Configuration spaces of points

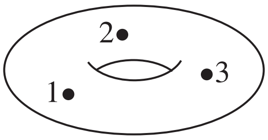
by **Ricardo Campos***

Given a topological space, one can consider its configuration space of n pairwise distinct points. We study the topological properties of such configuration spaces and address question of homotopy invariance.

1 Configuration spaces and definitions

Let X be a topological space and $n \geq 0$ be an integer. The *configuration space of n (non-overlapping) points on* X is the set

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$



$\in \mathrm{Conf}_3(\text{Torus})$

Notice that being a subset of X^n, the configuration space $\mathrm{Conf}_n(X)$ is itself a topological space. Configuration spaces describe the state of an entire system as a single point in a *higher dimensional space*.

It is clear that many situations can be expressed in terms of configuration spaces. For instance, in mechanics, where objects can often be assumed to be points and are not allowed to take the same place, the configuration of the system is a point on the configuration space.

What might be less clear is why we should be interested in $\mathrm{Conf}_n(X)$ not just as a set, but also as a topological space, and in its homotopical properties.

Here are some examples where the topology of configuration spaces appear:

- Imagine you have n small robots on a plane with obstacles. The surface of movement X can typically be represented as the complement of the obstacles. If we approximate the robots by points, their movement corresponds to a path in $\mathrm{Conf}_n(X)$. Turning the problem around, we can consider the path space on $\mathrm{Conf}_n(X)$, $\mathrm{Map}([0,1], \mathrm{Conf}_n(X))$ with the two projections

$$p\colon \mathrm{Map}([0,1], \mathrm{Conf}_n(X)) \to \\ \to \mathrm{Conf}_n(X) \times \mathrm{Conf}_n(X)$$

given by the initial and final configuration. A motion planning algorithm is essentially a section of the map p. Unless the configuration space is contractible (which is almost never the case) such a section does not exist globally. The topological complexity (surveyed in last years Bulletin [11]) is a homotopy invariant that allows us to construct not-very-discontinuous sections.

- In knot theory one wishes to classify all knots up to ambient isotopy, which corresponds to the connected components of the space of smooth embeddings $\mathrm{Emb}(S^1, \mathbb{R}^3)$. As a first approximation of the knot one can discretise the knot into many points, which gives a particular kind of configuration on $\mathbb{R}^3$. In fact, a drastic generalisation of this problem is the goal of understanding the homotopy type of the embedding space $\mathrm{Emb}(M, N)$ between two smooth

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manifolds