf{Shv}(\mathcal{L}) \overset{\mathrm{ev}_{(\mathcal{X},\mathcal{O}_\mathcal{X})}}{\longrightarrow} \mathcal{S}.$$

Note that for every $(n-1)$-étale map $(\mathcal{X},\mathcal{O}_\mathcal{X}) \to (\mathcal{Y},\mathcal{O}_\mathcal{Y})$, the tower $(F_\bullet)_{(\mathcal{X},\mathcal{O}_\mathcal{X})}$ factors as

$$(F_\bullet)_{(\mathcal{X},\mathcal{O}_\mathcal{X})} : \mathbf{N}(\mathbb{Z}^{\triangleright})^{op} \longrightarrow \mathsf{Shv}(\mathcal{L}) \overset{\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}}{\longrightarrow} \mathsf{Shv}\left(\mathcal{L}^{(n-1)\text{-ét}}_{/(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\right) \overset{\mathrm{ev}_{(\mathcal{X},\mathcal{O}_\mathcal{X})}}{\longrightarrow} \mathcal{S},$$

where $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}$ is the pullback of presheaves, which preserves sheaves by lemma 3.2.1.19. We claim that the proposition follows from the following assertion:

($*$) $F_\bullet$ is a Postnikov (pre)tower if and only if $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})} \circ F_\bullet$ is a Postnikov (pre)tower for all $(\mathcal{Y},\mathcal{O}_\mathcal{Y}) \in \mathcal{L}$.

Indeed, given a tower $F_\bullet : \mathbf{N}(\mathbb{Z}^{\triangleright}_{\geq 0})^{op} \to \mathsf{Shv}(\mathcal{L})$ the following are equivalent.

(a) $F_\bullet$ is a Postnikov tower.

(b) For all $(\mathcal{Y},\mathcal{O}_\mathcal{Y}) \in \mathcal{L}$, the tower $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})} \circ F_\bullet$ is a Postnikov tower.

(c) For all $(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}$, $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\circ F_\bullet|_{\mathbf{N}(\mathbb{Z}_\geq)^{op}}$ is a Postnikov pretower and $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\circ F_\bullet$ is a limit diagram.

(d) $F_\bullet|_{\mathbf{N}(\mathbb{Z}_\leq)^{op}}$ is a Postnikov pretower and for all $(\mathcal{X},\mathcal{O}_\mathcal{X})\in\mathcal{L}$, the diagram $(F_\bullet)_{(\mathcal{X},\mathcal{O}_\mathcal{X})}$ is a limit diagram.

(e) $F_\bullet|_{\mathbf{N}(\mathbb{Z}_\leq)^{op}}$ is a Postnikov pretower and $F_\bullet$ is a limit diagram.

We note that $(a)\Leftrightarrow(b)$ follows from $(*)$, $(b)\Leftrightarrow(c)$ is the case by assumption, $(c)\Leftrightarrow(d)$ follows from $(*)$ and the fact that limits are computed objectwise, as does $(d)\Leftrightarrow(e)$.
To prove $(*)$, we need to show that a map $\alpha: X\to Y$ of sheaves on $\mathcal{L}$ exhibits Y as an n-truncation of X if and only if for all $(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}$, the map $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}(\alpha)$ exhibits an n-truncation in $\mathsf{Shv}\left(\mathcal{L}^{(n-1)-\text{ét}}_{/(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\right)$. Since truncation in an ∞-category of sheaves on a small site is given by objectwise truncation of presheaves followed by sheafification, it suffices to verify that a map $\gamma: X'\to Y'$ of *pre*sheaves on $\mathcal{L}$ exhibits Y' as a sheafification of X' if and only if for all $(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}$, the map $\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}(\gamma)$ exhibits a sheafification in $\mathsf{PShv}\left(\mathcal{L}^{(n-1)-\text{ét}}_{/(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\right)$. This is the case by construction of the sheafification functor $L:\mathsf{PShv}(\mathcal{L})\to\mathsf{Shv}(\mathcal{L})$ in proposition 3.2.1.21. For the converse, we note that $\mathcal{Y}$ is a local subtopos of the slice topos $\mathsf{Shv}(\mathcal{L})_{/(\mathcal{Y},\mathcal{O}_\mathcal{Y})}$. ☐

Proposition 3.2.1.32. *Let $(\mathcal{G},\mathcal{L})$ be a geometry equipped with a* small *$\mathcal{G}$-scheme theory. Then the ∞-topos $\mathsf{Shv}(\mathcal{L})$ has enough points if and only if for every $(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}$, the ∞-topos $\mathcal{Y}$ has enough points.*

Proof. The if direction follows immediately from the fact that the collection of functors $\{\phi^*_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})}\}_{(\mathcal{Y},\mathcal{O}_\mathcal{Y})\in\mathcal{L}}$ is jointly conservative. For the converse, we note that $\mathcal{Y}$ is a local subtopos of the slice topos $\mathsf{Shv}(\mathcal{L})_{/(\mathcal{Y},\mathcal{O}_\mathcal{Y})}$. ☐

Example 3.2.1.33. Choose a geometric envelope $\mathcal{T}_{\mathrm{Diff}}\hookrightarrow\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$, then the functor $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diff}}}:\mathcal{T}_{\mathrm{Diff}}\to{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ is fully faithful, and the pair $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{T}_{\mathrm{Diff}}))$ is a good $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theory. Using that all (finite dimensional) manifolds have finite covering dimension, we see that $\mathsf{Shv}(M)$ has enough points for every manifold M. We note that the topology on $\mathcal{T}_{\mathrm{Diff}}$ induced from ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ is the étale topology, so we conclude that $\mathsf{Shv}(\mathcal{T}_{\mathrm{Diff}})=\mathsf{SmSt}$ has enough points. In particular, SmSt is hypercomplete and Postnikov towers converge.

Example 3.2.1.34. Consider the good $\mathcal{G}_{\mathrm{An}_\mathbb{C}}$-scheme theory given by derived analytic spaces. All analytic spaces have finite covering dimension, so using the same argument as in the previous remark, we find that the ∞-topos of derived analytic stacks has enough points.

3.2.2 Geometric contexts

Let $\mathcal{G}$ be a geometry and let $\mathcal{L}$ be an n-localic $\mathcal{G}$-scheme theory.

Definition 3.2.2.1. Let P be a property of morphisms in $\mathcal{L}$, then P is *local on the source for the $(n-1)$-étale topology* if the following conditions are satisfied.

(1) If in a composition
$$\mathfrak{X}\longrightarrow\mathfrak{Y}\longrightarrow\mathcal{Z},$$
the first map is $(n-1)$-étale and the second has the property P, then the composition has the property P.

(2) Suppose $\coprod_i\mathfrak{X}_i\to\mathfrak{X}$ is an $(n-1)$-étale covering and let $f:\mathfrak{X}\to\mathfrak{Y}$ be a map, then f has the property P if for each i, the composition $\mathfrak{X}_i\to\mathfrak{X}\to\mathfrak{Y}$ has the property P.

Definition 3.2.2.2. Let P be a property of morphisms in $\mathcal{L}$, then P is *local on the target for the $(n-1)$-étale topology* if the following conditions are satisfied.

(1) A pullback of a morphism in P along an $(n-1)$-étale map is in P.

(2) Let $f:\mathfrak{X}\to\mathfrak{Y}$ is a map in $\mathcal{L}$ and suppose that there is an $(n-1)$-étale cover $\coprod\mathfrak{Y}_i\to\mathfrak{Y}$ such that for each i, the induced map $\mathfrak{Y}_i\times_\mathfrak{Y}\mathfrak{X}\to\mathfrak{Y}_i$ lies in P, then f lies in P.

The following notion is adapted from the HAG contexts of [TV06] and that of [PY17]

Definition 3.2.2.3. A *geometric context* consists of a triple $(\mathcal{G},\mathcal{L},\mathcal{P})$ of a geometry together with an n-localic scheme theory, and a property $\mathcal{P}$ of morphisms that satisfies the following conditions.

G1. Morphisms in $\mathcal{P}$ are closed under taking pullbacks with any morphism in $\mathcal{L}$.

G2. For every $(n-1)$-étale covering $\{f_i:U_i\to X\}$, f_i is in $\mathcal{P}$.

G3. The property $\mathcal{P}$ is local on the source with respect to the $(n-1)$-étale topology.

Given a geometric context, one defines m-geometric stacks for $m \geq -1$ by induction, as follows. In what follows, we will restrict to 0-*localic scheme theories*, for simplicity.

Definition 3.2.2.4. Let $(\mathcal{G}, \mathcal{L}, \mathcal{P})$ be a geometric context where $\mathcal{L}$ is n-localic. We call a sheaf on $\mathcal{L}$ a *stack*.

(1) A (-1)-*geometric stack* is a stack representable by a 0-localic $\mathcal{L}$-scheme.

(2) A morphism of stacks $X \to Y$ is (-1)-*representable* if for any morphism $Z \to Y$ of stacks where Z lies in $\mathcal{L}$, the base change $X \times_Y Z$ is a (-1)-geometric stack.

(3) A morphism of stacks $X \to Y$ is an (-1)-$\mathcal{P}$ *morphism* if it is (-1)-representable and if for any morphism of stacks $Z \to Y$ where Z is a representable stack, the morphism $X \times_Y Z \to$ is in $\mathcal{P}$.

For $n \geq 0$, we say that

(1) A stack X *has an* n-$\mathcal{P}$ *atlas* if there is a collection $\{U_i\}_{i \in I}$ of representable stacks together with $(n-1)$-$\mathcal{P}$ morphisms $U_i \to X$ such that the induced map $\coprod_{i \in I} U_i \to X$ is an effective epimorphism.

(2) A stack X is n-$\mathcal{P}$-*geometric* if X has an n-$\mathcal{P}$ atlas and the diagonal map $X \to X \times X$ is $(n-1)$-representable.

(3) A morphism of stacks $X \to Y$ is n-*representable* if for any morphism of stacks $Z \to Y$ where Z is representable, the base change $X \times_Y Z$ is n-$\mathcal{P}$-geometric.

(4) A morphism of stacks $X \to Y$ is *an* n-$\mathcal{P}$ *morphism* if it is n-representable and if for any morphism $Z \to Y$ of stacks where Z is representable, there exists an n-$\mathcal{P}$ atlas $\{U_i \to X \times_Y Z\}$ such that for each i, the composite morphism $U_i \to Z$ is in $\mathcal{P}$.

The following proposition sums up the basic properties of n-$\mathcal{P}$-geometric stacks.

Proposition 3.2.2.5. *Let* $(\mathcal{G}, \mathcal{L}, \mathcal{P})$ *be a geometric context.*

(1) Any $(n-1)$*-representable morphism is* n*-representable.*

(2) Any $(n-1)$*-*$\mathcal{P}$ *morphism is* n*-*$\mathcal{P}$*.*

(3) n*-representable morphisms are stable by equivalences, compositions and pullbacks along any morphism of stacks.*

(4) n*-*$\mathcal{P}$ *morphisms are stable by equivalences, compositions and pullbacks along any morphism of stacks.*

Proof. See [TV06], prop 1.3.3.3. □

Proposition 3.2.2.6. *Let* $f : X \to Y$ *be an* n*-representable morphism of stacks. If* f *is an* m*-*$\mathcal{P}$ *morphism for* $m > n$*, then* f *is* n*-*$\mathcal{P}$*.*

Proof. See [TV06], prop. 1.3.3.6 □

Proposition 3.2.2.7. *Let* $(\mathcal{G}, \mathcal{L}, \mathcal{P})$ *be a geometric context and let* $f : X \to Y$ *be a morphism of stacks where* Y *is* n*-*$\mathcal{P}$*-geometric. Suppose that* Y *admits an* n*-*$\mathcal{P}$ *atlas* $\{U_i \to Y\}_{i \in I}$ *such that* $X \times_Y U_i$ *is* n*-*$\mathcal{P}$*-geometric for all* $i \in I$*. Then* X *is* n*-*$\mathcal{P}$*-geometric. Moreover, if for each* $i \in I$*, the map* $X \times_Y U_i \to U_i$ *is* n*-*$\mathcal{P}$*, then so is* f*.*

Proof. See [TV06], prop 1.3.3.4. □

Definition 3.2.2.8. Let $(\mathcal{G}, \mathcal{L}, \mathcal{P})$ be a geometric context. A groupoid object $X_\bullet \in \mathsf{Gpd}(\mathsf{Shv}(\mathcal{C}))$ is an n-$\mathcal{P}$ *groupoid* if X_0 and X_1 are small coproducts of n-$\mathcal{P}$-geometric stacks, and the degeneracy maps $d_0^0, d_1^0 : X_1 \to X_0$ are in n-$\mathcal{P}$.

Proposition 3.2.2.9. *Let* $(\mathcal{G}, \mathcal{L}, \mathcal{P})$ *be a geometric context, and let* $X \in \mathsf{Shv}(\mathcal{L})$ *be a stack. The following are equivalent:*

(1) X *is an* n*-*$\mathcal{P}$*-geometric stack.*

(2) X *has an* n*-*$\mathcal{P}$ *atlas.*

(3) There exists an $(n-1)$*-*$\mathcal{P}$ *groupoid* $X_\bullet$ *such that* $X \simeq \operatorname{colim}_{\mathbf{N}(\Delta)^{op}} X_\bullet$*.*

Proof. See [TV06], proposition 1.3.4.2. □

The following corollary is useful for establishing geometricity in situations where one is given an 'atlas' of a stack which is not affine.

Corollary 3.2.2.10. *Let X be an $(n-1)$-$\mathcal{P}$ geometric stack and let $f : X \to Y$ be an effective epimorphism that is $(n-1)$-representable and in $\mathcal{P}$. Then Y is n-$\mathcal{P}$ geometric.*

Proof. The assumptions easily imply that the Čech nerve of f is an $(n-1)$-$\mathcal{P}$ groupoid whose realization is equivalent to Y. □

Chapter 4

Derived C^∞-geometry: foundational aspects

In this chapter, we perform an in-depth study of the algebraic theory of *simplicial C^∞-rings*, the projectively generated presentable ∞-category $sC^\infty\mathsf{ring}$ associated to the category CartSp, which are to the C^∞-rings of the previous chapter as connective $\mathbb{E}_\infty$-algebras over a field k of characteristic 0 are to ordinary commutative k-algebras. The relevance of this theory to the derived C^∞-geometry we are in the business of developing is provided by the following result, which has appeared before in joint work with David Carchedi [CS19].

Theorem. *Let $C^\infty(\_):\mathcal{T}_{\mathrm{Diff}}\to sC^\infty\mathsf{ring}^{op}$ be the obvious functor carrying a manifold to its simplicial C^∞-ring of smooth functions. Then $C^\infty(\_)$ factors through the full subcategory $sC^\infty\mathsf{ring}_{\mathrm{fp}}\subset sC^\infty\mathsf{ring}$ spanned by compact objects, and the resulting functor lies in $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}},sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op})$ and there is a natural structure of a geometry on $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ such that $C^\infty(\_)$ exhibits a geometric envelope, i.e. the ∞-category $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ 2-represents the functor $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}},\_)$.*

As a corollary, a (local) $\mathcal{T}_{\mathrm{Diff}}$-structure on an ∞-topos $\mathcal{X}$ is just a sheaf of (local) simplicial C^∞-rings on $\mathcal{X}$. Note that, remarkably, we need not impose any condition on the C^∞-rings corresponding to the criterion for $\mathcal{T}_{\mathrm{Diff}}$-structures that pullbacks along admissible maps should be preserved. The fact that derived C^∞-geometry is controlled by an algebraic theory has many convenient consequences; for instance, there is a homological algebraic model for derived C^∞-rings due to Carchedi and Roytenberg [CR12b; CR12a]: there is a model category $C^\infty\mathbf{dga}$ which simply consists of commutative dg algebras over $\mathbb{R}$ such that the degree 0 elements admit the structure of a C^∞-ring. Let $C^\infty\mathsf{Alg}$ denote the localization of $C^\infty\mathbf{dga}$ at the weak equivalences, then there is a canonical equivalence

$$sC^\infty\mathsf{ring}\longrightarrow C^\infty\mathsf{Alg},$$

the *C^∞-Dold-Kan correspondence*. Since dg algebras tend to be easier to manipulate from the point-set point of view than simplicial algebras, the model structure of Carchedi-Roytenberg imports powerful computational machinery into the theory. Nevertheless, we will mostly stick with a simplicial and intrinsically ∞-categorical formulation because it allows for easy comparison with other contexts of derived geometry where differential graded models are unavailable. For instance, the obvious transformation of pregeometries

$$\mathcal{T}_{\mathrm{Diff}}\hookrightarrow\mathcal{T}_{\mathrm{Diffc}}$$

of manifolds into manifolds with corners and interior b-maps, determines for each ∞-topos a functor $p_{\mathcal{X}}:\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diffc}}}(\mathcal{X})\to\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X})$. By the theorem above, we can identify a (local) $\mathcal{T}_{\mathrm{Diff}}$-structure $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ with a sheaf of (local) simplicial C^∞-rings on $\mathcal{X}$. As it turns out, it is possible to completely characterize the fibres of $p_{\mathcal{X}}$ in a more or less algebraic manner.

Theorem. *The functor $p_{\mathcal{X}}$ is a presentable fibration and the fibre $p_{\mathcal{X}}^{-1}(\mathcal{O}_{\mathcal{X}})$ may be described as follows: applying the assignment $\mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{\geq0}),\_)$ objectwise on $\mathcal{X}$ determines an object $(\mathcal{O}_{\mathcal{X}})_{\geq0}$, the* positive elements *of $\mathcal{O}_{\mathcal{X}}$. This object lifts canonically to a sheaf of simplicial commutative monoids on $\mathcal{X}$, and the fibre $p_{\mathcal{X}}^{-1}(\mathcal{O}_{\mathcal{X}})$ is identified with the ∞-category of sheaves of* logarithmic structures *on $(\mathcal{O}_{\mathcal{X}})_{\geq0}$.*

The logarithmic structures in this theorem are (derived versions of) those of Fontaine-Illusie and Kato [Kat89; Ogu18]. This theorem follows from a characterization of the geometric envelope of $\mathcal{T}_{\mathrm{Diffc}}$.

Theorem. *Let $sC^\infty\mathsf{Log}$ be the ∞-category of simplicial C^∞-rings equipped with logarithmic structures on their positive elements, then the ∞-category $sC^\infty\mathsf{Log}_{\mathrm{fp}}^{op}$ is a geometric envelope of a pregeometry Morita equivalent to $\mathcal{T}_{\mathrm{Diffc}}$.*

The first sections of this chapter are devoted to the results described above. With this structure theory in hand, we can apply the results from the previous chapter and obtain a variety of affine derived objects associated to the geometries we mentioned above, and obtain without further effort theories of (geometric) derived stacks, of which we

will give a number of examples relevant to future work related to moduli problems in C^∞-geometry.
The last part of this chapter is devoted to sheaves of modules over sheaves of simplicial C^∞-rings. With the cotangent complex and deformation theory of derived manifolds, the subject of the next chapter, in mind, we compare to definitions: one intrinsic to $sC^\infty\mathsf{ring}$, the fibrewise stabilization of $\mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}) \to sC^\infty\mathsf{ring}$, and one algebraic, via the forgetful functor $sC^\infty\mathsf{ring} \to \mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}}$. As $sC^\infty\mathsf{ring}$ is monadic over $\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}}$, this comparison is quite a bit easier than the analogous one in derived analytic geometry [PY17].

4.1 C^∞-Rings and Derived Differential Geometry

In this section, our main goal is to verify that C^∞-rings in spaces provide models for the geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$, and derive some elementary consequences.

4.1.1 Lawvere theories

This subsection may be regarded as an elaboration on section 5.5.8 of [Lur17b].

Definition 4.1.1.1. A *Lawvere theory* is a small ∞-category T with finite products. A *transformation of Lawvere theories* is a functor $f:\mathrm{T}\to\mathrm{T}'$ that preserves finite products. We let $\mathsf{LawThy}\subset\mathsf{Cat}_\infty$ denote the subcategory whose objects are Lawvere theories and whose morphisms are transformations of Lawvere theories.
Let T be a Lawvere theory. A *set of sorts for* T is a (small) set S together with an injective function $i:S\hookrightarrow\mathrm{Ob}_{h\mathrm{T}}$, such that every object of T is equivalent to a product of objects in the image of i. A Lawvere theory with a specified set of sorts S is an *S-sorted Lawvere theory.* If the set S is the subset $\{1,\dots,n\}\subset\mathbb{N}$, we call an S-sorted Lawvere theory an n-sorted Lawvere theory.

Definition 4.1.1.2. Let $\mathcal{X}$ be an ∞-topos and let T be Lawvere theory. A T*-algebra in* $\mathcal{X}$ is a product preserving functor $F:\mathrm{T}\to\mathcal{X}$. The full subcategory of $\mathrm{Fun}(\mathrm{T},\mathcal{X})$ spanned by T-algebras in $\mathcal{X}$ is denoted $\mathsf{TAlg}(\mathcal{X})$. T-algebras in the ∞-topos of spaces are called *simplicial* T*-algebras* and the ∞-category thereof is denoted $s\mathsf{TAlg}$.

Obviously, the ∞-category $\mathsf{TAlg}(\mathcal{X})$ has all limits and sifted colimits (which are computed objectwise in $\mathcal{X}$). [Lur17b], prop. 5.5.8.10 shows that $\mathsf{TAlg}(\mathcal{X})$ a compactly generated presentable ∞-category; there exists an accessible localization $L:\mathrm{Fun}(\mathrm{T},\mathcal{X})\to\mathsf{TAlg}(\mathcal{X})$ that carries compact objects to compact objects. For any $\mathcal{X}$, we have a canonical equivalence

$$\mathsf{TAlg}(\mathcal{X})\simeq\mathsf{Shv}_{s\mathsf{TAlg}}(\mathcal{X})\simeq s\mathsf{TAlg}\otimes\mathcal{X}.$$

For $\mathcal{X}=\mathcal{S}$, [Lur17b], lem. 5.5.8.14 shows that $s\mathsf{TAlg}$ is projectively generated by the essential image of the Yoneda embedding $\mathrm{T}^{op}\hookrightarrow s\mathsf{TAlg}$. Clearly, a transformation of Lawvere theories $f:\mathrm{T}\to\mathrm{T}'$ induces for each ∞-topos $\mathcal{X}$ a functor $f^*:\mathrm{T}'\mathsf{Alg}(\mathcal{X})\to\mathsf{TAlg}(\mathcal{X})$ preserving small limits and small sifted colimits. For $\mathcal{X}=\mathcal{S}$, the relationship between Lawvere theories and projectively generated presentable ∞-categories can be made very precise. For the following proposition, we note that a left adjoint $f:s\mathsf{TAlg}\to s\mathrm{T}'\mathsf{Alg}$ admits a right adjoint g that preserves sifted colimits if and only if f carries compact projective objects to compact projective objects. The only 'if direction' is an immediate check and for the other direction, it suffices to show that for each $t\in\mathrm{T}$ the composition

$$s\mathrm{T}'\mathsf{Alg}\longrightarrow s\mathsf{TAlg}\subset\mathsf{PShv}(\mathrm{T}^{op})\xrightarrow{\mathrm{ev}_t}\mathcal{S}$$

preserves sifted colimits, but this functor is corepresented by $f(t)$ which is compact projective by assumption.

Proposition 4.1.1.3. *Let $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}\subset\mathsf{Pr}^{\mathrm{L}}$ be the subcategory whose objects are projectively generated presentable ∞-categories and whose morphisms are functors admitting a right adjoint that preserves sifted colimits. Let $\mathsf{LawThy}^{\mathrm{Idem}}\subset\mathsf{LawThy}$ denote the full subcategory spanned by idempotent complete Lawvere theories.*

(1) *The construction*

$$\mathrm{T}\longmapsto s\mathsf{TAlg}$$

extends to an equivalence of ∞-categories $\mathsf{LawThy}^{\mathrm{Idem}}\simeq\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$.

(2) *The ∞-category $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$ is presentable.*

(3) *The ∞-category $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$ is semiadditive.*

(4) *The subcategory inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}\subset\mathsf{Pr}^{\mathrm{L}}$ preserves colimits.*

Proof. We first construct the functor $\mathsf{LawThy}^{\mathrm{Idem}}\to\mathsf{Pr}^{\mathrm{L}}$. For $\mathcal{K}$ a class of small simplicial sets, let $\mathsf{Cat}_\infty(\mathcal{K})$ ($\widehat{\mathsf{Cat}}_\infty(\mathcal{K})$) denote the (very) large ∞-category of small (large) ∞-categories that admit colimits indexed by simplicial sets in $\mathcal{K}$ and functors that preserve colimits indexed by elements in $\mathcal{K}$. Let $\mathcal{P}$ be the collection of finite discrete simplicial sets together with the ∞-category Idem and let $\mathcal{P}'$ be the collection of all small simplicial sets. It follows from [Lur17b], prop. 5.3.6.2 that the inclusion $i:\widehat{\mathsf{Cat}}_\infty(\mathcal{P}')\subset\widehat{\mathsf{Cat}}_\infty(\mathcal{P})$ admits a left adjoint L. If $\mathcal{C}$ is a *small* ∞-category that admits finite coproducts, then [Lur17b], prop. 5.5.8.15 asserts that the Yoneda embedding

$$\mathcal{C}\longhookrightarrow\mathrm{Fun}^\pi(\mathcal{C}^{op},\mathcal{S})=s\mathcal{C}^{op}\mathsf{Alg}$$

exhibits a unit transformation for the adjunction $(L\dashv i)$ so we conclude that the restriction of L to the full subcategory $\mathsf{LawThy}^{\mathrm{Idem}}\cong\mathsf{Cat}_\infty(\mathcal{P})\subset\widehat{\mathsf{Cat}}_\infty(\mathcal{P})$ takes values in the full subcategory of $\mathsf{Pr}^{\mathrm{L}}\subset\widehat{\mathsf{Cat}}_\infty(\mathcal{P}')$ spanned by projectively generated presentable ∞-categories. To see that a transformation of Lawvere theories is carried to a morphism in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$, we note that for a coproduct preserving functor $f : \mathcal{C} \to \mathcal{D}$ between the opposite categories of two Lawvere theories, we have a commuting diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, hook, "j_{\mathcal{C}}"] & \mathcal{D} \arrow[d, hook, "j_{\mathcal{D}}"] \\ s\mathcal{C}^{op}\mathsf{Alg} \arrow[r, "L(f)"] & s\mathcal{D}^{op}\mathsf{Alg} \end{tikzcd}$$

of coproduct preserving functor where moreover $L(f)$ preserves all colimits. The right adjoint to $L(f)$ preserves sifted colimits if and only if $L(f)$ carries compact projective objects to compact projective objects, so since [Lur17b], prop. 5.5.8.25 implies every compact projective object in $s\mathcal{C}^{op}\mathsf{Alg}$ is a retract of one in the image of $j_{\mathcal{C}}$ we conclude using the diagram above and the stability of compact projectives under retracts. This concludes the construction of the desired functor.
Now (2), (3) and (4) follow from (1) and the following assertions.

(2′) The ∞-category $\mathsf{Cat}_\infty(\mathcal{P})$ is presentable.

(3′) The ∞-category $\mathsf{Cat}_\infty(\mathcal{P})$ is semiadditive.

(4′) The functor $L|_{\mathsf{Cat}_\infty(\mathcal{P})} : \mathsf{Cat}_\infty(\mathcal{P}) \to \widehat{\mathsf{Cat}}_\infty(\mathcal{P}')$ preserves small colimits.

Assertion (2′) follows from [Lur17a], lem. 4.8.4.2. To prove the semiadditivity of $\mathsf{Cat}_\infty(\mathcal{P})$, we note that the assignments $C \mapsto (C, \varnothing_{\mathcal{D}})$ and $D \mapsto (\varnothing_{\mathcal{C}}, D)$ where $\varnothing_{\mathcal{C}}$ and $\varnothing_{\mathcal{D}}$ are initial objects $\mathcal{C}$ and $\mathcal{D}$ determine fully faithful inclusions $\mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{D}$ and $\mathcal{D} \hookrightarrow \mathcal{C} \times \mathcal{D}$ left adjoint to the projections. Let $\mathcal{A}$ be an ∞-category admitting finite coproducts and suppose that we are given coproduct preserving functors $f : \mathcal{C} \to \mathcal{A}$ and $g : \mathcal{D} \to \mathcal{A}$. Since the inclusion $\mathcal{C} \subset \mathcal{C} \times \mathcal{D}$ is a left adjoint, the ∞-category $\mathcal{C}_{/(s,t)}$ admits a final object $(s, (s, \varnothing) \to (s, t))$ for any $(s, t) \in \mathcal{C} \times \mathcal{D}$, so the functor f admits a left Kan extension $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$. Similarly, g admits a left Kan extension $G : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$. Composing the inclusions $\mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{D}$ and $\mathcal{D} \hookrightarrow \mathcal{C} \times \mathcal{D}$ with $F \coprod G$ yields the functors f and g, and given any other functor $H : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ compatible with f and g, we have a natural transformation $F \coprod G \to H$. This natural transformation is an equivalence whenever H preserves binary coproducts since $\mathcal{C}$ and $\mathcal{D}$ generate $\mathcal{C} \times \mathcal{D}$ under binary coproducts. We conclude that $\mathcal{C} \times \mathcal{D}$ is a coproduct of $\mathcal{C}$ and $\mathcal{D}$ in the homotopy category $h\mathsf{Cat}_\infty(\mathcal{P})$. For (4′), we only have to show that the inclusion $\mathsf{Cat}_\infty(\mathcal{P}) \subset \widehat{\mathsf{Cat}}_\infty(\mathcal{P})$ preserves small colimits since L preserves colimits, but this is obvious.
We are left to prove (1). For any projectively generated presentably ∞-category $\mathcal{C}$, the full subcategory $\mathcal{C}_0$ spanned by compact projective objects is idempotent complete and admits finite coproducts, and [Lur17b], prop. 5.5.8.25 asserts that $\mathcal{C} \simeq s\mathcal{C}_0^{op}\mathsf{Alg}$, so the functor $L|_{\mathsf{Cat}_\infty(\mathcal{P})}$ is essentially surjective. Now let $\mathcal{C}$ and $\mathcal{D}$ be idempotent complete ∞-categories admitting finite coproducts. Let $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ denote the full subcategory spanned by finite coproduct preserving functors, $\mathrm{Fun}'(s\mathcal{C}^{op}\mathsf{Alg}, s\mathcal{D}^{op}\mathsf{Alg})$ the full subcategory spanned by colimit preserving functors whose right adjoint preserves sifted colimits, and $\mathrm{Fun}'(\mathcal{C}, s\mathcal{D}^{op}\mathsf{Alg})$ the full subcategory spanned by functors preserving finite coproducts and taking values in compact projective objects. We have a commuting diagram

$$\begin{tikzcd} \mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \arrow[rr, "\theta"] \arrow[dr, "\theta'"'] & & \mathrm{Fun}'(s\mathcal{C}^{op}\mathsf{Alg}, s\mathcal{D}^{op}\mathsf{Alg}) \arrow[dl, "\theta''"] \\ & \mathrm{Fun}'(\mathcal{C}, s\mathcal{D}^{op}\mathsf{Alg}) & \end{tikzcd}$$

where the diagonal functors are induced by the Yoneda embeddings for $\mathcal{C}$ and $\mathcal{D}$. It suffices to show that the diagonal functors are equivalences. [Lur17b], prop. 5.5.8.15 implies that θ'' is an equivalence and since [Lur17b], prop. 5.5.8.25 asserts that the Yoneda embedding $\mathcal{D} \hookrightarrow s\mathcal{C}^{op}\mathsf{Alg}$ is an equivalence on the full subcategory spanned by compact projective objects in virtue of the assumption that $\mathcal{D}$ is idempotent complete, we deduce that the functor θ' is also an equivalence. □

We will refer to the functor $\mathrm{T} \mapsto s\mathrm{T}\mathsf{Alg}$ as the *sifted colimit completion.*

Remark 4.1.1.4. It follows from proposition 4.1.1.3 that the sifted colimit completion carries a product $\mathrm{T} \times \mathrm{T}'$ to the coproduct $s\mathrm{T}\mathsf{Alg} \coprod s\mathrm{T}\mathsf{Alg}$ in Pr^{L}, which is the product in $(\mathsf{Pr}^{\mathrm{L}})^{op} \simeq \mathsf{Pr}^{\mathrm{R}}$. Since $\mathsf{Pr}^{\mathrm{R}} \subset \widehat{\mathsf{Cat}}_\infty$ preserves limits, the functor $\mathrm{T} \mapsto s\mathrm{T}\mathsf{Alg}$ carries the product $\mathrm{T} \times \mathrm{T}'$ to the product $s\mathrm{T}\mathsf{Alg} \times s\mathrm{T}'\mathsf{Alg}$ so that $s\mathrm{T}\mathsf{Alg} \times s\mathrm{T}'\mathsf{Alg}$ is generated under sifted colimits by the coproduct preserving functor

$$\mathrm{T}^{op} \times \mathrm{T}'^{op} \xrightarrow{j \times j} s\mathrm{T}\mathsf{Alg} \times s\mathrm{T}'\mathsf{Alg}.$$

Note that this may fail for other limits in $\mathsf{LawThy}^{\mathrm{Idem}}$; for instance, the fibre product $s\mathrm{T}'\mathsf{Alg} \times_{s\mathrm{T}\mathsf{Alg}} s\mathrm{T}''\mathsf{Alg}$ in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$ is the sifted colimit completion of the Lawvere theory $\mathrm{T}' \times_{\mathrm{T}} \mathrm{T}''$ (the fibre product in Cat_∞), which need not coincide

with the fibre product in $\widehat{\mathsf{Cat}}_\infty$.
In general, proposition 4.1.1.3 shows that a limit of $K \to \mathsf{LawThy}^{\mathrm{Idem}}$ is obtained by taking the limit of the composition $K \to \mathsf{LawThy}^{\mathrm{Idem}} \subset \mathsf{Cat}_\infty$, while the colimit is obtained by taking the limit of the diagram $K^{op} \to (\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}})^{op} \subset \mathsf{Pr}^{\mathrm{R}} \subset \widehat{\mathsf{Cat}}_\infty$ and extracting the compact projective objects in the resulting ∞-category.

Remark 4.1.1.5. The sifted colimit completion may also be obtained by using the self-enrichment of $\widehat{\mathsf{Cat}}_\infty$: the functor $\mathrm{Fun}^\pi(\_, \mathcal{S})$ determines a functor $\mathsf{LawThy}^{\mathrm{Idem}} \to (\mathsf{Pr}^{\mathrm{R}})^{op}$ which coincides with the sifted colimit completion after passage to adjoints. We will not prove this rigorously, but give a few hints on how to proceed. First, one can repeat the proof of proposition 4.1.1.3 (minus the semiadditivity result) to obtain an equivalence between the ∞-category $\mathsf{Cat}^{\mathrm{Idem}}_\infty$ of small idempotent complete ∞-categories and the ∞-category $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{cc}}$ whose objects are presentable ∞-categories admitting a small set of *completely compact* objects and whose morphisms are left adjoints that admit a right adjoint that admits a further left adjoint. This equivalence is implemented by the (small) colimit completion functor L which carries $\mathcal{C}$ to $\mathsf{PShv}(\mathcal{C})$. The construction $\mathrm{Fun}((\_)^{op}, \mathcal{S})$ determines another colimit preserving functor from Cat_∞ to $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{cc}}$. Composing $\mathrm{Fun}((\_)^{op}, \mathcal{S})$ with the inverse of L, we obtain a colimit preserving functor $\mathsf{Cat}_\infty \to \mathsf{Cat}^{\mathrm{Idem}}_\infty$. It is not hard to see that this functor carries the full subcategory $\mathbf{N}(\mathbf{\Delta})$ to itself, which implies that L and $\mathrm{Fun}((\_)^{op}, \mathcal{S})$ are in fact equivalent (both are the functor $\mathcal{C} \mapsto \mathrm{Idem}(\mathcal{C})$). Restricting L (or $\mathrm{Fun}((\_)^{op}, \mathcal{S})$) to $\mathsf{Cat}_\infty(\mathcal{P})$ yields a functor $\mathsf{Cat}_\infty(\mathcal{P}) \to \widehat{\mathsf{Cat}}_\infty$. Let $\mathcal{Q} \to \mathsf{Cat}_\infty(\mathcal{P})$ be a coCartesian fibration associated to this functor, then one readily verifies that the sifted colimit completion and the functor $\mathrm{Fun}^\pi((\_)^{op}, \mathcal{S})$ determine the same full subcategory of $\mathcal{Q}$.

Remark 4.1.1.6. It is observed in [Lur17b], rmk. 5.5.8.26 that the n'th truncation $\tau_{\le n} s\mathrm{T}\mathsf{Alg}$ is precisely the full subcategory of functors $\mathrm{T} \to \mathcal{S}$ taking values in n-truncated objects. Since we have an equivalence $\mathrm{Fun}(\mathrm{T}, \tau_{\le 0}\mathcal{S}) \simeq \mathbf{N}(\mathrm{Fun}(h\mathrm{T}, \mathsf{Set}))$ and the functor $\mathrm{T} \to h\mathrm{T}$ preserves and reflects finite products, the 1-category $\tau_{\le 0} s\mathrm{T}\mathsf{Alg}$ can be identified with $\mathbf{N}(h\mathrm{T}\mathsf{Alg})$, the nerve of the category of $h\mathrm{T}$-algebras and we have a fully faithful inclusion $\mathbf{N}(h\mathrm{T}\mathsf{Alg}) \hookrightarrow s\mathrm{T}\mathsf{Alg}$. In turn, this inclusion determines a morphism $sh\mathrm{T}\mathsf{Alg} \to s\mathrm{T}\mathsf{Alg}$ in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$. Quite often, this functor is not an equivalence, as the following example shows.

The Lawvere theories below are the basic ones we deal with in this work.

Example 4.1.1.7. Let $\mathcal{C}^\otimes$ be a symmetric monoidal projectively generated presentable ∞-category such that the tensor product commutes with colimits separately in each variable, then the ∞-category of $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ is projectively generated. To see this, we note that forgetful functor $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \to \mathcal{C}$ preserves limits and sifted colimits by [Lur17a], cor. 3.2.2.3 and 3.2.3.2 and admits a left adjoint, the free commutative algebra functor $\mathrm{Sym}^\bullet$. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory spanned by a collection of compact projective generators stable under coproducts, and let $F(\mathcal{C}_0) \subset \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ be the essential image of $\mathcal{C}_0$ under $\mathrm{Sym}^\bullet$, then it follows from [Lur17a], prop. 7.1.4.12 that the inclusion $F(\mathcal{C}_0) \subset \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ induces an equivalence $sF(\mathcal{C}_0)^{op}\mathsf{Alg} \simeq \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ (all this actually holds for algebras for an arbitrary ∞-operad in the symmetric monoidal ∞-category $\mathcal{C}^\otimes$). Now suppose that

(1) The full subcategory $\tau_{\le 0}\mathcal{C}$ is stable under the tensor product,

then $\tau_{\le 0}\mathcal{C} \subset \mathcal{C}$ is symmetric monoidal, and the fully faithful inclusion $\mathbb{E}_\infty\mathsf{Alg}(\tau_{\le 0}\mathcal{C}) \subset \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ can be identified with the nerve of the category $hF(\mathcal{C}_0)\mathsf{Alg}$ as the full subcategory spanned by 0-truncated objects, since the forgetful functor $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \to \mathcal{C}$ preserves and detects truncations, which follows from remark 4.1.1.6. We have a (strictly) commuting diagram

$$\begin{tikzcd} \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}) \arrow[r] & \mathcal{C} \\ \mathbb{E}_\infty\mathsf{Alg}(\tau_{\le 0}\mathcal{C}) \arrow[r] \arrow[u, hook] & \tau_{\le 0}\mathcal{C} \arrow[u, hook] \end{tikzcd}$$

of right adjoints. If we also suppose that

(2) For each $X \in \tau_{\le 0}$, the Σ_n-coinvariants of $X^{\otimes n}$ are 0-truncated,

then this diagram is horizontally left adjointable. If we moreover assume that

(3) The ∞-category $\mathcal{C}_0$ lies in the full subcategory $\tau_{\le 0}\mathcal{C} \subset \mathcal{C}$,

then $\mathbb{E}_\infty\mathsf{Alg}(\mathcal{C})$ is generated under sifted colimits by the image of the functor

$$\mathcal{C}_0 \xrightarrow{\mathrm{Sym}^\bullet} \mathbb{E}_\infty\mathsf{Alg}(\tau_{\le 0}\mathcal{C}) \hookrightarrow \mathbb{E}_\infty\mathsf{Alg}(\mathcal{C}).$$

If k is a field of containing $\mathbb{Q}$, then (1), (2) and (3) hold for $\mathcal{C} = \mathsf{Mod}^{\mathrm{cn}}_k$, the ∞-category of connective k-modules, so $\mathbb{E}_\infty\mathsf{Alg}_k$ can be identified with $s\mathrm{T}\mathsf{Alg}$, where T is the opposite of the category of discrete k-algebras of the form $k[x_1, \dots, x_n]$. We may also view T as the *discrete pregeometry* $\mathcal{T}^{\mathrm{disc}}_k$ whose objects are affine k-spaces $\mathbb{A}^n_k$ and whose morphisms are polynomial maps among them.
When $\mathcal{C}^\otimes = \mathcal{S}^\times$, only (1) and (3) hold. $\mathcal{S}$ is projectively generated by its full subcategory $\mathbf{N}(\mathsf{Fin})$ of finite discrete

spaces, and we can characterize its essential image under $\mathrm{Sym}^\bullet$ as a certain $(2,1)$-category $\mathcal{F} \subset \mathbb{E}_\infty \mathsf{Alg}(\mathcal{S}) = \mathsf{Mon}_{\mathbb{E}_\infty}$ whose objects are parametrized by $\mathbb{Z}_{\geq 0}$, and whose morphisms are disjoint unions of classifying spaces of symmetric groups. For instance, we have $\mathrm{Hom}_{\mathcal{F}}(0,0) \simeq \coprod_n B\Sigma_n$. Then $s\mathcal{F}^{op}\mathsf{Alg} \simeq \mathsf{Mon}_{\mathbb{E}_\infty}$, and using remark 4.1.1.6 and the diagram above, we deduce that the homotopy category of $\mathcal{F}$ must coincide with the Lawvere theory FCMon of free commutative monoids. We will let $s\mathsf{CMon}$ denote the ∞-category of *simplicial commutative monoids*, the algebras for the 1-sorted Lawvere theory $\mathbf{N}(\mathsf{FCMon})$. The transformation of Lawvere theories $\mathcal{F} \to \mathbf{N}(\mathsf{FCMon})$ induces a functor $s\mathsf{CMon} \to \mathsf{Mon}_{\mathbb{E}_\infty}$. This functor is not an equivalence, but it is conservative; in fact, it is both monadic and comonadic.

Example 4.1.1.8. The category CartSp whose set of objects is $\{\mathbb{R}^k;\, k \in \mathbb{Z}_{\geq 0}\}$ and whose morphisms are smooth maps is a Lawvere theory, generated under finite products by $\mathbb{R}$. A CartSp-algebra in an ∞-topos $\mathcal{X}$ is called a *C^∞-ring in $\mathcal{X}$*.

Example 4.1.1.9. The category CartSp_c whose set of objects is $\{\mathbb{R}^k \times \mathbb{R}^m_{\geq 0};\, k, m \in \mathbb{Z}_{\geq 0}\}$ and whose morphisms are *interior b-maps* is a 2-sorted Lawvere theory, generated under finite products by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$. A CartSp_c-algebra in an ∞-topos $\mathcal{X}$ is called a *C^∞-ring with pre-corners in $\mathcal{X}$*.

The obvious functors $\mathcal{T}_{\mathbb{R}}^{\mathrm{disc}} \hookrightarrow \mathsf{CartSp} \hookrightarrow \mathsf{CartSp}_c$ show that every C^∞-ring with pre-corners in $\mathcal{X}$ has an underlying C^∞-ring, and every C^∞-ring has an underlying commutative $\mathbb{R}$-algebra.
Anticipating the results in the next subsection, we develop here the theory of simplicial T-algebras a bit. The ∞-category $s\mathrm{T}\mathsf{Alg}$ is far from being an ∞-topos (colimits are not universal), but it does have a few redeeming features: as limits and sifted colimits are computed in the ∞-topos $\mathsf{PShv}(\mathrm{T}^{op})$ and geometric realizations are sifted colimits, we see that the ∞-category $s\mathrm{T}\mathsf{Alg}$ inherits the following properties from $\mathsf{PShv}(\mathrm{T}^{op})$.

Proposition 4.1.1.10. *Let* T *be a Lawvere theory.*

(1) *Sifted colimits are universal in* $s\mathrm{T}\mathsf{Alg}$.

(2) *Every groupoid object in* $s\mathrm{T}\mathsf{Alg}$ *is effective.*

(3) $s\mathrm{T}\mathsf{Alg}$ *has an* epi-mono factorization system*: there exists a factorization system* (S_L, S_R) *on* $s\mathrm{T}\mathsf{Alg}$ *such that* S_L *consists of effective epimorphisms and* S_R *consists of monomorphisms.*

(4) *For each small* sifted *simplicial set* K *and each natural transformation* $\overline{\alpha} : p \to q$ *between functors* $p, q : K^{\triangleright} \to s\mathrm{T}\mathsf{Alg}$ *the following holds: if* q *is a colimit diagram and* $\overline{\alpha}|_K$ *is a Cartesian transformation, then* p *is a colimit diagram if and only if* α *is a Cartesian transformation.*

Remark 4.1.1.11. The previous proposition shows that for every Lawvere theory T, the ∞-category $s\mathrm{T}\mathsf{Alg}$ is *differentiable* in the sense of [Lur17a], defn. 6.1.1.6. This observation will be important when we deal with modules of simplicial T-algebras. Using [Lur17b], prop 1.2.13.8, it's easy to show that the ∞-category $s\mathrm{T}\mathsf{Alg}_{/A}$ is also differentiable for any $A \in s\mathrm{T}\mathsf{Alg}$.

Remark 4.1.1.12. Let T be an S-sorted Lawvere theory, then we may associate to any simplicial T-algebra A a collection of homotopy sets as follows: for each object s of T in the image of $i : S \to \mathrm{Ob}_T$, there is a functor $\theta_s : s\mathrm{T}\mathsf{Alg} \to \mathcal{S}$ given by evaluating at s. It is customary to identify a simplicial T-algebra with the S-tuple of spaces $(\theta_s(A))_{s \in i(S)}$: we will usually denote the C^∞-ring of smooth functions on a manifold M (possibly with corners) as $C^\infty(M)$ and the C^∞-ring with pre-corners as $(C^\infty(M), C^\infty_b(M))$. For each $n \geq 0$ and $s \in S$, we denote by $\pi_n(A)_s$ the n'th homotopy set of $\theta_s(A)$ which is an abelian group for $n \geq 1$. By the previous remark, the homotopy sets $\pi_0(A)_s$ can be identified with $\tau_{\leq 0} A(s)$ and if T is a 1-category, the S-tuple $\pi_0(A) \coloneqq (\pi_0(A)_s)_{s \in i(S)}$ carries the structure of an ordinary T-algebra; we will use both notations in the sequel.

Remark 4.1.1.13. From the generating properties of the objects $s \in i(S)$ we deduce immediately that the functor

$$\theta : s\mathrm{T}\mathsf{Alg} \xrightarrow{\prod_{s \in i(S)} \theta_s} \prod_{s \in i(S)} \mathcal{S}$$

is conservative. Combining this observation with the fact that each θ_s preserves geometric realizations and [Lur17b], corollary 7.1.2.15, we see that a morphism $A \to B$ of simplicial T-algebras is an effective epimorphism if and only if the induced map $\pi_0(A) \to \pi_0(B)$ (with the notation from the previous remark) on sets is surjective.

Remark 4.1.1.14. By Yoneda, the space $\mathrm{Hom}_{s\mathrm{T}\mathsf{Alg}}(j(s), A)$ coincides with $\theta_s(A)$, where $j : \mathrm{T}^{op} \to s\mathrm{T}\mathsf{Alg}$ is the Yoneda embedding; it follows that there is a bijection of sets $\mathrm{Hom}_{hs\mathrm{T}\mathsf{Alg}}(j(s), A) \simeq \pi_0(A)_s$.

Definition 4.1.1.15. Let T be a Lawvere theory. A simplicial T-algebra A is

(1) *finitely generated* if the functor $\mathrm{Hom}_{s\mathrm{TAlg}}(A, \_) : s\mathsf{TAlg} \to \mathcal{S}$ corepresented by A preserves colimits of small filtered diagrams consisting only of monomorphisms.

(2) *finitely presented* or *compact* if the functor $\mathrm{Hom}_{s\mathrm{TAlg}}(A, \_) : s\mathsf{TAlg} \to \mathcal{S}$ corepresented by A preserves small filtered colimits, that is, if A is a compact object.

(3) *finitely presented and projective* or *compact projective* if the functor $\mathrm{Hom}_{s\mathrm{TAlg}}(A, \_) : s\mathsf{TAlg} \to \mathcal{S}$ corepresented by A preserves small sifted colimits.

(4) *almost finitely presented* if for all $n \geq 0$, $\tau_{\leq n} A$ is finitely presented in $\tau_{\leq n} s\mathsf{TAlg}$.

Remark 4.1.1.16. Let $\mathcal{C}$ be a compactly generated presentable ∞-category and let $\mathcal{J} : K \to \mathcal{C}$ be a small filtered diagram consisting only of monomorphisms. Then the map $\mathcal{J}(k) \to \mathrm{colim}_{k \in K'} \mathcal{J}(k')$ is a monomorphism for each $k \in K$. To see this, we write $\mathcal{C}$ as an accessible localization of an ∞-category $\mathsf{PShv}(\mathcal{C}_0)$ with the property that the inclusion $\mathcal{C} \subset \mathsf{PShv}(\mathcal{C}_0)$ preserves filtered colimits. As monomorphisms are detected in $\mathsf{PShv}(\mathcal{C}_0)$, we may replace $\mathcal{C}$ with $\mathsf{PShv}(\mathcal{C}_0)$; then we see that we may actually assume that $\mathcal{C} = \mathcal{S}$, in which case it is obvious. It follows that the map $\mathrm{colim}_{k \in K} \mathrm{Hom}_{\mathcal{C}}(C, \mathcal{J}(k)) \to \mathrm{Hom}_{\mathcal{C}}(C, \mathrm{colim}_{k \in K} \mathcal{J}(k))$ of spaces is also a monomorphism for any $C \in \mathcal{C}$. This latter map is thus an equivalence if and only if each map $C \to \mathrm{colim}_{k \in K} \mathcal{J}(k)$ factors through some $\mathcal{J}(k')$. Thus, C is finitely generated if and only if this latter condition is satisfied for all small filtered diagrams in $\mathcal{C}$ consisting only of monomorphisms.

Remark 4.1.1.17. As is standard in the theory of higher algebraic structures, there is a family of conditions of increasing strength indexed by the natural numbers between the condition of finite generation and that of finite presentation: let $n \in Z_{\geq -1}$, then we say that a simplicial T-algebra A is *finitely n-presented* if the functor $s\mathsf{TAlg} \to \mathcal{S}$ corepresented by A preserves colimits of small filtered diagrams that factor through the subcategory spanned by n-truncated morphisms.

In ordinary commutative algebra, an algebra A is finitely generated if there is some free algebra F on finitely many generators and a quotient map $F \to A$. The following proposition shows that the same principle can be applied to finitely generated T-algebras, with the caveat that an effective equivalence relation must be replaced by an effective groupoid.

Proposition 4.1.1.18. *Let* T *be an S-sorted Lawvere theory, and let A be a simplicial* T*-algebra. The following are equivalent.*

(1) A is finitely generated.

(2) There exists an object t of T *and an effective epimorphism $q : j(t) \to A$, where $j : \mathrm{T}^{op} \to s\mathsf{TAlg}$ is the Yoneda embedding.*

Proof. We start by proving that (1) ⇒ (2). Let A be finitely generated, and let $\mathrm{Sub}(A)$ be the (small) filtered poset of equivalence classes of subobjects of A. Let $\mathrm{Sub}'(A)$ be the subposet of $\mathrm{Sub}(A)$ spanned by subobjects of A that satisfy condition (2), which is nonempty (because every map $j(t) \to A$ factors as an effective epimorphism followed by a monomorphism) and is easily seen to be filtered. We claim that A is the colimit of the diagram

$$\mathcal{J} : \mathrm{Sub}'(A) \subset \mathrm{Sub}(A) \simeq \tau_{\leq -1} s\mathsf{TAlg}_{/A} \longrightarrow s\mathsf{TAlg}.$$

By proposition 4.1.1.24, the ∞-category $s\mathsf{TAlg}_{/A}$ is compactly generated so the ∞-categories $\tau_{\leq k} s\mathsf{TAlg}_{/A}$ are stable under filtered colimits for $k \geq -2$, so the map $\mathrm{colim}_{A_i \in \mathrm{Sub}'(A)} A_i \to A$ is a monomorphism, meaning that for each $s \in S$ (the minimal set of sorts), the map of spaces $\theta_s : \mathrm{colim}_{A_i \in \mathrm{Sub}'(A)} A_i(s) \to A(s)$ is an inclusion of connected components (here we use that the evaluation functors preserve filtered colimits). The evaluation functor $\theta : s\mathsf{TAlg} \to \prod_S \mathcal{S}$ of remark 4.1.1.13 is conservative, so it suffices to show that for all $s \in S$, the morphism $\mathrm{colim}_{A_i \in \mathrm{Sub}'(A)} A_i(s) \to A(s)$ is an equivalence. To see this, we only have to check that this morphism induces a surjection on connected components, meaning that every morphism $j(s) \to A$ factors through some $B \in \mathrm{Sub}'(A)$. This is the case as $j(s) \to A$ factors as an effective epimorphism followed by a monomorphism. Because A is finitely generated, we have $\mathrm{Hom}_{s\mathrm{TAlg}}(A, A) \simeq \mathrm{colim}_{A_i \in \mathrm{Sub}'(A)} \mathrm{Hom}_{s\mathrm{TAlg}}(A, A_i)$, so the identity map $A \to A$ factors as $A \xrightarrow{f} A_i \to A$ for some $A_i \in \mathrm{Sub}'(A)$. The map $A_i \to A$ is thus a monomorphism and an effective epimorphism, and therefore an equivalence.

Now we show that (2) ⇒ (1). Let $Y = \mathrm{colim}_{i \in I} Y_i$ be a colimit of a filtered diagram consisting only of monomorphisms. A map $A \to Y$ induces a map $j(t) \to Y$ which must factor through one of the Y_i's as $j(t)$ is a compact projective object in $s\mathsf{TAlg}$. Because $Y_i \to Y$ is a monomorphism and the class of effective epimorphisms is left orthogonal to the class of monomorphisms, we can find the dotted arrow that makes the diagram

$$\begin{tikzcd} j(t) \arrow[r] \arrow[d] & Y_i \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

commute, which proves that A is finitely generated, after remark 4.1.1.16. $\square$

Remark 4.1.1.19. Let A be a simplicial T-algebra. A surjective map $j(t) \to \pi_0(A)$ in $h\mathrm{T}$ determines an element in $\pi_0(A)(t) \simeq \mathrm{Hom}_{hs\mathsf{TAlg}}(j(t), A)$, so we get an effective epimorphism $j(t) \to A$ by remark 4.1.1.13, defined up to homotopy. It follows from the previous lemma that A is finitely generated if and only if $\pi_0(A)$ is finitely generated as an ordinary hT-algebra.

The following results give alternative characterizations of the full subcategory of finitely presented T-algebras.

Lemma 4.1.1.20. *Let* T *be a Lawvere theory. The full subcategory of finitely presented* T*-algebras is the smallest full subcategory of* $s\mathsf{TAlg}$ *that contains the essential image of the embedding* $\mathrm{T}^{op} \hookrightarrow s\mathsf{TAlg}$ *and is stable under finite colimits and retracts.*

Proof. Let $\mathcal{C}$ be the smallest full subcategory of $s\mathsf{TAlg}$ that contains the essential image of the fully faithful embedding $j : \mathrm{T}^{op} \hookrightarrow s\mathsf{TAlg}$ and is stable under finite colimits and retracts. Since $s\mathsf{TAlg}_{\mathrm{fp}}$ is stable under finite colimits and retracts and contains the objects of T^{op} as a set of compact projective generators of $s\mathsf{TAlg}$, we have $\mathcal{C} \subset s\mathsf{TAlg}_{\mathrm{fp}}$. To establish the other inclusion, we show that every finitely presented simplicial T-algebra is a retract of a finite colimit of objects in $j(\mathrm{T}^{op})$. Any simplicial T-algebra is a small colimit of free T-algebras, the objects in $j(\mathrm{T}^{op})$. By decomposing the index simplicial set K of a small colimit into the partially ordered set of finite simplicial subsets of K, we may write the colimit of K as a filtered colimit of finite colimits ([Lur17b], cor. 4.2.3.11). Applying this to a finitely presented simplicial T-algebra A, we have a filtered colimit $A = \mathrm{colim}_{\,i\in\mathcal{J}} A_i$, where each A_i is a finite colimit of free simplicial T-algebras. Because A is finitely presented, the identity map $A \to A$ factors trough some $A_i \to A$ which shows that the desired retraction exists. $\square$

Remark 4.1.1.21. For the example T = CartSp that will receive our attention in the coming sections, the finite colimits in the theorem above can be chosen to be of special type: every simplicial C^∞-ring A admits a presentation as a *cell object*, that is, a directed colimit of pushouts along maps of the form $\Sigma^m C^\infty(V) \to \mathbb{R}$ for V a (possibly infinite-dimensional) vector space.

Proposition 4.1.1.22. *Let* T *be a Lawvere theory. For each idempotent complete* ∞*-category* $\mathcal{C}$ *that admits finite limits, the restriction map*

$$\theta : \mathrm{Fun}^{\mathrm{lex}}((s\mathsf{TAlg}_{\mathrm{fp}})^{op}, \mathcal{C}) \longrightarrow \mathrm{Fun}^{\pi}(\mathrm{T}, \mathcal{C})$$

induced by the fully faithful embedding $j : \mathrm{T} \to (s\mathsf{TAlg}_{\mathrm{fp}})^{op}$ *is an equivalence.*

Proof. The Yoneda embedding $\mathcal{C} \hookrightarrow \mathsf{PShv}(\mathcal{C})$ induces a commuting diagram

$$\begin{tikzcd}
\mathrm{Fun}'(s\mathsf{TAlg}^{op}, \mathcal{C}) \arrow[r] \arrow[d, hook] & \mathrm{Fun}^{\mathrm{lex}}(s\mathsf{TAlg}^{op}_{\mathrm{fp}}, \mathcal{C}) \arrow[r] \arrow[d, hook] & \mathrm{Fun}^{\pi}(\mathrm{T}, \mathcal{C}) \arrow[d, hook] \\
\mathrm{Fun}'(s\mathsf{TAlg}^{op}, \mathsf{PShv}(\mathcal{C})) \arrow[r] & \mathrm{Fun}^{\mathrm{lex}}(s\mathsf{TAlg}^{op}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C})) \arrow[r, "j^*"] & \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))
\end{tikzcd}$$

where $\mathrm{Fun}'(s\mathsf{TAlg}^{op}, \mathcal{C})$ and $\mathrm{Fun}'(s\mathsf{TAlg}^{op}, \mathsf{PShv}(\mathcal{C}))$ denote full subcategories of functors preserving small limits. As $\mathsf{PShv}(\mathcal{C})$ admits small limits and the ∞-category $s\mathsf{TAlg}$ is compactly generated, left Kan extension induces an equivalence

$$\mathrm{Fun}^{\omega-\mathrm{cont}}(s\mathsf{TAlg}, \mathsf{PShv}(\mathcal{C})^{op}) \longrightarrow \mathrm{Fun}(s\mathsf{TAlg}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C})^{op}).$$

We first show that this functor restricts to an equivalence between functors $F : s\mathsf{TAlg} \to \mathsf{PShv}(\mathcal{C})^{op}$ preserving all colimits and right exact functors $f : s\mathsf{TAlg}_{\mathrm{fp}} \to \mathsf{PShv}(\mathcal{C})^{op}$. It is clear that if F preserves colimits, then the restriction $F|_{s\mathsf{TAlg}_{\mathrm{fp}}}$ is right exact. Now suppose that $f : s\mathsf{TAlg}_{\mathrm{fp}} \to \mathsf{PShv}(\mathcal{C})^{op}$ is right exact and let $F : s\mathsf{TAlg} \to \mathsf{PShv}(\mathcal{C})^{op}$ be a left Kan extension of f obtained by applying the inverse of the equivalence above. Let

$$F' : \mathsf{PShv}(s\mathsf{TAlg}_{\mathrm{fp}}) \longrightarrow \mathsf{PShv}(\mathcal{C})^{op}$$

be a left Kan extension of f along the Yoneda embedding $j : s\mathsf{TAlg}_{\mathrm{fp}} \hookrightarrow \mathsf{PShv}(s\mathsf{TAlg}_{\mathrm{fp}})$. We may identify $s\mathsf{TAlg}$ with the full subcategory $\mathrm{Ind}(s\mathsf{TAlg}_{\mathrm{fp}}) \subset \mathsf{PShv}(s\mathsf{TAlg}_{\mathrm{fp}})$ of right exact functors ([Lur17b], cor. 5.3.5.4), so that $F'|_{\mathrm{Ind}(s\mathsf{TAlg}_{\mathrm{fp}})}$ is identified with F. It follows from [Lur17b], prop. 5.5.2.9 and remark 5.5.2.10 that F' admits a right adjoint G. It suffices to prove that G factors through $\mathrm{Ind}(s\mathsf{TAlg}_{\mathrm{fp}})$, then G is also a right adjoint to $F'|_{\mathrm{Ind}(s\mathsf{TAlg}_{\mathrm{fp}})}$ which implies that F preserves colimits. But the value of G on some $X \in \mathsf{PShv}(\mathcal{C})^{op}$ is the presheaf

$$s\mathsf{TAlg}^{op}_{\mathrm{fp}} \xrightarrow{f^{op}} \mathsf{PShv}(\mathcal{C}) \xrightarrow{\mathrm{Hom}(X,\_)} \mathcal{S},$$

which is left exact as f^{op} is left exact. It follows that the lower horizontal functor

$$\mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}^{op}, \mathsf{PShv}(\mathcal{C})) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(s\mathrm{T}\mathsf{Alg}^{op}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C}))$$

is an equivalence, and the lower horizontal composition $\mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}^{op}, \mathsf{PShv}(\mathcal{C})) \to \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))$ is an equivalence by [Lur17b], prop. 5.5.8.13. To prove that the functor θ is an equivalence, it now suffices to show that for any left exact functors $f : s\mathrm{T}\mathsf{Alg}^{op} \to \mathsf{PShv}(\mathcal{C})$ such that $f|_{j(\mathrm{T})}$ takes values in the image $\mathcal{C}'$ of the Yoneda embedding $j : \mathcal{C} \to \mathsf{PShv}(\mathcal{C})$, then f also takes values in $\mathcal{C}'$. This follows because lemma 4.1.1.20 shows that every $A \in s\mathrm{T}\mathsf{Alg}^{op}_{\mathrm{fp}}$ is a retract of a finite limit of objects in $j(\mathrm{T})$ and $\mathcal{C}'$ is stable under finite limits and retracts in $\mathsf{PShv}(\mathcal{C})$ by assumption. □

Remark 4.1.1.23. The argument in the proof above can be used to show the following: let $\mathcal{C}$ be a κ-compactly generated ∞-category and let $\mathcal{C}_\kappa$ be the full subcategory spanned by κ-compact objects. Let $\mathcal{D}$ be a (not necessarily presentable) ∞-category that admits all small colimits. Then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves colimits if and only if it admits a right adjoint, and restriction along the inclusion $\mathcal{C}_\kappa \subset \mathcal{C}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\kappa\text{-rex}}(\mathcal{C}_\kappa, \mathcal{D})$$

where $\mathrm{Fun}^{\kappa\text{-rex}}(\mathcal{C}_\kappa, \mathcal{D})$ denotes the full subcategory spanned by functors preserving κ-small colimits.

We discuss slicing of ∞-categories of T-algebras.

Proposition 4.1.1.24. *Let $s\mathrm{T}\mathsf{Alg}$ be the ∞-category of algebras for a Lawvere theory* T*, and let $A \in s\mathrm{T}\mathsf{Alg}$. Then the overcategory $s\mathrm{T}\mathsf{Alg}_{/A}$ is equivalent to ∞-category of algebras for the Lawvere theory $(\mathrm{T}^{op}_{/A})^{op} := \mathrm{T}^{op} \times_{s\mathrm{T}\mathsf{Alg}^{op}} (s\mathrm{T}\mathsf{Alg}_{/A})^{op}$.*

Remark 4.1.1.25. It is easy to see (using [Lur17b], lem. 5.4.5.5 for instance) that $(\mathrm{T}^{op}_{/A})^{op}$ admits finite products and both the functors $(\mathrm{T}^{op}_{/A})^{op} \to \mathrm{T}$ and $(\mathrm{T}^{op}_{/A})^{op} \to s\mathrm{T}\mathsf{Alg}^{op}$ preserve finite products. Also note that because T is small and $s\mathrm{T}\mathsf{Alg}$ is locally small, the ∞-category $(\mathrm{T}^{op}_{/A})^{op}$ is essentially small.

Proof. In view of [Lur17b], prop. 5.5.8.22, it is sufficient to show that the fully faithful functor $\mathrm{T}^{op}_{/A} \hookrightarrow s\mathrm{T}\mathsf{Alg}_{/A}$ takes values in compact projective objects of $s\mathrm{T}\mathsf{Alg}_{/A}$ and that the essential image generates $s\mathrm{T}\mathsf{Alg}_{/A}$ under sifted colimits. Let $B \to A$ a morphism in $s\mathrm{T}\mathsf{Alg}$, then according to [Lur17b], lem. 5.5.8.13, we may choose a sifted diagram $K \to \mathrm{T}^{op} \to s\mathrm{T}\mathsf{Alg}$ with colimit B, determining a colimit diagram $K^{\triangleright} \to s\mathrm{T}\mathsf{Alg}$. Since the inclusion $K \star \Delta^0 \coprod_{\Delta^0} \Delta^1 \hookrightarrow K \star \Delta^1$ is inner anodyne, the map $K^{\triangleright} \to s\mathrm{T}\mathsf{Alg}$ lifts along the projection $p : s\mathrm{T}\mathsf{Alg}_{/A} \to s\mathrm{T}\mathsf{Alg}$ as a diagram $K^{\triangleright} \to \mathrm{T}^{op}_{/A} \to s\mathrm{T}\mathsf{Alg}_{/A}$ which is also a colimit diagram, as the right fibration p preserves and reflects colimits. It remains to be shown that each object of the form $j(t) \to A$ is compact projective. Let $\mathcal{J} : K \to s\mathrm{T}\mathsf{Alg}_{/A}$ be a sifted diagram, then we have a fibre sequence

$$\begin{tikzcd}
\mathrm{Hom}_{s\mathrm{T}\mathsf{Alg}_{/A}}(j(t), \operatorname{colim} \mathcal{J}) \arrow[r] \arrow[d] & \mathrm{Hom}_{s\mathrm{T}\mathsf{Alg}}(j(t), \operatorname{colim} \mathcal{J}) \arrow[d] \\
* \arrow[r] & \mathrm{Hom}_{s\mathrm{T}\mathsf{Alg}}(j(t), A)
\end{tikzcd}$$

where the fibre is taken at $j(t) \to A$, using that the functor p preserves and reflects colimits. We conclude using that $j(t)$ is compact projective in $s\mathrm{T}\mathsf{Alg}$ and the fact that colimits are universal in $\mathcal{S}$. □

Remark 4.1.1.26. Using that sifted colimits are universal in ∞-categories of algebras for Lawvere theories, it can be shown along the lines of the proof of Rezk descent for ∞-topoi ([Lur17b], section 6.1.3) that the functor $s\mathrm{T}\mathsf{Alg}^{op} \to \mathsf{Pr}^{\mathrm{L}} \subset \widehat{\mathsf{Cat}}_\infty$ associated to the Cartesian fibration $\mathrm{Fun}(\Delta^1, s\mathrm{T}\mathsf{Alg}) \to s\mathrm{T}\mathsf{Alg}$ preserves (co)sifted limits. As the ∞-category $\mathrm{T}^{op}_{/A}$ is sifted for each simplicial T algebra A, it follows that the functor

$$\left(\mathrm{T}^{op}_{/A}\right)^{op} \longrightarrow \widehat{\mathsf{Cat}}_\infty, \quad (j(t) \to A) \longmapsto s\mathrm{T}_{t/}\mathsf{Alg}$$

has limit $s\mathrm{T}\mathsf{Alg}_{/A}$.

Remark 4.1.1.27. Note that for any ∞-category $\mathcal{C}$ that admits binary coproducts, the projection $p : \mathcal{C}_{A/} \to \mathcal{C}$ admits a left adjoint given by taking the coproduct with A, for any object A of $\mathcal{C}$. If T is a Lawvere theory and $A \in s\mathrm{T}\mathsf{Alg}$ an object, consider the full subcategory of $s\mathrm{T}\mathsf{Alg}_{A/}$ spanned by objects of the form $A \coprod j(t)$ for $t \in \mathrm{T}^{op}$, denoted T^{op}_A. As taking the coproduct with A preserves colimits, T^{op}_A has finite coproducts so T_A is a Lawvere theory. The next proposition shows that the inclusion $\mathrm{T}^{op}_A \hookrightarrow s\mathrm{T}\mathsf{Alg}_{A/}$ induces an equivalence $s\mathrm{T}_A\mathsf{Alg} \to s\mathrm{T}\mathsf{Alg}_{A/}$, so we can think of morphisms $f : A \to B$ as simplicial algebras for the Lawvere theory T_A. In particular, the terminology and results of this section hold for maps of simplicial T-algebras. For instance, a map $f : A \to B$ of simplicial T-algebras is *finitely presented* if f is compact in $s\mathrm{T}\mathsf{Alg}_{A/}$, and this is the case if and only if f is a retract of a finite colimit of morphisms of the form $A \to A \coprod j(t)$, by lemma 4.1.1.20.

Proposition 4.1.1.28. *Let* T *be a Lawvere theory, then the inclusion* $\mathrm{T}_A^{op} \hookrightarrow s\mathsf{TAlg}_{A/}$ *of the previous remark induces an equivalence* $s\mathrm{T}_A\mathsf{Alg} \simeq s\mathsf{TAlg}_{A/}$.

We need the following lemma.

Lemma 4.1.1.29. *Let* K *be a weakly contractible simplicial set and let* $\mathcal{C}$ *be an* ∞*-category that admits* K*-indexed colimits. For each* $A \in \mathcal{C}$*, the* ∞*-category* $\mathcal{C}_{A/}$ *admits* K*-indexed colimits and the projection* $p : \mathcal{C}_{A/} \to \mathcal{C}$ *preserves and reflects* K*-indexed colimits.*

Proof. As categorical equivalences are left cofinal, we may suppose without loss that K is an ∞-category. Let $\tau : K \to \mathcal{C}_{A/}$ be a diagram, then an object $Z \in (\mathcal{C}_{A/})_{K/}$ is a colimit of τ if the projection $((\mathcal{C}_{A/})_{K/})_{Z/} \to (\mathcal{C}_{A/})_{K/}$ is a trivial Kan fibration. The diagram τ is equivalent to a diagram $\overline{\tau} : K^{\triangleleft} \to \mathcal{C}$ sending the cone vertex to A, and we have an isomorphism of simplicial sets $(\mathcal{C}_{A/})_{K/} \cong \mathcal{C}_{K^{\triangleleft}/}$, so Z is a colimit of τ in $\mathcal{C}_{A/}$ if and only if Z is a colimit of $\overline{\tau}$ in $\mathcal{C}$. We will be done once we show that Z is a colimit of $\overline{\tau}$ if and only if Z is a colimit of $p \circ \tau$. For this, it suffices to check that the inclusion $K \hookrightarrow K^{\triangleleft}$ is left cofinal. We need to show that $K \times_{K^{\triangleleft}} K^{\triangleleft}_{v/}$ is weakly contractible for every vertex $v \in K^{\triangleleft}$. If $v \in K$, then $K \times_{K^{\triangleleft}} K^{\triangleleft}_{v/} \cong K_{v/}$ which admits an initial object and is thus weakly contractible. If $v = \infty$, the cone vertex in $K^{\triangleleft}$, then $K \times_{K^{\triangleleft}} K^{\triangleleft}_{v/} \cong K$ which is weakly contractible by assumption. ☐

Proof of Proposition 4.1.1.28. In view of [Lur17b], prop 5.5.8.22, we need only check that for all $t \in \mathrm{T}^{op}$ the functor out of $s\mathsf{TAlg}_{A/}$ corepresented by $A \coprod j(t)$ preserves sifted colimits and that the collection of objects $\{A \coprod j(t)\}_{t \in \mathrm{T}^{op}}$ generates $s\mathsf{TAlg}_{A/}$ under sifted colimits. For the first assertion, we take a sifted diagram $\tau : K \to s\mathsf{TAlg}_{A/}$ and we observe that by adjunction $\mathrm{Hom}_{s\mathsf{TAlg}_{A/}}(A \coprod j(t), \mathrm{colim}\ \tau) \simeq \mathrm{Hom}_{s\mathsf{TAlg}}(j(t), p(\mathrm{colim}\ \tau))$, where $p : s\mathsf{TAlg}_{A/} \to s\mathsf{TAlg}$ is the projection. Since K is sifted and thus weakly contractible, $p(\mathrm{colim}\ \tau) \simeq \mathrm{colim}\ p \circ \tau$ by lemma 4.1.1.29. Now the conclusion follows because the functor $\mathrm{Hom}_{s\mathsf{TAlg}}(j(t), \_)$ preserves sifted colimits.
Now we check that $s\mathsf{TAlg}_{A/}$ is generated under sifted colimits by T_A. Note that the projection $p : s\mathsf{TAlg}_{A/} \to s\mathsf{TAlg}$ is conservative, preserves limits and sifted colimits by lemma 4.1.1.29 and is therefore monadic by Lurie's Barr-Beck theorem. The corresponding monad M_A is simply taking the coproduct with A, and it follows that every object $A \to B \in s\mathsf{TAlg}_{A/}$ is the colimit of its Bar resolution $\mathrm{Bar}_{M_A}(M_A, B)_\bullet$. Each term in the resolution is of the form $A \coprod X$ for some $X \in s\mathsf{TAlg}$, so if we write $X \simeq \mathrm{colim}\ \mathcal{J}$ for some sifted diagram $\mathcal{J} : K \to \mathrm{T}^{op}$, then $A \coprod X$ is the colimit of the diagram $K \xrightarrow{\mathcal{J}} \mathrm{T}^{op} \xrightarrow{A \coprod -} \mathrm{T}_A^{op} \hookrightarrow s\mathsf{TAlg}_{A/}$. ☐

Remark 4.1.1.30. For $f : A \to B$ a map of simplicial T-algebras, we have an adjunction

$$(f_! \dashv f^*) : \ s\mathsf{TAlg}_{A/} \rightleftarrows s\mathsf{TAlg}_{B/}$$

where the left adjoint $f_!$ is base change along f and the right adjoint f^* is the functor composing with f. The functor $f_!$ restricts to a coproduct preserving functor $f_!|_{\mathrm{T}_A^{op}} : \mathrm{T}_A^{op} \to \mathrm{T}_B^{op}$, and the adjunction above is obtained from the transformation of Lawvere theories $(f_!|_{\mathrm{T}_A^{op}})^{op} : \mathrm{T}_A \to \mathrm{T}_B$, in the sense that the functor f^* is given by composition with $(f_!|_{\mathrm{T}_A^{op}})^{op}$ when we think of simplicial T_A-algebras as product preserving presheaves on T_A^{op}.

4.1.2 Resolutions of diagrams and unramified transformations

When comparing two Lawvere theories T and T', or more generally two pregeometries $\mathcal{T}$ and $\mathcal{T}'$, a basic question that arises is the following:

- *when does a transformation* $f : \mathrm{T} \to \mathrm{T}'$ *preserve pushouts in* $s\mathrm{T}'\mathsf{Alg}$*?*

For pregeometries, the answer to this question depends on whether or not $\mathcal{T}$ and $\mathcal{T}'$ admit a well behaved theory of closed immersions. For Lawvere theories, the situation is simpler, as all Lawvere theories have well behaved effective epimorphisms. A transformation f then preserves certain pushout diagrams if the following condition is satisfied.

Definition 4.1.2.1. Let T and T′ be Lawvere theories. A morphism in T′ is a *graph inclusion* if it is equivalent to a morphism of the form $\mathrm{id}_X \times f : X \to X \times Y$ for some $f : X \to Y$. A transformation of Lawvere theories $f : \mathrm{T} \to \mathrm{T}'$ is *unramified* if for each graph inclusion $g : X \to X \times Y$ in T′ and each $Z \in \mathrm{T}'$, the diagram

$$\begin{tikzcd} f^*j(X \times Y) \arrow[r] \arrow[d] & f^*j(X \times Y \times Z) \arrow[d] \\ f^*j(X) \arrow[r] & f^*j(X \times Z) \end{tikzcd}$$

is a pushout in $s\mathsf{TAlg}$, where the upper horizontal map is induced by the projection $X \times Y \times Z \to X \times Y$.

The significance of this definition is explained by the following theorem.

Theorem 4.1.2.2. *Let $f : \mathrm{T} \to \mathrm{T}'$ be an unramified transformation of Lawvere theories. Then $f^* : s\mathrm{T}'\mathsf{Alg} \to s\mathrm{T}\mathsf{Alg}$ preserves pushouts along effective epimorphisms.*

The proof of this theorem requires some preparation: we need to resolve an arbitrary effective epimorphism by maps of the type appearing in definition 4.1.2.1. Such a resolution is constructed for an arbitrary pregeometry $\mathcal{T}$ in [Lur11a], sections 2 and 3, but we have no need of that generality, so we only treat the case of Lawvere theories, using somewhat different methods[1]. These resolutions turn out to be a remarkably powerful technical device in itself, for which we will find many uses. We formalize it in the following proposition.

Proposition 4.1.2.3 (Free resolutions of effective epimorphisms)**.** *Let T be a Lawvere theory. Let $\mathcal{C}$ be an ∞-category that admits sifted colimits and let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory stable under sifted colimits. Suppose we are given a functor $F : \mathrm{Fun}(\Delta^1, s\mathrm{T}\mathsf{Alg}) \to \mathcal{C}$ such that*

(1) *F preserves sifted colimits.*

(2) *For every graph inclusion $g : X \to X \times Y$ of free simplicial T-algebras, the object $F(j(g))$ lies in $\mathcal{C}_0$.*

Then F carries every effective epimorphism of $s\mathrm{T}\mathsf{Alg}$ into $\mathcal{C}_0$.

Now we consider the slightly more specialized situation of pushouts along effective epimorphisms.

Definition 4.1.2.4. A diagram $\tau : \Lambda^2_0 \to s\mathrm{T}\mathsf{Alg}$ in the ∞-category of simplicial T algebras is *elementary* if it is equivalent to a diagram of the form

$$\begin{array}{ccc} j(X \times Y) & \longrightarrow & j(X \times Y \times Z) \\ \big\downarrow & & \\ j(X) & & \end{array}$$

for some morphism graph inclusion $X \to X \times Y$ in T and some $Z \in \mathrm{T}$.

Proposition 4.1.2.5. *Let T be a Lawvere theory. Let $\mathcal{C}$ be an ∞-category that admits sifted colimits. Suppose we are given a functor $F : \mathrm{Fun}(\Delta^1, s\mathrm{T}\mathsf{Alg}) \to \mathcal{C}$ such that*

(1) *F preserves sifted colimits.*

(2) *F preserves the colimit of each elementary diagram of simplicial T-algebras.*

Then F preserves pushouts along effective epimorphisms.

Proof of proposition 4.1.2.3. Let T be a Lawvere theory, and I a small index set together with a functor $t_{\_} : I \to \mathrm{T}$ whose image minimally generates T under products. The functor

$$\mathrm{ev}_I : s\mathrm{T}\mathsf{Alg} \longrightarrow \mathrm{Fun}(I, \mathcal{S})$$

adjoint to the functor

$$s\mathrm{T}\mathsf{Alg} \times I = \coprod_{i \in I} s\mathrm{T}\mathsf{Alg} \overset{\coprod_{i\in I} \mathrm{ev}_{t_i}}{\longrightarrow} \mathcal{S}$$

is conservative and preserves limits and sifted colimits and is thus monadic. Let $\mathrm{Free}_{\mathrm{T}} : \mathrm{Fun}(I, \mathcal{S}) \to s\mathrm{T}\mathsf{Alg}$ be a left adjoint to ev_I, determined up to equivalence by $\mathrm{Free}_{\mathrm{T}}(*_i) = j(t_i)$ for $i \in I$, where $*_i : I \to \mathcal{S}$ carries i to the final space $*$ and all other indices to the initial empty space. Let K be a simplicial set, then the induced adjunction

$$\mathrm{Fun}(K, s\mathrm{T}\mathsf{Alg}) \rightleftarrows \mathrm{Fun}(K \times I, \mathcal{S})$$

[1] The construction in section 3 of [Lur11a] is a generalization of the one below if we view Lawvere theories as discrete pregeometries. The reason we offer an alternative proof is that the construction in *loc. cit.* does not appear to be completely correct: in construction 3.7 and remark 3.8, from the data of an ∞-topos $\mathcal{X}$ and a map $\alpha : \mathcal{O} \to \mathcal{O}'$ of local $\mathcal{T}$-structures on $\mathcal{X}$, a certain Cartesian fibration $p : \overline{\mathcal{E}} \to \mathcal{X}$ over an ∞-topos is constructed, which resolves the map α in a suitable sense. Informally, the ∞-category $p^{-1}(U)$ is given by pairs $(\mathcal{O}^F, S)$ of a 'free' $\mathcal{T}$-structure on U and a finite set S, a map $\mathcal{O}^F \to \mathcal{O}'|_U$ of local $\mathcal{T}$-structures on $\mathcal{X}_{/U}$ and a commuting diagram

$$\begin{array}{ccc} \coprod_S \mathcal{O}^F & \longrightarrow & \mathcal{O}^F \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}|_U & \xrightarrow{\alpha|_U} & \mathcal{O}'|_U \end{array}$$

where the upper horizontal morphism is the fold map. It is claimed that this fibre admits coproducts and is thus sifted. However, this does not appear to hold in general, since the only reasonable choice for a coproduct of a pair of data $(\mathcal{O}^F, S)$ and $(\mathcal{O}^{F'}, S')$ as above in this ∞-category is the pair $(\mathcal{O}^F \amalg \mathcal{O}^{F'}, S \amalg S')$, but there is no reason for the existence of a unique map $\coprod_{S \amalg S'} \mathcal{O}^F \amalg \mathcal{O}^{F'} \to \mathcal{O}|_U$ that makes the requisite diagram commute, since we are not in general given maps $\coprod_{S'} \mathcal{O}^F \to \mathcal{O}|_U$ and $\coprod_S \mathcal{O}^{F'} \to \mathcal{O}|_U$. For the argument to go through, it seems one needs to consider the ∞-category of finite tuples $\{(\mathcal{O}^F_1, S_1), (\mathcal{O}^F_2, S_2), \ldots\}$ equipped with data as above, but we will not attempt a formal construction at this point is again monadic; letting $K = \Delta^1$, we deduce that for each map $\alpha : A \to B$ of simplicial T-algebras, there exists an (ev_I-split) augmented simplicial object $\alpha_\bullet : \mathbf{N}(\mathbf{\Delta}_+^{op}) \times \Delta^1 \to s\mathsf{TAlg}$, the Bar resolution $\mathsf{Bar}_{\mathrm{ev}_I \circ \mathrm{Free}_{\mathrm{T}}}(\mathrm{ev}_I \circ \mathrm{Free}_{\mathrm{T}}, \alpha)$, such that $\alpha_{-1} = f$, $\alpha_\bullet : \mathbf{N}(\mathbf{\Delta}_+^{op}) \to \mathrm{Fun}(\Delta^1, s\mathsf{TAlg})$ is a colimit diagram and each α_n is the image of $\mathrm{ev}_I(\alpha_{n-1})$ under $\mathrm{Free}_{\mathrm{T}}$. Recall that we are given a sifted colimit preserving functor

$$F : \mathrm{Fun}(\Delta^1, s\mathsf{TAlg}) \longrightarrow \mathcal{C}$$

and a full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ stable under sifted colimits such that $F(j(g))$ lies in $\mathcal{C}_0$ for every graph inclusion. We now show that the proposition follows from the following two claims.

($*$) If α is an effective epimorphism, the map $\mathrm{Free}_{\mathrm{T}}(\mathrm{ev}_I(\alpha))$ is also an effective epimorphism.

($**$) If α is an effective epimorphism, then the object $F(\mathrm{Free}_{\mathrm{T}}(\mathrm{ev}_I(\alpha)))$ lies in $\mathcal{C}_0$.

Indeed, if α is an effective epimorphism, then ($*$) and the construction of $\alpha_\bullet$ guarantee that for every $n \geq 0$, the map α_n is an effective epimorphism. It follows from ($**$) that $F(\alpha_n)$ lies in $\mathcal{C}_0$ for each $n \geq 0$. Since F preserves sifted colimits and $\mathcal{C}_0$ is stable under sifted colimits, we conclude.

We prove ($*$). We have a strictly commuting diagram of right adjoints

$$\begin{tikzcd} s\mathsf{TAlg} \arrow[r, "\mathrm{ev}_I"] & \mathcal{S}^I \\ \tau_{\leq 0} s\mathsf{TAlg} \arrow[r, "\mathrm{ev}_I"] \arrow[u, hook] & \mathbf{N}(\mathsf{Set})^I \arrow[u, hook] \end{tikzcd}$$

hence a commuting diagram of left adjoints

$$\begin{tikzcd} s\mathsf{TAlg} \arrow[d, "\pi_0"] & \mathcal{S}^I \arrow[l, "\mathrm{Free}_{\mathrm{T}}"'] \arrow[d, "\pi_0"] \\ \tau_{\leq 0} s\mathsf{TAlg} & \mathbf{N}(\mathsf{Set})^I. \arrow[l, "\mathrm{Free}^0_{\mathrm{T}}"'] \end{tikzcd}$$

The fact that the truncation functor $\tau_{\leq 0} : s\mathsf{TAlg} \to \tau_{\leq 0} s\mathsf{TAlg}$ is given by composing product preserving functors with $\pi_0 : \mathcal{S} \to \mathbf{N}(\mathsf{Set})$ means precisely that the diagram of left adjoints above is horizontally right adjointable. Given a map $\alpha : A \to B$, it follows that the map $\pi_0(\mathrm{Free}_{\mathrm{T}}(\mathrm{ev}_I(\alpha)))$ is given by applying the free functor $\mathrm{Free}^0_{\mathrm{T}}$ to the map $\pi_0(\mathrm{ev}_I(\alpha))$. By assumption, this latter map is a surjection, so it suffices to observe that $\mathrm{Free}^0_{\mathrm{T}}$ carries surjections of I-indexed sets to (effective) epimorphisms of discrete simplicial T-algebras, as $\mathrm{Free}^0_{\mathrm{T}}$ can be identified with the free functor for discrete algebras for the Lawvere theory $h\mathrm{T}$.

We prove ($**$). The ∞-category $\mathrm{Fun}(I, \mathcal{S})$ is isomorphic to the nerve of the Kan-enriched category $\mathrm{Fun}(I, \mathsf{Kan}) = \prod_{i \in I} \mathsf{Kan}$. Let $\alpha : A \to B$ be a map in $s\mathsf{TAlg}$, then after applying a factorization as a trivial cofibration followed by a fibration, we may assume that the morphism $\mathrm{ev}_I(\alpha)$ in $\mathrm{Fun}(I, \mathcal{S})$ is an I-indexed collection $\{\mathrm{ev}_{t_i}(A) \to \mathrm{ev}_{t_i}(B)\}_{i \in I}$ of Kan fibrations between Kan complexes. If α is an effective epimorphism, then for each i, the map $\mathrm{ev}_{t_i}(A) \to \mathrm{ev}_{t_i}(B)$ is a surjection on connected components and thus a surjection in each simplicial degree since it is a Kan fibration. We may view $\mathrm{ev}_{t_i}(A)$ and $\mathrm{ev}_{t_i}(B)$ as constant bisimplicial objects, so that we can think of the collection $\{\mathrm{ev}_{t_i}(A) \to \mathrm{ev}_{t_i}(B)\}_{i \in I}$ as a morphism in the category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}^I_{\mathbf{\Delta}})$. Applying the diagonal functor

$$\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set}^I_{\mathbf{\Delta}}) \longrightarrow \mathsf{Set}^I_{\mathbf{\Delta}}$$

returns the morphism $\mathrm{ev}_I(\alpha)$ so we deduce from corollary 2.2.4.13 that $\mathrm{ev}_I(\alpha)$ is a colimit of the diagram

$$\{\mathrm{ev}_{t_i}(A) \to \mathrm{ev}_{t_i}(B)\}_{i \in I} : \mathbf{N}(\mathbf{\Delta}^{op}) \times \Delta^1 \longrightarrow \mathbf{N}(\mathsf{Set})^I \longrightarrow \mathbf{N}(\mathsf{Set}_{\mathbf{\Delta}})^I \longrightarrow \mathcal{S}^I,$$

where the last morphism implements localization at the weak equivalences. The functor $\mathbf{N}(\mathsf{Set}) \to \mathbf{N}(\mathsf{Set}_{\mathbf{\Delta}}) \to \mathcal{S}$ can be identified with the inclusion of 0-truncated spaces, so we conclude that the diagram above is in each simplicial degree n and for each $i \in I$ given by a surjective map

$$\coprod_{\mathrm{ev}_{t_i}(A)_n} * \longrightarrow \coprod_{\mathrm{ev}_{t_i}(B)_n} *$$

of discrete spaces. Since $\mathrm{Free}_{\mathrm{T}}$ preserves colimits, the map $\mathrm{Free}_{\mathrm{T}}(\mathrm{ev}_I(\alpha))$ is arises as the geometric realization of a simplicial diagram that is in each simplicial degree n given by

$$\coprod_i \coprod_{\mathrm{ev}_{t_i}(A)_n} j(t_i) \longrightarrow \coprod_i \coprod_{\mathrm{ev}_{t_i}(B)_n} j(t_i). \tag{4.1}$$

Using that F preserves sifted colimits and that $\mathcal{C}_0 \subset \mathcal{C}$ is stable under sifted colimits again, we are reduced to proving that F carries every morphism of the form (4.1) into $\mathcal{C}_0$. Writing I as a filtered colimit of its finite subsets, we may assume that I is finite (using that F preserves, and $\mathcal{C}_0 \subset \mathcal{C}$ is stable under, filtered colimits). Writing for each $i \in I$, the set $\mathrm{ev}_{t_i}(B)_n$ as a filtered colimit of its finite subsets, we may assume that $\mathrm{ev}_{t_i}(B)_n$ is finite. Writing for each $x \in \mathrm{ev}_{t_i}(B)_n$ the set $\mathrm{ev}_{t_i}(\alpha)_n^{-1}(x) \subset \mathrm{ev}_{t_i}(A)_n$ as a filtered colimit of its finite subsets, we may assume that $\mathrm{ev}_{t_i}(A)_n$ is finite. Now we observe that with all sets indexing the coproducts finite, the map (4.1) is a graph inclusion. □

Proof of proposition 4.1.2.5. Using the same notations as in the proof of proposition 4.1.2.3, we consider a diagram $\sigma : \Lambda^2_0 \to s\mathrm{T}\mathsf{Alg}$

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"] & C \\ B & \end{tikzcd}$$

where α is an effective epimorphism. Applying the functorial Bar construction of the monad $\mathrm{ev}_I \circ \mathrm{Free}_{\mathrm{T}}$, we obtain a ($\mathrm{ev}_I$-split) augmented simplicial object $\sigma_\bullet : \mathbf{N}(\mathbf{\Delta}_+^{op}) \to \mathrm{Fun}(\Lambda^2_0, s\mathrm{T}\mathsf{Alg})$ that is a colimit diagram with colimit σ and each σ_n is the image of $\mathrm{ev}_I(\sigma_{n-1})$ under $\mathrm{Free}_{\mathrm{T}}$. We see that it suffices to show that for each $n \geq 0$, the functor F preserves the colimit of the diagram σ_n; using assertion $(*)$ of the proof of proposition 4.1.2.3, it suffices to prove that F preserves the colimit of $\sigma_0 = \mathrm{Free}_{\mathrm{T}}(\mathrm{ev}_I(\sigma))$. We may assume that for each $i \in I$, the vertical morphism in the diagram

$$\begin{tikzcd} \mathrm{ev}_i(A) \arrow[r] \arrow[d] & \mathrm{ev}_i(C) \\ \mathrm{ev}_i(B) & \end{tikzcd}$$

is a Kan fibration, and thus a surjection in each simplicial degree, and the horizontal morphism a cofibration, that is, an injection in each simplicial degree. Using corollary 2.2.4.13, we see that the map $\mathrm{Free}_{\mathrm{T}}(\mathrm{ev}_I(\sigma))$ is given by a geometric realization of a simplicial diagram that is in each simplicial degree n given by

$$\begin{tikzcd} \coprod_i \coprod_{\mathrm{ev}_{t_i}(A)_n} j(t_i) \arrow[r] \arrow[d] & \coprod_i \coprod_{\mathrm{ev}_{t_i}(C)_n} j(t_i) \\ \coprod_i \coprod_{\mathrm{ev}_{t_i}(B)_n} j(t_i). & \end{tikzcd}$$

As in the proof of proposition 4.1.2.3, we may assume that all sets indexing the coproducts are finite, in which case the diagram above is elementary. □

Proof of theorem 4.1.2.2. Apply proposition 4.1.2.5 to the sifted colimit preserving functor $f^* : s\mathrm{T}'\mathsf{Alg} \to s\mathrm{T}\mathsf{Alg}$ induced by the unramified transformation of Lawvere theories. □

4.1.3 The geometry of finitely presented simplicial C^∞-rings

Here we specialize to the Lawvere theory CartSp and define the structure of a geometry on the full subcategory of finitely presented objects, following the outline of the previous chapter. The spectrum-global sections adjunction provided by proposition 3.1.1.2 turns out to be an equivalence on a full subcategory of $sC^\infty\mathsf{ring}$, the one that contains the C^∞-rings which we call *fair*, following Joyce [Joy12a]. We then go on and define a variety of derived affines corresponding to certain full subcategories of $sC^\infty\mathsf{ring}$, the central example being the ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}$ of affine derived manifolds of finite presentation, defined to be the essential image of $sC^\infty\mathsf{ring}_{\mathrm{fp}}$ under the spectrum functor. In the next subsection, we use the technology developed here to show on of the main results: the ∞-category of finitely presented simplicial C^∞-rings is a geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$, the more elaborate version of derived manifolds with corners following in a later subsection.
Using the results of the last subsection, we will show that the theory of simplicial C^∞-rings is controlled in large part by the underlying algebraic model; in this case given by the transformation of Lawvere theories $\mathcal{T}^{disc}_{\mathbb{R}} \to \mathbf{N}(\mathsf{CartSp})$. We write $(\_)^{\mathrm{alg}}$ for the functor induced by this transformation; it takes values in $s\mathsf{Cring}_{\mathbb{R}}$, the ∞-category of simplicial commutative $\mathbb{R}$-algebras, and is clearly conservative.

Notation 4.1.3.1. We reserve the symbol $\otimes^\infty$ for the pushout of simplicial C^∞-rings to distinguish it from the pushout of simplicial commutative $\mathbb{R}$-algebras.

We will occasionally abuse notation by identifying A^{alg} with a connective $\mathbb{E}_\infty$-algebra over $\mathbb{R}$ using the equivalence $s\mathsf{Cring}_{\mathbb{R}} \simeq \mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}}$. Also, for M a manifold, we will usually avoid writing $C^\infty(M)^{\mathrm{alg}}$, to avoid cluttering up notation; it will be clear from the context when we think of $C^\infty(M)$ as an $\mathbb{R}$-algebra.

Remark 4.1.3.2. Recall that for a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

of simplicial commutative algebras (over any ring), there is a convergent spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_*(A)}(\pi_* B, \pi_* C)_q \Rightarrow \pi_{p+q}(D). \tag{4.2}$$

See for instance, [Lur17a], prop. 7.2.1.19 and [Lur11b], corollary 4.1.14.

Remark 4.1.3.3. Recall the basic lemma of Hadamard: for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and any $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, there exists a collection of n smooth function $\{g_i\}$ on $\mathbb{R}^n$ such that $f(\mathbf{x}) - f(p) = \sum_{i=1}^n g_i(\mathbf{x})(x_i - p_i)$.

Our comparison of simplicial C^∞-rings with $\mathcal{T}_{\mathrm{Diff}}$-structures will require a number of preliminary results. The next lemma is a derived analogue of the fact that ideals of independent functions are point determined (see remark 3.1.3.10).

Lemma 4.1.3.4. *Let M be an m-dimensional manifold and let $\{f_1, \ldots, f_n\}$, $n \leq m$, be independent functions on M. Then the Koszul algebra $C^\infty(M)[y_1, \ldots, y_n]$ with $|y_i| = 1$ for $1 \leq i \leq n$ and $\partial y_i = f_i$, is a projective resolution of $C^\infty(Z(f_1, \ldots, f_n))$ in the category of differentially graded $C^\infty(M)$-modules.*

Proof. Clearly, the Koszul complex is a complex of projective $C^\infty(M)$-modules, so we should show that the complex is a resolution. Let C^∞_M denote the sheaf of C^∞ functions on M. Consider the sheaf of bounded differential graded C^∞_M-modules on M given by

$$\mathcal{F} : U \mapsto C^\infty(U)[y_1, \ldots, y_n], \quad \partial y_i = f_i|_U,\ 1 \leq i \leq n,$$

whose complex of global sections is the Koszul algebra $C^\infty(M)[y_1, \ldots, y_n]$. Proposition 2.2.5.37 implies that the homology presheaves are already sheaves, so in order to show that the higher homology of the Koszul complex vanishes, it suffices to give for each point $x \in M$ a neighbourhood basis $\{V_\beta\}$ of x in M such that $C^\infty(V_\beta)[y_1, \ldots, y_n]$ has vanishing homology in degrees > 0. The function $(f_1, \ldots, f_n) : M \to \mathbb{R}^n$ has full rank at $Z(f_1, \ldots, f_n)$, so it has full rank in some open neighbourhood $Z(f_1, \ldots, f_n) \subset V$. By the constant rank theorem, there is an open cover $\{U_\alpha\}$ of V such that $U_\alpha \cong \mathbb{R}^m$ and in these coordinates, the function $(f_1, \ldots, f_n)$ is the projection $(x_1, \ldots, x_n) : \mathbb{R}^m \to \mathbb{R}^n$ onto the first n coordinates. We have a cover $\{U_\alpha\} \coprod \{M \smallsetminus Z(f_1, \ldots, f_n)\}$ of M so each point in M has a neighbourhood basis on which $\mathcal{F}$ evaluates as either a complex of the form $C^\infty(V)[y_1, \ldots, y_n]$, $V \subset M \smallsetminus Z(f_1, \ldots, f_n)$, which is acyclic because all $f_i|_{M \smallsetminus Z(f_1, \ldots, f_n)}$ are invertible, or we have $C^\infty(U)[y_1, \ldots, y_n]$, where $U \subset \mathbb{R}^m$ is an open subset and $\partial y_i = x_i$, the projection onto the i'th coordinate. Applying Hadamard's lemma repeatedly, one finds that $C^\infty(U)/(x_1, \ldots, x_i) \cong C^\infty(U \cap (\{0\} \times \mathbb{R}^{m-i}))$ for $1 \leq i \leq n$. In particular, the zero locus of the function x_{i+1} has measure zero, so x_{i+1} is a nonzerodivisor of $C^\infty(U)/(x_1, \ldots, x_i)$. Thus, the sequence $(x_1, \ldots, x_n)$ is a regular sequence on $C^\infty(U)$ showing that the homology of the complex $C^\infty(U)[y_1, \ldots, y_n]$ is $C^\infty(U \cap (\{0\} \times \mathbb{R}^{m-n}))$ concentrated in degree 0. We are left to show that the zero'th homology of the Koszul complex is $C^\infty(Z(f_1, \ldots, f_n))$. This follows from the previous computation: as proposition 2.2.5.37 shows that the presheaf of zero'th homology is already a sheaf, the global sections are clearly given by $C^\infty(Z(f_1, \ldots, f_n))$. ☐

Lemma 4.1.3.5 ([Spi10] lemma 8.1, [Lur11a] lemma 11.10)**.** *The transformation of Lawvere theories $\mathcal{T}_{\mathbb{R}}^{disc} \to \mathbf{N}(\mathsf{CartSp})$ is unramified.*

Proof. We should prove that for any smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ and any $k \geq 0$, the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^{n+m}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^{n+m+k}) \arrow[d] \\ C^\infty(\mathbb{R}^n) \arrow[r] & C^\infty(\mathbb{R}^{n+k}) \end{tikzcd}$$

is a pushout in $s\mathsf{Cring}_{\mathbb{R}}$. We proceed by induction on m; for $m = 1$, there is nothing to prove. For $m = 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is some smooth function. As we work with discrete objects, the torsion spectral sequence collapses at the second page, so we should show that

$$\mathrm{Tor}_p^{C^\infty(\mathbb{R}^{n+1})}(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^{n+1+k})) = 0, \quad p \geq 1,$$

and that

$$\mathrm{Tor}_0^{C^\infty(\mathbb{R}^{n+1})}(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^{n+1+k})) \cong C^\infty(\mathbb{R}^n) \otimes_{C^\infty(\mathbb{R}^{n+1})} C^\infty(\mathbb{R}^{n+1+k}) \cong C^\infty(\mathbb{R}^{n+k}).$$

Denote the first n coordinates on $\mathbb{R}^{n+1}$ collectively by $\mathbf{x}$ and the last coordinate by y. The function $y - f(\mathbf{x})$ is a submersion and its zero locus is $\mathrm{Graph}(f) \cong \mathbb{R}^n$, so the ring $C^\infty(\mathbb{R}^n)$ admits a projective resolution as an $C^\infty(\mathbb{R}^{n+1})$-module of the form $C^\infty(\mathbb{R}^{n+1})[z]$, $\partial z = y - f(\mathbf{x})$, by lemma 4.1.3.4. The torsion groups are computed as the homology of

$$C^\infty(\mathbb{R}^{n+1})[z] \otimes_{C^\infty(\mathbb{R}^{n+1})} C^\infty(\mathbb{R}^{n+1+k}) \cong C^\infty(\mathbb{R}^{n+1+k})[z], \quad \partial z = y - f(\mathbf{x}).$$

By lemma 4.1.3.4 again, the complex on the right hand side is a resolution of $C^\infty(\mathbb{R}^{n+1+k})/(y - f(\mathbf{x}))$. Since the map $C^\infty(\mathbb{R}^{n+1+k}) \to C^\infty(\mathbb{R}^{n+k})$ given by restricting to the graph of $y - f(\mathbf{x})$ induces an isomorphism $C^\infty(\mathbb{R}^{n+1+k})/(y - f(\mathbf{x})) \to C^\infty(\mathbb{R}^{n+k})$, we are done.
Now suppose that for $m \leq l$, the statement is true for all n and k. Consider the diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^{n+l+1}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^{n+l+1+k}) \arrow[d] \\
C^\infty(\mathbb{R}^{n+1}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^{n+1+k}) \arrow[d] \\
C^\infty(\mathbb{R}^{n}) \arrow[r] & C^\infty(\mathbb{R}^{n+k})
\end{tikzcd}$$

where the upper square is a pushout by the induction hypothesis applied to $\mathbb{R}^{n+1}$. The large rectangle is a pushout if and only if the lower square is a pushout, so we reduce to the case $m = 1$, and we are done. □

Corollary 4.1.3.6. $(\_)^{\mathrm{alg}}$ *preserves pushouts along effective epimorphisms.*

Proof. Apply theorem 4.1.2.2 to the unramified transformation $\mathcal{T}_{\mathbb{R}}^{disc} \to \mathbf{N}(\mathsf{CartSp})$. □

Proving results by 'unramifiedness' using the corollary above unlocks the powerful techniques available in the framework of connective $\mathbb{E}_\infty$-algebras, and we will appeal to it many times in this work.

Lemma 4.1.3.7. *Let M, N be manifolds, then the natural map $C^\infty(M) \otimes^\infty C^\infty(N) \to C^\infty(M \times N)$ is an equivalence.*

Proof. Take an open submanifold $U \subset \mathbb{R}^n$, and let χ_U be a characteristic function for U. Denote by $\mathbf{x}$ the first n coordinates and by y the last coordinate on $\mathbb{R}^{n+1}$, let $f(\mathbf{x}, y) = \chi_U(\mathbf{x})y - 1$ and consider the colimit of the diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}) \arrow[r, "f^*"] \arrow[d, "\mathrm{ev}_0"'] & C^\infty(\mathbb{R}^{n+1}) \\
\mathbb{R} &
\end{tikzcd}$$

The left vertical map is an effective epimorphism, so by unramifiedness, we can compute this pushout in $s\mathsf{Cring}_{\mathbb{R}}$. Using the spectral sequence of remark 4.1.3.2, we see that the homotopy groups of the pushout are computed as the torsion groups $\mathrm{Tor}_k^{C^\infty(\mathbb{R})}(\mathbb{R}, C^\infty(\mathbb{R}^{n+1}))$. Using the projective resolution $C^\infty(\mathbb{R})[z]$, $\partial z = x$ of $\mathbb{R}$ as a $C^\infty(\mathbb{R})$-module, we find that the homotopy groups are given by the homology of the complex $C^\infty(\mathbb{R}^{n+1})[z]$, $\partial z = \chi_U y - 1$. Lemma 4.1.3.4 implies that this complex has homology $C^\infty(\mathbb{R}^{n+1})/(f) \cong C^\infty(U)$ concentrated in degree 0. Now for $U, V \in \mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}}$, with presentations $C^\infty(U) \cong C^\infty(\mathbb{R}^{n+1})/(f)$ and $C^\infty(V) \cong C^\infty(\mathbb{R}^{m+1})/(g)$, the coproduct $C^\infty(U) \otimes^\infty C^\infty(V)$ is the colimit of the diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^2) \arrow[r, "(f\times g)^*"] \arrow[d, "\mathrm{ev}_0"'] & C^\infty(\mathbb{R}^{n+1+m+1}) \\
\mathbb{R} &
\end{tikzcd}$$

Using unramifiedness, the torsion spectral sequence, and lemma 4.1.3.4 again, we find that the pushout above is the discrete C^∞-ring $C^\infty(U \times V)$.
For general manifolds M, N, we use that $\mathcal{T}_{\mathrm{Diff}} \simeq \mathrm{Idem}(\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}})$ to realize M and N as retracts of some U respectively V in $\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}}$. Then $M \times N$ is a retract of $U \times V$. $C^\infty(M) \otimes^\infty C^\infty(N)$ is a retract of $C^\infty(U) \otimes^\infty C^\infty(V)$ and $C^\infty(M \times N)$ is a retract of $C^\infty(U \times V)$. But as the natural map $C^\infty(U) \otimes^\infty C^\infty(V) \to C^\infty(U \times V)$ is an equivalence, $C^\infty(M) \otimes^\infty C^\infty(N)$ and $C^\infty(M \times N)$ split equivalent idempotents, so the natural map $C^\infty(M) \otimes^\infty C^\infty(N) \to C^\infty(M \times N)$ must be an equivalence. □

Remark 4.1.3.8. Notice that the proof of lemma 4.1.3.7 shows that the essential image of the functor $C^\infty(\_)$: $\mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}$ consists of retracts of pushouts of compact projective objects of $sC^\infty\mathsf{ring}$ which are thus compact.

Lemma 4.1.3.9. *The functor $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}^{op}$ sending a manifold M to the discrete simplicial C^∞-ring of smooth functions on M preserves transverse pullbacks of the form*

$$\begin{tikzcd} N \times_U M \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & U \end{tikzcd}$$

where U is an open submanifold of $\mathbb{R}^n$ for some $n \geq 1$.

Proof. We note that the pullback $N \times_U M$ is equivalent to the pullback

$$\begin{tikzcd} (M \times N) \times_{U \times U} U \arrow[r] \arrow[d] & N \times M \arrow[d, "g"] \\ U \arrow[r] & U \times U \end{tikzcd}$$

so, as the transformation $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}^{op}$ preserves binary products by lemma 4.1.3.7, we only have to deal with pullback diagrams of the form above. Because the map $C^\infty(U \times U) \to C^\infty(U)$ induced by the diagonal $U \to U \times U$ is an (effective) epimorphism and the fact that the transformation of Lawvere theories $\mathcal{T}_{\mathbb{R}}^{disc} \to \mathbf{N}(\mathsf{CartSp})$ is unramified, there is a natural equivalence

$$(C^\infty(U) \otimes^\infty_{C^\infty(U \times U)} C^\infty(N \times M))^{\mathrm{alg}} \simeq C^\infty(U) \otimes_{C^\infty(U \times U)} C^\infty(N \times M).$$

As $(\_)^{\mathrm{alg}}$ is conservative, it suffices to show that $C^\infty(U) \otimes_{C^\infty(U \times U)} C^\infty(N \times M)$ is 0-truncated and the natural map $\tau_{\leq 0}(C^\infty(U) \otimes_{C^\infty(U \times U)} C^\infty(N \times M)) \to C^\infty((M \times N) \times_{U \times U} U)$ is an equivalence. To see this, we note that we work with discrete objects, so the torsion spectral sequence (4.2) collapses at the second page and we have natural isomorphisms

$$\mathrm{Tor}_k^{C^\infty(U \times U)}(C^\infty(U), C^\infty(N \times M)) \cong \pi_k(C^\infty(U) \otimes_{C^\infty(U \times U)} C^\infty(N \times M))$$

for all $k \geq 0$. Since $U \subset \mathbb{R}^n$ is open, the diagonal embedding $U \to U \times U$ is cut out by n independent functions $\{f_1, \ldots, f_n\}$, so lemma 4.1.3.4 provides us with a projective resolution $C^\infty(U \times U)[y_1, \ldots, y_n]$ of $C^\infty(U)$ as a $C^\infty(U \times U)$-module. The torsion groups are computed as the homology of

$$C^\infty(U \times U)[y_1, \ldots, y_n] \otimes_{C^\infty(U \times U)} C^\infty(N \times M) \cong C^\infty(N \times M)[y_1, \ldots, y_n], \quad \partial y_i = f_i \circ g,\ 1 \leq i \leq n.$$

Because $g : N \times M \to U \times U$ is transverse to $U \to U \times U$, the functions $f_i \circ g$ are independent, so by lemma 4.1.3.4 again, this complex is a projective resolution of $C^\infty(Z(f_1 \circ g, \ldots, f_n \circ g))$. But $Z(f_1 \circ g, \ldots, f_n \circ g)$ is the image of the embedding $(M \times N)_{U \times U} U \to N \times M$, a closed submanifold. □

We will momentarily show that the functor $C^\infty(\_)$ preserves *all* transverse pullbacks. Now we show that $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ has a natural structure of a geometry. As the algebraic examples of the previous chapter, the admissibility structure on $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ is defined in terms of localization morphisms.

Definition 4.1.3.10. Let A be a simplicial C^∞-ring and let $a \in \pi_0(A)$. We say that a map $f : A \to B$ such that $f(a) \in \pi_0(B)$ is invertible is a *localization of A with respect to a* if for each $C \in sC^\infty\mathsf{ring}$, the map $\mathrm{Hom}_{sC^\infty\mathsf{ring}}(B, C) \to \mathrm{Hom}_{sC^\infty\mathsf{ring}}(A, C)$ given by composition with f induces a homotopy equivalence of Kan complexes

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}}(B, C) \xrightarrow{\simeq} \mathrm{Hom}^0_{sC^\infty\mathsf{ring}}(A, C),$$

where $\mathrm{Hom}^0_{sC^\infty\mathsf{ring}}(A, C)$ is the union of those connected components of $\mathrm{Hom}_{sC^\infty\mathsf{ring}}(A, C)$ spanned by those maps g such that $g(a)$ is invertible in the commutative $\mathbb{R}$-algebra $\pi_0(C)^{\mathrm{alg}}$.

In the case of an ordinary C^∞-ring A and some $a \in A$, the above definition reduces to the usual C^∞ localization $A[1/a]$ given up to equivalence by the pushout

$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r, "q_a"] \arrow[d] & A \arrow[d] \\ C^\infty(\mathbb{R} \setminus \{0\}) \arrow[r] & A[1/a] \end{tikzcd}$$

of C^∞-rings. The localization of a simplicial C^∞-ring admits a similar characterization, for which we will need the following definition.

Definition 4.1.3.11. (1) A map $f : A \to B$ of simplicial commutative rings is *strong* (in the sense of Toën-Vezzosi [TV06], definition 2.2.2.1.) if the natural map

$$\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_n(B)$$

is an isomorphism for all $n \geq 0$.

(2) A map $f : A \to B$ of simplicial C^∞-rings is *strong* if $f^{\mathrm{alg}} : A^{\mathrm{alg}} \to B^{\mathrm{alg}}$ is strong.

Before we give the desired characterization of localization morphisms, we recall the following easy lemma.

Lemma 4.1.3.12. *Let $U \hookrightarrow M$ be an open embedding of manifolds, then the induced map $C^\infty(M) \to C^\infty(U)$ is a flat map of commutative $\mathbb{R}$-algebras.*

Proof. Take a finite linear combination of 0 as $\sum_{i=1}^n g_i f_i|_U = 0$ with $g_i \in C^\infty(U)$ and $f_i \in C^\infty(M)$, then we should show that there exists a finite set of elements $\{h_j\}_j \subset C^\infty(U)$ and linear combinations $g_i = \sum_j h_j b_{ij}|_U$ with $b_{ij} \in C^\infty(M)$ such that $\sum_i f_i b_{ij} = 0$ for all j. We can write each g_i as a quotient g_i'/γ_i, where g_i' and γ_i are defined on M such that γ_i does not vanish on U. Now pick a characteristic function χ_U for U and set $h_i = 1/(\gamma_i \chi_U)$, $b_{ij} = 0$ if $i \neq j$ and $b_{ii} = \gamma_i g_i \chi_U$. □

Proposition 4.1.3.13. *Let A be a simplicial C^∞ ring and let $a \in \pi_0(A)$, and let $f : A \to B$ a map of simplicial C^∞-rings. The following are equivalent:*

(1) The map $f : A \to B$ exhibits B as a localization with respect to a.

(2) For every $n \geq 0$, the induced map

$$\pi_n(A^{\mathrm{alg}}) \otimes_{\pi_0(A^{\mathrm{alg}})} (\pi_0(A)[1/a])^{\mathrm{alg}} \to \pi_n(B^{\mathrm{alg}})$$

is an equivalence; that is, f is strong and the map of C^∞-schemes corresponding to $\pi_0(A) \to \pi_0(B)$ is an open inclusion.

(3) B fits into a pushout diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r, "q_a"] \arrow[d] & A \arrow[d, "f"] \\ C^\infty(\mathbb{R} \setminus \{0\}) \arrow[r] & B \end{tikzcd}$$

where q_a is the unique up to homotopy map associated to $a \in \pi_0(A)$ (note that as a consequence, localizations always exist).

Proof. First, we show that (1) is equivalent to (3). Let A be a simplicial C^∞-ring, and choose some $a \in \pi_0(A)$. By proposition 4.1.1.18 and an elementary cofinality argument, we can write A as a directed colimit of finitely generated subrings $\mathrm{colim}_{i \in \mathcal{J}} A_i \simeq A$ such that $a \in \pi_0(A_i)$ for all $i \in \mathcal{J}$. We claim that the map $\varphi : A \to \mathrm{colim}_{i \in J}(A_i[a^{-1}])$ is a localization. To see this, we let C be any simplicial C^∞-ring and $f \in \mathrm{Hom}_{sC^\infty\mathsf{ring}}(A, C)$ and we consider the homotopy pullback

$$\begin{tikzcd} K_f \arrow[r] \arrow[d] & \lim_{i \in J} \mathrm{Hom}_{sC^\infty\mathsf{ring}}(A_i[a^{-1}], C) \arrow[d] \\ \{f\} \arrow[r] & \mathrm{Hom}_{sC^\infty\mathsf{ring}}(A, C) \end{tikzcd}$$

of Kan complexes. The map φ is a localization if and only if for each C and each f, K_f is weakly contractible if $f(a)$ is invertible in $\pi_0(C)$ and empty if $f(a)$ is not invertible. The map f induces maps $f_i : A_i \to C$, and Kan complexes $K_{f_i} := \{f_i\} \times^h_{\mathrm{Hom}_{sC^\infty\mathsf{ring}}(A_i, C)} \mathrm{Hom}_{sC^\infty\mathsf{ring}}(A_i[a^{-1}], C)$, and we have an equivalence $\lim_{i \in \mathcal{J}} K_{f_i} \simeq K_f$. If $f(a)$ (and therefore $f_i(a)$) is not invertible, K_f is a limit of empty simplicial sets and also empty. If $f(a)$ (and therefore $f_i(a)$) is invertible, K_f is a limit of weakly contractible Kan complexes indexed by a weakly contractible simplicial set, and therefore also weakly contractible. Now if $A_i \to A_i[a^{-1}]$ satisfies (3), then $A \to \mathrm{colim}_{i \in J}(A_i[a^{-1}])$ satisfies (3), so we reduce to the case of finitely generated simplicial C^∞-rings. If A is finitely generated, we have an effective epimorphism $p : C^\infty(\mathbb{R}^n) \to A$ for some n by proposition 4.1.1.18, so the map $q_a : C^\infty(\mathbb{R}) \to A$ defining $a \in \pi_0(A)$ factors up to homotopy through p, which defines some $\widehat{a} \in C^\infty(\mathbb{R}^n)$. Consider the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^n) \arrow[r, "p"] \arrow[d] & A \arrow[d] \\ C^\infty(\mathbb{R} \setminus \{0\}) \arrow[r] & C^\infty(\mathbb{R}^n)[\hat{a}^{-1}] \arrow[r] & A[a^{-1}] \end{tikzcd}$$

It's easy to see that the right square is pushout, so if the left square is a pushout, the outer rectangle is a pushout as well, and we reduce to the case of free simplicial C^∞-rings, for which we already know that the localization is given by the pushout (3) in the truncated 1-category of C^∞-rings, which coincides with the pushout in $sC^\infty\mathsf{ring}$ by lemma 4.1.3.9.
Now we show that (3) and (2) are equivalent. First, we show that (3) implies (2). Since taking homotopy groups and tensor products commutes with filtered colimits, we may assume that we're dealing with finitely generated objects. The localization of a finitely generated object A is given by the pushout diagram above for some effective epimorphism $p: C^\infty(\mathbb{R}^n) \to A$. Let $U \subset \mathbb{R}^n$ be the open set where the function $\hat{a}$ is nonzero. By unramifiedness, $A[a^{-1}]$ is given by the pushout $A \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(U)$ of simplicial commutative rings. Moreover, since $U \to \mathbb{R}^n$ is an open inclusion, the map on smooth functions is flat by lemma 4.1.3.12, so applying the torsion spectral sequence we have an equivalence

$$\pi_n(A) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(U) \simeq \pi_n(A \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(U)) = \pi_n(A[a^{-1}]),$$

for all $n \geq 0$, so we have equivalences

$$\pi_n(A[a^{-1}]) \simeq \pi_n(A) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(U) \simeq \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A) \otimes_{C^\infty(\mathbb{R})} C^\infty(U) \simeq \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A[a^{-1}])$$

What remains to be shown is that (2) implies (3). If $f: A \to B$ satisfies (2), then there is an induced map $A[a^{-1}] \to B$ where $A[a^{-1}]$ is the pushout of (3). As we have just verified, this map induces an isomorphism on all homotopy groups so it is an equivalence as the functors taking homotopy groups are jointly conservative. □

Remark 4.1.3.14. Combining remark 4.1.3.8 with proposition 4.1.3.13 shows that the localization of an (almost) finitely presented simplicial C^∞-ring with respect to any $a \in \pi_0(A)$ is again (almost) finitely presented.

Corollary 4.1.3.15. *Let A be a simplicial C^∞-ring. The functor $\tau_{\leq 0} : (sC^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/A} \to \mathbf{N}(C^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/\pi_0 A}$ is an equivalence of ∞-categories.*

Proof. We show that $\tau_{\leq 0}$ is fully faithful and essentially surjective. For essential surjectivity, let $\pi_0(A) \to B$ be a localization morphism in $C^\infty\mathsf{ring}$ determined by some $a \in \pi_0(A)$, then proposition 4.1.3.13 immediately shows that B is isomorphic to the image under $\tau_{\leq 0}$ of the morphism $A \to A[a^{-1}]$. For fully faithfulness, note that the functor $\tau_{\leq 0}$ sends the Hom-spaces in $(sC^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/A}$ to their zero'th truncation. Thus, to show that $\tau_{\leq 0}$ is fully faithful, it suffices to show that the Hom-spaces of $(sC^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/A}$ are already discrete. Let $A \to B$ and $A \to C$ be localization morphisms. The space $\mathrm{Hom}_{(sC^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/A}}(C, B)$ is equivalent to the space $\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A/}}(B, C)$, but by [Lur17b], lem. 5.5.5.12, this space fits into a homotopy fibre sequence

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A/}}(B, C) \longrightarrow \mathrm{Hom}_{sC^\infty\mathsf{ring}}(B, C) \longrightarrow \mathrm{Hom}_{sC^\infty\mathsf{ring}}(A, C)$$

where the fibre is taken over the chosen morphism $A \to C$. Because $A \to B$ is a localization, the second map in the fibre sequence is an inclusion of connected components, so $\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A/}}(B, C)$ is empty or weakly contractible. □

Notation 4.1.3.16. We will denote $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ for the opposite category of the ∞-category of compact objects in $sC^\infty\mathsf{ring}$. To notationally distinguish a finitely presented simplicial C^∞-ring A from A as an object of $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$, we use the notation $\mathrm{Spec}\, A$ in the latter case (the next subsection will provide motivation for this notation).

We endow $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ with the structure of a geometry according to the following prescription:

(1) A map $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ is admissible if and only if there exists an element $b \in \pi_0(B)$ such that the image of b under f is invertible in $\pi_0(A)$ and the induced map $B[1/b] \to A$ is an equivalence.

(2) A collection $\{\mathrm{Spec}\, B[1/b_\alpha] \to \mathrm{Spec}\, B\}_{\alpha \in J}$ generates a covering sieve if and only if the germ determined ideal generated by the elements b_α in $\pi_0(B)$ contains the unit.

For the proof that the definition above defines a geometry, we first prove a lemma concerning strong morphisms.

Lemma 4.1.3.17. *(1) A retract of a strong morphism is strong.*

(2) In a diagram

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[dr, "g"] & \\ B \arrow[rr, "f"] & & A \end{tikzcd}$$

where h is strong, f is strong if and only if g is strong.

Proof. (1) A retraction diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ A' \arrow[r] & B' \arrow[r] & A' \end{tikzcd}$$

where f is a strong morphism induces for each $n \geq 0$ a diagram

$$\begin{tikzcd} \pi_n(A) \arrow[r] \arrow[d] & \pi_n(B) \arrow[r] \arrow[d] & \pi_n(A) \arrow[d] \\ \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A') \arrow[r] \arrow[d] & \pi_n(B) \otimes_{\pi_0(B)} \pi_0(B') \arrow[r] \arrow[d, "\cong"] & \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A') \arrow[d] \\ \pi_n(A') \arrow[r] & \pi_n(B') \arrow[r] & \pi_n(A') \end{tikzcd}$$

where both horizontal rectangles are retraction diagrams. The inverse of the indicated isomorphism yields a map $\pi_n(A') \to \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A')$ which is the inverse of $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(A') \to \pi_n(A')$.

(2) We are asked to prove that the map $\pi_n(B) \otimes_{\pi_0(B)} \pi_0(A) \to \pi_n(A)$ is an isomorphism if and only if the induced map $\pi_n(C) \otimes_{\pi_0(C)} \pi_0(A) \to \pi_n(A)$ is an isomorphism, given that $\pi_n(C) \otimes_{\pi_0(C)} \pi_0(B) \to \pi_n(B)$ is an isomorphism. The last isomorphism implies that the map $\pi_n(C) \otimes_{\pi_0(C)} \pi_0(A) \to \pi_n(B) \otimes_{\pi_0(B)} \pi_0(A)$ is also an isomorphism, so the desired statement reduces to the 2-out-of-3 property for isomorphisms in the commuting diagram

$$\begin{tikzcd} \pi_n(C) \otimes_{\pi_0(C)} \pi_0(A) \arrow[rr, "\cong"] \arrow[dr] & & \pi_n(B) \otimes_{\pi_0(B)} \pi_0(A) \arrow[dl] \\ & \pi_n(A) & \end{tikzcd}$$

$\square$

To see that we have indeed defined an admissibility structure on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$, we only have to check that admissible maps are stable under retracts and that in a diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[rr, "f"] \arrow[dr, "g"'] & & \operatorname{Spec} B \arrow[dl, "h"] \\ & \operatorname{Spec} C & \end{tikzcd}$$

where h is admissible, f is admissible if and only if g is admissible. Using characterization (2) of localization morphisms of proposition 4.1.3.13, and lemma 4.1.3.17, we reduce to the discrete case, which is handled in remark 3.1.3.22.

Remark 4.1.3.18. By definition of the admissibility structure on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ and the fact that effective epimorphisms are detected on connected components, the proof of proposition **??** shows that a simplicial C^∞-ring A is local as a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structure in spaces if and only if $\pi_0(A)$ is an (Archimedean) local C^∞ ring.

From the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ we deduce the existence of the spectrum functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} : \mathrm{Pro}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}) \simeq sC^\infty\mathsf{ring}^{op} \to {}^R\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ right adjoint to the global sections functor. Consider the following full subcategories of $sC^\infty\mathsf{ring}$.

(1) The full subcategory $sC^\infty\mathsf{ring}_{\mathrm{fp}}$ of *finitely presented* simplicial C^∞-rings spanned by compact objects.

(2) The full subcategory $sC^\infty\mathsf{ring}_{\mathrm{afp}}$ of *almost finitely presented* simplicial C^∞-rings spanned by almost compact objects.

(3) The full subcategory $sC^\infty\mathsf{ring}_{\mathrm{fg}}$ of *finitely generated* simplicial C^∞-rings spanned by almost finitely generated objects.

(4) The full subcategory $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ of *fair* simplicial C^∞-rings spanned by objects A for which $\pi_0(A)$ is a germ-determined C^∞-ring and $\pi_n(A)$ is a complete $\pi_0(A)$ module for all n.

Definition 4.1.3.19. (1) A $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structured ∞-topos $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is a *derived C^∞-scheme* if $\mathcal{X}$ is 0-localic and $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is a $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-scheme. We denote the ∞-category of derived C^∞-schemes by $\mathrm{d}C^\infty\mathrm{Sch}$.

(2) A 0-localic $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structured ∞-topos $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is a *fair affine derived manifold (an affine derived manifold of finite presentation/almost of finite presentation)* if $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ lies in the essential image of the spectrum functor $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}: sC^\infty\mathsf{ring}_{\mathrm{fair}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})$ $(sC^\infty\mathsf{ring}_{\mathrm{fp}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})/sC^\infty\mathsf{ring}_{\mathrm{afp}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}))$. We denote the ∞-category of fair affine derived manifolds (affine derived manifolds of finite presentation/almost of finite presentation) by $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}$ $(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}/\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{afp}})$.

(3) A derived C^∞-scheme $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is *locally fair (locally of finite presentation/locally almost of finite presentation* if $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is locally given by the spectrum of a fair simplicial C^∞-ring (a simplicial C^∞-ring of finite presentation/almost of finite presentation). The ∞-category of locally fair derived C^∞-schemes (derived C^∞-schemes locally of finite presentation/almost of finite presentation) is denoted $\mathrm{d}C^\infty\mathrm{Sch}_{\mathrm{fair}}$ $(\mathrm{d}C^\infty\mathrm{Sch}_{\mathrm{fp}}/\mathrm{d}C^\infty\mathrm{Sch}_{\mathrm{afp}})$.

Our next goal is to derive some fundamental properties of the spectrum-global sections adjunction for the geometry just defined. Let

$$\tau_{\le 0}: \mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}} \longrightarrow \mathcal{G}_{\mathrm{Diff}}$$

be the functor induced by truncation $sC^\infty\mathsf{ring} \to \tau_{\le 0}sC^\infty\mathsf{ring} \simeq \mathbf{N}(C^\infty\mathsf{ring})$. This functor preserves finite limits and carries admissibles to admissibles and coverings to coverings, so we deduce that $\tau_{\le 0}$ is a transformation of geometries. In the language of [Lur11b], the transformation $\tau_{\le 0}$ *exhibits* $\mathcal{G}_{\mathrm{Diff}}$ *as a 0-stub of* $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$, which means that

(1) for every 1-category that admits finite limits, composition with $\tau_{\le 0}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}_{\mathrm{Diff}},\mathcal{C}) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}},\mathcal{C}).$$

(2) the admissibility structure on $\mathcal{G}_{\mathrm{Diff}}$ is the coarsest one that makes $\tau_{\le 0}$ a transformation of geometries.

Property (1) follows easily from proposition 4.1.1.22, and (2) is an immediate consequence of corollary 4.1.3.15 and the definition of admissible coverings in both geometries. It follows that for any ∞-topos $\mathcal{X}$, composition with $\tau_{\le 0}$ induces an equivalence

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Diff}}}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\mathcal{X}) \cap \mathrm{Str}^{\le 0}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\mathcal{X})$$

between local $\mathcal{G}_{\mathrm{Diff}}$-structures on $\mathcal{X}$ and local $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structures on $\mathcal{X}$ that take 0-truncated values in $\mathcal{X}$. Since the functor ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}) \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})$ induced by $\tau_{\le 0}$ carries coCartesian morphisms to coCartesian morphisms, we deduce that this functor can be identified with the full subcategory inclusion ${}^{\mathrm{L}}\mathsf{Top}^{\le 0}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}) \subset {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})$ of $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structures $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}} \to \mathcal{X}$ that take 0-truncated values in $\mathcal{X}$ (as $\mathcal{X}$ varies). Our first order of business is the construction of a left adjoint to this inclusion. Naively, we might simply compose a $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structure with $\tau_{\le 0}: \mathcal{X} \to \mathcal{X}$, but this is guaranteed to fail as $\tau_{\le 0}$ does not preserve finite limits. Since $\tau_{\le 0}$ does preserve finite products, we can define the requisite functor as the upper horizontal arrow in the solid diagram

$$\begin{tikzcd}
\mathrm{Str}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\mathcal{X}) \arrow[r, dotted] \arrow[d, "\simeq"] & \mathrm{Str}_{\mathcal{G}_{\mathrm{Diff}}}(\mathcal{X}) \arrow[d, "\simeq"] \\
\mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}),\mathcal{X}) \arrow[r, "\tau_{\le 0}"] & \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}),\tau_{\le 0}\mathcal{X}),
\end{tikzcd}$$

that we will denote (somewhat abusively) also by $\tau_{\le 0}$. At this point, we need a description of what it means to be *local* for $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structures and morphisms of such as product preserving functors $\mathbf{N}(\mathsf{CartSp}) \to \mathcal{X}$.

Lemma 4.1.3.20. *Let $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}'_{\mathcal{X}}$ be $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structures on $\mathcal{X}$, then $\mathcal{O}_{\mathcal{X}}$ is local if and only if for each $\mathbb{R}^n$ and each good open cover $\{U_i \to \mathbb{R}^n\}$, the morphism $\coprod_i \mathcal{O}(U_i) \to \mathcal{O}(\mathbb{R}^n)$ is an effective epimorphism in $\mathcal{X}$. If $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}'_{\mathcal{X}}$ are local $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structures, then $\alpha: \mathcal{O}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$ is local if and only if for each open embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n$, the diagram*

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{X}}(\mathbb{R}^n) \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{X}}(\mathbb{R}^n) \arrow[d] \\
\mathcal{O}'_{\mathcal{X}}(\mathbb{R}^n) \arrow[r] & \mathcal{O}'_{\mathcal{X}}(\mathbb{R}^n)
\end{tikzcd}$$

is a pullback.

Proof. For every admissible cover $\mathfrak{U}$ of $\operatorname{Spec}_{\mathbb{R}} A \in \mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$, there exists an open cover $\mathfrak{V}$ of $\mathbb{R}^n$ and a morphism $f : \mathbf{Spec}\, A \to \mathbb{R}^n$ such that $\mathfrak{U}$ is the pullback of $\mathfrak{V}$ along f. Since every open cover of $\mathbb{R}^n$ is refined by a good one, we deduce that a $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structure is local if and only if it is local for good open covers of Cartesian spaces.
For the second claim, we consider an admissible map $U \to \operatorname{Spec}_{\mathbb{R}} A$, a map $f : \operatorname{Spec}_{\mathbb{R}} A \to \mathbb{R}^n$ and an open $V \subset \mathbb{R}^n$ such that U is obtained by pullback back V along f. We have a commuting cube

$$\begin{tikzcd}
 & \mathcal{O}_X(V) \arrow[rr] \arrow[dd] & & \mathcal{O}'_X(V) \arrow[dd] \\
\mathcal{O}_X(U) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{O}'_X(U) \arrow[ur] \arrow[dd] & \\
 & \mathcal{O}_X(\mathbb{R}^n) \arrow[rr] & & \mathcal{O}'_X(\mathbb{R}^n) \\
\mathcal{O}_X(\operatorname{Spec}_{\mathbb{R}} A) \arrow[ur] \arrow[rr] & & \mathcal{O}'_X(\operatorname{Spec}_{\mathbb{R}} A) \arrow[ur] &
\end{tikzcd}$$

wherein the faces on the side are pullbacks, so we may replace $\operatorname{Spec}_{\mathbb{R}} A$ with $\mathbb{R}^n$ and U by V. Since V admits a characteristic function, we may apply the same argument and replace $\mathbb{R}^n$ by $\mathbb{R}$, and V by $\mathbb{R} \setminus \{0\}$. If we can show that $\mathbb{R} \setminus \{0\}$ arises as the intersection of $\mathbb{R}$ inside some $\mathbb{R}^n$ with some open subset $W \subset \mathbb{R}^n$ such that W is diffeomorphic to $\mathbb{R}^n$ itself, we are done, but this is easy to arrange: choose a smooth bump function $\psi(x) : \mathbb{R} \to [0,1]$ whose value is equal to $-1/2$ on $(-1,1)$ and equal to $1/2$ on $(-\infty,-2) \cup (2,\infty)$ without local minima or maxima on $(-2,-1) \cup (1,2)$, then $\mathbb{R} \setminus \{0\}$ is diffeomorphic to the intersection of the graph of ψ with the open set $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}_{>0} \subset \mathbb{R}^2$. □

Lemma 4.1.3.21. *Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}_{\mathcal{X}}$ be a local $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structure on $\mathcal{X}$. Then $\tau_{\leq 0}\mathcal{O}_{\mathcal{X}}$ is a local $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structure and the map $\mathcal{O}_{\mathcal{X}} \to \tau_{\leq 0}\mathcal{O}_{\mathcal{X}}$ exhibits a unit transformation for the inclusion*

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Diff}}}(\mathcal{X}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\mathcal{X}) \cap \mathrm{Str}^{\leq 0}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\mathcal{X}) \subset \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\mathcal{X}).$$

Proof. Using lemma 4.1.3.20, we can use the same arguments as in propositions 3.3.3 and 3.3.5 of [Lur11b]: to see that $\tau_{\leq 0}\mathcal{O}_{\mathcal{X}}$ is local, it suffices to consider good open coverings of the form $\{U_i \to \mathbb{R}^n\}$. We have a commuting diagram

$$\begin{tikzcd}
\coprod_i \mathcal{O}_{\mathcal{X}}(U_i) \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{X}}(\mathbb{R}^n) \arrow[d] \\
\coprod_i \tau_{\leq 0}\mathcal{O}_{\mathcal{X}}(U_i) \arrow[r] & \tau_{\leq 0}\mathcal{O}_{\mathcal{X}}(\mathbb{R}^n)
\end{tikzcd}$$

It suffices to show that the upper horizontal and right vertical map are effective epimorphisms, which is the case by assumption and because effective epimorphisms are detected on sheaves of homotopy groups respectively.
To see that $\mathcal{O}_{\mathcal{X}} \to \tau_{\leq 0}\mathcal{O}_{\mathcal{X}}$ is local, we note that the proof of proposition 3.3.5 of [Lur11b] applies, since the geometry $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$ has the property that each admissible map is (-1)-truncated. The same argument as in the proof of (3) of proposition 3.3.3 of [Lur11b] shows that the map $\mathcal{O}_{\mathcal{X}} \to \tau_{\leq 0}\mathcal{O}_{\mathcal{X}}$ exhibits a unit transformation as claimed. □

Using the general yoga of coCartesian fibrations, it's easy to see that for each ∞-topos $\mathcal{X}$, the map $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}, \tau_{\leq 0}\mathcal{O}_{\mathcal{X}})$ exhibits a unit transformation for the functor ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}) \hookrightarrow {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})$ induced by $\tau_{\leq 0}$, that is, we may identify the *relative spectrum* $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}$ with the assignment $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}, \tau_{\leq 0}\mathcal{O}_{\mathcal{X}})$[2], so we have a commuting diagram

$$\begin{tikzcd}
sC^{\infty}\mathsf{ring} \arrow[r, "\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}"] \arrow[d, "\tau_{\leq 0}"] & {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}) \arrow[d, "\tau_{\leq 0} \circ -"] \\
\mathbf{N}(C^{\infty}\mathsf{ring}) \arrow[r, "\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}}"] & {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}})
\end{tikzcd}$$

[2] In proposition 3.11 of [Por15], it is claimed that for an arbitrary geometry $\mathcal{G}$ that is compatible with n-truncations, the relative spectrum $\mathbf{Spec}^{\tau_{\leq 0}\mathcal{G}}_{\mathcal{G}}$ associated to a transformation of geometries $\mathcal{G} \to \tau_{\leq n}\mathcal{G}$ exhibiting an n-stub coincides with the functor $\mathcal{O} \mapsto \tau_{\leq n}\mathcal{O}$, but this is false in general, since $\tau_{\leq n} : \mathcal{X} \to \mathcal{X}$ need not preserve finite limits. A version of this is true for geometries $\mathcal{G}$ that arise as the geometric envelope of a pregeometry $\mathcal{T}$: using proposition 3.3.3 of [Lur11b], it can be shown that the relative spectrum $\mathbf{Spec}^{\tau_{\leq n}\mathcal{G}}_{\mathcal{G}}$ coincides with the assignment $\mathcal{O} \mapsto \tau_{\leq n}\mathcal{O}$ *as functors* $\mathcal{T} \to \mathcal{X}$. Since we have equivalences $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ and $\mathrm{Str}^{\mathrm{loc},\leq n}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Str}^{\mathrm{loc},\leq n}_{\mathcal{T}}(\mathcal{X})$, this induces a functor $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc},\leq n}_{\mathcal{G}}(\mathcal{X})$, which *does not in general* coincide with the functor composing with $\tau_{\leq n} : \mathcal{X} \to \mathcal{X}$

of left adjoints. Invoking proposition 2.3.18 of [Lur11b], we deduce that if $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an (affine) $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme, then $(\mathcal{X}, \tau_{\le 0}\mathcal{O}_\mathcal{X})$ is an (affine) $\mathcal{G}_{\mathrm{Diff}}$-scheme. In particular, for A a simplicial C^∞-ring, the ∞-topos $\operatorname{Spec} A$ can be identified with $\mathsf{Shv}(X)$, with X the real spectrum of the C^∞-ring $\pi_0(A)$.

Theorem 4.1.3.22. *The adjunction*

$$
{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}) \underset{\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}}{\overset{\Gamma}{\rightleftarrows}} sC^\infty\mathsf{ring}^{op}
$$

induces an adjoint equivalence of ∞-categories

$$
\mathrm{d}C^\infty\mathsf{Aff}_{\mathrm{fair}} \simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{fair}}.
$$

Moreover, a 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an affine derived fair C^∞-scheme if and only if the following conditions are satisfied.

(1) $\mathcal{X}$ has enough points, and the topological space X underlying the 0-localic ∞-topos $\mathcal{X}$ is Hausdorff, Lindelöf and regular with respect to $\pi_0(\mathcal{O}_X)$.

(2) The global sections of $\pi_0(\mathcal{O}_X)$ are finitely generated.

Remark 4.1.3.23. Given a local C^∞-ringed space $(X, \mathcal{O}_X)$, we say that X is *regular with respect to* $\mathcal{O}_X$ if the global sections of $\mathcal{O}_X$ determine the topology of X in the following sense: X carries the initial topology with respect to the canonical map

$$
X \longrightarrow \operatorname{Spec}_\mathbb{R}\Gamma(\mathcal{O}_X).
$$

The theorem is proven by relating the adjunction $(\Gamma \dashv \mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}})$ to the adjunction on modules for all (sheaves of) homotopy groups. The following lemmas facilitate this strategy.

Lemma 4.1.3.24. *Let $\mathcal{C}_0 \subset {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}})$ denote the essential image of the full subcategory $C^\infty\mathsf{ring}_{\mathrm{fg}} \subset C^\infty\mathsf{ring}$ under $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}}$, and let $\mathcal{C} \subset {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ be the full subcategory spanned by those $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that $(\mathcal{X}, \tau_{\le 0}\mathcal{O}_\mathcal{X})$ lies in $\mathcal{C}_0$. Then $\Gamma|_\mathcal{C}$ takes values in $sC^\infty\mathsf{ring}_{\mathrm{fg}}$ and the diagram*

$$
\begin{tikzcd}
sC^\infty\mathsf{ring}_{\mathrm{fg}} & \mathcal{C} \arrow[l, "\Gamma"'] \\
\mathbf{N}(C^\infty\mathsf{ring}_{\mathrm{fg}}) \arrow[u, hook] & \mathcal{C}_0 \arrow[l, "\Gamma"] \arrow[u, hook]
\end{tikzcd}
$$

is vertically left adjointable.

Proof. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \in \mathcal{C}$, then it suffices to show that $\pi_0(\Gamma(\mathcal{O}_\mathcal{X}))$ is finitely generated as a C^∞-ring. To see this, we observe that $\Gamma(\mathcal{O}_\mathcal{X}) \to \Gamma(\tau_{\le 0}\mathcal{O}_\mathcal{X})$ exhibits a 0'th truncation in $sC^\infty\mathsf{ring}$: since $\mathcal{X} \simeq \mathsf{Shv}(\operatorname{Spec}_\mathbb{R} A)$ for some finitely generated $A \in C^\infty\mathsf{ring}$ and $\tau_{\le 0}\mathcal{X}$ is a fine sheaf of algebras, this follows from proposition 2.2.5.37. Since the unit map $A \to \Gamma(\tau_{\le 0}\mathcal{O}_\mathcal{X})$ of C^∞-rings coincides with the *fairification* of A which replaces the ideal $I \subset C^\infty(\mathbb{R}^n)$ defining A with the smallest germ determined ideal containing I, we see that $\Gamma(\tau_{\le 0}\mathcal{O}_\mathcal{X})$ is a finitely generated C^∞-ring. Now we observe that the statement that $\Gamma(\mathcal{O}_\mathcal{X}) \to \Gamma(\tau_{\le 0}\mathcal{O}_\mathcal{X})$ exhibits a 0'th truncation is simply a reformulation of the vertical left adjointability of the diagram above. ◻

Corollary 4.1.3.25. *The commuting diagram*

$$
\begin{tikzcd}
sC^\infty\mathsf{ring}_{\mathrm{fg}} \arrow[d, "\tau_{\le 0}"] & \mathcal{C} \arrow[l, "\Gamma"'] \arrow[d, "\tau_{\le 0}\circ -"] \\
\mathbf{N}(C^\infty\mathsf{ring}_{\mathrm{fg}}) & \mathcal{C}_0 \arrow[l, "\Gamma"]
\end{tikzcd}
$$

of ∞-categories is horizontally left adjointable. In particular, for any finitely generated simplicial C^∞-ring A, the 0'th truncation of the unit map $A \to \Gamma(\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} A)$ exhibits $\pi_0(\Gamma(\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} A))$ as the fairification of $\pi_0(A)$.

Now we turn to the behaviour of $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ on the higher homotopy groups.

Construction 4.1.3.26. Let A be a finitely generated simplicial C^∞-ring, and let $sC^\infty\mathsf{ring}'_{A/} \subset sC^\infty\mathsf{ring}_{A/}$ be the full subcategory spanned by those maps $A \to B$ for which the following condition is satisfied.

($*$) The map $f : A \to B$ induces an equivalence

$$(\operatorname{Spec} A, \tau_{\leq 0}\mathcal{O}_{\operatorname{Spec} A}) \longrightarrow (\operatorname{Spec} B, \tau_{\leq 0}\mathcal{O}_{\operatorname{Spec} B})$$

of $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$-structured ∞-topoi.

Then the spectrum functor takes values in the full subcategory

$${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})'_{(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})/} \subset {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})_{(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})/}$$

satisfying the condition that the projection to ${}^{\mathrm{L}}\mathsf{Top}_{\operatorname{Spec} A/}$ lands in the full subgroupoid of maps that are equivalences of ∞-topoi. This subgroupoid is a contractible Kan complex with initial object $\operatorname{Spec} A$, so the inclusion of the fibre

$$\mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)_{\mathcal{O}_{\operatorname{Spec} A}/} \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}(\operatorname{Spec} A)_{\mathcal{O}_{\operatorname{Spec} A}/} \longrightarrow {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})'_{(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})/}$$

is an equivalence of ∞-categories. The functor $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}|_{sC^\infty\mathsf{ring}'_{A/}}$ then takes values in the full subcategory

$$\mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)'_{\mathcal{O}_{\operatorname{Spec} A}/} \subset \mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)_{\mathcal{O}_{\operatorname{Spec} A}/}$$

spanned by those maps $\mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}$ that induce an equivalence after applying $\tau_{\leq 0}$, in view of condition ($*$). We claim that the global sections functor

$$\Gamma : \mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)'_{\mathcal{O}_{\operatorname{Spec} A}} \longrightarrow sC^\infty\mathsf{ring}_{A/}$$

takes values in the full subcategory $sC^\infty\mathsf{ring}'_{A/}$ of objects satisfying condition ($*$). To see this, note that for $\mathcal{O} \in \mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)'_{\mathcal{O}_{\operatorname{Spec} A}}$ the map $A \to \Gamma(\mathcal{O})$ is given by $A \to \Gamma(\mathcal{O}_{\mathbf{Spec}\, A}) \to \Gamma(\mathcal{O})$. The first map induces the fairification on π_0 by corollary 4.1.3.25, and the second map induces an isomorphism on π_0 by the commutativity of the diagram in corollary 4.1.3.25, which guarantees that condition ($*$) is satisfied. We conclude that the adjunction $(\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}} \dashv \Gamma)$ restricts to an adjunction

$$sC^\infty\mathsf{ring}'_{A/} \underset{\Gamma}{\overset{\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}}}{\rightleftarrows}} \mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)'_{\mathcal{O}_{\operatorname{Spec} A}/}$$

We have for all $n \geq 0$ a commuting diagram

$$\begin{tikzcd}
\mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\operatorname{Spec} A)'_{\mathcal{O}_{\operatorname{Spec} A}/} \arrow[r, "\Gamma"] \arrow[d, "\pi_n"] & sC^\infty\mathsf{ring}'_{A/} \arrow[d, "\pi_n"] \\
\mathrm{Mod}_{\pi_0(\mathcal{O}_{\operatorname{Spec} A})^{\mathrm{alg}}} \arrow[r, "\Gamma"] & \mathrm{Mod}_{\pi_0(A)^{\mathrm{alg}}}
\end{tikzcd}$$

where the left vertical functor is simply taking homotopy groups openwise on $\operatorname{Spec}_{\mathbb{R}} A$ because we work with fine sheaves.

Lemma 4.1.3.27. *For each $n \geq 0$, the diagram*

$$\begin{tikzcd}
\mathsf{Shv}^{\mathrm{loc}}_{sC^\infty\mathsf{ring}}(\mathcal{X})'_{\mathcal{O}_{\operatorname{Spec} A}/} \arrow[r, "\Gamma"] \arrow[d, "\pi_n"] & sC^\infty\mathsf{ring}'_{A/} \arrow[d, "\pi_n"] \\
\mathrm{Mod}_{\pi_0(\mathcal{O}_{\operatorname{Spec} A})^{\mathrm{alg}}} \arrow[r, "\Gamma"] & \mathrm{Mod}_{\pi_0(A)^{\mathrm{alg}}}
\end{tikzcd}$$

is horizontally left adjointable.

Proof. Since the functor Γ on modules is a fully faithful right adjoint participating in reflective localization, it suffices to argue that the right vertical map π_n carries the unit map $B \to \Gamma(\mathcal{O}_{\operatorname{Spec} B})$ to a localization map for each $f : A \to B \in sC^\infty\mathsf{ring}'_{A/}$. The unit map $B \to \Gamma(\mathcal{O}_{\operatorname{Spec} B})$ is identified with the global sections of the sheafification map for the presheaf defined by

$$sC^\infty\mathsf{ring}^{\mathrm{ad}}_{A/} \longrightarrow sC^\infty\mathsf{ring}, \quad U_a \mapsto B[f(a)^{-1}],$$

where $U_\alpha = \mathrm{ev}_a^{-1}(\mathbb{R} \setminus \{0\})$, with $\mathrm{ev}_a : \mathrm{Hom}_{C^\infty\mathsf{ring}}(\pi_0(A), \mathbb{R}) \to \mathbb{R}$ evaluating at $a \in \pi_0(A)$. Denote this presheaf by $\widetilde{\mathcal{O}_{\operatorname{Spec} B}}$, then the map $\pi_n(B) \to \pi_n(\Gamma(\mathcal{O}_{\operatorname{Spec} B}))$ of $\pi_0(A)$-modules is given by the global sections of the map of presheaves $\alpha : \pi_n(\widetilde{\mathcal{O}_{\operatorname{Spec} B}}) \to \pi_n(\mathcal{O}_{\operatorname{Spec} B})$. It follows from proposition 2.2.5.37 that the presheaf $\pi_n(\mathcal{O}_{\operatorname{Spec} B})$ is a sheaf and that α exhibits a sheafification, but this describes precisely the unit map of the adjunction $(M\mathbf{Spec}_{\pi_0(A)} \dashv \Gamma)$ at the object $\pi_0(B)$. $\square$

Proof of Theorem 4.1.3.22. First, note that for A a finitely generated simplicial C^∞-ring, the object $\Gamma\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}A$ is fair; this follows immediately from corollary 4.1.3.25 and lemma 4.1.3.27. If A is already fair, then the corollary and lemma imply that the map $\pi_n(A)\to\pi_n(\Gamma\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}A)$ is an equivalence for all $n\geq 0$; this proves the equivalence $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}\simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{fair}}$.
For the second assertion, if $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is an affine derived fair C^∞-scheme, then $(\mathcal{X},\mathcal{O}_{\mathcal{X}})=(\mathrm{Spec}\,A,\mathcal{O}_{\mathrm{Spec}\,A})$ for some fair simplicial C^∞-ring A, so by the first part of the proof $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ satisfies the condition in the statement of the theorem. For the converse, we take a 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured ∞-topos $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ satisfying the stated condition, then it follows from theorem 4.41 of [Joy12b] that the object $(\mathcal{X},\tau_{\leq 0}\mathcal{O}_{\mathcal{X}})$ lies in $\mathcal{C}_0$ so that $\Gamma(\tau_{\leq 0}\mathcal{O}_{\mathcal{X}})$ is a finitely generated simplicial C^∞-ring. We should show that the counit map

$$\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}\Gamma(\mathcal{O}_{\mathcal{X}})\longrightarrow(\mathcal{X},\mathcal{O}_{\mathcal{X}})$$

is an equivalence. It follows from corollary 4.1.3.25 that this counit map becomes an equivalence after applying $\tau_{\leq 0}$. Now the map

$$\mathcal{O}_{\mathbf{Spec}\,\Gamma(\mathcal{O}_{\mathcal{X}})}\longrightarrow\mathcal{O}_{\mathcal{X}}$$

is a counit for the adjunction in lemma 4.1.3.27, so the map

$$\pi_n(\mathcal{O}_{\mathbf{Spec}\,\Gamma(\mathcal{O}_{\mathcal{X}})})\longrightarrow\pi_n(\mathcal{O}_{\mathcal{X}})$$

of sheaves of $\pi_0(\mathcal{O}_{\mathbf{Spec}\,\Gamma(\mathcal{O}_{\mathcal{X}})})$-modules is a counit transformation and therefore an equivalence, as $\Gamma:\mathrm{Mod}_{\pi_0(\mathcal{O}_{\mathrm{Spec}\,A})}\to\mathrm{Mod}_{\pi_0(A)}$ is fully faithful. Since $\mathcal{X}$ is hypercomplete, we conclude. □

Remark 4.1.3.28. The functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ does not take values in hypercomplete ∞-topoi. To see this, note that for any finite $n\geq 0$, the characterization of affine fair derived C^∞-schemes of theorem 4.1.3.22 implies that the cube $[0,1]^n$ equipped with its usual sheaf of smooth functions, is an affine fair derived C^∞-scheme. It follows that the functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ sends the C^∞-ring $C^\infty([0,1]^n)$ to the cube $[0,1]^n$, so $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ sends the colimit of the diagram

$$C^\infty([0,1])\longrightarrow\ldots\longrightarrow C^\infty([0,1]^n)\longrightarrow C^\infty([0,1]^{n+1})\longrightarrow C^\infty([0,1]^{n+2})\longrightarrow\ldots$$

to the Hilbert cube.

Warning 4.1.3.29. The notion of *affineness* changes as we vary which adjectives we add to our derived C^∞-schemes. For instance, there are derived locally fair C^∞-schemes that are affine derived C^∞-schemes, yet not affine *as derived fair C^∞-schemes*. Similarly, there are derived C^∞-schemes locally almost of finite presentation that are affine derived fair C^∞-schemes, but not affine derived manifolds. The existence of such objects is essentially a consequence of the fact that the étale topology on derived C^∞-schemes is not quasi-compact.

We have defined affine derived manifolds in terms of almost finitely presented simplicial C^∞-rings. For this to be a sensible definition, we would expect that affine derived manifolds can be retrieved as the spectra of their global sections, that is, we like to have an inclusion $sC^\infty\mathsf{ring}_{\mathrm{afp}}\subset sC^\infty\mathsf{ring}_{\mathrm{fair}}$. Suppose for a moment that finite colimits in $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ are computed in $sC^\infty\mathsf{ring}$, then we would be able to conclude that $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ contains $sC^\infty\mathsf{ring}_{\mathrm{fp}}$ as a full subcategory. Indeed, theorem 4.1.3.22 shows that retracts in $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ are computed in $sC^\infty\mathsf{ring}$, and by lemma 4.1.1.20 it suffices to show that $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ contains the essential image of the Yoneda embedding $j:\mathbf{N}(\mathsf{CartSp})\to sC^\infty\mathsf{ring}$ and that $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ is stable under finite colimits and retracts.
In reality, the reflective subcategory $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ admits finite colimits by theorem 4.1.3.22, as $sC^\infty\mathsf{ring}_{\mathrm{fg}}$ admits finite colimits, but these are *not necessarily* computed in $sC^\infty\mathsf{ring}$. Thus, to prove the inclusion $sC^\infty\mathsf{ring}_{\mathrm{fp}}\subset sC^\infty\mathsf{ring}_{\mathrm{fair}}$, the strategy above does not work [3]. The proof below is based on proposition 4.1.3.32, stating that every simplicial C^∞-ring admits a 'cell decomposition' indexed by degree, and that for almost finitely presented objects, this cell decomposition is sufficiently finite.

Notation 4.1.3.30. Let V be a real vector space, possibly of infinite dimension. We write

$$C^\infty(V^\vee):=\operatorname*{colim}_{V'\subset V\,\dim V'<\infty}C^\infty((V')^\vee)$$

for the free simplicial C^∞-ring on V. Evaluation at $0\in V^\vee$ yields a map $C^\infty(V^\vee)\to\mathbb{R}$ of simplicial C^∞-rings, so $C^\infty(V^\vee)$ is augmented over the initial object in $sC^\infty\mathsf{ring}$, and we may consider the n-fold suspension $\Sigma^nC^\infty(V^\vee)$ with respect to the augmentation.

[3] This point was glossed over in [BN11]

Definition 4.1.3.31. Let $A \to B$ be a morphism of simplicial C^∞-rings. We say that B is a *good A-cell object* if B is a colimit of a sequential diagram

$$A = A_{-1} \xrightarrow{\phi_{-1}} A_0 \xrightarrow{\phi_0} A_1 \longrightarrow \dots,$$

where ϕ_{-1} is a pushout along a map of the form $A \to A \otimes^\infty C^\infty(V_{-1})$ for V_{-1} a possibly infinite dimensional vector space, and ϕ_n for $n \geq 0$ is a pushouts along a map of the form $A \otimes^\infty \Sigma^n C^\infty(V_n) \to A$ for V_n a possibly infinite dimensional vector space. A good A-cell object is

(1) *almost finite* if the dimension of the vector space V_n is finite for each $n \in \mathbb{Z}_{\geq -1}$.

(2) *finite* if it is almost finite and the directed colimit over $\mathbb{Z}_{\geq -1}$ in the definition may be replaced by a finite directed subset $\{n\}_{0 \leq n \leq k}$.

Proposition 4.1.3.32. *Let $A \to B$ be a morphism of simplicial C^∞-rings, then the following hold.*

(1) B is equivalent to a good A-cell object.

(2) If B is almost finitely presented over A, then B is equivalent to an almost finite good A-cell object.

(3) If B is finitely presented over A, then B is equivalent to a retract of a finite good A-cell object.

Cell decompositions which are guaranteed to exist by proposition 4.1.3.32 are among the most useful tools we develop in this work; they will appear again in later sections. We prove proposition 4.1.3.32 at the end of this section. Now we return to the question of fairness for almost finitely presented simplicial C^∞-rings.

Proposition 4.1.3.33. *Let A be an almost finitely presented simplicial C^∞-ring, then A is fair.*

Proof. Let A be almost finitely presented. We wish to show that A is fair. We first make the following observations.

(1) A is fair if and only if $\tau_{\leq n}A$ is fair for all $n \geq 0$.

(2) For every $n \geq 0$, there exists a finitely presented simplicial C^∞-ring A' such that $\tau_{\leq n}A$ is a retract of $\tau_{\leq n}A'$ ([Lur17b], cor. 5.5.7.4).

(3) As retracts are limits and the inclusion $sC^\infty\mathsf{ring}_{\mathrm{fair}} \hookrightarrow sC^\infty\mathsf{ring}$ preserves limits by theorem 4.1.3.22, retracts in $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ are computed in $sC^\infty\mathsf{ring}$.

Combining these facts, we may assume that A is finitely presented. Using proposition 4.1.3.32 and the stability of $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ under retracts again, we may also assume that A has a presentation as a finite good $\mathbb{R}$-cell object. Such a cell object is inductively obtained by pushouts of the form

$$\begin{tikzcd} \Sigma^{n-1}C^\infty(\mathbb{R}^n) \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\ \mathbb{R} \arrow[r] & A_n \end{tikzcd}$$

where $A_0 = C^\infty(\mathbb{R}^m)$, for some finite m. By unramifiedness, A^{alg} is given by the colimit of maps obtained by the same sequence of pushout diagrams in $s\mathsf{Cring}_{\mathbb{R}}$. We proceed by induction on the length k of the finite cell object, the case $k = 0$ being trivial.

Recall the left proper combinatorial model category structure on $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$ which presents the ∞-category $s\mathsf{Cring}_{\mathbb{R}}$. Lemma 4.1.3.4 implies that in the model category $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$, the morphism $C^\infty(\mathbb{R}^n) \to \mathbb{R}$ has a cofibrant replacement as $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)[y_1, \dots, y_n]$, with y_i in degree -1 and differential $\partial y_i = x_i$, the i'th coordinate function on $\mathbb{R}^n$. Since $\Sigma^{n-1}C^\infty(\mathbb{R}^k)^{\mathrm{alg}} \simeq \mathbb{R}[\epsilon_1, \dots, \epsilon_k]$ with $|\epsilon_i| = n - 1$ for $n > 1$, the map $\Sigma^{n-1}C^\infty(\mathbb{R}^k)^{\mathrm{alg}} \to \mathbb{R}$ can be replaced by a finite coproduct of copies of the generating cofibration $\mathbb{R}[\epsilon^i] \to \mathbb{R}[\epsilon^i, \epsilon^{i+1}]$. As the model category $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$ is left proper, it follows that A^{alg} is given by the (ordinary) colimit over a sequence of maps obtained by pushouts along the cofibrations we have just described. We have found that the object A^{alg} has a presentation in $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$ by a quasi-free object of the form

$$\tilde{A} = C^\infty(\mathbb{R}^m)[\epsilon_1^1, \dots, \epsilon_{l_1}^1, \epsilon_1^2, \dots, \epsilon_{l_2}^2, \dots, \epsilon_1^k, \dots, \epsilon_{l_k}^k]$$

where $|\epsilon_{j_i}^i| = i$ for $1 \leq i \leq k$ and some differential. As $\pi_0(A)$ is finitely presented and therefore fair as a C^∞-ring we only have to show that for all $n \geq 0$, $\pi_n(A)$ is a complete $\pi_0(A)$-module. Fix $n > 0$, and consider the truncated C^∞dga $\tau_{\leq(n+1)}\tilde{A}$, so that we have $H_n(\tau_{\leq(n+1)}\tilde{A}) \cong H_n(\tilde{A}) = \pi_n(A)$. As A is a *finite* good cell object, $\tilde{A}$ is a finitely generated free $C^\infty(\mathbb{R}^m)$-module in each degree, so $\tau_{\leq(n+1)}\tilde{A}$ is a finitely generated and free $C^\infty(\mathbb{R}^m)$-module. Now consider the presheaf of dg $C^\infty(\mathbb{R}^m)$-modules on $\mathbb{R}^m$ given by

$$\mathcal{F} := U \mapsto U \mapsto \tau_{\leq(n+1)}(C^\infty(U)[\epsilon_1^1, \dots, \epsilon_{l_1}^1, \epsilon_1^2, \dots, \epsilon_{l_2}^2, \dots, \epsilon_1^k, \dots, \epsilon_{l_k}^k]),$$

whose module of global sections is $\tau_{\le(n+1)}\tilde{A}$. This presheaf is a sheaf, precisely because $\tau_{\le(n+1)}\tilde{A}$ is a finitely generated free $C^\infty(\mathbb{R}^m)$-module[4]. By proposition 2.2.5.37 the homology groups of $\tau_{\le(n+1)}\tilde{A}$ are given by the global sections of the sheaves of homology groups of $\mathcal{F}$. This implies in particular that $H_n(\tilde{A})$ is a complete $C^\infty(\mathbb{R}^n)$-module. As the map $C^\infty(\mathbb{R}^m)\to\pi_0(A)$ is surjective, the module $H_n(\tilde{A})\otimes_{C^\infty(\mathbb{R}^m)}\pi_0(A)\cong H_n(\tilde{A})\cong\pi_n(A)$ is a complete $\pi_0(A)$-module by proposition 3.1.3.42. ☐

Corollary 4.1.3.34. *The equivalence*

$$(\Gamma\dashv\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}):\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}\simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{fair}}$$

restricts to an equivalence

$$(\Gamma\dashv\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}):\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{afp}}\simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{afp}}.$$

Remark 4.1.3.35. Let A be a simplicial C^∞-ring almost of finite presentation. It follows from proposition 4.1.3.33 and theorem 4.1.3.22 that the presheaf

$$U_a\longmapsto A[a^{-1}]$$

of simplicial C^∞-rings on the real spectrum of A is already a sheaf. There is another class of simplicial C^∞-rings for which this is true (which is incomparable with the class of almost finitely presented objects in the sense that neither class contains the other): any simplicial C^∞-ring A has an underlying $\mathbb{E}_\infty$-algebra object in the ∞-category of convenient vector spaces. If the locally convex topology on each $\pi_n(A)$ is Fréchet, then the structural presheaf is already a sheaf. If A is discrete and finitely generated, this class consists precisely of free C^∞-rings quotiented by near-point determined ideals, by Whitney's spectral theorem. Quotients by ideals generated by finitely many analytic functions are in this class, that is, C^∞-rings of functions on analytic sets, as are C^∞-rings of manifolds with corners. As a result, simplicial C^∞-rings in this class are fair.

Remark 4.1.3.36. Note that $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ sends admissible maps $A\to A[a^{-1}]$ to étale maps of $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured topoi, and the topology on $sC^\infty\mathsf{ring}_{\mathrm{afp}}$ coincides under $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ with the étale topology on $\mathsf{d}C^\infty\mathsf{Aff}$ coming from the restriction of the étale topology on ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$.

The rest of this subsection is devoted to the proof of proposition 4.1.3.32. The following lemmas are adapted from [Lur11a], lemmas 12.18 and 12.19.

Remark 4.1.3.37. The free C^∞-ring functor F^{C^∞} preserves colimits, so we have $\Sigma^nC^\infty(V^\vee)\simeq F^{C^\infty}(\mathrm{Sym}^\bullet(V[n]))$ for each $\mathbb{R}$-module V. The forgetful-free adjunction between $\mathbb{E}_\infty$-algebras and simplicial C^∞-rings now establishes the equivalence

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}}(\Sigma^nC^\infty(V^\vee),A)\simeq\mathrm{Hom}_{\mathsf{Mod}_{\mathbb{R}}}(V[n],A^{\mathrm{alg}})$$

for all $A\in sC^\infty\mathsf{ring}$.

Lemma 4.1.3.38. *Let V be a real vector space. The map $V[n]\to\Sigma^nC^\infty(V^\vee)^{\mathrm{alg}}$ corresponding to the identity $\Sigma^nC^\infty(V^\vee)\to\Sigma^nC^\infty(V^\vee)$ via the equivalences above induces an equivalence $\mathrm{Sym}^\bullet(V[n])\to\Sigma^nC^\infty(V^\vee)^{\mathrm{alg}}$ of $\mathbb{E}_\infty$-algebras over $\mathbb{R}$ for $n>0$.*

Proof. Since all forgetful and free functors involved commute with filtered colimits, we may write $V=\mathrm{colim}_{V'\subset V,\dim V'<\infty}$ and suppose that V is finite dimensional. We work by induction on n. For $n=1$, we are asked to prove that the natural map

$$\mathbb{R}\otimes_{\mathrm{Sym}^\bullet(V)}\mathbb{R}\to\mathbb{R}\otimes^\infty_{C^\infty(V^\vee)}\mathbb{R}\simeq\mathbb{R}\otimes_{C^\infty(V^\vee)^{\mathrm{alg}}}\mathbb{R}$$

is an equivalence (the last equivalence follows by unramifiedness). Suppose that V is 1-dimensional, then $\mathrm{Sym}^\bullet(V)\simeq\mathbb{R}[x]$ and we have a map of projective resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}[x] \arrow[r, "x"] \arrow[d] & \mathbb{R}[x] \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & C^\infty(\mathbb{R}) \arrow[r, "x"] & C^\infty(\mathbb{R}) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

where x denotes multiplication by the function $x\mapsto x$ on $\mathbb{R}$ which shows that $\mathrm{Tor}_i^{\mathbb{R}[x]}(\mathbb{R},\mathbb{R})\cong\mathrm{Tor}_i^{C^\infty(\mathbb{R})}(\mathbb{R},\mathbb{R})$ for all $i\ge0$, so we are done for $n=1$ and $\dim V=1$. For V k-dimensional, the map $\mathrm{Sym}^\bullet(V[1])\to\Sigma C^\infty(V^\vee)^{\mathrm{alg}}$ is simply the k-fold tensor product of the equivalence we have just established. The induction step for $n\ge1$ follows at once from unramifiedness. ☐

[4]The importance of the finiteness condition can be explained as follows: let V be an infinite dimensional real vector space, then the presheaf $\mathcal{F}':U\mapsto C^\infty(U)\otimes_{\mathbb{R}}V$ on $\mathbb{R}^m$ is *not* a sheaf. Indeed, let $\{e_\alpha\}_{\alpha\in A}$ be a (Hamel) basis for V, and take an infinite collection of disjoint opens $\{U_i\}_{i\in U}$ in $\mathbb{R}^m$ indexed by a set $I\subset A$. Then assigning to the open U_i the section $1\cdot e_i$ yields a collection of local sections which cannot be glued to a section on $\coprod_{i\in I}U_i\subset\mathbb{R}^m$. In fact, if B is a fair C^∞-ring, then $B\otimes_{\mathbb{R}}V$ is a complete B-module if and only if $\mathrm{Spec}_{\mathbb{R}}B$ is compact; see also [Joy12a], example 5.28 (d)

Lemma 4.1.3.39. *Let A be a simplicial C^∞-ring and let V be a vector space. Let $n > 0$ and $V[n] \to A^{\mathrm{alg}}$ be a map of $\mathbb{R}$-modules adjoint to a map $\varphi : V \otimes_{\mathbb{R}} A^{\mathrm{alg}}[n] \to A^{\mathrm{alg}}$ of A^{alg}-modules. By taking the symmetric algebra and the free simplicial C^∞-ring, $V[n] \to A^{\mathrm{alg}}$ is adjoint to a map $\Sigma^n C^\infty(V^\vee) \to A$. Consider the pushout diagram*

$$\begin{tikzcd} \Sigma^n C^\infty(V^\vee) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathbb{R} \arrow[r] & B \end{tikzcd}$$

Then there is a natural map $\mathrm{cofib}(\varphi) \to B^{\mathrm{alg}}$ *of A^{alg}-modules which has (2n+2)-connective cofibre.*

Proof. By unramifiedness and lemma 4.1.3.38, we have $B^{\mathrm{alg}} \simeq \mathbb{R} \otimes_{\mathrm{Sym}^\bullet(V[n])} A^{\mathrm{alg}}$. The composition

$$V \otimes_{\mathbb{R}} A^{\mathrm{alg}}[n] \xrightarrow{\varphi} A^{\mathrm{alg}} \longrightarrow B$$

of morphisms of A^{alg}-modules is homotopic to the composition

$$V \otimes_{\mathbb{R}} A^{\mathrm{alg}}[n] \longrightarrow \mathrm{Sym}^\bullet(V[n]) \longrightarrow B$$

which is nullhomotopic by construction, yielding the desired map $\mathrm{cofib}(\varphi) \to B$. Since taking cofibres commutes with tensor products, we have an equivalence

$$\mathrm{cofib}(V[n] \otimes_{\mathbb{R}} \mathrm{Sym}^\bullet(V[n]) \to \mathrm{Sym}^\bullet(V[n])) \otimes_{\mathrm{Sym}^\bullet(V[n])} A^{\mathrm{alg}} \simeq \mathrm{cofib}(V \otimes_{\mathbb{R}} A^{\mathrm{alg}}[n] \to A^{\mathrm{alg}}) = \mathrm{cofib}(\varphi).$$

One readily verifies that $\mathrm{cofib}(V[n] \otimes_{\mathbb{R}} \mathrm{Sym}^\bullet(V[n]) \to \mathrm{Sym}^\bullet(V[n]))$ has vanishing homotopy groups in degrees $0 < i \leq 2n$, so the map $\mathrm{cofib}(V[n] \otimes_{\mathbb{R}} \mathrm{Sym}^\bullet(V[n]) \to \mathrm{Sym}^\bullet(V[n])) \to \mathbb{R}$ has (2n+2)-connective cofibre, showing that the map

$$\mathrm{cofib}(V[n] \otimes_{\mathbb{R}} \mathrm{Sym}^\bullet(V[n]) \to \mathrm{Sym}^\bullet(V[n])) \otimes_{\mathrm{Sym}^\bullet(V[n])} A^{\mathrm{alg}} \simeq \mathrm{cofib}(\varphi) \to B^{\mathrm{alg}} \simeq \mathbb{R} \otimes_{\mathrm{Sym}^\bullet(V[n])} A^{\mathrm{alg}}$$

has $(2n+2)$-connective cofibre as well. □

Proof of Proposition 4.1.3.32. (1) Let $A \to B$ be a simplicial C^∞-ring. We will inductively define a sequence of n-connective maps $\psi_n : A_n \to B$ formed by pushouts as in definition 4.1.3.31. For the base step of the induction, choose an effective epimorphism $A \otimes^\infty C^\infty(\mathbb{R}^{J_0}) \to B$; for instance, J_0 may be the set of those generators of $\pi_0(B)$ that are not in the image of $\pi_0(A) \to \pi_0(B)$. Now let $n > 0$. Assuming we have constructed an $(n-1)$-connective map $\psi_{n-1} : A_{n-1} \to B$, we construct ψ_n. We have $\pi_j(A_{n-1}) \simeq \pi_j(B)$ for $j < (n-1)$. The algebraic fibre $\mathrm{fib}(\psi_{n-1}^{\mathrm{alg}})$ of the map $\psi_{n-1}^{\mathrm{alg}} : A_{n-1}^{\mathrm{alg}} \to B^{\mathrm{alg}}$ of connective $\mathbb{E}_\infty$-algebras over $\mathbb{R}$ fits into a long exact sequence

$$\ldots \to \pi_n(A_{n-1}^{\mathrm{alg}}) \to \pi_n(B^{\mathrm{alg}}) \to \pi_{n-1}(\mathrm{fib}(\psi_{n-1}^{\mathrm{alg}})) \to \pi_{n-1}(A_{n-1}^{\mathrm{alg}}) \to \pi_{n-1}(B^{\mathrm{alg}}) \to 0 \to \ldots$$

Choose a set J_n and a map $\mathbb{R}^{J_n} \otimes_{\mathbb{R}} A_{n-1}^{\mathrm{alg}}[n-1] \to \mathrm{fib}(\psi_{n-1}^{\mathrm{alg}})$ of A_{n-1}^{alg}-modules that induces a surjective map $\mathbb{R}^{J_n}[n-1] \otimes_{\mathbb{R}} \pi_0(A_{n-1}^{\mathrm{alg}}) \to \pi_{n-1}(\mathrm{fib}(\psi_{n-1}^{\mathrm{alg}}))$. The composition

$$\varphi : \mathbb{R}^{J_n} \otimes_{\mathbb{R}} A_{n-1}^{\mathrm{alg}}[n-1] \longrightarrow \mathrm{fib}(\psi_{n-1}^{\mathrm{alg}}) \longrightarrow A_{n-1}^{\mathrm{alg}}$$

in the ∞-category of A_{n-1}^{alg}-modules is adjoint to a map

$$\mathbb{R}^{J_n}[n-1] \longrightarrow A_{n-1}^{\mathrm{alg}}$$

of $\mathbb{R}$-modules. This map yields a map $\mathrm{Sym}^\bullet(\mathbb{R}^{J_n}[n-1]) \to A_{n-1}^{\mathrm{alg}}$ in $s\mathsf{Cring}_{\mathbb{R}}$, which is in turn adjoint to a map $\Sigma^{n-1} C^\infty((\mathbb{R}^{J_n})^\vee) \to A_{n-1}$ of simplicial C^∞-rings, with $\Sigma^{n-1} C^\infty((\mathbb{R}^{J_n})^\vee)$ the $(n-1)$'th suspension of $C^\infty((\mathbb{R}^{J_n})^\vee)$ at the basepoint $0 \in (\mathbb{R}^{J_n})^\vee$. Now we define A_n as the right pushout square in the diagram

$$\begin{tikzcd} \Sigma^{n-1} C^\infty((\mathbb{R}^{J_n})^\vee) \arrow[r] \arrow[d] & A \otimes^\infty \Sigma^{n-1} C^\infty((\mathbb{R}^{J_n})^\vee) \arrow[r, "f"] \arrow[d] & A_{n-1} \arrow[d] \\ \mathbb{R} \arrow[r] & A \arrow[r] & A_n \end{tikzcd}$$

where the left square and the outer rectangle are pushouts as well. The canonical nullhomotopy of the map $\mathbb{R}^{J_n} \otimes_{\mathbb{R}} A_{n-1}^{\mathrm{alg}}[n-1] \to \mathrm{fib}(\psi_{n-1}^{\mathrm{alg}}) \to B^{\mathrm{alg}}$ yields a homotopy between $\psi_{n-1} \circ f$ and $\Sigma^{n-1} C^\infty((\mathbb{R}^{J_n})^\vee) \to A \to B$, so we get a map $\psi_n : A_n \to B$. We check that ψ_n is n-connective: notice that the left and middle vertical maps in the diagram above induce surjections on connected components, so by unramifiedness, we have an equivalence $A_n^{\mathrm{alg}} \simeq \mathbb{R} \otimes_{\Sigma^{n-1} C^\infty((\mathbb{R}^{J_n})^\vee)^{\mathrm{alg}}} A_{n-1}^{\mathrm{alg}}$. For $n = 1$, we observe that $\pi_0(A_1) \simeq \pi_0(C^\infty((\mathbb{R}^{J_0})^\vee)/\pi_0(\mathrm{fib}(\psi_0^{\mathrm{alg}}))) \simeq \pi_0(B)$. For $n > 1$, lemma 4.1.3.39 provides us with a map $\mathrm{cofib}(\varphi) \to A_n$ with $(2n)$-connective cofibre. Comparing the π_{n-1}-terms in the long exact sequence associated with the fibre sequence $\mathrm{fib}(\psi_n^{\mathrm{alg}}) \to A_n^{\mathrm{alg}} \to B$ with those of the long exact sequence associated to the cofibre sequence of φ yields the desired connectivity estimate.

(2) Let A be a simplicial C^∞-ring and choose a good cell object $\{A_i\}_{i\in\mathbb{Z}_{\geq 0}}$ with an equivalence $\operatorname{colim}_{i\in\mathbb{Z}_{\geq 0}} A_i \simeq A$. We will show that if B is a finitely presented simplicial C^∞-ring, then any morphism $B \to A$ factors through a finite good cell object. The desired statement then follows by applying this to the identity morphism $A \to A$. Choose some morphism $f : B \to A$. We have $A \simeq \operatorname{colim}_{i\in\mathbb{Z}_{\geq 0}} A_i$, so f factors through some A_i. We prove by descending induction that f factors through a cell complex with finitely many cells in degrees greater than j for every $j \leq i$. For $j = i$, we use that

$$A_i = \mathbb{R} \otimes^\infty_{\Sigma^{i-1}C^\infty((\mathbb{R}^{J_i})^\vee)} A_{i-1} \simeq \operatorname*{colim}_{S\subset J_i,\,|S|<\infty} \mathbb{R} \otimes^\infty_{\Sigma^{i-1}C^\infty(\mathbb{R}^S)} A_{i-1},$$

to deduce that the map $B \to A_i$ factors through some $\mathbb{R} \otimes^\infty_{\Sigma^{i-1}C^\infty(\mathbb{R}^S)} A_{i-1}$ where S is a finite set. Now assume that $B \to A_i$ factors through a cell complex $\tilde{A}$ that is obtained from the object A_j, $j < i$, by attaching finitely many cells (in degrees $> j$). A_j is itself obtained as $\mathbb{R} \otimes^\infty_{\Sigma^{j-1}C^\infty((\mathbb{R}^{J_j})^\vee} A_{j-1}$, where J_j may be an infinite set. Just as in the case $i = j$, we have $A_j \simeq \operatorname{colim}_{S'\subset J_j,\,|S'|<\infty} C_{S'}$, where we write $C_{S'} := \mathbb{R} \otimes^\infty_{\Sigma^{j-1}C^\infty(\mathbb{R}^{S'})} A_{j-1}$. By assumption on $\tilde{A}$, we attach only finitely many cells (say n) in degree j, given by a pushout

$$\mathbb{R} \otimes^\infty_{\Sigma^j C^\infty(\mathbb{R}^n)} \operatorname*{colim}_{S'\subset J_j,\,|S'|<\infty} C_{S'}.$$

Because $\Sigma^j C^\infty(\mathbb{R}^n)$ is finitely presented, the map $\Sigma^j C^\infty(\mathbb{R}^n) \to \operatorname{colim}_{S'\subset J_j,\,|S'|<\infty} C_{S'}$ factors through some $C_{S'}$, so we can write the pushout above as the colimit $\operatorname{colim}_{S''\supset S',\,|S''|<\infty} \mathbb{R} \otimes^\infty_{\Sigma^j C^\infty(\mathbb{R}^n)} C_{S''}$. Now we repeat this argument for all cells of higher degrees, using finite presentation as there are only a finite number of cells left in each degree. We find that $\tilde{A}$ can be written as some filtered colimit $\operatorname{colim}_{k\in\mathcal{J}} \tilde{A}_k$, where each $\tilde{A}_k$ is a relative cell complex obtained by attaching a finite number of cells to the object A_{j-1}. Using compactness of B, we see that $B \to \tilde{A}$ factors through some $\tilde{A}_k$. This completes the induction step.

□

We observe that the construction of good cell objects in the proof of proposition 4.1.3.32 gives a bit more information.

Proposition 4.1.3.40. *Let $f : A \to B$ be a morphism of simplicial C^∞-rings, and suppose we have chosen a presentation*

$$A = A_{-1} \xrightarrow{\phi_{-1}} A_0 \xrightarrow{\phi_0} A_1 \longrightarrow \ldots \longrightarrow B$$

of B as a good A-cell object. If f is n-connective, then we may assume that $A = A_n$.

4.1.4 Simplicial C^∞-rings of finite presentation as a geometric envelope

Armed with the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$, we can complete our comparison of simplicial C^∞-rings with $\mathcal{T}_{\mathrm{Diff}}$-structures, and show that the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ is indeed a geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$. As a corollary, we find that the full subcategory of spectra of finitely presented simplicial C^∞-rings coincides with the *derived manifolds of finite presentation* we have already defined.

It's interesting to note how neatly the theory of derived differential geometry fits into the template of [Lur11b], sections 4.2 and 4.3 (which concern derived algebraic geometry for the Zariski and étale topology respectively). Although most of the non-formal arguments we use to show that our theory indeed follows this paradigm have to do with differential topology as opposed to algebra, many statements remain the same modulo replacing $\mathcal{T}_{\mathrm{Diff}}$ with $\mathcal{T}_{\mathrm{Zar}}(k)/\mathcal{T}_{\text{ét}}(k)$ and simplicial commutative rings with simplicial C^∞-rings; compare for instance theorem 4.1.4.6 with [Lur11b], proposition 4.2.3 and theorem 4.1.3.22 with [Lur11b], theorem 4.2.15.

Before we apply a formal argument, we need an improvement of lemma 4.1.3.9.

Proposition 4.1.4.1. *The functor $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}^{op}$ sending a manifold M to the discrete simplicial C^∞-ring of smooth functions on M is fully faithful, and preserves finite products and transverse pullbacks.*

Proof. Fully faithfulness follows because the functor in the proposition takes M to the ordinary C^∞-ring of smooth functions on M, which is a fully faithful functor (see [MR91]), followed by the fully faithful inclusion of discrete objects into $sC^\infty\mathsf{ring}$. The claim about finite products is proven in lemma 4.1.3.7, so we only have to show that pullback diagrams

$$\begin{tikzcd} Y \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

are preserved. Denote the pullback $Y \times_Z X$ by P. By theorem 2.8 of chapter 1 of [MR91], the map $\tau_{\leq 0}(C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X)) \to C^\infty(P)$ is an equivalence, so we only have to show that the higher homotopy groups vanish. Choose a cover $\{U_\alpha\}$ of Z such that each U_α is diffeomorphic to an open in $\mathbb{R}^{\dim Z}$, and let $\{V_\alpha\}$ denote the induced cover on P. Consider for each $n \geq 1$ the sheafification of the presheaf of $C^\infty(P)$-modules on P sending an open $W \subset P$ to $C^\infty(W) \otimes_{C^\infty(P)} \pi_n(C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X))$. For each $V_\alpha \subset Z$, we have

$$C^\infty(V_\alpha) \otimes_{C^\infty(P)} \pi_n(C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X)) \cong \pi_n(C^\infty(i^{-1}(U_\alpha)) \otimes^\infty_{C^\infty(U_\alpha)} C^\infty(p^{-1}(U_\alpha)))$$

by lemma 4.1.3.13. But by lemma 4.1.3.9, the transverse pullback $i^{-1}(U_\alpha) \times_{U_\alpha} p^{-1}(U_\alpha)$ is preserved by $C^\infty(\_)$ so the simplicial C^∞-ring $C^\infty(i^{-1}(U_\alpha)) \otimes^\infty_{C^\infty(U_\alpha)} C^\infty(p^{-1}(U_\alpha))$ is discrete. This implies that

$$\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X) = (\mathsf{Shv}(P), \mathcal{O}_P),$$

the manifold-theoretic intersection equipped with its local sheaf of C^∞-rings of smooth functions. Thus, the map $C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X) \to \Gamma\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X)$ coincides with the map $C^\infty(Y) \otimes^\infty_{C^\infty(Z)} C^\infty(X) \to C^\infty(P)$. But this morphism is an equivalence by proposition 4.1.3.33. $\square$

For an ∞-topos $\mathcal{X}$, there is a natural equivalence $C^\infty\mathsf{ring}(\mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(sC^\infty\mathsf{ring}^{op}_{\mathrm{fp}}, \mathcal{X}) = \mathrm{Str}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathcal{X})$ by proposition 4.1.1.22, and we write $\mathcal{O}_F$ for the $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structure associated to a C^∞-ring F in $\mathcal{X}$. We say that a C^∞-ring F in an ∞-topos $\mathcal{X}$ is *local* if $\mathcal{O}_F$ is a local $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structure in $\mathcal{X}$. The subcategory whose objects are local C^∞-rings in $\mathcal{X}$ and whose morphisms are local morphisms between them is denoted $C^\infty\mathsf{ring}^{\mathrm{loc}}(\mathcal{X}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathcal{X})$.
The following proposition shows that the theories of C^∞-rings and $\mathcal{T}_{\mathrm{Diff}}$-structures are equivalent.

Proposition 4.1.4.2. *Let $\mathcal{X}$ be an ∞-topos. Let $\iota^* : \mathrm{Str}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X}) \to C^\infty\mathsf{ring}(\mathcal{X})$ be the functor that sends a $\mathcal{T}_{\mathrm{Diff}}$-structure to its underlying C^∞-ring. Let $\iota_* : C^\infty\mathsf{ring}(\mathcal{X}) \to \mathrm{Fun}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{X})$ be a functor taking right Kan extensions as in the diagram*

$$\begin{tikzcd} \mathbf{N}(\mathsf{CartSp}) \arrow[r, "F"] \arrow[d, "\iota"] & \mathcal{X} \\ \mathcal{T}_{\mathrm{Diff}} \arrow[ur, "\iota_* F"'] & \end{tikzcd}$$

(1) ι_* takes values in $\mathcal{T}_{\mathrm{Diff}}$-structures on $\mathcal{X}$ and sends $C^\infty\mathsf{ring}^{\mathrm{loc}}(\mathcal{X})$ to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X})$.

(2) ι^* and ι_* define an equivalence of ∞-categories between $\mathrm{Str}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X})$ and $C^\infty\mathsf{ring}(\mathcal{X})$ that restricts to an equivalence on local objects and local morphisms.

Proof. (1) Let F be a C^∞-ring in $\mathcal{X}$. The right Kan extension j_*F of F along the opposite of the Yoneda embedding $j : \mathbf{N}(\mathsf{CartSp}) \hookrightarrow \mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op})^{op}$ preserves all small limits by [Lur17b], lemma 5.1.5.5. Applying the adjoint functor theorem ([Lur17b], cor. 5.5.2.9 and remark 5.5.2.10) to $(j_*F)^{op} : \mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op}) \to \mathcal{X}^{op}$, we obtain a left adjoint $L : \mathcal{X} \to \mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op})^{op}$ to j_*F. We show that for any $Y \in \mathcal{X}$, $L(Y)$ preserves finite products: the map $L(Y)(\mathbb{R}^n) \to L(Y)(\mathbb{R})^n$ is equivalent to the top horizontal map in the commuting diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op})^{op}}(L(Y), j(\mathbb{R}^n)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op})^{op}}(L(Y), j(\mathbb{R}))^n \arrow[d, "\simeq"] \\ \mathrm{Hom}_{\mathcal{X}}(Y, F(\mathbb{R}^n)) \arrow[r] & \mathrm{Hom}_{\mathcal{X}}(Y, F(\mathbb{R}))^n \end{tikzcd}$$

in $\mathcal{H}$. Since the vertical maps are equivalences and the lower horizontal map is an equivalence by assumption on F, the upper horizontal map is an equivalence as well, which shows that L lands in $sC^\infty\mathsf{ring}^{op}$. It follows that L is a left adjoint of $j_*F|_{sC^\infty\mathsf{ring}^{op}}$, so $j_*F|_{sC^\infty\mathsf{ring}^{op}}$ preserves small limits. Since the functor $sC^\infty\mathsf{ring}^{op} \to \mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op})^{op}$ is fully faithful, $j_*F|_{sC^\infty\mathsf{ring}^{op}}$ is a right Kan extension along the inclusion $\mathbf{N}(\mathsf{CartSp}) \to sC^\infty\mathsf{ring}^{op}$. This inclusion factors as in the diagram

$$\begin{tikzcd} \mathbf{N}(\mathsf{CartSp}) \arrow[r, "F"] \arrow[d, "\iota"'] & \mathcal{X} \\ \mathcal{T}_{\mathrm{Diff}} \arrow[ur, dotted, "\iota_* F"] \arrow[d, "C^\infty(\_)"'] & \\ sC^\infty\mathsf{ring}^{op} \arrow[uur, "j_*F|_{sC^\infty\mathsf{ring}^{op}}"'] & \end{tikzcd}$$

and we define ι_*F as the composition $j_*F|_{sC^\infty\mathsf{ring}^{op}} \circ C^\infty(\_)$. Since proposition 4.1.4.1 guarantees that $C^\infty(\_)$ preserves finite products and transverse pullbacks, the same is true for ι_*F, so ι_*F preserves in particular pullbacks

along admissible maps; that is, $\iota_* F$ is a $\mathcal{T}_{\mathrm{Diff}}$-structure. By proposition 4.1.4.1 again, $C^\infty(\_)$ is fully faithful, so $\iota_* F$ is a right Kan extension of F along ι.
To see that ι_* preserves local objects and local morphisms, we note that the assignment $F \mapsto \mathcal{O}_F$ sending a C^∞-ring in $\mathcal{X}$ to a left exact functor from $sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ is also a right Kan extension. Thus, $\iota_* F$ is canonically equivalent to the restriction of $\mathcal{O}_F$ to $\mathcal{T}_{\mathrm{Diff}} \subset sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$ viewed as a full subcategory via $C^\infty(\_)$. Now it is clear that locality is preserved.

(2) We check that ι_* and ι^* are mutually inverse to one another. It is clear that the counit $\iota^* \circ \iota_* \to \mathrm{id}$ is an equivalence, since we Kan extend along a full subcategory inclusion. To see that the unit $\mathrm{id} \to \iota_* \circ \iota^*$ is an equivalence, we first restrict to $\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}}$. Since any open submanifold U of $\mathbb{R}^n$ has a characteristic function, U fits into a pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathbb{R} \setminus \{0\} \arrow[d, hook] \\ \mathbb{R}^n \arrow[r] & \mathbb{R} \end{tikzcd}$$

where the vertical maps are admissible. Let $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X})$, then $\iota_*\iota^*\mathcal{O}$ lies also in $\mathrm{Str}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X})$, so $\mathcal{O}(U) \to \iota_*\iota^*\mathcal{O}(U)$ is an equivalence if $\mathcal{O} \to \iota_*\iota^*\mathcal{O}$ is an equivalence on $\mathbb{R}^n$, $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$. This is obviously true for $\mathbb{R}^n$ and $\mathbb{R}$. We can use the same argument to show that the equivalence also holds on $\mathbb{R}\setminus\{0\}$ if $\mathbb{R}\setminus\{0\}$ is diffeomorphic to a pullback of a diagram in $\mathbf{N}(\mathsf{CartSp})$ where one of the maps in the diagram is admissible, but this is easy to arrange: choose a smooth bump function $\psi(x) : \mathbb{R} \to [0,1]$ whose value is equal to $-1/2$ on $(-1,1)$ and equal to $1/2$ on $(-\infty,-2) \cup (2,\infty)$ without local minima or maxima on $(-2,-1) \cup (1,2)$, then $\mathbb{R} \setminus \{0\}$ is diffeomorphic to the intersection of the graph of ψ with the open set $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}_{>0} \subset \mathbb{R}^2$.
To show that $\mathcal{O}(M) \to \iota_*\iota^*\mathcal{O}(M)$ is an equivalence for any manifold M, we use that $\mathcal{T}_{\mathrm{Diff}} \simeq \mathrm{Idem}(\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}})$ to realize M as the splitting of an idempotent $U \to U$ in $\mathcal{T}_{\mathrm{Diff}}^{\mathrm{open}}$. Since $\mathcal{O}(U) \to \iota_*\iota^*\mathcal{O}(U)$ is an equivalence and $\mathcal{X}$ is idempotent complete, $\iota_*\iota^*\mathcal{O}(M)$ and $\mathcal{O}(M)$ split the same idempotent, so they are equivalent through the map $\mathcal{O}(M) \to \iota_*\iota^*\mathcal{O}(M)$. We will be done once we show that ι^* sends local $\mathcal{T}_{\mathrm{Diff}}$-structures to local C^∞-rings. The equivalence we have just established shows that a $\mathcal{T}_{\mathrm{Diff}}$-structure $\mathcal{O}$ is canonically equivalent to the restriction of $\mathcal{O}_{\iota^*\mathcal{O}}$ to $\mathcal{T}_{\mathrm{Diff}} \subset sC^\infty\mathsf{ring}_{\mathrm{fp}}^{op}$, confirming that ι^* preserves local objects and local morphisms.

□

Remark 4.1.4.3. If F is a *local* C^∞-ring in an ∞-topos $\mathcal{X}$, one can also prove that the left Kan extension $\iota_! F$ of F satisfies $\iota_! F(M) = \mathrm{colim}_{\mathbf{\Delta}^{op}} F(\check{\mathrm{C}}(h))$, where $\check{\mathrm{C}}(h)$ is the Čech nerve of a *good* open cover $h : \coprod_i U_i \to M$ by admissibles. From there, it is possible to prove that $\iota_! F$ is a local $\mathcal{T}_{\mathrm{Diff}}$-structure on $\mathcal{X}$ (the preservation of the required limits follows because colimits are universal in $\mathcal{X}$). Since local $\mathcal{T}_{\mathrm{Diff}}$-structures are determined by their values on $\mathbf{N}(\mathsf{CartSp})$, the left Kan extension functor $\iota_!$ is an equivalence with inverse ι^* when restricted to the subcategories of local objects, and it is therefore equivalent to ι_*.

Remark 4.1.4.4. The proof of proposition 4.1.4.2 can be amended to show that for any ∞-category $\mathcal{C}$ that admits *finite* limits and is idempotent complete, the restriction functor $\iota^* : \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{C}) \to \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}), \mathcal{C})$ is an equivalence: denote $\mathcal{C}' := \mathsf{PShv}(\mathcal{C})$, then $\mathcal{C}'$ has all small limits and for any diagram in $\mathcal{C}$ that has a limit in $\mathcal{C}$, this limit is also a limit in $\mathcal{C}'$. For $f \in \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}), \mathcal{C})$, we can consider f as a product preserving functor into $\mathcal{C}'$, and the arguments of proposition 4.1.4.2 show that $\iota_* f$ exists and lies in $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{C}')$. Because $\mathcal{C}$ admits finite limits and is idempotent complete, and $\iota_* f$ only creates retracts of pullbacks of objects in the essential image of f, $\iota_* f$ factors through $\mathcal{C}$. The counit of the adjunction is clearly an equivalence, and the unit is an equivalence by the same argument as in the proof of proposition 4.1.4.2.

Remark 4.1.4.5. Observe that proposition 4.1.4.1 and the argument of remark 4.1.4.4 allow us to prove the following slightly stronger assertion: let $\mathcal{C}$ be an ∞-category that admits finite limits and is idempotent complete, and let $\mathrm{Fun}^{\pitchfork}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{C})$ be the full subcategory of functors $\mathcal{T}_{\mathrm{Diff}} \to \mathcal{C}$ spanned by those that preserve finite products and transverse pullbacks, then the restriction map $\mathrm{Fun}^{\pitchfork}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{C}) \to \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}), \mathcal{C})$ is an equivalence.

The functor $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}^{op}$ factors through $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$, so we may state the following theorem.

Theorem 4.1.4.6. *The functor $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to \mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$ exhibits $\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}$ as a geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$.*

Proof. We should show that $C^\infty(\_)$ lies in $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})$, that for any idempotent complete ∞-category $\mathcal{C}$ that admits finite limits, composition with $C^\infty(\_)$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{C}),$$

and $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ is endowed with the coarsest admissibility structure that makes $C^\infty(\_)$ a transformation of pregeometries. Firstly, $C^\infty(\_)$ lies in $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ by proposition 4.1.4.1. Now let $\mathcal{C}$ be an idempotent complete ∞-category admitting finite limits, then we have a commuting diagram

$$\begin{tikzcd} & \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}_{\mathrm{Diff}}, \mathcal{C}) \arrow[dr, "\theta''"] & \\ \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathcal{C}) \arrow[ur, "\theta"] \arrow[rr, "\theta'"] & & \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}), \mathcal{C}) \end{tikzcd}$$

By proposition 4.1.1.22, the functor θ' is an equivalence and by remark 4.1.4.4, the functor θ'' is an equivalence. It follows that θ is an equivalence as well.
By proposition 4.1.3.13, every admissible map is a pullback of an admissible map in $\mathcal{T}_{\mathrm{Diff}}$, and just as in remark 3.1.3.24, every admissible covering in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ is pulled back from a covering in $\mathcal{T}_{\mathrm{Diff}}$. Consequently, the admissibility structure on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ is indeed the coarsest one that makes $C^\infty(\_)$ a transformation of pregeometries. □

Corollary 4.1.4.7. *Let $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})_{\leq n}$ be the opposite category of the $(n+1)$-category of compact objects in $\tau_{\leq n} sC^\infty\mathsf{ring}$ for $n \geq 0$. The inclusion $\mathcal{T}_{\mathrm{Diff}} \hookrightarrow (\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})_{\leq n}$ exhibits $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})_{\leq n}$ as an n-truncated geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$. In particular, the inclusion $\mathcal{T}_{\mathrm{Diff}} \hookrightarrow \mathbf{N}(C^\infty\mathsf{ring}_{\mathrm{fp}})^{op}$ exhibits $\mathbf{N}(C^\infty\mathsf{ring}_{\mathrm{fp}})^{op}$ as a 0-truncated geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$.*

Proof. Easy consequence of theorem 4.1.4.6 and remark 3.1.2.8. □

Proposition 4.1.4.2 shows composition with $C^\infty(\_)$ induces an equivalence between ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diff}})$ and ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$.

Corollary 4.1.4.8. *The spectrum functor $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diff}}} : \mathcal{T}_{\mathrm{Diff}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diff}})$ coincides with the composition $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} \circ C^\infty(\_)$.*

Proof. Easy consequence of theorem 4.1.4.6 and proposition 3.1.2.3. □

Corollary 4.1.4.9. *The adjoint equivalence*

$$(\Gamma \dashv \mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}) : \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{afp}} \simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{afp}}$$

restricts to an equivalence

$$\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}} \simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{fp}}.$$

Proof. $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}$ was defined as the smallest subcategory of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diff}})$ containing the essential image of $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diff}}}$ closed under retracts and finite limits. $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diff}}}$ factors through $C^\infty(\_) : \mathcal{T}_{\mathrm{Diff}} \to sC^\infty\mathsf{ring}^{op}_{\mathrm{fp}}$ via the fully faithful and limit preserving functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$, so $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}$ is equivalent, via $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$, to the smallest full subcategory of $sC^\infty\mathsf{ring}^{op}$ containing the essential image of $C^\infty(\_)$ that is stable under retracts and finite limits. Since any manifold is a retract of a limit of a transverse pullback diagram in $\mathbf{N}(\mathsf{CartSp})$, $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}$ is equivalent is to the smallest full subcategory of $sC^\infty\mathsf{ring}^{op}$ containing the essential image of the Yoneda embedding $j : \mathbf{N}(\mathsf{CartSp}) \to sC^\infty\mathsf{ring}^{op}$ that is stable under retracts and finite limits. Now the result follows from lemma 4.1.1.20. □

Notation 4.1.4.10. In the sequel, we will write **Spec** for the functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}} : sC^\infty\mathsf{ring} \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$.

4.1.5 C^∞dga's of finite presentation as a geometric envelope

Our first definition of the ∞-category of affine derived manifolds was conceptually satisfying, but computationally inconvenient. In this subsection, we are at the other end of the spectrum: we show that affine derived manifolds admit a presentation as C^∞dga's, which shows that the objects we are studying are not far removed from the d-manifolds of Joyce [Joy12b] or the dg-manifolds of Behrend-Liao-Xu [BLX20].

Definition 4.1.5.1. Let $\mathbf{cdga}^{\geq 0}_{\mathbb{R}}$ be the category of connective differentially graded algebras over $\mathbb{R}$ (grading conventions are homological). It comes with a canonical projection $\mathbf{cdga}^{\geq 0}_{\mathbb{R}} \to \mathsf{CAlg}_{\mathbb{R}}$ by restricting to degree 0, which is right adjoint to the obvious inclusion $\mathsf{CAlg}_{\mathbb{R}} \to \mathbf{cdga}^{\geq 0}_{\mathbb{R}}$. The category of *$C^\infty$ dga's*, denoted $C^\infty\mathbf{dga}$, is the pullback $\mathbf{cdga}^{\geq 0}_{\mathbb{R}} \times_{\mathsf{CAlg}_{\mathbb{R}}} C^\infty\mathsf{ring}$. Concretely, a *$C^\infty$ dga* is a connective differentially graded algebra $A_\bullet$, such that A_0 has the structure of a C^∞-ring compatible with its $\mathbb{R}$-algebra structure. A morphism of C^∞dga's is a homomorphism of connective dg algebras that restricts to a morphism of C^∞-rings in degree 0.

Remark 4.1.5.2. Occasionally, we will have to work with C^∞dga's whose underlying chain complex is not connective. The obvious inclusion $\mathsf{CAlg}_{\mathbb{R}} \to \mathbf{cdga}_{\mathbb{R}}$ has a right adjoint $\mathbf{cdga}_{\mathbb{R}} \to \mathsf{CAlg}_{\mathbb{R}}$ which takes $A_\bullet$ to $\ker(\partial_0) \subset A_0$. The category of *nonconnective C^∞ dga's*, denoted $C^\infty\mathbf{dga}^{\mathrm{nc}}$, is the pullback $\mathbf{cdga}_{\mathbb{R}} \times_{\mathsf{CAlg}} C^\infty\mathsf{ring}$ whose objects are cdga's $A_\bullet$ for which $\ker(\partial_0)$ is a C^∞-ring.

Remark 4.1.5.3. Consider the commuting diagram

$$\begin{tikzcd} \mathbf{cdga}_{\mathbb{R}}^{\geq 0} \arrow[r] \arrow[d] & \mathbf{Mod}_{\mathbb{R}}^{\geq 0} \arrow[d] \\ \mathsf{CAlg}_{\mathbb{R}} \arrow[r] & \mathrm{Vect}_{\mathbb{R}} \end{tikzcd}$$

of categories. By [Lur17a], cor. 3.2.3.2 the horizontal maps preserve and detect sifted colimits, and the right vertical map preserves all colimits, so the left vertical map preserves sifted colimits as well. Since the functor $C^\infty\mathsf{ring} \to \mathsf{CAlg}_{\mathbb{R}}$ preserves sifted colimits, it follows from [Lur17b], lem. 5.4.5.5 that the forgetful functors $C^\infty\mathbf{dga} \to \mathbf{cdga}_{\mathbb{R}}^{\geq 0}$ and $C^\infty\mathbf{dga} \to C^\infty\mathsf{ring}$ preserve sifted colimits. For nonconnective C^∞dga's, the same remarks hold for filtered colimits. It follows that both $C^\infty\mathbf{dga}$ and $C^\infty\mathbf{dga}^{\mathrm{nc}}$ are compactly generated.

Construction 4.1.5.4. There is a forgetful-free adjunction

$$(F_{\mathbf{dg}}^{C^\infty} \dashv (\_)_{\mathbf{dg}}^{\mathrm{alg}}) : \mathbf{cdga}_{\mathbb{R}}^{\geq 0} \rightleftarrows C^\infty\mathbf{dga} \ ,$$

where the right adjoint $(\_)_{\mathbf{dg}}^{\mathrm{alg}})$ takes a C^∞dga to its underlying connective cdga. The left adjoint $F_{\mathbf{dg}}^{C^\infty}$ takes a cdga $A_\bullet$ with $A_0 = 0$ to $A_\bullet$ as a C^∞dga, and it takes $\mathbb{R}[x_1,\ldots,x_n]$ in degree 0 to the free C^∞dga $C^\infty(\mathbb{R}^n)$ in degree 0. The forgetful functor $(\_)_{\mathbf{dg}}^{\mathrm{alg}}$ is conservative and preserves sifted colimits and is therefore monadic, by the classical Barr-Beck theorem.

Lemma 4.1.5.5 (Unramifiedness)**.** *Let*

$$\begin{tikzcd} A_\bullet \arrow[r, "f"] \arrow[d, "g"] & B_\bullet \arrow[d] \\ C_\bullet \arrow[r] & D_\bullet \end{tikzcd}$$

be a pushout diagram of possibly nonconnective C^∞dga's, and suppose that either f or g induces a surjection of C^∞-rings after applying the functor $\ker(\partial_0)$. Then the canonical map

$$(B_\bullet)_{\mathbf{dg}}^{\mathrm{alg}} \otimes_{(A_\bullet)_{\mathbf{dg}}^{\mathrm{alg}}} (C_\bullet)_{\mathbf{dg}}^{\mathrm{alg}} \longrightarrow (D_\bullet)_{\mathbf{dg}}^{\mathrm{alg}}$$

is an isomorphism.

Proof. Using the fact that the operation of taking hom sets commutes with limits of categories, it is not hard to see that a pushout $B_\bullet \otimes_{A_\bullet}^\infty C_\bullet$ of (possibly nonconnective) C^∞dga's fits into a pushout diagram

$$\begin{tikzcd} \ker(\partial_{B_0}) \otimes_{\ker(\partial_{A_0})} \ker(\partial_{C_0}) \arrow[r] \arrow[d] & \ker(\partial_{B_0}) \otimes_{\ker(\partial_{A_0})}^\infty \ker(\partial_{C_0}) \arrow[d] \\ B_\bullet \otimes_{A_\bullet} C_\bullet \arrow[r] & B_\bullet \otimes_{A_\bullet}^\infty C_\bullet \end{tikzcd}$$

of (nonconnective) cdga's. If either $\ker(\partial_{A_0}) \to \ker(\partial_{B_0})$ or $\ker(\partial_{A_0}) \to \ker(\partial_{C_0})$ is a surjection, the top horizontal map in this diagram is an equivalence by lemma 4.1.3.5. ☐

Proposition 4.1.5.6 (Carchedi-Roytenberg [CR12b; CR12a])**.** *There is a combinatorial model structure on $C^\infty\mathbf{dga}$ that is transferred along the adjunction $(F_{\mathbf{dg}}^{C^\infty} \dashv (\_)_{\mathbf{dg}}^{\mathrm{alg}})$. Specifically, a map f is a fibration respectively a weak equivalence if and only if $f_{\mathbf{dg}}^{\mathrm{alg}}$ is a fibration respectively a weak equivalence, and the set of generating (trivial) cofibrations is the image under $F_{\mathbf{dg}}^{C^\infty}$ of the set of generating (trivial) cofibrations in $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$. Explicitly, the set of generating cofibrations I contains the maps*

$$\mathbb{R} \longrightarrow C^\infty(\mathbb{R}), \quad C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})[\epsilon_1], \qquad \mathbb{R}[\epsilon_i] \longrightarrow \mathbb{R}[\epsilon_i,\epsilon_{i+1}],\ i \geq 1. \tag{4.3}$$

where $|\epsilon_i| = i$, and the differentials are given by $\partial\epsilon_i = \epsilon_{i+1}$ for $i \geq 2$ and $\partial\epsilon_1 = x$ for x the identity function on $\mathbb{R}$. The set of generating trivial cofibrations J contains the maps

$$\mathbb{R} \longrightarrow C^\infty(\mathbb{R})[\epsilon_1], \qquad \mathbb{R} \longrightarrow \mathbb{R}[\epsilon_i,\epsilon_{i+1}],\ i \geq 1. \tag{4.4}$$

Proof. The proof of Carchedi-Roytenberg uses Quillen's path object argument. We will argue somewhat differently as follows. It follows from remark 4.1.5.3 that $C^\infty\mathbf{dga}$ is compactly generated. Using adjointness, we see that a map g in $C^\infty\mathbf{dga}$ is a trivial fibration (i.e. a weak equivalence and a fibration) if and only if g satisfies the right lifting property with respect to the morphisms in the class I, and that g is a fibration if and only if it satisfies the right lifting property against the morphisms in the class J. Now it follows from the small object argument that all morphisms can be factored by a morphism in the class $\overline{J}$, the weak saturation of J, followed by a fibration. Similarly, all morphisms can be factored by a morphism in the class $\overline{I}$, the weak saturation of I, followed by a trivial fibration. By a standard argument, it is enough to show that every morphism in $\overline{J}$ is a weak equivalence as a morphism in $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$. Since the forgetful functor $(\_)^{\mathrm{alg}}_{\mathbf{dg}} : C^\infty\mathbf{dga} \to \mathbf{cdga}_{\mathbb{R}}$ preserves filtered colimits, weak equivalences in $C^\infty\mathbf{dga}$ are stable under retracts and transfinite compositions so we are reduced to showing that a pushout of any map in $C^\infty\mathbf{dga}$ along any of the generating trivial cofibrations of (4.4) is a weak equivalence. For the map $\mathbb{R} \to \mathbb{R}[\epsilon_i, \epsilon_{i+1}]$, this follows immediately from lemma 4.1.5.5: a pushout along this map is simply a coproduct with $\mathbb{R}[\epsilon_i, \epsilon_{i+1}]$ in $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$. For the case of $\mathbb{R} \to C^\infty(\mathbb{R})[\epsilon]$, we have to show that for any C^∞dga $A_\bullet$, the canonical map

$$f : A_\bullet \longrightarrow A_\bullet \otimes^\infty C^\infty(\mathbb{R})[\epsilon]$$

is a quasi-isomorphism, where $\otimes^\infty$ now denotes the tensor product in $C^\infty\mathbf{dga}$. The map f admits a retraction that fits into a pushout diagram

$$\begin{tikzcd} C^\infty(\mathbb{R})[\epsilon] \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ A_\bullet \otimes^\infty C^\infty(\mathbb{R})[\epsilon] \arrow[r] & A_\bullet \end{tikzcd}$$

of C^∞dga's. To show that f is a weak equivalence, it suffices to show the lower horizontal map in the diagram is one, but since the upper horizontal map is a weak equivalence and a surjection in degree 0, this is guaranteed by lemma 4.1.5.5 and the fact that the model category $\mathbf{cdga}_{\mathbb{R}}^{\geq 0}$ is left proper. ☐

We denote the ∞-category of (fibrant)-cofibrant C^∞dga's localized at the weak equivalences by $C^\infty\mathsf{Alg}$. Note that there is an obvious fully faithful and coproduct preserving functor $\mathbf{N}(\mathsf{CartSp})^{op} \hookrightarrow C^\infty\mathsf{Alg}$. This functor left Kan extends to yield a colimit preserving functor $\varphi : sC^\infty\mathsf{ring} \to C^\infty\mathsf{Alg}$.

Using that fibrations and trivial fibrations in $C^\infty\mathbf{dga}$ are stable under filtered colimits, we deduce that taking filtered colimits in $C^\infty\mathbf{dga}$ preserves trivial fibrations and thus (by Ken Brown's lemma and the fact that all objects are fibrant) all weak equivalences, so that filtered colimits are also homotopy colimits.

Theorem 4.1.5.7 (C^∞-Dold-Kan Correspondence)**.** *The functor $\varphi : sC^\infty\mathsf{ring} \to C^\infty\mathsf{Alg}$ induced by the fully faithful inclusion $\mathbf{N}(\mathsf{CartSp})^{op} \hookrightarrow C^\infty\mathsf{Alg}$ is an equivalence of ∞-categories.*

Proof. We have a commuting diagram

$$\begin{tikzcd} \mathbf{N}(\mathrm{Poly}_{\mathbb{R}})^{op} \arrow[r] \arrow[d] & \mathbf{N}(\mathsf{CartSp})^{op} \arrow[d] \\ \mathbb{E}_\infty\mathsf{Alg}_{\mathbb{R}} \arrow[r, "\mathbf{L}F^{C^\infty}_{\mathbf{dg}}"] & C^\infty\mathsf{Alg} \end{tikzcd}$$

of ∞-categories, where $\mathbf{L}F^{C^\infty}_{\mathbf{dg}}$ is the left derived functor of the free C^∞dga functor of construction 4.1.5.4. Passing to the sifted colimit completion ([Lur17b], cor. 5.3.6.10), we obtain a commuting diagram

$$\begin{tikzcd} s\mathsf{Cring}_{\mathbb{R}} \arrow[r, "F^{C^\infty}"] \arrow[d, "\simeq"] & sC^\infty\mathsf{ring} \arrow[d, "\varphi"] \\ \mathbb{E}_\infty\mathsf{Alg}_{\mathbb{R}} \arrow[r, "\mathbf{L}F^{C^\infty}_{\mathbf{dg}}"] & C^\infty\mathsf{Alg} \end{tikzcd}$$

of presentable ∞-categories and functors admitting right adjoints between them. Let $U : C^\infty\mathsf{Alg} \to sC^\infty\mathsf{ring}$ be a right adjoint to φ, and let $\mathcal{D} \subset sC^\infty\mathsf{ring}$ be the full subcategory spanned by those objects C for which the unit map $C \to U(\varphi(C))$ is an equivalence. It suffices to show that $\mathcal{D} = sC^\infty\mathsf{ring}$, and that U is conservative. Since φ is a left Kan extension along the functor $\mathbf{N}(\mathsf{CartSp})^{op} \hookrightarrow C^\infty\mathsf{Alg}$, the full subcategory $\mathbf{N}(\mathsf{CartSp})^{op} \subset sC^\infty\mathsf{ring}$ lies in $\mathcal{D}$. Passing to right adjoints in the diagram above, we have a diagram

$$\begin{tikzcd} C^\infty\mathsf{Alg} \arrow[rr, "U"] \arrow[dr, "G"'] & & sC^\infty\mathsf{ring} \arrow[dl, "(\_)^{\mathrm{alg}}"] \\ & \mathbb{E}_\infty\mathsf{Alg}_{\mathbb{R}} & \end{tikzcd}$$

where $G = (\_)^{\mathrm{alg}}_{\mathbf{dg}}$ is the right derived forgetful functor of construction 4.1.5.4. As G and $(\_)^{\mathrm{alg}}$ are both conservative, U is also conservative. Let $\mathcal{K} \coloneqq \{f_i : K_i \to C^\infty\mathsf{Alg}\}$ be the collection of small diagrams in $C^\infty\mathsf{Alg}$ such that

(a) G preserves colimits of diagrams in $\mathcal{K}$.

(b) $(\_)^{\mathrm{alg}}$ preserves colimits of diagrams in $\mathcal{K}$ after applying U.

Now note that, as $(\_)^{\mathrm{alg}}$ is conservative, U also preserves the colimits of the diagrams in $\mathcal{K}$. We observe the following:

(1) All filtered diagrams are in $\mathcal{K}$, since the underived functor $G : C^\infty\mathbf{dga} \to \mathbf{cdga}^{\geq 0}_{\mathbb{R}}$ preserves ordinary filtered colimits, which are also homotopy colimits as the model structure on $C^\infty\mathbf{dga}$ is combinatorial.

(2) Pushouts diagrams along the map

$$\mathbf{L}F^{C^\infty}_{\mathbf{dg}}(\mathbb{R}[x] \to \mathbb{R}) = \varphi(C^\infty(\mathbb{R}) \to \mathbb{R})$$

are in $\mathcal{K}$. To see that (a) holds, note that this map is modelled by the generating cofibration $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})[\epsilon]$ with $|\epsilon| = 1$, which is also a cofibration in $\mathbf{cdga}^{\geq 0}_{\mathbb{R}}$. As $\mathbf{cdga}^{\geq 0}_{\mathbb{R}}$ is left proper, this suffices. Observe that applying U to this map yields an effective epimorphism (because this can be checked by applying $(\_)^{\mathrm{alg}}$), so (b) follows by unramifiedness.

(3) Pushouts diagrams along the map

$$\mathbf{L}F^{C^\infty}_{\mathbf{dg}}(\mathbb{R}[\epsilon_n] \to \mathbb{R}) = \varphi(\Sigma^n C^\infty(\mathbb{R}) \to \mathbb{R})$$

where $n \geq 1$ and $|\epsilon_n| = n$ are in $\mathcal{K}$. Again, (a) holds because this map is modelled by the generating cofibration $\mathbb{R}[\epsilon_n] \to \mathbb{R}[\epsilon_n, \epsilon_{n+1}]$ and (b) holds because applying U yields an effective epimorphism.

It follows that U preserves the colimits described above, so $\mathcal{D} \subset sC^\infty\mathsf{ring}$ is stable under filtered colimits and pushouts along the maps $\Sigma^n C^\infty(\mathbb{R}) \to \mathbb{R}$ for $n \geq 0$. All good $\mathbb{R}$-cell objects in $sC^\infty\mathsf{ring}$ are constructed out of such colimits from the subcategory $\mathbf{N}(\mathsf{CartSp})^{op}$ so proposition 4.1.3.32 shows that we indeed have $\mathcal{D} = sC^\infty\mathsf{ring}$. □

Remark 4.1.5.8. In summary, we have very tractable models available for the geometric envelope of $\mathcal{T}_{\mathrm{Diff}}$, and for the category of affine derived manifolds, namely $\mathsf{d}C^\infty\mathsf{Aff} \simeq C^\infty\mathsf{Alg}^{op}_{\mathrm{afp}}$ and $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}} \simeq C^\infty\mathsf{Alg}^{op}_{\mathrm{fp}}$ by taking global sections and applying the smooth Dold-Kan correspondence.

Proposition 4.1.5.9. *The functor $sC^\infty\mathsf{ring} \to C^\infty\mathsf{Alg}^{\mathrm{nc}}$ induced by the right Quillen functor $C^\infty\mathbf{dga}^{\geq 0} \to C^\infty\mathbf{dga}$ is fully faithful. Moreover, there is a commuting diagram*

$$\begin{tikzcd} C^\infty\mathsf{Alg}^{\mathrm{nc}} \arrow[r, "(\_)^{\mathrm{alg}}"] \arrow[d, "\tau_{\geq 0}"] & \mathbb{E}_\infty\mathsf{Alg}^{\mathrm{nc}}_{\mathbb{R}} \arrow[d, "\tau_{\geq 0}"] \\ sC^\infty\mathsf{ring} \arrow[r, "(\_)^{\mathrm{alg}}"] & \mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}} \end{tikzcd}$$

in $\mathsf{Pr}^{\mathrm{R}}_\omega$ (compactly generated presentable ∞-categories and right adjoint continuous functors between them) which is $\tau_{\geq 0}$-left adjointable. The horizontal functors of this diagram are conservative.

Proof. We have a strictly commuting diagram of left derived functors

$$\begin{tikzcd} C^\infty\mathbf{dga}^{\mathrm{fc}} & \mathbf{cdga}^{\mathrm{fc}} \arrow[l] \\ C^\infty\mathbf{dga}^{\geq 0,\mathrm{fc}} \arrow[u] & \mathbf{cdga}^{\geq 0,\mathrm{fc}} \arrow[l] \arrow[u] \end{tikzcd}$$

between fibrant-cofibrant objects, using the fact that all objects in these model categories are fibrant. All functors in this diagram preserve weak equivalences, so taking the coherent nerve applying the fibrant replacement functor in the model category of marked simplicial sets, we obtain a (strictly) commuting diagram of ∞-categories. Each of these functors admits a right adjoint, and we obtain the desired diagram of ∞-categories commuting up to homotopy by passing the right adjoints. The unit of the adjunction on the left is an equivalence, so in order for the square to be $\tau_{\geq 0}$-left adjointable, it suffices to show that the unit of the adjunction on the right is also an equivalence, which is clear. □

We understand the model structure on C^∞dga's fairly well, so it is easy to compute explicit homotopy colimits presenting derived (non-transverse) intersections by taking cofibrant replacements.

Example 4.1.5.10 (Koszul C^∞dga's and derived zero loci of smooth functions)**.** Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. The derived zero locus $\mathrm{d}Z(f)$ of this function can be represented by a finitely presented C^∞dga, the homotopy pushout of the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^m) \arrow[r, "f^*"] \arrow[d, "\mathrm{ev}_0"'] & C^\infty(\mathbb{R}^n) \arrow[d] \\ \mathbb{R} \arrow[r] & \mathrm{d}Z(f) \end{tikzcd}$$

We can get a nice model for $\mathrm{d}Z(f)$ if we replace the map $C^\infty(\mathbb{R}^m) \to \mathbb{R}$ with the cofibration $C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)[e_1, \dots, e_m]$, with $\partial e_i = x^i$ for x^i, $1 \le i \le m$ the coordinate functions on $\mathbb{R}^m$. Since all objects in the diagram are cofibrant, the derived zero locus is modelled by the ordinary pushout of C^∞dga's $C^\infty(\mathbb{R}^n) \otimes^\infty_{C^\infty(\mathbb{R}^m)} C^\infty(\mathbb{R}^m)[e_1, \dots, e_m]$. We note that $C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)[e_1, \dots, e_m]$ is surjective in degree 0, so lemma 4.1.5.5 asserts that the derived zero locus is given by the tensor product $C^\infty(\mathbb{R}^n) \otimes_{C^\infty(\mathbb{R}^m)} C^\infty(\mathbb{R}^m)[e_1, \dots, e_m]$ of cdga's. This C^∞dga is isomorphic to the *Koszul C^∞dga* $C^\infty(\mathbb{R}^n)[e_1, \dots, e_m]$ with differential $\partial e_i = f_i$, through the map

$$C^\infty(\mathbb{R}^n) \otimes_{C^\infty(\mathbb{R}^m)} C^\infty(\mathbb{R}^m)[e_1, \dots, e_m] \to C^\infty(\mathbb{R}^n)[e_1, \dots, e_m], \quad h \otimes \left(g_0 + \sum_{i=1}^m g_i e_i\right) \mapsto hf^*(g_0) + \sum_{i=1}^m hf^*(g_i)e_i.$$

Example 4.1.5.11 (Kuranishi C^∞dga's and derived critical loci of smooth functions)**.** This example is largely a translation to the smooth setting of Vezzosi's notes on derived critical loci [Vez13]. Let $E \to M$ be a finite rank vector bundle over a manifold M. Generalizing the example above, we would like to find a convenient C^∞dga model for the derived zero locus of some smooth section $s : M \to E$. We start by taking a suitable cofibrant replacement of the map $0^* : C^\infty(E) \to C^\infty(M)$ given by pulling back along the zero section: consider the C^∞dga

$$C^\infty(E) \otimes_{C^\infty(M)} \Gamma(\Lambda^\bullet E^\vee), \quad \partial(f \otimes t)(x, v_x) = f(x, v_x)t|_x(v_x),\ x \in M,\ v \in E_x \text{ and } t \in \Gamma(\Lambda^\bullet E^\vee),\ |t| = 1$$

(as we explained in the previous example, it doesn't matter whether we take a pushout of cdga's or C^∞dga's here because the map $C^\infty(M) \to \Gamma(\Lambda^\bullet E^\vee)$ is an isomorphism in degree 0). We claim that the factorization

$$C^\infty(E) \to C^\infty(E) \otimes_{C^\infty(M)} \Gamma(\Lambda^\bullet E^\vee) \to C^\infty(M)$$

is a cofibration followed by a trivial fibration. Indeed, we note that this factorization is functorial in E so we only have to check the claim for E a trivial bundle by stability of cofibrations and trivial fibrations under retracts and the fact that any vector bundle is a retract of a trivial one. In the case of the trivial bundle $\mathbb{R}^n \times M \to M$, the factorization is simply

$$C^\infty(M \times \mathbb{R}^n) \to C^\infty(M \times \mathbb{R}^n)[e_1, \dots, e_n] \to C^\infty(M), \quad |e_i| = 1,\ \partial e_i = x^i,\ 1 \le i \le n.$$

Note that the first map is a pushout of a coproduct of generating cofibrations (and thus a cofibration), and the second map is a quasi-isomorphism by lemma 4.1.3.4 and degreewise surjective (and thus a trivial fibration). To compute the homotopy pushout $C^\infty(M) \otimes^{\infty,\mathbb{L}}_{C^\infty(E)} C^\infty(M)$ of C^∞dga's, it is not enough to replace $C^\infty(E) \xrightarrow{0^*} C^\infty(M)$ with the cofibration above, as the objects in the diagram are not cofibrant. However, since s^*, the pullback along the zero section, is an effective epimorphism, we may compute the homotopy colimit in $\mathbf{cdga}^{\ge 0}_{\mathbb{R}}$ by unramifiedness. To compute the homotopy colimit in connective cdga's, it suffices to replace 0^* with the cofibration above because the projective model structure on $\mathbf{cdga}^{\le 0}$ is left proper. We conclude that the derived zero locus of the smooth section $s : M \to E$ is computed by the ordinary pushout of connective cdga's

$$\begin{tikzcd} C^\infty(E) \arrow[r, "s^*"] \arrow[d] & C^\infty(M) \arrow[d] \\ C^\infty(E) \otimes_{C^\infty(M)} \Gamma(\Lambda^\bullet E^\vee) \arrow[r] & \mathrm{d}Z(s) \end{tikzcd}$$

(the vertical map induces an isomorphism in degree 0, so, by the same argument as in the previous example, this is also the pushout in C^∞dga's) so the derived zero locus is simply the tensor product

$$\mathrm{d}Z(s) = C^\infty(M) \otimes_{C^\infty(E)} C^\infty(E) \otimes_{C^\infty(M)} \Gamma(\Lambda^\bullet E^\vee) \simeq \Gamma(\Lambda^\bullet E^\vee),$$

with its obvious structure of a C^∞-ring in degree 0. One readily verifies that under this isomorphism, the differential maps to $\partial t = t(s)$, $t \in \Gamma(E^\vee)$. We call the C^∞dga $(\Gamma(\Lambda^\bullet E^\vee), \partial t = t(s))$ a *Kuranishi C^∞dga* for $\mathrm{d}Z(s)$[5]. We will later

[5]While it is customary -mainly in algebraic geometry- to name this complex also after Koszul, we have decided to invoke Kuranishi's name since in the study of moduli problems in differential geometry the spaces people consider (in the absence of stacky structures) are modelled on 'Kuranishi neighbourhoods'; zero loci of sections of a vector bundle (the so-called 'obstruction bundle'). The Kuranishi C^∞dga's we have just introduced are the homotopically correct objects that capture the derived geometry of zero loci of sections of vector bundles

see that any affine derived manifold X such that the cotangent complex $\mathbb{L}_X$ has Tor-amplitude $[-1,0]$ can be realized as a Kuranishi C^∞dga for some finite rank vector bundle $E \to M$.
Specializing to the case where the section s is the differential $df : M \to T^\vee M$ of a smooth function $f : M \to \mathbb{R}$, we obtain the *derived critical locus* $\mathrm{dCrit}(f) = \mathrm{d}Z(df) = (\Lambda^\bullet\Gamma(TM), \partial v = df(v))$, which is (-1)-shifted symplectic [Pan+11] (with associated P_0-structure the Schouten-Nijenhuis bracket of polyvector fields) and comes with a canonical Lagrangian fibration $\mathrm{dCrit}(f) \to M$ [Gra20].

Remark 4.1.5.12. While Koszul C^∞dga's are always cofibrant, Kuranishi C^∞dga's are usually not. For instance, $C^\infty(\mathbb{R} \setminus \{0\})$ is not cofibrant as a C^∞dga, since a lift of an invertible element along a surjection need not be invertible. However, there is an alternative model for $sC^\infty\mathsf{ring}$ in which this C^∞dga's is cofibrant: there is a localization functor on $C^\infty\mathbf{dga}$ that carries an object $A_\bullet$ to the pushout $A_\bullet \otimes^\infty_{A_0} \tilde{A}_0$, where $\tilde{A}_0$ is the germ of the zero locus of the zero'th differential on A. The essential image of this functor admits a model structure right transferred from $C^\infty\mathbf{dga}$ which is Quillen equivalent to $C^\infty\mathbf{dga}$ [Pri18].

4.1.6 Flatness of C^∞-completions and acyclicity of flat ideals

We have seen that resolving effective epimorphisms by morphisms dual to embeddings of graphs and some elementary properties of smooth functions lead to concrete ways of computing C^∞-tensor products $A \otimes^\infty_B C$, at least if one of the maps involved is an effective epimorphism. This result relates an operation induced by the extra C^∞-structure on our derived rings to the underlying homotopical algebra. In this technical subsection, we take up several other such problems which arise naturally in derived C^∞-geometry and are central to many constructions that follow in this work. The ideas in this subsection come from a variety of classical results on ideals of C^∞-functions due to Whitney, Lojasiewicz, Malgrange [Mal66] and Tougeron [Loj59; Mal66; Tou72].
Given a simplicial commutative $\mathbb{R}$-algebra, we may ask for a prescription for computing the homotopy groups of the free simplicial C^∞-ring $F^{C^\infty}(A)$ on A. We will show the following.

Proposition 4.1.6.1. *Let A be a simplicial commutative $\mathbb{R}$-algebra, then the unit map $A \to F^{C^\infty}(A)^{\mathrm{alg}}$ is flat (see definition 4.1.6.16) and thus induces for all $n \geq 0$ an equivalence*

$$\pi_n(A) \otimes_{\pi_0(A)} \pi_0(F^{C^\infty}(A)^{\mathrm{alg}}) \simeq \pi_n(F^{C^\infty}(A)^{\mathrm{alg}}).$$

In particular, the Beck-Chevalley transformation $F^{C^\infty} \circ i_n \to i_n \circ F^{C^\infty}_n$ is an equivalence, where i_n denotes the inclusions $\tau_{\leq n} sC^\infty\mathsf{ring} \subset sC^\infty\mathsf{ring}$ and $\tau_{\leq n} s\mathsf{Cring}_{\mathbb{R}} \subset s\mathsf{Cring}_{\mathbb{R}}$.

Another obvious question stems from the observation that unramifiedness tells us nothing about the homotopy groups of the coproduct $A \otimes^\infty B$ of two simplicial C^∞-rings, since the map $\mathbb{R} \to A$ is an effective epimorphism if and only if it is an equivalence. Proposition 4.1.6.1 gives a description of the homotopy groups of $A \otimes^\infty B$ when A and B lie in the essential image of the functor F^{C^∞}; indeed, in that case, we have isomorphisms

$$\pi_n(A \otimes^\infty B) \cong \pi_n(A \otimes B) \otimes_{\pi_0(A) \otimes \pi_0(B)} \pi_0(A \otimes^\infty B)$$

for all $n \geq 0$. It may then seem reasonable to expect that for any pair of simplicial C^∞-rings A and B, the canonical map

$$\pi_n(A \otimes B) \otimes_{\pi_0(A) \otimes \pi_0(B)} \pi_0(A \otimes^\infty B) \longrightarrow \pi_n(A \otimes^\infty B) \tag{4.5}$$

is an isomorphism. This assertion however is equivalent to an open problem in differential geometry.

Proposition 4.1.6.2. (1) *The following are equivalent.*

- (*a*) *For any $m, n \in \mathbb{Z}_{\geq 1}$, the map $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^{n+m})$ induced by the projections $\mathbb{R}^{n+m} \to \mathbb{R}^n$ and $\mathbb{R}^{n+m} \to \mathbb{R}^m$ onto the first n coordinates and the last m coordinates respectively is a flat map of commutative $\mathbb{R}$-algebras.*
- (*b*) *For any pair of simplicial C^∞-rings A and B, the canonical map*

$$A^{\mathrm{alg}} \otimes B^{\mathrm{alg}} \longrightarrow (A \otimes^\infty B)^{\mathrm{alg}}$$

is a flat map of simplicial commutative $\mathbb{R}$-algebras.

(2) *The following are equivalent.*

- (*a*) *For any $m, n \in \mathbb{Z}_{\geq 1}$, the map $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n+m})$ induced by the projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$ onto the first n coordinates is a flat map of commutative $\mathbb{R}$-algebras.*
- (*b*) *For any simplicial C^∞-ring A and any $m \in \mathbb{Z}_{\geq 1}$, the map $A^{\mathrm{alg}} \to (C^\infty(\mathbb{R}^m) \otimes^\infty A)^{\mathrm{alg}}$ is a flat map of simplicial commutative $\mathbb{R}$-algebras.*

(c) *For any simplicial C^∞-ring A and any $m \in \mathbb{Z}_{\geq 1}$, the map $A^{\mathrm{alg}} \to (C^\infty(\mathbb{R}^m) \otimes^\infty A)^{\mathrm{alg}}$ is strong, that is, the canonical map*

$$\pi_n(A) \otimes_{\pi_0(A)} \pi_0(A \otimes^\infty C^\infty(\mathbb{R}^m)) \longrightarrow \pi_n(A \otimes^\infty C^\infty(\mathbb{R}^n))$$

is an isomorphism for all $n \geq 0$.

(d) *For any $m \in \mathbb{Z}_{\geq 1}$ and any finitely generated ideal $I \subset C^\infty(\mathbb{R}^n)$, the first homotopy group of the coproduct $C^\infty(\mathbb{R}^n)/I \otimes^\infty C^\infty(\mathbb{R}^m)$ taken in $sC^\infty\mathsf{ring}$ vanishes.*

Clearly, the equivalent conditions of (1) imply those of (2). If we could establish the veracity of the conditions in proposition 4.1.6.4, we would also decide the question of *left properness* of the model category structure on $C^\infty\mathbf{dga}$ in the positive. Unfortunately, we haven't so far been able to prove that either of the conditions in this proposition are true, or provide a counterexample. Instead, we offer the following criterion for when the maps (4.5) are isomorphisms.

Proposition 4.1.6.3. *Let A and B be simplicial C^∞-rings and suppose that the coproduct $\pi_0(A) \otimes^\infty \pi_0(B)$ taken in $sC^\infty\mathsf{ring}$ is 0-truncated. Then the canonical map $A^{\mathrm{alg}} \otimes B^{\mathrm{alg}} \to (A \otimes^\infty B)^{\mathrm{alg}}$ is strong.*

We will prove this proposition in the next chapter, using obstruction theory along the Postnikov tower. In view of this result, it will be useful to identify some class of C^∞-rings whose coproduct in $sC^\infty\mathsf{ring}$ is 0-truncated.

Proposition 4.1.6.4. *Let $X \subset \mathbb{R}^n$ be a closed subset. Let $I \subset C^\infty(\mathbb{R}^m)$ be an ideal that is either principal or of the form $\mathfrak{m}_Y^\infty$ for some closed subset $Y \subset \mathbb{R}^m$. Then the unit map of the 0'th truncation functor*

$$C^\infty(\mathbb{R}^n)/\mathfrak{m}_X^\infty \otimes^\infty C^\infty(\mathbb{R}^m)/I \longrightarrow \tau_{\leq 0}(C^\infty(\mathbb{R}^n)/\mathfrak{m}_X^\infty \otimes^\infty C^\infty(\mathbb{R}^m)/I)$$

is an equivalence.

Propositions 4.1.6.1 and 4.1.6.4 are not quite obvious and will require some nontrivial facts about real analytic functions and Whitney functions. We will momentarily prove proposition 4.1.6.1 and give a proof of proposition 4.1.6.4 at the end of this subsection. First, we record a few consequences of these results. Note that proposition 4.1.6.4 asserts in particular that the theorem of Reyes-Van Quê remains true at the derived level.

Corollary 4.1.6.5. *The class of discrete simplicial C^∞-rings of Whitney functions is closed under coproducts in $sC^\infty\mathsf{ring}$: let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed subsets, and let $C^\infty(X;\mathbb{R}^n)$ and $C^\infty(Y;\mathbb{R}^m)$ be the discrete simplicial C^∞-rings of Whitney functions on X and Y respectively, then the canonical map*

$$C^\infty(X;\mathbb{R}^n) \otimes^\infty C^\infty(Y;\mathbb{R}^m) \longrightarrow C^\infty(X \times Y;\mathbb{R}^{n+m})$$

is an equivalence, where the tensor product is the coproduct of simplicial C^∞-rings.

As it turns out, this is an essential result for the development of derived logarithmic C^∞-geometry and derived C^∞-geometry with corners. In particular, proposition 4.1.6.4 guarantees that the notion of the 'subspace of positive elements' of a simplicial C^∞-ring is well behaved (it is canonically endowed with the structure of homotopy coherent commutative monoid, or a Γ-*object* in the sense of Segal, as we will show later). Recall that a category CartSp_c of Cartesian spaces with corners has as objects the Cartesian spaces with corners $\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$ and as morphisms the interior b-maps. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ correspond to closed quadrants of the form $\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}$ and $\mathbb{R}^l \times \mathbb{R}^{m-l}_{\geq 0}$, then proposition 4.1.6.4 implies that the composition

$$\mathbf{N}(\mathsf{CartSp}_c) \xrightarrow{j} sC^\infty\mathsf{ring}_c \xrightarrow{\iota_c^*} sC^\infty\mathsf{ring}$$

preserves coproducts, where ι_c is the fully faithful morphism of Lawvere theories $\mathsf{CartSp} \to \mathsf{CartSp}_c$. As ι_c^* is a sifted colimit completion of ι_c, we find that ι_c^* preserves all small colimits which then implies the following result.

Corollary 4.1.6.6. *Let $sC^\infty\mathsf{ring}_{pc}$ be an ∞-category of simplicial C^∞-ring with pre-corners, and consider the adjunction*

$$sC^\infty\mathsf{ring} \underset{\iota_c^*}{\overset{\iota_{c!}}{\rightleftarrows}} sC^\infty\mathsf{ring}_{pc}$$

Then the functor ι_c^* carrying a C^∞-ring with pre-corners to the underlying C^∞-ring is a left adjoint. The right adjoint is given by the functor $\iota_{c*} : sC^\infty\mathsf{ring} \to sC^\infty\mathsf{ring}_{pc}$ obtained by adjunction from the functor

$$\mathsf{CartSp}_c^{op} \times sC^\infty\mathsf{ring} \xrightarrow{j^{op}\times\mathrm{id}} sC^\infty\mathsf{ring}_{pc}^{op} \times sC^\infty\mathsf{ring} \xrightarrow{\iota_c^*\times\mathrm{id}} sC^\infty\mathsf{ring}^{op} \times sC^\infty\mathsf{ring} \xrightarrow{\mathrm{Hom}_{sC^\infty\mathsf{ring}}} \mathcal{S} \tag{4.6}$$

which is on objects given by the formula

$$\iota_{c*}(A)(\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}) = \mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}), A).$$

Proof. The existence of a right adjoint to ι_c^* is a consequence of the adjoint functor theorem. The composition $\mathsf{PShv}(\mathsf{CartSp}_c) \xrightarrow{L} sC^\infty\mathsf{ring}_{pc} \xrightarrow{\iota_c^*} sC^\infty\mathsf{ring}$ where L is a left adjoint to the inclusion preserves colimits and is therefore a left Kan extension of $i_c^* \circ j^{op} : \mathsf{CartSp}_c^{op} \to sC^\infty\mathsf{ring}$ along the Yoneda embedding, and we can identify the functor obtained via adjunction from (4.6) as a right adjoint to $\iota_c^* \circ L$. Since this right adjoint factors through $sC^\infty\mathsf{ring}_{pc}$ by corollary 4.1.6.5, it is also right adjoint to i_c^*. □

This corollary admits a somewhat surprising corollary itself.

Proposition 4.1.6.7. *The functor $\iota_c^* : sC^\infty\mathsf{ring}_{pc} \to sC^\infty\mathsf{ring}$ is a presentable fibration.*

Proof. Clearly, ι_c^* is a categorical fibration, so it suffices to show that ι_c^* is a Cartesian and coCartesian fibration with presentable fibres. We use the following formal argument, the proof of which is easy and left to the reader.

(∗) Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration among ∞-categories, and suppose that $\mathcal{C}$ admits pushouts and that p preserves pushouts, and that p admits a fully faithful left adjoint. Then p is a coCartesian fibration and an edge $e : \Delta^1 \to \mathcal{C}$ is p-coCartesian if and only if the diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ obtained from e by applying the counit transformation is a pushout.

Applying (∗) and its dual to ι_c^*, we deduce that ι_c^* is a Cartesian and coCartesian fibration. For the assertion regarding presentability, we first note that the fibres of ι_c^* are accessible as the ∞-category of accessible ∞-categories and accessible functors between them is stable under pullbacks in $\widehat{\mathsf{Cat}}_\infty$ (note that the functor $\Delta^0 \hookrightarrow \mathcal{C}$ classifying some object $C \in \mathcal{C}$ preserves colimits of weakly contractible diagrams for any ∞-category $\mathcal{C}$; in particular, this functor is κ-accessible for any regular cardinal κ). The presentability of the fibres now follows from the following formal argument.

(∗∗) Let $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration among ∞-categories and K a simplicial set. Let $f : K \to \mathcal{C}_D$ be a diagram in the fibre over some object $D \in \mathcal{D}$. Let $i_D : \mathcal{C}_D \subset \mathcal{C}$ denote the inclusion, and suppose that the induced diagram $i_D f : K \to \mathcal{C}$ admits a colimit and that p preserves the colimit of $i_D f$. Then the diagram f admits a colimit.

We prove (∗∗). Let C denote a colimit of $i_D f$ and denote $D' = p(C)$ so that we have a map $D \to D'$ for each $k \in K$. Pick one such map $e : D \to D'$. We have a diagram

$$\begin{tikzcd} K \arrow[r, "i_D f"] \arrow[d, hook] & \mathcal{C} \arrow[d] \\ K^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D} \end{tikzcd}$$

wherein the diagonal carries the cone point to C. Since the lower horizontal functor $K^{\triangleright} \to \mathcal{D}$ is a colimit diagram, the square is also a p-colimit diagram. It follows from [Lur17b], prop. 4.3.1.9 that the object C is a p-colimit of the diagram $e_! f : K \to \mathcal{C}_{D'} \subset \mathcal{C}$. Since D' is a colimit of the constant diagram with domain K on D, there is a map $e' : D' \to D$ such that $e' \circ e \simeq \mathrm{id}_D$, so using [Lur17b], prop. 4.3.1.10, we deduce that $e'_!(C)$ is a colimit of the diagram $e'_! e_! f \simeq f : K \to \mathcal{C}_D$. □

We give one final application of proposition 4.1.6.4, answering another question about the interaction of C^∞-geometry and the categorical structure of $sC^\infty\mathsf{ring}$. Corollary 4.1.6.5 asserts that the class of C^∞-rings of Whitney functions is closed under coproducts; we may also ask whether the class of C^∞-rings of Whitney functions on closed sets in a given $\mathbb{R}^n$ is closed under intersections. We verify this is the case for a class of self-intersections.

Proposition 4.1.6.8. *Let $\{X_i \subset \mathbb{R}\}_{i \in I}$ be a finite collection of closed subsets and let $X = \prod_i X_i \subset \mathbb{R}^n$ be the product as a closed subset of $\mathbb{R}^n$. Let $p : C^\infty(\mathbb{R}^n) \to C^\infty(X; \mathbb{R}^n)$ be the quotient map onto the discrete simplicial C^∞-rings of Whitney functions on X. Then the commuting diagram*

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, "p"] \arrow[d, "p"] & C^\infty(X;\mathbb{R}^n) \arrow[d, "\mathrm{id}"] \\ C^\infty(X;\mathbb{R}^n) \arrow[r, "\mathrm{id}"] & C^\infty(X;\mathbb{R}^n) \end{tikzcd}$$

is a pushout in the ∞-category $sC^\infty\mathsf{ring}$.

Remark 4.1.6.9. It follows from unramifiedness that the diagram above is also a pushout in $s\mathsf{Cring}_{\mathbb{R}}$.

For the proof, we recall the following notion.

Definition 4.1.6.10. Let $X, Y \subset \mathbb{R}^n$ be closed subsets. The sets X and Y are *regularly situated* if either $X \cap Y = \varnothing$ or for each $x_0 \subset X \cap Y$, there is a neighbourhood $x_0 \in V$ in $\mathbb{R}^n$ for which there are constants $C \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{\geq 0}$ such that for each $x \in V \cap X$, we have the inequality

$$Cd(x, X \cap Y)^{\lambda} \leq d(x, Y),$$

where $d(\_,\_)$ denotes the Euclidean distance on $\mathbb{R}^n$.

Example 4.1.6.11. If $X \subset Y$, then X and Y are regularly situated. In particular, two copies of the same set X are regularly situated.

Example 4.1.6.12. Let $X, Y \subset \mathbb{R}^n$ be subanalytic closed sets, then X and Y are regularly situated. This is proven by Bierstone-Milman [BM88].

In the next chapter, we will give a characterization of the condition of being regularly situated for $X, Y \subset \mathbb{R}^n$ in terms of the derived intersection of the locally finitely generated C^∞-schemes $(X, C^\infty_{(X;\mathbb{R}^n)})$ and $(Y, C^\infty_{(Y;\mathbb{R}^n)})$.

Definition 4.1.6.13. Given a closed set $X \subset \mathbb{R}^n$, the space $\mathcal{M}(X;\mathbb{R}^n)$ of smooth functions $f : \mathbb{R}^n \smallsetminus X \to \mathbb{R}$ that have the property that for any compact $K \subset \mathbb{R}^n$ and any multi-index $k \in \mathbb{Z}^n_{\geq 0}$, there exist constants $C, \alpha \in \mathbb{R}_{>0}$ such that for each $x \in K \smallsetminus K \cap X$ the inequality

$$|D^k(f)(x)| \leq Cd(x, X)^{-\alpha}$$

is satisfied, is the space of *multipliers for the ideal* $\mathfrak{m}^\infty_X$: for any $\varphi \in \mathcal{M}(X;\mathbb{R}^n)$ and any $f \in \mathfrak{m}^\infty_X$, the function $f\varphi$ defined on $\mathbb{R}^n \smallsetminus X$ uniquely extends to a C^∞-function (still denoted $f\varphi$) on $\mathbb{R}^n$ that is flat on X.

We will require the following result.

Lemma 4.1.6.14 (Tougeron's Multiplier Lemma)**.** *Let $X, Y \subset \mathbb{R}^n$ be closed and regularly situated, then there exists a multiplier φ for the ideal $\mathfrak{m}^\infty_{X\cap Y}$ that equals 0 in a neighbourhood of $X \smallsetminus X \cap Y$ and equals 1 in a neighbourhood of $Y \smallsetminus X \cap Y$.*

Proof. Lemme 4.5 of [Tou72]. ☐

Proof of proposition 4.1.6.8. Applying corollary 4.1.6.5, we may assume that $n = 1$. It is obvious that the diagram in the statement of the proposition is a pushout after applying the 0'th truncation functor $\tau_{\leq 0}$, so it suffices to argue that the higher homotopy groups vanish. Since $sC^\infty\mathsf{ring}$ is a coCartesian symmetric monoidal ∞-category, the pushout $C^\infty(X;\mathbb{R}) \otimes^\infty_{C^\infty(\mathbb{R})} C^\infty(X;\mathbb{R})$ is a colimit of the two sided Bar construction $\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R}))_\bullet$, the simplicial object

$$\ldots \; C^\infty(X) \otimes^\infty C^\infty(\mathbb{R})^{\otimes^\infty 2} \otimes^\infty C^\infty(X) \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; C^\infty(X) \otimes^\infty C^\infty(\mathbb{R}) \otimes^\infty C^\infty(X) \;\rightrightarrows\; C^\infty(X) \otimes^\infty C^\infty(X).$$

It follows from corollary 4.1.6.5 that $\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R}))_k \simeq C^\infty(X \times \mathbb{R}^k \times X)$ and the face maps are induced by the various inclusions of small diagonals $X \times \mathbb{R}^m \times X \hookrightarrow X \times \mathbb{R}^k \times X$ for $m < k$. As geometric realizations are sifted, the colimit $|\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R}))_\bullet|$ may be computed in the ∞-category $\mathsf{Mod}_\mathbb{R}$, where it becomes a geometric realization of a simplicial object in the heart. By the stable Dold-Kan correspondence, the homotopy groups of $|\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R}))_\bullet|$ as $\mathbb{R}$-vector spaces are computed by the spectral sequence associated to the filtered object determined by $\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R}))_\bullet$ now viewed as a simplicial object in $\mathbb{R}$-vector spaces, which collapses at the first page to the unnormalized chain complex $C(\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R})))_\bullet$. It thus suffices to show that the higher homology groups of the *normalized* chain complex $N(\mathsf{Bar}_{C^\infty(\mathbb{R})}(C^\infty(X;\mathbb{R}), C^\infty(X;\mathbb{R})))_\bullet$ vanish. This will be accomplished by constructing for each cycle in degrees ≥ 1 an explicit boundary. Unraveling the definitions, we need to show the following.

(∗) Let $k \geq 1$ and let $F(x, z_1, \ldots, z_k, y)$ be a Whitney function on $X \times \mathbb{R}^k \times X$ such that for all $1 < j \leq k$, the Whitney function $F(x, z_1, \ldots, z_j, z_j, \ldots, z_{k-1}, y)$ on $X \times \mathbb{R}^{k-1} \times X$ vanishes (if $j = k$, then we have $F(x, z_1, \ldots, z_{k-1}, y, y)$ i.e. we restrict the penultimate coordinate to X). If $F(x, x, z_2, \ldots, z_k, z)$ also vanishes, then there exists a Whitney function $\widehat{F}(x, z_1, \ldots, z_{k+1}, y)$ on $X \times \mathbb{R}^{k+1} \times X$ such that for all $1 \leq j \leq k+1$, the Whitney function $\widehat{F}(x, z_1, \ldots, z_j, z_j, \ldots, z_k, y)$ vanishes and $\widehat{F}(x, x, z_2, \ldots, z_{k+1}, y) = F$.

Let F be a cycle of degree k and let $\Delta_i \subset \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}$ for $0 \leq i \leq k$ denote the small diagonal

$$\Delta_i := \{(x_0, z_1, \ldots, z_k, z_{k+1}) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R};\, z_i = z_{i+1}\}.$$

Let f be any representative of F. Since F lies in the joint kernel of all face maps d_i for $0 < i \leq k$, the restriction $f|_{\Delta_i}$ is flat on $X \times \mathbb{R}^{k-1} \times X$ for all $0 < i \leq k$, and since the differential of the normalized chain is the 0'th face maps, we see that $f|_{\Delta_0}$ is also flat on $X\times\mathbb{R}^{k-1}\times X$. The inclusion $\Delta_0\subset\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}$ admits a smooth deformation retraction r defined by

$$(z_0,z_1,\ldots,z_k,z_{k+1})\longmapsto(1/2(z_0+z_1),z_2,\ldots,z_k,z_{k+1}).$$

Pulling back functions along the composition

$$\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}\xrightarrow{r}\Delta_0\hookrightarrow\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}$$

yields an operator $r^*(\_)|_{\Delta_0}:C^\infty(\mathbb{R}\times\mathbb{R}^k\times\mathbb{R})\to C^\infty(\mathbb{R}\times\mathbb{R}^k\times\mathbb{R})$ that we denote $(\_)_{\Delta_0}$. By the vanishing properties of F, the function f_{Δ_0} is flat on the closed subset

$$(X\times\mathbb{R}^k\times X)\cap\Delta_0\cong X\times\mathbb{R}^{k-1}\times X.$$

Now we claim that the sets Δ_0 and $X\times\mathbb{R}^k\times X$ are regularly situated: indeed, for any $p=(z_0,z_1,\ldots,z_k,z_{k+1})$, the distance $d(p,\Delta_0)$ is $1/\sqrt{2}d(z_0,z_1)$ but if $p\in X\times\mathbb{R}^k\times X$, then $d(p,(X\times\mathbb{R}^k\times X)\cap\Delta_0)$ is also $1/\sqrt{2}d(z_0,z_1)$ which immediately implies that two sets in question are regularly situated. Tougeron's multiplier lemma provides a function φ on $\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}\smallsetminus(X\times\mathbb{R}^k\times X)\cap\Delta_0$ that is 1 in a neighbourhood of $\Delta_0\smallsetminus(X\times\mathbb{R}^k\times X)\cap\Delta_0$ and 0 in a neighbourhood of $X\times\mathbb{R}^k\times X\smallsetminus(X\times\mathbb{R}^k\times X)\cap\Delta_0$. The function φf_{Δ_0} is then a C^∞ function on $\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}$ that is flat on $X\times\mathbb{R}^k\times X$ and equals f on Δ_0. Now consider the function $\tilde{f}:=f-\varphi f_{\Delta_0}$, then the Whitney jet of $\tilde{f}$ is F and $\tilde{f}$ vanishes along Δ_0. It follows from Hadamard's lemma that $\tilde{f}$ may be written as

$$\tilde{f}=(z_0-z_1)g(z_0,z_1\ldots,z_k,z_{k+1}).$$

It follows from the construction of the Hadamard quotient g that $g|_{\Delta_i}$ is flat on $X\times\mathbb{R}^{k-1}\times X$ for $i\geq 1$. Now define the function $\widehat{f}:\mathbb{R}\times\mathbb{R}^{k+1}\times\mathbb{R}\to\mathbb{R}$ via the formula

$$(z_0,z_1,\ldots,z_{k+1},z_{k+2})\longmapsto(z_1-z_2)g(z_0,z_2,z_3,\ldots,z_{k+1},z_{k+2}),$$

then one readily verifies that the Whitney jet of $\widehat{f}$ at $X\times\mathbb{R}^{k+1}\times X$ is a cycle of degree $k+1$ the boundary of which is F. □

Remark 4.1.6.15. In the next chapter, proposition 4.1.6.8 will play a crucial role in the computation of the cotangent complex of C^∞-rings of Whitney functions of the form $C^\infty(X;\mathbb{R}^n)$; it will follow rather trivially that the cotangent complex of a simplicial C^∞-ring of the form $C^\infty(X;\mathbb{R}^n)$ for $X=\prod_i X_i$ with $X_i\subset\mathbb{R}$ closed (such as rings of smooth functions on closed quadrants of the form $C^\infty(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0})$) is *free on n generators*. More precisely, the relative cotangent complex of the map $C^\infty(\mathbb{R}^n)\to C^\infty(X;\mathbb{R}^n)$ vanishes, which should be viewed as an articulation of the idea that seen through the lens of deformation theory, the simplicial C^∞-rings $C^\infty(X;\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$ are equivalent. We will give a number of consequences of this result in the next chapter; for instance, we will deduce that the map $\mathrm{Hom}(C^\infty(X;\mathbb{R}),A)\to\mathrm{Hom}(C^\infty(\mathbb{R}),A)$ of spaces is an inclusion of connected components, and if a map $A\to B$ of simplicial C^∞-rings exhibits B as an m-truncation of A, then the induced map

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(X;\mathbb{R}^n),A)\longrightarrow\mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(X;\mathbb{R}^n),B)$$

exhibits an m-truncation of spaces. Proofs will be provided in the next chapter. In fact, there unfortunately are several instances in this chapter where we use that $C^\infty(X;\mathbb{R}^n)$ has a free cotangent complex. Since a detailed discussion of the cotangent complex is not in order at this point, we ask the reader to tolerate a small amount of nonlinear logical interdependency and recognize that no circular reasoning occurs.

We proceed with the proof of proposition 4.1.6.1. Recall the following definition.

Definition 4.1.6.16. A morphism $f:A\to B$ of simplicial commutative $\mathbb{R}$-algebras is *flat* (respectively *faithfully flat*) if $\pi_0(f)$ is a flat (respectively faithfully flat) morphism of commutative $\mathbb{R}$-algebras and f is strong. A morphism $f:A\to B$ of simplicial C^∞-rings is *flat* (respectively *faithfully flat*) if f^{alg} is flat (respectively faithfully flat).

We start by recording the following permanence properties of flat morphisms.

Proposition 4.1.6.17. *Let* $\mathrm{Flat}\subset\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}})$ *be the full subcategory spanned by flat morphisms. Then the following hold.*

(1) *The full subcategory* $\mathrm{Flat}\subset\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}})$ *is stable under composition.*

(2) *If $f:A\to B$ is flat, and $g:A\to C$ is any morphism in $s\mathsf{Cring}_{\mathbb{R}}$ then the base change $B\to A\otimes_B C$ is flat.*

(3) $\mathrm{Flat}\subset\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}})$ *is stable under filtered colimits.*

(4) $\mathrm{Flat}\subset\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}})$ *is stable under retracts.*

(5) $\mathrm{Flat} \subset \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ *is stable under finite products.*

(6) *If A is a coherent simplicial $\mathbb{R}$-algebra, then the ∞-category Flat_A is stable under arbitrary small products.*

Proof. All of these assertions are standard. For instance, in the case of (3), we let $f : K \to \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ be a filtered diagram of flat morphisms. Let $\overline{f} : K^{\triangleright} \to s\mathsf{Cring}_{\mathbb{R}}$ be a colimit diagram of the composition

$$K \xrightarrow{f} \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \xrightarrow{\mathrm{ev}_0} s\mathsf{Cring}_{\mathbb{R}},$$

then we have a commuting square

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \arrow[d, "\mathrm{ev}_0"] \\ K^{\triangleright} \arrow[r, "\overline{f}"] & s\mathsf{Cring}_{\mathbb{R}} \end{tikzcd}$$

of ∞-categories and it is easy to see that an ev_0-colimit of this diagram is a colimit of f. All the fibres of the coCartesian fibration ev_0 admit colimits, and for each map $g : A \to B$, the coCartesian pushforward functor $g_!$ can be identified with the base change functor along g which preserves colimits, so f admits an ev_0-colimit. We may compute this colimit by taking a coCartesian transformation $F : K \times \Delta^1 \to \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ such that $F|_{K\times\{0\}} = f$ and $\mathrm{ev}_0 \circ F|_{K\times\{1\}}$ is constant on $\overline{f}(-\infty)$, and taking the colimit of $F|_{K\times\{1\}}$ in $(s\mathsf{Cring}_{\mathbb{R}})_{\overline{f}(-\infty)/}$. Since base change preserves flatness, $F|_{K\times\{1\}}$ is a filtered diagram of flat morphisms, whose colimit may be computed in $\mathsf{Mod}_{\overline{f}(-\infty)}$; it follows that this colimit is flat. Now (4) follows from (3) since the ∞-category Idem classifying idempotents is filtered. ☐

We have need of the following classical result of Malgrange.

Proposition 4.1.6.18 (Malgrange [Mal66]). *Let $\{f_1, \dots, f_n\}$ be a collection of real analytic functions on $\mathbb{R}^n$, then the finitely generated ideal $(f_1, \dots, f_n)$ is closed.*

As an immediate corollary, we have the following.

Corollary 4.1.6.19. *Let $x \in \mathbb{R}^n$, then the local morphism of local $\mathbb{R}$-algebras $\mathcal{O}_x^{\mathrm{an}} \to C^\infty(\mathbb{R}^n)_x$ is faithfully flat.*

Proposition 4.1.6.20. *For every integer $n \geq 0$, the map of $\mathbb{R}$-algebras*

$$\mathbb{R}[x_1, \dots, x_n] \longrightarrow C^\infty(\mathbb{R}^n)$$

determined by the n coordinate functions is flat.

Proof. Clearly, we may suppose that $n \geq 1$. Consider the composition

$$\varphi : \mathbb{R}[x_1, \dots, x_n] \longrightarrow C^\infty(\mathbb{R}^n) \longrightarrow \prod_{x\in\mathbb{R}^n} C^\infty(\mathbb{R}^n)_x$$

where the second map is induced by the quotient maps $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/\mathfrak{m}_x^g \simeq C^\infty(\mathbb{R}^n)_x$ sending a smooth function to its germ at x as x ranges over $\mathbb{R}^n$. We now prove the proposition under the assumption that the map φ is flat. Let I be an ideal of $\mathbb{R}[x_1, \dots, x_n]$, then we should show that the top horizontal map in the commuting diagram

$$\begin{tikzcd} I \otimes_{\mathbb{R}[x_1,\dots,x_n]} C^\infty(\mathbb{R}^n) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^n) \arrow[d] \\ I \otimes_{\mathbb{R}[x_1,\dots,x_n]} \prod_{x\in\mathbb{R}^n} C^\infty(\mathbb{R}^n)_x \arrow[r] & \prod_{x\in\mathbb{R}^n} C^\infty(\mathbb{R}^n)_x \end{tikzcd}$$

is injective. By assumption, the lower horizontal map is injective, so it suffices to show that the left vertical map is injective. Unwinding the definitions, we observe that it suffices to show the following.

($*$) Let $\{P_i\}_{i\in J}$ and $\{f_i\}_{i\in J}$ be finite collections of real polynomials in n variables and smooth functions in n variables respectively. Suppose that there exist a finite index set K together with a K-indexed collection $\{Q_{ik}\}$ of real polynomials in n variables for each $i \in J$ such that the following hold.

  (a) $\sum_i Q_{ik} P_i = 0$ for each $k \in K$.

  (b) For each $x \in \mathbb{R}^n$ there exists an open neighbourhood $x \in U_x$ and a K-indexed collection $\{g_k^x\}$ of smooth functions on U_x such that $f_i = \sum_k g_k^x Q_{ik}$ on U_x.

  Then there exists a K-indexed collection $\{g_k\}$ of smooth functions on $\mathbb{R}^n$ such that $f_i = \sum_k g_k Q_{ik}$.

To prove $(*)$, let $\{\psi_x\}_{x\in\mathbb{R}^n}$ be a partition of unity subordinate to the cover $\{U_x \to \mathbb{R}^n\}_{x\in\mathbb{R}^n}$, and define $g_k := \sum_{x\in\mathbb{R}^n} \psi_x g_k^x$, then it is easy to see that $f_i = \sum_k g_k Q_{ik}$ holds for all $i \in J$. We are left to prove that φ is a flat morphism. Since $\mathbb{R}[x_1,\ldots,x_n]$ is Noetherian hence coherent, (6) of proposition 4.1.6.17 guarantees that it suffices to show that for each $x \in \mathbb{R}^n$, the map $\mathbb{R}[x_1,\ldots,x_1] \to C^\infty(\mathbb{R}^n)_x$ is flat. As localizations are flat, we only have to show that the local homomorphism $\mathcal{O}^{\mathrm{reg}}(\mathbb{R}^n)_x \to C^\infty(\mathbb{R}^n)_x$ is flat, where $\mathcal{O}^{\mathrm{reg}}(\mathbb{R}^n)_x$ is the local ring of regular functions at x. We have a factorization

$$\mathcal{O}^{\mathrm{reg}}(\mathbb{R}^n)_x \longrightarrow \mathcal{O}^{\mathrm{an}}(\mathbb{R}^n)_x \longrightarrow C^\infty(\mathbb{R}^n)_x$$

of local ring homomorphisms where $\mathcal{O}^{\mathrm{an}}(\mathbb{R}^n)_x$ is the local ring of real analytic functions on $\mathbb{R}^n$ at x. The first map is a local morphism between Noetherian local rings that becomes an equivalence after formal completion at the maximal ideals and is thus faithfully flat, and the second map is faithfully flat by corollary 4.1.6.19. $\square$

Remark 4.1.6.21. We will use corollary 4.1.6.19 again in the next section, to prove the more powerful result that the unit of the relative spectrum functor sending a *derived real analytic space* to the corresponding derived C^∞-scheme is faithfully flat.

Remark 4.1.6.22. In the proof of proposition 4.1.6.20 we use that for every $x \in \mathbb{R}^n$ the map $\mathcal{O}_x^{\mathrm{reg}} \to C^\infty(\mathbb{R}^n)_x$ is faithfully flat; beware however that the map $\mathbb{R}[x_1,\ldots,x_n] \to C^\infty(\mathbb{R}^n)$ is *not* faithfully flat because $\mathbb{R}$ is not algebraically closed. The maximal ideals of $\mathbb{R}[x_1,\ldots,x_n]$ with residue field $\mathbb{C}$, such as $(x_1^2+1, x_2,\ldots,x_n)$, have the property that multiplying such an ideal with the module $C^\infty(\mathbb{R}^n)$ recovers all of $C^\infty(\mathbb{R}^n)$ and therefore do not lie in the image of the induced map on maximal ideal spectra.

The following proposition is yet another consequence of the resolution theorem for effective epimorphism.

Proposition 4.1.6.23. *Let $f : A \to B$ be an effective epimorphism of simplicial commutative $\mathbb{R}$-algebras, then the natural diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ F^{C^\infty}(A)^{\mathrm{alg}} \arrow[r] & F^{C^\infty}(B)^{\mathrm{alg}} \end{tikzcd}$$

is a pushout in $s\mathsf{Cring}_{\mathbb{R}}$.

Proof. Applying the unit transformation $s\mathsf{Cring}_{\mathbb{R}} \times \Delta^1 \to s\mathsf{Cring}_{\mathbb{R}}$ of the adjunction $(F^{C^\infty} \dashv (\_)^{\mathrm{alg}})$ yields a functor

$$\mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \longrightarrow \mathrm{Fun}(\Delta^1\times\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$$

carrying a map $A \to B$ to the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ F^{C^\infty}(A)^{\mathrm{alg}} \arrow[r] & F^{C^\infty}(B)^{\mathrm{alg}}. \end{tikzcd}$$

As the comonad $F^{C^\infty}(\_)^{\mathrm{alg}}$ preserves sifted colimits, this functor preserves sifted colimits. Since the full subcategory of $\mathrm{Fun}(\Delta^1\times\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ spanned by pushout diagrams is stable under colimits, proposition 4.1.2.3 asserts that we may suppose that $A \to B$ is a graph inclusion. We wish to show that the diagram

$$\begin{tikzcd} \mathbb{R}[x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m}] \arrow[r] \arrow[d] & \mathbb{R}[x_1,\ldots,x_n] \arrow[d] \\ C^\infty(\mathbb{R}^{n+m}) \arrow[r] & C^\infty(\mathbb{R}^n) \end{tikzcd}$$

is a pushout in $s\mathsf{Cring}_{\mathbb{R}}$. It is not hard to see that as in the proof of lemma 4.1.3.5, an inductive argument reduces us to the case $m = 1$. Then the horizontal maps in the diagram above are induced by the inclusion of the graph of a polynomial $P(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ in n variables. The $\mathbb{R}$-algebra $\mathbb{R}[x_1,\ldots,x_n]$ has a projective resolution $\mathbb{R}[x_1,\ldots,x_n,x_{n+1},\epsilon]$ with $\partial\epsilon = x_{n+1} - P(\mathbf{x})$ as a $\mathbb{R}[x_1,\ldots,x_n,x_{n+1}]$-module, so using that the torsion spectral sequence of the pushout collapses at the second page, we see that the homotopy groups of the pushout are given by the homology of the complex $C^\infty(\mathbb{R}^{n+1})[\epsilon]$. Lemma 4.1.3.4 asserts that the homology is indeed $C^\infty(\mathbb{R}^n)$, concentrated in degree zero. $\square$

Corollary 4.1.6.24. *For every simplicial commutative $\mathbb{R}$-algebra A, the unit map $A \to F^{C^\infty}(A)^{\mathrm{alg}}$ of the free C^∞-ring monad is flat.*

Proof. The class of simplicial commutative $\mathbb{R}$-algebras which satisfies the conclusion of the corollary is stable under filtered colimits, so we may suppose that A is of finite type over $\mathbb{R}$. Invoking proposition 4.1.1.18, we can find an effective epimorphism $\mathbb{R}[x_1,\dots,x_n]\to A$, so that proposition 4.1.6.23 provides a pushout diagram

$$\begin{tikzcd} \mathbb{R}[x_1,\dots,x_n] \arrow[r] \arrow[d] & A \arrow[d] \\ C^\infty(\mathbb{R}^n) \arrow[r] & F^{C^\infty}(A)^{\mathrm{alg}} \end{tikzcd}$$

of simplicial commutative $\mathbb{R}$-algebras. Since flatness is stable under base change, we are done by proposition 4.1.6.20. $\square$

Corollary 4.1.6.25. *For every simplicial commutative ring A and for every maximal ideal $\mathfrak{m}$ in $\pi_0(A)$ with residue field $\mathbb{R}$, the localization of the unit map $A_{\mathfrak{m}}\to F^{C^\infty}(A)^{\mathrm{alg}}_{\mathfrak{m}}$ is faithfully flat.*

We proceed with the proof of proposition 4.1.6.2. We will need a prelimenary result that is of independent interest, relating pushouts of simplicial C^∞-rings with pushouts of simplicial commutative $\mathbb{R}$-algebras.

Proposition 4.1.6.26. *Let $\alpha:\Lambda^2_0\times\Delta^1\to sC^\infty\mathsf{ring}$ be natural transformation from a diagram*

$$A\longleftarrow X\longrightarrow B$$

to a diagram

$$C\longleftarrow Y\longrightarrow D.$$

Suppose that for each $i\in\Lambda^2_0$, the map $\alpha|_{\{i\}\times\Delta^1}:\Delta^1\to sC^\infty\mathsf{ring}$ is an effective epimorphism, then the natural diagram

$$\begin{tikzcd} A^{\mathrm{alg}}\otimes_{X^{\mathrm{alg}}}B^{\mathrm{alg}} \arrow[r] \arrow[d] & C^{\mathrm{alg}}\otimes_{Y^{\mathrm{alg}}}D^{\mathrm{alg}} \arrow[d] \\ (A\otimes^\infty_X B)^{\mathrm{alg}} \arrow[r] & (C\otimes^\infty_Y D)^{\mathrm{alg}} \end{tikzcd}$$

in $s\mathsf{Cring}_{\mathbb{R}}$ is a pushout.

Proof. In a coCartesian symmetric monoidal ∞-category $\mathcal{C}^{\amalg}$ that admits geometric realizations of simplicial objects, the pushout $X\coprod_Y Z$ is a colimit of the two sided Bar construction $\mathsf{Bar}_Y(X,Z)_\bullet$. In simplicial degree n, the Bar construction is given by

$$X\coprod\underbrace{Y\coprod\dots\coprod Y}_{n-\text{times}}\coprod Z$$

so in order to prove that the diagram given in the proposition is a pushout, it suffices to prove that diagrams of the form

$$\begin{tikzcd} A^{\mathrm{alg}}\otimes X^{\mathrm{alg}}\otimes\dots\otimes X^{\mathrm{alg}}\otimes B^{\mathrm{alg}} \arrow[r] \arrow[d] & C^{\mathrm{alg}}\otimes Y^{\mathrm{alg}}\otimes\dots\otimes Y^{\mathrm{alg}}\otimes D^{\mathrm{alg}} \arrow[d] \\ (A\otimes^\infty X\otimes^\infty\dots\otimes^\infty X\otimes^\infty B)^{\mathrm{alg}} \arrow[r] & (C\otimes^\infty Y\otimes^\infty\dots\otimes^\infty Y\otimes^\infty D)^{\mathrm{alg}} \end{tikzcd}$$

are pushouts. With an easy inductive argument we may reduce to the case where the coproducts are binary, that is, we may assume that $X=Y=\mathbb{R}$. Denote the effective epimorphisms by $\alpha:A\to C$ and $\beta:B\to D$. Consider the composition

$$\phi:\mathrm{Fun}(\Delta^1,sC^\infty\mathsf{ring})\times\{\beta\}\longrightarrow\mathrm{Fun}(\partial\Delta^1,\mathrm{Fun}(\Delta^1,sC^\infty\mathsf{ring}))\longrightarrow\mathrm{Fun}((\partial\Delta^1)^{\triangleright},\mathrm{Fun}(\Delta^1,sC^\infty\mathsf{ring})),$$

where the second functor is a functor taking colimits, a section of the trivial fibration provided by [Lur17b], prop. 4.3.2.15. Now consider the restriction functor

$$\mathrm{Fun}(\partial\Delta^1\star\Delta^1,\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))\longrightarrow\mathrm{Fun}((\partial\Delta^1)^{\triangleright},\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))$$

induced by the full subcategory inclusion $i:(\partial\Delta^1)^{\triangleright}=\partial\Delta^1\star\Delta^{\{1\}}\subset\partial\Delta^1\star\Delta^1$. Let $\mathrm{Fun}'(\partial\Delta^1\star\Delta^1,\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))\subset\mathrm{Fun}(\partial\Delta^1\star\Delta^1,\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))$ be the full subcategory spanned by functors which are left Kan extensions along i. The restriction map $\mathrm{Fun}'(\partial\Delta^1\star\Delta^1,\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))\to\mathrm{Fun}((\partial\Delta^1)^{\triangleright},\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))$ is a trivial fibration by [Lur17b], prop. 4.3.2.15. Choosing a section of this fibration and composing with the restriction $\Delta^1\subset\partial\Delta^1\star\Delta^1$ yields a functor

$$\chi:\mathrm{Fun}((\partial\Delta^1)^{\triangleright},\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))\longrightarrow\mathrm{Fun}(\Delta^1,\mathrm{Fun}(\Delta^1,s\mathsf{Cring}_{\mathbb{R}}))\cong\mathrm{Fun}(\Delta^1\times\Delta^1,s\mathsf{Cring}_{\mathbb{R}}).$$

Composing $(\_)^{\mathrm{alg}} \circ \phi$ with χ then gives a functor

$$\mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}) \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}})$$

which carries a map $f : R \to S$ of simplicial C^∞-rings to the diagram

$$\begin{tikzcd} R^{\mathrm{alg}} \otimes B^{\mathrm{alg}} \arrow[r, "f^{\mathrm{alg}} \otimes \beta^{\mathrm{alg}}"] \arrow[d] & S^{\mathrm{alg}} \otimes D^{\mathrm{alg}} \arrow[d] \\ (R \otimes^\infty B)^{\mathrm{alg}} \arrow[r, "(f \otimes^\infty \beta)^{\mathrm{alg}}"] & (S \otimes^\infty D)^{\mathrm{alg}} \end{tikzcd}$$

Note that this functor preserves sifted colimits and that the collection of pushout diagrams is closed under colimits in $\mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}})$. Invoking proposition 4.1.2.3, we see that it is sufficient to argue that the natural commuting diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^{p+q}) \otimes B^{\mathrm{alg}} \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^p) \otimes D^{\mathrm{alg}} \arrow[d] \\ (C^\infty(\mathbb{R}^{p+q}) \otimes^\infty B)^{\mathrm{alg}} \arrow[r] & (C^\infty(\mathbb{R}^p) \otimes^\infty D)^{\mathrm{alg}} \end{tikzcd}$$

is a pushout for every effective epimorphism $C^\infty(\mathbb{R}^{p+q}) \to C^\infty(\mathbb{R}^p)$ induced by a smooth map $\mathbb{R}^p \to \mathbb{R}^q$. Now we apply this argument again to the effective epimorphism $B \to D$ to reduce to the case

$$\begin{tikzcd} C^\infty(\mathbb{R}^{p+q}) \otimes C^\infty(\mathbb{R}^{k+l}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^p) \otimes C^\infty(\mathbb{R}^k) \arrow[d] \\ C^\infty(\mathbb{R}^{p+q+k+l}) \arrow[r] & C^\infty(\mathbb{R}^{p+k}) \end{tikzcd}$$

By induction, we may assume that $q = l = 1$. The upper horizontal map is then induced by taking graphs of functions $f : \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}$. Applying lemma 4.1.3.4, $C^\infty(\mathbb{R}^p)$ has a resolution $C^\infty(\mathbb{R}^{p+1})[z_1]$ as a $C^\infty(\mathbb{R}^{p+1})$-module, and similarly $C^\infty(\mathbb{R}^k)$ has a resolution $C^\infty(\mathbb{R}^{k+1})[z_2]$ as a $C^\infty(\mathbb{R}^{k+1})$-module. Computing the torsion groups of the pushout using this resolution shows that the homotopy groups of the pushout are given by the homology of the complex $C^\infty(\mathbb{R}^{p+k+1+1})[z_1, z_2]$, which is by lemma 4.1.3.4 isomorphic to the algebra of functions on the graph of the function $f \times g : \mathbb{R}^{p+k} \to \mathbb{R}^{1+1}$, concentrated in degree zero. □

Proof of proposition 4.1.6.2. We prove (1). Note that $(b) \Rightarrow (a)$ is obvious. For the other direction, observe that the functor $sC^\infty\mathsf{ring} \times sC^\infty\mathsf{ring} \to \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ carrying a pair (A, B) to the map $A^{\mathrm{alg}} \otimes B^{\mathrm{alg}} \to (A \otimes^\infty B)^{\mathrm{alg}}$ preserves sifted colimits and that flat maps are stable under filtered colimits, so we may assume that both A and B are finitely generated. Choose effective epimorphism $C^\infty(\mathbb{R}^n) \to A$ and $C^\infty(\mathbb{R}^m) \to B$, then proposition 4.1.6.26 provides a pushout diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m) \arrow[r] \arrow[d] & A^{\mathrm{alg}} \otimes B^{\mathrm{alg}} \arrow[d] \\ C^\infty(\mathbb{R}^{n+m}) \arrow[r] & (A \otimes^\infty B)^{\mathrm{alg}} \end{tikzcd}$$

and we conclude by stability of flat morphisms under base change. We prove (2). For $(a) \Rightarrow (b)$, we repeat the proof of (1) with $C^\infty(\mathbb{R}^m)$ in place of B. $(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$ are obvious. For $(d) \Rightarrow (a)$, we note that if R is a commutative ring and M is a (discrete) R-module, then M is flat if and only if $\mathrm{Tor}_1^R(M, R/I) \cong 0$ for every finitely generated ideal $I \subset R$. Indeed, for any ideal $I \subset R$, we have a fibre sequence $I \to R \to R/I$ of discrete $\mathbb{R}$-modules, so using the long exact sequence associated to the fibre sequence

$$I \otimes_R M \longrightarrow M \longrightarrow R/I \otimes_R M$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(M, R/I) \longrightarrow I \otimes_R M \longrightarrow M \longrightarrow R/I \otimes_R M$$

and the vanishing of $\mathrm{Tor}_1^R(M, R/I)$ is equivalent to the injectivity of the map $I \otimes_R M \to M$. Applying this to the $C^\infty(\mathbb{R}^n)$-module $C^\infty(\mathbb{R}^{n+m})$ for $n, m \geq 1$, we deduce that in order to establish (1), it suffices to show that the derived tensor product $C^\infty(\mathbb{R}^{n+m}) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^n)/I$ has vanishing first homotopy group for every finitely generated ideal I. We can identify this derived tensor product with the pushout

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^n)/I \arrow[d] \\ C^\infty(\mathbb{R}^{n+m}) \arrow[r] & C^\infty(\mathbb{R}^{n+m}) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^n)/I \end{tikzcd}$$

of simplicial commutative $\mathbb{R}$-algebras. Using proposition 4.1.6.26, this pushout is equivalent to the underlying simplicial commutative $\mathbb{R}$-algebra of the coproduct $C^\infty(\mathbb{R}^n)/I \otimes^\infty C^\infty(\mathbb{R}^m)$ of simplicial C^∞-rings, which we assume has vanishing first homotopy group. $\square$

We now turn to the proof of proposition 4.1.6.4. In the absence of a general flatness result, we proceed by constructing, for certain ideals of $C^\infty(\mathbb{R}^n)$, small explicit resolutions that are acyclic for the base change induced by a projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$ onto the first n coordinates. The remainder of the results in this section hinges on the following lemma, due to Tougeron (for a single manifold), and extended to the form below by Reyes-van Quê (who attribute this generalization to Calderón).

Lemma 4.1.6.27 (Tougeron's Flat Function Lemma)**.** *Let $X \subset M$ and $Y \subset N$ be closed subsets of manifolds M and N, and let Let I be a countable set and let $\{\phi_i\}_{i\in I}$ be a set of functions on $M \times N$ that lie in $\mathfrak{m}^\infty_{X\times Y}$, that is, functions that are flat on $X \times Y$. Then there exists a characteristic function φ_X for $M \smallsetminus X$ and a characteristic function φ_Y for $N \smallsetminus Y$ that are flat on X and Y respectively such that the functions $\{\phi_i\}_{i\in I}$ are divisible by $\varphi_X + \varphi_Y$.*

Proof. See [Tou72] or [QR82]. $\square$

The following lemma shows that flat ideals in C^∞-rings of smooth functions on manifolds behave for many purposes just as principal ideals.

Lemma 4.1.6.28. *Let M and N be manifolds and let $X \subset M$ be a closed subset in a manifold. Denote by $I \coloneqq \mathfrak{m}^\infty_{X\times N}$ the closed ideal of functions flat on $X\times N$ viewed as a $C^\infty(M\times N)$-module, and let $K \subset \mathrm{Sub}_{\mathrm{fg}}(I)$ be the full subcategory of the filtered poset of finitely generated ideals contained in I spanned by principal ideals contained in I generated by functions depending only on coordinates in M. Then K is filtered and the inclusion $K \subset \mathrm{Sub}_{\mathrm{fg}}(I)$ is left cofinal.*

Remark 4.1.6.29. As the $C^\infty(M \times N)$-module $\mathfrak{m}^\infty_{X\times N}$ is a colimit of the diagram $\mathrm{Sub}_{\mathrm{fg}}(I)$, it follows that for every closed $X \subset M$, the diagram $K^{\triangleright} \to \mathrm{Mod}_{C^\infty(M\times N)}$ sending the cone to $\mathfrak{m}^\infty_{X\times N}$ is a colimit diagram.

Proof. We prove that the inclusion is left cofinal. According to [Lur17b], thm. 4.1.3.1, we need to show that the poset $K_{J/} \coloneqq K \times_{\mathrm{Sub}_{\mathrm{fg}}(I)} \mathrm{Sub}_{\mathrm{fg}}(I)_{J/}$ is weakly contractible for every finitely generated ideal $J \subset I$. It suffices to show that $K_{J/}$ is filtered. Let $\{f_1,\ldots,f_n\}$ be a collection of functions that are flat on X generating the ideal J, then it follows from Tougeron's flat function lemma that there exists a function φ_X flat on X and strictly positive outside X on M that divides each f_i as a function on $M \times N$, so that $(f_1,\ldots,f_n) \subset (\varphi_X) \subset I$, that is, $K_{J/}$ is nonempty. Similarly, if we have a finite collection of functions $\{\varphi^j_X\}$ such that $I \subset (\varphi^j_X)$ for all j, then we apply the flat function lemma to the collection $\{\varphi^j_X\}$ to find a function φ'_X such that $(\varphi^j_X) \subset (\varphi'_X)$ for all j, so $K_{J/}$ is indeed filtered. The same argument shows that K itself is filtered, which concludes the proof. $\square$

Proposition 4.1.6.30. *Let I belong to either of the following classes of ideals of $C^\infty(\mathbb{R}^n)$.*

(1) *Principal ideals.*

(2) *Ideals of the form $\mathfrak{m}^\infty_X$ for $X \subset \mathbb{R}^n$ closed.*

Then as a $C^\infty(\mathbb{R}^n)$-module, I is $({}_{-}\otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}))$-acyclic and the map

$$I \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^m) \longrightarrow C^\infty(\mathbb{R}^{n+m})$$

is a monomorphism, where the base change is induced by the projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$ for any $m \in \mathbb{Z}_{\geq 0}$.

Proof. (1) Let $g \in C^\infty(\mathbb{R}^n)$ be nonzero, then $hg = 0$ if and only if $h \in \mathfrak{m}^0_{\mathrm{Supp}(g)}$. Since $\mathrm{Supp}(g) \subset \overline{\mathrm{Supp}(g)^\circ}$, we have the equality $\mathfrak{m}^0_{\mathrm{Supp}(g)} = \mathfrak{m}^\infty_{\mathrm{Supp}(g)}$, which establishes the fibre sequence

$$\mathfrak{m}^\infty_{\mathrm{Supp}(g)} \longrightarrow C^\infty(\mathbb{R}^n) \xrightarrow{1\mapsto g} (g)$$

of discrete $C^\infty(\mathbb{R}^n)$-modules. From this fibre sequence, we obtain a fibre sequence

$$\mathfrak{m}^\infty_{\mathrm{Supp}(g)} \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^{n+m}) \longrightarrow (g) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m})$$

of connective $C^\infty(\mathbb{R}^{n+m})$-modules, because base change is right t-exact. The long exact sequence associated to this last fibre sequence yields equivalences

$$\mathrm{Tor}_k^{C^\infty(\mathbb{R}^n)}(\mathfrak{m}^\infty_{\mathrm{Supp}(g)}, C^\infty(\mathbb{R}^{n+m})) \cong \mathrm{Tor}_{k+1}^{C^\infty(\mathbb{R}^n)}((g), C^\infty(\mathbb{R}^{n+m}))$$

for all $k \geq 1$. Let $k > 1$ and suppose for the sake of induction that we have proven that for all $1 \leq j < k$ and all $g' \in C^\infty(\mathbb{R}^n)$, the torsion group $\mathrm{Tor}_j^{C^\infty(\mathbb{R}^n)}((g'), C^\infty(\mathbb{R}^{n+m}))$ vanishes, then in view of lemma 4.1.6.28, the torsion groups $\mathrm{Tor}_j^{C^\infty(\mathbb{R}^n)}(\mathfrak{m}_X^\infty, C^\infty(\mathbb{R}^{n+m}))$ also vanish for any $X \subset \mathbb{R}^n$ since the Tor functors commute with filtered colimits. Taking $X = \mathrm{Supp}(g)$, the isomorphism above shows that also for $j = k$, the torsion group $\mathrm{Tor}_j^{C^\infty(\mathbb{R}^n)}((g), C^\infty(\mathbb{R}^{n+m}))$ vanishes for all $g \in C^\infty(\mathbb{R}^n)$. It remains to prove the base case $k = 1$: we have an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{C^\infty(\mathbb{R}^n)}((g), C^\infty(\mathbb{R}^{n+m})) \longrightarrow \mathfrak{m}^\infty_{\mathrm{Supp}(g)} \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^{n+m}) \longrightarrow (g) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \longrightarrow 0,$$

so it suffices to show that the map $\mathfrak{m}^\infty_{\mathrm{Supp}(g)} \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \to C^\infty(\mathbb{R}^{n+m})$ is a monomorphism, but lemma 4.1.6.28 implies that this map is a filtered colimit of maps of the form

$$(h) \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^{n+m}) \tag{4.7}$$

for (h) a principal ideal of the commutative ring $C^\infty(\mathbb{R}^n)$. Since the collection of monomorphisms is stable under filtered colimits in a Grothendieck abelian category, we are reduced to showing that each of the maps (4.7) is a monomorphism. We are required to show that if $h(\mathbf{x})f(\mathbf{x},\mathbf{y}) = 0$ as a function on $\mathbb{R}^{n+m}$, then there exists a decomposition $f(\mathbf{x},\mathbf{y}) = \sum_i k_i(\mathbf{x})l_i(\mathbf{x},\mathbf{y})$ such that $k_i(\mathbf{x})h(\mathbf{x}) = 0$. This follows at once from the flat function lemma applied to $\mathfrak{m}^\infty_{\mathrm{Supp}(h)\times\mathbb{R}^m}$.

(2) Combine (1), lemma 4.1.6.28 and the fact that acyclic modules are stable under filtered colimits, as are monomorphism in a Grothendieck abelian category.

□

Proof of proposition 4.1.6.4. Let I be an ideal of $C^\infty(\mathbb{R}^n)$ of the form given in the statement of the proposition, then we claim that the following diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \longrightarrow & C^\infty(\mathbb{R}^n)/I \\ \downarrow & & \downarrow \\ C^\infty(\mathbb{R}^{n+m}) & \longrightarrow & C^\infty(\mathbb{R}^{n+m})/I \end{array}$$

of simplicial C^∞-ring is a pushout. The upper horizontal map is an effective epimorphism, so it suffices to show that the associated diagram of simplicial $\mathbb{R}$-algebras is a pushout. Clearly, the diagram above becomes a pushout after taking the 0'th truncation, so it suffices to show that the higher homotopy groups of the pushout $C^\infty(\mathbb{R}^n)/I \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m})$ vanish. We have a fibre sequence

$$I \longrightarrow C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)/I$$

of discrete $C^\infty(\mathbb{R}^n)$-modules, so we get a fibre sequence

$$I \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^n)/I \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m})$$

of connective $C^\infty(\mathbb{R}^{n+m})$-modules. By proposition 4.1.6.30, $\mathrm{Tor}_n^{C^\infty(\mathbb{R}^n)}(I, C^\infty(\mathbb{R}^{n+m}))$ vanishes for all $n \geq 1$ and the map $\mathrm{Tor}_0^{C^\infty(\mathbb{R}^n)}(I, C^\infty(\mathbb{R}^{n+m})) \to C^\infty(\mathbb{R}^{n+m})$ is a monomorphism, so the long exact sequence associated to the fibre sequence above guarantees the vanishing of $\mathrm{Tor}_n^{C^\infty(\mathbb{R}^n)}(C^\infty(\mathbb{R}^n)/I, C^\infty(\mathbb{R}^{n+m}))$ for all $n \geq 1$. Using the pushout diagram just established, we see that the coproduct $C^\infty(\mathbb{R}^n)/I \otimes^\infty C^\infty(\mathbb{R}^m)/J$ fits into a pushout diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}^{n+m}) & \longrightarrow & C^\infty(\mathbb{R}^{n+m})/I \\ \downarrow & & \downarrow \\ C^\infty(\mathbb{R}^{n+m})/\mathfrak{m}^\infty_{\mathbb{R}^n\times Y} & \longrightarrow & C^\infty(\mathbb{R}^n)/I \otimes^\infty C^\infty(\mathbb{R}^m)/\mathfrak{m}^\infty_{\mathbb{R}^n\times Y} \end{array}$$

By unramifiedness, this is also a pushout in $s\mathsf{Cring}_{\mathbb{R}}$. Since I is principal or flat, we have the acyclic resolution

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow I \longrightarrow C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^{n+m})/I$$

so it suffices to show that the map

$$I \otimes_{C^\infty(\mathbb{R}^{n+m})} C^\infty(\mathbb{R}^{n+m})/\mathfrak{m}^\infty_{\mathbb{R}^n\times Y} \longrightarrow C^\infty(\mathbb{R}^{n+m})/\mathfrak{m}^\infty_{\mathbb{R}^n\times Y}$$

is a monomorphism (i.e. injective). Using lemma 4.1.6.28, we may reduce to the case where $I = (h)$ is principal, with $h(\mathbf{x},\mathbf{y}) = h(\mathbf{x})$. To show injectivity, we take some object $h(\mathbf{x}) \otimes f(\mathbf{x},\mathbf{y}) \in I \otimes_{C^\infty(\mathbb{R}^{n+m})} C^\infty(\mathbb{R}^{n+m})/\mathfrak{m}^\infty_{\mathbb{R}^n\times Y}$ (since (h) has a generator, all objects in the tensor product over $C^\infty(\mathbb{R}^{n+m})$ are pure tensors) and suppose that $h(\mathbf{x})f(\mathbf{x},\mathbf{y}) \in \mathfrak{m}^\infty_{\mathbb{R}^n\times Y}$. Then every iterated derivative $D^\alpha_{\mathbf{y}}f$ of f with respect to the $\mathbf{y}$-coordinates also has the property that $h(\mathbf{x})D^\alpha_{\mathbf{y}}f(\mathbf{x},\mathbf{y}) \in \mathfrak{m}^\infty_{\mathbb{R}^n\times Y}$, and a straightforward inductive argument reveals that every iterated derivative of f with respect to the $\mathbf{x}$-coordinates vanishes on $\mathrm{Supp}(h)\times Y$. Thus, we conclude that $f(\mathbf{x},\mathbf{y}) \in \mathfrak{m}^\infty_{\mathrm{Supp}(h)\times Y}$ which implies by the flat function lemma that $f(\mathbf{x},\mathbf{y})$ can be written as $g(\mathbf{x},\mathbf{y})(\varphi(\mathbf{x}) + \varphi'(\mathbf{y}))$ with $\varphi(\mathbf{x}) \in \mathfrak{m}^\infty_{\mathrm{Supp}(h)}$ and $\varphi'(\mathbf{y}) \in \mathfrak{m}^\infty_Y$. Then $h(\mathbf{x}) \otimes f(\mathbf{x},\mathbf{y}) = h(\mathbf{x})\varphi(\mathbf{x}) \otimes g(\mathbf{x},\mathbf{y}) + h(\mathbf{x}) \otimes g(\mathbf{x},\mathbf{y})\varphi'(\mathbf{y}) = 0$. □

4.1.7 Variant: derived real analytic geometry

It follows from proposition 4.1.6.20 that for any pair A, B of simplicial commutative rings, the n'th homotopy group of the coproduct $F^{C^\infty}(A) \otimes^\infty F^{C^\infty}(B)$ is isomorphic to $\pi_n(A \otimes B) \otimes_{\pi_0(A)\otimes\pi_0(B)} F^{C^\infty}(\pi_0(A) \otimes^\infty \pi_0(B))$. In this section, we extend this result to the case where the simplicial C^∞-ring are dual to derived C^∞-schemes free on derived real analytic spaces. We will give (a sketch of) a proof of the following result.

Definition 4.1.7.1. Let $\mathcal{T}_{\mathrm{An}_\mathbb{R}}$ be the ∞-category defined as the nerve of the category of open subsets of Euclidean space and real analytic maps between them. We endow this ∞-category with the structure of a pregeometry as follows.

(1) A map $f : U \to V$ of open submanifolds of Euclidean spaces is admissible if f is equivalent to an open inclusion of real analytic manifolds.

(2) A family of admissibles $\{U_i \to V\}_i$ generates a covering sieve if and only if the topological spaces underlying the real analytic manifolds U_i cover the topological space underlying V The admissible coverings define a pretopology on $\mathcal{T}_{\mathrm{An}_\mathbb{R}}$ whose associated topology we call the *étale topology.*

Let $\mathcal{G}^{\mathrm{der}}_{\mathrm{An}_\mathbb{R}}$ denote a geometric envelope for $\mathcal{T}_{\mathrm{An}_\mathbb{R}}$.

Definition 4.1.7.2. A *derived real analytic space* is a 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{An}_\mathbb{R}}$-scheme locally of finite presentation.

Remark 4.1.7.3. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a derived real analytic space, then there exists an effective epimorphism $\coprod U_i \to 1_\mathcal{X}$ such that

Remark 4.1.7.4. We cannot follow [Lur11a] in the complex analytic setting and define a real analytic space as a $\mathcal{T}_{\mathrm{An}_\mathbb{R}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that there exists an effective epimorphism $\coprod_i U_i \to 1_\mathcal{X}$ satisfying the following conditions.

(1) For each i, the ∞-topos $\mathcal{X}_{/U_i}$ is the ∞-topos of sheaves on a topological space X_i.

(2) For each i, $(X_i, \pi_0(\mathcal{O}_\mathcal{X}|_{U_i}))$ is a real analytic space,

(3) for each $n \geq 0$, $\pi_n(\mathcal{O}_\mathcal{X}|_{U_i})$ is a coherent sheaf of $\pi_0(\mathcal{O}_\mathcal{X}|_{U_i})$-modules.

The reason for this is the fact that Oka's coherence theorem fails for real analytic spaces (while it holds true for real analytic manifolds), which has the effect that the full subcategory of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{An}_\mathbb{R}})$ spanned by the objects just described is not stable under finite limits.

Remark 4.1.7.5. The failure of Oka's coherence theorem can be controlled by conditions on the analytic ideals in question. Let X be a germ of an analytic set at 0 defined by an ideal $I \subset \mathcal{O}^{\mathrm{an}}(\mathbb{R}^n)_0$, then X is coherent at X if and only if the ideal $IC^\infty(\mathbb{R}^n)_x \cong I \otimes_{\mathcal{O}^{\mathrm{an}}(\mathbb{R}^n)_0} C^\infty(\mathbb{R}^n)_x$ (the isomorphism follows from flatness of $C^\infty(\mathbb{R}^n)_0$ over $\mathcal{O}^{\mathrm{an}}(\mathbb{R}^n)_0$) coincides with the ideal $\mathfrak{m}^0_X$ of germs of functions at 0 vanishing on X.

There is an obvious transformation of pregeometries $\mathcal{T}_{\mathrm{An}_\mathbb{R}} \to \mathcal{T}_{\mathrm{Diff}}$ which induces a transformation of geometries

$$\varphi_{\mathrm{An}_\mathbb{R}} : \mathcal{G}^{\mathrm{der}}_{\mathrm{An}_\mathbb{R}} \longrightarrow \mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}.$$

We denote by $\mathbf{Spec}^{\mathrm{Diff}}_{\mathrm{An}_\mathbb{R}}$ the associated spectrum functor. The first goal of this section is to prove the following theorem.

Proposition 4.1.7.6. *Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a derived real analytic space. Then the unit map*

$$(\mathcal{X}, \mathcal{O}_\mathcal{X}) \longrightarrow \varphi^*_{\mathrm{An}_\mathbb{R}} \mathbf{Spec}^{\mathrm{Diff}}_{\mathrm{An}_\mathbb{R}}(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

is a faithfully flat map of $\mathbb{E}_\infty$-ringed ∞-topoi.

The techniques in this section are similar to the ones used in the previous section with the additional complication that we have to work relative over the ∞-category of ∞-topoi, because we do not have a convenient description of the geometric envelope $\mathcal{G}_{\mathrm{An}_\mathbb{R}}$.

Remark 4.1.7.7. A treatment of derived real analytic geometry proper as in [Lur11a] will necessitate using the results of sections 2 and 3 of *loc. cit.*, some proofs of which do not seem to be correct as stated. We will gloss over this point since we do not doubt that the results are true.

Proof of proposition 4.1.7.6. It is not hard to see that for a derived real analytic space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, the unit map

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow \varphi^*_{\mathrm{An}_{\mathbb{R}}} \mathbf{Spec}^{\mathrm{Diff}}_{\mathrm{An}_{\mathbb{R}}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

induces an equivalence on the underlying ∞-topoi. This implies, together with the fact that the question of flatness is local, that it suffices to show that for each local $\mathcal{G}^{\mathrm{der}}_{\mathrm{An}_{\mathbb{R}}}$-structure $\mathcal{O}$ on $\mathcal{S}$, the unit $\mathcal{O} \to GF(\mathcal{O})$ of the adjunction

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{S}) \underset{F}{\overset{G}{\rightleftarrows}} \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{An}_{\mathbb{R}}}}(\mathcal{S})$$

determines a faithfully flat map of local simplicial commutative $\mathbb{R}$-algebras. In fact, both ∞-categories in this adjunction are presentable and projectively generated: the locality condition on morphisms in both ∞-categories is always satisfied since any morphisms between local $\mathbb{R}$-algebras with residue field $\mathbb{R}$ is local. Now the full subcategory of $\mathrm{Str}_{\mathcal{T}}(\mathcal{S})$ spanned by local $\mathcal{T}$-structures is presentable as a consequence of the factorization system constructed in section 1.3 of [Lur11b]. Proposition of 3.3.1 of [Lur11b] asserts that the composition

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S}) \subset \mathrm{Fun}(\mathcal{T}, \mathcal{S}) \longrightarrow \prod_{v \in \mathcal{T}} \mathcal{S},$$

which is obviously conservative, preserves limits and sifted colimits so we conclude that $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{S})$ and $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{An}_{\mathbb{R}}}}(\mathcal{S})$ are projectively generated. It also follows from proposition 3.3.1 of [Lur11b] that the functor G preserves limits and sifted colimits, and is thus by proposition 4.1.1.3 determined by a functor between Lawvere theories. Now one can use the resolution theorem for effective epimorphisms 4.1.2.3 in Lawvere theories and apply the arguments of proposition 4.1.6.23 and corollary 4.1.6.24 to reduce to the case of rings of germs of analytic functions on $\mathbb{R}^n$, which follows from corollary 4.1.6.19. ☐

Now we wish to show the following.

Theorem 4.1.7.8. *Let A and B be local simplicial C^∞-rings in the image of the functor F from the proof above, that is, A and B are local simplicial C^∞-rings of germs of affine $\mathcal{G}^{\mathrm{der}}_{\mathrm{An}_{\mathbb{R}}}$-schemes, then the canonical map $A \otimes B \longrightarrow A \otimes^\infty B$ is strong.*

Since we can identify $A \otimes^\infty B$ with $F(A' \coprod B')$ for some pair $A', B' \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{An}_{\mathbb{R}}}}(\mathcal{S})$, it suffices to show that the map

$$A' \otimes B' \longrightarrow A' \coprod B'$$

is faithfully flat. To see this, we can repeat the proof of proposition 4.1.6.26 for the Lawvere theory $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{An}_{\mathbb{R}}}}(\mathcal{S})$ to reduce to the case where A' and B' are rings of germs of real analytic functions on Cartesian spaces, that is, we should show that the map

$$\mathcal{O}(\mathbb{R}^n)_0 \otimes \mathcal{O}(\mathbb{R}^m)_0 \longrightarrow \mathcal{O}(\mathbb{R}^{n+m})_0$$

is faithfully flat, but this is a consequence of the fact that $\mathcal{O}(\mathbb{R}^n)_0$ is coherent.

Corollary 4.1.7.9. *Let $I \subset C^\infty(\mathbb{R}^n)$ and $J \subset C^\infty(\mathbb{R}^m)$ be ideals of analytic functions, then the sheafified homotopy groups of $C^\infty(\mathbb{R}^n)/I \otimes^\infty C^\infty(\mathbb{R}^m)/J$ vanish.*

We give no further indication of the flatness or nonflatness of the map $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^{n+m})$, but we believe that the tools used in the preceding sections can be extended to larger classes of ideals. Recall the following notion.

Definition 4.1.7.10. A finitely generated ideal $I = (f_1, \ldots, f_k) \in C^\infty(\mathbb{R}^n)$ is a *Lojasiewicz ideal* if either of the following equivalent conditions are satisfied.

(1) $\mathfrak{m}^\infty_{Z(I)} \subset I$.

(2) The function $f_1^2 + \ldots + f_k^2$ satisfies *Lojasiewicz inequality*: for all compact $K \subset \mathbb{R}^n$ there exists a constant $C \in \mathbb{R}_{>0}$ and a constant $\alpha \in \mathbb{R}_{\geq 0}$ such that

$$f_1^2(x) + \ldots + f_k^2(x) \geq C d(x, Z(I))^\alpha, \quad \forall x \in K$$

where $d(x, Z(I))$ is the Euclidean distance between x and $Z(I)$.

It follows immediately from characterization (1) and Whitney's spectral theorem that finitely generated closed ideals are Lojasiewicz. In fact a finitely generated ideal of $C^\infty(\mathbb{R})$ is closed if and only if it is Lojasiewicz, and in this case it can be shown that the ideal in question is necessarily principal. It is possible to show that if $I \subset C^\infty(\mathbb{R}^n)$ is Lojasiewicz, then the map

$$I \otimes_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^{n+m}) \longrightarrow C^\infty(\mathbb{R}^{n+m})$$

is injective.

Conjecture 4.1.7.11. *Let $I \subset C^\infty(M)$ be a Lojasiewicz ideals and let $Y \subset N$ be a closed subset, then the unit map*

$$C^\infty(M)/I \otimes^\infty C^\infty(N)/\mathfrak{m}_Y^\infty \longrightarrow \tau_{\leq 0}(C^\infty(M)/I \otimes^\infty C^\infty(N)/\mathfrak{m}_Y^\infty)$$

is an equivalence.

4.1.8 Corners and logarithmic structures

In this section we use the results of the previous subsections to define *derived manifolds with corners* and study their basic properties. We define derived manifolds with corners along the lines of our initial definition of derived manifolds (without corners) and derived real analytic spaces, in accordance with the general philosophy outlined in the previous subsection.

Definition 4.1.8.1. The ∞-category of *derived manifold with corners locally of finite presentation* is the full subcategory of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diffc}})$ that contains the essential image of the spectrum functor $\mathbf{Spec}^{\mathcal{T}_{\mathrm{Diffc}}} : \mathcal{T}_{\mathrm{Diffc}} \hookrightarrow {}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diffc}})$ and is stable under finite limits and retracts, and is also stable under colimits in the ∞-category ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diffc}})^{-1-\mathrm{et}}$.

The goal in this subsection will be the introduction of a tractable geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ that yields the same structured spaces, such that ∞-category of 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-schemes of finite presentation coincides with that of the ∞-category of derived manifolds with corners described above. To construct such a geometry, it seems natural to consider the ∞-category $sC^\infty\mathsf{ring}_{pc}$ of algebras for the Lawvere theory of Cartesian spaces with corners. In the 1-categorical setting, such a theory has been developed by Joyce and Francis-Staite [JF19].

Remark 4.1.8.2. For technical reasons, the geometry we will construct does not come equipped with a functor $\mathcal{T}_{\mathrm{Diffc}} \to \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ exhibiting a geometric envelope. We will instead introduce another pregeometry $\mathcal{T}'_{\mathrm{Diffc}}$ together with functors

$$\mathcal{T}_{\mathrm{Diffc}} \longleftarrow \mathcal{T}'_{\mathrm{Diffc}} \hookrightarrow \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$$

where the left arrow is a Morita equivalence of pregeometries and the right arrow exhibits a geometric envelope.

At first glance, one might be tempted to define $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ as the compact objects of $sC^\infty\mathsf{ring}_{pc}$, exactly analogous to how $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ was introduced, but this turns out to be too naive: the functor

$$(C^\infty(\_), C^\infty_b(\_)) : \mathcal{T}_{\mathrm{Diffc}} \longrightarrow sC^\infty\mathsf{ring}_{pc}^{op}$$

does not preserve pullbacks along open inclusions. Consider, for instance, the following pullback

$$\begin{tikzcd} \mathbb{R} \arrow[r, equal] \arrow[d, hook, "\exp"] & \mathbb{R} \arrow[d, hook, "\exp"] \\ \mathbb{R}_{\geq 0} \arrow[r] & \mathbb{R} \end{tikzcd}$$

in $\mathcal{T}_{\mathrm{Diffc}}$. The vertical maps are admissible, so the diagram

$$\begin{tikzcd} (C^\infty(\mathbb{R}), C^\infty_b(\mathbb{R})) \arrow[r] \arrow[d, "\exp^*"] & (C^\infty(\mathbb{R}_{\geq 0}), C^\infty_b(\mathbb{R}_{\geq 0})) \arrow[d, "\exp^*"] \\ (C^\infty(\mathbb{R}), C^\infty_b(\mathbb{R})) \arrow[r, equal] & (C^\infty(\mathbb{R}), C^\infty_b(\mathbb{R})) \end{tikzcd}$$

should be a pushout diagram in $sC^\infty\mathsf{ring}_{pc}$. Let C denote the pushout of the diagram above, and let $\varphi : C \to (C^\infty(\mathbb{R}), C^\infty_b(\mathbb{R}))$ be the canonical morphism; we should verify whether or not this morphism is an equivalence. Corollary 4.1.6.6 shows that the map $C^\infty(\mathbb{R}) \to \mathrm{ev}_{\mathbb{R}}(C)$ is an equivalence of spaces. Since evaluation at $\mathbb{R}_{\geq 0}$ preserves sifted colimits, the space $\mathrm{ev}_{\mathbb{R}_{\geq 0}}(C)$ may be computed as the colimit of the simplicial object $\mathsf{Bar}_{C^\infty_b(\mathbb{R})}(C^\infty_b(\mathbb{R}), C^\infty_b(\mathbb{R}_{\geq 0}))_\bullet$ which takes the form

$$\ldots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} C^\infty_b(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_{\geq 0}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} C^\infty_b(\mathbb{R} \times \mathbb{R}^1 \times \mathbb{R}_{\geq 0}) \rightrightarrows C^\infty_b(\mathbb{R} \times \mathbb{R}_{\geq 0}).$$

In each simplicial level, we have the space of interior b-maps on a manifold of the form $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, which has one connected boundary component whose defining function is the last coordinate. All face maps preserve this boundary defining function, so it follows from lemma 4.1.8.37 that this simplicial object may be written as a product

$$\mathbb{Z}_{\geq 0} \times \Big(\ldots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} C^\infty_{>0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_{\geq 0}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} C^\infty_{>0}(\mathbb{R} \times \mathbb{R}^1 \times \mathbb{R}_{\geq 0}) \rightrightarrows C^\infty_{>0}(\mathbb{R} \times \mathbb{R}_{\geq 0}) \Big).$$

where $\mathbb{Z}_{\geq 0}$ is a constant simplicial object. The simplicial object in parentheses is equivalent to

$$\dots \; C^\infty(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}_{\geq 0}) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; C^\infty(\mathbb{R}\times\mathbb{R}^1\times\mathbb{R}_{\geq 0}) \;\substack{\longrightarrow\\\longrightarrow}\; C^\infty(\mathbb{R}\times\mathbb{R}_{\geq 0}),$$

whose colimit can be identified with the pushout $C^\infty(\mathbb{R})\otimes^\infty_{C^\infty(\mathbb{R})}C^\infty(\mathbb{R}_{\geq 0})\simeq C^\infty(\mathbb{R})$. Since sifted colimits commute with products, we find that $\mathrm{ev}_{\mathbb{R}_{\geq 0}}(C)\simeq\mathbb{Z}_{\geq 0}\times C^\infty_{>0}(\mathbb{R})$, and the map $\mathrm{ev}_{\mathbb{R}_{\geq 0}}(\varphi)$ is identified with the projection $\mathbb{Z}_{\geq 0}\times C^\infty_{>0}(\mathbb{R})\to C^\infty_{>0}(\mathbb{R})=C^\infty_b(\mathbb{R})$, which is not an equivalence. We could correct for the fact that the pullback diagram above is not preserved simply by localizing at the morphism $\varphi:(C^\infty(\mathbb{R}),C^\infty_b(\mathbb{R}))\to C$, but for the purposes of defining a transformation of pregeometries $\mathcal{T}_{\mathrm{Diffc}}\to\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ this appears too myopic; it seems we have to localize at all comparison maps arising from applying $(C^\infty(\_),C^\infty_b(\_))$ to admissible pullbacks $\mathcal{T}_{\mathrm{Diffc}}$. Fortunately, it turns out that localizing at φ already yields the correct ambient ∞-category. It will be convenient to introduce a different but equivalent localization.

Definition 4.1.8.3. Consider the image of the map

$$\mathbb{R}_{>0}\hookrightarrow\mathbb{R}_{\geq 0}$$

under the functor $(C^\infty(\_),C^\infty_b(\_)):\mathcal{T}^{op}_{\mathrm{Diffc}}\hookrightarrow sC^\infty\mathsf{ring}_{pc}$. Denote by ϵ the counit $\epsilon:\iota_{c!}\iota_c^*\to\mathrm{id}$ and define an object $\mathcal{A}\in sC^\infty\mathsf{ring}_{pc}$ together with a map $\phi:\iota_{c!}\iota_c^*(C^\infty(\mathbb{R}_{>0}),C^\infty_b(\mathbb{R}_{>0}))\to\mathcal{A}$ via the pushout diagram

$$\begin{tikzcd}
\iota_{c!}\iota_c^*(C^\infty(\mathbb{R}_{\geq 0}),C^\infty_b(\mathbb{R}_{\geq 0})) \arrow[r, "\epsilon"] \arrow[d] & (C^\infty(\mathbb{R}_{\geq 0}),C^\infty_b(\mathbb{R}_{\geq 0})) \arrow[d] \\
\iota_{c!}\iota_c^*(C^\infty(\mathbb{R}_{>0}),C^\infty_b(\mathbb{R}_{>0})) \arrow[r, "\phi"] & \mathcal{A}.
\end{tikzcd}$$

We let $S=\{\phi\}$, the one element set containing the morphism ϕ. The ∞-category of *simplicial C^∞-rings with corners*, denoted $sC^\infty\mathsf{ring}_c$, is the presentable ∞-category of S-local objects of $sC^\infty\mathsf{ring}_{pc}$.

Remark 4.1.8.4. Unraveling the definition, a simplicial C^∞-ring with pre-corners (A,A_c) is S-local just in case the upper horizontal map in the pullback diagram

$$\begin{tikzcd}
A_c\times_{A_{\geq 0}}A_{>0} \arrow[r] \arrow[d] & A_{>0} \arrow[d] \\
A_c \arrow[r] & A_{\geq 0}
\end{tikzcd}$$

of spaces is an equivalence, where we use the notation $A_{\geq 0}:=\mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{\geq 0}),A)$ and $A_{>0}:=\mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{>0}),A)$ for A a simplicial C^∞-ring. This subsection will be concerned with the simplicial commutative monoid structure on the space $A_{\geq 0}$ induced by the homotopy coherent C^∞-operations. To this end, it turns out to be crucial to establish, as we will in a moment, that the right vertical map $A_{>0}\to A_{\geq 0}$ -an inclusion of connected components- coincides with the largest subgroup contained in the simplicial commutative monoid $A_{\geq 0}$. Contrary to the 1-categorical case, this is not immediate and depends on a computation of the cotangent complex of $C^\infty(\mathbb{R}_{\geq 0})$ which is deferred to the next chapter (see remark 4.1.6.15).

Remark 4.1.8.5. On $(C^\infty(\mathbb{R}),C^\infty_b(\mathbb{R}))$ and $(C^\infty(\mathbb{R}_{>0}),C^\infty_b(\mathbb{R}_{>0}))$, the counit $\iota_{c!}\iota_c^*\to\mathrm{id}$ is an equivalence, so for each $(A,A_c)\in sC^\infty\mathsf{ring}_{pc}$, there is a diagram

$$\begin{tikzcd}
A_c\times_{A_{\geq 0}}A_{>0} \arrow[r] \arrow[d] & A_{>0} \arrow[r, equal] \arrow[d] & A_{>0} \arrow[d] \\
A_c \arrow[r] & A_{\geq 0} \arrow[r] & A
\end{tikzcd}$$

Here, the right square is a pullback. Let $S'=\{\varphi\}$, the one element set containing the map $\varphi:(C^\infty(\mathbb{R}),C^\infty_b(\mathbb{R}))\to C$ from the discussion above. Unwinding the definitions, we see that (A,A_c) is S-local if and only if it is S'-local.

Since the forgetful functor $\iota_c^*:sC^\infty\mathsf{ring}_{pc}\to sC^\infty\mathsf{ring}$ preserves colimits and carries the counit $\epsilon:\iota_{c!}\iota_c^*\to\mathrm{id}$ to the identity, ι_c^* carries the map ϕ of definition 4.1.8.3 to an equivalence. From the universal property of cocontinuous localizations, we deduce that ι_c^* factors via a left adjoint $sC^\infty\mathsf{ring}_c\to sC^\infty\mathsf{ring}$. This functor coincides with the composition $sC^\infty\mathsf{ring}_c\hookrightarrow sC^\infty\mathsf{ring}_{pc}\to sC^\infty\mathsf{ring}$, which is a right adjoint. Note that both adjoints of this functor are fully faithful, so the argument of proposition 4.1.6.7 grants the following result.

Proposition 4.1.8.6. *The functor $sC^\infty\mathsf{ring}_c\to sC^\infty\mathsf{ring}$ is a presentable fibration. Moreover, the inclusion $sC^\infty\mathsf{ring}_c\hookrightarrow sC^\infty\mathsf{ring}_{pc}$ preserves Cartesian edges, and the localization $L:sC^\infty\mathsf{ring}_{pc}\to sC^\infty\mathsf{ring}_c$ preserves coCartesian edges.*

Remark 4.1.8.7. As corollary 4.1.8.33 asserts, the localization $sC^\infty\mathsf{ring}_c \subset sC^\infty\mathsf{ring}_{pc}$ is ω-accessible, that is, $sC^\infty\mathsf{ring}_c \subset sC^\infty\mathsf{ring}_{pc}$ is stable under filtered colimits. In particular, every compact object in $sC^\infty\mathsf{ring}_c$ is a retract of an object in the image of L. Since $\iota_{c!}L$ is equivalent to L and idempotents may be lifted along coCartesian fibrations, we deduce that for any compact object (A, A_c) in $sC^\infty\mathsf{ring}_c$, there is some A'_c such that (A, A'_c) is compact in $sC^\infty\mathsf{ring}_{pc}$.

We will use the ∞-category $sC^\infty\mathsf{ring}_c$ to define a derived geometry generated by manifolds with corners. To this end, we first make an observation concerning finitely generated and compact objects in $sC^\infty\mathsf{ring}_{pc}$.

Proposition 4.1.8.8. *The following hold true.*

(1) *The functor ι_c^* carries finitely generated objects of $sC^\infty\mathsf{ring}_{pc}$ to finitely generated objects of $sC^\infty\mathsf{ring}$.*

(2) *The functor ι_c^* carries finitely presented objects of $sC^\infty\mathsf{ring}_{pc}$ into the full subcategory $sC^\infty\mathsf{ring}_{\mathrm{fair}} \subset sC^\infty\mathsf{ring}$.*

Proof. For (1), we need to show that the right adjoint ι_{c*} preserves colimits of filtered diagrams consisting only of monomorphisms. From the general theory of algebraic theories it is enough to check that the functors $\mathrm{ev}_{\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}}\iota_{c*} : sC^\infty\mathsf{ring} \to \mathcal{S}$ have this property, but these functors are corepresented by the finitely generated objects $C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0})$. For (2), we suppose that (A, A_c) is finitely presented in $sC^\infty\mathsf{ring}_{pc}$. Consider a finite presentation of $(\pi_0(A), \pi_0(A_c))$, that is, a coequalizer diagram

$$(C^\infty(\mathbb{R}^p \times \mathbb{R}^q_{\geq 0}), C^\infty_b(\mathbb{R}^p \times \mathbb{R}^q_{\geq 0})) \rightrightarrows (C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}), C^\infty_b(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})) \longrightarrow (\pi_0(A), \pi_0(A_c)).$$

As $\iota_c^* : C^\infty\mathsf{ring}_c \to C^\infty\mathsf{ring}$ preserves colimits, we have a coequalizer diagram

$$C^\infty(\mathbb{R}^p \times \mathbb{R}^q_{\geq 0}) \rightrightarrows C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}) \longrightarrow \pi_0(A).$$

Since we have an epimorphism of C^∞-rings $C^\infty(\mathbb{R}^{p+q}) \to C^\infty(\mathbb{R}^p \times \mathbb{R}^q_{\geq 0})$, we also have a coequalizer diagram

$$C^\infty(\mathbb{R}^{p+q}) \rightrightarrows C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}) \longrightarrow \pi_0(A)$$

which shows that $\pi_0(A)$ is finitely presented over $C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})$ *as a C^∞-ring*. Since the latter object is free in $sC^\infty\mathsf{ring}_{pc}$, the map $C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}) \to \pi_0(A)$ lifts to a map $f : C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}) \to A$ in $sC^\infty\mathsf{ring}$, and as we have just verified, $\pi_0(f)$ is finitely presented. It follows from corollary 5.0.0.3 that the cotangent complexes of both $C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})$ and A are perfect, so $\mathbb{L}_f$ is also perfect. Invoking proposition 5.1.1.8, we deduce that A is finitely presented over $C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})$. Invoking proposition 4.1.3.32, we deduce that A admits a presentation as a retract of a finite good cell object over $C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})$. Since $sC^\infty\mathsf{ring}_{\mathrm{fair}}$ is stable under retracts, we may suppose that A is a finite good cell object over $C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})$, that is, in the model category $C^\infty\mathbf{dga}$, the object A admits a presentation as

$$C^\infty(\mathbb{R}^n \times \mathbb{R}^m_{\geq 0})[\epsilon^1_1, \ldots, \epsilon^1_{n_1}, \ldots, \epsilon^k_1, \ldots, \epsilon^k_{n_k}]$$

with $|\epsilon^j_i| = j$ and some differential. Now considering this Koszul complex and its truncations as sheaves over $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$, the same argument as the one used in proposition 4.1.3.33 shows that the homotopy groups of A are complete $\pi_0(A)$-modules. □

Recall that *fairness* implies that if (A, A_c) is a compact object of $sC^\infty\mathsf{ring}_{pc}$, then the underlying simplicial C^∞-ring A has the property that $\pi_0(A)$ is finitely generated and germ determined, and for each $n \geq 1$, the object $\pi_n(A)$ has the property that module of global sections of the sheafification of the presheaf

$$U_a \longmapsto \pi_n(A) \otimes_{\pi_0(A)} \pi_0(A)[a^{-1}]$$

coincides with $\pi_n(A)$. More briefly, for (A, A_c) compact, the unit map

$$A \longrightarrow \Gamma\mathbf{Spec}\,A$$

is an equivalence.

Definition 4.1.8.9. Let $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ be the opposite of the full subcategory of $sC^\infty\mathsf{ring}_c^{op}$ spanned by compact objects. We define the notions of an admissible morphism and admissible covering in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ as follows.

(1) A morphism $f : \mathrm{Spec}\,(A, A_c) \to \mathrm{Spec}\,(B, B_c)$ is admissible if and only if there exists some $b \in \pi_0(B)$ such that the underlying map $B \to A$ of simplicial C^∞-rings exhibits A as a localization of B by b and f is a coCartesian morphism for the fibration $sC^\infty\mathsf{ring}_c \to sC^\infty\mathsf{ring}$.

(2) A collection of morphisms $\{\mathrm{Spec}\,(B_i, B_{ic}) \to \mathrm{Spec}\,(B, B_c)\}$ generates a covering sieve if and only if the underlying collection $\{\mathrm{Spec}\, B_i \to \mathrm{Spec}\, B\}$ of morphisms among fair (cf. proposition 4.1.8.8 and the preceding remark) simplicial C^∞-rings generates a covering sieve for the étale topology on $sC^\infty\mathsf{ring}^{op}_{\mathrm{fair}} \simeq \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}$.

Let $\mathcal{T}'_{\mathrm{Diffc}} \subset \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ be the smallest full subcategory of $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ that contains the objects $(C^\infty(\mathbb{R}^n), C^\infty_b(\mathbb{R}^n))$ for all n and satisfies the following condition: should $f : \mathrm{Spec}\,(A, A_c) \to \mathrm{Spec}\,(B, B_c)$ be admissible and $\mathrm{Spec}\,(B, B_c) \in \mathcal{T}'_{\mathrm{Diffc}}$, then also $\mathrm{Spec}\,(A, A_c) \in \mathcal{T}'_{\mathrm{Diffc}}$ (hence we require that the inclusion $\mathcal{T}'_{\mathrm{Diffc}} \subset \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ is a categorical fibration).

Remark 4.1.8.10. Suppose that $(A, A_c) \to (B, B_c)$ is (the opposite of) an admissible morphism in $sC^\infty\mathsf{ring}_c$, then it might not be a priori clear that (B, B_c) is a compact object of $sC^\infty\mathsf{ring}_c$. To see this is the case, we note that the assumption that $A \to B$ is a localization of simplicial C^∞-rings provides a map $C^\infty(\mathbb{R}) \to A$ and an equivalence $B \simeq C^\infty(\mathbb{R} \smallsetminus \{0\}) \otimes^\infty_{C^\infty(\mathbb{R})} A$. We have an adjoint map $L\iota_{c!}(C^\infty(\mathbb{R})) \to (A, A_c)$ and we can form the pushout

$$\begin{tikzcd}
L\iota_{c!}(C^\infty(\mathbb{R})) \arrow[r] \arrow[d] & L\iota_{c!}(C^\infty(\mathbb{R} \smallsetminus \{0\})) \arrow[d] \\
(A, A_c) \arrow[r] & (C, C_c).
\end{tikzcd}$$

Since $sC^\infty\mathsf{ring}_c \to sC^\infty\mathsf{ring}$ preserves colimits, the map $A \to C$ coincides with $A \to B$. Because the upper horizontal map is coCartesian by the description of coCartesian edges in proposition 4.1.6.7, the lower horizontal map is coCartesian as well, as all colimits in $sC^\infty\mathsf{ring}_c$ are relative colimits. It follows that $(C, C_c) \simeq (B, B_c)$, that is, being admissible in $sC^\infty\mathsf{ring}_c$ is equivalent to fitting into a pushout diagram as above, which shows that (B, B_c) is compact if (A, A_c) is, since $L\iota_{c!}$ preserves compact objects.

The main results of this subsection are summarized in the following theorem.

Theorem 4.1.8.11. *(i) The ∞-category $sC^\infty\mathsf{ring}_c$ is compactly generated, that is, the canonical functor* $\mathrm{Pro}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}) \to sC^\infty\mathsf{ring}^{op}_c$ *is an equivalence.*

(ii) Definition 4.1.8.9 furnishes the structure of a geometry on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ and the structure of a pregeometry on $\mathcal{T}'_{\mathrm{Diffc}}$ such that the inclusion $\mathcal{T}'_{\mathrm{Diffc}} \subset \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ is a transformation of pregeometries.

(iii) The geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ 2-represents the functor

$$\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \_) : \mathsf{Cat}^{\mathrm{lex},\mathrm{Idem}}_\infty \longrightarrow \mathsf{Cat}_\infty.$$

(iv) The functor $(C^\infty_{\_}, C^\infty_{b\_}) : \mathcal{T}_{\mathrm{Diffc}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})$ is fully faithful and preserves pullbacks along admissible maps.

(v) Denote by $\mathbf{Spec}_c$ the functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}$, then $\mathbf{Spec}_c : \mathcal{T}'_{\mathrm{Diffc}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})$ takes values in the essential image of $(C^\infty_{\_}, C^\infty_{b\_})$ and determines a Morita equivalence of pregeometries

$$\mathcal{T}'_{\mathrm{Diffc}} \longrightarrow \mathcal{T}_{\mathrm{Diffc}}.$$

The proof of this theorem will require a number of prelemenaries. First observe that the geometry structure on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ makes reference to the coCartesian morphisms of ι^*_c, which involve the formation of certain pushouts in $sC^\infty\mathsf{ring}_c$ and are more difficult to characterize explicitly than its Cartesian morphism, which are obtained by taking certain pullbacks in $sC^\infty\mathsf{ring}_c$ and are therefore detectable on the underlying spaces. To improve our understanding of the fibres of $sC^\infty\mathsf{ring}_c \to sC^\infty\mathsf{ring}$ and its coCartesian morphisms, we will establish a structural result of independent interest which relates simplicial C^∞-rings with corners to an algebraic model for C^∞-geometry with corners and more general singularities. The latter theory is a derived and differential geometric version of *logarithmic geometry* in the sense of Fontaine-Illusie, Kato and Ogus [Kat89; Ogu18].

Remark 4.1.8.12. While we make no use of this perspective, the theory of positive logarithmic C^∞-geometry we expose in this subsection could have been developed entirely in a model categorical setting, as is done by Sagave, Schürg and Vezzosi and Bhatt [SSV16; Bha12], at the cost of rendering many arguments significantly more cumbersome. In particular, it is not hard to see that the equivalence of theorem 4.1.8.24 is induced by a Quillen equivalence between combinatorial model categories. We leave it as an exercise for the sufficiently industrious reader to make the necessary translations.

Remark 4.1.8.13. Apart from the works of Sagave-Schürg-Vezzosi and Bhatt, the derived antecedents of this section include the work on logarithmic structures for $\mathbb{E}_\infty$-ring spectra and applications to THH of Rognes, Sagave and Schlichtkrull [Rog09; RSS15]. In differential geometry, the origins of logarithmic ideas trace back to the *b-geometry* of Melrose [Mela], made explicit in the work of Kottke-Melrose [KM11], and especially that of Gillam-Molcho [GM15].

Notation 4.1.8.14. As in our notation, the set $\mathbb{N}$ does not contain 0, we write $\mathbb{Z}_{\geq 0}$ for the free commutative monoid on one generator. The commutative product in a generic commutative monoid is written additively $(\_+\_)$, while the product in a commutative monoid coming from a commutative algebra is written multiplicatively (by juxtaposing the elements being multiplied).

Construction 4.1.8.15. Let $\mathsf{CartSp}_{\geq 0} \subset \mathsf{CartSp}_c$ be the full subcategory spanned by the objects of the form $\mathbb{R}^n_{\geq 0}$; this determines a (1-sorted) Lawvere theory. Recall the notation FCMon for the category of finitely generated and free commutative monoids. We define a functor $\theta : \mathsf{FCMon}^{op} \to \mathsf{CartSp}_{\geq 0}$ as follows.

(1) θ carries the free commutative monoid $\mathbb{Z}^n_{\geq 0}$ to $\mathbb{R}^n_{\geq 0}$.

(2) θ carries a morphism $f : \mathbb{Z}^n_{\geq 0} \leftarrow \mathbb{Z}^m_{\geq 0}$ determined by an m-tuple $\{(k^i_1, \ldots, k^i_n) \in \mathbb{Z}^n\}_{1 \leq i \leq m}$ to the smooth map $\mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ given by
$$(x_1, \ldots, x_n) \longmapsto \left(\prod_{1 \leq j \leq n} x_j^{k^1_j}, \ldots, \prod_{1 \leq j \leq n} x_j^{k^m_j} \right).$$

Restricting along θ induces a product preserving functor $\theta^* : \mathrm{Fun}(\mathbf{N}(\mathsf{CartSp}_{\geq 0}), \mathcal{S}) \to \mathrm{Fun}(\mathbf{N}(\mathsf{FCMon}^{op}), \mathcal{S})$, resulting in a functor
$$\theta^* : \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}_{\geq 0}), \mathcal{S}) \longrightarrow s\mathsf{CMon}$$
which fits into a commuting diagram
$$\begin{tikzcd} \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}_{\geq 0}), \mathcal{S}) \arrow[rr, "\theta^*"] \arrow[dr, "\mathrm{ev}_{\mathbb{R}_{\geq 0}}"'] & & s\mathsf{CMon} \arrow[dl, "\mathrm{ev}_{\mathbb{Z}_{\geq 0}}"] \\ & \mathcal{S}. & \end{tikzcd}$$

The diagonal morphisms in this diagram are conservative and preserve limits and sifted colimits, so the same is true for θ^*. Composing θ^* with the functor induced by the product preserving full subcategory inclusion $\iota_{\geq 0} : \mathsf{CartSp}_{\geq 0} \hookrightarrow \mathsf{CartSp}_c$ yields a limit and sifted colimit preserving functor $\theta^* \iota^*_{\geq 0} : sC^{\infty}\mathsf{ring}_{pc} \to s\mathsf{CMon}$. Corollary 4.1.6.6 provides a right adjoint ι_{c*} to the functor $\iota^*_c : sC^{\infty}\mathsf{ring}_{pc} \to sC^{\infty}\mathsf{ring}$ induced by the inclusion $\iota_c : \mathsf{CartSp} \hookrightarrow \mathsf{CartSp}_c$. The composite functor $\theta^* \iota^*_{\geq 0} \iota_{c*}$ carries simplicial C^{∞}-rings to simplicial commutative monoids, and we will denote this functor by $(\_)_{\geq 0} : sC^{\infty}\mathsf{ring} \to s\mathsf{CMon}$. We define the presentable ∞-category of *positive prelog simplicial C^{∞}-rings* as the cone in the pullback diagram
$$\begin{tikzcd} sC^{\infty}\mathsf{PLog} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, s\mathsf{CMon}) \arrow[d, "\mathrm{ev}_1"] \\ sC^{\infty}\mathsf{ring} \arrow[r, "(\_)_{\geq 0}"] & s\mathsf{CMon} \end{tikzcd}$$
among presentable ∞-categories and functors that admit left adjoints between them. An object of $sC^{\infty}\mathsf{PLog}$ consists of a pair $(A, M \to A_{\geq 0})$ where A is a simplicial C^{∞}-ring and $M \to A_{\geq 0}$ is a map of simplicial commutative monoids. We define a functor $sC^{\infty}\mathsf{ring}_{pc} \to sC^{\infty}\mathsf{PLog}$ as follows. Composing the unit transformation $\mathrm{id} \to \iota_{c*}\iota^*_c$ with $\theta^*\iota^*_{\geq 0}$ yields a functor
$$sC^{\infty}\mathsf{ring}_{pc} \longrightarrow \mathrm{Fun}(\Delta^1, sC^{\infty}\mathsf{ring}_{pc}) \longrightarrow \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$$
which participates as the top horizontal map in the strictly commuting diagram
$$\begin{tikzcd} sC^{\infty}\mathsf{ring}_{pc} \arrow[r] \arrow[d, "\iota^*_c"] & \mathrm{Fun}(\Delta^1, s\mathsf{CMon}) \arrow[d, "\mathrm{ev}_1"] \\ sC^{\infty}\mathsf{ring} \arrow[r, "(\_)_{\geq 0}"] & s\mathsf{CMon} \end{tikzcd}$$
among ∞-categories; hence we obtain an induced functor
$$\begin{tikzcd} sC^{\infty}\mathsf{ring}_{pc} \arrow[rr, "\Xi"] \arrow[dr, "\iota^*_c"'] & & sC^{\infty}\mathsf{PLog} \arrow[dl, "p"] \\ & sC^{\infty}\mathsf{ring} & \end{tikzcd}$$
which is given on objects by the assignment $(A, A_c) \mapsto (A, A_c \to A_{\geq 0})$. From the description of ι^*_c-Cartesian edges in proposition 4.1.6.7 and the fact that $sC^{\infty}\mathsf{ring}_{pc} \to s\mathsf{CMon}$ preserves limits we immediately deduce that Ξ carries ι^*_c-Cartesian edges to p-Cartesian edges.

Remark 4.1.8.16. The functor Ξ of construction 4.1.8.15 does not take ι_c^*-coCartesian edges to p-coCartesian edges and therefore merely induces a lax natural transformation between straightened functors $\mathrm{St}^{+,\mathrm{co}}(\iota_c^*) \Rightarrow \mathrm{St}^{+,\mathrm{co}}(p)$. By the results of [Hau+20], straightening/unstraightening yields equivalences

$$\mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)_{\mathrm{lax}} \simeq \mathsf{biCart}_{\mathcal{C}}^{(\mathrm{co})lax} \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathsf{Cat}_\infty)_{\mathrm{colax}}.$$

For each map $A \to B$ of simplicial C^∞-rings, requisite the 2-morphism is given by the Beck-Chevalley transformation.

To aid our analysis, we recall some facts about simplicial abelian groups and simplicial commutative monoids.

Lemma 4.1.8.17. *Consider $s\mathsf{CMon}$ with its coCartesian symmetric monoidal structure and $\mathcal{S}$ with its Cartesian symmetric monoidal structure, then the forgetful functor $s\mathsf{CMon} \to \mathcal{S}$ induced by evaluation at $\mathbb{Z}_{\geq 0}$ has a canonical symmetric monoidal structure.*

Proof. We sketch two proofs. According to [Lur17a], thm. 2.3.4.18, the functor $f : s\mathsf{CMon} \to \mathsf{Mon}_{\mathbb{E}_\infty}$ classifies an ∞-operad map $s\mathsf{CMon}^{\amalg} \to \mathcal{S}^{\times}$ lifting the functor evaluating at $\mathbb{Z}_{\geq 0}$. Unwinding the definitions, this functor is symmetric monoidal if and only if f preserves finite coproducts, which is the case.
For another argument, it is not hard to see that the functor of 1-categories $\mathsf{FCMon} \to \mathsf{Set}$ has a canonical symmetric monoidal structure, and the relevant symmetric monoidal functor can be obtained by symmetric monoidal left Kan extension. □

Proposition 4.1.8.18. *A simplicial commutative monoid A is* grouplike *if the commutative monoid $\pi_0(A)$ is a (necessarily abelian) group. Let $s\mathsf{CMon}^{\mathrm{gp}}$ denote the full subcategory spanned by the grouplike commutative monoids.*

(1) *The full subcategory inclusion $s\mathsf{CMon}^{\mathrm{gp}} \subset s\mathsf{CMon}$ admits a right adjoint (that we will denote $(\_)^{\times}$, the* ∞-group of units*).*

(2) *The full subcategory inclusion $s\mathsf{CMon}^{\mathrm{gp}} \subset s\mathsf{CMon}$ admits a left adjoint (that we will denote $(\_)^{\mathrm{gp}}$, the* group completion*).*

(3) *Let $\mathsf{FAb} \subset s\mathsf{CMon}^{\mathrm{gp}}$ denote the full subcategory spanned by finitely generated free abelian groups, which is an idempotent complete Lawvere theory. Let $s\mathsf{Ab}$ be the ∞-category of algebras for this theory, then the inclusion $\mathsf{FAb} \subset s\mathsf{CMon}^{\mathrm{gp}}$ induces an equivalence of ∞-categories $s\mathsf{Ab} \simeq s\mathsf{CMon}^{\mathrm{gp}}$.*

(4) *There is a functor $\mathcal{S}\mathrm{p}^{\mathrm{cn}} \to s\mathsf{Ab}$ in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$ fitting into a pushout diagram*

$$\begin{tikzcd} \mathsf{Mon}_{\mathbb{E}_\infty} \arrow[r] \arrow[d, "(\_)^{\mathrm{gp}}"] & s\mathsf{CMon} \arrow[d, "(\_)^{\mathrm{gp}}"] \\ \mathcal{S}\mathrm{p}^{\mathrm{cn}} \arrow[r] & s\mathsf{Ab} \end{tikzcd}$$

in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Proj}}$.

Proof. (1) To see that the inclusion $s\mathsf{CMon}^{\mathrm{gp}} \subset s\mathsf{CMon}$ admits a right adjoint, let $\pi_0(A)^{\times} \subset \pi_0(A)$ be the submonoid on the invertible elements of $\pi_0(A)$, that is, the largest subgroup contained in $\pi_0(A)$, and consider the pullback diagram

$$\begin{tikzcd} A^{\times} \arrow[r] \arrow[d] & A \arrow[d] \\ \pi_0(A)^{\times} \arrow[r] & \pi_0(A), \end{tikzcd}$$

in $s\mathsf{CMon}$, then $A^{\times}$ is clearly grouplike and for each grouplike simplicial commutative monoid B, the map of spaces

$$\mathrm{Hom}_{s\mathsf{CMon}}(B, A^{\times}) \longrightarrow \mathrm{Hom}_{s\mathsf{CMon}}(B, A)$$

is a pullback of the map of sets

$$\mathrm{Hom}_{\mathsf{CMon}}(\pi_0(B), \pi_0(A)^{\times}) \longrightarrow \mathrm{Hom}_{\mathsf{CMon}}(\pi_0(B), \pi_0(A)),$$

which is a bijection as the operation $(\_)^{\times} : \mathsf{CMon} \to \mathsf{Ab}$ is right adjoint to the inclusion of abelian groups into commutative monoids. Thus, the inclusion of connected components $A^{\times} \to A$ exhibits a colocalization.

(2) Consider the full subcategory $s\mathsf{CMon}^{\geq 1} \subset s\mathsf{CMon}$ spanned by objects A for which the underlying space is 1-connective, which is stable under colimits. It follows from the previous lemma that the underlying space functor $s\mathsf{CMon} \to \mathcal{S}$ carries the initial object to a final object, but as the underlying space functor reflects limits, we conclude that $s\mathsf{CMon}$ is pointed, so we have a suspension/looping adjunction

$$s\mathsf{CMon} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} s\mathsf{CMon}.$$

It follows from the previous lemma that the underlying space of ΣA is the colimit of the Bar construction $|\mathsf{Bar}_A(*,*)_\bullet|$ which is 1-connective. Unwinding the definitions, we can identify the functor

$$s\mathsf{CMon} \longrightarrow \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{S}), \quad A \longmapsto \mathsf{Bar}_A(*,*)_\bullet$$

with the composition

$$s\mathsf{CMon} \longrightarrow \mathsf{Mon}_{\mathbb{E}_\infty} \longrightarrow \mathsf{Mon}_{\mathbb{E}_1} \subset \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}^{op}), \mathcal{S}).$$

We conclude that A is a grouplike simplicial commutative monoid if and only if the simplicial object $\mathsf{Bar}_A(*,*)_\bullet$ is a group object. Since all groups are effective in $\mathcal{S}$ and in $s\mathsf{CMon}$, the augmented simplicial object $|\mathsf{Bar}_A(*,*)_\bullet|$ is a Čech nerve. The functor Ω factor as

$$s\mathsf{CMon} \xrightarrow{\simeq} \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}_+^{op}), s\mathsf{CMon})' \subset \mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}_+^{op}), s\mathsf{CMon}) \xrightarrow{\mathrm{ev}_{[1]}} s\mathsf{CMon},$$

where $\mathrm{Fun}(\mathbf{N}(\mathbf{\Delta}_+^{op}), s\mathsf{CMon})'$ denotes the full subcategory spanned by Čech nerves $U_\bullet$ such that $U_0 \simeq *$. The first equivalence restricts to one $s\mathsf{CMon}^{\geq 1} \simeq \mathsf{Grp}^+(s\mathsf{CMon})$ between 1-connective objects and Čech nerves $U_\bullet$ with $U_0 \simeq *$ that are colimit diagrams. Let $U_\bullet$ be a Čech nerve with $U_0 \simeq *$, then $U_\bullet|_{\mathbf{N}(\mathbf{\Delta}^{op})}$ is a group object in sCMon; then $\pi_0(U_1)$ is a group so that U_1 is grouplike, since group object in commutative monoids are abelian groups by the classical Eckmann-Hilton argument. It follows that the adjunction $(\Sigma \dashv \Omega)$ restricts to give an adjunction

$$s\mathsf{CMon}^{\mathrm{gp}} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} s\mathsf{CMon}^{\geq 1}$$

which is an equivalence: if A is grouplike, then the Bar construction $U_\bullet := \mathsf{Bar}_A(*,*)_\bullet$ is a Čech nerve so the canonical map $A = U_1 \to * \times_{U_{-1}} *$ is an equivalence. Conversely, let B be a 1-connective object and $V_\bullet$ the Čech nerve of $* \to B$, then we should show that the canonical map $\Sigma V_1 \to B$ is an equivalence. Let $V'_\bullet$ denote a right Kan extension of the diagram $W : \mathbf{N}(\mathbf{\Delta}_+^{op})_{\leq 1} \to \mathsf{sCMon}$ given by

$$V_1 \rightrightarrows * \longrightarrow \Sigma V_1$$

along the inclusion $\mathbf{N}(\mathbf{\Delta}_+^{op})_{\leq 1} \subset \mathbf{N}(\mathbf{\Delta}_+^{op})$, then we have an induced map $\alpha : V'_\bullet \to V_\bullet$ which restricts to the identity on $\mathbf{N}(\mathbf{\Delta}^{op})_{\leq 1}$. Because V_1 is grouplike, the diagram W is a right Kan extension of $W|_{\mathbf{N}(\mathbf{\Delta}_+^{op})_{\leq 0}}$, which implies by [Lur17b], prop. 4.3.2.8 that $V'_\bullet$ is a Čech nerve. We conclude that $\alpha|_{\mathbf{N}(\mathbf{\Delta}^{op})}$ is a morphism of group objects such that α_1 is the identity, but this implies that α is an equivalence. It follows that the composition

$$s\mathsf{CMon} \xrightarrow{\Sigma} s\mathsf{CMon}^{\geq 1} \underset{\simeq}{\xrightarrow{\Omega}} s\mathsf{CMon}^{\mathrm{gp}}$$

is a left adjoint to the inclusion.

(3) It follows from (1), (2) and [Lur17a], prop. 7.1.4.12 that it suffices to argue that the essential image of $(\_)^{\mathrm{gp}}$ on FCMon consists of finitely generated free abelian groups. On underlying spaces, we can identify the map $\Sigma\mathbb{Z}_{\geq 0} \to \Sigma\mathbb{Z}$ induced by the inclusion $\mathbb{Z}_{\geq 0} \to \mathbb{Z}$ with the map $\beta : B\mathcal{C} \to B\mathcal{D}$ of classifying spaces, where $\mathcal{C}$ and $\mathcal{D}$ are the single object categories with space of morphisms $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}$ respectively. We can identify the fibre product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{*/}$ with the poset category $\mathbb{Z}$, which has contractible classifying space, so we deduce that β is an equivalence by Quillen's theorem A. It follows that the unit of the group completion $\mathbb{Z}_{\geq 0} \to (\mathbb{Z}_{\geq 0})^{\mathrm{gp}}$ is equivalent to $\mathbb{Z}_{\geq 0} \to \mathbb{Z}$.

(4) Consider the functor $s\mathsf{CMon} \to \mathsf{Mon}_{\mathbb{E}_\infty}$ induced by the transformation of algebraic theories $\mathcal{F} \to \mathbf{N}(\mathsf{FCMon})$, then $A \in s\mathsf{CMon}$ is grouplike if and only if the associated $\mathbb{E}_\infty$-space is grouplike, but we have an equivalence $\mathsf{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}} \simeq \mathcal{S}\mathrm{p}^{\mathrm{cn}}$ (by [Lur17a], rmk. 5.2.6.26 for instance), so we have a pullback diagram

$$\begin{tikzcd} s\mathsf{Ab} \arrow[r] \arrow[d] & s\mathsf{CMon} \arrow[d] \\ \mathcal{S}\mathrm{p}^{\mathrm{cn}} \arrow[r] & \mathsf{Mon}_{\mathbb{E}_\infty} \end{tikzcd}$$

of ∞-categories and conservative functors preserving limits and colimits.

□

Remark 4.1.8.19. We have seen that for any simplicial C^∞-ring A, the space $A_{\geq 0} := \mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{\geq 0}), A)$ admits a natural structure of a simplicial commutative monoid. The simplicial C^∞-ring A evidently also admits the structure of a simplicial commutative monoid; the requisite forgetful functor $(\_)^{\mathsf{Mon}} : sC^\infty\mathsf{ring} \to s\mathsf{CMon}$ can be defined in two (naturally equivalent) ways: we can define a functor $\mathsf{FCMon}^{op} \to \mathsf{CartSp}$ via the same formulae that appear in construction 4.1.8.15, or we can take the composition $sC^\infty\mathsf{ring} \overset{(\text{-})^{\mathrm{alg}}}{\to} \mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}} \to s\mathsf{CMon}$ where the second functor is induced by the lax monoidal functor $\mathsf{Mod}^{\mathrm{cn}}_{\mathbb{R}} \to \mathcal{S}$. The functors $(\_)_{\geq 0}$ and $(\_)^{\mathsf{Mon}}$ may be combined by defining a functor $\Theta : \mathsf{FCMon}^{op} \times \Delta^1 \to \mathsf{CartSp}_c$ as follows.

(1) Θ carries the object $(\mathbb{Z}^n_{\geq 0}, 0)$ to $\mathbb{R}^n_{\geq 0}$ and the object $(\mathbb{Z}^m_{\geq 0}, 1)$ to $\mathbb{R}^m$.

(2) Θ carries a morphism $f : (\mathbb{Z}^n_{\geq 0}, 0) \leftarrow (\mathbb{Z}^m_{\geq 0}, 0)$ determined by an m-tuple $\{(k^i_1, \ldots, k^i_n) \in \mathbb{Z}^n\}_{1 \leq i \leq m}$ to the map $\mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ given by
$$(x_1, \ldots, x_n) \longmapsto \left(\prod_{1 \leq j \leq n} x_j^{k^1_j}, \ldots, \prod_{1 \leq j \leq n} x_j^{k^m_j} \right).$$
The morphisms $(\mathbb{Z}^n_{\geq 0}, 1) \leftarrow (\mathbb{Z}^m_{\geq 0}, 1)$ and $(\mathbb{Z}^n_{\geq 0}, 0) \leftarrow (\mathbb{Z}^m_{\geq 0}, 1)$ are carried to morphisms $\mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{R}^n_{\geq 0} \to \mathbb{R}^m$ respectively, defined by the same formula.

Composing ι_{c*} with $\Theta^* : \mathrm{Fun}(\mathbf{N}(\mathsf{CartSp}_c), \mathcal{S}) \to \mathrm{Fun}(\mathbf{N}(\mathsf{FCMon}^{op}) \times \Delta^1, \mathcal{S})$ yields a natural transformation $sC^\infty\mathsf{ring} \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$ that lifts, for each $A \in sC^\infty\mathsf{ring}$, the map of spaces $A_{\geq 0} \to A$ induced by the map $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}_{\geq 0})$ to a map of simplicial commutative monoids $A_{\geq 0} \to A^{\mathsf{Mon}}$. In remark 4.1.6.15 we argued that the natural map
$$\pi_0(A_{\geq 0}) \longrightarrow \pi_0(A)_{\geq 0} = \mathrm{Hom}_{C^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{\geq 0}), \pi_0(A))$$
is an equivalence. Since the map $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}_{\geq 0})$ is a regular epimorphism of C^∞-rings, we have an injection $\pi_0(A)_{\geq 0} \hookrightarrow \pi_0(A)$, which is obtained by applying the functor taking connected components to the map $A_{\geq 0} \to A$ and the isomorphism $\pi_0(A_{\geq 0}) \cong \pi_0(A)_{\geq 0}$. We conclude that the commutative monoid structure on $\pi_0(A)$ restricts to one on the subset $\pi_0(A_{\geq 0})$, and this latter structure then coincides with the one coming from the simplicial commutative monoid structure on $A_{\geq 0}$ defined in construction 4.1.8.15. We use this observation to identify the group of units of $\pi_0(A_{\geq 0})$: the group $\pi_0(A)^\times \hookrightarrow \pi_0(A)$ of invertible elements coincides with the map $\mathrm{Hom}_{C^\infty\mathsf{ring}}(C^\infty(\mathbb{R} \smallsetminus \{0\}), A) \to \mathrm{Hom}_{C^\infty\mathsf{ring}}(C^\infty(\mathbb{R}), A)$ by definition of the localization. Thus, if $x \in \pi_0(A_{\geq 0})$ is invertible as an element in $\pi_0(A)$ we have a commuting diagram
$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}_{\geq 0}) \arrow[d, "x"] \\ C^\infty(\mathbb{R} \smallsetminus \{0\}) \arrow[r] & \pi_0(A) \end{tikzcd}$$
of C^∞-rings, so the map classifying x factors through the pushout $C^\infty(\mathbb{R}_{>0}) \to \pi_0(A)$, which shows that the inverse of x lies in the submonoid $\pi_0(A_{\geq 0})$. It follows that the group of units $\pi_0(A_{\geq 0})^\times$ is given by a pullback $\pi_0(A_{\geq 0}) \times_{\pi_0(A)} \pi_0(A)^\times$, so the monomorphism $A^\times_{\geq 0} \to A_{\geq 0}$ fits as the left vertical map into a pullback diagram
$$\begin{tikzcd} A^\times_{\geq 0} \arrow[r] \arrow[d] & \mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R} \smallsetminus \{0\}), A) \arrow[d] \\ A_{\geq 0} \arrow[r] & \mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}), A). \end{tikzcd}$$
As a result, this map coincides with the map $\mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{>0}), A) \to \mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{\geq 0}), A)$. Thus, the map $A_{>0} \to A_{\geq 0}$ of remark 4.1.8.4 coincides with the inclusion of the ∞-group of units $A^\times_{\geq 0} \hookrightarrow A_{\geq 0}$.

Remark 4.1.8.20. We give one more application of remark 4.1.6.15. Abusing notation, we denote $(\_)_{\geq 0} : C^\infty\mathsf{ring} \to \mathsf{CMon}$ for the functor given by $A \mapsto \mathrm{Hom}_{C^\infty\mathsf{ring}}(C^\infty(\mathbb{R}_{\geq 0}), A)$, and define a category $C^\infty\mathsf{PLog}$ as the pullback $C^\infty\mathsf{ring} \times_{\mathsf{CMon}} \mathrm{Fun}(\Delta^1, \mathsf{CMon})$. We have a diagram
$$\begin{tikzcd} \mathbf{N}(C^\infty\mathsf{ring}) \arrow[r, "(\_)_{\geq 0}"] \arrow[d, hook] & \mathbf{N}(\mathsf{CMon}) \arrow[d, hook] \\ sC^\infty\mathsf{ring} \arrow[r, "(\_)_{\geq 0}"] & s\mathsf{CMon} \end{tikzcd}$$
which commutes up to canonical homotopy, determining a fully faithful functor $g : \mathbf{N}(C^\infty\mathsf{PLog}) \to sC^\infty\mathsf{PLog}$. The vertical maps admit left adjoint functors denoted by π_0 and the associated Beck-Chevalley transformation at an object $A \in sC^\infty\mathsf{ring}$ is obtained by applying π_0 to the map of simplicial commutative monoids $f : A_{\geq 0} \to \pi_0(A)_{\geq 0}$ induced by the unit map $A \to A_{\geq 0}$. It follows from remark 4.1.6.15 that f exhibits a 0-truncation, so we have an

equivalence $\pi_0((\_)_{\geq 0}) \simeq \pi_0(\_)_{\geq 0}$ which provides a left adjoint $\pi_0 : sC^\infty\mathsf{PLog} \to \mathbf{N}(C^\infty\mathsf{PLog})$ to g. This adjunction is equivalent to the 0'th truncation $\tau_{\leq 0}$. To see this, it suffices to show that an object $(A, M \to A_{\geq 0})$ is 0-truncated if and only if it lies in the essential image of g which is easily seen to consist of those objects $(B, N \to B_{\geq 0})$ where B is a 0-truncated simplicial C^∞-ring and N is a 0-truncated simplicial commutative monoid. The 'only if' direction follows immediately from the fact that both $p : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{ring}$ and $sC^\infty\mathsf{PLog} \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$ preserve limits. For the 'if' direction, we suppose that A and M are 0-truncated, then we have for any $(B, N \to B_{\geq 0}) \in sC^\infty\mathsf{PLog}$ and any map $f : B \to A$ a fibre sequence

$$\mathrm{Hom}_{(s\mathsf{CMon})_{/B_{\geq 0}}}(N, M \times_{A_{\geq 0}} B_{\geq 0}) \longrightarrow \mathrm{Hom}_{sC^\infty\mathsf{PLog}}((B, N \to B_{\geq 0}), (A, M \to A_{\geq 0})) \longrightarrow \mathrm{Hom}_{sC^\infty\mathsf{ring}}(B, A)$$

since $p : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{ring}$ is a Cartesian fibration. To conclude that $\mathrm{Hom}_{sC^\infty\mathsf{PLog}}((B, N \to B_{\geq 0}), (A, M \to A_{\geq 0}))$ is 0-truncated, it suffices to argue that the base and the fibre spaces are 0-truncated. As A is 0-truncated, the base space is also 0-truncated and as $M \to A_{\geq 0}$ is a 0-truncated morphism, the map $M \times_{A_{\geq 0}} B_{\geq 0} \to B_{\geq 0}$ is too so the fibre is also 0-truncated.
Using an analogous argument, it can be shown that $\Xi : sC^\infty\mathsf{ring}_{pc} \to sC^\infty\mathsf{PLog}$ takes n-truncations to n-truncations for all $n \geq 0$, that is, the relevant Beck-Chevalley map provides an equivalence $\tau_{\leq n} \circ \Xi \simeq \Xi^n \circ \tau_{\leq n}$, where Ξ^n is the functor $\tau_{\leq n} sC^\infty\mathsf{ring}_{pc} \to \tau_{\leq n} sC^\infty\mathsf{PLog}$ induced by Ξ.

Definition 4.1.8.21. Let A be a simplicial commutative monoid and let $M \in (s\mathsf{CMon})_{/A}$ be a prelog structure on A, then M is a *log structure* on A if the upper horizontal map in the pullback diagram

$$\begin{tikzcd} M \times_A A^\times \arrow[r] \arrow[d] & A^\times \arrow[d] \\ M \arrow[r] & A \end{tikzcd}$$

is an equivalence, where the right vertical map is the counit of the coreflective embedding $s\mathsf{Ab} \subset s\mathsf{CMon}$, that is, the inclusion of connected components determined by the invertible elements in the commutative monoid $\pi_0(A)$. We denote by $\mathsf{Log}_A \subset (s\mathsf{CMon})_{/A}$ the full subcategory spanned by log structures and $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog}$ the full subcategory spanned by objects $(A, M \to A_{\geq 0})$ such that the prelog structure M is a log structure on $A_{\geq 0}$.

Remark 4.1.8.22. A prelog structure $M \to A$ is a log structure if and only if the canonical maps $M^\times \to A^\times$ and $M^\times \to M \times_A A^\times$ are both equivalences.

The following proposition is an immediate consequence of remarks 4.1.8.4 and 4.1.8.19.

Proposition 4.1.8.23. *The functor $\Xi : sC^\infty\mathsf{ring}_{pc} \to sC^\infty\mathsf{PLog}$ restricted to $sC^\infty\mathsf{ring}_c$ takes values in $sC^\infty\mathsf{Log}$. Denoting the resulting functor $sC^\infty\mathsf{ring}_c \to sC^\infty\mathsf{PLog}$ by Ξ_{Log}, the commuting diagram*

$$\begin{tikzcd} sC^\infty\mathsf{ring}_c \arrow[r, "\Xi_{\mathsf{Log}}"] \arrow[d, hook] & sC^\infty\mathsf{Log} \arrow[d, hook] \\ sC^\infty\mathsf{ring}_{pc} \arrow[r, "\Xi"] & sC^\infty\mathsf{PLog} \end{tikzcd}$$

is a homotopy pullback diagram of ∞-categories.

The construction $(A, A_c) \mapsto (A, A_c \to A_{\geq 0})$ implemented by the functors Ξ and Ξ_{Log} is obviously conservative. Ξ and Ξ_{Log} also preserve limits and sifted colimits (as we will show shortly) so we might like to interpret them as forgetful functors. The notion of a simplicial C^∞-ring with corners appears prima facie strictly more structured than a positive prelog simplicial C^∞-ring, as Ξ forgets the C^∞ information contained in A_c. When we restrict to logarithmic structures however, we see that there is no loss of information at all.

Theorem 4.1.8.24. *The functor $\Xi_{\mathsf{Log}} : sC^\infty\mathsf{ring}_c \to sC^\infty\mathsf{Log}$ is an equivalence of ∞-categories.*

As we will see, this result grants us control over the coCartesian morphisms of ι_c^*, which reduces the computation of limits and colimits in $sC^\infty\mathsf{ring}_c$ to limits and colimits in $sC^\infty\mathsf{ring}$ and in ∞-categories of log structures. The proof of theorem 4.1.8.24 requires a few prelimenaries. Our first order of business is to understand the relative left adjoint to the inclusion $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog}$. The following result is familiar from the usual theory of log structures on monoids, albeit that the proof is somewhat more involved since we do not take recourse to point-set arguments.

Proposition 4.1.8.25. *Denote by p_{Log} the composition $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog} \xrightarrow{p} sC^\infty\mathsf{Log}$.*

(1) *The functor p_{Log} is a Cartesian fibration and the inclusion $sC^\infty\mathsf{Log} \hookrightarrow sC^\infty\mathsf{PLog}$ carries Cartesian edges to Cartesian edges.*

(2) *For each simplicial commutative monoid A, the fully faithful inclusion $\mathsf{Log}_A \subset (s\mathsf{CMon})_{/A}$ preserves sifted colimits and admits a left adjoint.*

(3) *The inclusion $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog}$ admits a left adjoint relative to $sC^\infty\mathsf{ring}$.*

Proof. The proof of (1) amounts to the assertion that if $M \to A$ is a log structure on a simplicial commutative monoid A and $B \to A$ is any morphism of simplicial commutative monoids, then $B \times_A M$ is a log structure on B, which is a straightforward check. For (2), let $M \to A$ be a log structure, and consider the pushout diagram of prelog structures over A:

$$\begin{tikzcd} M \times_A A^\times \arrow[r] \arrow[d] & A^\times \arrow[d] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

It suffices to show that N is a log structure over A and that restriction along the morphism f induces, for each log structure $M' \to A$, an equivalence

$$\mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(N, M') \xrightarrow{\simeq} \mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(M, M').$$

The following assertion will enjoy verification at the end of the proof.

($*$) The diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ N \arrow[r] & A. \end{tikzcd}$$

is a pullback square of simplicial commutative monoids.

We have a diagram

$$\begin{tikzcd} N^\times \arrow[r, "g"] \arrow[d] & N \times_A A^\times \arrow[r, "h"] \arrow[d] & A^\times \arrow[d] \\ N \arrow[r, equal] & N \arrow[r] & A. \end{tikzcd}$$

Both maps $N^\times \to N$ and $N \times_A A^\times \to N$ are inclusions of connected components, so the map $N^\times \to N \times_A A^\times$ is one as well. We first show that the map $h \circ g : N^\times \to A^\times$ is an equivalence. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ N^\times \arrow[r] & N \times_A A^\times \arrow[r] \arrow[d] & A^\times \arrow[d] \\ & N \arrow[r] & A. \end{tikzcd}$$

The right upper square is a pullback diagram since the right outer rectangle is one, by ($*$). Because the map $N^\times \to N \times_A A^\times$ is an inclusion of connected components, the upper rectangle is also a pullback diagram of simplicial abelian groups, and therefore also a pullback diagram of connective spectra. Since the map $N^\times \to A^\times$ is an effective epimorphism (i.e. 0-connective), the upper rectangle is also a pullback diagram of spectra. Then it is a pushout diagram, so the map $N^\times \to A^\times$ is an equivalence. It follows that $A^\times$ is a retract of $N \times_A A^\times$. Choose an element $x \in \pi_0(N \times_A A^\times)$, then $h(x)$ is invertible in $\pi_0(A^\times)$ so admits an inverse y. Consider the element $z := g((h \circ g)^{-1}(y)) \in \pi_0(A)$, then $h(x + z) = h(x) + h(g((h \circ g)^{-1}(y))) = h(x) + y$, which is the unit. By ($*$), we have $h^{-1}(0) = 0$, so z is an inverse of x. It follows that g and therefore also h is an equivalence. We now proceed by showing that the map on morphism spaces induced by restricting along f induces an equivalence for each log structure $M' \to A$. The relevant map is the the upper horizontal one in a pullback diagram

$$\begin{tikzcd} \mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(N, M') \arrow[r] \arrow[d] & \mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(M, M') \arrow[d] \\ \mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(A^\times, M') \arrow[r] & \mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(M \times_A A^\times, M') \end{tikzcd}$$

of spaces, so it suffices to argue that the lower horizontal map is an equivalence. In fact, we claim that both the domain and codomain of this map are weakly contractible. Note that both $A^\times$ and $M \times_A A^\times$ lie in the image of the functor $(s\mathsf{CMon})_{/A^\times} \to (s\mathsf{CMon})_{/A}$. From the adjunction $(s\mathsf{CMon})_{/A^\times} \leftrightarrows (s\mathsf{CMon})_{/A}$ we have for any $L \to A^\times$ an equivalence $\mathrm{Hom}_{(s\mathsf{CMon})_{/A}}(L, M') \simeq \mathrm{Hom}_{(s\mathsf{CMon})_{/A^\times}}(L, M' \times_A A^\times)$. As M' is a log structure, the map $M' \times_A A^\times \to A^\times$ is an equivalence, so $M' \times_A A^\times$ is a final object in the ∞-category of prelog structures over $A^\times$, which proves our claim. We have constructed a left adjoint to the inclusion $\mathsf{Log}_A \subset (s\mathsf{CMon})_{/A}$, so (2) follows from the observation that this inclusion is stable under sifted colimits, as sifted colimits are universal in $s\mathsf{CMon}$.

It is an immediate consequence of (1), (2) and HA. prop. 7.3.2.6 that the inclusion $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog}$ admits a left adjoint relative to $sC^\infty\mathsf{ring}$.

We are left to prove assertion $(*)$. The diagram

$$\begin{tikzcd}
M \times_A A^\times \arrow[r] \arrow[d] & A^\times \arrow[d] \\
M \arrow[r] & A
\end{tikzcd}$$

of simplicial commutative monoids induces an $M \times_A A^\times$-bilinear (in the sense of [Lur17a], section 4.4.4) map $M \times A^\times \to A$ which is encoded by the simplicial object $\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet$ being equipped with an augmentation to A. The map $N \to A$ can be identified with the canonical map $|\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet| \to A$. We have morphisms of simplicial objects

$$\mathsf{Bar}_{M \times_A A^\times}(M, A^\times)_\bullet \xleftarrow{\alpha} \mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \xrightarrow{\beta} \mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, 0)_\bullet$$

induced by the $(M \times_A A^\times)$-module morphisms $M \times_A A^\times \to M$ and $A^\times \to 0$; in particular, for each $[n] \in \mathbf{\Delta}$, we have maps of spaces

$$M \times (M \times_A A^\times)^{\times n} \times A^\times \longleftarrow M \times_A A^\times \times (M \times_A A^\times)^{\times n} \times A^\times \longrightarrow M \times_A A^\times \times (M \times_A A^\times)^{\times n} \times *,$$

where the left map is an inclusion of connected components and the right map projects away the factor $A^\times$. The map $\alpha : \mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \to \mathsf{Bar}_{M \times_A A^\times}(M, A^\times)_\bullet$ fits as the left vertical map into a diagram

$$\begin{tikzcd}
\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \arrow[r] \arrow[d, "\alpha"] & A^\times \arrow[d] \\
\mathsf{Bar}_{M \times_A A^\times}(M, A^\times)_\bullet \arrow[r] & A.
\end{tikzcd}$$

Since both vertical maps are inclusions of connected components in each simplicial level, it follows from an easy check on connected components that this diagram is a pullback diagram of simplicial objects. Since colimits are universal in spaces, it suffices to show that the colimit of the simplicial object defined as the cone in the pullback diagram

$$\begin{tikzcd}
\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \times_{A^\times} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \arrow[r] & A^\times
\end{tikzcd}$$

of simplicial objects is contractible. The map $A^\times \to 0$ induces a commuting diagram

$$\begin{tikzcd}
\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \arrow[r] \arrow[d, "\beta"] & A^\times \arrow[d] \\
\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, 0)_\bullet \arrow[r] & 0.
\end{tikzcd}$$

Since the left vertical map projects away the factor $A^\times$ in each simplicial level, this diagram is a pullback diagram. It follows that the composite map

$$\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, A^\times)_\bullet \times_{A^\times} \longrightarrow \mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, 0)_\bullet$$

is an equivalence, as it is a pullback along the map $0 \to 0$. We conclude by observing that the augmented simplicial object $\mathsf{Bar}_{M \times_A A^\times}(M \times_A A^\times, 0)_\bullet \to 0$ is a colimit diagram. $\square$

Remark 4.1.8.26. The proof above gives an explicit description of the value of the left adjoint $sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{Log}$ on a prelog structure $M \to A_{\geq 0}$ as the pushout

$$\begin{tikzcd} M \times_{A_{\geq 0}} A_{\geq 0}^\times \arrow[r] \arrow[d] & A_{\geq 0}^\times \arrow[d] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

If a log structure $N \to A_{\geq 0}$ fits into a pushout diagram as above, we say that f *exhibits N as a logification of M* (with respect to some simplicial commutative monoid $A_{\geq 0}$). We denote the resulting left adjoint, the *logification functor* by $L_{\mathsf{Log}} : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{Log}$. Also note that in virtue of theorem 4.1.8.24, a map $(A, A_c) \to (B, B_c)$ is coCartesian precisely if $A_c \to B_c$ exhibits a logification in the ∞-category of prelog structures over $B_{\geq 0}$.

At this point, we give criteria for the recognition of limits and colimits in $sC^\infty\mathsf{PLog}$. we need the following lemma concerning pushforwards of relative colimits along coCartesian edges.

Lemma 4.1.8.27. *Let $p : \mathcal{C} \to \mathcal{D}$ be a categorical fibration, let K be a simplicial set and let $\overline{\mathcal{J}}_0 : K^\triangleright \to \mathcal{C}$ be a p-colimit diagram. Denote $D = p\overline{\mathcal{J}}_0(\infty)$ and let $e : D \to D'$ be a map in $\mathcal{D}$, which induces a diagram $h : K \star \Delta^1 \to \mathcal{D}$ since the projection $\mathcal{D}_{/e} \to \mathcal{D}_{/D}$ is a trivial Kan fibration. Denote by $\mathcal{J}$ the restriction $\overline{\mathcal{J}}_0|_K$ and suppose we are given a diagram*

$$\begin{tikzcd} K \arrow[r, "\mathcal{J}"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ K \star \Delta^1 \arrow[r, "h"'] \arrow[ur, "\overline{\mathcal{J}}"'] & \mathcal{D}. \end{tikzcd}$$

such that $\overline{\mathcal{J}}|_{K \star \Delta^{\{0\}}} = \overline{\mathcal{J}}_0$. Then the diagram $\overline{\mathcal{J}}_1 := \overline{\mathcal{J}}|_{K \star \Delta^{\{1\}}} : K^\triangleright \to \mathcal{C}$ is a p-colimit diagram if and only if $\overline{\mathcal{J}}|_{\Delta^1}$ is a p-coCartesian lift of e starting at $\overline{\mathcal{J}}_0(\infty)$.

Remark 4.1.8.28. This result is somewhat orthogonal to proposition 4.1.3.9 of [Lur17b], where instead the cone point is fixed and the diagram $K \to \mathcal{C}$ is moved.

Proof. Since we have isomorphisms of simplicial sets $K \times_{K \star \Delta^1} K \star \Delta^1_{/i} \cong K$ for $i \in \Delta^1$ and the functors

$$K^\triangleright = K \star \Delta^{\{0\}} \hookrightarrow K \star \Delta^1 \xrightarrow{\overline{\mathcal{J}}} \mathcal{C}$$

and

$$K^\triangleright = K \star \Delta^{\{1\}} \hookrightarrow K \star \Delta^1 \xrightarrow{\overline{\mathcal{J}}} \mathcal{C}$$

coincide with the functors $\overline{\mathcal{J}}_0$ and $\overline{\mathcal{J}}_1$ respectively, we see that $\overline{\mathcal{J}}$ is a p-left Kan extension of $\mathcal{J}$ if and only if $\overline{\mathcal{J}}_1$ is a p-colimit diagram. By transitivity of p-left Kan extensions ([Lur17b], prop. 4.3.2.8), $\overline{\mathcal{J}}$ is a p-left Kan extension of $\mathcal{J}$ if and only if $\overline{\mathcal{J}}$ is a p-left Kan extension of $\overline{\mathcal{J}}_0$, which in turn is equivalent to the diagram

$$\begin{tikzcd} K^\triangleright \arrow[r, "\overline{\mathcal{J}}_0"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ (K^\triangleright)^\triangleright \arrow[r] \arrow[ur, "\overline{\mathcal{J}}"'] & \mathcal{D}. \end{tikzcd}$$

being a p-colimit diagram. Since the inclusion $\Delta^0 \hookrightarrow K^\triangleright$ of the cone point is left cofinal, we deduce that the right square in the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, hook] \arrow[d, hook] & K^\triangleright \arrow[r, "\overline{\mathcal{J}}_0"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \Delta^1 \arrow[r, hook] & (K^\triangleright)^\triangleright \arrow[r] \arrow[ur, "\overline{\mathcal{J}}"'] & \mathcal{D}. \end{tikzcd}$$

is a p-colimit diagram if and only if the outer rectangle is, which corresponds precisely to the edge $\overline{\mathcal{J}}|_{\Delta^1} : \Delta^1 \to \mathcal{C}$ being p-coCartesian. □

Lemma 4.1.8.29. *The following hold true.*

(1) *The functor $p : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{ring}$ preserves all limits and colimits.*

(2) *The functor*

$$\mathrm{ev}_{\mathsf{PLog}} : sC^\infty\mathsf{PLog} \longrightarrow \mathrm{Fun}(\Delta^1, s\mathsf{CMon}) \xrightarrow{\mathrm{ev}_{\{0\}}} s\mathsf{CMon}$$

preserves all limits and colimits.

Proof. Only the statements involving colimits are not immediate. Since p is a presentable fibration over a presentable base, p preserves colimits. Now choose a small diagram $f : K \to sC^\infty\mathsf{PLog}$, then we wish to show that the map $\mathrm{colim}\,\mathrm{ev}_{\mathsf{PLog}} \circ f \to \mathrm{ev}_{\mathsf{PLog}}(\mathrm{colim}\, f)$ is an equivalence. Let G denote the functor $sC^\infty\mathsf{PLog} \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$. Since an edge $\Delta^1 \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$ is $\mathrm{ev}_{\{1\}}$-coCartesian if and only if the composition $\Delta^1 \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon}) \xrightarrow{\mathrm{ev}_{\{0\}}} s\mathsf{CMon}$ is an equivalence, we are required to show that the map $\mathrm{colim}\, Gf \to G(\mathrm{colim}\, f)$ is a coCartesian edge of $\mathrm{Fun}(\Delta^1, s\mathsf{CMon})$. Choose a colimit diagram $\overline{pf} : K^{\triangleright} \to sC^\infty\mathsf{ring}$ extending $K \xrightarrow{f} sC^\infty\mathsf{PLog} \xrightarrow{p} sC^\infty\mathsf{ring}$ and choose an $\mathrm{ev}_{\{1\}}$-colimit as the dotted lift in the diagram

$$\begin{tikzcd} K \arrow[r, "Gf"] \arrow[d, hook] & \mathrm{Fun}(\Delta^1, s\mathsf{CMon}) \arrow[d, "\mathrm{ev}_{\{1\}}"] \\ K^{\triangleright} \arrow[ur, dotted, "\widehat{Gf}"] \arrow[r, "(-)_{\geq 0}\overline{pf}"'] & s\mathsf{CMon}, \end{tikzcd}$$

then the induced diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, hook] & sC^\infty\mathsf{PLog} \arrow[d, "p"] \\ K^{\triangleright} \arrow[ur, dotted] \arrow[r, "\overline{pf}"'] & sC^\infty\mathsf{ring}, \end{tikzcd}$$

is a p-colimit diagram and, since $\overline{pf}$ is a colimit diagram, the dotted lift is also a colimit diagram extending f. Choose a colimit diagram $\overline{Gf} : K^{\triangleright} \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$, then we have a diagram $K \star \Delta^1 \to \mathrm{Fun}(\Delta^1, s\mathsf{CMon})$, unique up to contractible ambiguity, such that the restriction to K equals Gf and the restriction to Δ^1 is a map $\overline{Gf}(\infty) \to \widehat{Gf}(\infty)$ which we can identify with the canonical map $\mathrm{colim}\, Gf \to G(\mathrm{colim}\, f)$. Moreover, as $\mathrm{ev}_{\{1\}}$ preserves colimits, the diagram $\mathrm{ev}_{\{1\}}\overline{Gf}$ is a colimit diagram, so $\overline{Gf}$ is an $\mathrm{ev}_{\{1\}}$-colimit diagram as well. We obtain a commuting diagram

$$\begin{tikzcd} K \arrow[r, "Gf"] \arrow[d, hook] & \mathrm{Fun}(\Delta^1, s\mathsf{CMon}) \arrow[d, "\mathrm{ev}_{\{1\}}"] \\ K \star \Delta^1 \arrow[ur, dotted] \arrow[r] & s\mathsf{CMon}. \end{tikzcd}$$

such that the diagonal map becomes an $\mathrm{ev}_{\{1\}}$-colimit diagram when restricted to both $K \star \Delta^{\{0\}}$ and $K \star \Delta^{\{1\}}$, so lemma 4.1.8.27 guarantees that the diagonal map restricted to Δ^1 is an $\mathrm{ev}_{\{1\}}$-coCartesian edge. ◻

Corollary 4.1.8.30. *The functor* $p \times \mathrm{ev}_{\mathsf{PLog}} : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{ring} \times s\mathsf{CMon}$ *is conservative and preserves all limits and colimits.*

Corollary 4.1.8.31. *The inclusion* $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog}$ *preserves filtered colimits; in other words, the localization* L_{Log} *is* ω*-accessible.*

Proof. Let $\mathcal{J} : K \to sC^\infty\mathsf{Log}$ be filtered diagram and denote by $(A, M \to A_{\geq 0})$ a colimit of $\mathcal{J}$. According to the proof of lemma 4.1.8.29, we have a commuting diagram

$$\begin{tikzcd} \mathrm{colim}_{\,i \in \mathcal{J}} M_i \arrow[r, "\simeq"] \arrow[d] & M \arrow[d] \\ \mathrm{colim}_{\,i \in \mathcal{J}} (A_i)_{\geq 0} \arrow[r] & A_{\geq 0} \end{tikzcd}$$

in $s\mathsf{CMon}$. Since the functor $s\mathsf{CMon}_{/(A^\times_{\geq 0} \to A_{\geq 0})} \to s\mathsf{CMon}_{/A_{\geq 0}}$ is fully faithful and for each $i \in K$, the composition $(A_i)^\times_{\geq 0} \to A_{\geq 0}$ factors through $A^\times_{\geq 0}$, there is a map $\mathrm{colim}_{\,i \in \mathcal{J}} (A_i)^\times_{\geq 0} \to A^\times_{\geq 0}$ fitting into a commuting diagram

$$\begin{tikzcd} \mathrm{colim}_{\,i} M_i \times_{(A_i)_{\geq 0}} (A_i)^\times_{\geq 0} \arrow[r, "\simeq"] \arrow[d] & \mathrm{colim}_{\,i} (A_i)^\times_{\geq 0} \arrow[r] \arrow[d, "\gamma"] & A^\times_{\geq 0} \arrow[d] \\ \mathrm{colim}_{\,i} M_i \arrow[r] & \mathrm{colim}_{\,i} (A_i)_{\geq 0} \arrow[r] & A_{\geq 0} \end{tikzcd}$$

where the indicated map γ is an inclusion of connected components. Here, the upper left map is a colimit of equivalences and therefore an equivalence. As filtered colimits commute with finite limits in $s\mathsf{CMon}$, the left square is a pullback. It suffices to show that the right square is a pullback and that the upper right map is an equivalence. Using the natural transformation $A_{\geq 0} \to A$ of remark 4.1.8.19, we deduce the existence of a commuting diagram

$$\begin{tikzcd}
\operatorname{colim}_i (A_i)^{\times}_{\geq 0} \arrow[r] \arrow[d, "\gamma"] & A^{\times}_{\geq 0} \arrow[d] \\
\operatorname{colim}_i (A_i)_{\geq 0} \arrow[r] \arrow[d, "\delta"] & A_{\geq 0} \arrow[d] \\
\operatorname{colim}_i A_i \arrow[r, equal] & A
\end{tikzcd}$$

As the composition $\delta \circ \gamma$ is also an inclusion of components, it is clear that the outer square is a pullback so that the upper horizontal map is an equivalence. We will be done once we show that $x \in \operatorname{colim}_i (A_i)_{\geq 0}$ lies in the image of γ if and only if x is invertible in A. Suppose x factors through some $(A_i)_{\geq 0}$. The 'only if' direction is obvious, and in the other direction we see that there must be some j such that $x^{-1} \in A_j$. Choose an upper bound k for $\{i, j\} \subset K$, then x is invertible in A_k and therefore also in $(A_k)_{\geq 0}$ since we have $(A_k)^{\times}_{\geq 0} \simeq (A_k)_{\geq 0} \times_{A_k} A_k^{\times}$, so x lies in the image of γ as required. □

Proposition 4.1.8.32. *The functor Ξ of construction 4.1.8.15 has the following properties.*

(1) *Ξ is conservative.*

(2) *Ξ preserves limits and sifted colimits.*

(3) *Ξ is monadic.*

(4) *Let Υ be a left adjoint to Ξ, then for each $(A, M \to A_{\geq 0})$, the unit map $(A, M \to A_{\geq 0}) \to \Xi\Upsilon(A, M \to A_{\geq 0})$ maps to an equivalence under p.*

(5) *Ξ is a left Kan extension and a p-left Kan extension of its restriction to the image of $j : \mathsf{CartSp}_c^{op} \hookrightarrow s\mathsf{Cring}_c$.*

Proof. Consider the functor $\rho : sC^\infty\mathsf{PLog} \to \mathcal{S} \times \mathcal{S}$ obtained by taking the product of the functors

$$sC^\infty\mathsf{PLog} \xrightarrow{p} sC^\infty\mathsf{ring} \xrightarrow{\mathrm{ev}_{\{\mathbb{R}\}}} \mathcal{S}$$

and

$$sC^\infty\mathsf{PLog} \xrightarrow{\mathrm{ev}_{\mathsf{PLog}}} s\mathsf{CMon} \xrightarrow{\mathrm{ev}_{\mathbb{Z}_{\geq 0}}} \mathcal{S}.$$

Then ρ is conservative and preserves limits and sifted colimits by lemma 4.1.8.29. We have a commuting diagram

$$\begin{tikzcd}
sC^\infty\mathsf{ring}_{pc} \arrow[rr, "\Xi"] \arrow[dr, "\mathrm{ev}_{\mathbb{R}} \times \mathrm{ev}_{\mathbb{R}_{\geq 0}}"'] & & sC^\infty\mathsf{PLog} \arrow[dl, "\rho"] \\
& \mathcal{S} \times \mathcal{S} &
\end{tikzcd}$$

of ∞-categories. Since the left diagonal map is conservative and preserves limits and sifted colimits, we deduce (1) and (2). Note that (3) is an immediate consequence (1) and (2), the presentability of both $sC^\infty\mathsf{ring}_{pc}$ and $sC^\infty\mathsf{PLog}$ and Lurie's Barr-Beck theorem. To prove (4), [Lur17a], prop. 7.3.2.6 guarantees that it suffices to show that for each $A \in sC^\infty\mathsf{ring}$, the functor Ξ_A between the fibres at A admits a left adjoint since Ξ preserves Cartesian edges. Suppose $q : \mathcal{C} \to \mathcal{D}$ is a presentable fibration and $D \in \mathcal{D}$ an object, then a diagram $K^{\triangleright} \to \mathcal{C}_D$ where K is weakly contractible is a colimit diagram if and only if it is a q-colimit diagram if and only if it is a colimit diagram in $\mathcal{C}$. As Ξ preserves sifted colimits, it follows that Ξ_A also preserves sifted colimits for each $A \in sC^\infty\mathsf{ring}$. To conclude that Ξ_A admits a left adjoint, it suffices to prove that Ξ_A preserves limits, by the adjoint functor theorem and the presentability of the fibres. This follows from the following relative version of assertion $(**)$ of proposition 4.1.6.7, the proof of which uses the same techniques and is left to the reader.

- $(*)$ Let $p : \mathcal{C} \to \mathcal{D}$ and $q : \mathcal{C}' \to \mathcal{D}$ be coCartesian fibrations among ∞-categories and let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism in $\mathsf{coCart}_{\mathcal{D}}$. Let K be a simplicial set and let $g : K \to \mathcal{C}_D$ be a diagram in the fibre over some object $D \in \mathcal{D}$. Let $i_D : \mathcal{C}_D \subset \mathcal{C}$ denote the inclusion, and suppose that the induced diagram $i_D g : K \to \mathcal{C}$ admits a colimit and that p and f preserve the colimit of $i_D g$. Then the diagram g admits a colimit and the functor $f_D : \mathcal{C}_D \to \mathcal{C}'_D$ preserves this colimit.

Note that (5) follows immediately from [Lur17b], prop. 5.5.8.15. $\square$

Corollary 4.1.8.33. *The localization $sC^\infty\mathsf{ring}_c \subset sC^\infty\mathsf{ring}_{pc}$ is an ω-accessible localization. In particular, $sC^\infty\mathsf{ring}_c$ is compactly generated.*

Proof. It suffices to show that the inclusion $sC^\infty\mathsf{ring}_c \subset sC^\infty\mathsf{ring}_{pc}$ preserves filtered colimits. To see this, combine propositions 4.1.8.23 and 4.1.8.32 and corollary 4.1.8.31. $\square$

The functor $(\_)_{\geq 0} : sC^\infty\mathsf{ring} \to s\mathsf{CMon}$ does not preserve filtered colimits (it only preserves κ-filtered colimits for regular cardinals κ for which $C^\infty(\mathbb{R}_{\geq 0})$ is κ-compact in $sC^\infty\mathsf{ring}$; such a cardinal is necessarily uncountable by Tougeron's flat function lemma), so we cannot conclude that $sC^\infty\mathsf{PLog}$ is compactly generated solely from the knowledge that it arises as a pullback of compactly generated presentable ∞-categories. Nevertheless, we have the following result.

Proposition 4.1.8.34. *The ∞-category $sC^\infty\mathsf{PLog}$ is the ∞-category of algebras for a 2-sorted Lawvere theory (in particular, $sC^\infty\mathsf{PLog}$ is compactly generated). More precisely, consider the wide subcategory $\mathsf{CartSp}^{\triangleright}_c \subset \mathsf{CartSp}$ whose morphisms are interior b-maps $f : \mathbb{R}^n \times \mathbb{R}^k_{\geq 0} \to \mathbb{R}^m \times \mathbb{R}^j_{\geq 0}$ that satisfy the following condition.*

- $(*)$ *f pulls back every boundary defining function of $\mathbb{R}^m \times \mathbb{R}^j_{\geq 0}$ to a product of boundary defining functions on $\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$.*

We may repeat construction 4.1.8.15 for $\mathsf{CartSp}^{\triangleright}_c$, which results in a functor $\Xi^{\triangleright}$. Then the functor $\Xi^{\triangleright}$ induces an equivalence

$$sC^\infty\mathsf{ring}^{\triangleright}_{pc} \xrightarrow{\simeq} sC^\infty\mathsf{PLog}.$$

Proof. It follows from proposition 4.1.1.3 that the ∞-category $sC^\infty\mathsf{ring} \times s\mathsf{CMon}$ is the ∞-category of algebras for the 2-sorted Lawvere theory $\mathsf{CartSp} \times \mathsf{FCMon}^{op}$. It follows from corollary 4.1.8.30 and [Lur17a], prop. 7.1.4.12 that $sC^\infty\mathsf{PLog}$ is generated under sifted colimits by the essential image of the map

$$\mathbf{N}(\mathsf{CartSp})^{op} \times \mathbf{N}(\mathsf{FCMon}) \overset{j}{\hookrightarrow} sC^\infty\mathsf{ring} \times s\mathsf{CMon} \xrightarrow{F} sC^\infty\mathsf{PLog},$$

which consists of compact projective objects, where F is a left adjoint to $p \times \mathrm{ev}_{\mathsf{PLog}}$. Let $\mathrm{T}^{op} \subset sC^\infty\mathsf{PLog}$ denote this essential image which is equivalent to its full subcategory spanned by objects of the form $(C^\infty(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k), \mathbb{Z}^n_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k))$, then T is a 2-sorted Lawvere theory and the full subcategory inclusion $\mathrm{T}^{op} \subset sC^\infty\mathsf{PLog}$ induces an equivalence $s\mathrm{T}\mathsf{Alg} \simeq sC^\infty\mathsf{PLog}$. We are left to show that the functor $\Xi^{\triangleright}$ is an equivalence. Since $\Xi^{\triangleright}$ is a right adjoint that preserves sifted colimits, its left adjoint $V : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{ring}^{\triangleright}_{pc}$ carries T^{op} into the full subcategory $\mathcal{C}_0 \subset sC^\infty\mathsf{ring}^{\triangleright}_{pc}$ spanned by compact projective objects, which contains $\mathbf{N}(\mathsf{CartSp}^{\triangleright})$. It suffices to show that the resulting functor $\mathrm{T} \to \mathcal{C}_0^{op}$ factors through $\mathbf{N}(\mathsf{CartSp}^{\triangleright})$ as an equivalence. To see it is essentially surjective, note that the diagram

$$\begin{tikzcd}
sC^\infty\mathsf{ring}^{\triangleright}_{pc} \arrow[rr, "\Xi^{\triangleright}"] \arrow[dr, "\iota_c^* \times \mathrm{ev}_{\mathbb{R}_{\geq 0}}"'] & & sC^\infty\mathsf{PLog} \arrow[dl, "p \times \mathrm{ev}_{\mathsf{PLog}}"] \\
& sC^\infty\mathsf{ring} \times s\mathsf{CMon}. &
\end{tikzcd}$$

induces a diagram

$$\begin{tikzcd}
\mathcal{C}_0^{op} & & \mathrm{T} \arrow[ll] \\
& \mathbf{N}(\mathsf{CartSp}) \times \mathbf{N}(\mathsf{FCMon}^{op}). \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

so we conclude using that $\mathbf{N}(\mathsf{CartSp}) \times \mathbf{N}(\mathsf{FCMon}^{op}) \to \mathcal{C}_0^{op}$ factors through $\mathbf{N}(\mathsf{CartSp}^{\triangleright}_c)$ as an essentially surjective functor. For fully faithfulness, we note that proposition 4.1.8.32 establishes that $\Xi^{\triangleright}$ is a right adjoint relative to $sC^\infty\mathsf{ring}$, so we have a natural equivalence $\iota_c^* \circ V \simeq p$ which yields for each pair of objects $A := (C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), \mathbb{Z}^k_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}))$ and $B := (C^\infty(\mathbb{R}^m \times \mathbb{R}^j_{\geq 0}), \mathbb{Z}^j_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^m \times \mathbb{R}^j_{\geq 0}))$ a commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}_{sC^\infty\mathsf{ring}^{\triangleright}_c}(j(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), j(\mathbb{R}^m \times \mathbb{R}^j_{\geq 0})) \arrow[dr] & & \mathrm{Hom}_{sC^\infty\mathsf{PLog}}(A, B) \arrow[ll] \arrow[dl] \\
& \mathrm{Hom}_{sC^\infty\mathsf{ring}}(C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), C^\infty(\mathbb{R}^m \times \mathbb{R}^j_{\geq 0})) &
\end{tikzcd}$$

of 0-truncated spaces (the upper right space is 0-truncated by remark 4.1.8.20). The result will thus be established if we can argue that on connected components, both diagonal maps are injective and have the same image, which is a direct inspection. $\square$

Remark 4.1.8.35. In view of proposition 4.1.8.34, the functor $\Xi : sC^\infty\mathsf{ring}_{pc} \to sC^\infty\mathsf{PLog}$ can be identified with the functor $sC^\infty\mathsf{ring}_{pc} \to sC^\infty\mathsf{ring}^{\triangleright}_{pc}$ given by the subcategory inclusion $\mathsf{CartSp}^{\triangleright}_c \to \mathsf{CartSp}_c$.

Remark 4.1.8.36. The reasoning applied in this subsection is valid in algebraic (derived) logarithmic geometry as well, showing that the ∞-category of simplicial prelog rings over some commutative ring k is also projectively generated.

The main observation not of formal nature underlying theorem 4.1.8.24 is contained in the following lemma.

Lemma 4.1.8.37. *Let M be a manifold with faces and let $H_1(M)$ be the set of connected boundary components. Consider the map $e_M : \mathbb{Z}_{\geq 0}^{H_1(M)} \to C^\infty_{\geq 0}(M)$ of commutative monoids induced by the map of sets $H^1(M) \to C^\infty_{\geq 0}(M)$ carrying the boundary component S to a function defining S. Then e_M takes values in the submonoid of interior b-maps and the commuting triangle*

$$\begin{tikzcd}
\mathbb{Z}_{\geq 0}^{H_1(M)} \arrow[rr] \arrow[dr, "e_M"'] & & C^\infty_b(M) \arrow[dl] \\
& C^\infty_{\geq 0}(M) &
\end{tikzcd}$$

exhibits $C^\infty_b(M) \subset C^\infty_{\geq 0}(M)$ as the logification of $e_M : \mathbb{Z}_{\geq 0}^{H_1(M)} \to C^\infty_{\geq 0}(M)$.

Proof. Clearly, functions defining boundary components on M are interior b-maps. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C^\infty_{>0}(M) \arrow[r, equal] \arrow[d] & C^\infty_{>0}(M) \arrow[d] \\
\mathbb{Z}_{\geq 0}^{H_1(M)} \arrow[r] & C^\infty_b(M) \arrow[r] & C^\infty_{\geq 0}(M)
\end{tikzcd}$$

of commutative monoids. It is easy to see that both squares are pullbacks, so it suffices to show that the left square is also a pushout of simplicial commutative monoids; that is, the maps $Z_{\geq 0}^{H_1(M)} \to C^\infty_b(M)$ and $C^\infty_{>0}(M) \to C^\infty_b(M)$ exhibit $C^\infty_b(M)$ as a coproduct of $\mathbb{Z}_{\geq 0}^{H_1(M)}$ and $C^\infty_{>0}(M)$. The symmetric monoidal structure on $s\mathsf{CMon}$ is coCartesian and the symmetric monoidal structure on $\mathcal{S}$ is Cartesian, so after unwinding definitions, we are reduced to producing an equivalence of spaces $C^\infty_b(M) \simeq \mathbb{Z}_{\geq 0}^{H_1(M)} \times C^\infty_{>0}(M)$ (which is just a bijection of sets in this case) such that the induced maps $C^\infty_{>0}(M) \to C^\infty_{>0}(M)$ and $\mathbb{Z}_{\geq 0}^{H_1(M)} \to \mathbb{Z}_{\geq 0}^{H_1(M)}$ are equivalent to the identity, and the maps $C^\infty_{>0}(M) \to \mathbb{Z}_{\geq 0}^{H_1(M)}$ and $\mathbb{Z}_{\geq 0}^{H_1(M)} \to C^\infty_{>0}(M)$ are equivalent to the zero morphism. We get the desired bijection of sets $C^\infty_b(M) \cong \mathbb{Z}_{\geq 0}^{H_1(M)} \times C^\infty_{>0}(M)$ from the observation that every interior b-map $f : M \to \mathbb{R}_{\geq 0}$ can be written as $h_{S_1}^{m_1} \dots h_{S_n}^{m_n} g$ with a unique $g \in C^\infty_{>0}(M)$ and a unique tuple $(h_S)_{H_1(M)} \in \mathbb{Z}_{\geq 0}^{H_1(M)}$, the indicated coefficients associated to the $\{S_j\}$ being the only ones that are nonzero. $\square$

Corollary 4.1.8.38. *The composition $sC^\infty\mathsf{ring}_{pc} \xrightarrow{\Xi} sC^\infty\mathsf{PLog} \xrightarrow{L_{\mathsf{Log}}} sC^\infty\mathsf{Log}$ preserves all colimits.*

Proof. As L_{Log} preserves colimits and Ξ preserves sifted colimits, the composition $L_{\mathsf{Log}}\Xi$ is a left Kan extension of its restriction to the essential image of the Yoneda embedding $j : \mathbf{N}(\mathsf{CartSp}_c)^{op} \hookrightarrow sC^\infty\mathsf{ring}_{pc}$, so it suffices to show that the composition $L_{\mathsf{Log}}\Xi j$ preserves coproducts. Contemplate the commuting diagrams

$$\begin{tikzcd}
(C^\infty(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k), \mathbb{Z}^n_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k)) \arrow[r, "\alpha"] \arrow[d] & (C^\infty(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}), \mathbb{Z}^{n+m}_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l})) \arrow[d, "\gamma"] \\
(C^\infty(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k), C^\infty_b(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k) \to C^\infty_{\geq 0}(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k)) \arrow[r, "\alpha'"] & (C^\infty(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}), C^\infty_b(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}) \to C^\infty_{\geq 0}(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}))
\end{tikzcd}$$

and

$$\begin{tikzcd}
(C^\infty(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}), \mathbb{Z}^{n+m}_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l})) \arrow[d, "\gamma"] & (C^\infty(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^l), \mathbb{Z}^m_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^l)) \arrow[l, "\beta"'] \arrow[d] \\
(C^\infty(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}), C^\infty_b(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l}) \to C^\infty_{\geq 0}(\mathbb{R}^{n+m}_{\geq 0} \times \mathbb{R}^{k+l})) & (C^\infty(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^l), C^\infty_b(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^l) \to C^\infty_{\geq 0}(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^l)). \arrow[l, "\beta'"]
\end{tikzcd}$$

We wish to show that the maps α' and β' exhibit a coproduct in $sC^\infty\mathsf{Log}$. Using corollary 4.1.8.30, we deduce that the maps α and β exhibit a coproduct in $sC^\infty\mathsf{PLog}$ as the underlying diagram of C^∞-rings and the underlying diagram of (finitely generated free) simplicial commutative monoids exhibit a coproduct. In virtue of lemma 4.1.8.37, the vertical maps in the diagrams above exhibit logifications, so we conclude by observing that logification, as a left adjoint, preserves coproducts. $\square$

Lemma 4.1.8.39. *The composition*

$$sC^\infty\mathsf{ring} \xrightarrow{\iota_{c!}} sC^\infty\mathsf{ring}_{pc} \xrightarrow{\Xi} sC^\infty\mathsf{PLog} \xrightarrow{L_{\mathsf{Log}}} sC^\infty\mathsf{Log}$$

is equivalent to the composition

$$sC^\infty\mathsf{ring} \xrightarrow{s} sC^\infty\mathsf{PLog} \xrightarrow{L_{\mathsf{Log}}} sC^\infty\mathsf{Log},$$

where s is a left adjoint to $p : sC^\infty\mathsf{PLog} \to sC^\infty\mathsf{ring}$.

Proof. Consider the full subcategory $\mathcal{C} \subset \mathrm{Fun}_{sC^\infty\mathsf{ring}}(sC^\infty\mathsf{ring}, sC^\infty\mathsf{Log})$ spanned by sections F satisfying the following conditions.

(1) F preserves sifted colimits.

(2) For each $n \geq 0$, F carries the object $C^\infty(\mathbb{R}^n)$ to an initial object in the fibre over $C^\infty(\mathbb{R}^n)$.

Sections satisfying (1) are precisely left Kan extensions of their restriction along the full subcategory inclusion $\mathbf{N}(\mathsf{CartSp})^{op} \subset sC^\infty\mathsf{ring}$ so this restriction induces an equivalence between $\mathcal{C}$ and the full subcategory of

$$\mathrm{Fun}_{\mathbf{N}(\mathsf{CartSp})^{op}}(\mathbf{N}(\mathsf{CartSp})^{op}, \mathbf{N}(\mathsf{CartSp})^{op} \times_{sC^\infty\mathsf{ring}} sC^\infty\mathsf{Log})$$

spanned by sections f that carry each object of $\mathbf{N}(\mathsf{CartSp})^{op}$ to an initial object in the fibre. The projection $q : \mathbf{N}(\mathsf{CartSp})^{op} \times_{sC^\infty\mathsf{ring}} sC^\infty\mathsf{Log} \to \mathbf{N}(\mathsf{CartSp})^{op}$ is a Cartesian fibration, so each such functor is a left adjoint to q. It follows that the set of equivalence classes of objects of $\mathcal{C}$ consists of a single element. We conclude by observing that both functors in the statement of the lemma satisfy (1) and (2). □

Lemma 4.1.8.40. *The functor $\Xi_{\mathsf{Log}} : sC^\infty\mathsf{ring}_c \to sC^\infty\mathsf{Log}$ preserves all colimits.*

Proof. It suffices to argue that $L_{\mathsf{Log}}\Xi$ carries the set $S = \{\phi\}$ of definition 4.1.8.3 into the set of equivalences of $sC^\infty\mathsf{Log}$, as it then follows from the universal property of cocontinuous localizations that the functor $L_{\mathsf{Log}}\Xi$ factors through $sC^\infty\mathsf{ring}_c$ as a colimit preserving functor. Since Ξ restricted to $sC^\infty\mathsf{ring}_c$ takes values in $sC^\infty\mathsf{Log}$, we consequently deduce that Ξ_{Log} is equivalent to $L_{\mathsf{Log}}\Xi$ and therefore preserves colimits.
The functor $L_{\mathsf{Log}}\Xi : s\mathsf{Cring}_{pc} \to sC^\infty\mathsf{Log}$ preserves colimits by corollary 4.1.8.38 so it carries the pushout diagram

$$\begin{tikzcd}
\iota_{c!}\iota_c^*(C^\infty(\mathbb{R}_{\geq 0}), C_b^\infty(\mathbb{R}_{\geq 0})) \arrow[r, "\epsilon"] \arrow[d] & (C^\infty(\mathbb{R}_{\geq 0}), C_b^\infty(\mathbb{R}_{\geq 0})) \arrow[d] \\
\iota_{c!}\iota_c^*(C^\infty(\mathbb{R}_{>0}), C_b^\infty(\mathbb{R}_{>0})) \arrow[r] & \mathcal{A}
\end{tikzcd}$$

of definition 4.1.8.3 to a pushout diagram in $sC^\infty\mathsf{Log}$. It follows from lemma 4.1.8.39 that the pushout diagram above is carried to a pushout diagram

$$\begin{tikzcd}
(C^\infty(\mathbb{R}_{\geq 0}), C_{\geq 0}^\infty(\mathbb{R}_{\geq 0})^\times \to C_{\geq 0}^\infty(\mathbb{R}_{\geq 0})) \arrow[r] \arrow[d] & (C^\infty(\mathbb{R}_{\geq 0}), C_b^\infty(\mathbb{R}_{\geq 0}) \to C_{\geq 0}^\infty(\mathbb{R}_{\geq 0})) \arrow[d] \\
(C^\infty(\mathbb{R}_{>0}), C_{\geq 0}^\infty(\mathbb{R}_{>0})^\times \to C_{\geq 0}^\infty(\mathbb{R}_{>0})) \arrow[r] & L_{\mathsf{Log}}\Xi(\mathcal{A}),
\end{tikzcd}$$

where the left vertical map is a coCartesian morphism between initial log structures. Since the functor $sC^\infty\mathsf{Log} \to sC^\infty\mathsf{ring}$ preserves colimits, the map on underlying simplicial C^∞-ring of the lower horizontal map in the diagram above is an equivalence. Since the left vertical map is a p_{Log}-coCartesian edge and the diagram is a p_{Log}-pushout, the right vertical map is also p_{Log}-coCartesian. Therefore, we are reduced to verifying that the logification of $(C^\infty(\mathbb{R}_{>0}), C_b^\infty(\mathbb{R}_{\geq 0}) \to C_{\geq 0}^\infty(\mathbb{R}_{>0}))$ is the initial log structure. Consider the pullback diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & C_{>0}^\infty(\mathbb{R}_{>0}) \arrow[d] \\
C_b^\infty(\mathbb{R}_{\geq 0}) \arrow[r] & C_{\geq 0}^\infty(\mathbb{R}_{>0}).
\end{tikzcd}$$

Recalling the description of the logification functor, we wish to show that the map $C_{>0}^\infty(\mathbb{R}_{>0}) \to C_{>0}^\infty(\mathbb{R}_{>0}) \coprod_M C_b^\infty(\mathbb{R}_{\geq 0})$ is an equivalence. It is sufficient to argue that the left vertical map in the diagram above is an equivalence, which is equivalent to the assertion that if $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is an interior b-map, then the restriction $f|_{\mathbb{R}_{>0}}$ factors through $\mathbb{R}_{>0} \hookrightarrow \mathbb{R}_{\geq 0}$, but this holds by definition of interior b-maps. □

Proof of theorem 4.1.8.24. Let F denote a left adjoint to Ξ_{Log}. Since Ξ_{Log} is conservative, it suffices to argue that the unit transformation $\mathrm{id} \to \Xi_{\mathsf{Log}} F$ is an equivalence. Since both Ξ_{Log} and F preserve colimits and the objects $L_{\mathsf{Log}}(C^\infty(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k), \mathbb{Z}^n_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k))$ generate $sC^\infty\mathsf{Log}$ under sifted colimits, we need only check that the unit is an equivalence on this collection of objects. It follows from (the proof of) lemma 4.1.8.40 that Ξ carries the strong saturation $\overline{S}$ of the set $S = \{\phi\}$ to the set of maps in $sC^\infty\mathsf{PLog}$ that become an equivalence after applying L_{Log}. In particular, for any localization $X \to L(X)$ in $sC^\infty\mathsf{ring}_{pc}$, the map $\Xi(X) \to \Xi L(X) \simeq \Xi_{\mathsf{Log}} L(X)$ in $sC^\infty\mathsf{PLog}$, whose codomain lies in $sC^\infty\mathsf{Log}$, becomes an equivalence upon logifying and is therefore also a localization, that is, the diagram

$$\begin{tikzcd} sC^\infty\mathsf{ring}_c \arrow[r, "\Xi_{\mathsf{Log}}"] \arrow[d, hook] & sC^\infty\mathsf{Log} \arrow[d, hook] \\ sC^\infty\mathsf{ring}_{pc} \arrow[r, "\Xi"] & sC^\infty\mathsf{PLog} \end{tikzcd}$$

is vertically left adjointable. Then the resulting commuting diagram

$$\begin{tikzcd} sC^\infty\mathsf{ring}_c \arrow[r, "\Xi_{\mathsf{Log}}"] & sC^\infty\mathsf{Log} \\ sC^\infty\mathsf{ring}_{pc} \arrow[r, "\Xi"] \arrow[u, "L"] & sC^\infty\mathsf{PLog} \arrow[u, "L_{\mathsf{Log}}"'] \end{tikzcd}$$

is tautologically vertically right adjointable, and therefore also horizontally left adjointable, that is, the logification functor carries unit transformations of the lower adjunction to unit transformations of the upper one. It follows from proposition 4.1.8.34 that the object $(C^\infty(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k), C^\infty_b(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k))$ together with the triangle

$$\begin{tikzcd} \mathbb{Z}^n_{\geq 0} \arrow[rr] \arrow[dr, "e_n"'] & & C^\infty_b(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k) \arrow[dl] \\ & C^\infty_{\geq 0}(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k) & \end{tikzcd}$$

is a unit transformation at $(C^\infty(\mathbb{R}^n_{\geq 0} \times \mathbb{R}^k), \mathbb{Z}^n_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k))$. This map exhibits a logification by lemma 4.1.8.37 and is therefore carried to an equivalence by L_{Log}. □

We now turn to the proof of theorem 4.1.8.11.

Lemma 4.1.8.41. *Let* T *be a Lawvere theory and let* $s\mathrm{T}\mathsf{Alg}$ *be the associated* ∞*-category of algebras. Let* S *be small set of morphisms in* $s\mathrm{T}\mathsf{Alg}$ *and denote by* $s\mathrm{T}\mathsf{Alg}[S^{-1}] \subset s\mathrm{T}\mathsf{Alg}$ *the strongly reflective full subcategory spanned by* S*-local objects. Let* $\mathcal{C}$ *be an idempotent complete* ∞*-category that admits finite limits and denote by* $\mathrm{Fun}^\pi(\mathrm{T}, \mathcal{C})[S^{-1}] \subset \mathrm{Fun}^\pi(\mathrm{T}, \mathcal{C})$ *the full subcategory spanned by functors* $F : \mathrm{T} \to \mathcal{C}$ *for which the following condition is satisfied.*

(∗) *For each object* $C \in \mathcal{C}$*, the composition*

$$\mathrm{T} \xrightarrow{F} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(C,-)} \mathcal{S}$$

is S*-local in* $s\mathrm{T}\mathsf{Alg}$.

Suppose that the inclusion $s\mathrm{T}\mathsf{Alg}[S^{-1}] \subset s\mathrm{T}\mathsf{Alg}$ *preserves filtered colimits, then restriction along the functor* $\mathrm{T}^{op} \overset{j}{\hookrightarrow} s\mathrm{T}\mathsf{Alg} \overset{L}{\to} s\mathrm{T}\mathsf{Alg}[S^{-1}]$ *induces an equivalence*

$$\mathrm{Fun}^{\mathrm{lex}}(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^\pi(\mathrm{T}, \mathcal{C})[S^{-1}].$$

Proof. The Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathsf{PShv}(\mathcal{C})$ induces a commuting diagram

$$\begin{tikzcd} \mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}, \mathcal{C}) \arrow[r] \arrow[d, hook] & \mathrm{Fun}^{\mathrm{lex}}(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathcal{C}) \arrow[r] \arrow[d, hook] & \mathrm{Fun}^\pi(\mathrm{T}, \mathcal{C}) \arrow[d, hook] \\ \mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}, \mathsf{PShv}(\mathcal{C})) \arrow[r] & \mathrm{Fun}^{\mathrm{lex}}(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C})) \arrow[r, "(Lj)^*"] & \mathrm{Fun}^\pi(\mathrm{T}, \mathsf{PShv}(\mathcal{C})) \end{tikzcd}$$

where $\mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}, \mathcal{C})$ and $\mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}, \mathsf{PShv}(\mathcal{C}))$ denote full subcategories of functors preserving small limits. As $\mathsf{PShv}(\mathcal{C})$ admits small limits and the ∞-category $s\mathrm{T}\mathsf{Alg}[S^{-1}]$ is compactly generated in virtue of the assumption that the inclusion $s\mathrm{T}\mathsf{Alg}[S^{-1}] \subset s\mathrm{T}\mathsf{Alg}$ preserves filtered colimits, the lower left horizontal restriction

$$\mathrm{Fun}'(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}, \mathsf{PShv}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(s\mathrm{T}\mathsf{Alg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C}))$$

is an equivalence after remark 4.1.1.23. The composition $\mathrm{Fun}'(s\mathsf{TAlg}[S^{-1}]^{op}, \mathsf{PShv}(\mathcal{C})) \to \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))$ factors via the restriction

$$r : \mathrm{Fun}'_S(s\mathsf{TAlg}, \mathsf{PShv}(\mathcal{C})) \longrightarrow \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C})),$$

where $\mathrm{Fun}'_S(s\mathsf{TAlg}, \mathsf{PShv}(\mathcal{C}))$ is the full subcategory spanned by limit preserving functors $F : s\mathsf{TAlg}^{op} \to \mathsf{PShv}(\mathcal{C})$ carrying the set S to into the set of equivalences of $\mathsf{PShv}(\mathcal{C})$. This is the case for such a functor F if and only if for each $C \in \mathcal{C}$, the functor $\mathrm{ev}_C \circ F : s\mathsf{TAlg}^{op} \to \mathcal{S}$ carries the set S into the set of equivalences in $\mathcal{S}$, but since $\mathrm{ev}_C \circ F$ preserves limits and is therefore representable, this corresponds to the associated representing object $A \in s\mathsf{TAlg}$ being S-local. Let $\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}]$ be the full subcategory spanned by limit preserving functors F such that $\mathrm{ev}_C \circ F : s\mathsf{TAlg}^{op} \to \mathcal{S}$ carries the set S into the set of equivalences in $\mathcal{S}$. Since the representing object A of $\mathrm{ev}_C \circ F$ may be identified with the functor $\mathrm{ev}_C \circ F \circ j$, we conclude that the restriction r takes values in $\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}]$ and determines an equivalence $\mathrm{Fun}'_S(s\mathsf{TAlg}, \mathsf{PShv}(\mathcal{C})) \simeq \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}]$. It follows that the restriction $(Lj)^*$ factors up to homotopy through $\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}]$ and is an equivalence onto its essential image, but as $\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}] \subset \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))$ is a replete full subcategory, $(Lj)^*$ itself factors through $\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}]$ and determines an equivalence $\mathrm{Fun}^{\mathrm{lex}}(s\mathsf{TAlg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C})) \simeq \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}]$. Since we have an isomorphisms of simplicial sets $\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}] \times_{\mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))} \mathrm{Fun}^{\pi}(\mathrm{T}, \mathcal{C}) \cong \mathrm{Fun}^{\pi}(\mathrm{T}, \mathcal{C})[S^{-1}]$, we deduce that restriction along jL induces the top horizontal map in the commuting diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{lex}}(s\mathsf{TAlg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathcal{C}) \arrow[r] \arrow[d, hook] & \mathrm{Fun}^{\pi}(\mathrm{T}, \mathcal{C})[S^{-1}] \arrow[d, hook] \\
\mathrm{Fun}^{\mathrm{lex}}(s\mathsf{TAlg}[S^{-1}]^{op}_{\mathrm{fp}}, \mathsf{PShv}(\mathcal{C})) \arrow[r, "\simeq"] & \mathrm{Fun}^{\pi}(\mathrm{T}, \mathsf{PShv}(\mathcal{C}))[S^{-1}].
\end{tikzcd}$$

By assumption on $\mathcal{C}$, the essential image of the Yoneda embedding is stable under finite limits and retracts in $\mathsf{PShv}(\mathcal{C})$, so using that every object of $s\mathsf{TAlg}[S^{-1}]^{op}_{\mathrm{fp}}$ is a retract of a finite limit of objects in the essential image of $\mathrm{T} \to s\mathsf{TAlg} \xrightarrow{L} s\mathsf{TAlg}[S^{-1}]$ we conclude that the top horizontal map is an equivalence. $\square$

Proof of theorem 4.1.8.11 (i), (ii), (iii). We verify the claims made in the statement of the theorem.

(i) The ∞-category $sC^{\infty}\mathsf{ring}_c$ is compactly generated. This was checked in corollary 4.1.8.33.

(ii) Definition 4.1.8.9 determines the structure of a geometry on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$. We need to check that admissible morphisms are stable under pullbacks, retracts and that, if g is admissible and h another map with codomain the domain of g, then h is admissible if and only if $g \circ h$ is admissible. Since localizations are stable under pushouts of simplicial C^{∞}-rings and the functor ι_c^* preserves colimits, it suffices to show that a pushout in $sC^{\infty}\mathsf{ring}_c$ along a ι_c^*-coCartesian morphism is again ι_c^*-coCartesian. This is the case since all colimits in $sC^{\infty}\mathsf{ring}_c$ are ι_c^*-colimits. Similarly, we know that localizations of morphisms of simplicial C^{∞}-rings are stable under retracts, so we conclude that admissible morphisms in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ are stable under retracts from the observation that coCartesian morphisms are (which in turn follows from the fact that pullback squares are stable under retracts). Repeating this line of argument once more, we obtain the last verification from the corresponding verification for localizations, together with [Lur17b], prop. 2.4.1.7.

(iii) The inclusion $\mathcal{T}'_{\mathrm{Diffc}} \to \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ exhibits a geometric envelope. Choose an idempotent complete ∞-category admitting finite limits, then we have a commuting diagram

$$\begin{tikzcd}
 & \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{C}) \arrow[dr, "\theta''"] & \\
\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}, \mathcal{C}) \arrow[ur, "\theta"] \arrow[rr, "\theta'"] & & \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}_c), \mathcal{C})[S^{-1}].
\end{tikzcd}$$

Note that the restriction functor θ'' indeed takes values in $\mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}_c), \mathcal{C})[S^{-1}]$: composing with the functor $\mathrm{Hom}_{\mathcal{C}}(C, \_) : \mathcal{C} \to \mathcal{S}$, we may replace $\mathcal{C}$ by $\mathcal{S}$ and $\mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}_c), \mathcal{C})[S^{-1}]$ by $sC^{\infty}\mathsf{ring}_c$. We note that $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{S})$ is an ω-accessible localization of $\mathsf{PShv}(\mathcal{T}'^{op}_{\mathrm{Diffc}})$ and that restriction along $\mathbf{N}(\mathsf{CartSp}_c) \to \mathcal{T}'_{\mathrm{Diffc}}$ induces the functor $\theta'' : \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{S}) \to sC^{\infty}\mathsf{ring}_{pc}$ which preserves limits and filtered colimits. To conclude that θ'' factors through $sC^{\infty}\mathsf{ring}_c$, we need to show that its left adjoint F carries S' of remark 4.1.8.5 into the set of equivalences of $\mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{S})$. We have a commuting diagram

$$\begin{tikzcd}
\mathbf{N}(\mathsf{CartSp})^{op} \arrow[r] \arrow[d, "j"] & \mathcal{T}'^{op}_{\mathrm{Diffc}} \arrow[d, "j"] \\
\mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op}) \arrow[r] \arrow[d] & \mathsf{PShv}(\mathcal{T}'^{op}_{\mathrm{Diffc}}) \arrow[d] \\
sC^\infty\mathsf{ring}_{pc} \arrow[r, "F"] & \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{S})
\end{tikzcd}$$

where the lower square is obtained by passing to left adjoints in the square

$$\begin{tikzcd}
\mathsf{PShv}(\mathbf{N}(\mathsf{CartSp})^{op}) & \mathsf{PShv}(\mathcal{T}'^{op}_{\mathrm{Diffc}}) \arrow[l] \\
sC^\infty\mathsf{ring}_{pc} \arrow[u, hook] & \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{S}) \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

where the horizontal functors are induced by pulling back along $\mathbf{N}(\mathsf{CartSp}) \to \mathcal{T}'_{\mathrm{Diffc}}$. It follows that F carries the map in S' to the lower horizontal one in the pushout diagram

$$\begin{tikzcd}
j(\mathbb{R}) \arrow[r] \arrow[d, "j(\exp)"'] & j(\mathbb{R}_{\geq 0}) \arrow[d] \\
j(\mathbb{R}) \arrow[r, "\phi"] & \mathcal{A},
\end{tikzcd}$$

but the Yoneda embedding $j : \mathcal{T}'^{op}_{\mathrm{Diffc}} \to \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{S})$ preserves pushouts along admissible maps, so ϕ is indeed an equivalence. It follows from lemma 4.1.8.41 that the functor θ' is an equivalence, so it suffices to show that the functor θ'' is an equivalence. By replacing $\mathcal{C}$ with the ∞-category of presheaves on $\mathcal{C}$, we may suppose that $\mathcal{C}$ is an ∞-topos. Invoking [Lur17b], prop. 4.3.2.15, it suffices to show the following.

(1) The right Kan extension of each functor $F \in \mathrm{Fun}^{\pi}(\mathbf{N}(\mathsf{CartSp}_c), \mathcal{X})[S^{-1}]$ along $\mathbf{N}(\mathsf{CartSp}_c) \hookrightarrow \mathcal{T}'_{\mathrm{Diffc}}$ is a $\mathcal{T}'_{\mathrm{Diffc}}$-structure.

(2) Every $\mathcal{T}'_{\mathrm{Diffc}}$-structure is a right Kan extension of its restriction to $\mathbf{N}(\mathsf{CartSp}_c)$.

In fact, (2) is obvious from the definition of $\mathcal{T}'_{\mathrm{Diffc}}$. To establish (1), consider the diagram

$$\begin{tikzcd}
\mathbf{N}(\mathsf{CartSp}_c) \arrow[r, "F"] \arrow[d, hook] & \mathcal{X} \\
\mathcal{T}'_{\mathrm{Diffc}} \arrow[d, hook] \arrow[ur, dotted] & \\
sC^\infty\mathsf{ring}^{op}_c \arrow[d, hook] \arrow[uur, dotted] & \\
sC^\infty\mathsf{ring}^{op}_{pc} \arrow[uuur, "\widehat{F}"'] &
\end{tikzcd}$$

where $\widehat{F}$ is a right Kan extension of F along the vertical inclusion. It follows from lemma 4.1.8.41 that $\widehat{F}$ preserves limits and carries S into the set of equivalences of $\mathcal{X}$, so the composition $\mathcal{T}'_{\mathrm{Diffc}} \hookrightarrow sC^\infty\mathsf{ring}_{pc} \xrightarrow{\widehat{F}} \mathcal{X}$ is a $\mathcal{T}'_{\mathrm{Diffc}}$-structure. Since the vertical maps are fully faithful, this functor is also a right Kan extension of F.

$\square$

Before we complete the proof of theorem 4.1.8.11, we remark on the discrepancy between $\mathcal{T}_{\mathrm{Diffc}}$ and $\mathcal{T}'_{\mathrm{Diffc}}$. The pregeometry $\mathcal{T}'_{\mathrm{Diffc}}$ is *not* equivalent to $\mathcal{T}_{\mathrm{Diffc}}$, nor does the functor $(C^\infty(\_), C^\infty_b(\_)) : \mathcal{T}_{\mathrm{Diffc}} \to sC^\infty\mathsf{ring}_c$ take values in $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})^{op}$. Indeed, we have the following alternative.

Lemma 4.1.8.42. *Let M be a manifold with faces, then $(C^\infty(M), C^\infty_b(M))$ is a compact object in $sC^\infty\mathsf{ring}_c$ if and only if M has finitely many connected boundary components.*

Proof sketch. First, consider M a manifold with faces with infinitely many boundary components. Lemma 4.1.8.37 shows that there is an equivalence $C^\infty_b(M) \cong \mathbb{Z}^{M_1(M)} \coprod C^\infty_{\geq 0}(M)^\times$, where $M_1(M)$ is the set of connected boundary components of M. It follows that the sharpening of $C^\infty_b(M)$ is infinitely generated so, as the simplicial commutative monoid associated to any finitely generated simplicial C^∞-rings with corners has finitely generated sharpening, the object $(C^\infty(M), C^\infty_b(M))$ cannot be compact in $sC^\infty\mathsf{ring}_c$. The converse follows from the following assertions.

(∗) For every manifold with faces M, there exists an interior b-map $M \hookrightarrow \mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$ which is a a *p-embedding* of manifolds with corners (see [Mela]).

(∗∗) Let $S \subset M$ be a p-embedded submanifold, then S admits a tubular neighbourhood.

To prove (∗), we use the boundary flowout map $M \hookrightarrow M^\circ$ to embed M into its interior, and then apply the Whitney embedding theorem to embed M° into $\mathbb{R}^n$ for some $n >> 1$ resulting in a closed embedding $f : M \hookrightarrow \mathbb{R}^n$. Choose a finite complete set of boundary defining functions $\{\rho_H\}_{H\in M_1(M)}$, then the map $f \prod_{H\in M_1(M)} \rho_H : M \to \mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$ with $k = |M_1(M)|$ is a embedding. The fact that every p-embedded submanifold admits a tubular neighbourhood is proven verbatim as in the case without corners. □

Remark 4.1.8.43. A similar argument as the one presented in the previous lemma yields that every object $(C^\infty(U), M \to C^\infty_{\geq 0}(U))$ in $\mathcal{T}'_{\mathrm{Diffc}}$ must have finitely generated sharpening, but it is certainly possible that as an open subset $U \subset \mathbb{R}^n_{\geq 0} \times \mathbb{R}^k$ has infinitely many boundary components, so admissible morphisms in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ may not 'create' sufficiently many boundary defining functions. Both these issues disappear when we apply the spectrum functor $\mathbf{Spec}_c$, since every manifold with faces may always be covered by opens that admit an embedding $U \to \mathbb{R}^n_{\geq 0} \times \mathbb{R}^k$ onto a *connected* open subset.

To complete the proof of theorem 4.1.8.11, we investigate the spectrum functor associated to the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$. We note that the results in this subsection hold for any truncation of $sC^\infty\mathsf{ring}_c$; in particular, the category $C^\infty\mathsf{ring}_c$ is compactly generated and the functor $C^\infty\mathsf{ring}_c \to C^\infty\mathsf{ring}$ is a presentable fibration equivalent to $C^\infty\mathsf{Log} \to C^\infty\mathsf{ring}$. Let $\mathcal{G}_{\mathrm{Diffc}}$ be the opposite of the category $\mathbf{N}(\tau_{\leq 0} sC^\infty\mathsf{ring}_c)_{\mathrm{fp}} \simeq \mathbf{N}(C^\infty\mathsf{ring}_c)_{\mathrm{fp}}$, and endow $\mathcal{G}_{\mathrm{Diffc}}$ with the structure of a geometry according to definition 4.1.8.9 (this indeed defines a geometry by the proof of theorem 4.1.8.11 (ii)). The truncation functor

$$\tau_{\leq 0} : \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}} \longrightarrow \mathcal{G}_{\mathrm{Diffc}}$$

exhibits a 0-stub: this follows from lemma 4.1.8.41 and the following result.

Lemma 4.1.8.44. *Let (A, A_c) be a simplicial C^∞-ring with corners, then the functor*

$$(sC^\infty\mathsf{ring}^{op}_c)^{\mathrm{ad}}_{/(A,A_c)} \longrightarrow \mathbf{N}(C^\infty\mathsf{ring}^{op}_c)^{\mathrm{ad}}_{/\pi_0(A,A_c)}$$

is an equivalence.

Proof. By definition of admissible morphisms in $sC^\infty\mathsf{ring}_c$ and $C^\infty\mathsf{ring}_c$, we have a commuting diagram

$$\begin{tikzcd}
(sC^\infty\mathsf{ring}^{op}_c)^{\mathrm{ad}}_{/(A,A_c)} \arrow[r] \arrow[d] & \mathbf{N}(C^\infty\mathsf{ring}^{op}_c)^{\mathrm{ad}}_{/\pi_0(A,A_c)} \arrow[d] \\
(sC^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/A} \arrow[r] & \mathbf{N}(C^\infty\mathsf{ring}^{op})^{\mathrm{ad}}_{/\pi_0(A)}
\end{tikzcd}$$

among ∞-categories. The lower horizontal functor is an equivalence by corollary 4.1.3.15, and the vertical functors are Cartesian fibrations with contractible fibres. □

Corollary 4.1.8.45. (1) *The functor $\mathcal{T}'_{\mathrm{Diffc}} \hookrightarrow \mathcal{G}_{\mathrm{Diffc}}$ exhibits a 0-truncated geometric envelope.*

(2) *The functor ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diffc}}) \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})$ is fully faithful and its essential image consists of those $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which*

$$\mathcal{O}_{\mathcal{X}} : \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}} \longrightarrow \mathcal{X}$$

is a local $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-structure taking 0-truncated values in $\mathcal{X}$.

(3) *The relative spectrum $\mathbf{Spec}^{\mathcal{G}_{\mathrm{Diffc}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}$ admits the following description: given a $\mathcal{T}'_{\mathrm{Diffc}}$-structure $\mathcal{O}_{\mathcal{X}}$, consider the $\mathcal{T}'_{\mathrm{Diffc}}$-structure $\tau^{\mathcal{X}}_{\leq 0}\mathcal{O}_{\mathcal{X}}$, then the map*

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow (\mathcal{X}, \tau^{\mathcal{X}}_{\leq 0}\mathcal{O}_{\mathcal{X}})$$

is a unit transformation.

Remark 4.1.8.46. For (A, A_c) a simplicial C^∞-ring with corners and $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ its spectrum, the object $(\mathcal{X}, \tau^{\mathcal{X}}_{\leq 0}\mathcal{O}_\mathcal{X})$ can be identified with the value of the spectrum functor constructed by Joyce and Francis-Staite in [JF19] on $(\pi_0(A), \pi_0(A_c))$.

Proof of theorem 4.1.8.11 (iv), (v). We can identify the full subcategory of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})$ spanned by objects $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that $\mathcal{X}$ is the category of sheaves on a topological space and $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is 0-truncated with the 1-category of topological spaces equipped with sheaves of local C^∞-rings with corners, and the 1-category $\mathcal{T}_{\mathrm{Diffc}}$ admits a fully faithful embedding into this 1-category via the assignment

$$M \longmapsto (\mathsf{Shv}(M), C^\infty_M, C^\infty_b(M)).$$

The nontrivial part of the proof consists in showing that the functor $\mathcal{T}_{\mathrm{Diffc}} \to {}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})$ preserves pullbacks along admissible maps. Since this is local question, it suffices to consider admissible maps of the form $U \subset \mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$, where U is a connected open subsets. Unwinding the definitions, it is enough to show that for such connected open subsets, the map

$$(C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), C^\infty_b(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0})) \longrightarrow (C^\infty(U), C^\infty_b(U))$$

is a coCartesian morphism in $sC^\infty\mathsf{ring}_c$. It follows from theorem 4.1.8.24 that it suffices to show that the morphism is a logification. This follows from the fact that the composite map

$$\mathbb{Z}^k_{\geq 0} \longrightarrow C^\infty_b(\mathbb{R}^k) \longrightarrow C^\infty_b(U)$$

where the first functor exhibits a logification and specifies the boundary defining functions, factors via a projection $\mathbb{Z}^k_{\geq 0} \to \mathbb{Z}^S_{\geq 0}$, where $S \subset \{1, \dots, k\}$ is the subset determining the boundary defining functions on U (which indeed form a subset as U is connected).
The spectrum functor $\mathbf{Spec}_c$ restricted to $\mathcal{T}'_{\mathrm{Diffc}}$ factors through the full subcategory $\mathcal{T}_{\mathrm{Diffc}}$, since each object $\mathbf{Spec}(U)$ is a pullback of some $\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}$ along $\mathbb{R} \smallsetminus \{0\} \hookrightarrow \mathbb{R}$ and $\mathcal{T}_{\mathrm{Diffc}}$ is stable under pullbacks by admissible maps, as we have just verified. Thus, we have a functor

$$\mathbf{Spec}_c : \mathcal{T}'_{\mathrm{Diffc}} \longrightarrow \mathcal{T}_{\mathrm{Diffc}}$$

which is now clearly a transformation of pregeometries. Let $\mathcal{T}$ be the categorical mapping cylinder of the functor $\mathbf{Spec}_c$ defined as follows.

(1) An object of $\mathcal{T}$ is either an object of $\mathcal{T}'_{\mathrm{Diffc}}$ or an object of $\mathcal{T}_{\mathrm{Diffc}}$.

(2) Morphism sets are given by

$$\mathrm{Hom}_\mathcal{T}(M, N) = \begin{cases} \mathrm{Hom}_{\mathcal{T}'_{\mathrm{Diffc}}}(M, N) & M, N \in \mathcal{T}'_{\mathrm{Diffc}} \\ \mathrm{Hom}_{\mathcal{T}_{\mathrm{Diffc}}}(M, N) & M, N \in \mathcal{T}_{\mathrm{Diffc}} \\ \mathrm{Hom}_{\mathcal{T}_{\mathrm{Diffc}}}(\mathbf{Spec}_c(M), N) & M \in \mathcal{T}'_{\mathrm{Diffc}},\ N \in \mathcal{T}_{\mathrm{Diffc}} \\ \varnothing & M \in \mathcal{T}_{\mathrm{Diffc}},\ N \in \mathcal{T}'_{\mathrm{Diffc}}. \end{cases}$$

There are obvious full subcategory inclusions $i : \mathcal{T}'_{\mathrm{Diffc}} \subset \mathcal{T}$ and $j : \mathcal{T}_{\mathrm{Diffc}} \subset \mathcal{T}$. Note that the latter admits a retraction $r : \mathcal{T} \to \mathcal{T}_{\mathrm{Diffc}}$ such that $r \circ i = \mathbf{Spec}_c$ defined on objects by $r(M) = M$ if $M \in \mathcal{T}_{\mathrm{Diffc}}$ and $r(M) = \mathbf{Spec}_c(M)$ if $M \in \mathcal{T}'_{\mathrm{Diffc}}$. Let $\mathrm{Fun}'(\mathcal{T}, \mathcal{X}) \subset \mathrm{Fun}(\mathcal{T}, \mathcal{X})$ be the full subcategory spanned by functors $F : \mathcal{T} \to \mathcal{X}$ such that

(a) For each $M \in \mathcal{T}'_{\mathrm{Diffc}}$, the map $F(M) \to F(\mathbf{Spec}_c(M))$ is an equivalence in $\mathcal{X}$.

(b) The restriction $F|_{\mathcal{T}_{\mathrm{Diffc}}}$ is a local $\mathcal{T}_{\mathrm{Diffc}}$-structure on $\mathcal{X}$.

We note that (a) is equivalent to the assertion that F is a right Kan extension of $F|_{\mathcal{T}_{\mathrm{Diffc}}}$, so it follows from [Lur17b], prop. 4.3.2.15 that the restriction

$$\mathrm{Fun}(\mathcal{T}, \mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{T}_{\mathrm{Diffc}}, \mathcal{X})$$

induces a trivial Kan fibration $\mathrm{Fun}'(\mathcal{T}, \mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diffc}}}(\mathcal{X})$. Choose a section s of this trivial fibration, then the map $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X}) \to \mathrm{Fun}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{X})$ factors as

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X}) \xrightarrow{s} \mathrm{Fun}'(\mathcal{T}, \mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{X})$$

where the second functor is induced by restriction along i. Thus, our work will be done once we show that the restriction

$$\mathrm{Fun}(\mathcal{T}, \mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{T}'_{\mathrm{Diffc}}, \mathcal{X})$$

induces a trivial fibration $\mathrm{Fun}'(\mathcal{T}, \mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'_{\mathrm{Diffc}}}(\mathcal{X})$. In view of [Lur17b], prop. 4.3.2.15, it suffices to show that

(i) A functor $F : \mathcal{T} \to \mathcal{X}$ lies in $\mathrm{Fun}'(\mathcal{T}, \mathcal{X})$ if and only if its restriction $F|_{\mathcal{T}'_{\mathrm{Diffc}}}$ is a local $\mathcal{T}'_{\mathrm{Diffc}}$-structure and F is a left Kan extension of $F|_{\mathcal{T}'_{\mathrm{Diffc}}}$.

(ii) Every functor $F_0 \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'_{\mathrm{Diffc}}}(\mathcal{X})$ admits a left Kan extension along $i : \mathcal{T}'_{\mathrm{Diffc}} \hookrightarrow \mathcal{T}$.

We note that the (essential) smallness of $\mathcal{T}$ and the presentability of $\mathcal{X}$ guarantee that (ii) is satisfied, so we show (i). We first show the 'if' direction. Let $F : \mathcal{T} \to \mathcal{X}$ be a left Kan extension of its restriction to $\mathcal{T}'_{\mathrm{Diffc}}$, which we assume is a $\mathcal{T}'_{\mathrm{Diffc}}$-structure. Let $M \in \mathcal{T}'_{\mathrm{Diffc}}$, then we are required to show that the map $F(M) \to F(\mathbf{Spec}_c M)$ is an equivalence. Consider the full subcategory $\mathcal{C} \subset \mathcal{T}'_{\mathrm{Diffc}} \times_{\mathcal{T}'_{\mathrm{Diffc}}} \mathcal{T}_{/\mathbf{Spec}_c M}$ spanned by pairs $(N, \mathbf{Spec}_c N \to \mathbf{Spec}_c M)$ for which the counit map $\Gamma\mathbf{Spec}_c N \to N$ is an isomorphism. Since $F|_{\mathcal{T}'_{\mathrm{Diffc}}}$ is a local $\mathcal{T}'_{\mathrm{Diffc}}$-structure and there is a covering sieve , it follows from [Lur17b], prop. 4.3.2.7 that the object $F(\mathbf{Spec}_c M)$ is a colimit of $\mathcal{C} \to \mathcal{X}$. Since $F(M)$ is a colimit of $\mathcal{C}_{/M}$, it suffices to show that the projection $\mathcal{C}_{/M} \to \mathcal{C}$ is left cofinal, but this functor is an equivalence. We wish to show that $F|_{\mathcal{T}_{\mathrm{Diffc}}}$ is a local $\mathcal{T}_{\mathrm{Diffc}}$-structure, but this follows from [Lur11b], lemma 1.2.14. Conversely, we assume that $F \in \mathrm{Fun}'(\mathcal{T}, \mathcal{X})$, then $F|_{\mathcal{T}'_{\mathrm{Diffc}}}$ is clearly a local $\mathcal{T}'_{\mathrm{Diffc}}$-structure since $\mathcal{T}'_{\mathrm{Diffc}} \to \mathcal{T}_{\mathrm{Diffc}}$ is a transformation of pregeometries, so we need only show that F is a left Kan extension of $F|_{\mathcal{T}'_{\mathrm{Diffc}}}$. This follows from the same cofinality argument just given. $\square$

Remark 4.1.8.47. It is an immediate consequence of theorem 4.1.8.11 (and the general theory of geometries and pregeometries) that there are preferred equivalences between

(i) The ∞-category ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{T}_{\mathrm{Diffc}})$ of ∞-topoi equipped with local $\mathcal{T}_{\mathrm{Diffc}}$-structures.

(ii) The ∞-category ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})$ of ∞-topoi equipped with local $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-structures.

(iii) The ∞-category of ∞-topoi equipped with local simplicial C^{∞}-rings with corners.

(iv) The ∞-category of ∞-topoi equipped with local sheaves of positive log simplicial C^{∞}-rings.

These equivalence restrict to one between the ∞-category of derived manifolds with corners locally of finite presentation and the ∞-category of 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-schemes locally of finite presentation. A 1-categorical version of this result was obtained by Francis-Staite in her recent thesis [Fra19]; she compared the positive log differentiable spaces of Gillam-Molcho with interior C^{∞}-schemes with corners; both classes of objects form full subcategories of all of the equivalent four ∞-categories described above.

Remark 4.1.8.48. In applications to moduli theory, such as the construction of representing stacks for elliptic moduli problems later in this work, derived manifolds with corners will usually be locally given by a retract of the zero set of a section of a vector bundle over a manifold with faces. Such derived manifolds with corners will have the simplest possible nontrivial corners/log structures: they have free sharpening.

In the introduction to this chapter, we made the following claim: given any ∞-topos $\mathcal{X}$, the functor $\mathrm{Str}_{\mathcal{T}_{\mathrm{Diffc}}}(\mathcal{X}) \to \mathrm{Str}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X})$ induced by the obvious transformation of pregeometries is a presentable fibration and under the equivalence $\mathrm{Str}_{\mathcal{T}_{\mathrm{Diff}}}(\mathcal{X}) \simeq \mathsf{Shv}_{sC^{\infty}\mathsf{ring}}(\mathcal{X})$, the fibre over $\mathcal{O}_{\mathcal{X}}$ can be identified with the ∞-category of sheaves of log structures on $(\mathcal{O}_{\mathcal{X}})_{\geq 0}$. We now substantiate that claim. Consider the left adjoint to the presentable fibration $p_{\mathsf{Log}} : sC^{\infty}\mathsf{Log} \to sC^{\infty}\mathsf{ring}$, a section of p_{Log} which carries each $A \in sC^{\infty}\mathsf{ring}$ to the object $(A, A_{>0})$. We have already seen that $(A, A_{>0})$ is compact in $sC^{\infty}\mathsf{ring}_c$ if A is compact in $sC^{\infty}\mathsf{ring}$ and it follows immediately from the definitions that the assignment $A \mapsto (A, A_{>0})$ determines a transformation of geometries

$$s_c : \mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}} \longrightarrow \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}.$$

We will end this subsection with an analysis of this transformation of geometries and its associated relative spectrum functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$, which completes our discussion of the structure theory of derived C^{∞}-geometry with corners.

Proposition 4.1.8.49. (1) *The functor $s_c^* : \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathcal{X})$ induced by composition with s_c is a presentable fibration. The fibre over a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structure $\mathcal{O}_{\mathcal{X}}$ can be identified with the ∞-category of log structures on $(\mathcal{O}_{\mathcal{X}})_{\leq 0}$.*

(2) *The functor s_c^* admits a left adjoint carries each $\mathcal{O}_{\mathcal{X}}$ to an initial object in the fibre over $\mathcal{O}_{\mathcal{X}}$.*

(3) *The functor ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}) \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ is a presentable fibration. The fibre over a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ can be identified with the ∞-category of log structures on $(\mathcal{O}_{\mathcal{X}})_{\leq 0}$.*

(4) *The relative spectrum $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ may be identified with the section of ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}) \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ that carries each $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ to an initial object in the fibre over $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$; in particular, $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ is fully faithful.*

Proof. (1) It is an immediate consequence of definition of the geometry structure on $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ that a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structure $\mathcal{O}_{\mathcal{X}}$ is a local if and only if $s_c^*(\mathcal{O}_{\mathcal{X}})$ is local, and a morphism $f:\mathcal{O}_{\mathcal{X}}\to\mathcal{O}'_{\mathcal{X}}$ is local if and only if $s_c^*(f)$ is local; in other words, the functor s_c^* fits into a pullback diagram

$$\begin{tikzcd}
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}(\mathcal{X}) \arrow[r] \arrow[d, "s_c^*"] & \mathrm{Str}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}(\mathcal{X}) \arrow[d, "s_c^*"] \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathcal{X}) \arrow[r] & \mathrm{Str}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathcal{X})
\end{tikzcd}$$

among ∞-categories. It follows from [Lur17b], rmk. 5.2.6.4 and theorem 4.1.8.24 that the functor $s_c^*:\mathrm{Str}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}(\mathcal{X})\to\mathrm{Str}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathcal{X})$ can be identified with the functor $p_{\mathsf{Log}}:\mathsf{Shv}_{sC^\infty\mathsf{Log}}(\mathcal{X})\to\mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X})$ given by composing with p_{Log}. This functor is a presentable fibration with fibres given by $\mathsf{Log}((\mathcal{O}_{\mathcal{X}})_{\geq 0})$.

(2) Since s_c^* is a presentable fibration, it admits a left adjoint carrying each $\mathcal{O}_{\mathcal{X}}$ to an initial object in the fibre.

(3) We have a commuting diagram

$$\begin{tikzcd}
{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}) \arrow[rr, "r"] \arrow[dr, "q_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}"'] & & {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}) \arrow[dl, "q_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}"] \\
& {}^{\mathrm{L}}\mathsf{Top} &
\end{tikzcd}$$

where $q_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}$ and $q_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ are coCartesian fibrations and r is an inner fibration. To prove that r is a coCartesian fibration, we employ lemma 1.4.14 of [Lur09] and show that for each algebraic morphism $f^*:\mathcal{X}\to\mathcal{Y}$, the induced functor $\mathsf{Shv}_{sC^\infty\mathsf{ring}_c}(\mathcal{X})\to\mathsf{Shv}_{sC^\infty\mathsf{ring}_c}(\mathcal{Y})$ carries coCartesian morphisms to coCartesian morphisms. By theorem 4.1.8.24, this amounts to the verification that f^* takes logifications to logifications. Recalling the explicit form of the logification functor, this follows easily from the fact that algebraic morphisms preserve finite limits and colimits

(4) Since the functor r is a presentable fibration, it admits a left adjoint carrying each $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ to an initial object in the fibre ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}})_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\simeq\mathsf{Log}((\mathcal{O}_{\mathcal{X}})_{\geq 0})$.

□

4.2 Derived C^∞-Stacks

Our study of the relation between simplicial C^∞-rings and derived geometry in the axiomatic setup of pregeometries has yielded the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$ which controls derived C^∞-geometry. The constructions of section 4.1 provide us with several 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theories and $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-scheme theories.

Proposition 4.2.0.1. *The following ∞-categories are saturated 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theories:*

(1) The ∞-category $\mathcal{T}_{\mathrm{Diff}}$ of manifolds.

(2) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}\simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{fp}}$.

(3) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{afp}}\simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{afp}}$.

(4) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}\simeq sC^\infty\mathsf{ring}^{op}_{\mathrm{fair}}$.

Proof. It is clear that $\mathcal{L}1$ and $\mathcal{L}4$ are satisfied for these full subcategories, so it remains to check condition $\mathcal{L}2'$. Suppose that we have a (-1)-étale map $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)\to(\mathcal{X},\mathcal{O}_{\mathcal{X}})$, where $(\mathcal{X},\mathcal{O}_{\mathcal{X}})=\mathbf{Spec}\,A$ for A fair, then the object $U\in\mathrm{Sub}(1_{\mathcal{X}})$ corresponds to some localization $A\to A[1/a]$. Applying the spectrum functor to this map yields a (-1)-étale map $(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})\to(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ associated to the same object U. The ∞-category of (-1)-étale maps over $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is equivalent to the poset $\mathrm{Sub}(1_{\mathcal{X}})$ so we have an equivalence $(\mathcal{X}_{/U},\mathcal{O}_{\mathcal{X}}|_U)\simeq(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$. By theorem 4.1.3.22, the map $A[1/a]\to\Gamma(\mathcal{O}_{\mathcal{Y}})$ exhibits a reflection onto the fair simplicial C^∞-rings so that $(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})\simeq\mathbf{Spec}\,\Gamma(\mathcal{O}_{\mathcal{Y}})$ is an affine fair derived C^∞-scheme. If A is (almost) finitely presented, then $A[1/a]\simeq\Gamma(\mathcal{O}_{\mathcal{Y}})$ is also (almost) finitely presented. If A is the ring of smooth functions on a manifold, then so is $A[1/a]$. □

Remark 4.2.0.2. Suppose that $\mathcal{L}$ is a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theory, then the ∞-category $\mathcal{L}$ is also a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theory via the relative spectrum $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$ and the two étale topologies $\mathcal{L}$ inherits (as a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theory and a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-scheme theory) coincide.

Proposition 4.2.0.3. *The following ∞-categories are saturated 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-scheme theories:*

(1) The ∞-category $\mathcal{T}_{\mathrm{Diff}}$ of manifolds.

(2) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}$.

(3) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{afp}}$.

(4) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}$.

(5) The ∞-category $\mathcal{T}_{\mathrm{Diffc}}$ of manifolds with faces and interior b-maps among them.

(6) The ∞-category $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fpc}} = \mathbf{Spec}_c(sC^\infty\mathsf{ring}_{\mathrm{fpc}})$.

Proof. We view the first four ∞-categories as $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-scheme theories by identifying them with their essential image under the fully faithful functor $\mathbf{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$. It is again clear that $\mathcal{L}1$ and $\mathcal{L}4$ are satisfied for these full subcategories. The same argument employed in the proof of proposition 4.2.0.1 may be used to show that $\mathcal{L}2'$ holds for these four ∞-categories. □

The theory of higher geometric stacks may be applied to these good scheme theories, and the stacks thus obtained are the main objects of interest in this work.

Notation 4.2.0.4. The ∞-topos $\mathsf{Shv}(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}})$ of sheaves on the site of affine derived manifolds of finite presentation is denoted $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$. The objects in this ∞-topos are called *derived C^∞-stacks locally of finite presentation.* Similarly, we denote the ∞-topoi $\mathsf{Shv}(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{afp}})$ and $\mathsf{Shv}(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fpc}})$ by $\mathsf{d}C^\infty\mathsf{St}_{\mathsf{lafp}}$ and $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfpc}}$. The objects in these ∞-topos are called *derived C^∞-stacks locally almost of finite presentation* and *derived C^∞-stacks with corners locally of finite presentation.* We also have the ∞-category $\mathsf{Shv}(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}})$ (which is not accessible and so not an ∞-topos) of locally fair derived C^∞-stacks. We will often abbreviate 'locally of finite presentation' with *lfp* and 'locally almost of finite presentation' with *lafp*.

We now extract several consequences of the general theory of $\mathcal{G}$-scheme theories we have set up in chapter 2.

Proposition 4.2.0.5. *The sheaf ∞-topoi associated to all of the scheme theories considered in proposition 4.2.0.3 have enough points.*

Proof. This is an immediate consequence of proposition 3.2.1.32 and the fact that the sheaf topoi of the underlying topological spaces of affine derived fair C^∞-schemes are locally of finite homotopy dimension and therefore have enough points by [Lur17b], cor. 7.2.1.17. □

In particular all these ∞-topoi are hypercomplete and Postnikov towers converge.
The general theory of chapter 2 allows us to compare different $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-scheme theories. For instance, we have a commuting square of fully faithful embeddings

$$\begin{tikzcd} \mathcal{T}_{\mathrm{Diff}} \arrow[r, hook, "\iota"] \arrow[d, hook, "s_c"] & \mathcal{G}_{\mathrm{Diff}} \arrow[d, hook, "s_c"] \\ \mathcal{T}_{\mathrm{Diffc}} \arrow[r, hook, "\iota_c"] & \mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}} \end{tikzcd}$$

Proposition 4.2.0.6. *The square above induces a square of fully faithful left adjoint functors*

$$\begin{tikzcd} \mathsf{SmSt} \arrow[r, hook, "\iota"] \arrow[d, hook, "s_{c!}"] & \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}} \arrow[d, hook, "s_{c!}"] \\ \mathsf{SmSt}_c \arrow[r, hook, "\iota_{c!}"] & \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfpc}}. \end{tikzcd}$$

Proof. This is an immediate consequence of proposition 3.2.1.29. □

Remark 4.2.0.7. The functor $s_{c!}$ on the right is left exact and thus an algebraic morphism. The functors $\iota_!$ and $\iota_{c!}$ are *not* algebraic morphisms; if they were, they would be essentially surjective and therefore equivalences, which is not possible. According to proposition 3.1.0.29 and theorem 4.1.4.6 however, $\iota_!$ and $\iota_{c!}$ do preserve pullbacks along strongly étale and strongly submersive maps in SmSt and SmSt_c with respect to the pregeometry structures on $\mathcal{T}_{\mathrm{Diff}}$ and $\mathcal{T}_{\mathrm{Diffc}}$.

Remark 4.2.0.8. In the sequel, we will restrict to the 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme theories obtained in proposition 4.2.0.1, hence we will only consider schemes and stacks which are at least locally finitely generated (recall that the fairness condition does not play a deep role in the theory; it is just a byproduct of the fact that taking the global sections of the spectrum of a module over a C^∞-ring is a localization, but need not be an equivalence). There are two reasons for this choice:

(1) The real spectra of finitely generated C^∞-rings are locally of finite homotopy dimension, which implies that the t-structures on the ∞-categories of sheaves of modules are particularly well behaved (they are *excellent* in the sense of [Lur11c], definition 6.9, by proposition 2.2.5.19).

(2) The inverse function theorem holds for morphisms $f : A \to B$ with $\pi_0(f)$ finitely presented if A (and therefore also B) is finitely generated (theorem 5.1.3.17).

In the next subsections, we will give some distinguished families derived C^∞-stacks. We will treat geometric (i.e. Artin and Deligne-Mumford stacks), but also nongeometric examples such as the geometric stack classifiers and mapping stacks.

4.2.1 Geometric C^∞-stacks

In what follows, we will use the 0-localic scheme theory $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathcal{L})$ for notational convenience, but the definitions below make sense for all the 0-localic scheme theories in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$ we have constructed.
We introduce two geometric contexts for the pairs $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}})$.

Definition 4.2.1.1. (1) A map $f : \mathbf{Spec}\, A \to \mathbf{Spec}\, B$ of affine derived fair C^∞-schemes is *étale* if it is an equivalence, up to localization on A and B. More precisely, f is *étale* if there is an admissible covering $\{U_i \to \mathbf{Spec}\, A\}_{i\in I}$ of A such that for each $i \in I$, the composition $U_i \to \mathbf{Spec}\, B$ is admissible.

(2) A map $f : \mathbf{Spec}\, A \to \mathbf{Spec}\, B$ of affine derived fair C^∞-schemes is *submersive* if there is an admissible covering $\{U_i \to \mathbf{Spec}\, A\}_{i\in I}$ of A such that for each $i \in I$ there is an admissible map $V_i \to \mathbf{Spec}\, B$ and an equivalence $U_i \simeq V_i \times \mathbb{R}^n$ such that the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & \mathbf{Spec}\, A \arrow[d, "f"] \\ V_i \arrow[r] & \mathbf{Spec}\, B \end{tikzcd}$$

commutes, where the left vertical map is the projection $U_i \simeq V_i \times \mathbb{R}^n \to V_i$.

Proposition 4.2.1.2. *Let $\mathcal{P}_{et}$ be the subcategory of $\mathsf{d}C^\infty\mathsf{Aff}$ spanned by étale morphisms, and let $\mathcal{P}_{lis}$ be the subcategory of $\mathsf{d}C^\infty\mathsf{Aff}$ spanned by submersive morphisms. These subcategories define geometric contexts $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}, \mathcal{P}_{et})$ and $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}, \mathcal{P}_{lis})$ in the sense of definition 3.2.2.3.*

Proof. We only have to check that conditions $\mathbf{G_1}$, $\mathbf{G_2}$ and $\mathbf{G_3}$ are satisfied. For $\mathcal{P}_{et}$, $\mathbf{G_1}$ follows by the stability of admissible maps under pullbacks, and $\mathbf{G_2}$ and $\mathbf{G_3}$ are obvious. For $\mathcal{P}_{lis}$, $\mathbf{G_1}$ is a consequence of [Lur11b], prop. 3.1.8, and $\mathbf{G_2}$ and $\mathbf{G_3}$ are again obvious. ☐

Definition 4.2.1.3. A *derived n-Deligne-Mumford C^∞-stack locally of finite presentation* is an n-geometric stack for the geometric context $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}, \mathcal{P}_{et})$. A *derived n-Artin C^∞-stack locally of finite presentation* is an n-geometric stack for the geometric context $(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}, \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}, \mathcal{P}_{lis})$.

Remark 4.2.1.4. Usually, it will be understood that the underlying ∞-site is $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fp}}$ or one of its variants. When this is the case, we sometimes abuse terminology and call a derived C^∞-stack just a *derived stack*.

In the next subsection, we discuss how $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-schemes of higher locality may be interpreted as geometric C^∞-stacks.

4.2.2 Localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-schemes and Deligne-Mumford stacks

In this subsection, we establish a claim we made in the introduction: higher Deligne-Mumford C^∞-stacks, defined inductively as sheaves on $\mathsf{d}C^\infty\mathsf{Aff}$, can equivalently by described as $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-schemes, if we allow for schemes whose underlying 'spaces' are ∞-topoi that are not simply sheaves on a topological space. Recall that an ∞-topos is *n-localic* for $0 \leq n \leq \infty$ if for every n-topos $\mathcal{Y}$, the natural restriction map $\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X}) \to \mathrm{Fun}_*(\tau_{\leq n-1}\mathcal{Y}, \tau_{\leq n-1}\mathcal{X})$ is an equivalence.

Definition 4.2.2.1. For $\mathcal{G}$ a geometry, we say that a $\mathcal{G}$-scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *n-localic* if the underlying ∞-topos $\mathcal{X}$ is n-localic. We write $\mathrm{Sch}^{n-\mathrm{loc}}_{\mathrm{fp}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ for the ∞-categories of n-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-schemes locally of finite presentation.

Remark 4.2.2.2. We establish the results in this section only for derived Deligne-Mumford C^∞-stacks locally of finite presentation without corners, but this is only for notational convenience; the results hold for all the $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diffc}}$-scheme theories considered in this section.

We need the following prelimenary result.

Proposition 4.2.2.3. *Let $(\mathcal{X},\mathcal{O}_\mathcal{X})$ be a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme, and denote $X := j_{\mathrm{Sch}}(\mathcal{X},\mathcal{O}_\mathcal{X})$. Then for all $n\geq 0$, the following are equivalent.*

(1) *For every finitely presented discrete simplicial C^∞-ring A, $X(A)$ is n-truncated.*

(2) *The ∞-topos $\mathcal{X}$ is n-localic.*

Proof. (1) $\Rightarrow$ (2). It suffices to show that each $U\in\mathcal{X}$ such that $(\mathcal{X}_{/U},\mathcal{O}_\mathcal{X}|_U)$ is an affine derived manifold of finite presentation is $(n-1)$-truncated, by proposition 2.2.3.3. Let $\mathcal{X}_0\subset\mathcal{X}$ be the full subcategory spanned by those $V\in\mathcal{X}$ such that $\mathrm{Hom}_\mathcal{X}(V,U)$ is $(n-1)$-truncated, then we should show that $\mathcal{X}_0=\mathcal{X}$. Since $\mathcal{X}_0$ is stable under colimits, we may suppose that $(\mathcal{X}_{/V},\mathcal{O}_\mathcal{X}|_V)$ is an affine derived manifold of finite presentation. Consider the truncation $(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V)$, then the space $\mathrm{Hom}_\mathcal{X}(V,U)$ can be identified with the fibre of the map

$$\mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})}((\mathcal{X}_{/U},\mathcal{O}_\mathcal{X}|_U),(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V))\longrightarrow \mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})}((\mathcal{X},\mathcal{O}_\mathcal{X}),(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V))$$

at the morphism

$$(\mathcal{X},\mathcal{O}_\mathcal{X})\longrightarrow(\mathcal{X}_{/V},\mathcal{O}_\mathcal{X}|_V)\longrightarrow(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V)$$

where the first map is étale and the second map exhibits a truncation. The space $\mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})}((\mathcal{X}_{/U},\mathcal{O}_\mathcal{X}|_U),(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V))$ is 0-truncated because $(\mathcal{X}_{/U},\mathcal{O}_\mathcal{X}|_U)$ is affine, and the space $\mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})}((\mathcal{X},\mathcal{O}_\mathcal{X}),(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V))$ coincides with $X(\Gamma(\mathcal{X}_{/V},\tau_{\leq 0}\mathcal{O}_\mathcal{X}|_V))$ which is n-truncated by assumption, so we conclude that $\mathrm{Hom}_\mathcal{X}(V,U)$ is indeed $(n-1)$-truncated.
(2) $\Rightarrow$ (1). Let A be 0-truncated, then we have a fibre sequence

$$\mathrm{Hom}_{\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}(\mathrm{Spec}_\mathbb{R} A)}(f^*\mathcal{O}_\mathcal{X},\mathcal{O}_{\mathbf{Spec}\,A})\longrightarrow \mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})}((\mathcal{X},\mathcal{O}_\mathcal{X}),\mathbf{Spec}\,A)\longrightarrow \mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}}(\mathcal{X},\mathrm{Spec}_\mathbb{R} A)$$

where the fibre is 0-truncated because A is, and the space $\mathrm{Hom}_{{}^{\mathrm{L}}\mathsf{Top}}(\mathcal{X},\mathrm{Spec}_\mathbb{R} A)$ is n-truncated because $\mathcal{X}$ is n-localic. ☐

Lemma 4.2.2.4. *Let X be a derived n-Artin C^∞-stack locally of finite presentation. Then for each finitely presented discrete simplicial C^∞-ring A, the space $X(A)$ is $(n+1)$-truncated.*

Proof. This is proven by induction on the degree of geometricity. If X is (-1)-geometric, X is representable by a 0-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme, in which case proposition 4.2.2.3 yields the desired statement. Suppose that X is an n-Artin stack for $n\geq 0$. Clearly, it suffices to show that all connected components of $X(A)$ are $(n+1)$-truncated. Differently put, we should show that for each $x\in hX(A)=\pi_0(X(A))$, the homotopy fibre at x of the projection $p:X(A)\to\pi_0(X(A))$ is $(n+1)$-truncated. Since X is a sheaf, a choice of cover $\{\mathbf{Spec}\,A_i\to\mathbf{Spec}\,A\}$ allows us to write the homotopy fibre $p^{-1}(x)$ as a limit $\lim_{i\in\mathcal{J}}K_i$, where each K_i is the connected component of x in a space of the form $X(A_{i_1}\otimes^\infty_A\ldots\otimes^\infty_A A_{i_n})$. There is an effective epimorphism $\coprod_j U_j\to X$ defining an n-submersive atlas on X, so, using that k-truncated spaces are stable under limits, we may assume that the map $\mathbf{Spec}\,A\to X$ defining $x\in\pi_0(X(A))$ factors through $\coprod_j U_j$. Now the fibre of $\coprod_j U(A)\to X(A)$ at x is equivalent to the fibre of $\coprod_j U_j\times_X\mathbf{Spec}\,A(A)\to\mathrm{Hom}_{C^\infty\mathsf{ring}}(A,A)$ at the identity, which is n-truncated because the induction hypothesis asserts that $\coprod_j U_j\times_X\mathbf{Spec}\,A(A)$ is n-truncated. Using the long exact sequence, we find that the connected component of $x\in X(A)$ is $(n+1)$-truncated. ☐

Theorem 4.2.2.5. *For $n\geq 0$, the fully faithful functor $j_{\mathrm{Sch}}:\mathrm{Sch}_{\mathrm{lfp}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})\to\mathrm{Fun}(sC^\infty\mathsf{ring}_{\mathrm{fp}},\mathcal{S})$ induces an equivalence*

$$\mathrm{Sch}^{n-\mathrm{loc}}_{\mathrm{lfp}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})\simeq\mathsf{dSmDM}^{n-1}_{\mathrm{fp}}$$

Proof. This is proven by induction on the degree of geometricity. We prove the first equivalence; the proof for the second equivalence is the same. For $n=0$, the equivalence is definitional. We assume that the equivalence holds for $0\leq k<n$. Let $(\mathcal{X},\mathcal{O}_\mathcal{X})$ be an n-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme locally of finite presentation. Choose an effective epimorphism $\coprod_i U_i\to 1_\mathcal{X}$ such that each $(\mathcal{X}_{/U_i},\mathcal{O}_\mathcal{X}|_{U_i})$ is equivalent to the spectrum of a finitely presented simplicial C^∞-ring. Since $\mathfrak{X}$ is equivalent to the ∞-category of sheaves on an n-category, we may choose for each U_i a small collection of morphisms $\{V_{\beta_i}\to U_i\}$ such that

$$\coprod_{\beta_i}V_{\beta_i}\longrightarrow U_i$$

is an effective epimorphism and each V_{i_β} is $(n-1)$-truncated in $\mathfrak{X}$. Since $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i})$ is the spectrum of a finitely presented simplicial C^∞-ring A, there exists for each V_{β_i} a small collection of morphisms $\{W_{\gamma_{\beta_i}} \to V_{\beta_i}\}$ such that

$$\coprod_{\gamma_{\beta_i}} W_{\gamma_{\beta_i}} \longrightarrow V_{\beta_i}$$

is an effective epimorphism (in $\mathfrak{X}$ and in $\mathfrak{X}_{/U_i}$), each map $W_{\gamma_{\beta_i}} \to V_{\beta_i}$ is a (-1)-truncated and each $(\mathcal{X}_{/W_{\gamma_{\beta_i}}}, \mathcal{O}_{\mathcal{X}}|_{W_{\gamma_{\beta_i}}})$ is equivalent to the spectrum of a finitely presented simplicial C^∞-ring. Replacing the collection $\{U_i\}$ by $\{W_{\gamma_{\beta_i}}\}$, we may suppose that each U_i is an $(n-1)$-truncated object in $\mathfrak{X}$. Taking the Čech nerve of the morphism

$$h : \coprod_i (\mathfrak{X}_{/U_i}, \mathcal{O}_{\mathfrak{X}}|_{U_i}) \longrightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

yields an effective groupoid in $\mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})_{\mathrm{et}}$ which is carried to an effective groupoid in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lafp}}$ by the functor j_{Sch}. The object in simplicial degree 1 is a coproduct of objects of the form $j_{\mathrm{Sch}}(\mathfrak{X}_{/U_i \times U_j}, \mathcal{O}_{\mathfrak{X}}|_{U_i \times U_j})$. Since U_i and U_j are $(n-1)$-truncated, the morphism $U_i \times U_j \to U_i$ is $(n-1)$-truncated. Since the ∞-topos $\mathfrak{X}_{/U_i}$ is 0-localic, the ∞-topos $\mathfrak{X}_{/U_i \times U_j}$ is $(n-1)$-localic. The inductive hypothesis now guarantees that the object $j_{\mathrm{Sch}}(\check{C}(h)_1)$ is an $(n-1)$-Deligne-Mumford stack locally of finite presentation. Since j_{Sch} carries étale maps between $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-schemes to étale maps of Deligne-Mumford stacks, the simplicial object $j_{\mathrm{Sch}}(\check{C}(h)_\bullet)$ is an a n-étale groupoid presentation of the object $j_{\mathrm{Sch}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.
We are left to show that every $(n-1)$-Deligne-Mumford stack X is represented by an n-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme locally of finite presentation. Choose an $(n-1)$-étale atlas $h : \coprod_i U_i \to X$, then the Čech nerve of h is a simplicial diagram consisting of $(n-2)$-Deligne-Mumford stacks and étale face maps between them, so the induction hypothesis implies that there is a simplicial diagram $C_\bullet$ in $\mathrm{Sch}_{\mathrm{afp}}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ and an equivalence $j_{\mathrm{Sch}}(C_\bullet) \simeq \check{C}(h)_\bullet$. Using once again that j_{Sch} preserves colimits of the simplicial diagram $C_\bullet$, we find that X is representable by a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme locally of finite presentation. Since X is $(n-1)$-Artin, $X(A)$ is n-truncated for any discrete simplicial C^∞-ring A of finite presentation by lemma 4.2.2.4, so proposition 4.2.2.3 asserts that X is representable by an n-localic $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-scheme locally of finite presentation. □

Proposition 4.2.2.6. *Let $n \geq -1$ and let X be a derived C^∞-stack locally of finite presentation. Then the following are equivalent.*

(1) *X is a derived n-Deligne-Mumford C^∞-stack.*

(2) *There exists a collection of n-Deligne-Mumford stacks $\{U_i\}_{i \in I}$ and a collection of n-étale morphisms $\{f_i : U_i \to X\}_{i \in I}$ such that $\coprod_{i \in I} U_i \to X$ is an effective epimorphism, and for each pair $i, j \in I$, the map $U_i \times_X U_j \to U_i$ is an open substack inclusion.*

(3) *There exists a collection of n-Deligne-Mumford stacks $\{U_i\}_{i \in I}$ and a collection of n-étale morphisms $\{f_i : U_i \to X\}_{i \in I}$ such that $\coprod_{i \in I} U_i \to X$ is an effective epimorphism, and for each pair $i \in I$, the map $U_i \times_X U_i \to U_i$ is an open substack inclusion.*

(4) *There exists a collection of n-Deligne-Mumford stacks $\{U_i\}_{i \in I}$ and a collection of n-étale morphisms $\{f_i : U_i \to X\}_{i \in I}$ such that $\coprod_{i \in I} U_i \to X$ is an effective epimorphism, and for each pair $i \in I$, the map $U_i \times_X U_i \to U_i$ is an equivalence.*

Proof. (1) ⇒ (2) follows if we take the covering $U_i \to X$ to consist of the single map $\mathrm{id} : X \to X$ and clearly (2) ⇒ (3). We have a retraction

$$U_i \longrightarrow U_i \times_X U_i \longrightarrow U_i$$

which implies that the second map is an effective epimorphism. An open substack inclusion is a monomorphism in the ∞-topos $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lafp}}$, which implies that the second map is an equivalence, and this establishes (3) ⇒ (4).
We turn to the proof of (4) ⇒ (1). Since $\coprod_i U_i$ is an n-Deligne-Mumford stack, the object X is an $(n+1)$-Deligne-Mumford stack. It suffices to show that for each discrete simplicial C^∞-ring A of finite presentation, the space $X(A)$ is n-truncated. Since X is a sheaf, we may suppose that $\mathbf{Spec}\, A \to X$ factors through some U_i, then the homotopy fibre of $U_i(A) \to X(A)$ at x may be identified with the homotopy fibre of $U_i(A) \times_{X(A)} U_i(A) \longrightarrow X(A)$ which is an equivalence by assumption. It follows that $X(A)$ is n-truncated. □

Remark 4.2.2.7. Since hypercompleteness is a local property on any ∞-topos, every derived Deligne-Mumford C^∞-stack is hypercomplete.

4.2.3 Moduli of geometric stacks

In the previous subsections, we have constructed ∞-categories $\mathsf{d}C^\infty\mathsf{DM}_n$ and $\mathsf{d}C^\infty\mathsf{Ar}_n$ of Deligne-Mumford and Artin stacks. Now we show that these ∞-categories can be enhanced to sheaves of ∞-categories on affine derived manifolds.

Lemma 4.2.3.1. *Let $\mathcal{G}$ be a geometry and $\mathcal{L}$ a small $\mathcal{G}$-scheme theory. Let P be a property of morphisms in $\mathsf{Shv}(\mathcal{L})$ stable under pullbacks and arbitrary small coproducts, and denote by $\mathcal{O}^P_{\mathcal{L}} \subset \mathrm{Fun}(\Delta^1, \mathsf{Shv}(\mathcal{L}))$ the full subcategory spanned by P, which determines a Cartesian fibration $\mathcal{O}^P_{\mathcal{L}} \to \mathsf{Shv}(\mathcal{L})$. Let $\mathcal{O}^P_{\mathcal{L}} : \mathsf{Shv}(\mathcal{L})^{op} \to \widehat{\mathsf{Cat}}_\infty$ be the straightening of this fibration, then the following are equivalent.*

(1) *P is stable under small coproducts, and for any morphism $f : X \to Y$, if for all morphisms $j(\mathfrak{Y}) \to Y$ with $\mathfrak{Y} \in \mathcal{L}$ the morphism $j(\mathfrak{Y}) \times_Y X \to X$ has the property P, then f has the property P.*

(2) *The functor $\mathcal{O}^P_{\mathcal{L}}$ preserves limits.*

Proof. It follows from [Lur17b], lem. 6.1.3.5 that (2) is equivalent the following condition: suppose $\alpha : f \to g$ is a Cartesian transformation between diagrams $f, g : K \to \mathsf{Shv}(\mathcal{L})$ and suppose that $\alpha(k)$ has the property P for each $k \in K$, then $\operatorname{colim} f \to \operatorname{colim} g$ has the property P. This is a reformulation of (1). □

Example 4.2.3.2. Let P be the property of being an n-representable morphism for the submersive/étale geometric context, then P satisfies condition (1) of lemma 4.2.3.1 and therefore induces $\widehat{\mathsf{Cat}}_\infty$-valued sheaves

$$\mathsf{d}C^\infty\mathsf{Ar}_n / \mathsf{d}C^\infty\mathsf{DM}_n : \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{lfp}} \longrightarrow \widehat{\mathsf{Cat}}_\infty$$

which carry an affine $\mathbf{Spec}\, A$ to the slice categories $(\mathsf{d}C^\infty\mathsf{Ar}_n)_{/\mathbf{Spec}\, A} / (\mathsf{d}C^\infty\mathsf{DM}_n)_{/\mathbf{Spec}\, A}$.

Example 4.2.3.3. Let P be the property of being n-representable and locally of finite presentation in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{fair}}$, then P satisfies condition (1) of lemma 4.2.3.1 and there induces a $\widehat{\mathsf{Cat}}_\infty$-valued sheaf $\mathsf{d}C^\infty\mathsf{Ar}_{n\mathrm{lfp}}$.

Sheaves of the form $\mathcal{O}^P_{\mathcal{L}}$ are generically obtained via the following result.

Lemma 4.2.3.4. *Let P be a property of morphisms of affine derived manifolds (of finite presentation, with or without corners...) that is stable under pullbacks and local on the target with respect to the étale topology. Say that a morphism $f : X \to Y$ of $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$ ($\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfpc}}$,...) has the property n-P if f is n-representable and for each map $\mathbf{Spec}\, A \to Y$, the pullback $\mathbf{Spec}\, A \times_Y X$ admits an n-submersive atlas $\coprod_i U_i \to \mathbf{Spec}\, A \times_Y X$ such that for each i, the composite map $U_i \to \mathbf{Spec}\, A$ has the property P. Then the property n-P is stable under pullbacks and satisfies condition (1) of lemma 4.2.3.1.*

Remark 4.2.3.5. Note that for a property P of morphisms in $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{lfp}}$ that is stable under pullbacks and local on the target, the property (-1)-P of the lemma above does not in general coincide with P; this is true precisely if P is also local on the source.

Example 4.2.3.6. Let P be the property of being an n-representable submersion, then P satisfies condition (1) of lemma 4.2.3.1 and therefore induces a functor

$$\mathsf{Mfds} : \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{lfp}} \longrightarrow \widehat{\mathsf{Cat}}_\infty,$$

the *moduli space of manifolds.*

Remark 4.2.3.7. In the previous subsection, we have considered a variety of properties of morphisms stable under base change and local on the target for the étale topology on a $\mathcal{G}$-scheme theory $\mathcal{L}$. These properties were in fact local on the target by design: it is possible to show the following variant of lemma 4.2.3.1: let P be a property of morphisms stable under pullback and let $\overline{P}$ be the smallest property of morphisms containing P that is stable under pullback and satisfies condition (1) of lemma 4.2.3.1, then the full subcategory inclusion $\mathcal{O}^P_{\mathcal{L}} \subset \mathcal{O}^{\overline{P}}_{\mathcal{L}}$ induces a morphism $\mathcal{O}^P_{\mathcal{L}} \to \mathcal{O}^{\overline{P}}_{\mathcal{L}}$ of $\widehat{\mathsf{Cat}}_\infty$-valued presheaves on $\mathcal{L}$ that exhibits $\mathcal{O}^{\overline{P}}_{\mathcal{L}}$ as a sheafification of $\mathcal{O}^P_{\mathcal{L}}$.

4.2.4 Weil restrictions

Let $\mathcal{C}$ be a presentable ∞-category such that colimits are universal in $\mathcal{C}$ (which is the case, for instance, if $\mathcal{C}$ is an ∞-topos). Given a morphism $f : X \to Y$ in $\mathcal{C}$, we have an adjunction

$$\mathcal{C}_{/Y} \underset{f_!}{\overset{f^*}{\rightleftarrows}} \mathcal{C}_{/X}$$

where f^* takes pullbacks along f and $f_!$ composes with f. Under the assumption that colimits in $\mathcal{C}$ are universal, it follows from the adjoint functor theorem that f^* admits a right adjoint f_*, which takes a morphism $Z \to X$ to an object $f_*(Z) \to Y$ which comes equipped with a map $\alpha : f_*(Z) \times_Y X \to Z$ in $\mathcal{C}_{/X}$ satisfying the universal property that for each $Y' \to Y$, composition with α induces an equivalence

$$\mathrm{Hom}_{\mathcal{C}_{/Y}}(Y', f_*(Z)) \simeq \mathrm{Hom}_{\mathcal{C}_{/X}}(Y' \times_Y X, Z)$$

of spaces. In this situation, the object $f_*(Z)$ is known as the *Weil restriction of Z along f*. The following example is central to the construction of moduli spaces.

Example 4.2.4.1 (Mapping stacks)**.** Let $X \in \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$ and let $f : X \to *$ be the canonical map to the final object in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfpc}}$, then we denote by $\mathsf{Map}_X(X,\_)_{\mathsf{d}C^\infty} : (\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}})_{/X} \to \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$ the Weil restriction along f. For $p : Y \to X$ a map, we call the object $\mathsf{Map}_X(X,Y)_{\mathsf{d}C^\infty}$ the *mapping stack of sections of* p. By construction, it comes equipped with an evaluation functor $\mathrm{ev} : \mathsf{Map}_X(X,Y)_{\mathsf{d}C^\infty} \times X \to Y$. Composing the functor f^* with the mapping stack of sections yields a functor

$$\mathsf{Map}(X,\_)_{\mathsf{d}C^\infty} : \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}} \longrightarrow \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$$

right adjoint to the composition $f_! f^*$ which takes products with X. One readily verifies that for each $p : Y \to X$, the mapping stack of sections fits into a pullback diagram

$$\begin{tikzcd}
\mathsf{Map}_X(X,Y)_{\mathsf{d}C^\infty} \arrow[r] \arrow[d] & * \arrow[d, "\mathrm{id}"] \\
\mathsf{Map}(X,Y)_{\mathsf{d}C^\infty} \arrow[r] & \mathsf{Map}(X,X)_{\mathsf{d}C^\infty}
\end{tikzcd}$$

where the lower horizontal map is obtained by adjunction from the map $\mathsf{Map}(X,Y) \times X \xrightarrow{\mathrm{ev}} Y \xrightarrow{p} X$.
We may also consider the Weil restriction along maps $f : X \to Z$. It is a consequence of corollary 2.2.0.13 that for each map $p : Y \to X$, the cone in the pullback diagram

$$\begin{tikzcd}
\mathsf{Map}_X(Y,Y)_{\mathsf{d}C^\infty} \arrow[r] \arrow[d] & * \arrow[d, "\mathrm{id}"] \\
\mathsf{Map}(Y,Y)_{\mathsf{d}C^\infty} \arrow[r] & \mathsf{Map}(Y,X)_{\mathsf{d}C^\infty}
\end{tikzcd}$$

is an associative algebra object of $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$.

Remark 4.2.4.2. For the ∞-topoi SmSt, SmSt_c and $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfpc}}$, we denote the mapping stack of sections by $\mathsf{Map}_X(X,\_)_{\mathsf{Sm}}$, $\mathsf{Map}_X(X,\_)_{\mathsf{Sm}_c}$ and $\mathsf{Map}_X(X,\_)_{\mathsf{d}C^\infty_c}$ respectively. Note that the functors $\iota_!$ and $\iota_{c!}$ do *not* commute with taking mapping stacks. In fact, determining when $\iota_!$ (and $\iota_{c!}$) takes the mapping stack $\mathsf{Map}_X(X,Y)_{\mathsf{Sm}}$ to $\mathsf{Map}_{\iota_!(X)}(\iota_!(X), \iota_!(Y))_{\mathsf{d}C^\infty}$ is of crucial importance in the construction of differential geometric moduli spaces.

The following example is the central geometric input in the construction of virtual fundamental cycles [BF97; Kha19; DJK21].

Example 4.2.4.3 (Deformation to the normal bundle)**.** Let $X \in \mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$, then the map $* \overset{0}{\hookrightarrow} \mathbb{R}$ induces a closed immersion $i : X \hookrightarrow X \times \mathbb{R}$. Let $f : Y \to X$ be a map, then we denote the Weil restriction of f along i by $D_{Y/X}$. This object comes equipped with a map $D_{Y/X} \to X \times \mathbb{R}$. The basic properties of this stack are summarized thus (a proof will appear in upcoming work).

(1) Suppose that f is n-representable, then the map $D_{Y/X} \to Y \times \mathbb{R}$ is $(n+1)$-representable. If f is a locally closed immersion, then this map is n-representable.

(2) For every object $Y \in \mathsf{dSmSt}_{/X}$ and any map $X' \to X$, the canonical map

$$D_{Y \times_X X'/X'} \longrightarrow D_{Y/X} \times_{X \times \mathbb{R}} X' \times \mathbb{R}$$

in $(\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}})_{/Y' \times \mathbb{R}}$ is an equivalence.

(3) The object $D_{Y/X} \times_{X \times \mathbb{R}} X \times \mathbb{R} \setminus \{0\}$ is equivalent to $Y \times \mathbb{R} \setminus \{0\}$.

(4) The object $D_{Y/X} \times_{X \times \mathbb{R}} X \times \{0\}$ is equivalent to $\mathbf{T}[1]Y/X$, the shifted normal bundle stack $\mathbb{V}(\mathbb{L}_{Y/X}[-1])$, the *linear stack over* X defined by the looped relative cotangent complex.

The stack $D_{Y/X}$ sometimes coincides with a more classical object. For instance, when $N \hookrightarrow M$ is a closed embedding of manifolds, the stack $D_{N/M}$ is representable by a manifold that can be explicitly constructed using a tubular neighbourhood of N inside M, or by choosing local coordinates for M and N and constructing the deformation to the normal bundle out of local data as in Kashiwara-Schapira [KS90]. On the other hand, if $Y \to X$ is the map $M \to *$ for M a manifold, then the map $D_M \to \mathbb{R}$ is a smooth 0-Artin stack that is represented by the *tangent groupoid* of M, introduced by Connes [Con94].

The next example is familiar from the formal and synthetic geometry of differential equations.

Example 4.2.4.4 (Jet spaces)**.** Let M be a manifold, viewed as an object in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$, then, as we show in chapter 5, one can construct an effective epimorphism $M \to M_{\mathrm{dR}}$ that fits into a pullback diagram

$$\begin{tikzcd} \widehat{M \times M} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M \arrow[d, "\pi"] \\ M \arrow[r, "\pi"] & M_{\mathrm{dR}} \end{tikzcd}$$

where $\widehat{M \times M}$ is the *formal completion* of $M \times M$ along the diagonal $\Delta \subset M \times M$ which may be represented by the C^∞-ring of Whitney functions $C^\infty(M \times M)/\mathfrak{m}_\Delta^\infty$. Let $f : X \to M$ be an arbitrary map of derived C^∞-stacks, then the *infinite Jet bundle* $J_M^\infty(X)$ of f is the pullback along π of the Weil restriction of X along π, that is, $J_M^\infty(X) \simeq \pi^*\pi_*X$.

Remark 4.2.4.5. In the examples above, we considered Weil restrictions in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$, but one can evidently perform these constructions in larger ∞-topoi as well. If $f_* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism, and $X \to Y \xrightarrow{\theta} Z$ are morphisms in $\mathcal{X}$, then f_* carries the Weil restriction $\theta_*(X)$ of X along θ to the Weil restriction of $f_*(X)$ along $f_*(\theta)$. In particular, Weil restrictions in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfp}}$ are obtained from Weil restrictions in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{lfpc}}$, and even $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{fair}}$.

In (derived) algebraic geometry, there are satisfactory conditions which guarantee that Weil restrictions are representable by derived Artin stacks, based on Lurie's version of Artin's representability theorem. For instance, if $\theta : \mathfrak{X} \to \mathfrak{Y}$ is a strongly proper morphism of derived Deligne-Mumford stacks of finite Tor amplitude and locally almost of finite presentation and $\mathfrak{U} \to \mathfrak{X}$ is a morphism of derived Deligne-Mumford stacks locally almost of finite presentation, then the Weil restriction $\theta_*(\mathfrak{U})$ is a derived Artin stack. In differential geometry, the state of affairs is somewhat more complicated: the mapping stacks of example 4.2.4.1 are only representable in very trivial cases, when the source and target are both representable and one of which is a 0-dimensional manifold, for instance. On the other hand, as asserted in example 4.2.4.3, the operation D_- always carries n-representable morphisms to $(n+1)$-representable morphisms. When $E \to M$ is a surjective submersion of manifolds, the infinite jet space $J_M^\infty(E)$ is not quite representable, but it is a limit of the finite jet spaces $J_M^k(E)$ which are.

4.3 Quasi-Coherent Modules

In this section, we define modules for simplicial C^∞-rings and simplicial C^∞-rings with corners and sheaves of modules of such. We adhere to the philosophy that the theory of *modules* of objects in any presentable ∞-category $\mathcal{C}$ should be given by the ∞-category of *parametrized stable objects of* $\mathcal{C}$ (see [Lur17a], chapter 7.3). This perspective allows for a uniform treatment of C^∞-derivations and log C^∞-derivations and the associated cotangent modules that we explore in the next chapter.

Definition 4.3.0.1. Let $A \in sC^\infty\mathsf{ring}$ be a simplicial C^∞-ring. The ∞-category $sC^\infty\mathsf{ring}_{/A}$ is presentable, so we may consider its stabilization $\mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{/A})$. We write Mod_A for this ∞-category and call it *the ∞-category of A-modules.*

Remark 4.3.0.2. The functor $\Omega^\infty : \mathsf{Mod}_A \to sC^\infty\mathsf{ring}_{/A}$ given by evaluation at the 0-sphere is accessible and preserves small limits, so it admits a left adjoint Σ^∞_+. In the next section, we will identify the A-module $\Sigma^\infty_+(B{\to}A)$ with the *(absolute) cotangent complex* $\mathbb{L}_B \otimes_B A$ of A. As part of this identification, it will become clear that the infinite loop space $\Omega^\infty(M)$ should be considered as the *split square-zero extension of A by M*. For this reason, we will write $\Omega^\infty(M) = A \oplus M$; while the direct sum notation does not strictly make sense in simplicial C^∞-rings, we actually do have a coproduct of the underlying $\mathbb{R}$-modules. Mapping A into a trivial square zero extension $A \oplus M$ over A is equivalent to giving a *derivation* from A into M; this is well known in the discrete case of commutative algebras for instance, where one can check this via explicit formulae. In the case of homotopy algebras, it becomes unwieldy to write down equational presentations of derivations that take into account higher coherence data, so the space of derivations $\mathrm{Der}_A(M)$ is taken to be $\mathrm{Hom}_{sC^\infty\mathsf{ring}_{/A}}(A, A \oplus M)$ by definition. With these interpretations in place, we automatically have the equivalence

$$\mathrm{Der}_A(M) \simeq \mathrm{Hom}_{\mathsf{Mod}_A}(\mathbb{L}_A, M),$$

showing that the cotangent complex corepresents linear *C^∞-derivations.*

Before moving on to more tractable notions of A-modules for a simplicial C^∞-ring A, we need some concepts from the calculus of functors.

Definition 4.3.0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with finite limits, and let $F : \mathcal{C} \to \mathcal{D}$ and $f : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{S}\mathrm{p}(\mathcal{D})$ be functors. A natural transformation $\alpha : F \circ \Omega^\infty_{\mathcal{C}} \to \Omega^\infty_{\mathcal{D}} \circ f$ *exhibits f as a (Goodwillie) derivative of F* if for each $g : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{S}\mathrm{p}(\mathcal{D})$, composition with α induces a homotopy equivalence

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{S}\mathrm{p}(\mathcal{C}),\mathcal{S}\mathrm{p}(\mathcal{D}))}(f, g) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{S}\mathrm{p}(\mathcal{C}),\mathcal{S}\mathrm{p}(\mathcal{D}))}(F \circ \Omega^\infty_{\mathcal{C}}, \Omega^\infty_{\mathcal{D}} \circ g)$$

of Kan complexes. In this situation, f is determined up to equivalence by F, and we denote this functor by ∂F.

The following is [Lur17a], prop. 6.2.1.9.

Proposition 4.3.0.4. *Let $\mathcal{C}$ be an ∞-category that has finite colimits, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \mathcal{C} \to \mathcal{D}$ be reduced functor that preserves filtered colimits. Then F admits a Goodwillie derivative $\partial F : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{S}\mathrm{p}(\mathcal{D})$ given by the formula* $\mathrm{colim}_m \Omega^m_{\mathcal{D}} \circ F \circ \Sigma^m_{\mathcal{C}}$.

It will turn out to be convenient to characterize the (fibrewise) stabilization by a universal property.

Definition 4.3.0.5. Let $\mathcal{D}$ be a presentable ∞-category. A categorical fibration $v : \mathcal{D}' \to \mathcal{D}$ *exhibits $\mathcal{D}'$ as a stable envelope of $\mathcal{D}$* if the following conditions are satisfied.

(1) $\mathcal{D}'$ is stable and presentable.

(2) v admits a left adjoint.

(3) For every stable presentable ∞-category $\mathcal{E}$, composition with v induces an equivalence

$$\mathrm{Fun}^R(\mathcal{E}, \mathcal{D}') \xrightarrow{\simeq} \mathrm{Fun}^R(\mathcal{E}, \mathcal{D}),$$

where for each pair of ∞-categories $\mathcal{C}, \mathcal{C}'$, $\mathrm{Fun}^R(\mathcal{C}, \mathcal{C}')$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by those functors that admit a left adjoint.

Let $p : \mathcal{E} \to \mathcal{C}$ be a presentable fibration. A commuting diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{C} & \end{tikzcd}$$

of ∞-categories where r is a categorical fibration *exhibits r as a stable envelope of p* if the following conditions are satisfied.

(1) q is a presentable fibration.

(2) r sends q-Cartesian edges to p-Cartesian edges.

(3) For each $C \in \mathcal{C}$, the induced map $\mathcal{E}'_C \to \mathcal{E}_C$ on the fibres exhibits $\mathcal{E}'_C$ as a stable envelope of $\mathcal{E}_C$.

Remark 4.3.0.6. A stable envelope of a presentable ∞-category $\mathcal{C}$ is determined up to equivalence, and unsurprisingly, the stabilization $\mathcal{S}\mathrm{p}(\mathcal{C})$ is a stable envelope. Indeed, the stabilization of a presentable ∞-category is presentable, and the suspension spectrum functor Σ^∞_+ is a left adjoint to the functor $\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$. The universal property follows at once from [Lur17a], prop. 1.4.4.5. It now follows from the construction of the fibrewise stabilization that $\mathcal{S}\mathrm{p}(p)$ is a stable envelope of the presentable fibration $p : \mathcal{E} \to \mathcal{C}$.

The stable envelope of a presentable fibration enjoys the following universal property ([Lur17a], prop. 7.3.1.7).

Proposition 4.3.0.7. *Let $p : \mathcal{E} \to \mathcal{C}$ be a presentable fibration, and let $r : \mathcal{E}' \to \mathcal{E}$ be a stable envelope of p. Then for each presentable fibration $\mathcal{D} \to \mathcal{C}$ whose fibres are stable, composition with r induces an equivalence*

$$\mathrm{Fun}^R_{\mathcal{C}}(\mathcal{D}, \mathcal{E}') \xrightarrow{\simeq} \mathrm{Fun}^R_{\mathcal{C}}(\mathcal{D}, \mathcal{E}),$$

where $\mathrm{Fun}^R_{\mathcal{C}}(\mathcal{D}, \mathcal{E}')$ denotes the full subcategory spanned by those functors $\mathcal{D} \to \mathcal{E}'$ over $\mathcal{C}$ that preserve Cartesian edges and admit a left adjoint on each fibre, and similarly for $\mathrm{Fun}^R_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$.

4.3.1 Comparisons of notions of modules

The goal of this subsection is to prove that the rather abstractly defined stable ∞-category $\mathsf{Mod}_A = \mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{/A})$ is equivalent to the ∞-category of A^{alg}-modules, which admits much more concrete descriptions. This is familiar in the 1-categorical setting, where we have an equivalence $\mathsf{Mod}^\heartsuit_{A^{\mathrm{alg}}} \simeq \mathsf{Ab}(C^\infty\mathsf{ring}_{/A})$, where the left hand side is the category of abelian group objects of $C^\infty\mathsf{ring}_{/A}$. The equivalence is exhibited by sending an A^{alg}-module to the *trivial square zero extension of A by M*. This C^∞-ring has as underlying $\mathbb{R}$-module the object $A \oplus M$, and the C^∞-operations are the more or less obvious ones that satisfy the requirement that on M, the kernel of the augmentation morphism $A \oplus M \to A$, they be square zero. We proved the following in chapter 1.

Lemma 4.3.1.1. *Let T be a multisorted Lawvere theory, let A be a simplicial T-algebra, and consider the stabilization $\mathcal{S}\mathrm{p}(s\mathrm{T}\mathsf{Alg}_{/A})$ endowed with the accessible t-structure of remark 2.1.4.9. There is a canonical equivalence*

$$\mathcal{S}\mathrm{p}(s\mathrm{T}\mathsf{Alg}_{/A})^\heartsuit \simeq \mathsf{Ab}(\tau_{\leq 0} s\mathrm{T}\mathsf{Alg}_{/\pi_0(A)})$$

of categories.

We will construct an ∞-categorical equivalence between $\mathsf{Mod}_{A^{\mathrm{alg}}}$ and $\mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{/A})$ using the calculus of functors of [Lur17a], chapter 6 via the functor taking the free C^∞-ring of an A-module and the functor taking the augmentation ideal of a simplicial C^∞-ring over A. Along the way, we find a very concrete model for the cotangent complex, which will be the central object of study in the next section.
In this section we will write the suspension functor on $(s\mathsf{Cring}_{\mathbb{R}})_{B//B}$ as Σ_B and the loop functor as Ω_B. The forgetful functor $(\_)^{\mathrm{alg}} : sC^\infty\mathsf{ring}_{A//A} \longrightarrow (s\mathsf{Cring}_{\mathbb{R}})_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$ has a left adjoint that we denote

$$F^{C^\infty}_A : (s\mathsf{Cring}_{\mathbb{R}})_{A^{\mathrm{alg}}//A^{\mathrm{alg}}} \longrightarrow sC^\infty\mathsf{ring}_{A//A},$$

which is given by the composition

$$(s\mathsf{Cring}_{\mathbb{R}})_{A^{\mathrm{alg}}//A^{\mathrm{alg}}} \xrightarrow{F^{C^\infty}} sC^\infty\mathsf{ring}_{F(A^{\mathrm{alg}})//F(A^{\mathrm{alg}})} \xrightarrow{\epsilon_!} sC^\infty\mathsf{ring}_{A//A}$$

where F^{C^∞} is the left adjoint of $(\_)^{\mathrm{alg}} : sC^\infty\mathsf{ring} \to s\mathsf{Alg}_{\mathbb{R}}$ and $\epsilon_!$ is the functor taking pushouts along the counit map $\epsilon : F(A^{\mathrm{alg}}) \to A^{\mathrm{alg}}$ ([Lur17b], prop 5.2.5.1). The following result is straightforward using unramifiedness and the Barr-Beck theorem.

Proposition 4.3.1.2. *Let A be a simplicial C^∞-ring, then the functor*

$$(\_)^{\mathrm{alg}} : sC^\infty\mathsf{ring}_{A//A} \longrightarrow (s\mathsf{Cring}_{\mathbb{R}})_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$$

induces an equivalence

$$\mathcal{S}\mathrm{p}\left(sC^\infty\mathsf{ring}_{A//A}\right) \simeq \mathcal{S}\mathrm{p}\left((s\mathsf{Cring}_{\mathbb{R}})_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}\right)$$

Proof. Using [Lur17a], cor. 6.2.2.17, it suffices to show that $(\_)^{\mathrm{alg}}$ is conservative and preserves sifted colimits (so that $(\_)^{\mathrm{alg}}$ exhibits $sC^\infty\mathsf{ring}_{A//A}$ as monadic over $(s\mathsf{Cring}_{\mathbb{R}})_{A^{\mathrm{alg}}//A^{\mathrm{alg}}}$) and that the unit map induces an equivalence $\partial\,\mathrm{id} \to \partial((\_)^{\mathrm{alg}} \circ F_A^{C^\infty})$. The first two assertions are immediate using [Lur17b], lem 1.2.13.8, lemma 4.1.1.29 and the fact that sifted ∞-categories are weakly contractible. To prove the last assertion, we note that it suffices to show that the unit induces an equivalence on the essential image of $\Sigma_{A^{\mathrm{alg}}}$, since the unit map will then induce equivalences

$$\Omega^i_{A^{\mathrm{alg}}} \circ \Sigma^i_{A^{\mathrm{alg}}} \simeq \Omega^i_{A^{\mathrm{alg}}} \circ (\_)^{\mathrm{alg}} \circ F_A^{C^\infty} \circ \Sigma^i_{A^{\mathrm{alg}}}$$

for all $i \geq 1$, which proves the proposition. We may write any simplicial commutative A^{alg}-algebra as a good A^{alg}-cell object: a sequential colimit

$$A^{\mathrm{alg}} = A^{\mathrm{alg}}_{-1} \longrightarrow A^{\mathrm{alg}} \otimes_{\mathbb{R}} \mathrm{Sym}^\bullet(V_{-1}) = A^{\mathrm{alg}}_0 \longrightarrow A^{\mathrm{alg}}_1 \longrightarrow A^{\mathrm{alg}}_2 \longrightarrow \ldots$$

of pushouts of maps of the form

$$\varphi_k : A^{\mathrm{alg}} \otimes_{\mathbb{R}} \Sigma^k_{\mathbb{R}}\mathrm{Sym}^\bullet(V_k) \longrightarrow A^{\mathrm{alg}}$$

for V_k a vector space and $k \geq 0$. Since $\Sigma_{A^{\mathrm{alg}}}$ preserves colimits, the functor $\_ \otimes_{\mathbb{R}} A^{\mathrm{alg}}$ intertwines $\Sigma_{\mathbb{R}}$ and $\Sigma_{A^{\mathrm{alg}}}$ which implies that the essential image of $\Sigma_{A^{\mathrm{alg}}}$ consists of good A^{alg}-cell objects with $A^{\mathrm{alg}}_0 = A^{\mathrm{alg}}_{-1} = A^{\mathrm{alg}}$ and $A^{\mathrm{alg}}_1 = A^{\mathrm{alg}} \otimes \Sigma_{\mathbb{R}}\mathrm{Sym}^\bullet(V)$. Sequential colimits are preserved by $F_A^{C^\infty}$ and $(\_)^{\mathrm{alg}}$, and the functor $F_A^{C^\infty}$ takes the map φ_k to the map

$$A \otimes^\infty \Sigma^k C^\infty(V^\vee) \longrightarrow A$$

which is an effective epimorphism, so pushouts along this map are also preserved by $(\_)^{\mathrm{alg}}$ by unramifiedness. It follows that we need only check that the unit is an equivalence on objects of the form $A^{\mathrm{alg}} \otimes_{\mathbb{R}} \Sigma^k_{\mathbb{R}}\mathrm{Sym}^\bullet(V)$ for $k \geq 1$. This follows from lemma 4.1.3.38 and unramifiedness applied to the effective epimorphism $\mathbb{R} \to \Sigma^k C^\infty(V^\vee)$. □

We have equivalences $(s\mathsf{Cring}_{\mathbb{R}})_{A//A} \simeq (\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}})_{A//A} \simeq (\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_A)^{\mathrm{aug}}$ so that we have the A-augmentation ideal functor

$$I_A : (s\mathsf{Cring}_{\mathbb{R}})_{A//A} \longrightarrow \mathsf{Mod}^{\mathrm{cn}}_A$$

with left adjoint $\mathrm{Sym}^\bullet_A$.

Proposition 4.3.1.3. *Let A be a simplicial commutative $\mathbb{R}$-algebra, then the functor*

$$I_A : (s\mathsf{Cring}_{\mathbb{R}})_{A//A} \longrightarrow \mathsf{Mod}^{\mathrm{cn}}_A$$

induces an equivalence

$$\mathcal{S}\mathrm{p}\left((s\mathsf{Cring}_{\mathbb{R}})_{A//A}\right) \simeq \mathcal{S}\mathrm{p}\left(\mathsf{Mod}^{\mathrm{cn}}_A\right)$$

Proof. Using [Lur17a], cor. 6.2.2.17, it suffices to show that I_A is conservative and preserves sifted colimits and that the unit map induces an equivalence $\partial\,\mathrm{id} \to \partial(I_A \circ \mathrm{Sym}^\bullet_A)$. Since the inclusions $(\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}})_{A//A} \to (\mathbb{E}_\infty\mathsf{Alg}_{\mathbb{R}})_{A//A}$ and $\mathsf{Mod}^{\mathrm{cn}}_A \hookrightarrow \mathsf{Mod}_A$ are conservative and preserve colimits, it suffices to check that the functor

$$I_A : \mathbb{E}_\infty\mathsf{Alg}^{\mathrm{aug}}_A \longrightarrow \mathsf{Mod}_A$$

(the augmentation ideal functor on possibly nonconnective A-augmented A-algebras) is conservative and preserves sifted colimits. This follows from [Lur17a], lem. 7.3.4.12, since the functor I_A is the composition of the forgetful functor $(\mathbb{E}_\infty\mathsf{Alg})^{\mathrm{aug}}_A \to \mathsf{Mod}_A$, which is conservative and preserves sifted colimits, with the pullback functor along the unit map $0 \to A$, which is also conservative and preserves sifted colimits by [Lur17a], lem. 7.3.4.11. To prove that the unit induces an equivalence on derivatives, we recall that the unit map is given by

$$\mathrm{id} = \mathrm{Sym}^1_A \longrightarrow \coprod_{n\geq 1} \mathrm{Sym}^n_A.$$

[Lur17a], lem. 7.3.4.8 implies that natural transformation $\coprod_{n\geq 1}\partial\mathrm{Sym}^n_A \to \partial\coprod_{n\geq 1}\mathrm{Sym}^1_A$ is an equivalence, so we only have to show that $\partial\mathrm{Sym}^n_A$ nullhomotopic for $n \geq 2$. Using the explicit colimit formula for the derivative of a reduced functor, we need only show that the functor $\Omega^i \circ \mathrm{Sym}^n_A \circ \Sigma^i$ takes values in i-connective modules for $n \geq 2$, but this functor clearly takes values in $i(n-1)$-connective modules. □

Corollary 4.3.1.4. *Let A be a simplicial C^∞-ring. For any morphism of simplicial C^∞-rings $B \to A$, there is a canonical equivalence of ∞-categories*

$$\mathsf{Mod}_{A^{\mathrm{alg}}} \simeq \mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{B//A})$$

Proof. Follows from propositions 4.3.1.2 and 4.3.1.3, together with the right completeness of the t-structure on $\mathsf{Mod}_{A^{\mathrm{alg}}}$ and the equivalence $\mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{S}\mathrm{p}(\mathcal{C}_{C/})$ for any ∞-category $\mathcal{C}$ with finite limits and any object $C \in \mathcal{C}$. □

Remark 4.3.1.5. Let A be a simplicial C^∞-ring, and let M be an A-module, which we can identify with a chain complex of A^{alg}-modules via corollary 4.3.1.4. We call the object $\Omega^\infty_A(M)$ the *trivial square zero extension of A by M*. The chain rule gives a (homotopy) commuting diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{A//A}) \arrow[r, "\partial(I_{A^{\text{alg}}}\circ(_)^{\text{alg}})"] \arrow[d, "\Omega^\infty_A"] & \mathcal{S}\mathrm{p}(\mathsf{Mod}^{\text{cn}}_{A^{\text{alg}}}) \arrow[d, "\Omega^\infty"] \\
sC^\infty\mathsf{ring}_{A//A} \arrow[r, "I_{A^{\text{alg}}}\circ(_)^{\text{alg}}"] & \mathsf{Mod}^{\text{cn}}_{A^{\text{alg}}}
\end{tikzcd}$$

for which the lower horizontal functor fits into a split fibre sequence

$$I_{A^{\text{alg}}}\circ(\_)^{\text{alg}} \longrightarrow \rho_{A^{\text{alg}}}\circ(\_)^{\text{alg}} \longrightarrow \underline{A^{\text{alg}}}$$

where $\underline{A^{\text{alg}}}$ is the constant functor on the free A^{alg}-module and

$$\rho_{A^{\text{alg}}}:(s\mathsf{Cring}_{\mathbb{R}})_{A^{\text{alg}}//A^{\text{alg}}} \longrightarrow (\mathsf{Mod}_{A^{\text{alg}}})_{/A^{\text{alg}}}$$

is the underlying A^{alg}-module functor. Using the commuting diagram above and the fact that the functor $\Omega^\infty:\mathcal{S}\mathrm{p}(\mathsf{Mod}^{\text{cn}}_{A^{\text{alg}}})\to\mathsf{Mod}^{\text{cn}}_{A^{\text{alg}}}$ is identified with the connective cover functor, we deduce that the functor $\rho_{A^{\text{alg}}}\circ(\_)^{\text{alg}}\circ\Omega^\infty_A:\mathsf{Mod}_A\to\mathsf{Mod}_A$ fits into a split fibre sequence

$$\tau_{\geq 0}\longrightarrow \rho_{A^{\text{alg}}}\circ(\_)^{\text{alg}}\circ\Omega^\infty_A\longrightarrow\underline{A^{\text{alg}}}$$

so $\Omega^\infty_A(M)\simeq A\oplus\tau_{\geq0}M$ as A-modules. The underlying $\mathbb{E}_\infty$-algebra of the square zero extension is the *algebraic* square zero extension of A^{alg} by M, and the algebra structure on $A\oplus\tau_{\geq0}M$ is described as in [Lur17a], rmk. 7.3.4.16: the multiplication map $m:(A\oplus\tau_{\geq0}M)\otimes(A\oplus\tau_{\geq0}M)\to A\oplus\tau_{\geq0}M$ is the multiplication on A on the summand $A\otimes A$, the A-action on $\tau_{\geq0}M$ on the summand $A\otimes\tau_{\geq0}M$, and nullhomotopic on the summand $\tau_{\geq0}M\otimes\tau_{\geq0}M$; this description also determines the graded $\mathbb{R}$-algebra structure on the graded $\mathbb{R}$-module $\pi_*(A\oplus\tau_{\geq0}M)\cong\pi_*(A)\oplus\pi_*(\tau_{\geq0}M)$.

Remark 4.3.1.6. The analysis of the previous remark shows that the composition $\Omega^\infty_A|_{\mathsf{Mod}^{\text{cn}}_A}:\mathsf{Mod}^{\text{cn}}_A\to sC^\infty\mathsf{ring}_{A//A}\to\mathsf{Mod}^{\text{cn}}_A$ is equivalent to the coproduct $\mathrm{id}\oplus\underline{A}$, where $\underline{A}$ is the constant functor on A. This functor obviously preserves colimits, and the functor $sC^\infty\mathsf{ring}_{A//A}\to\mathsf{Mod}^{\text{cn}}_A$ is conservative and preserves sifted colimits, so we deduce that the functor $\Omega^\infty_A|_{\mathsf{Mod}^{\text{cn}}_A}$ also preserves sifted colimits.

4.3.2 Tangent categories

The purpose of this section is to study *sheaves of modules* on derived C^∞-schemes. Clearly, this will require an understanding of the functoriality of the assignment

$$A\longmapsto\mathsf{Mod}_A,$$

for A a simplicial C^∞-ring. The description $\mathsf{Mod}_A:=\mathcal{S}\mathrm{p}(sC^\infty\mathsf{ring}_{/A})$ immediately suggests a way to achieve this: we could stabilize the fibration $\mathrm{ev}_1:\mathrm{Fun}(\Delta^1,sC^\infty\mathsf{ring})\to sC^\infty\mathsf{ring}$ fibrewise; this yields a presentable fibration $T_{sC^\infty\mathsf{ring}}\to sC^\infty\mathsf{ring}$ over $sC^\infty\mathsf{ring}$, the *tangent category* of $sC^\infty\mathsf{ring}$, the straightening of which yields a functor

$$sC^\infty\mathsf{ring}\longrightarrow\mathsf{Pr}^{\mathrm{L}},\quad A\longmapsto\mathsf{Mod}_A.$$

Alternatively, the inclusion of ∞-operads $\mathrm{Comm}^\otimes\hookrightarrow\mathrm{MComm}^\otimes$ yields a coCartesian fibration $p_M:\mathsf{Mod}_{\text{alg}}\to\mathbb{E}_\infty\mathsf{Alg}_{\mathbb{R}}$ which we may pull back along the functor

$$sC^\infty\mathsf{ring}\xrightarrow{(\_)^{\text{alg}}}s\mathsf{Cring}_{\mathbb{R}}\simeq\mathbb{E}_\infty\mathsf{Alg}^{\text{cn}}_{\mathbb{R}}\hookrightarrow\mathbb{E}_\infty\mathsf{Alg}_{\mathbb{R}}.$$

The resulting coCartesian fibration $p_M:\mathsf{Mod}\to sC^\infty\mathsf{ring}$ yields a second functor $\mathsf{Mod}_{\_}:sC^\infty\mathsf{ring}\to\mathsf{Pr}^{\mathrm{L}}$ which coincides with the straightening of the tangent category on objects, by corollary 4.3.1.4. The goal of this subsection is to show that the fibrations $\mathsf{Mod}\to sC^\infty\mathsf{ring}$ and $T_{sC^\infty\mathsf{ring}}\to sC^\infty\mathsf{ring}$ are equivalent, and deduce some elementary consequences of this parametrized stabilization construction.

Definition 4.3.2.1. Let $\mathcal{C}$ be a presentable ∞-category , then a commuting diagram

$$\begin{tikzcd}
T_{\mathcal{C}} \arrow[rr, "f"] \arrow[dr] & & \mathrm{Fun}(\Delta^1,\mathcal{C}) \arrow[dl, "\mathrm{ev}_1"] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $T_{\mathcal{C}}$ as a tangent category of $\mathcal{C}$ if the commuting diagram exhibits $T_{\mathcal{C}}$ as a stable envelope of ev_1.

Remark 4.3.2.2. Let $\mathcal{C}$ be a presentable ∞-category, and let $P_*(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those maps $\sigma : \Delta^2 \to \mathcal{C}$ such that $\sigma|_{\Delta^{\{0,2\}}}$ is an equivalence. The projection $p_* : P_*(\mathcal{C}) \to \mathcal{C}$ given by evaluating at $\{2\}$ is a presentable fibration whose fibre over an object $A \in \mathcal{C}$ is equivalent to $\mathcal{C}^{A//A}$. We call P_* a *pointed envelope* of $\mathcal{C}$. The map $\mathrm{ev}_{\Delta^{\{1,2\}}} : P_*(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ over $\mathcal{C}$ preserves Cartesian edges and therefore induces a map $\mathcal{S}\mathrm{p}(p_*) \to T_{\mathcal{C}}$ over $\mathcal{C}$ preserving Cartesian edges. Fibrewise, this map is an equivalence, induced by the map $\mathcal{C}^{A//A} \to \mathcal{C}^{/A}$. Invoking [Lur17b], cor. 2.4.4.4, we have an equivalence $\mathcal{S}\mathrm{p}(p_*) \to T_{\mathcal{C}}$.

Lemma 4.3.2.3. *There is a canonical equivalence* $T_{sC^\infty\mathsf{ring}} \simeq sC^\infty\mathsf{ring} \times_{s\mathsf{Cring}_{\mathbb{R}}} T_{s\mathsf{Cring}_{\mathbb{R}}}$

Proof. For any functor $f : \mathcal{C} \to \mathcal{D}$ and any presentable fibration $p : \mathcal{E} \to \mathcal{D}$, there is an equivalence $\mathcal{S}\mathrm{p}(p) \times_{\mathcal{D}} \mathcal{C} \to \mathcal{S}\mathrm{p}(f^*p)$. Now we need only remark that the functor $\mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}) \to sC^\infty\mathsf{ring} \times_{s\mathsf{Cring}_{\mathbb{R}}} \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ preserves Cartesian edges and, after stabilizing the fibres, induces the equivalence of proposition 4.3.1.2, so we conclude by invoking [Lur17b], cor. 2.4.4.4. □

Definition 4.3.2.4. The ∞-category Mod is the pullback $sC^\infty\mathsf{ring} \times_{s\mathsf{Cring}_{\mathbb{R}}} \mathsf{Mod}_{\mathrm{alg}}$. We have a presentable fibration $p_M : \mathsf{Mod} \to sC^\infty\mathsf{ring}$.

We first give criteria for the recognition of limits and colimits in Mod.

Proposition 4.3.2.5. *The following hold true.*

(1) *The functor $p_M : \mathsf{Mod} \to sC^\infty\mathsf{ring}$ carrying a pair (A, M) to A preserves all limits and colimits.*

(2) *The functor $q : \mathsf{Mod} \to \mathsf{Mod}_{\mathbb{R}}$ carrying a pair (A, M) to M preserves limits and sifted colimits.*

Proof. As p_M is a presentable fibration over a presentable base, p_M preserves all limits and colimits. The functor q factors as

$$\mathsf{Mod} \xrightarrow{\rho} \mathsf{Mod}_{\mathrm{alg}} \longrightarrow \mathsf{Mod}_{\mathbb{R}}$$

where the second functor evaluates at the colour $\mathfrak{m}$; this functor preserves limits and sifted colimits by [Lur17a], cor. 3.2.2.3 and prop. 3.2.3.1, so it suffices to show that the functor ρ preserves sifted colimits. Using the argument employed in lemma 4.1.8.29, we see that for each diagram $f : K \to \mathsf{Mod}$, the comparison map $e : \mathrm{colim}\, \rho \circ f \to \rho(\mathrm{colim}\, f)$ is a coCartesian edge of the fibration $p_M : \mathsf{Mod}_{\mathrm{alg}} \to s\mathsf{Cring}_{\mathbb{R}}$. In case K is sifted, applying the map p_M to e yields an equivalence in $s\mathsf{Cring}_{\mathbb{R}}$ since $(\_)^{\mathrm{alg}}$ preserves sifted colimits, so we conclude that in this case e is an equivalence as well. □

Corollary 4.3.2.6. *Let $\mathsf{Mod}^{\mathrm{cn}}$ be the full subcategory spanned by objects (A, M) for which M is a connective A^{alg}-module, which admits a presentable fibration $p_M^{\mathrm{cn}} : \mathsf{Mod}^{\mathrm{cn}} \to sC^\infty\mathsf{ring}$ then $\mathsf{Mod}^{\mathrm{cn}}$ is projectively generated. Let $\mathrm{T} = \mathbf{N}(\mathsf{VectCartSp})$ be the nerve of the category of (trivial) finite rank vector bundles over Cartesian spaces with linear fibre preserving maps over smooth base maps, which is a 2-sorted Lawvere theory, then there is a canonical equivalence*

$$s\mathrm{T}\mathsf{Alg} \simeq \mathsf{Mod}^{\mathrm{cn}}.$$

Proof. We may identify $\mathsf{Mod}^{\mathrm{cn}}$ with the pullback $\mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}} \times_{s\mathsf{Cring}_{\mathbb{R}}} sC^\infty\mathsf{ring}$ where $\mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}}$ is the ∞-category $\mathsf{Alg}_{\mathrm{MComm}}(\mathsf{Mod}^{\mathrm{cn}}_{\mathbb{R}})$. It follows from proposition 4.3.2.5 that the functor $p_M \times q : \mathrm{Mod}^{\mathrm{cn}} \to sC^\infty\mathsf{ring} \times \mathsf{Mod}^{\mathrm{cn}}_{\mathbb{R}}$, which is clearly conservative, preserves limits and sifted colimits. Proposition 4.1.1.3 implies that $sC^\infty\mathsf{ring} \times \mathsf{Mod}_{\mathbb{R}}$ is projectively generated by the 2-sorted Lawvere theory $\mathbf{N}(\mathsf{CartSp}) \times \mathbf{N}(\mathsf{FMod}_{\mathbb{R}})$ where $\mathbf{N}(\mathsf{FMod}_{\mathbb{R}})$ is the nerve of the category of finite dimensional real vector spaces, so it follows from [Lur17a], prop 7.1.4.12 that $\mathsf{Mod}^{\mathrm{cn}}$ is projectively generated by the essential image of the left adjoint of $p_M^{\mathrm{cn}} \times q$, which consists of objects of the form $(C^\infty(\mathbb{R}^n), V)$ for V a finite rank free $C^\infty(\mathbb{R}^n)$-module, which is readily identified with the opposite of $\mathbf{N}(\mathsf{VectCartSp})$. □

Remark 4.3.2.7. Consider the pullbacks $sC^\infty\mathsf{ring}_{pc} \times_{sC^\infty\mathsf{ring}} \mathsf{Mod}$ and $sC^\infty\mathsf{ring}_{pc} \times_{sC^\infty\mathsf{ring}} \mathsf{Mod}^{\mathrm{cn}}$, then one can also show that the $sC^\infty\mathsf{ring}_{pc} \times_{sC^\infty\mathsf{ring}} \mathsf{Mod}^{\mathrm{cn}}$ is the ∞-category of algebras of the 3-sorted Lawvere theory of finite rank vector bundles over Cartesian spaces with corners. The same remark holds mutatis mutandis for $sC^\infty\mathsf{PLog} \times_{sC^\infty\mathsf{ring}} \mathsf{Mod}^{\mathrm{cn}}$.

Theorem 4.3.2.8. *There is a canonical equivalence*

$$T_{sC^\infty\mathsf{ring}} \longrightarrow \mathsf{Mod}$$

of presentable fibrations over $sC^\infty\mathsf{ring}$.

Proof. In view of lemma 4.3.2.3, it suffices to produce an equivalence $\varphi : T_{s\mathsf{Cring}_{\mathbb{R}}} \to \mathsf{Mod}_{\mathrm{alg}}$.
As the map $p_M^{\mathrm{cn}} : \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}} \to s\mathsf{Cring}_{\mathbb{R}}$ is a presentable fibration, all p-limits exists in $\mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}}$. Consider p-limits of the following form:

$$\begin{tikzcd} \Delta^2 \arrow[r] \arrow[d, hook] & \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}} \arrow[d, "p"] \\ (\Delta^2)^{\triangleleft} \arrow[r] \arrow[ur, dotted] & s\mathsf{Cring}_{\mathbb{R}} \end{tikzcd}$$

Identify $(\Delta^2)^{\triangleleft} = \Delta^1 \times \Delta^1$ and let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}})$ spanned by those diagrams that are p-limits of their restriction to Δ^2, then it follows from (the dual of) theorem 2.1.1.4 that the projection map

$$\pi : \mathcal{E} \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \times_{\mathrm{Fun}(\Delta^2, s\mathsf{Cring}_{\mathbb{R}})} \mathrm{Fun}(\Delta^2, \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}})$$

is a trivial Kan fibration. Now consider p-left Kan extensions of the form

$$\begin{tikzcd} \Delta^{\{1,2\}} \arrow[r] \arrow[d, hook] & \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}} \arrow[d, "p"] \\ \Delta^2 \arrow[r] \arrow[ur, dotted] & s\mathsf{Cring}_{\mathbb{R}} \end{tikzcd}$$

then applying theorem 2.1.1.4 again, we deduce that the map

$$\pi' : \mathcal{E}' \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \times_{\mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})} \mathrm{Fun}(\Delta^1, \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}})$$

is a trivial Kan fibration, where $\mathcal{E}' \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}})$ is the full subcategory spanned by those commuting squares

$$\begin{tikzcd} (A, M) \arrow[r] \arrow[d] & (B, N) \arrow[d] \\ (A', M') \arrow[r] & (B', N') \end{tikzcd}$$

that are p-limit diagrams and such that (A', M') is a p-initial object in $\mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}}$, or equivalently, such that M' is a zero object in Mod_A. Now note that the functor $s : s\mathsf{Cring} \to \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}}$ sending A to the pair (A, A) yields a section of the projection

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \times_{\mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}})} \mathrm{Fun}(\Delta^1, \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}}) \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}}).$$

Let $\mathcal{D}$ denote the essential image of this section, then we have a trivial fibration $(\pi')^{-1}(\mathcal{D}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ which we can restrict further to a trivial fibration

$$\overline{\mathcal{E}} \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}) \times_{\mathrm{Fun}(\Delta^2, s\mathsf{Cring})} s\mathsf{Cring}_{\mathbb{R}} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}})$$

over the full subcategory $\overline{\mathcal{D}} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, s\mathsf{Cring}_{\mathbb{R}})$ spanned by commuting diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & B \arrow[d] \\ A \arrow[r, "\mathrm{id}"'] & A \end{tikzcd}$$

Explicitly, the ∞-category $\overline{\mathcal{E}}$ is the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}})$ spanned by those commuting diagrams

$$\begin{tikzcd} (A, M) \arrow[r] \arrow[d] & (B, B) \arrow[d] \\ (A, M') \arrow[r] & (A, A) \end{tikzcd}$$

that are p-limit diagrams and such that M' is a zero object of $\mathsf{Mod}_A^{\mathrm{cn}}$. Choose a section $r : \overline{\mathcal{D}} \to \overline{\mathcal{E}}$ of this fibration, then we have a commuting diagram

$$\begin{tikzcd} \overline{\mathcal{D}} \arrow[r, "r"] \arrow[dr] & \overline{\mathcal{E}} \arrow[r, "\mathrm{ev}_{(\{0\},\{0\})}"] \arrow[d] & \mathsf{Mod}_{\mathrm{alg}}^{\mathrm{cn}} \arrow[dl] \\ & s\mathsf{Cring}_{\mathbb{R}} & \end{tikzcd}$$

where the right horizontal functor evaluates at the top left corner of a square diagram. Note that over each simplicial commutative $\mathbb{R}$-algebra A, the fibre $\overline{\mathcal{D}}_A$ is the ∞-category $(s\mathsf{Cring}_{\mathbb{R}})^{A//A}$, and the map $(r \circ \mathrm{ev}_{(\{0\},\{0\})})_A$ on the fibres may be identified with the augmentation ideal functor I_A. The ∞-category $\overline{\mathcal{D}}$ is a pointed envelope of $s\mathsf{Cring}_{\mathbb{R}}$ in the sense of remark 4.3.2.2, so we can consider the diagram

$$\tilde{\phi} : T_{s\mathsf{Cring}_{\mathbb{R}}} \xrightarrow{\Omega^{\infty}_{*}} \overline{\mathcal{D}} \xrightarrow{r} \overline{\mathcal{E}} \longrightarrow \mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}}$$

of fibrations over $s\mathsf{Cring}_{\mathbb{R}}$. Note that Ω^{∞}_{*} preserves Cartesian edges by construction. To show that the map $r \circ \mathrm{ev}_{(\{0\},\{0\})}$ also preserves Cartesian edges, we observe that a Cartesian edge in $\overline{\mathcal{D}}$ is given by a cube

$$\begin{tikzcd}[row sep=small, column sep=small]
 & A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
A \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & B \arrow[ur] & \\
 & A' \arrow[rr] & & A' \\
A \arrow[ur] \arrow[rr] & & A \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where all the faces except the front and back ones are pullbacks. After applying the functor r and the underling $\mathbb{R}$-module functor $\mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}} \to \mathsf{Mod}_{\mathbb{R}}$, we obtain the following cube

$$\begin{tikzcd}[row sep=small, column sep=small]
 & M' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
M \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & B \arrow[ur] & \\
 & 0 \arrow[rr] & & A' \\
0 \arrow[ur] \arrow[rr] & & A \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

The right face is still a pullback, and the front and back faces are also pullbacks. It follows that the left face is a pullback as well, so we deduce that $M \to M'$ is an equivalence, which by [Lur17a], cor. 3.4.3.4 is equivalent to the assertion that the edge $(A, M) \to (A', M')$ is p_M-Cartesian in $\mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}}$. It follows from the universal property of stable envelopes that $\tilde{\phi}$ factors as in the diagram

$$\begin{tikzcd}
T_{s\mathsf{Cring}_{\mathbb{R}}} \arrow[r, "\phi"] \arrow[dr, "p_T"'] & \mathcal{S}\mathrm{p}(p^{\mathrm{cn}}_M) \arrow[r, "\Omega^{\infty}_{*}"] \arrow[d, "p"] & \mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}} \arrow[dl, "p^{\mathrm{cn}}_M"] \\
 & s\mathsf{Cring}_{\mathbb{R}} &
\end{tikzcd}$$

where ϕ sends p_T-Cartesian edges to p-Cartesian edges. By [Lur17b], cor. 2.4.4.4, ϕ is an equivalence if for each $A \in s\mathsf{Cring}_{\mathbb{R}}$, the induced map $\phi_A : \mathcal{S}\mathrm{p}((s\mathsf{Cring}_{\mathbb{R}})_{A//A}) \to \mathcal{S}\mathrm{p}(\mathsf{Mod}^{\mathrm{cn}}_A)$ is an equivalence. This is the case as the construction of the stabilized fibration $\mathcal{S}\mathrm{p}(\tilde{p}_2)$ ensures that the map ϕ_A is the Goodwillie derivative of the augmentation ideal functor which is an equivalence by proposition 4.3.1.3. We finish the proof by showing that the functor $\psi : \mathsf{Mod}_{\mathrm{alg}} \to \mathcal{S}\mathrm{p}(p^{\mathrm{cn}}_M)$, induced from the functor $\mathsf{Mod}_{\mathrm{alg}} \to \mathsf{Mod}^{\mathrm{cn}}_{\mathrm{alg}}$ (which preserves Cartesian edges and admits fibrewise left adjoints) is an equivalence. This follows again from [Lur17b], cor. 2.4.4.4 as the functor induced on the fibres in this case is the functor $\mathcal{S}\mathrm{p}(\mathsf{Mod}^{\mathrm{cn}}_A) \to \mathsf{Mod}_A$ that exhibits Mod_A as the right completion of $\mathsf{Mod}^{\mathrm{cn}}_A$. Choosing a homotopy inverse of ψ (in the coCartesian model structure on $(\mathsf{Set}_{\Delta})^{+}_{/s\mathsf{Cring}_{\mathbb{R}}}$) and composing with ϕ yields a functor

$$\varphi : T_{s\mathsf{Cring}_{\mathbb{R}}} \longrightarrow \mathsf{Mod}_{\mathrm{alg}}$$

implementing the desired equivalence. □

We may repeat the constructions of theorem 4.3.2.8 with $sC^{\infty}\mathsf{ring}$ in place of $s\mathsf{Cring}_{\mathbb{R}}$, which yields a parametrized trivial square zero extension functor $\mathsf{Mod} \simeq T_{sC^{\infty}\mathsf{ring}} \to P_{*}(sC^{\infty}\mathsf{ring}) \simeq \overline{\mathcal{D}}$.

Lemma 4.3.2.9. *The functor*

$$\mathsf{Mod}^{\mathrm{cn}} \subset T_{sC^\infty\mathsf{ring}} \xrightarrow{\Omega^\infty_*} \overline{\mathcal{D}} \overset{\mathrm{ev}_{(\{1\},\{0\})}}{\longrightarrow} sC^\infty\mathsf{ring}$$

preserves sifted colimits.

Proof. The functor $\overline{\mathcal{D}} \overset{\mathrm{ev}_{\{1\}\times\{0\}}}{\longrightarrow} sC^\infty\mathsf{ring}$ takes a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & B \arrow[d] \\ A \arrow[r, "\mathrm{id}"'] & A \end{tikzcd}$$

to the object B. Let $\theta_M : sC^\infty\mathsf{ring} \to \mathsf{Mod}_{\mathbb{R}}$ denote the underlying $\mathbb{R}$-module functor, and consider the composition

$$\overline{\mathcal{D}} \overset{\mathrm{ev}_{\{1\}\times\{0\}}}{\longrightarrow} s\mathsf{Cring}_{\mathbb{R}} \xrightarrow{\theta_M} \mathsf{Mod}_{\mathbb{R}},$$

which is equivalent to the composition

$$\overline{\mathcal{D}} \xrightarrow{r} \overline{\mathcal{E}} \overset{\mathrm{ev}_{\{1\}\times\{0\}}}{\longrightarrow} \mathsf{Mod} \xrightarrow{q} \mathsf{Mod}_{\mathbb{R}}.$$

We observe that by construction of the equivalence $T_{s\mathsf{Cring}_{\mathbb{R}}} \simeq \mathsf{Mod}_{\mathrm{alg}}$, the functor

$$\mathsf{Mod} \xrightarrow{\Omega^\infty_*} \overline{\mathcal{D}} \xrightarrow{r} \overline{\mathcal{E}} \xrightarrow{\mathrm{ev}_{(\{0\},\{0\})}} \mathsf{Mod}$$

is the one taking fibrewise connective covers, so it follows from the construction of the ∞-category $\overline{\mathcal{C}}$ that the functor $\theta_M \circ \mathrm{ev}_{(\{0\},\{1\})} \circ \Omega^\infty_*$ fits into a fibre sequence of functors

$$q \circ \tau_{\geq 0} \longrightarrow \theta_M \circ \mathrm{ev}_{(\{0\},\{1\})} \circ \Omega^\infty_* \longrightarrow \theta_M \circ p_M$$

The last map admits a section, so we find that $\theta_M \circ \mathrm{ev}_{(\{0\},\{1\})} \circ \Omega^\infty_*$ is equivalent to $q \circ \tau_{\geq 0} \oplus \theta_M \circ p_M$. It follows from proposition 4.3.2.5 that this functor restricted to $\mathsf{Mod}^{\mathrm{cn}}$ preserves sifted colimits. ☐

Invoking [Lur17b], prop. 5.5.8.15, we have the following.

Corollary 4.3.2.10. *The functor* $\mathsf{Mod}^{\mathrm{cn}} \subset T_{sC^\infty\mathsf{ring}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ *is a left Kan extension of its restriction to* $\mathbf{N}(\mathsf{VectCartSp})^{op}$.

With this result in hand, we can show that the functor $T_{sC^\infty\mathsf{ring}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ is in fact the derived functor of a right Quillen functor.

Construction 4.3.2.11. Let **Mod** be the category defined as follows.

(1) Objects are pairs (A, M) where A is a C^∞-dga and M is an $A^{\mathrm{alg}}_{\mathbf{dg}}$-module.

(2) Morphisms are pairs $(f, \alpha) : (A, M) \to (B, N)$ where $f : A \to B$ is a morphism of C^∞dga's and $\alpha : f_!(M) = M \otimes_A B \to N$ is a map of $B^{\mathrm{alg}}_{\mathbf{dg}}$-modules.

The obvious projection $\mathbf{Mod} \to C^\infty\mathbf{dga}$ is a biCartesian fibration over a presentable base with presentable fibres; it follows from the main result of [GHN15] that **Mod** is presentable. Let $\mathbf{Mod}_{\mathrm{alg}}$ be the category whose objects are pairs (A, M) where A is a nonnegatively graded cdga over $\mathbb{R}$ and M is an A-module, and morphisms are defined similarly as in $\mathbf{Mod}_{\mathrm{alg}}$, then we may identify the category **Mod** with the pullback $\mathbf{Mod}_{\mathrm{alg}} \times_{\mathbf{cdga}^{\geq 0}_{\mathbb{R}}} C^\infty\mathbf{dga}$. The forgetful functor $(\_)^{M\mathrm{alg}}_{\mathbf{dg}} : \mathbf{Mod} \to \mathbf{Mod}_{\mathrm{alg}}$ admits a left adjoint $F^{MC^\infty}_{\mathbf{dg}}$ defined by the assignment

$$(A, M) \longmapsto (F^{C^\infty}_{\mathbf{dg}}(A), M \otimes_A F^{C^\infty}_{\mathbf{dg}}(A)).$$

The category $\mathbf{Mod}_{\mathrm{alg}}$ admits a proper combinatorial model structure in which

(W) weak equivalences are maps $(A, M) \to (B, N)$ for which the map $A \to B$ is an equivalence of cdga's and the underlying map on dg $\mathbb{R}$-modules $M \to N$ is an equivalence.

(F) fibrations are maps $(A, M) \to (B, N)$ such that $A \to B$ is a fibration of cdga's and and the underlying map on dg $\mathbb{R}$-modules $M \to N$ is a fibration, i.e. both maps on underlying $\mathbb{R}$-modules are degreewise surjective.

(C) cofibrations are maps $(A, M) \to (B, N)$ such that $A \to B$ is a cofibration of cdga's and the map $M \otimes_A B \to N$ is a cofibration of B-modules.

The category **Mod** likewise admits a right proper combinatorial model structure right transferred along the adjunction

$$\mathbf{Mod}_{\mathrm{alg}} \underset{(-)^{M\,\mathrm{alg}}_{\mathbf{dg}}}{\overset{F^{MC^\infty}_{\mathbf{dg}}}{\rightleftarrows}} \mathbf{Mod}.$$

This model structure is an example of a *model fibration* in the sense of [HP14], and the existence of the desired model structure can be deduced from theorem 3.0.12 of *loc. cit.* using the classical Grothendieck construction. As the model structure on $\mathbf{Mod}_{\mathrm{alg}}$ is also obtained via the Grothendieck construction, one readily verifies that the model structure on **Mod** is right transferred from $\mathbf{Mod}_{\mathrm{alg}}$ which implies it is combinatorial, since the one on $\mathbf{Mod}_{\mathrm{alg}}$ is. There is a canonical functor $\mathbf{Mod} \to \mathsf{Mod}$ which exhibits a localization, by proposition 2.1.4 of [Hin16]. Now we define a right Quillen functor

$$\mathbf{SZ} : \mathbf{Mod}^{\geq 0} \longrightarrow \mathrm{Fun}(\Delta^1, C^\infty\mathbf{dga}),$$

where we endow the target with the projective model structure. For M a nonnegatively graded dg A-module, define a structure of a C^∞dga on the dg $\mathbb{R}$-module $A \oplus M$ as follows.

(1) The cdga structure on $A \oplus M$ is that of the square zero extension of A by M.

(2) In degree 0, the C^∞-ring structure on $A_0 \oplus M_0$ is defined as follows. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, then we have a map $f_* : A_0^n \to A_0^n$. We define $f_* : (A_0 \oplus M_0)^n \to A_0 \oplus M_0$ by setting

$$f_*((a_i, m_i)_{1\leq i\leq n}) = \left(f_*((a_i)_{1\leq i\leq n}), \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)_* (a_i) m_i \right)$$

The obvious map $A \oplus M \to A$ of C^∞dga's determines the functor $\mathbf{SZ} : \mathbf{Mod}^{\geq 0} \to \mathrm{Fun}(\Delta^1, C^\infty\mathbf{dga})$, which is clearly right Quillen (the adjoint may be constructed explicitly as in remark 4.3.2.14 or using the adjoint functor theorem as $\mathbf{SZ}$ preserves filtered colimits). We define a right Quillen functor

$$\mathbf{SZ}_{\geq 0} : \mathbf{Mod} \longrightarrow \mathrm{Fun}(\Delta^1, C^\infty\mathbf{dga})$$

as the composition of right Quillen functors $\mathbf{SZ} \circ \tau_{\geq 0}$.

Corollary 4.3.2.12. *The functor* $\mathbf{RSZ}_{\geq 0} : \mathsf{Mod} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ *is equivalent to the canonical functor* $T_{sC^\infty\mathsf{ring}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ *defining the tangent category.*

Proof. In view of corollary 4.3.2.10 and [Lur17b], prop. 5.5.8.15, it suffices to show that the functor **RSZ** preserves sifted colimits and coincides with the functor Ω^∞_* on $\mathbf{N}(\mathsf{VectCartSp})$. For the first point, it suffices to show that the composition

$$\mathsf{Mod}^{\mathrm{cn}} \xrightarrow{\mathbf{RSZ}} sC^\infty\mathsf{ring} \xrightarrow{\theta_M} \mathsf{Mod}_{\mathbb{R}}$$

preserves sifted colimits. The second functor is the right derived functor of the forgetful functor $G : C^\infty\mathbf{dga} \to \mathbf{Mod}_{\mathbb{R}}$. We have $\mathbf{R}G \circ \mathbf{RSZ} \simeq \mathbf{R}(G \circ \mathbf{SZ})$, but the composition $G \circ \mathbf{SZ}$ is the functor

$$(A, M) \longmapsto A \oplus M,$$

whose right derived functor coincides with the functor

$$q \circ \tau_{\geq 0} \oplus \theta_M \circ p_M$$

of lemma 4.3.2.9, which preserves sifted colimits by proposition 4.3.2.5. On the 0-truncated objects $\mathrm{Mod} \subset \mathsf{Mod}$, the functor $\Omega^{\infty\mathrm{alg}}_*$ coincides with the classical square zero extension functor by [Lur17a], rmk 7.3.4.16, so there is an equivalence of functors $\mathbf{SZ}^{\mathrm{alg}}|_{\mathrm{Mod}} \to \Omega^{\infty\mathrm{alg}}_*|_{\mathrm{Mod}}$. For each pair (A, M) of a C^∞-ring and a discrete module, this induces a map of C^∞-rings

$$F^{C^\infty}_A(\mathbf{SZ}(A, M)^{\mathrm{alg}}) \longrightarrow \Omega^\infty_*(A, M)$$

functorial in the pair (A, M). We also have the counit map

$$F^{C^\infty}_A(\mathbf{SZ}(A, M)^{\mathrm{alg}}) \longrightarrow \mathbf{SZ}(A, M)$$

and it suffices to show that this map is an equivalence for $(A, M) = (C^\infty(R^n), V)$ where V is a free finite rank $C^\infty(\mathbb{R}^n)$-module. Suppose V is a rank k-module, then the algebraic square zero extension $\mathbf{SZ}(C^\infty(\mathbb{R}^n), V)^{\mathrm{alg}}$ can be identified with the coproduct $C^\infty(\mathbb{R}^n) \otimes \mathbb{R}[x_1, \ldots, x_k]/(x_i x_j)_{1\leq i,j\leq k}$ and the equivalence

$$F^{C^\infty}_{C^\infty(\mathbb{R}^n)}(C^\infty(\mathbb{R}^n) \otimes \mathbb{R}[x_1, \ldots, x_k]/(x_i x_j)_{1\leq i,j\leq k}) \longrightarrow C^\infty(\mathbb{R}^n) \oplus V$$

is a consequence of Hadamard's lemma. ☐

Remark 4.3.2.13. From the argument in the previous proof, one can deduce that for each $A \in sC^\infty\mathsf{ring}$ and $M \in \mathsf{Mod}_A$, the trivial square zero extension $\Omega^\infty_A(M)$ is naturally equivalent to $F_A^{C^\infty}(\Omega^\infty_{A^{\mathrm{alg}}} M)$.

Remark 4.3.2.14. The functor $\mathbf{SZ}$ is right Quillen and thus admits a left adjoint, the *parametrized C^∞-Kähler differentials*. Let A be a C^∞dga, then we define a dg A-module Ω^1_A as the universal A-module that comes equipped with a map $d_{\mathrm{dR}} : A \to \mathbf{SZ}(\Omega^1_A)$ over A, that is, for each A-module M, composition with d_{dR} induces a bijection

$$\mathrm{Hom}_{\mathbf{Mod}_{A^{\mathrm{alg}}_{\mathbf{dg}}}}(\Omega^1_A, M) \xrightarrow{\cong} \mathrm{Hom}_{C^\infty\mathbf{dga}_{/A}}(A, \mathbf{SZ}(M)).$$

When A is a cofibrant object of the form $C^\infty(\mathbb{R}^n)[\epsilon_1, \ldots, \epsilon_k]$, with $d\epsilon_j = \sum_m f_m \epsilon_{j_m}$ then Ω^1_A is a quasi-free A-module generated by symbols $d_{\mathrm{dR}} x_i$ in degree 0 for each coordinate function $x_i : \mathbb{R}^n \to \mathbb{R}$, and $d_{\mathrm{dR}}\epsilon_i$ in degree $|\epsilon_i|$, with differential given by

$$d(a d_{\mathrm{dR}}\epsilon_j) = da\, d_{\mathrm{dR}}\epsilon_j + (-1)^{|a|} a\, d_{\mathrm{dR}}(\sum_m f_m \epsilon_{j_m}) = da\, d_{\mathrm{dR}}\epsilon_j + \sum_m (-1)^{|a|} a\, \epsilon_{j_m} d_{\mathrm{dR}} f_m + \sum_m (-1)^{|a|} a\, f_m d_{\mathrm{dR}} \epsilon_{j_m}$$

for $a \in A$ a homogeneous element, where $d_{\mathrm{dR}} f$ for f a smooth function is $\sum_i \frac{df}{dx_i} d_{\mathrm{dR}} x_i$. Now the left adjoint to $\mathbf{SZ}$ is given by the functor

$$(A \to B) \longmapsto \Omega^1_A \otimes_A B.$$

There is an analogous construction one can perform for trivial square zero extensions of positive prelog simplicial C^∞-rings and thereby also for simplicial C^∞-rings with corners.

Construction 4.3.2.15. Let $\mathsf{Mod}_{\mathsf{PLog}}$ denote the pullback $\mathsf{Mod} \times_{sC^\infty\mathsf{ring}} sC^\infty\mathsf{PLog}$ and $\mathsf{Mod}^{\mathrm{cn}}_{\mathsf{PLog}}$ the pullback $\mathsf{Mod}^{\mathrm{cn}} \times_{sC^\infty\mathsf{ring}} sC^\infty\mathsf{PLog}$, and consider the functor λ given by

$$\mathsf{Mod}_{\mathsf{PLog}} \longrightarrow \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}) \xrightarrow{(-)_{\geq 0}} \mathrm{Fun}(\Delta^1, \mathsf{sCMon})$$

carrying a triple $(A, M, N \to A_{\geq 0})$ to the map $(A \oplus \tau_{\geq 0} M)_{\geq 0} \to A_{\geq 0}$. Using the natural transformation of remark 4.1.8.19, we obtain a natural commuting diagram

$$\begin{tikzcd}
(A \oplus \tau_{\geq 0} M)_{\geq 0} \arrow[r] \arrow[d] & A_{\geq 0} \arrow[d] \\
(A \oplus \tau_{\geq 0} M)^{\mathsf{CMon}} \arrow[r] & A
\end{tikzcd}$$

of simplicial commutative monoids. It follows from the fact that the relative cotangent complex $\mathbb{L}_{\mathbb{R}_{\geq 0}/\mathbb{R}}$ vanishes that this diagram is a pullback. The underlying space of the object $(A \oplus \tau_{\geq 0} M)^{\mathsf{CMon}}$ coincides with the underlying space of $A \oplus \tau_{\geq 0} M$, which is simply the product $A \times \tau_{\geq 0} M$, so the same holds for $(A \oplus \tau_{\geq 0} M)_{\geq 0}$. The ∞-category $\mathsf{Mod}^{\mathrm{cn}}_{sC^\infty\mathsf{PLog}}$ is projectively generated by the discrete full subcategory $\mathbf{N}(\mathsf{VectCartSp}^{\triangleright}_{\mathsf{c}})$ spanned by objects of the form $(C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), V, \mathbb{Z}^k_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}))$ where V is a finitely generated and free $C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0})$-module. It follows from the previous analysis that λ carries such an object to the map

$$C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \oplus V \longrightarrow C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}).$$

We may view V as a simplicial commutative monoid by remembering the additive structure it inherits as a connective $\mathbb{R}$-module; this is simply the restriction to $\mathbf{N}(\mathsf{VectCartSp}^{\triangleright}_{\mathsf{c}})^{op}$ of a functor

$$(\_)^{\mathrm{add}} : \mathsf{Mod}^{\mathrm{cn}}_{\mathsf{PLog}} \longrightarrow \mathsf{Mod}^{\mathrm{cn}}_{\mathbb{R}} \longrightarrow s\mathsf{CMon}$$

that preserves limits and sifted colimits. Now we define a functor (of 1-categories)

$$\omega : \mathsf{VectCartSp}^{\triangleright op}_c \longrightarrow \mathrm{Fun}(\Delta^1, C^\infty\mathsf{PLog})$$

by carrying a triple $(C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), V, \mathbb{Z}^k_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}))$ to the map

$$(C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \oplus V, \mathbb{Z}^k_{\geq 0} \times V^{\mathrm{add}} \to C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \oplus V) \longrightarrow (C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}), \mathbb{Z}^k_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}))$$

where

$$\mathbb{Z}^k_{\geq 0} \times V^{\mathrm{add}} \longrightarrow C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \oplus V$$

is the coproduct of the map

$$\mathbb{Z}^k_{\geq 0} \longrightarrow C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \longrightarrow C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \oplus V$$

and the map

$$V^{\mathrm{add}} \longrightarrow C^\infty_{\geq 0}(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0}) \oplus V, \quad v \longmapsto (1, v).$$

We define a functor $\mathsf{Mod}^{\mathrm{cn}}_{\mathsf{Plog}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog})$ as an ev_1-left Kan extension

$$\begin{tikzcd}
\mathbf{N}(\mathsf{VectCartSp}_{\mathsf{c}}^{\triangleright})^{op} \arrow[r, "\omega"] \arrow[d, hook] & \mathbf{N}(\mathrm{Fun}(\Delta^1, C^\infty\mathsf{PLog})) \arrow[r, hook] & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \arrow[d, "\mathrm{ev}_1"] \\
\mathsf{Mod}^{\mathrm{cn}}_{\mathsf{Plog}} \arrow[rru, dotted] \arrow[rr] & & sC^\infty\mathsf{PLog}
\end{tikzcd}$$

obtaining a (strictly) commuting diagram

$$\begin{tikzcd}
\mathsf{Mod}^{\mathrm{cn}}_{\mathsf{Plog}} \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \arrow[dl, "\mathrm{ev}_1"] \\
& sC^\infty\mathsf{PLog} &
\end{tikzcd}$$

of ∞-categories, where the vertical maps are presentable fibrations. From the fact that weakly contractible colimits in the fibres of a presentable fibration $\mathcal{C} \to \mathcal{D}$ are equivalently colimits in $\mathcal{C}$, we deduce that the diagonal filler in the square above is also an absolute left Kan extension and thus preserves sifted colimits. We have a functor Ω^∞_{*pc} : $\mathsf{Mod}_{\mathsf{PLog}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog})$ by composing the functor just constructed with the functor $\tau_{\geq 0} : \mathsf{Mod}_{\mathsf{PLog}} \to \mathsf{Mod}^{\mathrm{cn}}_{\mathsf{PLog}}$, the relative right adjoint to the inclusion of fibrewise connective objects.

Proposition 4.3.2.16. (1) *The functor Ω^∞_{*pc} preserves limits and restricted to $\mathsf{Mod}^{\mathrm{cn}}_{\mathsf{PLog}}$ preserves also sifted colimits.*

(2) *Ω^∞_{*pc} carries Cartesian edges to Cartesian edges.*

(3) *Ω^∞_{*pc} preserves fibrewise limits and restricted to connective objects, also fibrewise sifted colimits.*

(4) *Ω^∞_{*pc} carries the full subcategory $\mathsf{Mod}_{\mathsf{Log}} = sC^\infty\mathsf{Log}\times_{sC^\infty\mathsf{ring}}\mathsf{Mod} \subset \mathsf{Mod}_{\mathsf{PLog}}$ into the full subcategory $\mathrm{Fun}(\Delta^1, sC^\infty\mathsf{Log})$ and the horizontal functor in the resulting commuting diagram*

$$\begin{tikzcd}
\mathsf{Mod}_{\mathsf{Log}} \arrow[rr, "\Omega^\infty_{*c}"] \arrow[dr] & & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{Log}) \arrow[dl, "\mathrm{ev}_1"] \\
& sC^\infty\mathsf{Log} &
\end{tikzcd}$$

preserves Cartesian edges and (fibrewise) limits and (fibrewise) filtered colimits.

Proof. (1) By construction, Ω^∞_{*pc} preserves sifted colimits restricted to fibrewise connective objects. To see that Ω^∞_{*pc} also preserves limits, it suffices to observe that the functors

$$(A, M, N \to A_{\geq 0}) \longmapsto A_{\geq 0} \oplus \tau_{\geq 0} M,$$

and

$$(A, M, N \to A_{\geq 0}) \longmapsto N \times \tau_{\geq 0} M^{\mathrm{add}},$$

preserve limits.

(2) Let $(A, M, N \to A_{\geq 0}) \to (A', M', N' \to A'_{\geq 0})$ be a Cartesian edge, which amounts to the assertion that the map $M \to M'$ is an equivalence of $\mathbb{R}$-modules. Unwinding the definitions, we are required to show that the diagram

$$\begin{tikzcd}
N \times \tau_{\geq 0} M^{\mathrm{add}} \arrow[r] \arrow[d] & N \arrow[d] \\
N' \times \tau_{\geq 0} M'^{\mathrm{add}} \arrow[r] & N'
\end{tikzcd}$$

is a pullback square, which is obvious.

(3) We are left with the case of limits, which follows from (1), (2) and assertion $(*)$ of proposition 4.1.8.32.

(4) We need only check that Ω^∞_{*pc} preserves the subcategory of logarithmic objects; the other properties follow immediately from (1), (2) and (3) and the fact that the inclusion $sC^\infty\mathsf{Log} \subset sC^\infty\mathsf{PLog}$ preserves filtered colimits. We need to show that the upper horizontal map in the pullback diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & (A_{\geq 0} \oplus \tau_{\geq 0} M)^\times \arrow[d] \\ N \times \tau_{\geq 0} M^{\mathrm{add}} \arrow[r] & A_{\geq 0} \oplus \tau_{\geq 0} M \end{tikzcd}$$

is an equivalence. Since $N \to A_{\geq 0}$ is a log structure, it suffices to show that the diagrams

$$\begin{tikzcd} A^\times_{\geq 0} \arrow[r] \arrow[d] & (A_{\geq 0} \oplus \tau_{\geq 0} M)^\times \arrow[d] \\ A_{\geq 0} \arrow[r] & A_{\geq 0} \oplus \tau_{\geq 0} M \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r] \arrow[d] & A_{\geq 0} \arrow[d] \\ N \times \tau_{\geq 0} M^{\mathrm{add}} \arrow[r] & A_{\geq 0} \oplus \tau_{\geq 0} M \end{tikzcd}$$

are pullbacks. Since the relative cotangent complex $\mathbb{L}_{\mathbb{R}_{>0}/\mathbb{R}_{\geq 0}}$ vanishes, the map $(A_{\geq 0} \oplus M)^\times \to A_{\geq 0} \oplus \tau_{\geq 0} M$ coincides with the map $A^\times \oplus \tau_{\geq 0} M \to A_{\geq 0} \oplus \tau_{\geq 0} M$, and the left square is then readily seen to be a pullback. To see that the right square is a pullback, we note that the lower horizontal map factors through $A_{\geq 0} \times \tau_{\geq 0} M^{\mathrm{add}}$, so we may assume that $N = A_{\geq 0}$. Clearly, we may also suppose that M is connective. Using the fact that $A_{\geq 0} \to A$ is an inclusion of components, we may replace $A_{\geq 0}$ by A so that we need to show that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[d] \\ A \times M^{\mathrm{add}} \arrow[r] & A \oplus M \end{tikzcd} \tag{4.8}$$

which is natural in A and M, is a pullback. First, we show that we may suppose that A is discrete. Choose a colimit diagram $f : \mathbf{N}(\mathbf{\Delta}^{op}_+) \to sC^\infty\mathsf{ring}$ carrying the cone point to A such that $f([n])$ is a (possibly infinitely generated) free C^∞-ring for $[n] \in \mathbf{N}(\mathbf{\Delta}^{op})$, and choose a Cartesian lift F as in the diagram

$$\begin{tikzcd} {[-1]} \arrow[r, "{(A,M)}"] \arrow[d, hook] & \mathsf{Mod}^{\mathrm{cn}} \arrow[d] \\ \mathbf{N}(\mathbf{\Delta}^{op}_+) \arrow[r, "f"] \arrow[ur, dotted, "F"] & sC^\infty\mathsf{ring} \end{tikzcd}$$

then it follows from proposition 4.3.2.5 that F is a colimit diagram. Amalgamating the composition of F with the functors Ω^∞_* and $(\_)^{\mathrm{add}}$, we obtain commuting square of simplicial objects

$$\begin{tikzcd} f \arrow[r] \arrow[d] & f \arrow[d] \\ f \times M^{\mathrm{add}} \arrow[r] & \Omega^\infty_* \circ F = f \oplus M. \end{tikzcd}$$

with colimit the diagram (4.8). The map $f \to f \oplus M$ is a Cartesian transformation and therefore a realization fibration, so in order to prove that (4.8) is a pullback it suffices to show that the diagram above is a pullback; that is, we may suppose that A is discrete. Now consider the diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & A \arrow[d] \\ M^{\mathrm{add}} \arrow[r] & A \times M^{\mathrm{add}} \arrow[r] & A \oplus M \end{tikzcd}$$

in which both squares are pullbacks. Since the map $Q \to A$ induces a bijection on connected components, it suffices to show that Q is discrete. This is an immediate consequence of the Mayer-Vietoris sequence associated to the left pullback.

□

4.3.3 Geometries of modules

In the previous subsections, we constructed for every simplicial C^∞-ring A a stable ∞-category of A-modules in a functorial manner. In this subsection, we replace A by $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, a local $\mathcal{G}^{\text{der}}_{\text{Diff}}$-structured ∞-topos, and Mod_A by an ∞-category $\mathsf{QCoh}_X \coloneqq \mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$. Let A be a simplicial C^∞-ring, then, just as in the discrete case (see definition-proposition 3.1.3.38), there is a spectrum functor for modules $M\mathbf{Spec}_A : \mathsf{Mod}_A \to \mathsf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}$ taking an A-module M to a sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules. Contrary to the algebraic case, this functor is *not* fully faithful; instead, it is essentially surjective. It's adjoint Γ^{Mod} is fully faithful, and the full subcategory $\Gamma^{\mathsf{Mod}}(\mathsf{QCoh}_{\mathbf{Spec}\, A}) \subset \mathsf{Mod}_A$ is strongly reflective, determining a full subcategory $\mathsf{Mod}_A^{\text{cplt}}$ of *complete modules.*

To work efficiently with the various spectrum and global sections functors for simplicial C^∞-rings and modules thereof, we find it convenient to employ the language of module geometries, following Lurie in [Lur11d].

Proposition 4.3.3.1. *Mod is compactly generated. Moreover, an object (A, M) is compact in Mod if and only if A is finitely presented and M is a perfect A-module; that is, A is compact in $sC^\infty\mathsf{ring}$ and M is compact in Mod_A.*

Proof. This proof is identical to the one of proposition 2.2.2 of [Lur11d], replacing the ∞-category of spectra with the ∞-category of $\mathbb{R}$-modules. □

Notation 4.3.3.2. We write Perf for the full subcategory spanned by compact objects in Mod. By proposition 4.3.3.1, the coCartesian fibration $p : \mathsf{Mod} \to sC^\infty\mathsf{ring}$ restricts to a coCartesian fibration $p : \mathsf{Perf} \to sC^\infty\mathsf{ring}_{\text{fp}}$. We denote $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$ for the opposite ∞-category of Perf; by taking the opposite of p, we have a Cartesian fibration $q : \mathcal{G}^{\mathsf{Mod}}_{\text{Diff}} \to \mathcal{G}^{\text{der}}_{\text{Diff}}$. Objects of $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$ will be denoted as pairs $(\operatorname{Spec} A, M)$ with $\operatorname{Spec} A \in \mathcal{G}^{\text{der}}_{\text{Diff}}$ and M a perfect A-module.

We endow $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$ with the structure of a geometry according to the following prescription:

(1) A map $f : (\operatorname{Spec} A, M) \to (\operatorname{Spec} B, N)$ is admissible if and only if f is q-Cartesian and $q(f)$ is admissible in $\mathcal{G}^{\text{der}}_{\text{Diff}}$.

(2) A collection $\{(\operatorname{Spec} B[1/b_\alpha], N_\alpha) \to (\operatorname{Spec} B, N)\}_{\alpha \in J}$ generates a covering sieve if and only if the collection $\{\operatorname{Spec} B[1/b_\alpha] \to \operatorname{Spec} B\}_{\alpha \in J}$ generates a covering sieve in $\mathcal{G}^{\text{der}}_{\text{Diff}}$.

This indeed defines a geometry by proposition 2.2.6 of [Lur11d].

Remark 4.3.3.3. Let $\mathcal{T}^{\mathsf{Mod}}_{\text{Diff}} \subset \mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$ be the discrete full subcategory spanned by objects of the form

$$(\operatorname{Spec} C^\infty(N), M)$$

where N is a manifold and M is a finitely generated projective $C^\infty(N)$-module. Note that $\mathcal{T}^{\mathsf{Mod}}_{\text{Diff}}$ is nothing but $\mathbf{N}(\text{Vect})$, the category of finite dimensional vector bundles with globally bounded rank on manifolds. We endow $\mathcal{T}^{\mathsf{Mod}}_{\text{Diff}}$ with the structure of a pregeometry as follows: a map between vector bundles $E \to U$ and $F \to N$ as in the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & F \arrow[d] \\ U \arrow[r, "f"] & N \end{tikzcd}$$

where $\tilde{f}$ is fibrewise linear, is admissible if this diagram is a pullback and $f : U \to N$ is an open embedding. Also, let $\mathcal{G}^{\mathsf{Mod}^{\text{cn}}}_{\text{Diff}} \subset \mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$ be the full subcategory spanned by pairs $(\operatorname{Spec} A, M)$ where M is a connective perfect A-module. This ∞-category inherits the structure of a geometry from $\mathcal{G}^{\text{der}}_{\text{Diff}}$ and it can be shown that inclusion $\mathcal{T}^{\mathsf{Mod}}_{\text{Diff}} \hookrightarrow \mathcal{G}^{\mathsf{Mod}^{\text{cn}}}_{\text{Diff}}$ exhibits $\mathcal{G}^{\mathsf{Mod}^{\text{cn}}}_{\text{Diff}}$ as a geometric envelope of $\mathcal{T}^{\mathsf{Mod}}_{\text{Diff}}$. The proof goes along along the lines of the one of theorem 4.1.4.6, using that $\mathcal{G}^{\mathsf{Mod}^{\text{cn}}}_{\text{Diff}}$ is the ∞-category of simplicial algebras for the Lawvere theory generated by the objects $(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n)^m)$. By remark 3.1.2.8, the category Perf^{op} from remark 3.1.3.39 consisting of pairs $(\operatorname{Spec} A, M)$ where A is a C^∞-ring and M a (discrete, not differentially graded) A-module of finite presentation, is a 0-truncated geometric envelope of $\mathcal{T}^{\mathsf{Mod}}_{\text{Diff}}$. We leave the details of the proof to the motivated reader, since we won't need these results.

Recall that a $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$-structure on an ∞-topos $\mathcal{X}$ can be canonically identified with an $\text{Ind}((\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}})^{op})$-valued sheaf on $\mathcal{X}$. As Mod is compactly generated, a $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$-structure on $\mathcal{X}$ is precisely a Mod-valued sheaf on $\mathcal{X}$. Let $\mathsf{RingTop}_{\mathsf{Mod}}$ be the ∞-category of (possibly non-local) $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$-structured ∞-topoi (this is the same thing as the ∞-category ${}^{\text{R}}\mathsf{Top}((\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}})_{disc})$ of *local* $(\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}})_{disc}$-structured ∞-topoi, where $(\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}})_{disc}$ is the discrete geometry underlying $\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}$), and let $\mathsf{RingTop}_{\text{d}C^\infty}$ be the ∞-category of (possibly non-local) $\mathcal{G}^{\text{der}}_{\text{Diff}}$-structured ∞-topoi.

Proposition 4.3.3.4. *The ∞-category ${}^{\text{R}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}})$ fits into a pullback diagram*

$$\begin{tikzcd} {}^{\text{R}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\text{Diff}}) \arrow[r] \arrow[d, "\tilde{q}"] & \mathsf{RingTop}_{\mathsf{Mod}} \arrow[d, "q"] \\ {}^{\text{R}}\mathsf{Top}(\mathcal{G}^{\text{der}}_{\text{Diff}}) \arrow[r] & \mathsf{RingTop}_{\text{d}C^\infty} \end{tikzcd}$$

of ∞-categories, where the right vertical map is the obvious forgetful functor and the lower horizontal map is the inclusion of the subcategory of local objects and local morphisms. Moreover, the left vertical map $\tilde{q}$ is a Cartesian fibration.

Proof. We claim that it suffices to show that the map q is a Cartesian fibration. Supposing for a moment that we have verified this, it then follows that $\tilde{q}$ is also a Cartesian fibration and that the pullback of the diagram above is a pullback of simplicial sets. Then the ∞-category ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})\times_{\mathsf{RingTop}_{\mathrm{d}C^\infty}}\mathsf{RingTop}_{\mathsf{Mod}}$ is the subcategory of $\mathsf{RingTop}_{\mathsf{Mod}}$ of those objects and morphisms that lie in ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ after applying the forgetful functor to $\mathsf{RingTop}_{\mathrm{d}C^\infty}$. Differently put, ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})\times_{\mathsf{RingTop}_{\mathrm{d}C^\infty}}\mathsf{RingTop}_{\mathsf{Mod}}\subset\mathsf{RingTop}_{\mathsf{Mod}}$ is the subcategory whose

(*a*) objects are triples $(\mathcal{X},\mathcal{O}_{\mathcal{X}},F)$ such that $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is a local $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structure.

(*b*) morphism are those $\alpha:(\mathcal{X},\mathcal{O}_{\mathcal{X}},F)\to(\mathcal{Y},\mathcal{O}'_{\mathcal{Y}},F')$ such that the underlying morphism $f^*\circ\mathcal{O}_{\mathcal{X}}\to\mathcal{O}_{\mathcal{Y}}$ is a local morphism of $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structures on $\mathcal{Y}$.

Proposition 2.2.7 of [Lur11d] now shows that ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})\times_{\mathsf{RingTop}_{\mathrm{d}C^\infty}}\mathsf{RingTop}_{\mathsf{Mod}}\subset\mathsf{RingTop}_{\mathsf{Mod}}$ coincides with ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}})$. Now we prove that q is Cartesian: it suffices to show that the conditions of lemma 1.4.14 of [Lur09] are satisfied for the triangle of ∞-categories

$$\begin{tikzcd}
\mathsf{RingTop}_{\mathsf{Mod}} \arrow[rr, "q"] \arrow[dr, "p"'] & & \mathsf{RingTop}_{\mathrm{d}C^\infty} \arrow[dl, "p'"] \\
& {}^{\mathrm{R}}\mathsf{Top} &
\end{tikzcd}$$

Clearly, q is an inner fibration. For any ∞-topos, the induced map on the fibre is identified with the map $q_{\mathcal{X}}:\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X})^{op}\to\mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X})^{op}$, which is a Cartesian fibration. We are are required to show that for each geometric morphism $f^*:\mathcal{X}\to\mathcal{Y}$, the induced functor $\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X})\to\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{Y})$ carries $q_{\mathcal{X}}$-coCartesian morphisms to $q_{\mathcal{Y}}$-coCartesian morphisms. Using [Lur17a], prop. 4.6.2.17, this amounts to the following assertion: suppose that $(\mathcal{O}_{\mathcal{X}},\mathcal{F})\to(\mathcal{O}'_{\mathcal{X}},\mathcal{F}')$ is a morphism in $\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X})$ such that the induced map

$$\mathcal{F}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}'_{\mathcal{X}}\longrightarrow\mathcal{F}'$$

of sheaves of $\mathbb{R}$-modules on $\mathcal{X}$ is an equivalence, then the map

$$f^*(\mathcal{F})\otimes_{f^*(\mathcal{O}_{\mathcal{X}})}f^*(\mathcal{O}'_{\mathcal{X}})\longrightarrow f^*(\mathcal{F}')$$

is an equivalence, which follows immediately from the fact that $f^*:\mathsf{Shv}_{\mathsf{Mod}_{\mathbb{R}}}(\mathcal{X})\to\mathsf{Shv}_{\mathsf{Mod}_{\mathbb{R}}}(\mathcal{Y})$ is symmetric monoidal and preserves colimits. □

In view of the proposition above, we will identify the objects of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}})$ with triples $(\mathcal{X},\mathcal{O}_{\mathcal{X}},F)$, with $\mathcal{X}$ an ∞-topos, $\mathcal{O}_{\mathcal{X}}$ a sheaf of local simplicial C^∞-rings on $\mathcal{X}$ and F a sheaf of $\mathcal{O}$-modules.

Definition 4.3.3.5. Let us write $\tilde{q}:{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}})\to{}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ for the projection of proposition 4.3.3.4. Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured ∞-topos, then we denote the fibre $\tilde{q}^{-1}(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ by $\mathsf{Mod}_{\mathcal{O}_{\mathcal{X}}}$ and we call it *the ∞-category of $\mathcal{O}_{\mathcal{X}}$-modules.* For $\mathfrak{X}=(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured ∞-topos, we will also use the notation $\mathsf{QCoh}_{\mathfrak{X}}$ for $\mathsf{Mod}_{\mathcal{O}_{\mathcal{X}}}$.

Remark 4.3.3.6. Note that for an ∞-topos $\mathcal{X}$, we have a diagram of ∞-categories

$$\begin{tikzcd}
\mathsf{Mod}_{\mathcal{O}_{\mathcal{X}}} \arrow[r] \arrow[d] & \mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X}) \arrow[r] \arrow[d] & \mathsf{RingTop}_{\mathsf{Mod}} \arrow[d] \\
* \arrow[r, "\mathcal{O}_{\mathcal{X}}"] & \mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X}) \arrow[r] \arrow[d] & \mathsf{RingTop}_{\mathrm{d}C^\infty} \arrow[d] \\
& * \arrow[r, "\mathcal{X}"] & {}^{\mathrm{R}}\mathsf{Top}
\end{tikzcd}$$

where all squares are pullbacks. It follows that $\mathsf{Mod}_{\mathcal{O}_{\mathcal{X}}}$ is the fibre at $\mathcal{O}_{\mathcal{X}}$ of the presentable fibration $\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X})\to\mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X})$.

Remark 4.3.3.7. Since the map $\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X})\to\mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}}}(\mathcal{X})$ is isomorphic to $\mathsf{Mod}(\mathsf{Shv}_{\mathsf{Mod}_{\mathbb{R}}}(\mathcal{X}))\to\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}(\mathsf{Shv}_{\mathsf{Mod}_{\mathbb{R}}}(\mathcal{X}))$, we may also identify both maps with the tangent category $T_{\mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}}}(\mathcal{X})}\to\mathsf{Shv}_{\mathbb{E}_\infty\mathsf{Alg}^{\mathrm{cn}}_{\mathbb{R}}}(\mathcal{X})$. It follows the map $\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X})\to\mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X})$ may also be identified with the tangent category.

The following is just a restatement of proposition 2.2.5.26.

Proposition 4.3.3.8. *Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}$-structured ∞-topos.*

(1) *The ∞-category $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ is stable and presentable. Moreover, $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ admits an accessible* t*-structure.*

(2) *The forgetful functor $\theta : \mathsf{Mod}_{\mathcal{O}_\mathcal{X}} \to \mathsf{Shv}_{\mathsf{Mod}_\mathbb{R}}(\mathcal{X})$ is conservative and preserves small limits and colimits.*

(3) *The forgetful functor θ is* t*-exact, and the* t*-structure on $\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}$ of point (1) can be identified with the pair $(\theta^{-1}(\mathsf{Shv}_{\mathsf{Mod}_\mathbb{R}}(\mathcal{X})_{\leq 0}), \theta^{-1}(\mathsf{Shv}_{\mathsf{Mod}_\mathbb{R}}(\mathcal{X})_{\geq 0}))$.*

(4) *The* t*-structure of point (1) is right complete.*

(5) *Suppose that $\mathcal{X}$ is hypercomplete, then the* t*-structure of point (1) is left complete.*

Remark 4.3.3.9. The coCartesian fibration $\tilde{q} : {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}) \to {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})$ is a presentable fibration: it follows from the description of coCartesian edges of $\tilde{q}$ that the coCartesian pushforward determined by a map $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ can be identified with the composition

$$\mathsf{Mod}_{\mathcal{O}_\mathcal{X}} \xrightarrow{f^*} \mathsf{Mod}_{f^*\mathcal{O}_\mathcal{X}} \xrightarrow{-\otimes_{f^*\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{Y}} \mathsf{Mod}_{\mathcal{O}_\mathcal{Y}}.$$

The second map clearly preserves colimits, and the first map can be identified with the fibre at $\mathcal{O}_\mathcal{X}$ of the diagram

$$\begin{tikzcd}
\mathsf{Shv}_{\mathsf{Mod}}(\mathcal{X}) \arrow[r] \arrow[d, "q_\mathcal{X}"] & \mathsf{Shv}_{\mathsf{Mod}}(\mathcal{Y}) \arrow[d, "q_\mathcal{X}"] \\
\mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X}) \arrow[r] & \mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{Y})
\end{tikzcd}$$

induced by f^*. It suffices to show that the upper horizontal map carries $q_\mathcal{X}$-colimits to $q_\mathcal{Y}$-colimits, but this follows because f^* preserves colimits and carries $q_\mathcal{X}$-coCartesian edges to $q_\mathcal{Y}$-coCartesian edges. As a result, the functor $\tilde{q}$ admits a left adjoint, a section which carries each pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ to the triple $(\mathcal{X}, \mathcal{O}_\mathcal{X}, 0)$, which is the initial object in the fibre. It follows from [Lur17b], rmk. 5.2.6.4 that the functor $\tilde{q}$ is the map induced by composition with the transformation of geometries

$$s_0 : \mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}} \longrightarrow \mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}},$$

which carries A to the pair $(A, 0)$, so we find that the section described above coincides with the relative spectrum $\mathbf{Spec}^{\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}}_{\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}}}$.

Proposition 4.3.3.10. *The global sections functor $\Gamma^{\mathsf{Mod}} : {}^{R}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}) \to \mathsf{Mod}^{op}$ admits a right adjoint $\mathbf{Spec}^{\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}}$.*

Proof. This is construction 3.1.1.1 for the geometry $\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}$ together with proposition 3.1.1.2. ☐

Lemma 4.3.3.11. *Let A be a fair simplicial C^∞-ring, then the following diagram*

$$\begin{tikzcd}
\mathsf{Mod}_{\mathcal{O}_{\mathcal{S}\mathrm{p}\,A}} \arrow[r, "\Gamma"] \arrow[d, "\pi_n"] & \mathsf{Mod}_A \arrow[d, "\pi_n"] \\
\mathsf{Mod}^\heartsuit_{\pi_0(\mathcal{O}_{\mathbf{Spec}\,A})} \arrow[r, "\Gamma"] & \mathsf{Mod}^\heartsuit_{\pi_0(A)}
\end{tikzcd}$$

which commutes up to homotopy in virtue of proposition 2.2.5.37 is Γ-left adjointable.

Proof. This is proven exactly as in lemma 4.1.3.27. ☐

Proposition 4.3.3.12. *The unit of the adjunction $\mathrm{id} \to \mathbf{Spec}^{\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}} \circ \Gamma^{\mathsf{Mod}}$ is an equivalence when restricted to the full subcategory of ${}^{\mathrm{R}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}})$ spanned by objects $(\mathcal{X}, \mathcal{O}_\mathcal{X}, \mathcal{F}_M)$, where $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an affine fair derived C^∞-scheme.*

Proof. By theorem 4.1.3.22, the unit induces an equivalence

$$(\mathcal{X}, \mathcal{O}_\mathcal{X}) \xrightarrow{\simeq} (\mathrm{Spec}\,\Gamma(\mathcal{X}), \mathcal{O}_{\mathrm{Spec}\,\Gamma(\mathcal{X})}),$$

so, given a sheaf $\mathcal{F}$ of $\mathcal{O}_\mathcal{X}$-modules, it suffices to show that the natural map $\epsilon : M\mathbf{Spec}_A\,\Gamma(\mathcal{F}) \to \mathcal{F}$ of sheaves of $\mathcal{O}_\mathcal{X}$-modules is an equivalence. As the t-structure on $\mathsf{Shv}_{\mathsf{Mod}_{\mathcal{O}_\mathcal{X}}}(\mathcal{X})$ is left and right complete, it suffices to show that ϵ induces an equivalence on all sheaves of homotopy groups. We should show that the canonical map

$$\pi_n(M\mathbf{Spec}_A\,\Gamma(\mathcal{F})) \longrightarrow \pi_n(\mathcal{F})$$

is an equivalence, but, as the square of lemma 4.3.3.11 is left adjointable, the map above is equivalent to the counit map

$$M\mathbf{Spec}_{\pi_0(A)}\,\Gamma(\pi_n\mathcal{F}) \longrightarrow \pi_n(\mathcal{F})$$

which is an equivalence by proposition 5.20 of [Joy12a]. ☐

The previous proposition states that a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules for $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ an affine derived manifold can always be retrieved as the spectrum of its global sections.

Corollary 4.3.3.13. *Let A be a fair simplicial C^∞-ring. The full subcategory $\Gamma^{\mathsf{Mod}}(\mathsf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}) \subset \mathsf{Mod}_A$ is strongly reflective.*

Definition 4.3.3.14. Let A be a fair simplicial C^∞-ring. The stable presentable ∞-category of *complete modules* is $\Gamma^{\mathsf{Mod}}(\mathsf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}})$. We denote it $\mathsf{Mod}_A^{\mathrm{cplt}}$.

Remark 4.3.3.15. Let A be a fair simplicial C^∞-ring. The stable ∞-category $\mathsf{Mod}_A^{\mathrm{cplt}}$ inherits a t-structure from $\mathsf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}$ via the spectrum-global sections equivalence. Both the inclusion functor $\mathsf{Mod}_A^{\mathrm{cplt}} \to \mathsf{Mod}_A$ and the localization functor $\mathsf{Mod}_A \to \mathsf{Mod}_A^{\mathrm{cplt}}$ are t-exact. It follows that the t-structure on $\mathsf{Mod}_A^{\mathrm{cplt}}$ is simply given by $(\mathsf{Mod}_A^{\mathrm{cplt}} \cap \mathsf{Mod}_A^{\leq 0}, \mathsf{Mod}_A^{\mathrm{cplt}} \cap \mathsf{Mod}_A^{\geq 0})$. We can also conclude that the heart $\mathsf{QCoh}^\heartsuit(A)$ can be identified with the abelian category of complete $\pi_0(A)$-modules.

Proposition 4.3.3.16. *Let A be a fair simplicial C^∞-ring, then an A-module M is complete if and only if $\pi_n(M)$ is a complete $\pi_0(A)$-module for all $n \in \mathbb{Z}$.*

Proof. Suppose M is complete. The functor $\mathsf{QCoh}_A \hookrightarrow \mathsf{Mod}_A \xrightarrow{\pi_n} \mathrm{Mod}_{\pi_0(A)}$ coincides with the functor $\pi_n : \mathsf{QCoh}_A \to \mathsf{QCoh}_A^\heartsuit$, and we know that $\mathsf{QCoh}_A^\heartsuit$ is the abelian category of complete $\pi_0(A)$-modules. Conversely, suppose that $\pi_n(M)$ is a complete $\pi_0(A)$-module for all $n \in \mathbb{Z}$. We should verify that the map $\eta : M \to \Gamma(M\mathbf{Spec}_A M)$ is an equivalence. Since the t-structure on Mod_A is left and right complete, it suffices to show that η induces an equivalence $\eta : \pi_n M \to \pi_n \Gamma(M\mathbf{Spec}_A M)$ for each $n \in \mathbb{Z}$. Since the square of lemma 4.3.3.11 is left adjointable, this map is equivalent to the unit map $\pi_n M \to \Gamma M\mathbf{Spec}_{\pi_0(A)} \pi_n(M)$, which is an equivalence because $\pi_n M$ is complete. □

To show that there is always a good supply of complete modules, we recall the following definition. Let A be an $\mathbb{E}_\infty$-algebra, then an A-module M is *almost perfect* if $M \in \mathsf{Mod}_A^{\geq k}$ for some $k \leq 0$ and M is almost compact as an object of $\mathsf{Mod}_A^{\geq k}$.

Proposition 4.3.3.17. *Let A be a fair simplicial C^∞-ring. If M is an almost perfect A-module, then M is complete.*

Proof. Let M be an almost perfect A-module. Since $\mathsf{Mod}_A^{\mathrm{cplt}} \subset \mathsf{Mod}_A$ is a stable full subcategory and M is eventually connective, we may assume that M is connective. By proposition 4.3.3.16, M is complete if and only if $\tau_{\leq n} M$ is complete for all $n \geq 0$, and by proposition 4.3.3.13, the full subcategory $\mathsf{Mod}_A^{\mathrm{cplt}} \subset \mathsf{Mod}_A$ is stable under retracts. Since there exists for each $n \geq 0$ a perfect connective A-module M' such that $\tau_{\leq n} M' \simeq \tau_{\leq n} M$, we may assume that M is perfect. To show that M is complete, it suffices to prove that $\mathsf{Mod}_A^{\mathrm{cplt}} \subset \mathsf{Mod}_A$ is a stable full subcategory containing A that is closed under retracts. The only nonobvious thing is the verification that $A \in \mathsf{Mod}_A^{\mathrm{cplt}}$, but by proposition 4.3.3.16, A is itself a complete module over A because A is fair. □

It follows from proposition 4.3.3.4 and remark 4.3.3.6 that the restriction ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathsf{Mod}}) \times_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})} \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}^{op} \to \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}^{op}$ is a presentable fibration. Unstraightening this fibration, we obtain a functor

$$\mathsf{QCoh} : \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}^{op} \longrightarrow \mathsf{Pr}^{\mathrm{L}}.$$

Theorem 4.3.3.18. *The functor QCoh is a sheaf of presentable ∞-categories on $\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}$ for the étale topology.*

Proof. Using remark 3.2.1.22, it suffices to show that for each $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}}$, the pullback $\phi^*_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\mathsf{QCoh} : \mathcal{X}^{op} \to \mathsf{Pr}^{\mathrm{L}}$ is a sheaf. The functor

$${}^{\mathrm{L}}\mathsf{Top} \longrightarrow \mathsf{Pr}^{\mathrm{L}}$$

obtained by unstraightening the coCartesian fibration $\mathsf{RingTop}_{\mathsf{Mod}}^{op} \to {}^{\mathrm{L}}\mathsf{Top}$ can be identified with the functor

$${}^{\mathrm{L}}\mathsf{Top} \xrightarrow{\mathrm{Fun}^{\mathbf{R}}(\mathsf{Mod}^{op}, \_)} \mathsf{Pr}^{\mathrm{L}}$$

which is simply the Lurie tensor product of presentable ∞-categories with Mod, which preserves colimits separately in each variable, and similarly unstraightening the fibration $\mathsf{RingTop}_{\mathsf{d}C^\infty} \to {}^{\mathrm{L}}\mathsf{Top}$ produces the colimit preserving functor

$$\_ \otimes sC^\infty\mathsf{ring} : {}^{\mathrm{L}}\mathsf{Top} \longrightarrow \mathsf{Pr}^{\mathrm{L}}.$$

The functor $\phi^*_{(\mathcal{X},\mathcal{O}_{\mathcal{X}})}\mathsf{QCoh}$ is obtained as the unstraightening of the functor

$$\mathcal{X}^{op} \times_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathrm{der}})} {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}_{\mathrm{Diff}}^{\mathsf{Mod}}) \longrightarrow \mathcal{X}^{op}$$

but this functor fits into a pullback diagram

$$\begin{tikzcd} \mathcal{X}^{op} \times_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})} {}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}) \arrow[r] \arrow[d] & \mathcal{X}^{op} \times_{{}^{\mathrm{L}}\mathsf{Top}} \mathsf{RingTop}^{op}_{\mathsf{Mod}} \arrow[d] \\ \mathcal{X}^{op} \arrow[r] & \mathcal{X}^{op} \times_{{}^{\mathrm{L}}\mathsf{Top}} \mathsf{RingTop}^{op}_{\mathsf{d}C^\infty} \end{tikzcd}$$

of coCartesian fibrations over $\mathcal{X}^{op}$. It is now sufficient to show that the functors $\mathcal{X}^{op} \to \mathsf{Pr}^{\mathrm{L}}$ given by

$$U \longmapsto \mathcal{X}_{/U} \otimes sC^\infty\mathsf{ring}, \quad U \longmapsto \mathcal{X}_{/U} \otimes \mathsf{Mod}$$

preserve limits. Since the functor $\mathcal{X}_{/U} \to \mathcal{X}_{/V}$ induced by a map $V \to U$ also admits a left adjoint given by Weil restriction, we may consider both functors as taking values in Pr^{R}. Then we need to show that the functors $\mathcal{X} \to \mathsf{Pr}^{\mathrm{L}}$ on opposite categories preserves colimits, but these functors are compositions of the opposite of the functor $\mathcal{X}^{op} \to \mathsf{Pr}^{\mathrm{R}}$, $U \mapsto \mathcal{X}_{/U}$ which preserves limits by descent, and the functors $\_ \otimes sC^\infty\mathsf{ring}$ and $\_ \otimes \mathsf{Mod}$, which preserve colimits. □

Remark 4.3.3.19. By right Kan extending QCoh along the Yoneda embedding, we have a functor $\mathsf{d}C^\infty\mathsf{St}^{op} \to \mathsf{Pr}^{\mathrm{R}}$ which we abusively also denote QCoh. For X a derived stack, we call QCoh_X the ∞*-category of quasi-coherent sheaves on* X. By definition of the right Kan extension, we have

$$\mathsf{QCoh}_X = \lim_{\mathbf{Spec}\,A \to X \in (\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}})^{op}_{X/}} \mathsf{QCoh}(\mathbf{Spec}\,A). \tag{4.9}$$

An object in the limit is a Cartesian section of the Cartesian fibration classified by the diagram $(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}})^{op}_{X/} \to \mathsf{Pr}^{\mathrm{L}}$, that is, the data of a complete module M_A for each $\mathbf{Spec}\,A \in \mathsf{d}C^\infty\mathsf{Aff}$ together with, for each homotopy commutative diagram

$$\begin{tikzcd} \mathbf{Spec}\,A \arrow[rr, "f"] \arrow[dr] & & \mathbf{Spec}\,B \arrow[dl] \\ & X & \end{tikzcd}$$

of derived stacks an equivalence $f^*M_B \simeq M_A$, and these equivalences are themselves compatible up to coherent higher homotopies. The previous theorem implies that $\mathsf{QCoh} : \mathsf{d}C^\infty\mathsf{St}^{op} \to \mathsf{Pr}^{\mathrm{R}}$ takes colimits of derived stacks to limits of ∞-categories, so for many stacks, QCoh admits a simpler description than the formula (4.9). In particular, if a derived n-Artin stack X is represented by a derived Lie n-groupoid X_*, we have an equivalence

$$\mathsf{QCoh}_X \xrightarrow{\simeq} \lim_{\mathbf{N}(\Delta)} \mathsf{QCoh}(X_*).$$

We note that by theorem 4.3.3.18, we have given *two* definitions of the ∞-category of quasi-coherent sheaves on a derived Deligne-Mumford C^∞-stack X: viewing X as a structured topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, we have the ∞-category $\mathsf{Mod}_{\mathcal{O}_{\mathcal{X}}}$ of definition 4.3.3.14, and, viewing X as a sheaf on $\mathsf{d}C^\infty\mathsf{Aff}$ via the functor j_{Sch}, we have the ∞-category $\mathsf{QCoh}(X)$ of remark 4.3.3.19. These two ∞-categories can be canonically identified, via the following analogue of proposition 2.7.18 of [Lur11d].

Proposition 4.3.3.20. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived Deligne-Mumford C^∞-stack and denote by $X = j_{\mathrm{Sch}}(\mathfrak{X})$ the associated sheaf, then there is a canonical equivalence $\mathsf{QCoh}(X) \simeq \mathsf{Mod}_{\mathcal{O}_{\mathcal{X}}}$.*

Proof. The proof of theorem 4.3.3.18 applies to show that for any $n \geq 0$, the functor

$$\mathsf{QCoh} : \mathsf{DMSt}_n^{op} \longrightarrow \mathsf{Pr}^{\mathrm{L}}$$

obtained by unstraightening the fibration ${}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathsf{Mod}}_{\mathrm{Diff}}) \times_{{}^{\mathrm{L}}\mathsf{Top}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Diff}})} \mathsf{DMSt}_n^{op}$ is a sheaf. Since this functor restricts to the functor QCoh already defined on affines, we conclude by invoking the equivalence $\mathsf{Shv}(\mathsf{DMSt}_n) \simeq \mathsf{Shv}(\mathsf{d}C^\infty\mathsf{Aff}_{\mathrm{fair}})$. □

4.3.4 Local properties of quasi-coherent modules

Here we introduce of variety of subclasses of quasi-coherent sheaves.

Remark 4.3.4.1. Recall that a connective module M of a connective $\mathbb{E}_\infty$-ring A is called *strong* if the natural maps $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_n(M)$ are isomorphisms for all $n \geq 0$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules for $\mathcal{O}_{\mathcal{X}}$ a sheaf of connective $\mathbb{E}_\infty$-rings, we say that $\mathcal{F}$ is strong if the map $\pi_n(\mathcal{O}_{\mathcal{X}}) \otimes_{\pi_0(\mathcal{O}_{\mathcal{X}})} \pi_0(\mathcal{F}) \to \pi_n(\mathcal{F})$ is an isomorphism in $\tau_{\leq 0}\mathcal{X}$.

Definition 4.3.4.2. Let $\mathbf{Spec}\,A$ be an affine fair derived C^∞-scheme and M a complete A-module.

(1) M is *n-connective* if $\pi_k(M) = 0$ for $k < n$.

(2) M is *eventually connective* if there exists some $n << 0$ such that M is n-connective.

(3) M is *n-truncated* if $\pi_k(M) = 0$ for $k > n$.

(4) M is *truncated* if there exists some $n >> 0$ such that M is n-truncated.

(5) M *has* Tor*-amplitude in* $[n, m]$ for $n \leq m$ if for every discrete A-module N, the homotopy group $\pi_i(M \otimes_A N)$ vanishes if i does not lie in the interval $[n, m]$.

(6) M is *flat* if the sheaf of modules $\mathcal{F}_M$ associated to M is a flat $\mathcal{O}_{\mathbf{Spec}\,A}$-module; that is, $\mathcal{F}_M$ is strong and $\pi_0(\mathcal{F}_M)$ is a flat $\pi_0(\mathcal{O}_{\mathbf{Spec}\,A})$-module.

(7) M is *dualizable* if the sheaf of modules $\mathcal{F}_M$ associated to M is a dualizable $\mathcal{O}_{\mathbf{Spec}\,A}$-module.

(8) M is *locally projective* if M is connective (that is, 0-connective) and there is an admissible covering $\{\iota_\alpha : U_\alpha \to \mathbf{Spec}\,A\}$ such that $\iota_\alpha^* M$ is a projective object of $\mathsf{Mod}^{\leq 0}_{\mathcal{O}_{U_\alpha}}$, that is, the functor $\mathsf{Mod}^{\leq 0}_{\mathcal{O}_{U_\alpha}} \to \mathcal{S}$ corepresented by $\iota_\alpha^* MM$ preserves geometric realizations.

(9) M is a *vector bundle locally of finite rank* if there is an admissible covering $\{\iota_\alpha : U_\alpha \to \mathbf{Spec}\,A\}$ such that $\iota_\alpha^* M$ is a free rank n $\mathcal{O}_{U_\alpha}$-module for some $n < \infty$. The full subcategory spanned by vector bundles locally of finite rank is denoted $\mathsf{Vect}(A)$.

(10) M is *locally perfect* if there is an admissible covering $\{\iota_\alpha : U_\alpha \to \mathbf{Spec}\,A\}$ such that $\iota_\alpha^* M$ is perfect in $\mathsf{Mod}_{\mathcal{O}_{U_\alpha}}$. The full subcategory of Mod_A spanned by locally perfect A-modules is denoted $\mathsf{Perf}(A)$.

Remark 4.3.4.3. Heuristically, if a property P on quasi-coherent modules is defined in terms of the vanishing of certain homotopy groups, then it will be local for the étale topology. On the other hand, if a property comes as some sort of finiteness condition, we have to sheafify.

Definition 4.3.4.4. Let P be a property for complete modules. We say that the property P is *stable under base change* if the following condition holds.

($*$) If (A, M) has the property P and $A \to B$ is a map of fair simplicial C^∞-rings, then $M \otimes_A M$ has the property P.

We say that P is *local for the étale topology* if the following conditions hold.

(1) If (A, M) has the property P and $f : A \to B$ is an admissible map, then $(B, B \otimes_A M)$ has the property P.

(2) If $\{\mathbf{Spec}\,A_i \to \mathbf{Spec}\,A\}$ is an admissible covering, M an A-module, and for each i, $A_i \otimes_A M$ has the property P, then M has the property P.

Proposition 4.3.4.5. *Let P be a property P for complete modules that is stable under base change and local for the étale topology. Let $\mathsf{QCoh}_P \subset \mathsf{QCoh}$ be the full subfunctor spanned by modules that have the property P, then QCoh_P is a subsheaf.*

Proof. Let $\{f_i : U_i \to \mathbf{Spec}\,A\}$ be an admissible covering of an affine fair derived C^∞-scheme and let

$$h : \mathbf{N}(\mathbf{\Delta}^{op}) \longrightarrow \mathrm{d}C^\infty\mathrm{Sch}_{\mathrm{fair}}$$

be the Čech nerve of the map $\coprod U_i \to \mathbf{Spec}\,A$, then it follows from theorem 4.3.3.18 and [Lur17b], cor. 3.3.3.3 that we may identify the ∞-category $\mathsf{Mod}_A^{\mathrm{cplt}}$ with the ∞-category of coCartesian sections of the coCartesian fibration $\mathsf{QCoh} \times_{\mathrm{d}C^\infty\mathrm{Sch}^{op}_{\mathrm{fair}}} \mathbf{N}(\mathbf{\Delta})$. Since the collection of fully faithful functors is stable under limits, we can identify the limit of the functor

$$U_i \times_{\mathbf{Spec}\,A} \ldots \times_{\mathbf{Spec}\,A} U_j \longmapsto \mathsf{QCoh}_P(U_i \times_{\mathbf{Spec}\,A} \ldots \times_{\mathbf{Spec}\,A} U_j)$$

with the full subcategory of $\mathsf{Mod}_A^{\mathrm{cplt}}$ spanned by modules M for which $f_i^* M$ has the property P for each i, but by locality of P, this is precisely the subcategory of modules that have the property P. ☐

Remark 4.3.4.6. The inclusion $\widehat{\mathcal{S}} \subset \widehat{\mathsf{Cat}}_\infty$ admits a right adjoint denoted $(\_)^\simeq$ taking the maximal subgroupoid. Suppose that P is a property of complete modules stable under base change and local for the étale topology such that QCoh_P has essentially small fibres, then $\mathsf{QCoh}_P^\simeq$ determines an object in $\mathsf{d}C^\infty\mathsf{St}_{\mathrm{fair}}$.

Proposition 4.3.4.7. *All the properties of quasi-coherent modules of definition 4.3.4.2 are local for the étale topology. All the properties* except the ones of being (n)-truncated *are stable under base change.*

Proof. The properties defined by the vanishing of certain sheaves of homotopy groups are local as the homotopy groups of complete modules are complete. Only the property of being dualizable requires proof. This is an immediate consequence of [Lur17a], prop. 4.6.1.11. □

Definition 4.3.4.8. Let A be a fair simplicial C^∞-ring, and let M be a finitely generated A-module. For each $x : \mathbb{R}$-point $A \to \mathbb{R}$, the *rank of* M *at* x is the dimension of the $\mathbb{R}$-module $\pi_0(M \otimes_A \mathbb{R})$.

Remark 4.3.4.9. For a general locally finitely generated $M \in \mathsf{Mod}_A$, the rank function $\mathrm{rk}_{\pi_0(M)} : \operatorname{Spec} A \to \mathbb{N}$ is upper-semicontinuous, and locally constant if M is locally free (i.e. if M is a vector bundle). If a locally free module M has constant finite rank k, it is called a *rank* k *vector bundle*. The full subcategory of $\mathsf{Mod}_A^{\mathrm{cplt}}$ spanned by rank k vector bundles is denoted $\mathsf{Vect}_k(A)$. The property of being a rank k vector bundle is clearly stable under base change and local for the étale topology.

Proposition 4.3.4.10 (Serre-Swan)**.** *Let A be a fair simplicial C^∞-ring, and let M be an A-module. M is locally finitely generated and locally projective if and only if M is a vector bundle locally of finite rank.*

Proof. M is locally finitely generated and locally projective if and only if $\pi_0(M)$ is locally finitely generated and locally projective over $\pi_0(A)$ and M is strong. Fix an $\mathbb{R}$-point $* \to \mathbf{Spec}\, A$, and let n be the dimension of the real vector space $\pi_0(M) \otimes_{\pi_0(A)} \mathbb{R}$, which is finite because $\pi_0(M) \otimes_{\pi_0(A)} \pi_0(A)[1/a]$ is finitely presented over $\pi_0(A[1/a])$, for some a such that $x(a) \neq 0$. Nakayama's lemma implies that $\pi_0(M)_x$, the stalk at X of the module spectrum of $\pi_0(M)$, is free of rank n as a module over the local C^∞-ring $\pi_0(X)_x$, so after localizing to a neighbourhood of x, $\pi_0(M)$ is free. Now we conclude, since a connective module N over a connective $\mathbb{E}_\infty$-ring is free if and only if N is strong and $\pi_0(N)$ is free.
For the converse, the problem is local for the étale topology. Thus, we may suppose that M is a trivial vector bundle, in which case the result is obvious. □

Proposition 4.3.4.11 (Dualizable is locally perfect)**.** *Let A be a fair simplicial C^∞-ring and let M be a complete A-module, then M is dualizable if only if M is locally perfect.*

Proof. First suppose that M is locally perfect. Since dualizability is a local property, we may assume that M is perfect, in which case the result follows form [Lur17a], prop. 7.2.4.4. Conversely, if M is dualizable, then for each real point $x : A \to \mathbb{R}$, the module M_x is a dualizable A_x-module, where A_x is the fair simplicial C^∞-ring of germs at x. Since A_x is a ring of germs, every module arises as the global sections of its associated sheaf of modules, so we have $\mathsf{Mod}_{A_x}^{\mathrm{cplt}} \simeq \mathsf{Mod}_{A_x}$. It follows that M_x is dualizable as an object in the symmetric monoidal ∞-category Mod_{A_x}; invoking [Lur17a], prop. 7.2.4.4, we deduce that M_x is perfect, but this implies that there is some $a \in A$ such that $x(A) \neq 0$ and $A[a^{-1}] \otimes_A M$ is perfect. □

Remark 4.3.4.12. Let X be a derived C^∞-stack, then proposition 4.3.4.5 we have a full subcategory $\mathsf{QCoh}_X^{\geq 0} \subset \mathsf{QCoh}_X$ of connective objects. Since the inclusion $\mathsf{Mod}_A^{\mathrm{cplt}\geq 0} \subset \mathsf{Mod}_A$ is a morphism in Pr^{L}, so is the functor $\mathsf{QCoh}_X^{\geq 0} \subset \mathsf{QCoh}_X$. This full subcategory is closed under extensions and thus determines an accessible t-structure on QCoh_X by [Lur17a], prop 1.4.4.11. If $X = j_{\mathrm{Sch}}(\mathcal{X}, \mathcal{O}_\mathcal{X})$ for some derived Deligne-Mumford C^∞-stack $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, then this t-structure coincides with the one constructed in proposition 4.3.3.8, so the coconnective objects of QCoh_X coincide with the truncated objects. If X is an Artin stack however, we do not know whether an object $\mathcal{F} \in \mathsf{QCoh}_X$ lies in $\mathsf{QCoh}_X^{\leq 0}$ if and only if for each map $f : \mathbf{Spec}\, A \to X$, the pullback $f^*\mathcal{F}$ is 0-truncated; this is essentially equivalent to the flatness of submersions.

In [Lur17a], cor. 7.2.2.19, it is proven that pulling back 0-equivalences of $\mathbb{E}_1$-rings induces an equivalence on the homotopy categories of projective modules. Along the same lines, we have the following vector bundle extension lemma.

Lemma 4.3.4.13. *Let $f : A \to B$ be an effective epimorphism of fair simplicial C^∞-ring, then the functor $f_! := \_ \otimes_A B$ induces a full functor*

$$h\mathsf{Vect}(A) \longrightarrow h\mathsf{Vect}(B).$$

Moreover, for each $P \in h\mathsf{Vect}(B)$, there is a localization $A \to A[1/a]$ and some $P' \in h\mathsf{Vect}(A)$ such that $f_! P' \cong P$.

Proof. First, we show that the functor is full. Let M be a vector bundle on $\mathbf{Spec}\, A$ and write $N = f_!(M)$, then we should show that the map

$$\mathrm{Ext}^0_A(M, M) \longrightarrow \mathrm{Ext}^0_B(N, N)$$

is a surjection. We may assume that M is a free rank k module, in which case the map above can be identified with the surjection $\pi_0(A)^{k^2} \to \pi_0(B)^{k^2}$. Now choose a finitely generated projective B-module P, then we will show that after localizing near $\mathbf{Spec}\,A$, we can find a finitely generated projective module that pulls back to P. We may choose a free rank k B-module N and an idempotent $e : N \to N$ such that P is the colimit of the diagram

$$N \xrightarrow{e} N \xrightarrow{e} N \ldots$$

Let M denote a free rank k A-module, then using fullness of the functor above induced by the functor $f_!$, we may choose some $\tilde{e} : N \to N$ such that $f_!(F) \simeq P$, where F is the colimit of the diagram

$$M \xrightarrow{\tilde{e}} M \xrightarrow{\tilde{e}} M \ldots$$

It remains to be shown that F is finitely generated and projective after localizing near $\mathrm{Spec}_{\mathbb{R}}\, B$. Since F is flat (as flat objects are stable under filtered colimits) and localizations are flat maps, it suffices to show that $\pi_0(F)$ is finitely generated and projective over some localization of $\pi_0(A)$. Consider the map

$$\pi_0(M) \xrightarrow{\pi_0(\tilde{e}) - \pi_0(\tilde{e}^2)} \pi_0(M),$$

then at each point of $\mathrm{Spec}_{\mathbb{R}}\, B$, pulling back this map to $\mathbb{R}$ yields the zero map because $\pi_0(e)$ is an idempotent. Since $\pi_0(M)$ is finitely generated and free, Nakayama's lemma tells us that each point $x \in \mathrm{Spec}_{\mathbb{R}}\, B$ has a neighbourhood $U_x \subset \mathrm{Spec}_{\mathbb{R}}\, A$ on which the map $\pi_0(\tilde{e})$ becomes an idempotent. Now we take $U := \bigcup_x U_x$, then U has a characteristic element $a \in \pi_0(A)$ and $F' := F \otimes_A A[1/a]$ is a retract of a free rank k module and $F' \otimes_{A[1/a]} B \simeq P$. $\square$

Chapter 5

The Cotangent Complex

To any morphism $f : A \to B$ among C^∞-rings, we may associate a *module of relative C^∞-Kähler differentials*, denoted $\Omega^1_{B/A}$, which classifies A-linear C^∞-derivations $d : B \to M$, for M a B-module, that is, we have a canonical isomorphism

$$\mathrm{Der}(A/B, M) \coloneqq \mathrm{Hom}_{C^\infty\mathsf{ring}_{A//B}}(B, B \oplus^\infty M) \simeq \mathrm{Hom}_{\mathrm{Mod}_B}(\Omega^1_{B/A}, M),$$

where $B \oplus^\infty M$ denotes the square-zero extension equipped with its canonical structure of a C^∞-ring. Taking C^∞-derivations here is crucial: the usual *algebraic* module of relative Kähler differentials $(\Omega^1_{B^{\mathrm{alg}}/A^{\mathrm{alg}}})^{\mathrm{alg}}$ of f^{alg} is far too large, blind as it is to relations between elements involving smooth functions that cannot be reduced to regular functions. It can be shown that $(\Omega^1_{C^\infty(\mathbb{R}^n)})^{\mathrm{alg}}$ is uncountably generated, while $\Omega^1_{C^\infty(\mathbb{R}^n)}$ is free on n generators. In certain cases, this difference disappears however. If $f : A \to B$ is a surjection, dual to a closed immersion of affine C^∞-schemes, the module of relative C^∞-Kähler differentials vanishes (as do the relative algebraic Kähler differentials), and we will show that the map

$$I/I^2 \longrightarrow A \otimes_B \Omega^1_A, \qquad [f] \longmapsto 1 \otimes d_{\mathrm{dR}} f$$

determines an exact sequence

$$I/I^2 \longrightarrow A \otimes_B \Omega^1_A \longrightarrow \Omega^1_B \longrightarrow 0,$$

which shows that the algebraic conormal module I/I^2 of f^{alg} is already the correct object from the perspective of C^∞-geometry. When passing from the classical C^∞-derivations to the derivations constructed in the previous chapter, we recover the *cotangent complex* in derived C^∞-geometry. We establish a number of properties of the assignment $(A \to B) \mapsto \mathbb{L}_{B/A}$ that practicioners of derived geometry will be familiar with. For instance, for each $n \geq 1$, there exists a derivation $d : \mathbb{L}_{\tau_{\leq n-1}A} \to \pi_n(A)[n+1]$ such that the map $\tau_{\leq n} \to \tau_{\leq n-1}A$ fits into a pullback diagram

$$\begin{tikzcd}
\tau_{\leq n}A \arrow[r] \arrow[d] & \tau_{\leq n-1}A \arrow[d, "\eta_d"] \\
\tau_{\leq n-1}A \arrow[r, "\eta_0"] & \tau_{\leq n-1}A \oplus \pi_n(A)[n+1]
\end{tikzcd}$$

The cotangent complex detects local equivalences:

Theorem (Inverse function theorem)**.** *Let $f : A \to B$ be a morphism between fair simplicial C^∞-rings such that $\pi_0(f)$ is finitely presented, then f is étale if and only if $\mathbb{L}_f$ vanishes.*

It follows from the inverse function theorem that an effective epimorphism $C^\infty(\mathbb{R}^n) \to A$ among finitely presented simplicial C^∞-rings whose cotangent complex vanishes must be an equivalence. If we were doing derived algebraic geometry over a Noetherian ring R, this continues to hold if A is only assumed to be of finite type. In this case, a finite type R-algebra with a free cotangent complex is necessarily a localization of a free and finitely generated R-algebra. In C^∞-geometry, it occurs often that an object is finitely generated but not finitely presented; when dealing with manifolds with corners, for instance. As it turns out, we can characterize to an extent the finitely generated simplicial closed C^∞-rings whose cotangent complex is free.

Theorem. *Let $f : C^\infty(\mathbb{R}^n) \to A$ be an effective epimorphism, let $I = \ker \pi_0(f)$ be jet determined and suppose that there are n closed sets $X_i \subset \mathbb{R}$ such that $Z(I) = \prod_i X_i$. Then the following are equivalent.*

(1) $\mathbb{L}_f \simeq 0$ *in* Mod_A.

(2) $I = \mathfrak{m}^\infty_{Z(I)}$ *and the unit map of the 0'th truncation*

$$A \longrightarrow \pi_0(A) \simeq C^\infty(\mathbb{R}^n)/I = C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)}$$

is an equivalence.

This result holds for all closed sets, but we will not develop the tools to prove this here. The theorem explains the role that discrete simplicial C^∞-rings of Whitney functions play in the theory: even though they are very far from being free in the ∞-category $sC^\infty\mathsf{ring}$, they are up to a topological condition (closure in the Fréchet topology) precisely the objects that are *formally smooth* if we remove the assumption of being of finite presentation. Applying the theorem to the point determined ideal of functions that vanish in some half space of $\mathbb{R}^k$, we obtain the cotangent complex of simplicial C^∞-rings of manifolds with corners.

Corollary 5.0.0.1. *Let $A = C^\infty(\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k})$ viewed as a discrete simplicial C^∞-ring. Then $\mathbb{L}_A$ is free on n generators.*

Applying the theorem to the ideal of functions that have all derivatives vanishing on a linear hyperplane, we obtain the cotangent complex for power series algebras as simplicial C^∞-rings.

Corollary 5.0.0.2. *Let $A = C^\infty(\mathbb{R}^k)[[x_1, \ldots, x_{n-k}]]$ be an algebra of power series of smooth functions, viewed as a finitely generated discrete simplicial C^∞-ring. Then $\mathbb{L}_A$ is free on n generators.*

The previous two corollaries constitute a convincing case that the somewhat abstract procedure we will engage in to define the cotangent complex of a simplicial C^∞-ring yields the correct generalization of the cotangent bundle. One of the most technically convenient corollaries of theorem 5.1.1.26 is the following.

Corollary 5.0.0.3. *Let A be the underlying simplicial C^∞-ring of an affine derived manifold with corners. Then $\mathbb{L}_A$ is perfect.*

Proof. Let $\mathcal{C}$ be the full subcategory of $sC^\infty\mathsf{ring}$ spanned by objects with perfect cotangent complex, which is stable under finite colimits and retracts. The functor

$$\mathbf{N}(\mathsf{CartSp}_c) \longrightarrow sC^\infty\mathsf{ring}^{op}, \qquad \mathbb{R}^n \times \mathbb{R}^k_{\geq 0} \longmapsto C^\infty(\mathbb{R}^n \times \mathbb{R}^k_{\geq 0})$$

preserves products by corollary 4.1.6.6. By lemma 4.1.1.20, there is an essentially unique right exact functor $sC^\infty\mathsf{ring}_c^{\mathrm{fp}} \to \mathcal{C}$ extending the one above. By lemma 4.1.1.20, the composition $sC^\infty\mathsf{ring}_c^{\mathrm{fp}} \to \mathcal{C} \to sC^\infty\mathsf{ring}$ is equivalent to the functor taking the underlying simplicial C^∞-ring of a derived manifold with corners. □

The relevance of this result lies therein that in the presence of perfection of the cotangent complex, a larger set of tools for manipulating atlases becomes available.

5.1 The Relative Cotangent Complex

Construction 5.1.0.1. Let $\mathcal{C}$ be a presentable ∞-category . Recall that we have defined the tangent category of $\mathcal{C}$ as a stable envelope of the arrow ∞-category of $\mathcal{C}$, fitting into a diagram

$$\begin{tikzcd} T_{\mathcal{C}} \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[dl, "\mathrm{ev}_{\{1\}}"] \\ & \mathcal{C} & \end{tikzcd}$$

of fibrations over $\mathcal{C}$. At each $A \in \mathcal{C}$, the fibre of the functor G at A can be identified with $\Omega^\infty_A : \mathcal{S}\mathrm{p}(\mathcal{C}_{/A}) \to \mathcal{C}_{/A}$, which admits a left adjoint $(\Sigma^\infty_+)_A$. By [Lur17a] 7.3.2.6, these left adjoints assemble into a functor F left adjoint to G. The *cotangent complex functor* is the composition

$$\mathbb{L} : \mathcal{C} \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{F} T_{\mathcal{C}},$$

where the first map is the diagonal embedding.
A diagram $\sigma : \Delta^1 \times \Delta^1 \to T_{\mathcal{C}}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a *relative cofibre sequence* if it is a p-colimit diagram and the diagram $p \circ \sigma$ factors through the projection $\Delta^1 \times \Delta^1 \to \Delta^1$ so that the vertical maps become identities. The full subcategory $\mathcal{E}$ of $\mathrm{Fun}(\Delta^1 \times \Delta^1, T_{\mathcal{C}}) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by relative cofibre sequences admits a trivial Kan fibration over $\mathrm{Fun}(\Delta^1, T_{\mathcal{C}})$ by restricting to the top morphism in the diagram. We let s be a section of this fibration, defined up to contractible ambiguity. The *relative cotangent complex functor* is the composite

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{\mathbb{L}} \mathrm{Fun}(\Delta^1, T_{\mathcal{C}}) \xrightarrow{s} \mathcal{E} \xrightarrow{\mathrm{ev}_\infty} T_{\mathcal{C}},$$

where the last morphism evaluates a relative cofibre sequence at the cocone $\{1\} \times \{1\} \in \Delta^1 \times \Delta^1$.

Definition 5.1.0.2. For $sC^\infty\mathsf{ring}$, the tangent category is the presentable fibration $p : \mathsf{Mod} \to sC^\infty\mathsf{ring}$. For A a simplicial C^∞-ring, the *cotangent complex* $\mathbb{L}_A \in \mathsf{Mod}_A \simeq T_{sC^\infty\mathsf{ring}} \times_{sC^\infty\mathsf{ring}} \{A\}$ of A is the value of the cotangent complex functor at A. For a morphism $f : A \to B$ of simplicial C^∞-rings, the *relative cotangent complex* $\mathbb{L}_f \in \mathsf{Mod}_B$ (also denoted $\mathbb{L}_{B/A}$ if the morphism is clear from the context) is the value of the relative cotangent complex functor at f.

Remark 5.1.0.3. By definition, the relative cotangent complex of a morphism $A \to B$ of simplicial C^∞-rings fits into a p-colimit diagram $\mathcal{J} : \Delta^1 \times \Delta^1 \to T_{sC^\infty\mathsf{ring}}$

$$\begin{tikzcd} \mathbb{L}_A \arrow[r] \arrow[d] & \mathbb{L}_B \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_{B/A} \end{tikzcd}$$

Denote $K^\triangleright := \Delta^1 \times \Delta^1$, and let $q : K^\triangleright \times \Delta^1 \to K^\triangleright$ be the natural transformation that collapses $K^\triangleright$ to its cocone and consider the composition

$$K^\triangleright \times \Delta^1 \xrightarrow{q} K^\triangleright \xrightarrow{\mathcal{J}} T_{sC^\infty\mathsf{ring}} \xrightarrow{p} sC^\infty\mathsf{ring}.$$

Then we have a commuting diagram

$$\begin{tikzcd} K^\triangleright \arrow[r] \arrow[d, hook] & T_{sC^\infty\mathsf{ring}} \arrow[d, "p"] \\ K^\triangleright \times \Delta^1 \arrow[r] \arrow[ur, dotted] & sC^\infty\mathsf{ring} \end{tikzcd}$$

and there is a (unique up to contractible ambiguity) dotted coCartesian lift $K^\triangleright \times \Delta^1 \to T_{sC^\infty\mathsf{ring}}$ such that $K^\triangleright \times \{0\}$ is the diagram $\mathcal{J}$. This lift exhibits a coCartesian transformation between $\mathcal{J}$ and a diagram $\mathcal{J}' : \Delta^1 \times \Delta^1 \to T_{sC^\infty\mathsf{ring}} \times_{sC^\infty\mathsf{ring}} \{B\} \simeq \mathsf{Mod}_B$. By [Lur17b], prop 4.3.1.9, the diagram $\mathcal{J}'$ is a cofibre sequence

$$\begin{tikzcd} f_!\mathbb{L}_A \arrow[r] \arrow[d] & \mathbb{L}_B \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_{B/A} \end{tikzcd}$$

in Mod_B.

Remark 5.1.0.4. In a similar vein, the same proofs of [Lur17a] prop. 7.3.3.5, cor. 7.3.3.6 and proposition 7.3.3.7 show that

(1) for a commuting triangle

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ A \arrow[ur] \arrow[rr] & & C \end{tikzcd}$$

of simplicial C^∞-rings, there is a cofibre sequence

$$\begin{tikzcd} f_!\mathbb{L}_{B/A} \arrow[r] \arrow[d] & \mathbb{L}_{C/A} \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_{C/B} \end{tikzcd}$$

in Mod_C

(2) for a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ A' \arrow[r] & B' \end{tikzcd}$$

of simplicial C^∞-rings, there is an equivalence

$$f_!\mathbb{L}_{B/A} \xrightarrow{\simeq} \mathbb{L}_{B'/A'}$$

Definition 5.1.0.5. Let A be a simplicial C^∞-ring. The functor of *($\mathbb{R}$-linear) A-derivations* is the mapping space

$$\mathrm{Der}(A,\_) \coloneqq \mathrm{Hom}_{sC^\infty\mathsf{ring}_{/A}}(A,\Omega^\infty_A(\_)) : \mathsf{Mod}_A \longrightarrow \mathcal{S}.$$

For a map $B \to A$ a map of simplicial C^∞-rings, the functor of *(B-linear) A-derivations* is the mapping space

$$\mathrm{Der}_B(A,\_) \coloneqq \mathrm{Hom}_{sC^\infty\mathsf{ring}_{B//A}}(A,\Omega^\infty_A(\_)) : \mathsf{Mod}_A \longrightarrow \mathcal{S}.$$

By definition, the cotangent complex of A corepresents A-derivations. What is not obvious from the definition, is that the relative cotangent complex of a map $B \to A$ corepresents B-linear A-derivations. Nevertheless, this is true; this assertion is an easy corollary of the following result

Lemma 5.1.0.6. *Let $f : B \to A$ be a map of simplicial C^∞-rings. The relative cotangent complex is the cotangent complex of f obtained by applying construction 5.1.0.1 to the presentable ∞-category $sC^\infty\mathsf{ring}_{B/}$.*

Proof. This is [Lur17a] prop. 7.3.3.8 and prop 7.3.3.14. □

For any map $B \to A$ of simplicial C^∞-rings, it is straightforward to characterize the functor

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}_{/A}}(B,\Omega^\infty_A(\_)) : \mathsf{Mod}_A \longrightarrow \mathcal{S}$$

in terms of the cotangent complex.

Proposition 5.1.0.7. *Let $\mathcal{C}$ be a presentable ∞-category. Then the functor $F : \mathrm{Fun}(\Delta^1,\mathcal{C}) \to T_{\mathcal{C}}$ takes a morphism $f : X \to Y$ to the object $f_!\mathbb{L}_X$.*

Proof. The functor F is a relative left adjoint to the horizontal functor in the diagram

$$\begin{tikzcd} T_{\mathcal{C}} \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathrm{Fun}(\Delta^1,\mathcal{C}) \arrow[dl, "\mathrm{ev}_{\{1\}}"] \\ & \mathcal{C} & \end{tikzcd}$$

exhibiting $T_{\mathcal{C}}$ as a stable envelope of $\mathrm{ev}_{\{1\}} : \mathrm{Fun}(\Delta^1,\mathcal{C}) \to \mathcal{C}$, so F takes $\mathrm{ev}_{\{1\}}$-coCartesian edges to p-coCartesian edges. For any morphism $f : X \to Y$, the square

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is an $\mathrm{ev}_{\{1\}}$-coCartesian morphism in $\mathrm{Fun}(\Delta^1,\mathcal{C})$, so it follows that the morphism $\mathbb{L}_X \to F(f)$ obtained as the image under F of the square above is p-coCartesian in the tangent category, and thus induces an equivalence $f_!\mathbb{L}_X \simeq F(f)$. □

Corollary 5.1.0.8. *Let $\Delta^2 \to sC^\infty\mathsf{ring}$ be a commuting triangle*

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ A \arrow[rr, "g"] & & C \end{tikzcd}$$

viewed as a morphism in $sC^\infty\mathsf{ring}_{B/}$. Then the functor

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}_{B//C}}(A,\Omega^\infty_C(\_)) : \mathsf{Mod}_C \longrightarrow \mathcal{S}$$

is corepresented by the object $g_!\mathbb{L}_{A/B}$.

Proof. By definition of the tangent category of $sC^\infty\mathsf{ring}_{B/}$ and the relative left adjoint F to $G : T_{sC^\infty\mathsf{ring}_{B/}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}_{B/})$, the functor $\mathrm{Hom}_{sC^\infty\mathsf{ring}_{B//C}}(A,\Omega^\infty_C(\_))$ is corepresented by the object $F(g)$, so we conclude by invoking proposition 5.1.0.7. □

Notation 5.1.0.9. Recall that for any fair simplicial C^∞-ring A, we have a localization $(\_)^{\mathrm{cplt}}$ denote the functor of *completion* on A-modules. Accordingly, for $f : A \to B$ a functor of simplicial C^∞-rings, we let $\mathbb{L}^{\mathrm{cplt}}_{B/A}$ denote the *complete* or *quasi-coherent cotangent complex.*

Remark 5.1.0.10. We will see later that the complete cotangent complex of a fair simplicial C^∞-ring A coincides with the cotangent complex associated to the adjoint of the infinite loop space functor $\mathcal{S}\mathrm{p}(\mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X})_{/\mathcal{O}_\mathcal{X}}) \to \mathsf{Shv}_{sC^\infty\mathsf{ring}}(\mathcal{X})_{/\mathcal{O}_\mathcal{X}}$, where $(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \mathbf{Spec}\, A$.

Notation 5.1.0.11. For $f : A \to B$ of morphism of simplicial commutative $\mathbb{R}$-algebras. We denote the relative cotangent complex of f, obtained from the presentable ∞-category $s\mathsf{Cring}_\mathbb{R}$ via construction 5.1.0.1, by $\mathbb{L}^{\mathrm{alg}}_f$, or $\mathbb{L}^{\mathrm{alg}}_{B/A}$. This cotangent complex is discussed in [TV06], and [Lur17a] sections 7.3 and 7.4.

The following result -another formal consequence of unramifiedness- underlies a number of important computations of cotangent complexes.

Proposition 5.1.0.12. *Let $f : B \to A$ be a morphism of simplicial commutative $\mathbb{R}$-algebras, then there is a canonical equivalence $\mathbb{L}^{\mathrm{alg}}_{A/B} \otimes_A F(A) \simeq \mathbb{L}_{F(A)/F(B)}$.*

Proof. Denote by $F^A : (s\mathsf{Cring}_\mathbb{R})_{A//A} \to sC^\infty\mathsf{ring}_{F(A)//F(A)}$ the functor induced by the left adjoint to $(\_)^{\mathrm{alg}}$. F^A has a right adjoint itself (given by pulling back along the unit map); consequently, there is a commuting diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{p}\big((s\mathsf{Cring}_\mathbb{R})_{A//A}\big) \arrow[r, "\partial F^A"] & \mathcal{S}\mathrm{p}\big(sC^\infty\mathsf{ring}_{F(A)//F(A)}\big) \\
(s\mathsf{Cring}_\mathbb{R})_{A//A} \arrow[u, "\Sigma^\infty_+"] \arrow[r, "F^A"] & sC^\infty\mathsf{ring}_{F(A)//F(A)} \arrow[u, "\Sigma^\infty_+"']
\end{tikzcd}$$

Since F^A sends the object $A \otimes_B A$ of $(s\mathsf{Cring}_\mathbb{R})_{A//A}$ to $F(A) \otimes^\infty_{F(B)} F(A)$ in $sC^\infty\mathsf{ring}_{F(A)//F(A)}$, and the relative cotangent complex is identified with the object $\partial(\_)^{\mathrm{alg}}(\Sigma^\infty_+(F(A) \otimes^\infty_{F(B)} F(A)))$, we conclude that there is a canonical equivalence $\partial(\_)^{\mathrm{alg}} \circ \partial F^A(\mathbb{L}^{\mathrm{alg}}_{A/B}) \simeq \mathbb{L}_{F(A)/F(B)}$. The chain rule yields a canonical equivalence of functors $\partial((\_)^{\mathrm{alg}} \circ F^A) \simeq \partial(\_)^{\mathrm{alg}} \circ \partial F^A$; we wish to compare the functor $(\_)^{\mathrm{alg}} \circ F^A$ to the pushforward $g_!$ along the unit map $g : A \to F(A)^{\mathrm{alg}}$, the derivative of which implements the base change functor $\_ \otimes_A F(A)$. We define a natural transformation $\alpha : g_! \to (\_)^{\mathrm{alg}} \circ F^A$ as follows: consider $g_!$ and $(\_)^{\mathrm{alg}} \circ F^A$ as functors $(s\mathsf{Cring}_\mathbb{R})_{A//A} \to (s\mathsf{Cring}_\mathbb{R})_{/F(A)}$. The functor $(\_)^{\mathrm{alg}} \circ F^A$ is reduced, but $g_!$ is not, so we must first pass to the coreduction of $g_!$ as exposed in [Lur17a] section 6.2.3. Choose a natural transformation $\beta : \underline{A} \to g_!$, where $\underline{A}$ is the constant functor on the object A. Recall that $\mathrm{cored}(g_!)$ fits into a pushout diagram of functors

$$\begin{tikzcd}
\underline{A} \arrow[r] \arrow[d, "g"] & g_! \arrow[d] \\
\underline{F(A)^{\mathrm{alg}}} \arrow[r] & \mathrm{cored}(g_!)
\end{tikzcd}$$

The unit transformation induces a natural transformation $g_! \to (\_)^{\mathrm{alg}} \circ F^A$ which gives us a natural transformation $\alpha : \mathrm{cored}(g_!) \to (\_)^{\mathrm{alg}} \circ F^A$. Now it suffices to show the following:

($*$) The natural transformation α induces an equivalence

$$\mathrm{colim}_i \Omega^i_{F(A)} \circ \mathrm{cored}(g_!) \circ \Sigma^i_A \xrightarrow{\simeq} \mathrm{colim}_i \Omega^i_{F(A)} \circ (\_)^{\mathrm{alg}} \circ F^A \circ \Sigma^i_A.$$

To prove the assertion above, it clearly suffices to show that α induces an equivalence on the full subcategory spanned by objects in the essential image of Σ_A. We argue as in the proof of proposition 4.3.1.2: the essential image of Σ_A consists of good A-cell objects of the form

$$A_0 = A \longrightarrow A_1 = A \otimes_\mathbb{R} \Sigma_\mathbb{R}\mathrm{Sym}^\bullet(V) \longrightarrow A_2 \longrightarrow \dots$$

where each map $A_k \to A_{k+1}$ is a pushout along a map of the form $A \otimes_\mathbb{R} \Sigma^k_\mathbb{R}\mathrm{Sym}^\bullet(V_k) \to A$. As in the proof of proposition 4.3.1.2, the functors $g_!$ and $(\_)^{\mathrm{alg}} \circ F^A$ preserve the colimits that assemble a good cell object (sequential colimits and certain pushouts), so we only have to show that α induces an equivalence on objects of the form $A \otimes_\mathbb{R} \Sigma^k_\mathbb{R}\mathrm{Sym}^\bullet(V)$ for $k \geq 1$. Unwinding definitions, we must show that the following diagram in $s\mathsf{Cring}_\mathbb{R}$ is a pushout:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] & F(A)^{\mathrm{alg}} \arrow[d] \\
A \otimes_\mathbb{R} \Sigma^k_\mathbb{R}\mathrm{Sym}^\bullet(V) \arrow[r] & (F(A) \otimes^\infty \Sigma^k C^\infty(V^\vee))^{\mathrm{alg}}
\end{tikzcd}$$

or equivalently, using pasting of pushout squares, that the object $(F(A)\otimes^{\infty}\Sigma^k C^{\infty}(V^{\vee}))^{\mathrm{alg}}$ exhibits a coproduct of $F(A)^{\mathrm{alg}}$ and $\Sigma^k_{\mathbb{R}}\mathrm{Sym}^{\bullet}(V)$ in $s\mathsf{Cring}_{\mathbb{R}}$. We have a commuting diagram

$$\begin{tikzcd} & F(A)^{\mathrm{alg}}\otimes_{\mathbb{R}}(\Sigma^k C^{\infty}(V^{\vee}))^{\mathrm{alg}} \arrow[dr, "r"] & \\ F(A)^{\mathrm{alg}}\otimes_{\mathbb{R}}\Sigma^k_{\mathbb{R}}\mathrm{Sym}^{\bullet}(V) \arrow[ur, "f"] \arrow[rr, "h"] & & (F(A)\otimes^{\infty}\Sigma^k C^{\infty}(V^{\vee}))^{\mathrm{alg}} \end{tikzcd}$$

The map f is an equivalence by lemma 4.1.3.38 and the map r is an equivalence by unramifiedness applied to the effective epimorphism $\mathbb{R}\to\Sigma^k C^{\infty}(V^{\vee})$ of simplicial C^{∞}-rings. □

Corollary 5.1.0.13. *Let V a real vector space, let n be a nonnegative integer and let $A := \Sigma^n(C^{\infty}(V^{\vee}))$, then there is a canonical equivalence $\mathbb{L}_A \simeq A\otimes_{\mathbb{R}} V[n]$.*

Corollary 5.1.0.14. *Let $f : A\to B$ be a localization of simplicial C^{∞}-rings, then $\mathbb{L}_f$ vanishes.*

Proof. By proposition 4.1.3.13, the map f is a pushout of the map $h : C^{\infty}(\mathbb{R})\to C^{\infty}(\mathbb{R}\smallsetminus\{0\})$, so by point (2) of remark 5.1.0.4 it suffices to show that $\mathbb{L}_h$ vanishes. But h is the map obtained by applying the free C^{∞}-ring functor to the map $h' : \mathbb{R}[x]\to\mathbb{R}[x,x^{-1}]$ that inverts x (algebraically). This last map is an étale map of simplicial $\mathbb{R}$-algebras, so the algebraic cotangent complex $\mathbb{L}^{\mathrm{alg}}_{h'}$ vanishes and the result follows from proposition 5.1.0.12. □

Example 5.1.0.15. Let $A = \mathbb{R}[x_1,\dots,x_n]/I$ be a finite type $\mathbb{R}$-algebra that is not lci at some $x\in Z(I)$ (i.e. the localization of the ideal I at at the maximal ideal $\mathfrak{m}_x$ of $\mathbb{R}[x_1,\dots,x_n]$ determined by x is not generated by a regular sequence over the regular local ring $\mathbb{R}[x_1,\dots,x_n]_{\mathfrak{m}_x}$), then the cotangent complex $\mathbb{L}^{\mathrm{alg}}_A$ is not left bounded at x by Avramov's theorem, so it follows (recall that the map $A_x\to F^{C^{\infty}}(A)^{\mathrm{alg}}_x$ is faithfully flat by corollary 4.1.6.25) that the cotangent complex of the finitely presented C^{∞}-ring $F^{C^{\infty}}(A)\simeq F_0^{C^{\infty}}(A) = C^{\infty}(\mathbb{R}^n)/I$ is not left bounded in $\mathsf{Mod}_{F^{C^{\infty}}(A)}$ either.

Proposition 5.1.0.16. *Let $f : B\to A$ be an effective epimorphism of simplicial C^{∞}-rings, then there is a canonical equivalence $\mathbb{L}^{\mathrm{alg}}_{B^{\mathrm{alg}}/A^{\mathrm{alg}}}\to\mathbb{L}_{B/A}$.*

Proof. We have a diagram of right adjoints

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, sC^{\infty}\mathsf{ring}) \arrow[r, "(-)^{\mathrm{alg}}"] & \mathrm{Fun}(\Delta^1, s\mathsf{Cring}_{\mathbb{R}}) \\ \mathsf{Mod} \arrow[u] \arrow[r] & \mathsf{Mod}_{\mathrm{alg}} \arrow[u] \end{tikzcd}$$

and passing to left adjoints vertically determines the Beck-Chevalley transformation carrying $A\to B$ to $\mathbb{L}_A\otimes_A B\to\mathbb{L}^{\mathrm{alg}}_A\otimes_{A^{\mathrm{alg}}} B^{\mathrm{alg}}$. This natural transformation induces a natural transformation

$$(A\longrightarrow B)\longmapsto(\mathbb{L}^{\mathrm{alg}}_{B^{\mathrm{alg}}/A^{\mathrm{alg}}}\longrightarrow\mathbb{L}_{B/A}).$$

Both functors $(A\to B)\mapsto\mathbb{L}^{\mathrm{alg}}_{B^{\mathrm{alg}}/A^{\mathrm{alg}}}$ and $(A\to B)\mapsto\mathbb{L}_{B/A}$ preserve sifted colimits so invoking proposition 4.1.2.3, we may suppose that $A\to B$ is of the form $C^{\infty}(\mathbb{R}^{n+m})\to C^{\infty}(\mathbb{R}^n)$ induced by the inclusion of a graph of a polynomial function $P : \mathbb{R}^n\to\mathbb{R}^m$. In this case, lemma 4.1.3.4 shows that there is a pushout diagram

$$\begin{tikzcd} \mathbb{R}[x_1,\dots,x_{n+m}] \arrow[r] \arrow[d] & \mathbb{R}[x_1,\dots,x_n] \arrow[d] \\ C^{\infty}(\mathbb{R}^{n+m}) \arrow[r] & C^{\infty}(\mathbb{R}^n), \end{tikzcd}$$

the upper horizontal map also being induced by P, so that the map $\mathbb{L}^{\mathrm{alg}}_{B^{\mathrm{alg}}/A^{\mathrm{alg}}}\to\mathbb{L}_{B/A}$ coincides with the map

$$\mathbb{L}^{\mathrm{alg}}_{\mathbb{R}[x_1,\dots,x_n]/\mathbb{R}[x_1,\dots,x_{n+m}]}\otimes_{\mathbb{R}[x_1,\dots,x_n]} C^{\infty}(\mathbb{R}^n)\longrightarrow\mathbb{L}_{C^{\infty}(\mathbb{R}^n)/C^{\infty}(\mathbb{R}^{n+m})},$$

which is an equivalence by proposition 5.1.0.12. □

In many cases, cotangent complexes are not readily obtained by first computing some algebraic cotangent complex. In such situations, the following result is often useful.

Proposition 5.1.0.17. *Let $f : A \to B$ be a morphism of simplicial C^∞-rings, then the map*

$$\pi_0(\mathbb{L}_A \otimes_A B) \cong \pi_0(\mathbb{L}_A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_0(\mathbb{L}_B)$$

is canonically isomorphic to the map

$$\Omega^1_{\pi_0(A)} \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \Omega^1_{\pi_0(B)}.$$

Proof. Recall that there is a commuting diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Mod} \arrow[r] \arrow[d, "\mathbf{sz}"] & \mathsf{Mod}^{\mathrm{cn}} \arrow[d, "\Omega^\infty"] \\
\mathrm{Fun}(\Delta^1, C^\infty\mathsf{ring}) \arrow[r] & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})
\end{tikzcd}$$

The lower horizontal map has a left adjoint given by 0'th truncation, and both vertical maps have left adjoints given by the smooth Kähler differentials functor and the cotangent complex functor respectively. The upper horizontal map has a left adjoint given by 0'th truncation. The associated diagram of left adjoints then commutes up to homotopy, and this homotopy applied to the map f furnishes the desired isomorphism. □

Remark 5.1.0.18. Present a simplicial C^∞-ring by some C^∞dga A, then it follows from remark 4.3.2.14 that the cotangent complex of A is the value of the left derived functor of the relative Kähler differentials evaluated on the identity $A \to A$. A cofibrant replacement of this map in the arrow category is simply a cofibrant replacement $A \to \tilde{A}$, so the cotangent complex of A may be identified with the cofibrant dg A-module $\Omega^1_{\tilde{A}} \otimes_{\tilde{A}} A$.

It is a consequence of proposition 5.1.0.12 that the cotangent complex of a free simplicial C^∞-ring is free. Another way to prove this is to observe that the parametrized square zero extension functor $T_{sC^\infty\mathsf{ring}} \to \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ factors via the connective cover functor $\tau_{\geq 0} : \mathsf{Mod} \to \mathsf{Mod}^{\mathrm{cn}}$. On connective modules, taking square zero extensions preserves limits and sifted colimits, so the adjoint carries compact projective objects of $\mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ to compact projective objects of $\mathsf{Mod}^{\mathrm{cn}}$, which we identified with $\mathbf{N}(\mathsf{CartSpVect})$. We can apply this argument to the *log cotangent complex*, a derived and positive C^∞ version of Gabber's cotangent complex [Ols05], which we now construct.

Definition 5.1.0.19. Recall from construction 4.3.2.15 the functor Ω^∞_{*pc} fitting into a commuting diagram

$$\begin{tikzcd}
\mathsf{Mod}_{\mathsf{Plog}} \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \arrow[dl, "\mathrm{ev}_1"] \\
& sC^\infty\mathsf{PLog} &
\end{tikzcd}$$

It follows from proposition 4.3.2.16 that Ω^∞_{*pc} admits a left adjoint F relative to $sC^\infty\mathsf{PLog}$. Let $(A, M \to A_{\geq 0})$ be a positive prelog simplicial C^∞-ring, then the *log-cotangent complex functor* is the composition

$$\mathbb{L} : sC^\infty\mathsf{PLog} \longrightarrow \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \xrightarrow{F} \mathsf{Mod}_{\mathsf{Plog}}.$$

Similarly, the relative log-cotangent complex is the functor

$$\mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \xrightarrow{\mathbb{L}} \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \longrightarrow \mathcal{E} \xrightarrow{\mathrm{ev}_\infty} \mathsf{Mod}_{\mathsf{Plog}},$$

where $\mathcal{E}$ is the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{Mod}_{\mathsf{Plog}}) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, sC^\infty\mathsf{PLog})} \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog})$ spanned by relative cofibre sequences.

Proposition 4.3.2.16 also provides a functor Ω^∞_{*c} fitting into a commuting diagram

$$\begin{tikzcd}
\mathsf{Mod}_{\mathsf{Log}} \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{Log}) \arrow[dl, "\mathrm{ev}_1"] \\
& sC^\infty\mathsf{Log} &
\end{tikzcd}$$

admitting a relative left adjoint, allowing us to define a cotangent complex for simplicial C^∞-rings with corners. We have a commuting diagram of left adjoints

$$\begin{tikzcd}
\mathsf{Mod}_{\mathsf{Log}} & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{Log}) \arrow[l] \\
\mathsf{Mod}_{\mathsf{PLog}} \arrow[u, "L_{\mathsf{Log}}"] & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \arrow[l, "F"] \arrow[u, "L_{\mathsf{Log}}"]
\end{tikzcd} \tag{5.1}$$

so it follows that the cotangent complex of a simplicial C^∞-ring with corners (A, A_c) coincides with $\mathbb{L}_{(A,A_c\to A_{\geq 0})}$. We also have a commuting diagram of right adjoints

$$\begin{tikzcd} \mathsf{Mod}_{\mathsf{PLog}} \arrow[r, "\Omega^\infty_{*pc}"] \arrow[d, "p"] & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{PLog}) \arrow[d, "p"] \\ \mathsf{Mod} \arrow[r, "\Omega^\infty_*"] & \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}). \end{tikzcd} \tag{5.2}$$

Passing to horizontal left adjoints determines a Beck-Chevalley transformation $\mathbb{L}_A \to \mathbb{L}_{(A,M\to A_{\geq 0})}$ for $(A, M \to A_{\geq 0}) \in sC^\infty\mathsf{PLog}$, and passing to left adjoints in the entire square shows that this Beck-Chevalley map induces equivalences $\mathbb{L}_A \simeq \mathbb{L}_{(A,0\to A_{\geq 0})} \simeq \mathbb{L}_{(A,A_{>0})}$.

Proposition 5.1.0.20. *The log-cotangent complex has the following properties.*

(1) *The object $\mathbb{L}_{(C^\infty(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}),(C^\infty_b(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}))}$ is free on $n+k$ generators.*

(2) *The analogues of remarks 5.1.0.4 and 5.1.0.3 hold for the relative log-cotangent complex.*

(3) *For (A, A_c) a 0-truncated simplicial C^∞-ring with corners, the object $\pi_0(\mathbb{L}_{(A,A_c)})$ coincides with the module of* b-Kähler differentials *constructed in section 7 of [JF19].*

(4) *Let $f : (A, A_c) \to (B, B_c)$ an admissible map of simplicial C^∞-rings with corners, then the relative log-cotangent complex of f vanishes.*

Proof. (2) and (3) are entirely formal and left as an exercise to the reader. For (1), note that proposition 4.3.2.16 asserts that the functor Ω^∞_{*pc} preserves limits and sifted colimits restricted to connective objects so that the adjoint carries compact projectives to compact projectives. Since $(C^\infty(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}), (C^\infty_b(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}))$ is the logification of the compact projective object $(C^\infty(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}), \mathbb{Z}^k_{\geq 0} \to C^\infty_{\geq 0}(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}))$, we conclude using the commuting diagram (5.1) that $\mathbb{L}_{(C^\infty(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}),(C^\infty_b(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}))}$ is a free $C^\infty(\mathbb{R}^n\times\mathbb{R}^k_{\geq 0})$-module. It follows from (3) that the module is generated by $n+k$ elements. For (4), we note that f is a pushout of $(C^\infty(\mathbb{R}), C^\infty_b(\mathbb{R})) \to (C^\infty(\mathbb{R}\smallsetminus\{0\}), C^\infty_b(\mathbb{R}\smallsetminus\{0\}))$, so we conclude using (2), the vanishing of cotangent complexes of localizations of simplicial C^∞-rings and the fact that for initial log structures, the log-cotangent complex coincides with the cotangent complex of the underlying simplicial C^∞-rings. □

Remark 5.1.0.21. Let (A, A_c) be a simplicial C^∞-ring with corners. Dualizing the map $\mathbb{L}_A \to \mathbb{L}_{(A,A_c)}$ determines an object $\mathbb{T}_{(A,A_c)} \to \mathbb{T}$ in $(\mathsf{Mod}_A)_{/\mathbb{T}_A}$. When (A, A_c) is a manifold with corners M, the module $\mathbb{T}_{(A,A_c)}$ is the locally free *b-tangent sheaf* which is locally on $\mathbb{R}^n\times\mathbb{R}^k_{\geq 0}$ spanned differentials $\{\frac{\partial}{\partial x_i}, x_j\frac{\partial}{\partial x_j}\}$ where the x_i are coordinate functions on $\mathbb{R}^n$ and the x_j are coordinate functions on $\mathbb{R}^k_{\geq 0}$ and the map $\mathbb{T}_{(A,A_c)} \to \mathbb{T}_A$ determines a submodule. The commutator bracket of vector fields on $\mathbb{T}_A$ restricts to $\mathbb{T}_{(A,A_c)}$ and determines the structure of a Lie algebroid on $\mathbb{T}_{(A,A_c)}$. When (A, A_c) is not (log) smooth, $\mathbb{T}_{(A,A_c)}$ still admits the structure of a Lie algebroid under suitable conditions on A. Suppose that A is truncated, that is, there is some n such that $A \simeq \tau_{\leq n}A$, then the fundamental theorem of (parametrized) derived deformation theory asserts that Koszul duality for Lie algebroids ([Nui19] induces a canonical equivalence of ∞-categories

$$\mathsf{FMP}_A \simeq \mathsf{LieAlgd}_A$$

between *formal moduli problems* over A and Lie algebroids over A. Let $\tilde{A}$ be a small extension of A, so that $\tilde{A}$ is given be a finite sequence

$$\tilde{A} \longrightarrow A_{n-1} \longrightarrow \ldots \longrightarrow A_1 \longrightarrow A,$$

where each map is a square zero extension by a shifted copy of A. A *deformation of* (A, A_c) *to* $\tilde{A}$ is a pair $((B, B_c), \alpha)$ where (B, B_c) lies over $(\tilde{A}, \tilde{A}_c) := (\tilde{A}, \tilde{A}_{\geq 0} \times_{A_{\geq 0}} A_c)$ and α is an equivalence $(B, B_c)\coprod_{(\tilde{A},\tilde{A}_c)}(A, A_c) \simeq (A, A_c)$ in $sC^\infty\mathsf{ring}_c$. Consider the functor informally given by

$$sC^\infty\mathsf{ring}^{\mathrm{sm}}_{/A} \longrightarrow \mathcal{S}, \quad \tilde{A} \longmapsto \{\text{Deformations of } (A, A_c) \text{ to } (\tilde{A})\},$$

then it can be shown that this functor is a formal moduli problem whose tangent complex coincides with the anchored module $\mathbb{T}_{(A,A_c)} \to \mathbb{T}_A$, so that the equivalence above endows $\mathbb{T}_{(A,A_c)}$ with a Lie bracket.

5.1.1 Connectivity and finiteness of cotangent complexes

First we establish useful results asserting that connectivity and finiteness is preserved by taking cotangent complexes.

Proposition 5.1.1.1 (Hurewicz theorem for simplicial C^∞-rings)**.** *Let $f : A \to B$ be a map of simplicial C^∞-rings. If $\mathrm{cofib}(f)$ is n-connective, then there is a canonical $(2n)$-connective map $B \otimes_A \mathrm{cofib}(f) \to \mathbb{L}_{B/A}$ of B-modules.*

We need an easy lemma.

Lemma 5.1.1.2. *If a map $f : A \to B$ of simplicial C^∞-rings is an n-equivalence (i.e. f induces an isomorphism on the k'th homotopy group for $k \leq n$), then $\mathbb{L}_{B/A}$ is $(n+1)$-connective.*

Proof. It suffices to show that for any connective n-truncated B-module M, the map $\mathrm{Hom}_{\mathsf{Mod}_B}(\mathbb{L}_B, M) \to \mathrm{Hom}_{\mathsf{Mod}_B}(\mathbb{L}_A \otimes_A B, M)$ is an equivalence. By proposition 5.1.0.7 and corollary 5.1.0.8, this map is equivalent to the map

$$\theta : \mathrm{Hom}_{sC^\infty\mathsf{ring}_{/B}}(B, B \oplus M) \longrightarrow \mathrm{Hom}_{sC^\infty\mathsf{ring}_{/B}}(A, B \oplus M)$$

As M is n-truncated, the unit map $M \to M \otimes_B \tau_{\leq n} B$ of B-modules is an equivalence. Differently put, in the tangent category, the coCartesian lift of the map $B \to \tau_{\leq n} B$ starting at M is also Cartesian, so we have a pullback diagram

$$\begin{tikzcd} B \oplus M \arrow[r] \arrow[d] & \tau_{\leq n} B \oplus M \arrow[d] \\ B \arrow[r] & \tau_{\leq n} B \end{tikzcd}$$

and we deduce that the map θ is equivalent to the map

$$\theta' : \mathrm{Hom}_{sC^\infty\mathsf{ring}_{/\tau_{\leq n} B}}(B, \tau_{\leq n} B \oplus M) \longrightarrow \mathrm{Hom}_{sC^\infty\mathsf{ring}_{/\tau_{\leq n} B}}(A, \tau_{\leq n} B \oplus M).$$

But as both $\tau_{\leq n} B$ and $\tau_{\leq n} B \oplus M$ are clearly n-truncated, the assumption that $f : A \to B$ is an n-equivalence ensures that θ' is an equivalence. $\square$

Proof of proposition 5.1.1.1. The argument proceeds as in [Lur17a] thm 7.4.3.12; we refer to Higher Algebra where the proof is the same, and provide details where our argument differs.
We say that a map $f : A \to B$ is *n-good* if $\mathrm{fib}(\epsilon_f)$ is $(2n)$-connective. The following assertions hold.

(1) If in a commuting triangle

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd}$$

f and g are n-good and f and g are $(n-1)$-connective, then h is n-good. This is proven as in [Lur17a] thm 7.4.3.12.

(2) If in a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

of simplicial C^∞-rings the map f is n-good, then f' is n-good. As in [Lur17a] thm 7.4.3.12.

(3) Let V be a real vector space. If $k \geq n-1$, then the map $\Sigma^k C^\infty(V^\vee) \to \mathbb{R}$ is n-good. To prove this, first consider the fibre sequence

$$\mathbb{L}_{\Sigma^k C^\infty(V^\vee)} \otimes_{\Sigma^k C^\infty(V^\vee)} \mathbb{R} \longrightarrow 0 \longrightarrow \mathbb{L}_{\mathbb{R}/\Sigma^k C^\infty(V^\vee)}$$

of $\mathbb{R}$-modules provided by remark 5.1.0.3, yielding an equivalence $\mathbb{L}_{\mathbb{R}/\Sigma^k C^\infty(V^\vee)} \simeq V[k+1]$. The domain of the map ϵ_f is given by the cofibre

$$\mathrm{cofib}(\Sigma^k C^\infty(V^\vee) \to \mathbb{R}) \otimes_{\Sigma^k C^\infty(V^\vee)} \mathbb{R} \simeq \mathrm{cofib}(\mathbb{R} \to \Sigma^{k+1} C^\infty(V^\vee))$$

(here we use unramifiedness). Using lemma 4.1.3.38, we identify the underlying map of simplicial $\mathbb{R}$-algebras of the map $\mathbb{R} \to \Sigma^{k+1} C^\infty(V^\vee)$ with the map $\mathbb{R} \to \mathrm{Sym}^\bullet(V[k+1])$, whose cofibre is $\coprod_{n=1}^\infty \mathrm{Sym}^n(V[k+1])$. The map

$$\epsilon_f : \coprod_{n=1} \mathrm{Sym}^n(V[k+1]) \longrightarrow V[k+1]$$

is equivalent to the identity on the first summand and nullhomotopic on all the other summands. Now we are done, since $\coprod_{n=2}^\infty \mathrm{Sym}^n(V[k+1])$ is a $(2n)$-connective object if $k \geq n-1$.

(4) If $f : A \to B$ is a $(2n-1)$-equivalence, then f is n-good. This is true because $B \otimes_A \operatorname{cofib}(f)$ is $(2n)$-connective by a straightforward application of the torsion spectral sequence, and $\mathbb{L}_{B/A}$ is $(2n)$-connective by lemma 5.1.1.2.

Now proposition 4.1.3.32 yields a sequence of simplicial C^∞-rings over B

$$A = A_n \longrightarrow A_{n+1} \longrightarrow A_{n+2} \longrightarrow \dots$$

where each $A_k \to A_{k+1}$ is obtained as a pushout

$$\begin{tikzcd} \Sigma^k C^\infty(V^\vee) \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ A_k \arrow[r] & A_{k+1} \end{tikzcd}$$

and $A_n \to B$ is n-connective. By point (4), the map $A_{2n+1} \to B$ is n-good. By point (1), it suffices to show that $A_k \to A_{k+1}$ is n-good for $k \leq 2n$. By point (2) we are reduced to proving that $\Sigma^k C^\infty(V^\vee) \to \mathbb{R}$ is n-good for $k \geq n$. This is the conclusion of point (3). □

The following corollaries are proven exactly as [Lur17a] cor. 7.4.3.2 until 7.3.4.5.

Corollary 5.1.1.3. *Let $f : A \to B$ be a morphism of simplicial C^∞-rings. If f has n-connective cofibre for some $n \geq 0$, then the relative cotangent complex $\mathbb{L}_{B/A}$ is n-connective. The converse holds if $\pi_0(f)$ is an isomorphism of C^∞-rings.*

Corollary 5.1.1.4. *A map $f : A \to B$ of simplicial C^∞-rings is an equivalence if and only if the underlying map $\pi_0(A) \to \pi_0(B)$ is an equivalence and $\mathbb{L}_{B/A}$ vanishes.*

Corollary 5.1.1.5. *Let $f : A \to B$ be a map of simplicial C^∞-rings that has an n-connective cofibre for some $n \geq 0$, then the induced map $\mathbb{L}_A \to \mathbb{L}_B$ also has n-connective cofibre.*

Remark 5.1.1.6. Let $f : A \to B$ be a surjection of C^∞-rings, viewed as discrete simplicial C^∞-rings, then proposition 5.1.0.17 and corollary 5.1.1.3 show that $\Omega^1_{\pi_0(B)/\pi_0(A)} \cong 0$ and that the map $\pi_0(\operatorname{fib}(f) \otimes_A B) \to \pi_1(\mathbb{L}_{B/A})$ is an isomorphism. Because B is discrete, the module $\pi_0(\operatorname{fib}(f) \otimes_A B)$ is canonically identified with I/I^2 where $I = \ker(f)$, and we have the classical conormal exact sequence

$$I/I^2 \longrightarrow \Omega^1_{\pi_0(A)} \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \Omega^1_{\pi_0(B)} \longrightarrow 0$$

for C^∞-rings.

Remark 5.1.1.7. Suppose that $f : A \to B$ is a map between fair simplicial C^∞-rings which has n-connective cofibre for $n \geq 1$. Then f is always an effective epimorphism, so lemma 3.1.3.42 shows that B is a complete A-module. There is a $(2n-1)$-connective map of A-modules $\operatorname{cofib}(f) \to \mathbb{L}_{B/A}$. Since $\operatorname{cofib}(f)$ is complete, there is also a $(2n-1)$-connective map $\operatorname{cofib}(f) \to \mathbb{L}^{\mathrm{cplt}}_{B/A}$. It follows that corollaries 5.1.1.3, 5.1.1.4 and 5.1.1.5 hold in the situation described above with $\mathbb{L}^{\mathrm{cplt}}_{B/A}$ in place of $\mathbb{L}_{B/A}$.

The connectivity estimates we have just proven are very powerful, particularly because they allow us to put any morphism $f : A \to B$ of simplicial C^∞-ring into standard, starting from the map $\pi_0(A) \to \pi_0(B)$ and the the relative cotangent complex. This result has two very important consequences:

(1) A map $f : A \to B$ is (almost) finitely presented if and only if $\pi_0(f)$ is finitely presented and $\mathbb{L}_{B/A}$ is (almost) perfect.

(2) Any affine derived manifold **Spec** A whose cotangent complex has Tor-amplitude in $[0, n]$ admits a presentation as a dg-manifold

The construction of the standard form of a morphism is explained in the proof of the following result.

Proposition 5.1.1.8. *Let $f : A \to B$ be a morphism of simplicial C^∞-rings.*

(1) If f is of finite presentation, then $\mathbb{L}_{B/A}$ is perfect. The converse is true if $\pi_0(f)$ is finitely presented.

(2) If f is almost of finite presentation, then $\mathbb{L}_{B/A}$ is almost perfect. The converse is true if $\pi_0(f)$ is finitely presented.

Proof. We prove the forward implications. Using corollary 5.1.0.8, we see that the cotangent complex functor

$$
sC^\infty\mathsf{ring}_{A//B} \longrightarrow \mathsf{Mod}_B, \qquad \begin{tikzcd} & C \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & B \end{tikzcd} \longmapsto \mathbb{L}_{C/A} \otimes_C B
$$

is a left adjoint. Suppose B is finitely presented in the ∞-category of A-algebras, which is generated under sifted colimits by objects of the form $A \otimes^\infty C^\infty(\mathbb{R}^n)$ by proposition 4.1.1.28. Invoking lemma 4.1.1.20, it suffices to show that the object $\mathbb{L}_{A\otimes^\infty C^\infty(\mathbb{R}^n)/A} \otimes_{A\otimes^\infty C^\infty(\mathbb{R}^n)} B$ is perfect, but it follows from corollary 5.1.0.13 that this object is free on n generators.
If B is almost finitely presented over A, then proposition 4.1.3.32 provides a map $f_n : B' \to B$ in $sC^\infty\mathsf{ring}_{A//B}$ where B' is finitely presented over A and f_n n-connective. It follows that the relative cotangent complex $\mathbb{L}_{B/B'}$ is $(n+1)$-connective, which implies that the map

$$
\mathbb{L}_{B'/A} \otimes_{B'} B \longrightarrow \mathbb{L}_{B/A}
$$

is n-connective, using the fibre sequence of remark 5.1.0.4.
Now suppose that $\pi_0(f)$ is of finite presentation and that $\mathbb{L}_f$ is almost perfect. First, we prove (2): following the proof of [Lur17a] thm. 7.4.3.18, we construct a sequence of simplicial C^∞-rings over B

$$
A = A(-1) \longrightarrow A(0) \longrightarrow A(1) \longrightarrow A(2) \longrightarrow \dots
$$

such that each map $f_n : A(n) \to B$ is n-connective and $A(n)$ is of finite presentation over A. To construct $A(0)$, we choose an effective epimorphism $g : C^\infty(\mathbb{R}^n) \otimes^\infty A \to B$ (which exists because $\pi_0(B)$ is finitely generated over $\pi_0(A)$) and consider the kernel $I := \ker(\pi_0(f))$ as a finitely generated $C^\infty(\mathbb{R}^n) \otimes^\infty \pi_0(A)$-module. The map $\pi_0(\mathrm{fib}(f)) \to I$ is a surjection, so we can choose a map $M \to \mathrm{fib}(f)$ where M is a finitely generated and free $C^\infty(\mathbb{R}^n) \otimes^\infty A$-module (on k generators say) that induces a surjection $\pi_0(M) \to I$. Now we take the free simplicial C^∞-ring over A of the module M and define $A(0)$ as the pushout diagram

$$
\begin{tikzcd}
C^\infty(\mathbb{R}^k) \otimes^\infty A \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^n) \otimes^\infty A \arrow[d, "g"] \\
A \arrow[r] & A(0)
\end{tikzcd}
$$

There is a canonical map $A(0) \to B$ which is a 0-equivalence by construction.
Now we assume that we have constructed an n-connective map $f_n : A(n) \to A$ for $n \geq 0$ (but as we have just explained, we may assume that $A(n) \to B$ is a 0-equivalence for *all* $n \geq 0$). Proposition 5.1.1.1 shows that we have an isomorphism $\pi_n(\mathrm{fib}(f_n)) \to \pi_{n+1}(\mathbb{L}_{B/A(n)})$. Using that $\mathbb{L}_{B/A}$ is almost perfect and $\mathbb{L}_{A(n)/A}$ is perfect (by the previous part of the proof), we see that $\mathbb{L}_{B/A(n)}$ is almost perfect. Since $\mathbb{L}_{B/A(n)}$ is $(n+1)$-connective, the module $\pi_{n+1}(\mathbb{L}_{B/A(n)})$ is finitely presented; choose a finite set J_n of generators of $\pi_n(\mathrm{fib}(f_n))$ as a $\pi_0(A(n))$-module, then we have a map $\mathbb{R}^{J_n} \otimes_{\mathbb{R}} A(n)[n] \to \mathrm{fib}(f_n)$ of $A(n)$-modules, which induces a surjective map on the n'th homotopy group. We now define $A(n+1)$ by forming a pushout diagram

$$
\begin{tikzcd}
F^{C^\infty}_{A(n)}(\mathrm{Sym}^\bullet_{A(n)}(\mathbb{R}^{J_n} \otimes_{\mathbb{R}} A(n)[n])) \arrow[r] \arrow[d] & A(n) \arrow[d] \\
A(n) \arrow[r] & A(n+1)
\end{tikzcd}
$$

The map $\mathbb{R}^{J_n} \otimes_{\mathbb{R}} A(n)[n] \to B$ is nullhomotopic, which yields a map $A(n+1) \to B$. Notice that we have a diagram of shape $\Delta^2 \times \Delta^1$, where both squares are pushouts

$$
\begin{tikzcd}
\Sigma^n C^\infty((\mathbb{R}^{J^n})^\vee) \arrow[r] \arrow[d] & \Sigma^n C^\infty((\mathbb{R}^{J^n})^\vee) \otimes^\infty A(n) \arrow[r] \arrow[d] & A(n) \arrow[d] \\
\mathbb{R} \arrow[r] & A(n) \arrow[r] & A(n+1)
\end{tikzcd}
$$

It is clear that $A(n+1)$ is of finite presentation over A, so to finish the construction, we need to show that the induced map $A(n+1) \to B$ is $(n+1)$-connective. First, note that by the diagram above, $A(n) \to A(n+1)$ is an effective epimorphism, which implies that $A(n+1) \to B$ is a 0-equivalence since $A(n) \to B$ is a 0-equivalence by assumption.

In light of corollary 5.1.1.3, it suffices to show that the relative cotangent complex $\mathbb{L}_{B/A(n+1)}$ is $(n+2)$-connective. We have a fibre sequence

$$\mathbb{L}_{A(n+1)/A(n)} \otimes_{A(n+1)} B \longrightarrow \mathbb{L}_{B/A(n)} \longrightarrow \mathbb{L}_{B/A(n+1)},$$

which, using the pushout diagram above and remark 5.1.0.4, we can identify with a fibre sequence

$$\mathbb{R}^{J_n} \otimes_{\mathbb{R}} B[n+1] \longrightarrow \mathbb{L}_{B/A(n)} \longrightarrow \mathbb{L}_{B/A(n+1)}.$$

By assumption, $\mathbb{L}_{B/A(n)}$ is n-connective. Using the long exact sequence, it suffices to show that the map $\pi_{n+1}(\mathbb{R}^{J_n} \otimes_{\mathbb{R}} B[n+1]) \to \pi_{n+1}(\mathbb{L}_{B/A(n)})$ is surjective. For this, we just have to note that the map $\mathbb{R}^{J_n} \otimes_{\mathbb{R}} B[n+1] \to \mathbb{L}_{B/A(n)}$ is the shift of the map $\mathbb{R}^{J_n} \otimes_{\mathbb{R}} B[n] \to \mathbb{L}_{B/A(n)}[-1]$ that we have constructed above, which we have chosen so as to induce a surjection on the n'th homotopy group.
Now suppose that $\mathbb{L}_{B/A}$ is perfect. It suffices to show that for some large enough k, the map $A(k) \to B$ is an equivalence. The proof of this fact is word for word the same as the proof of [Lur17a] thm. 7.4.3.18. ☐

Remark 5.1.1.9. Proposition 5.1.1.8 is false if $\pi(f)$ is not assumed to be finitely presented. For a counterexample, let M be a manifold with boundary, then corollary 5.0.0.3 shows that $\mathbb{L}_M$ is a perfect (in fact, finitely generated and projective), yet $C^\infty(M)$ is not even finitely 1-presented.

Combining propositions 5.1.1.8 and 4.3.3.17 shows that for almost finitely presented simplicial C^∞-rings, the cotangent complex is a quasi-coherent module. However, if A is a fair simplicial C^∞-ring such that $\pi_0(A)$ is finitely presented, we can perform the constructions of proposition 5.1.1.8 with the quasi-coherent cotangent complex instead of the cotangent complex. Since for fair simplicial C^∞-rings, the cotangent complex controls the connectivity and finiteness properties as explained in remark 5.1.1.7, we have the following corollary.

Corollary 5.1.1.10. *Let A be a simplicial C^∞-ring such that $\pi_0(A)$ is finitely presented, then the following are equivalent.*

(1) *A is of finite presentation.*

(2) *$\mathbb{L}_A$ is perfect and is equivalent to $\mathbb{L}_A^{\mathrm{cplt}}$.*

(3) *$\mathbb{L}_A^{\mathrm{cplt}}$ is perfect.*

The same holds when 'finite presentation' and 'perfect' is replaced with 'almost of finite presentation' and 'almost perfect' respectively.

A more careful construction of the object $A(0)$ yields the following result (see also theorem 4.34 of [Joy12b]).

Proposition 5.1.1.11. *Let A be a fair simplicial C^∞-ring such that $\pi_0(A)$ is finitely presented, and such that $\mathbb{L}_A$ (equivalently $\mathbb{L}_A^{\mathrm{cplt}}$) is perfect and has* Tor*-amplitude in $[-1,0]$, then there is an open submanifold $U \subset \mathbb{R}^n$, a vector bundle $E \to U$ with a section $s : U \to E$ and an equivalence $\mathrm{dZ}(s) \simeq A$.*

Proof. Choose an effective epimorphism $f : C^\infty(\mathbb{R}^n) \to A$, then the object $\mathbb{L}_{A/C^\infty(\mathbb{R}^n)}$ also has Tor-amplitude in $[-1,0]$ and is moreover 1-connective. It follows that $\mathbb{L}_{A/C^\infty(\mathbb{R}^n)}[-1]$ is connective, perfect and has Tor-amplitude 0, and is therefore finitely generated and projective. Using the vector bundle extension lemma 4.3.4.13 we may choose an open set $\mathrm{Spec}_{\mathbb{R}} B \subset U \subset \mathbb{R}^n$, a finitely generated projective $C^\infty(U)$-module P such that $P \otimes_{C^\infty(U)} A \simeq \mathbb{L}_{A/C^\infty(\mathbb{R}^n)}[-1]$. In particular, we can identify P with the module of sections of a vector bundle $E \to U$.
Now let $I := \ker(\pi_0(f))$, then we have surjections $\pi_1(\mathbb{L}_{A/C^\infty(\mathbb{R}^n)}) \simeq \pi_0(\mathrm{fib}(f)) \otimes_{C^\infty(\mathbb{R}^n)} \pi_0(A) \to I/I^2$. Identifying P with an object in the abelian category of finitely presented $C^\infty(U)$-modules, we have a surjection $P \to I/I^2$ which lifts to map $P \to I$, by projectivity of P. Because the inclusion $I \hookrightarrow C^\infty(U)$ becomes the zero map after tensoring with $\mathbb{R}$, the map $I \to I/I^2$ becomes an isomorphism after tensoring with $\mathbb{R}$. Thus, using the assumption that I is locally finitely generated, we may choose a locally finite cover $\{V_\alpha\}$ of U such that on each V_α, the map $P \to I$ induces a surjection $P \otimes_{C^\infty(U)} C^\infty(V_\alpha) \to I \otimes_{C^\infty(U)} C^\infty(V_\alpha)$. Using that I and P are closed under locally finite sums, we see that $P \to I$ is in fact a surjection. The composite map $P \to I \to C^\infty(U)$ is a section of the vector bundle $E \to U$, so we can form the pushout

$$\begin{tikzcd} C^\infty(E) \arrow[r] \arrow[d] & C^\infty(U) \arrow[d] \\ C^\infty(U) \arrow[r] & \widetilde{A} \end{tikzcd}$$

By construction, there is a canonical map $\widetilde{A} \to A$ that induces an isomorphism on connected components. The first map in the fibre sequence

$$\mathbb{L}_{\widetilde{A}/C^\infty(U)} \otimes_{\widetilde{A}} A \longrightarrow \mathbb{L}_{A/C^\infty(U)} \longrightarrow \mathbb{L}_{A/\widetilde{A}}$$

is an equivalence, so it follows from corollary 5.1.1.4 that $\widetilde{A} \simeq A$. ☐

Definition 5.1.1.12. A map of simplicial C^∞-rings $f : A \to B$ is *n-quasi-smooth* if $\mathbb{L}_{B/A}$ has Tor-amplitude in $[0, n]$. A simplicial C^∞-ring A is n-quasi smooth if the map $\mathbb{R} \to A$ is n-quasi-smooth.

Corollary 5.1.1.13. *Let* $\mathbf{Spec}\, A$ *be an affine derived manifold of finite presentation. Then* $\mathbf{Spec}\, A$ *is the derived zero locus of a section of a vector bundle on a manifold if and only if* A *is* 1*-quasi-smooth.*

We have the following permanence properties of n-quasi-smooth morphisms.

Proposition 5.1.1.14. *Let* $\mathsf{QS}_n \subset \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ *be the class of* n*-quasi-smooth morphisms.*

(1) QS_n *is stable under composition; that is,* $\mathsf{QS}_n \subset \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ *determines a full subcategory.*

(2) *If* $f : A \to B$ *is* n*-quasi-smooth, and* $g : A \to C$ *is any morphism in* $s\mathsf{Cring}_{\mathbb{R}}$ *then the base change* $B \to A \otimes_B C$ *is* n*-quasi-smooth.*

(3) QS_n *is stable under retracts.*

Proof. Let $i : A \to B$ be a map of finitely presented simplicial C^∞-rings that admits a retraction r. If B is n-quasi-smooth for $n \geq 0$, then A is also n-quasi-smooth: for $n = 0$, this follows because $\mathcal{T}_{\mathrm{Diff}}$ is idempotent complete. For $n > 0$, we can observe that $\mathbb{L}_A$ is a retract of $r_!\mathbb{L}_B$, and that Tor-amplitude is stable under retracts and base change by flat maps (r is flat because it is a retract). ☐

As a consequence of this proposition, we deduce that the subcategory of $sC^\infty\mathsf{ring}$ whose objects are n-quasi-smooth derived manifolds and whose morphisms are n-quasi-smooth morphisms is stable under finite colimits.

Remark 5.1.1.15. Combining propositions 5.1.1.14 and 5.1.1.11, we deduce that if $\mathbf{Spec}\, A \to \mathbf{Spec}\, B$ is a 1-quasi-smooth morphism of finitely presented affine derived manifolds and $\mathbf{Spec}\, C \to \mathbf{Spec}\, B$ is a map with 1-quasi-smooth finitely presented domain, then there exists an affine Kuranishi model (V, E, s) and an equivalence $\mathbf{Spec}\, C \times_{\mathbf{Spec}\, B} \mathbf{Spec}\, A \simeq \mathrm{dZ}(s)$. This was observed by Fukaya-Oh-Ohta-Ono ([Fuk+00], Appendix A), who show that on the pullback $\mathbf{K} \times_N \mathbf{K}'$ of sets, there exists a Kuranishi structure provided that the maps $\mathbf{K} \to N$ and $\mathbf{K}' \to N$ are *weakly submersive*, which means precisely that the induced map of affine derived manifolds is 1-quasi-smooth. Fukaya-Oh-Ohta-Ono actually prove this when $\mathbf{K}'$ and $\mathbf{K}$ are 1-quasi-smooth derived orbifolds. To generalize to this case, we first consider an intersection

$$[\mathbf{Spec}\, A/G] \times_N \mathbf{K}'$$

where N is a manifold, $\mathbf{K}'$ is a 1-quasi-smooth derived orbifold, $\mathbf{Spec}\, A$ is 1-quasi-smooth and finitely presented and G is a finite group. In this case (and even if G is an arbitrary group object in $\mathsf{d}C^\infty\mathsf{St}$), we have an equivalence

$$[\mathbf{Spec}\, A/G] \times_N \mathbf{K}' \simeq [\mathbf{Spec}\, A/G] \times_{N \times \mathrm{B}G} \mathbf{K}' \times \mathrm{B}G,$$

so it follows from our analysis of the affine case, together with general yoga of realization fibrations that $[\mathbf{Spec}\, A/G] \times_N \mathbf{K}'$ is a 1-quasi-smooth derived orbifold. In the general case, we observe that $\mathbf{K} \times_N \mathbf{K}'$ admits a 0-étale atlas by objects of the form $[\mathbf{Spec}\, A/G] \times_N \mathbf{K}'$ satisfying the conditions of proposition 4.2.2.6, so that $\mathbf{K} \times_N \mathbf{K}'$ is also a 1-quasi-smooth derived orbifold.

Remark 5.1.1.16. It is also true that a morphism f of affine derived manifolds is submersive if and only if $\mathbb{L}_f$ has Tor-amplitude 0 (equivalently, using [Lur17a] prop. 7.2.4.23, if $\mathbb{L}_f$ is projective), but proving that assertion obviously requires more differential-topological input than we have used so far. We will deduce this result as a consequence of the derived inverse function theorem 5.1.3.17.

Proposition 5.1.1.17. *Let* $f : A \to B$ *be an étale morphism of fair simplicial* C^∞*-rings (that is,* f *is a localization up to localizations on* B*). Then the quasi-coherent relative cotangent complex* $\mathbb{L}^{\mathrm{cplt}}_{B/A}$ *vanishes.*

Proof. The vanishing of $\mathbb{L}^{\mathrm{cplt}}_{B/A}$ is local on $\mathbf{Spec}\, B$. By assumption, there is a cover $\{B \to B[1/b]\}$ such that each composition $A \to B \to B[1/b]$ is admissible; point (1) of remark 5.1.0.4 provides a cofibre sequence

$$\mathbb{L}_{B/A} \otimes_B B[1/b] \longrightarrow \mathbb{L}_{B[1/b]/A} \longrightarrow \mathbb{L}_{B[1/b]/B},$$

but since the relative cotangent complex vanishes on localizations by corollary 5.1.0.14, the second and third term in this sequence are zero, so $\mathbb{L}_{B/A} \otimes_B B[1/b]$ vanishes as well. ☐

Corollary 5.1.1.18. *Let* $U \subset \mathbf{Spec}\, A$ *be an admissible map of affine derived manifolds. The cotangent complex* $\mathbb{L}_A \in \mathsf{QCoh}(A)$ *restricted to* U *is naturally equivalent to the cotangent complex of* U*.*

Corollary 5.1.1.19. *Let* N *be a manifold, viewed as an affine derived manifold. The cotangent complex of* N *has vanishing homotopy groups in degrees other than 0, and* $\pi_0(\mathbb{L}_N) \cong T_N^\vee$ *in* $\mathsf{Mod}^\heartsuit_{\mathcal{O}_N}$*, the abelian category of* $\mathcal{O}_N$*-modules.*

Proof. Since $\pi_0(\mathbb{L}_{C^\infty(N)}) = \Omega^1_{C^\infty(N)}$ is the module of sections of the cotangent sheaf of N, we only have to show that the higher homotopy groups vanish. Because taking global sections commutes with taking homotopy groups, it suffices to check this locally on N. By corollary 5.1.1.18, we have for each open inclusion $i : U \hookrightarrow N$ an equivalence $i^*\mathbb{L}_{C^\infty(N)} \simeq \mathbb{L}_{C^\infty(U)}$, so it suffices to prove the statement for $N = \mathbb{R}^n$, in which case the statement follows from corollary 5.1.0.13. □

Definition 5.1.1.20. Let $\mathbf{Spec}\, A$ be an affine derived manifold of finite presentation. Then $\mathbb{L}_A$ has finite Tor-amplitude, so that for each $\mathbb{R}$-point $x : A \to \mathbb{R}$, the object $\mathbb{L}_A \otimes_A \mathbb{R}$ is an $\mathbb{R}$-module with finitely many nonzero homotopy groups all of finite dimension. The *virtual dimension of* $\mathbf{Spec}\, A$ *at* x is the Euler characteristic of $\mathbb{L}_A \otimes_A \mathbb{R}$.

Proposition 5.1.1.21. *Let* $\mathbf{Spec}\, A$ *be an affine derived manifold of finite presentation, then the virtual dimension of* $\mathbf{Spec}\, A$ *is locally constant on* $\mathrm{Spec}_{\mathbb{R}}\, A$.

Proof. By proposition 5.1.1.8, we may suppose after localizing that $A = A(n)$, the object appearing in the construction of proposition 5.1.1.8 for the map $\mathbb{R} \to A$, where we attach a simplicial C^∞-ring of the form $\Sigma^{n-1}C^\infty(\mathbb{R}^k)$ to $A(n-1)$. The object $A(0)$ clearly has constant virtual dimension, so by induction we may assume that $A(n-1)$ has constant virtual dimension m in a neighbourhood of a point $x : * \to \mathrm{Spec}_{\mathbb{R}}\, A(n) \hookrightarrow \mathrm{Spec}_{\mathbb{R}}\, A(n-1)$. Now the equivalence $\mathbb{L}_{A(n)} \simeq \mathbb{L}_{A(n-1)/\Sigma^{n-1}C^\infty(\mathbb{R}^k)} \otimes_{A(n-1)} A(1)$ and the fact that $\mathbb{L}_{A(n-1)}$ has Tor-amplitude $[-n+1, 0]$ show that there are equivalences $\pi_r(\mathbb{L}_A(n)) \simeq \pi_r(\mathbb{L}_{A(n-1)})$ for $r < n-1$ and that we have an exact sequence

$$0 \longrightarrow \pi_n(\mathbb{L}_A(n)) \otimes_{A(n)} \mathbb{R} \longrightarrow \mathbb{R}^k \longrightarrow \pi_{n-1}(\mathbb{L}_{A(n-1)} \otimes_{A(n-1)} \mathbb{R}) \longrightarrow \pi_{n-1}(\mathbb{L}_A(n) \otimes_{A(n)} \mathbb{R}) \longrightarrow 0,$$

showing that the virtual dimension at x is $m \pm k$, and thus it is so for all points in some neighbourhood. □

Definition 5.1.1.22. Let A be a fair simplicial C^∞-ring. For an $\mathbb{R}$-point $x : A \to \mathbb{R}$, we call

$$\mathrm{embdim}_x\, A := \dim \pi_0(\mathbb{L}_A) \otimes_A \mathbb{R}$$

the *embedding dimension of* A *at* x. This is an upper semicontinuous function on the real spectrum of $\pi_0(A)$. The *embedding dimension of* A is

$$\mathrm{embdim}\, A := \sup_{x: A \to \mathbb{R}} \mathrm{embdim}_x\, A \in [0, \infty)$$

Lemma 5.1.1.23. *Let* A *be a fair simplicial* C^∞*-ring, and suppose that* A *has embedding dimension* n *at an* $\mathbb{R}$*-point* $x : A \to \mathbb{R}$*, then there exists a closed immersion* $\mathbf{Spec}\, A \supset U \to \mathbb{R}^n$ *for* $x \in U \to \mathbf{Spec}\, A$ *some admissible map.*

Proof. Choose an effective epimorphism $f : C^\infty(\mathbb{R}^k) \to A$, then we have the conormal exact sequence

$$I/I^2 \longrightarrow \Omega^1_{C^\infty(\mathbb{R}^n)} \otimes_{C^\infty(\mathbb{R}^n)} \pi_0(A) \longrightarrow \Omega^1_{\pi_0(A)} \longrightarrow 0.$$

Nakayama's lemma tells us that after localizing near x, may lift a basis of $\pi_0(\mathbb{L}_A) \otimes_A \mathbb{R}$ and choose n generators $\{b_1, \ldots, b_n\}$ in the module $\Omega^1_{\pi_0(A)}$. Consider the differentials $\{d_{\mathrm{dR}} x_i\}_{1 \le i \le k}$ for x_i the coordinate functions on $\mathbb{R}^k$, then these differentials also generate $\Omega^1_{\pi_0(A)}$ and we can write $b_j = \sum_i K_{ij} d_{\mathrm{dR}} x_i$ as an equation in $\pi_0(\mathbb{L}_A)$. Let K denote the matrix with coefficients K_{ij} in $\pi_0(A)$ and let $\{a_{ij}\}_{1 \le i \le n}$ for $1 \le j \le n-k$ be a linearly independent collection of real vectors in the null space of K_{ij} at x, then the finitely generated submodule of $\Omega^1_{\pi_0(A)}$ generated by the images of the differentials $\sum_i a_{ij} d_{\mathrm{dR}} x_i$ becomes the zero vector space after base change along $x : A \to \mathbb{R}$, so Nakayama's lemma asserts that after localizing near x, we may suppose that the elements $\sum_i a_{ij} d_{\mathrm{dR}} x_i$ lie in the kernel of the map $\Omega^1_{C^\infty(\mathbb{R}^n)} \otimes_{C^\infty(\mathbb{R}^n)} \pi_0(A) \to \Omega^1_{\pi_0(A)}$. It follows from the conormal exact sequence that we can find $n-k$ functions g_i on $\mathbb{R}^k$ in I that are independent at x. Localizing near x, we may assume that the functions $\{g_i\}$ determine a submersion $\mathbb{R}^k \to \mathbb{R}^{k-n}$ and a pushout diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^{n-k}) \arrow[r] \arrow[d, "0"] & C^\infty(\mathbb{R}^k) \arrow[d] \\
\mathbb{R} \arrow[r] & C^\infty(\mathbb{R}^n),
\end{tikzcd}$$

then we have an effective epimorphism $f' : C^\infty(\mathbb{R}^n) \to A$ such that $\Omega^1_{\pi_0(f')}$ vanishes at x. □

We now give the results on cotangent complexes of C^∞-rings of Whitney functions that we have alluded to.

Lemma 5.1.1.24. *Let* A *be a closed fair* C^∞*-ring, and suppose that as a simplicial* C^∞*-ring,* $\pi_0(\mathbb{L}_A)$ *is a free module* $\pi_0(A)$*-module of finite rank. Then for each* $x \in \mathrm{Spec}_{\mathbb{R}}\, A$*, there exists a localization* $A[a^{-1}]$ *containing* x *and a manifold* M *with a closed subset* $X \subset M$ *such that* $A \cong C^\infty(M)/\mathfrak{m}^\infty_X$.

Proof. Using lemma 5.1.1.23, we may suppose that we have an effective epimorphism $C^\infty(\mathbb{R}^n) \to A$ where n is the embedding dimension of A. Since $\pi_0(\mathbb{L}_A)$ is free of rank n, the second map in the conormal exact sequence

$$I/I^2 \longrightarrow \Omega^1_{C^\infty(\mathbb{R}^n)} \otimes_{C^\infty(\mathbb{R}^n)} \pi_0(A) \longrightarrow \Omega^1_{\pi_0(A)} \longrightarrow 0$$

is an isomorphism, so the first map, which takes $[f]$ to $d_{\mathrm{dR}} f$, is the zero morphism. It follows that at all x points of $Z(I)$, the jets of the partial derivatives $\{\frac{\partial f}{\partial x_i}\}_i$ at x are contained in the ideal generated by the partial derivative of f at x. Using the assumption that I is closed, we deduce that the functions $\{\frac{\partial f}{\partial x_i}\}_i$ are contained in I. By induction, we conclude that all higher partial derivatives of f are contained in I, which implies that $I \subset \mathfrak{m}^\infty_{Z(I)}$. Since I is closed, we also have $\mathfrak{m}^\infty_{Z(I)} \subset I$. □

Remark 5.1.1.25. Even when $\pi_1(\mathbb{L}_A)$ vanishes, so that $I = I^2$ it is necessary to impose that I be closed to deduce that $C^\infty(\mathbb{R}^n)/I$ is a ring of Whitney functions. To prove this, take a closed set $X \subset \mathbb{R}^n$ and define a sequence of subsets $\varnothing = I_{-1} \subset I_0 \subset I_1 \subset I_2 \subset \ldots \subset \mathfrak{m}^\infty_X$ where I_k has cardinality $\sum_{i=0}^k 2^i$ as follows. Choose an arbitrary nonzero $f \in \mathfrak{m}^\infty_X$, and set $I_0 := \{f\}$. Suppose that I_n has been defined, then we define I_{n+1} by choosing for each $g \in I_n \setminus I_{n-1}$ a factorization $g = \varphi h$ with $\varphi, h \in \mathfrak{m}^\infty_X$, using Tougeron's flat function lemma, and adjoining φ and h to I_n. Consider the ideal I generated by the set $\bigcup_n I_n \subset \mathfrak{m}^\infty_X$. By construction, we have $I = I^2$, but I is not closed: by Whitney's spectral theorem, the closure of I is $\mathfrak{m}^\infty_X$, but as I is countably generated, the flat function lemma provides a principal ideal $(\psi) \subset \mathfrak{m}^\infty_X$ such that $I \subset (\psi)$.

Theorem 5.1.1.26. *Let $f : C^\infty(\mathbb{R}^n) \to A$ be an effective epimorphism, let $I = \ker \pi_0(f)$ be jet determined and suppose that there are n closed subsets $X_i \subset \mathbb{R}$ such that $Z(I) = \prod_i X_i$. Then the following are equivalent.*

(1) $\mathbb{L}_f \simeq 0$ *in* Mod_A.

(2) $I = \mathfrak{m}^\infty_{Z(I)}$ *and the unit map of the 0'th truncation*

$$A \longrightarrow \pi_0(A) \simeq C^\infty(\mathbb{R}^n)/I = C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)}$$

is an equivalence.

Proof. We first show (2) ⇒ (1). Proposition 4.1.6.8 shows that there is a pushout diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, "f"] \arrow[d] & A \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "\mathrm{id}"] & A \end{tikzcd}$$

so point (2) of remark 5.1.0.4 provides an equivalence $\mathbb{L}_f \simeq \mathbb{L}_{\mathrm{id}} = 0$. Now assume (1), then $\pi_0(\mathbb{L}_f) = 0$ so lemma 5.1.1.24 shows that $I = \mathfrak{m}^\infty_{Z(I)}$. Applying point (1) of remark 5.1.0.4 to the composition

$$C^\infty(\mathbb{R}^n) \longrightarrow A \longrightarrow C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)}$$

yields a fibre sequence

$$\mathbb{L}_f \otimes_A C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)} \longrightarrow \mathbb{L}_{C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)}/C^\infty(\mathbb{R}^n)} \longrightarrow \mathbb{L}_{C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)}/A}.$$

By assumption, $\mathbb{L}_f = 0$ and $\mathbb{L}_{C^\infty(\mathbb{R}^n)/\mathfrak{m}^\infty_{Z(I)}/C^\infty(\mathbb{R}^n)} = 0$ by the proof of (2) ⇒ (1), so we conclude using corollary 5.1.1.4. □

Remark 5.1.1.27. The previous proposition in fact holds without any restriction on the closed subsets, but the proof of this fact depends on an alternative calculation of the cotangent complex of simplicial C^∞-rings of the form $C^\infty(X;\mathbb{R}^n)$ which uses the HKR filtration on the universal S^1-equivariant simplicial C^∞-ring $C^\infty(X;\mathbb{R}^n) \otimes^\infty_{C^\infty(X;\mathbb{R}^n)\otimes^\infty C^\infty(X;\mathbb{R}^n)} C^\infty(X;\mathbb{R}^n)$, which can be identified, using corollary 4.1.6.5, with the continuous Hochschild homology of $C^\infty(X;\mathbb{R}^n)$ as a Fréchet algebra. The vanishing of the higher homotopy groups of $\mathbb{L}_{C^\infty(X;\mathbb{R}^n)}$ is then a consequence of excision for continuous Hochschild homology as proven by Meyer [Mey10]. We will come back to this result in later work, since a discussion of Hochschild homology in functional analytic settings is not in order at this point.

5.1.2 Application: derived intersection of regularly situated sets

In this short subsection, we apply some of the ideas developed in this chapter so far to give an alternative characterization of the condition of being regularly situated for two suitable closed sets X and Y in some $\mathbb{R}^n$: X and Y are regularly situated if and only if the derived intersection of the finitely generated affine C^∞-schemes $(X, C^\infty_{(X;\mathbb{R}^n)})$ and $(Y, C^\infty_{(Y;\mathbb{R}^n)})$ coincides with $(X\cap Y, C^\infty_{(X\cap Y;\mathbb{R}^n)})$. More precisely, our goal is to prove the following result.

Proposition 5.1.2.1. *Let $X, Y \subset \mathbb{R}^n$ be closed subsets of the form $\prod_i X_i$ and $\prod_i Y_i$ for $\{X_i\}$ and $\{Y_i\}$ tuples of n closed subsets of $\mathbb{R}$, and let $p : C^\infty(\mathbb{R}^n) \to C^\infty(X;\mathbb{R}^n)$ and $q : C^\infty(\mathbb{R}^n) \to C^\infty(Y;\mathbb{R}^n)$ be the quotient maps onto the discrete simplicial C^∞-rings of Whitney functions on X and Y, then the following are equivalent.*

(1) *X and Y are regularly situated, that is, either $X\cap Y = \varnothing$ or for each $x_0 \subset X\cap Y$, there is a neighbourhood $x_0 \in V$ in $\mathbb{R}^n$ for which there are constants $C \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{\geq 0}$ such that for each $x \in V\cap X$, we have the inequality*

$$Cd(x, X\cap Y)^\lambda \leq d(x, Y),$$

where $d(\_,\_)$ denotes the Euclidean distance on $\mathbb{R}^n$.

(2) *Either $X\cap Y = \varnothing$ or the ideal $p(\mathfrak{m}^\infty_Y) \subset C^\infty(X;\mathbb{R}^n)$ of those Whitney functions F on X which admit a representative $f : \mathbb{R}^n \to \mathbb{R}$ that is flat on Y, is closed for the Fréchet topology on $C^\infty(X;\mathbb{R}^n)$.*

(3) *Either $X\cap Y = \varnothing$ or the ideal $q(\mathfrak{m}^\infty_X) \subset C^\infty(Y;\mathbb{R}^n)$ of those Whitney functions G on Y which admit a representative $g : \mathbb{R}^n \to \mathbb{R}$ that is flat on X, is closed for the Fréchet topology on $C^\infty(Y;\mathbb{R}^n)$.*

(4) *The commuting diagram*

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, "p"] \arrow[d, "q"] & C^\infty(X;\mathbb{R}^n) \arrow[d] \\ C^\infty(Y;\mathbb{R}^n) \arrow[r] & C^\infty(X\cap Y;\mathbb{R}^n) \end{tikzcd}$$

is a pushout in the category $C^\infty\mathsf{ring}$.

(5) *The commuting diagram*

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, "p"] \arrow[d, "q"] & C^\infty(X;\mathbb{R}^n) \arrow[d] \\ C^\infty(Y;\mathbb{R}^n) \arrow[r] & C^\infty(X\cap Y;\mathbb{R}^n) \end{tikzcd}$$

is a pushout in the ∞-category $sC^\infty\mathsf{ring}$.

Proof. The proofs of $(1 \Rightarrow 2)$ and $(1 \Rightarrow 3)$, $(4 \Rightarrow 2)$ and $(4 \Rightarrow 3)$, and $(2 \Rightarrow 1)$ and $(3 \Rightarrow 1)$ are identical, so we only do one of each.

$(1 \Rightarrow 2)$ It is a result of Łojasiewicz [Łoj59] that the condition of being regularly situated for X, Y can be reformulated as follows: the chain complex

$$0 \longrightarrow C^\infty(X\cup Y;\mathbb{R}^n) \xrightarrow{\delta} C^\infty(X;\mathbb{R}^n) \oplus C^\infty(Y;\mathbb{R}^n) \xrightarrow{\pi} C^\infty(X\cap Y;\mathbb{R}^n) \longrightarrow 0$$

of $\mathbb{R}$-modules is exact, where δ is the map

$$F \longmapsto (F|_X, F|_Y)$$

and π is the map

$$(F, G) \longmapsto F|_{X\cap Y} - G|_{X\cap Y}.$$

These maps are continuous so $\ker\pi$ is closed in $C^\infty(X;\mathbb{R}^n) \oplus C^\infty(Y;\mathbb{R}^n)$ and $\ker\pi \cap C^\infty(X;\mathbb{R}^n) \oplus \{0\}$ is closed in $C^\infty(X;\mathbb{R}^n)$. The space $\ker\pi \cap C^\infty(X;\mathbb{R}^n) \oplus \{0\}$ coincides with the closed ideal $\mathfrak{m}^\infty_{X\cap Y}$ of Whitney functions on X that are flat on $X\cap Y$. Under the assumption that the chain complex above is exact, this ideal coincides with the subspace $\operatorname{im}\delta \cap C^\infty(X;\mathbb{R}^n) \oplus \{0\}$ of Whitney functions on X that can be extended to a Whitney function on $Y\cup X$ flat on Y, which is precisely the space $p(\mathfrak{m}^\infty_Y)$ of functions that admit a representative that is flat on Y.

(2 ⇒ 1) Let $(F,G)\in\ker\pi$, then we should show that $(F,G)\in\operatorname{im}\delta$. We may assume that $G=0$, otherwise we use that the map $C^\infty(X\cup Y;\mathbb{R}^n)\to C^\infty(X;\mathbb{R}^n)$ is a surjection to lift G to some $\overline{G}$ and replace (F,G) by $(F,G)-\delta\overline{G}$. It follows that it suffices to argue that the inclusion $\operatorname{im}\delta\cap C^\infty(X;\mathbb{R}^n)\oplus\{0\}\subset\ker\delta\cap C^\infty(X;\mathbb{R}^n)\oplus\{0\}$ is an equality. Since π is continuous and $\operatorname{im}\delta\cap C^\infty(X;\mathbb{R}^n)\oplus\{0\}$ is closed by assumption, it suffices to show that the inclusion is dense. Let $f\in C^\infty(\mathbb{R}^n)$ be a lift of $F\in\ker\delta\cap C^\infty(X;\mathbb{R}^n)\oplus\{0\}$, then $f\in\mathfrak{m}^\infty_{X\cap Y}$. By Whitney's spectral theorem, the ideal $\mathfrak{m}^\infty_{X\cap Y}$ is the closure of the ideal $\mathfrak{m}^g_{X\cap Y}$, so there exists a sequence $\{f_m\}_m\subset\mathfrak{m}^g_{X\cap Y}$ converging to f, and therefore the associated sequence of Whitney jets $\{F_m\}_m$ converges to F. Suppose that f_m vanishes in a neighbourhood U_m of $X\cap Y$, then $X\smallsetminus U_m$ and $Y\smallsetminus U_m$ are disjoint closed sets, so we may modify f_m by a function φ_m that equals 1 in a neighbourhood of $X\smallsetminus U_m$ and equals 0 in a neighbourhood of $Y\smallsetminus U_m$ and conclude that the sequence $\{F_m\}_m$ lies in $\operatorname{im}\delta\cap C^\infty(X;\mathbb{R}^n)\oplus\{0\}$.

(1 ⇒ 4) The diagram in the statement of the proposition is a pushout if and only if the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^{2n}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^n) \arrow[d] \\ C^\infty(X;\mathbb{R}^n)\otimes^\infty C^\infty(Y;\mathbb{R}^n) \arrow[r] & C^\infty(X\cap Y;\mathbb{R}^n) \end{tikzcd}$$

is a pushout, where the upper diagonal map is the fold map. The maps $C^\infty(X;\mathbb{R}^n)\to C^\infty(X\cap Y,\mathbb{R}^n)$ and $C^\infty(Y;\mathbb{R}^n)\to C^\infty(X\cap Y,\mathbb{R}^n)$ both factor through the map $\Delta^*:C^\infty(X\times Y;\mathbb{R}^n)\to C^\infty(X\cap Y;\mathbb{R}^{2n})$ induced by restricting Whitney functions to the diagonal. It follows from corollary 4.1.6.5 that we are reduced to proving that the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^{2n}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^n) \arrow[d] \\ C^\infty(X\times Y;\mathbb{R}^{2n}) \arrow[r] & C^\infty(X\cap Y;\mathbb{R}^n) \end{tikzcd}$$

is a pushout of C^∞-rings. Using unramifiedness, and the projective resolution $C^\infty(\mathbb{R}^{2n})[e_1,\dots,e_n]$ with $\partial e_i=x_i-y_i$ for $C^\infty(\mathbb{R}^n)$ as a $C^\infty(\mathbb{R}^{2n})$-module, we are reduced to proving that the canonical map $C^\infty(X\times Y;\mathbb{R}^{2n})/(\{x_i-y_i\}_i)\to C^\infty(X\cap Y)$ is an equivalence. If X and Y were open sets, this would follow from Hadamard's lemma, but this result does not hold for Whitney functions in general. Thus, given a Whitney function F on $X\times Y$ that vanishes when restricted to the diagonal $X\times Y\cap\mathbb{R}^n\subset\mathbb{R}^{2n}$, we need to show that there exists a representative $f:\mathbb{R}^{2n}\to\mathbb{R}$ of F which vanishes on the entire diagonal $\mathbb{R}^n\subset\mathbb{R}^{2n}$. To this end, we make the following claim. Let $\Delta\subset\mathbb{R}^{2n}$ be the diagonal consisting of points $(x_1,\dots,x_n,y_1,\dots,y_n)$ for which $x_i=y_i$ for $1\le i\le n$.

(∗) The sets $X\times Y$ and Δ are regularly situated in $\mathbb{R}^{2n}$.

We prove (∗). Let $p=(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2n}$, then $d(p,\Delta_{k+1})=1/\sqrt{2}d(\mathbf{x},\mathbf{y})$. The distance $d(p,X\times Y\cap\Delta)$ is given by

$$d(p,X\times Y\cap\Delta)\le\inf_{q\in X\cap Y}d(\mathbf{x},q)+d(\mathbf{y},q)\le\inf_{q\in X\cap Y}d(\mathbf{x},\mathbf{y})+2d(\mathbf{y},q)=d(\mathbf{x},\mathbf{y})+2d(\mathbf{y},X\cap Y).$$

Since X and Y are regularly situated, we have for some constants $C>0$ and $\lambda\ge0$ an inequality

$$Cd(\mathbf{y},X\cap Y)\le d(\mathbf{y},X)\le d(\mathbf{y},\mathbf{x}).$$

We may without loss suppose that $0\le\lambda\le1$, then we have

$$Cd(p,X\times Y\cap\Delta)^\lambda\le C(d(\mathbf{x},\mathbf{y})+2d(\mathbf{y},X\cap Y))^\lambda\le Cd(\mathbf{x},\mathbf{y})+2Cd(\mathbf{y},X\cap Y)^\lambda\le C'd(\mathbf{x},\mathbf{y})$$

which confirms our claim. Now let F be a Whitney function on $X\times Y$ and suppose that $F|_{X\times Y\cap\Delta}$ vanishes. Choose a representative $f:\mathbb{R}^{2n}\to\mathbb{R}$ whose Whitney jet is F, then $f|_\Delta\in\mathfrak{m}^\infty_{X\times Y\cap\Delta}\subset C^\infty(\Delta)$. We may view $f|_\Delta$ as a function on $\mathbb{R}^{2n}$ constant in the directions orthogonal to Δ, so that $f|_\Delta\in\mathfrak{m}^\infty_{X\times Y\cap\Delta}$ as a function in $C^\infty(\mathbb{R}^{2n})$. Since $X\times Y$ and Δ are regularly situated, lemme 4.5 of [Tou72] provides a multiplier $\varphi:\Delta\smallsetminus X\times Y\cap\Delta\to\mathbb{R}$ for the ideal $\mathfrak{m}^\infty_{X\times Y\cap\Delta}$ that is equal to 0 in a neighbourhood of $X\times Y\smallsetminus X\times Y\cap\Delta$ and is equal to 1 is a neighbourhood of $\Delta\smallsetminus X\times Y\cap\Delta$. Since φ is a multiplier for $\mathfrak{m}^\infty_{X\times Y\cap\Delta}$, the function $f|_\Delta\varphi$ defined on $\mathbb{R}^{2n}\smallsetminus X\times Y\cap\Delta$ extends uniquely to a C^∞-function on $\mathbb{R}^{2n}$ which is flat on $X\times Y\cap\Delta$, and by construction of φ, this function is also flat on $X\times Y$. Now consider

$$\tilde{f}:=f-f|_\Delta\varphi,$$

then the Whitney jet of $\tilde{f}$ is F as $f|_\Delta\varphi\in\mathfrak{m}^\infty_{X\times Y}$ and $\tilde{f}$ vanishes on Δ by construction of φ. It follows from Hadamard's lemma that we may write $\tilde{f}=\sum_i g_i(x_i-y_i)$ so that F lies in the ideal $(\{x_i-y_i\}_i)\subset C^\infty(X\times Y;\mathbb{R}^{2n})$, where we now understand x_i and y_i as Whitney functions on $X\times Y$. We conclude that the map $C^\infty(X\times Y;\mathbb{R}^{2n})\to C^\infty(X\times Y\cap\Delta;\Delta)\cong C^\infty(X\cap Y;\mathbb{R}^n)$ coincides with the projection $C^\infty(X\times Y;\mathbb{R}^{2n})\to C^\infty(X\times Y;\mathbb{R}^{2n})/(\{x_i-y_i\}_i)$.

(4 ⇒ 5) Since the 0'th truncation of the pushout A is $C^\infty(X \cap Y; \mathbb{R}^n)$ by assumption, it suffices to show that the pushout is 0-truncated, but since $\pi_0(A)$ is jet determined and the relative cotangent complex of the composition $C^\infty(\mathbb{R}^n) \to A$ vanishes by theorem 5.1.1.26 and the transitivity fibre sequence of point (1) of remark 5.1.0.4, another application of theorem 5.1.1.26 grants the result.

(5 ⇒ 4) Obvious.

(4 ⇒ 2) If the diagram in the statement of the proposition is a pushout, unramifiedness shows that it is also a pushout of commutative $\mathbb{R}$-algebras, so by standard commutative algebra we know that the map $q' : C^\infty(X; \mathbb{R}^n) \to C^\infty(X \cap Y; \mathbb{R}^n)$ is a quotient by the ideal $p(\mathfrak{m}_Y^\infty)$. However, the map q' is a morphism of C^∞-rings, and thus continuous for the natural topologies on the domain and codomain, which are both Fréchet. Hence $\ker q' = p(\mathfrak{m}_Y^\infty)$ is closed.

□

Remark 5.1.2.2. The preceding proposition in fact holds, and its proof is valid, for any pair X and Y of closed sets in $\mathbb{R}^n$ given the generalization of theorem 5.1.1.26 mentioned in the previous subsection.

5.1.3 Square zero extensions, Postnikov towers and obstruction theory

Given two simplicial C^∞-rings A and B, the problem of finding a morphism $A \to B$ can be broken down into two stages.

(1) Construct a morphism $\pi_0(A) \to \pi_0(B)$, which is a problem in classical C^∞-geometry.

(2) Find a way to lift the map $A \to \pi_0(A) \to \pi_0(B)$ along the map $B \to \tau_{\leq 0} B = \pi_0(B)$.

Stage (1) may be easy or entirely intractable depending on the situation at hand, and naturally, one cannot expect to discover a general method for finding maps between C^∞-rings. In this subsection, we focus on problem (2), which does admit a surprising degree of systematization. First, we may observe that this problem decomposes into an infinite series of lifting problems

$$\begin{tikzcd} & \vdots \arrow[d] \\ & \tau_{\leq n} B \arrow[d] \\ & \vdots \arrow[d] \\ & \tau_{\leq 1} B \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] \arrow[uur, dotted] \arrow[uuur, dotted] & \pi_0(B) \end{tikzcd}$$

along the Postnikov tower of B, so problem (2) may be recast as the following question: what data is required to lift a map $A \to \tau_{\leq n} B$ to a map $A \to \tau_{\leq (n+1)} B$? This question admits a satisfactory answer in terms of the cotangent complex of A.

Definition 5.1.3.1. Unstraightening the square zero extension functor

$$\mathsf{Mod}^{\mathrm{cn}} \longrightarrow \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring}) \longrightarrow sC^\infty\mathsf{ring}$$

carrying (A, M) to $A \oplus M$ determines a functor $p : \mathcal{M}_T \to \Delta^1 \times sC^\infty\mathsf{ring}$ such that the composition $\mathcal{M}_T \to \Delta^1 \times sC^\infty\mathsf{ring} \to sC^\infty\mathsf{ring}$ is a biCartesian fibration associated to the absolute cotangent complex, the *tangent correspondence* of $sC^\infty\mathsf{ring}$. We can identify an $\mathbb{R}$-linear derivation $A \to M$ with a functor $\Delta^1 \to \mathcal{M}_T$ fitting into a commuting diagram

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & \mathcal{M}_T \arrow[d] \\ \Delta^1 \times \{A\} \arrow[r] & \Delta^1 \times sC^\infty\mathsf{ring} \end{tikzcd}$$

We let $\mathsf{Der}(sC^\infty\mathsf{ring})$ denote the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{M}_T) \times_{\mathrm{Fun}(\Delta^1,\Delta^1 \times sC^\infty\mathsf{ring}} sC^\infty\mathsf{ring}$ of derivations.
Let $\overline{\mathsf{Der}}(sC^\infty\mathsf{ring})$ be the full subcategory of $\mathrm{Fun}(\Delta^1\times\Delta^1, \mathcal{M}_T) \times_{\mathrm{Fun}(\Delta^1\times\Delta^1,\Delta^1\times sC^\infty\mathsf{ring})} \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$ spanned by pullback diagrams

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[d, "\eta"] \\ 0 \arrow[r] & M \end{tikzcd}$$

where the upper horizontal map belongs to $sC^\infty\mathsf{ring}$ and the lower horizontal one to Mod, and the lower left corner is p-initial. The projection map $\overline{\mathsf{Der}}(sC^\infty\mathsf{ring}) \to \mathsf{Der}(sC^\infty\mathsf{ring})$ is a trivial Kan fibration, so we may choose a section s and consider the map

$$\Phi : \mathsf{Der}(sC^\infty\mathsf{ring}) \xrightarrow{s} \overline{\mathsf{Der}}(sC^\infty\mathsf{ring}) \longrightarrow \mathrm{Fun}(\Delta^1, sC^\infty\mathsf{ring})$$

Let A be a simplicial C^∞-ring, and let $M \in \mathsf{Mod}_A$ be an A-module. Given a derivation $d : A \to M$ determined by a map $\mathbb{L}_A \to M$ of A-modules, or equivalently a map $d_\eta : A \to A \oplus \tau_{\geq 0} M$ the value $\Phi(A)$ is a map $\tilde{A} \to A$ fitting into a Cartesian square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[d, "\eta_d"] \\ A \arrow[r, "\eta_0"] & A \oplus \tau_{\geq 0} M. \end{tikzcd}$$

A morphism $\tilde{A} \to A$ of simplicial C^∞-rings is a *square zero extension of A by $M[-1]$* if there exists a derivation $d : \mathbb{L}_A \to M$ such that $\tilde{A}$ fits into a Cartesian square as above.
The functor Φ admits a left adjoint Ψ, which carries a map $\tilde{A} \to A$ to the derivation $(A, d : \mathbb{L}_A \to \mathbb{L}_{A/\tilde{A}})$; it is easy to verify that the commuting square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A_d[\mathbb{L}_{A/\tilde{A}}] \arrow[d] \\ A \arrow[r, equal] & A \end{tikzcd}$$

exhibits a unit transformation.

Since a square zero extension by M is defined up to equivalence by the A-module M and the derivation d, we will usually denote it $A_d[M]$. Knowing that a given map $A' \to A$ is a square zero extension gives a powerful method of constructing maps from another simplicial C^∞-ring B into A'; indeed, given a map $f : B \to A$, then f lifts to a map f' as in the commuting diagram

$$\begin{tikzcd} & A' \arrow[dr] & \\ B \arrow[ur, "f'"] \arrow[rr, "f"] & & A \end{tikzcd}$$

if and only if the induced map $f^*\mathbb{L}_B \to \mathbb{L}_A \to M[1]$ is nullhomotopic. By the cofibre sequence of remark 5.1.0.3, the existence of such a homotopy is in turn equivalent to the condition that $\mathbb{L}_A \to M[1]$ factors through the relative cotangent complex $\mathbb{L}_A \to \mathbb{L}_{B/A}$. Reasoning like this allows one to split the problem of finding maps between derived manifolds, or more generally simplicial C^∞-rings, into two parts: a 'global' one, that is, defining the map $B \to A$, and an 'infinitesimal' one having to do with the cotangent complex. Note also that the 'infinitesimal' part is purely algebraic: it asks only that the obstruction class in $\mathrm{Ext}^0_{\mathsf{Mod}_A}(f^*\mathbb{L}_B, M[1])$ vanishes, where the Ext group is the 0'th homology of the complex of homomorphisms in the dg-category of dg A^{alg}-modules. If the obstruction vanishes, the set of connected components of the space of lifts is a torsor for $\mathrm{Ext}^0_{\mathsf{Mod}_A}(f^*\mathbb{L}_B, M)$. We formulate these ideas more precisely in the following proposition.

Proposition 5.1.3.2. *Let $f : A \to B$ be a morphism of simplicial C^∞-rings, and let M be a connective module of a simplicial C^∞-ring C equipped with a derivation $d : \mathbb{L}_C \to M[1]$. Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C_d[M] \arrow[d] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

Then this diagram determines a point $x \in \mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1])$ such that there is a canonical equivalence

$$\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C_d[M]) \simeq \Omega_{x,0}\mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1]).$$

Proof. Since the ∞-category Λ^2_2 is weakly contractible, it follows from from the dual of lemma 4.1.1.29 and the dual of [Lur17b], prop. 1.2.13.8 that the inner fibration $sC^\infty\mathsf{ring}_{A//C} \to sC^\infty\mathsf{ring}$ preserves and reflects pullbacks. The square zero extension $C_d[M]$ is a limit of such a diagram, so we have a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C_d[M]) \arrow[r] \arrow[d] & \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C) \arrow[d] \\
\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C) \arrow[r] & \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C \oplus M[1])
\end{tikzcd}$$

Clearly, the map $* \to \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C)$ specifiying $g : B \to C$ is a weak homotopy equivalence, so the lower horizontal map determines a point $x \in \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C \oplus M[1])$ which is given by a composition $B \overset{g}{\to} C \overset{\eta_d}{\to} C \oplus M[1]$, and the right vertical map determines a point $y \in \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C \oplus M[1])$ which is given by a composition $B \overset{g}{\to} C \overset{\eta_0}{\to} C \oplus M[1]$. Note that there is commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(C, C \oplus M[1]) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C \oplus M[1]) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathsf{Mod}_C}(F(C), M[1]) \arrow[r] & \mathrm{Hom}_{\mathsf{Mod}_C}(F(B), M[1])
\end{tikzcd}$$

in $\mathcal{H}$, where F is the left adjoint to the functor taking trivial square zero extensions. By definition, $F(C) = \mathbb{L}_{C/A}$ and by corollary 5.1.0.8, $F(B) \simeq g_!\mathbb{L}_{B/A}$. It follows from the commutativity of the diagram above that the point $y \in \mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C \oplus M[1])$ corresponds to a composition $g_!\mathbb{L}_{B/A} \to \mathbb{L}_{C/A} \overset{0}{\to} M[1]$, so that y is in fact equivalent to the zero map. Now it follows that the space $\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C_d[M])$ fits into a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}_{sC^\infty\mathsf{ring}_{A//C}}(B, C_d[M]) \arrow[r] \arrow[d] & * \arrow[d, "0"] \\
* \arrow[r, "x"] & \mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1])
\end{tikzcd}$$

as desired. □

Remark 5.1.3.3. In the situation of proposition 5.1.3.2, we have that the space of dotted lifts

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & C_d[M] \arrow[d] \\
B \arrow[r, "g"] \arrow[ur, dotted] & C
\end{tikzcd}$$

extending the square to a 3-simplex is equivalent to $\Omega_{x,0}\mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1])$ (see [Lur17b], lem. 5.2.8.22).

The previous lemma classifies the extension problem for maps along square zero extensions in terms of the cotangent complex. The relevance to problem (2) mentioned above is the content of the following proposition.

Proposition 5.1.3.4. *Let A be a simplicial C^∞-ring. For each $n \geq 0$, the map $\tau_{\leq(n+1)}A \to \tau_{\leq n}A$ is a square zero extension. Moreover, the relative cotangent complex $\mathbb{L}_{\tau_{\leq n}A/\tau_{\leq(n+1)}A}$ is $(n+2)$-connective and we have a canonical isomorphism $\pi_{n+1}(A) \simeq \pi_{n+2}(\mathbb{L}_{\tau_{\leq n}A/\tau_{\leq(n+1)}A})$.*

The proposition asserts that the Postnikov tower of a simplicial C^∞-ring is a sequence of square zero extensions. Before we give the proof, we need a definition.

Definition 5.1.3.5. A map $f : A' \to A$ of simplicial C^∞-rings is an *n-connective extension* for $n \geq 0$ if $\mathrm{fib}(f)$ is n-connective. We say that f is an *n-small extension* if

(1) $\mathrm{fib}(f)$ is n-connective.

(2) $\mathrm{fib}(f)$ is $(2n)$-truncated in $\mathsf{Mod}^{cn}_{\mathbb{R}}$.

(3) The multiplication map
$$\mathrm{fib}(f) \otimes_{A'} \mathrm{fib}(f) \longrightarrow \mathrm{fib}(f)$$
is nullhomotopic.

We let $\mathrm{Fun}_{n-\mathrm{sm}}(\Delta^1, sC^\infty\mathsf{ring})$ be the full subcategory spanned by n-small extensions.

Proposition 5.1.3.6. *Let* $\mathsf{Der}_{n\text{-sm}}(sC^\infty\mathsf{ring}) \subset \mathsf{Der}(sC^\infty\mathsf{ring})$ *be the full subcategory of those derivations* $A \to M$ *such that* $M[-1]$ *is* n*-connective and* $2n$*-truncated. Then the functor* Φ *induces an equivalence* $\mathsf{Der}_{n\text{-sm}}(sC^\infty\mathsf{ring}) \simeq \mathrm{Fun}_{n\text{-sm}}(\Delta^1, sC^\infty\mathsf{ring})$.

Proof. One readily verifies that Φ carries $\mathsf{Der}_{n\text{-sm}}(sC^\infty\mathsf{ring})$ into $\mathrm{Fun}_{n\text{-sm}}(\Delta^1, sC^\infty\mathsf{ring})$, determining a functor $\Phi_{n\text{-sm}}$. This functor admits a left adjoint $\Psi_{n\text{-sm}}$ given by the assignment

$$(\tilde{A} \to A) \longmapsto (A, d : \mathbb{L}_A \to \tau_{\leq 2n+1}\mathbb{L}_{\tilde{A}/A}).$$

The functor $\Phi_{n\text{-sm}}$ is clearly conservative, so we are reduced to verifying that the map $\tilde{A} \to A_d[\tau_{\leq 2n+1}\mathbb{L}_{A/A'}]$ is an equivalence. We have morphisms of fibre sequences

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d, "h"] & \tilde{A} \arrow[r] \arrow[d] & A \arrow[d, equal] \\
\mathbb{L}_{A/A'}[-1] \arrow[r] \arrow[d, "g"] & A_d[\mathbb{L}_{A/A'}] \arrow[r] \arrow[d] & A \arrow[d, equal] \\
\tau_{\leq(2n+1)}\mathbb{L}_{A/A'}[-1] \arrow[r] & A_d[\tau_{\leq(2n+1)}\mathbb{L}_{A/A'}] \arrow[r] & A
\end{tikzcd}$$

Since the map g is a $(2n)$-equivalence, it suffices to check that h is a $(2n)$-equivalence. Now h factors as

$$\mathrm{fib}(f) \xrightarrow{h'} \mathrm{fib}(f) \otimes_{\tilde{A}} A \xrightarrow{h''} \mathbb{L}_{A/\tilde{A}}[-1],$$

where the map h'' is $(2n+1)$-connective as a consequence of the assumption that $\mathrm{fib}(f)$ is n-connective (so that $\mathrm{cofib}(f)$ is $(n+1)$-connective) and proposition 5.1.1.1, so we only have to show that the map $h' : \mathrm{fib}(f) \longrightarrow \mathrm{fib}(f) \otimes_{\tilde{A}} A$ is a $(2n)$-equivalence. We have a fibre sequence

$$\mathrm{fib}(f) \otimes_{\tilde{A}} \mathrm{fib}(f) \longrightarrow \mathrm{fib}(f) \otimes_{\tilde{A}} \tilde{A} \simeq \mathrm{fib}(f) \xrightarrow{h'} \mathrm{fib}(f) \otimes_{\tilde{A}} A$$

where the first map is the multiplication on $\mathrm{fib}(f)$. Since $\mathrm{fib}(f)$ is n-connective, the groups $\pi_k(\mathrm{fib}(f) \otimes_{\tilde{A}} \mathrm{fib}(f))$ vanish for $k < 2n$, so the fibre sequence above shows that h' is $(2n)$-connective. Thus, the second map in the exact sequence

$$\pi_{2n}(\mathrm{fib}(f) \otimes_{\tilde{A}} \mathrm{fib}(f)) \longrightarrow \pi_{2n}(\mathrm{fib}(f)) \longrightarrow \pi_{2n}(\mathrm{fib}(f) \otimes_{\tilde{A}} A)$$

is a surjection. By assumption, the multiplication on $\mathrm{fib}(f)$ is nullhomotopic, so this map is also an injection and we conclude that h' is a $(2n)$-equivalence. □

Proof of proposition 5.1.3.4. According to proposition 5.1.3.6, we only have to show that each map $\tau_{\leq(n+1)}A \to \tau_{\leq n}A$ is an $(n+1)$-small extension, but the fibre of this map can be identified with the object $\pi_{n+1}(A)[n+1]$ which is obviously $(n+1)$-connective, $(2n+2)$-truncated and has vanishing multiplication. Proposition 5.1.3.6 also provides us with an isomorphism $\pi_{n+1}(A) \simeq \pi_{n+2}(\mathbb{L}_{\tau_{\leq n}A/\tau_{\leq(n+1)}A})$. □

Remark 5.1.3.7. Proposition 5.1.3.6 could have been proven using proposition 5.1.0.16 to reduce to the algebraic situation, and using the results in [Lur17a] section 7.4.2. We have opted to give a more elementary proof, which does not rely on Dunn-Lurie additivity (observe that the proof also works for $\mathbb{E}_\infty$-algebra objects in any presentably symmetric monoidal stable ∞-category $\mathcal{C}$ equipped with a t-structure such that the tensor product carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$).

We now have tools to construct Postnikov towers of simplicial C^∞-rings if we are given a cotangent complex. For instance, we have the following result on liftings of étale mappings.

Proposition 5.1.3.8. *Let* $f : A \to B(0)$ *be a morphism of fair simplicial* C^∞*-rings where* $B(0)$ *is* 0*-truncated. If the induced morphism* $\pi_0(A) \to B(0)$ *is étale, then there exists an object* $B \in sC^\infty\mathsf{ring}_{A//B(0)}$ *such that the map* $B \to B(0)$ *induces an equivalence* $\pi_0(B) \cong B(0)$ *in* $C^\infty\mathsf{ring}$ *and the map* $A \to B$ *is étale.*

We recall for the reader's convenience the following easy lemma.

Lemma 5.1.3.9. *Let* $f : A \to B$ *be a* 0*-equivalence of simplicial* C^∞*-rings, and let* M *be a connective* A*-module. If* $M \otimes_A B$ *is* n*-connective for* $n \geq 1$*, then* M *is* n*-connective.*

Proof. Suppose that $M \otimes_A B$ is n-connective for some $n \geq 1$, then the second page of the torsion spectral sequence yields

$$0 = \pi_0(M \otimes_A B) \cong \pi_0(M) \otimes_{\pi_0(A)} \pi_0(A) \cong \pi_0(M)$$

and M is 1-connective. Now suppose for the sake of induction that M is k-connective for $k \leq n-1$, then $M[-k]$ is connective so the torsion spectral sequence again gives equivalences

$$0 = \pi_k(M \otimes_A B) \cong \pi_0(M[-k] \otimes_A B) \cong \pi_k(M)$$

so M is $(k+1)$-connective. □

Proof of proposition 5.1.3.8. We inductively define a tower of simplicial C^∞-rings

$$\ldots \longrightarrow B(n) \longrightarrow B(n-1) \longrightarrow \ldots \longrightarrow B(1) \longrightarrow B(0)$$

under A with the following properties:

(1) $B(n)$ is n-truncated.

(2) The $B(n)$-module $\mathbb{L}^{\mathrm{cplt}}_{B(n)/A}$ is $(n+1)$-connective.

(3) The map $\tau_{\leq(n-1)}B(n) \to B(n-1)$ is an equivalence.

Note that for $n = 0$, $B(0)$ satisfies the conditions above since $\pi_0(\mathbb{L}^{\mathrm{cplt}}_{B(0)/A}) \cong \Omega^{1\mathrm{cplt}}_{B(0)/\pi_0(A)} = 0$ because $\pi_0(A) \to B(0)$ is an equivalence. Suppose that $A \to B(n-1)$ is already defined and satisfies the conditions above, then we have a derivation $d : \mathbb{L}^{\mathrm{cplt}}_{B(n-1)} \to \tau_{\leq(n+1)}\mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A}$ induced by the canonical derivation $\mathbb{L}^{\mathrm{cplt}}_{B(n-1)} \to \mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A}$ so we may define $B(n)$ as the square zero extension $B(n-1)_d[\tau_{\leq(n+1)}\mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A}[-1]]$. There is a fibre sequence of $B(n)$-modules

$$\tau_{\leq(n+1)}\mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A}[-1] \longrightarrow B(n) \longrightarrow B(n-1).$$

Since $B(n-1)$ satisfies conditions (1) through (3) above, it follows that $\pi_n(B(n)) \cong \tau_{\leq(n+1)}\mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A}[-n-1] \cong \pi_{n+1}(\mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A})$, that $B(n)$ is n-truncated and that the map $\tau_{\leq(n-1)}B(n) \to B(n-1)$ is an equivalence. It follows from proposition 5.1.3.4 that the second map in the fibre sequence

$$\mathbb{L}^{\mathrm{cplt}}_{B(n)/A} \otimes_{B(n)} B(n-1) \longrightarrow \mathbb{L}^{\mathrm{cplt}}_{B(n-1)/A} \longrightarrow \mathbb{L}^{\mathrm{cplt}}_{B(n-1)/B(n)}$$

is $(n+1)$-connective, so that $\mathbb{L}^{\mathrm{cplt}}_{B(n)/A} \otimes_{B(n)} B(n-1)$ is $(n+1)$-connective. Since the map $B(n-1) \to B(n)$ is n-connective, it follows from lemma 5.1.3.9 that $\mathbb{L}^{\mathrm{cplt}}_{B(n)/A}$ is also $(n+1)$-connective. Now we have constructed $B(n)$ satisfying conditions (1) through (3) above. Conditions (1) and (3), together with [Lur17b], prop. 5.5.6.26 and the fact that Postnikov towers are convergent in $sC^\infty\mathsf{ring}$ guarantee that we have a Postnikov tower

$$B := \lim_k B(k) \longrightarrow \ldots \longrightarrow B(n) \longrightarrow B(n-1) \longrightarrow \ldots \longrightarrow B(1) \longrightarrow B(0)$$

and for all $n \geq 0$, we see that condition (2) and the fibre sequence

$$\mathbb{L}^{\mathrm{cplt}}_{B/A} \otimes_B B(n) \longrightarrow \mathbb{L}^{\mathrm{cplt}}_{B(n)/A} \longrightarrow \mathbb{L}^{\mathrm{cplt}}_{B(n)/B}$$

establish that $\mathbb{L}^{\mathrm{cplt}}_{B/A} \otimes_B B(n)$ is n-connective. The map $B \to B(n)$ is $(n+1)$-connective, so by lemma 5.1.3.9 again, $\mathbb{L}^{\mathrm{cplt}}_{B/A}$ is also n-connective for all n, that is $\mathbb{L}^{\mathrm{cplt}}_{B/A} = 0$. Now we conclude using corollary 5.1.1.4. □

Remark 5.1.3.10. The proof of proposition 5.1.3.8 applies to lift atlases on stacks. By a standard procedure in derived geometry, given a derived stack and $U(0) \to \tau_{\leq 0}X$ an étale or submersive atlas in the category of stacks on $C^\infty\mathsf{ring}_{\mathrm{fair}}$, then $U(0)$ can be lifted to an atlas $U \to X$ provided that X has a cotangent complex and that X interacts nicely with the constructions of proposition 5.1.3.8; that is, X should be compatible with square zero extensions and Postnikov towers.

The main objective of the of this section is to describe local properties of morphisms in terms of differential data. We have already done this for the properties of being (almost) finitely presented. Our next goal is to show analogues of the inverse function and constant rank theorems.

Definition 5.1.3.11. A map $f : A \to B$ of simplicial C^∞-rings is

(1) *formally étale* if $\mathbb{L}_{B/A} \simeq 0$.

(2) *formally submersive* if $\mathbb{L}_{B/A}$ is a projective B-module.

Remark 5.1.3.12. In algebraic geometry, it standard to also introduce the condition of being *formally unramified* or equivalently, *formally immersive* on a morphism $f: A \to B$ by demanding that $\pi_0(\mathbb{L}_{B/A})$ vanishes, but we will not make much use of this terminology.

We will start by rephrasing the conditions of being formally étale and submersive in terms of lifting properties against infinitesimal extensions of objects. Since such extensions abound in derived geometry, the usefulness of this reformulation can hardly be overstated.

Definition 5.1.3.13. A map $g: \tilde{C} \to C$ of simplicial C^∞-rings is a *nilpotent extension* if g is an effective epimorphism and $\pi_0(g): \pi_0(\tilde{C}) \to \pi_0(C)$ has nilpotent kernel.

Nilpotent extensions appear much more frequently than square-zero extensions, but when proving theorems in practice, it usually suffices to only consider the latter case, as the following lemma shows.

Lemma 5.1.3.14. *Let $g: \tilde{C} \to C$ be a nilpotent extension, then there exists a sequence*

$$\ldots \to C(k) \to \ldots \to C(1) \to C(0) := C$$

as an object of $\mathrm{Fun}(\mathbf{N}(\mathbb{Z}_{\geq 0}^{\triangleleft})^{op}, sC^\infty\mathsf{ring}_{\tilde{C}/})$ *with limit $\tilde{C}$, where each $C(k+1) \to C(k)$ is a square zero extension.*

Proof. Denote $I = \ker(\pi_0(f))$ and suppose that $I^n = 0$, then we have a sequence

$$\pi_0(\tilde{C}) \longrightarrow \pi_0(\tilde{C})/I^{n-1} \longrightarrow \pi_0(\tilde{C})/I^{n-2} \longrightarrow \ldots \longrightarrow \pi_0(C)$$

of length n where the fibre of each map is of the form I^{k-1}/I^k for $n \geq k \geq 1$. Since I^{k-1}/I^k is 0-connective, 0-truncated and has vanishing multiplication, we conclude that each map in the sequence above is a 0-small extension, so proposition 5.1.3.4 shows that the sequence above is a sequence of square zero extensions. Setting $C(k) = C \times_{\pi_0(C)} \pi_0(\tilde{C})/I^{n-k}$ for $n \geq k \geq 0$, we see that each map $C(k) \to C(k-1)$ is a square zero extension by the module $I^{(k-1)}/I^k$ in the range $n \geq k \geq 0$. Now define a $(k+1)$-connective map $f_k: \tilde{C} \to C(k+n)$ inductively as follows: supposing that $f_k: \tilde{C} \to C(k+n)$ has been defined for $k \geq 0$, proposition 5.1.1.1 tells us $\mathbb{L}_{C(k+n)/\tilde{C}}$ is k-connective and we have an equivalence $\pi_k(\mathbb{L}_{C(k+n)/\tilde{C}}) \simeq \pi_{k-1}(\mathrm{fib}(\tilde{C} \to C(k+n)))$. Let M denote the $C(k+n)$-module $\pi_{k-1}(\mathrm{fib}(\tilde{C} \to C(k+n)))$, then we have a derivation $d: \mathbb{L}_{C(k+n)} \to M[k]$ and it suffices to define $C(k+n+1)$ as the square zero extension $C(k+n)_d[M[k]]$. ☐

Proposition 5.1.3.15. *Let $f: A \to B$ be a map of simplicial C^∞-rings.*

(1) *The following are equivalent.*

  (a) *f is formally submersive.*

  (b) *f has the left lifting property with respect to all nilpotent extensions.*

  (c) *f has the left lifting property with respect to all square zero extensions.*

(2) *The following are equivalent.*

  (a) *f is formally étale.*

  (b) *Let $\tilde{C} \to C$ be a nilpotent extension, then the space of dotted lifts in the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \tilde{C} \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & C \end{tikzcd}$$

  is weakly contractible.

  (c) *Let $C_d[M] \to C$ be a square zero extension, then the space of dotted lifts in the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C_d[M] \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & C \end{tikzcd}$$

  is weakly contractible.

Proof. (1) It is clear that $(b) \Rightarrow (c)$. We show that $(a) \Rightarrow (c)$, that $(c) \Rightarrow (b)$, and that $(b) \Rightarrow (a)$. Suppose that f is formally submersive. Consider a commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C_d[M] \arrow[d] \\ B \arrow[r, "g"'] \arrow[ur, dotted] & C \end{tikzcd}$$

where $C_d[M] \to C$ is the square zero extension of C by M. By proposition 5.1.3.2, the obstruction to finding a solution to this lifting problem is a nonzero element of the abelian group $\mathrm{Ext}^0_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1])$. As $g_!\mathbb{L}_{B/A}$ is a retract of a free C-module, the space $\mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1])$ is a retract of a connected space, and is therefore connected; this proves $(a) \Rightarrow (c)$. To show $(c) \Rightarrow (b)$, let $\tilde{C} \to C$ be a nilpotent extension, and choose a sequence

$$\ldots \to C(k) \to \ldots \to C(1) \to C(0) := C$$

with limit $\tilde{C}$, where each $C(k+1) \to C(k)$ is a square zero extension. By assumption, we may solve each successive lifting problem, so we obtain the desired solution by passing to the limit.
We prove $(b) \Rightarrow (a)$. Condition (b) and proposition 5.1.3.2 imply that for any connective B-module M, the abelian group $\mathsf{Ext}^1_{\mathsf{Mod_B}}(\mathbb{L}_{B/A}, M) \cong \mathsf{Ext}^0_{\mathsf{Mod_B}}(\mathbb{L}_{B/A}, M[1])$ vanishes. This proves that $\mathbb{L}_{B/A}$ is projective by appealing to [Lur17a] prop. 7.2.2.6, point (2).

(2) Assume that f is formally étale, then f is formally submersive so the obstruction to the existence of a dotted lift in the commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C_d[M] \arrow[d] \\ B \arrow[r, "g"'] \arrow[ur, dotted] & C \end{tikzcd}$$

vanishes by the previous part of the proof, and the space of such lifts is equivalent to

$$\Omega_0 \mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M[1]) \simeq \mathrm{Hom}_{\mathsf{Mod}_C}(g_!\mathbb{L}_{B/A}, M).$$

This space is weakly contractible as $g_!\mathbb{L}_{B/A}$ is initial by assumption, which proves $(a) \Rightarrow (c)$. Conversely, taking $g = \mathrm{id} : B \to B$ in the diagram above, condition (b) tells us that the space $\mathrm{Hom}_{\mathsf{Mod}_B}(\mathbb{L}_{B/A}, M)$ is weakly contractible for any connective B-module M, so that $\mathbb{L}_{B/A}$ is an initial connective B-module, that is, $\mathbb{L}_{B/A} \simeq 0$. This proves $(c) \Rightarrow (a)$. The implications $(b) \Leftrightarrow (c)$ are easy to prove writing a nilpotent extension as a sequence of square zero extensions.

$\square$

Corollary 5.1.3.16. *Let A be a simplicial C^∞-ring. The truncation functor $\tau_{\leq 0}$ induces an equivalence $sC^\infty\mathsf{ring}^{ét}_{A/} \simeq \mathbf{N}\left(C^\infty\mathsf{ring}^{ét}_{\pi_0(A)/}\right)$ of ∞-categories.*

Proof. Proposition 5.1.3.8 immediately yields essential surjectivity of the functor

$$\tau_{\leq 0} : sC^\infty\mathsf{ring}^{ét}_{A/} \longrightarrow \mathbf{N}\left(C^\infty\mathsf{ring}^{ét}_{\pi_0(A)/}\right).$$

We thus show fully faithfulness. We have a commuting diagram of spaces

$$\begin{tikzcd} \mathrm{Hom}_{sC^\infty\mathsf{ring}^{ét}_{A/}}(B,C) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{N}\left(C^\infty\mathsf{ring}^{ét}_{\pi_0(A)/}\right)}(\pi_0(B),\pi_0(C)) \arrow[d, equal] \\ \mathrm{Hom}_{sC^\infty\mathsf{ring}^{ét}_{A/}}(B,\pi_0(C)) \arrow[r] & \mathrm{Hom}_{\mathbf{N}\left(C^\infty\mathsf{ring}^{ét}_{\pi_0(A)/}\right)}(\pi_0(B),\pi_0(C)) \end{tikzcd}$$

where the lower horizontal arrow is an equivalence by adjunction. Thus, it suffices to show that the fibre of the left vertical map is weakly contractible, but this fibre is exactly the space of dotted lifts in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \pi_0(C) \end{tikzcd}$$

which is weakly contractible by virtue of proposition 5.1.3.15 and the fact that $C \to \pi_0(C)$ is a nilpotent extension. $\square$

One of the main results of this section is the following theorem.

Theorem 5.1.3.17 (Inverse Function Theorem)**.** *Let $f : A \to B$ be a morphism of fair simplicial C^∞-rings such that the induced map $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is finitely presented. Then f is étale if and only if f is formally étale.*

Remark 5.1.3.18. Note that in the situation of theorem 5.1.3.17, the map f is itself of finite presentation by proposition 5.1.1.8.

The proof goes along the lines of the one for the inverse function theorem in algebraic geometry (for the étale topology), supplemented by the usual implicit function theorem in differential geometry. First, we recall from standard commutative algebra the Fitting invariant of a finitely presented module:

Lemma 5.1.3.19 (The Fitting Ideal Lemma)**.** *Let k be a commutative ring, and let M be a finitely presented k-module. Choose a presentation $k^p \overset{K}{\to} k^q \to M$ and define for each $0 \le n \le q$ the* Fitting ideal $\mathrm{Fit}_n(M)$ *as the ideal of k generated by the $(q-n)\times(q-n)$-minors of the matrix K. Then $\mathrm{Fit}_n(M)$ does not depend on the presentation of M.*

Proof. See e.g. [Eis95], chapter 21. □

It follows from the Fitting ideal lemma that $\mathrm{Fit}_0(0) = k$, since the identity map $k \overset{\mathrm{id}}{\to} k$ gives a presentation of 0. As a first step to theorem 5.1.3.17, we prove that under some finiteness conditions, a formally unramified morphism is locally a closed immersion.

Lemma 5.1.3.20. *Let $f : A \to B$ be a morphism of fair simplicial C^∞-rings such that the induced map $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is finitely presented. If $\pi_0(\mathbb{L}_{B/A}) = 0$, then there exist finitely many elements $\{a_i\} \in \pi_0(B)$ that generate the unit ideal such that each of the induced maps $A \to B[a_i^{-1}]$ factors as*

$$A \xrightarrow{f_i'} A_i \xrightarrow{f_i''} B[a_i^{-1}],$$

where f_i' is étale and f_i'' is an effective epimorphism.

Proof. By assumption, $\pi_0(B)$ is finitely presented over $\pi_0(A)$ which is finitely generated, so if we write $\pi_0(A) = C^\infty(\mathbb{R}^n)/I$, then we may write $\pi_0(B)$ as the quotient of $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/I$ by some finitely generated ideal $J = (h_1, \dots, h_l)$. The $\pi_0(B)$-module $\pi_0(\mathbb{L}_{B/A})$ is computed as the module of relative smooth Kähler differentials $\Omega^1_{\pi_0(B)/\pi_0(A)}$, which in this case is the quotient of the free $\pi_0(B)$-module generated by the elements $\{d_{\mathrm{dR}}x_i\}_{1\le i\le m}$, the de Rham differentials of the coordinate functions on $\mathbb{R}^m$, by the submodule generated by the elements $\{d_{\mathrm{dR}}h_j\}_{1\le j\le l}$. Since $\Omega^1_{\pi_0(B)/\pi_0(A)} = 0$, the Jacobian matrix $\left\{\frac{\partial h_j}{\partial x_i}\right\}_{i,j}$ must have rank m, so $l \ge m$. Also, we may conclude that the collection $\{a_i\}$ of $m\times m$-minors of the Jacobian generates the unit ideal in $\pi_0(B)$; this follows because the description of the module of smooth Kähler differentials implies that the ideal generated by the collection of $m\times m$-minors $\{a_i\}$ is the 0'th Fitting ideal of $\Omega^1_{\pi_0(B)/\pi_0(A)}$. Each minor a_i is determined by m functions out of the collection $\{h_1, \dots, h_l\}$ and we may assume without loss that a_i is determined by the first m functions. Now note that the map $\pi_0(A) \to \pi_0(B)[a_i^{-1}]$ factors as

$$\pi_0(A) \cong C^\infty(\mathbb{R}^n)/I \xrightarrow{r_i'} C^\infty(\mathbb{R}^n \times \mathbb{R}^m)[a_i^{-1}]/(I, h_1, \dots, h_m) \xrightarrow{r_i''} C^\infty(\mathbb{R}^n \times \mathbb{R}^m)[a_i^{-1}]/(I, J) \cong \pi_0(B)[a_i^{-1}].$$

The second map is a quotient map and thus a surjection. We claim that the first map is étale; observe that we have localized to the open set U where the the Jacobian matrix of the map $(h_1, \dots, h_m) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is invertible; thus the functions $(h_1, \dots, h_m)$ are independent, and we have an isomorphism $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)[a_i^{-1}]/(h_1, \dots, h_m) \cong C^\infty(Z(h_1, \dots, h_m) \cap U)$. Shrinking U if necessary, the implicit function theorem now yields a smooth function $g : \mathbb{R}^n \supset V \to \mathbb{R}^m$ from some open V such that $V \cong \mathrm{Graph}(g) = Z(h_1, \dots, h_m) \cap U$. Thus, we have a map of C^∞-rings $C^\infty(Z(h_1, \dots, h_m) \cap U) \to C^\infty(V)$ that is inverse to the map $C^\infty(V) \to C^\infty(Z(h_1, \dots, h_m) \cap U)$. After we take the quotient by the ideal I, we see that r_i' is étale. Now we may define A_i via the procedure of proposition 5.1.3.8 from the map $r_i'' := \pi_0(A) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^m)[a_i^{-1}]/(I, h_1, \dots, h_m)$ to obtain an étale A-algebra A_i in an essentially unique way, and since the map $B[a_i^{-1}] \to \pi_0(B)[a_i^{-1}]$ is a nilpotent extension, the space of dotted lifts in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_i'"'] & B[a_i^{-1}] \arrow[d] \\ A_i \arrow[r] \arrow[ur, dotted] & \pi_0(B)[a_i^{-1}] \end{tikzcd}$$

is weakly contractible by proposition 5.1.3.15, so we obtain the desired factorization. □

Remark 5.1.3.21. As the collection $\{a_i\}$ of $m\times m$-minors of the Jacobian in the proof above is finite, it generates a germ determined ideal. Since the lemma asserts that the collection $\{a_i\}$ generates the unit ideal, it follows that the admissible morphisms $B \to B[a_i^{-1}]$ determine an admissible covering of B.

Lemma 5.1.3.22. *Let $f : A \to B$ be an effective epimorphism of simplicial C^∞-rings such that the induced map $\pi_0(f)$ is finitely presented. If $\pi_1(\mathbb{L}_{B/A})$ vanishes, then there exists an element $D \in \pi_0(A)$ that becomes invertible in $\pi_0(B)$ such that the induced map $\pi_0(A)[1/D] \to \pi_0(B)$ is an isomorphism.*

Proof. Since f is an effective epimorphism, proposition 5.1.1.1 shows that the relative cotangent complex is 1-connective, and that there is an isomorphism $\pi_0(\mathrm{fib}(f)) \otimes_{\pi_0(A)} \pi_0(B) \cong \pi_1(\mathbb{L}_{B/A})$. Write $I = \ker(\pi_0(f))$, then we have a surjection

$$\pi_0(\mathrm{fib}(f)) \longrightarrow I$$

of $\pi_0(A)$-modules, and therefore also a surjection

$$\pi_0(\mathrm{fib}(f)) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow I/I^2.$$

It follows that $I/I^2 = 0$ so that $I = I^2$. Since $\pi_0(f)$ is finitely presented, the ideal I is generated by finitely many elements $(g_1, \ldots, g_n)$, and we may write $g_i = \sum_j K_{ij} g_j$, where $K_{ij} \in I$. Now consider the matrix K with entries K_{ij}, and let $D \in \pi_0(A)$ denote the determinant of the matrix $H : \mathrm{id} - K$. The matrix H maps to the identity in $\pi_0(B)$ so that D becomes invertible in $\pi_0(B)$. To see that the canonical map $\pi_0(A)[1/D] \to \pi_0(B)$ is an isomorphism, it suffices to show that a map of C^∞-rings $\pi_0(A) \to C$ sends I to zero if and only if it inverts D, which is clear. □

Remark 5.1.3.23. When the assumptions of lemma 5.1.3.22 are satisfied, the map $\pi_0(A) \to \pi_0(B)$ exhibits a localization of $\pi_0(A)$ with respect to the element D constructed in the proof as an $\mathbb{R}$-algebra *and* as a C^∞-ring. This happens, for instance, when $\mathrm{Spec}_{\mathbb{R}} A$ has multiple connected components, and the map $A \to B$ takes the quotient by the ideal generated by characteristic functions of some, but not all, of those components.

Remark 5.1.3.24. For the conclusions of lemma 5.1.3.22, it does *not* suffice to demand that the map $\pi_0(\mathbb{L}_A \otimes_B A) \to \pi_0(\mathbb{L}_B)$ induces an isomorphism. For instance, let $A = C^\infty(\mathbb{R})$ and take a function g on $\mathbb{R}$ such that g and g' are zero on $\mathbb{R}_{\le 0}$ and nonzero on $\mathbb{R}_{>0}$, then the map $A \to A/(g) = B$ is not étale, but does induce an isomorphism $\pi_0(\mathbb{L}_A \otimes_B A) \to \pi_0(\mathbb{L}_B)$.

Proof of theorem 5.1.3.17. The fact that an étale map has vanishing quasi-coherent relative cotangent complex is the content of proposition 5.1.1.17. For the other direction, let $f : A \to B$ be a morphism between fair simplicial C^∞-rings such that $\mathbb{L}_{B/A}$ vanishes and $\pi_0(f)$ is finitely presented. Then lemma 5.1.3.20 provides us with an admissible covering $\{B \to B[1/b_i]\}_i$ such that each of the induced maps $A \to B[1/b_i]$ factors as an étale map followed by an effective epimorphism. Using that the relative cotangent complex vanishes for étale maps, we may replace B by $B[1/b_i]$ and assume that $f : A \to B$ is an effective epimorphism. Lemma 5.1.3.22 asserts that the underlying map $\pi_0(f)$ is a localization. Now it follows from corollary 5.1.1.4 that f is also a localization. □

The inverse function theorem is most useful in the case of finitely presented maps between simplicial C^∞-rings. In this case, having the cotangent complex $\mathbb{L}_{B/A}$ of $f : A \to B$ vanish at a point $x : B \to \mathbb{R}$ implies that after localizing near x, B and A are equivalent, as the following proposition shows.

Proposition 5.1.3.25. *Let A be a fair simplicial C^∞-ring and let M be an almost perfect A-module.*

(1) *Suppose that for an $\mathbb{R}$-point $x : A \to \mathbb{R}$ the $\mathbb{R}$-module $M \otimes_A \mathbb{R}$ is n-connective. Then there exists some $a \in \pi_0(A)$ such that $x(a) \neq 0$ and $M \otimes_A A[1/a]$ is n-connective.*

(2) *Suppose that M is perfect and that for an $\mathbb{R}$-point $x : A \to \mathbb{R}$ the $\mathbb{R}$-module $M \otimes_A \mathbb{R}$ is a zero object. Then there exists some $a \in \pi_0(A)$ such that $x(a) \neq 0$ and $M \otimes_A A[1/a] \simeq 0$.*

Proof. (1) Since M is almost perfect, M is eventually connective, so we only have to treat the case of M connective and $n > 0$, the case $n = 0$ being trivial. Now lemma 5.1.3.9 tells us that if suffices to show that there exists some $a \in \pi_0(A)$ such that $M \otimes_A \pi_0(A)[1/a]$ is n-connective. Write $M' := M \otimes_A \pi_0(A)$, and suppose for the sake of induction that for some $0 \le k < n$ we have found some $a \in \pi_0(A)$ such that $x(a) \neq 0$ and $M'_k := M' \otimes_{\pi_0(A)} \pi_0(A)[1/a]$ is k-connective. Our goal is to produce an element $a' \in \pi_0(A)$ such that $x(a') \neq 0$ and $M' \otimes_{\pi_0(A)} \pi_0(A)[1/a']$ is $(k+1)$-connective.
$M'_k[-k]$ is connective, so the torsion spectral sequence yields an isomorphism

$$\pi_0(M'_k[-k] \otimes_{\pi_0(A)[1/a]} \mathbb{R}) \cong \pi_k(M'_k) \otimes_{\pi_0(A)[1/a]} \mathbb{R},$$

but clearly, $\pi_0(M'_k[-k] \otimes_{\pi_0(A)[1/a]} \mathbb{R}) \cong \pi_k(M \otimes_A \mathbb{R}) = 0$ by assumption. Because M'_k is almost perfect and k-connective, $\pi_k(M'_k)$ is finitely presented, so if we write $\pi_0(A)_x$ for the local C^∞-ring of germs at $x : A \to \mathbb{R}$, then the stalk $\pi_k(M'_k)_x$ is finitely presented over $\pi_0(A)_x$. Since $\pi_k(M'_k)\otimes_{\pi_0(A)[1/a]}\mathbb{R}$ is the quotient of $\pi_k(M'_k)_x$ by the maximal ideal of $\pi_0(A)_x$, it follows from Nakayama's lemma that $\pi_k(M'_k)_x = 0$, so there exists some $a' \in \pi_0(A)[1/a]$ such that $x(a') \neq 0$ and

$$0 = \pi_k(M'_k)\otimes_{\pi_0(A)[1/a]}\pi_0(A)[1/a'] \cong \pi_k(M'\otimes_{\pi_0(A)}\pi_0(A)[1/a']),$$

which completes the induction step. After n steps this procedure yields an element $a \in \pi_0(A)$ such that $x(a) \neq 0$ and $M'\otimes_{\pi_0(A)}\pi_0(A)[1/a]$ is n-connective.

(2) Again, we may suppose that M is connective, and by lemma 5.1.3.9 and the fact that Mod_A is left complete, it suffices to show that there exists some $a \in \pi_0(A)$ such that $M\otimes_A \pi_0(A)[1/a]$ vanishes. We note that because M is perfect, we may assume that M has Tor-amplitude in $[0, n-1]$ for some $n \geq 1$, so $M\otimes_A \pi_0(A)$ is $(n-1)$-truncated. Since $M\otimes_A\mathbb{R}$ is in particular n-connective, (1) shows that we may find an element $a \in \pi_0(A)$ with $x(a)\neq 0$ such that $M\otimes_A\pi_0(A)[1/a]$ vanishes, being $(n-1)$-truncated and n-connective.

□

Remark 5.1.3.26. In the previous proof, we use the following fact. Suppose that M is a connective A-module. Suppose furthermore that M k-connective for some $k \in \mathbb{Z}_{>0}$ and almost compact as an object of $\mathsf{Mod}_A^{\geq n}$ for some $0 \leq n < k$, then M is also almost compact as an object of $\mathsf{Mod}_A^{\geq k}$. This implies that $\pi_k(M)$ is finitely presented which permits the application of Nakayama's lemma. Note that although the hypothesis of Nakayama's lemma would allow for $\pi_k(M)$ to be finitely generated, it does not suffice that M is merely finitely generated as an object of $\mathsf{Mod}_A^{\geq n}$, since this does not imply that M is finitely generated as an object of $\mathsf{Mod}_A^{\geq k}$. In general, if M is connective, (1) holds for a fixed $n \geq 0$ provided that M is *finitely* $(n-2)$*-presented*.

Corollary 5.1.3.27. *Let $f : \mathbf{Spec}\,B \to \mathbf{Spec}\,A$ be a finitely presented morphism of affine fair derived C^∞-schemes, and suppose that for an $\mathbb{R}$-point $x : B \to \mathbb{R}$, the $\mathbb{R}$-module $\mathbb{L}_{B/A}\otimes_B\mathbb{R}$ vanishes. Then there exists some $b \in \pi_0(B)$ such that $x(b)\neq 0$ and the induced map $\mathbf{Spec}\,B[1/b]\to\mathbf{Spec}\,A$ is étale.*

Corollary 5.1.3.28. *Let $\mathbf{Spec}\,A$ be an affine derived manifold of finite presentation, and suppose that for an $\mathbb{R}$-point $x : A\to\mathbb{R}$, the $\mathbb{R}$-module $\mathbb{L}_A\otimes_A\mathbb{R}$ is free. Then there exists some $a\in\pi_0(A)$ such that $x(a)\neq 0$ and $\mathbf{Spec}\,A[1/a]$ is a manifold.*

Proof. The cotangent complex is perfect, so as in the proof of corollary 5.1.1.23, we may choose after localizing near x, a finite collection $\{a_1,\dots,a_n\}\subset\pi_0(A)$ such that the differentials $\{d_{\mathrm{dR}}a_1,\dots,d_{\mathrm{dR}}a_n\}$ form a basis of $\pi_0(\mathbb{L}_A\otimes_A\mathbb{R})\simeq\mathbb{L}_A\otimes_A\mathbb{R}$. The elements $\{a_i\}$ determine a map $f:\mathbf{Spec}\,A\to\mathbb{R}^n$ such that the perfect complex $\mathbb{L}_f$ vanishes at x. By proposition 5.1.3.25, $\mathbb{L}_f$ vanishes after localizing on $\mathbf{Spec}\,A$, rendering f étale by the inverse function theorem. □

Corollary 5.1.3.29. *Let $X\subset M$ be a closed subset in a manifold (X can be a manifold with corners, for instance) then the C^∞-ring of Whitney functions $C^\infty(X;M)$ is not finitely presented in $C^\infty\mathsf{ring}$.*

Proof. If $C^\infty(X;M)$ were finitely presented in $C^\infty\mathsf{ring}$, then $C^\infty(X;M)$ would be finitely presented in $sC^\infty\mathsf{ring}$, since $\mathbb{L}_M$ is perfect. Because $\mathbb{L}_M$ is projective, corollary 5.1.3.28 would imply that M is a manifold, a contradiction. □

Remark 5.1.3.30. We say that a point x in $(X,\mathcal{O}_X)$, a derived manifold locally of finite presentation, is *smooth* if $x^*\mathbb{L}_X$ is free, which is the case if and only if the dimension of $\pi_0(x^*\mathbb{L}_X)$ coincides with the virtual dimension of $(X,\mathcal{O}_X)$ near the point x. In view of the previous corollary, the *smooth locus* (the collection of all smooth points) of a derived manifold locally of finite presentation $(X,\mathcal{O}_X)$ is open and forms a manifold, which is all of X if and only if $\mathbb{L}_X$ is locally free. In particular, if $(X,\mathcal{O}_X)=\mathbf{Spec}\,A$ is affine, the fact that open sets of X are in bijection with localizations of A shows that we may choose an element $\chi_{\mathsf{Sm}}\in A$ whose localization corresponds to the smooth locus. Of course, this characteristic function for the smooth locus need not be nonzero. Conversely, χ_{Sm} is invertible if and only if $\mathbb{L}_A$ is projective.

Remark 5.1.3.31. Proposition 5.1.3.25 is quite powerful in a variety of situations. For instance, it can be used to easily check nondegeneracy of shifted presymplectic structures on Artin stacks locally of finite presentation. In the study of elliptic moduli problems, proposition 5.1.3.25 allows one to reduce a number of local questions on moduli stacks (Is a given map between moduli stack étale? When is the quotient of the derived manifold of solutions of a certain PDE by a Lie group -which is a priori a derived Artin stack- actually a derived orbifold?) to pointwise questions, which are often easy to handle by linear elliptic theory. In other approaches to moduli spaces of elliptic equations that do not develop the geometry of the unperturbed derived moduli stacks, such questions usually have to resolved *before* passing to the finite dimensional moduli spaces, which may involve heavier infinite dimensional analysis.

Proposition 5.1.3.32. *Let $f : A\to B$ be a morphism of fair simplicial C^∞-ring such that $\pi_0(f)$ is finitely presented. Then f is submersive if and only if f is formally submersive.*

Proof. Suppose that f is submersive; we wish to show that the perfect B-module $\mathbb{L}_{B/A}$ is projective, or equivalently, flat. This is local on $\mathbf{Spec}\, B$, so we may assume after localizing on A and B that f is the canonical inclusion $A \to A \otimes^{\infty} C^{\infty}(\mathbb{R}^n)$ for some $n \geq 0$. Point (2) of remark 5.1.0.4 shows that $\mathbb{L}_{B/A} \simeq g_! \mathbb{L}_{C^{\infty}(\mathbb{R}^n)}$ where g is the map $g : C^{\infty}(\mathbb{R}^n) \to A \otimes^{\infty} C^{\infty}(\mathbb{R}^n)$. Thus, $\mathbb{L}_{B/A}$ is finitely generated and free.
Conversely, suppose that f is formally submersive. The map $\pi_0(f)$ is finitely presented, so after localizing on A and B, we may assume that $\mathbb{L}_{B/A}$ is free and finitely generated. In particular, the module of relative smooth Kähler differentials $\Omega^1_{\pi_0(B)/\pi_0(A)}$ is a free $\pi_0(B)$-module, say of rank k. We have an exact sequence

$$\Omega^1_{\pi_0(A)} \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \Omega^1_{\pi_0(B)} \xrightarrow{p} \Omega^1_{\pi_0(B)/\pi_0(A)}$$

where the map p admits a section s because $\Omega^1_{\pi_0(B)/\pi_0(A)}$ is free. Choosing an isomorphism $\Omega^1_{\pi_0(B)/\pi_0(A)} \cong \pi_0(B)^k$, let $\{b_1, \ldots, b_k\}$ be the images under s of the canonical generators of $\pi_0(B)^k$. Choose a finite set of differentials $\{d_{\mathrm{dR}} a_i\}_{1 \leq i \leq m}$ with $m \geq k$ that generate $\Omega^1_{\pi_0(B)}$ and consider the matrix K whose entries are defined by the equation $b_j = \sum_i K_{ij} d_{\mathrm{dR}} a_i$. The collection of $k \times k$-minors of this matrix coincides with the 0'th fitting ideal associated to the presentation

$$\pi_0(B)^m \xrightarrow{K} \pi_0(B)^k \longrightarrow 0,$$

and therefore generates the unit ideal of $\pi_0(B)$ by the Fitting ideal lemma; thus, after replacing B by a localization, we may assume that there are k functions $\{a_i\}_{1 \leq i \leq k} \subset \pi_0(B)$ such that the differentials $\{d_{\mathrm{dR}} a_i\}$ become generators for $\Omega^1_{\pi_0(B)/\pi_0(A)}$ after applying the map p. These functions determine a map $A \otimes^{\infty} C^{\infty}(\mathbb{R}^k) \to B$ so that point (1) of remark 5.1.0.4 provides a fibre sequence

$$\mathbb{L}_{A \otimes^{\infty} C^{\infty}(\mathbb{R}^k)/A} \otimes_{A \otimes^{\infty} C^{\infty}(\mathbb{R}^k)} B \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/A \otimes^{\infty} C^{\infty}(\mathbb{R}^k)},$$

and using point (2) of remark 5.1.0.4 we can identify the first map in this fibre sequence with the map $B^k \to \mathbb{L}_{B/A}$ determined by the differentials $d_{\mathrm{dR}} a_i$. This map is an equivalence by construction, which shows that $\mathbb{L}_{B/A \otimes^{\infty} C^{\infty}(\mathbb{R}^k)}$ vanishes. Now we conclude by invoking theorem 5.1.3.17. □

Corollary 5.1.3.33. *Let $f : \mathbf{Spec}\, B \to \mathbf{Spec}\, A$ be a finitely presented morphism of affine fair derived C^{∞}-schemes, and suppose that for an $\mathbb{R}$-point $x : B \to \mathbb{R}$, the $\mathbb{R}$-module $\mathbb{L}_{B/A} \otimes_A \mathbb{R}$ is free. Then there exists some $b \in \pi_0(B)$ such that $x(b) \neq 0$ and the induced map $\mathbf{Spec}\, B[1/b] \to \mathbf{Spec}\, A$ is submersive.*

Proof. In view of proposition 5.1.3.32, it suffices to show at after localizing near x, the relative cotangent complex is free. Using Nakayama's lemma, we may choose after localizing near x a finitely generated free B-module N and a map $g : N \to \mathbb{L}_{B/A}$ such that the base change $\mathrm{fib}(g) \otimes_B \mathbb{R}$ along $x : \mathbb{R} \to B$ vanishes. Since $\mathrm{fib}(g)$ is a perfect complex, proposition 5.1.3.25 implies that g becomes an equivalence after localizing near x once more. □

Corollary 5.1.3.34. *An affine derived manifold $\mathbf{Spec}\, A$ is a manifold if and only if $\mathbb{L}_A$ is a projective A-module.*

Corollary 5.1.3.35. *An affine derived closed C^{∞}-scheme is of the form $\mathbf{Spec}\, A$ for A a C^{∞}-ring of Whitney functions if and only if $\mathbb{L}_A$ is a projective A-module.*

Corollary 5.1.3.36. *Let $Y \subset \mathbb{R}^m$ be a closed subset, viewed as a C^{∞}-scheme equipped with the sheaf $C^{\infty}_{\mathbb{R}^n}/\mathfrak{m}^{\infty}_Y|_Y$. Then for any affine derived C^{∞}-scheme $(X, \mathcal{O}_X)$ and any morphism $(X, \tau_{\leq 0}\mathcal{O}_X) \to N$ of C^{∞}-schemes, there exists an extension $(X, \mathcal{O}_X) \to N$ fitting into a commuting diagram*

$$\begin{tikzcd}
(X, \tau_{\leq 0}\mathcal{O}_X) \arrow[r] \arrow[d] & Y \\
(X, \mathcal{O}_X) \arrow[ur, dotted] &
\end{tikzcd}$$

Proof. This is a reformulation of propositions 5.1.3.32 and 5.1.3.15 for the map $\mathbb{R} \to C^{\infty}(Y; \mathbb{R}^m)$. □

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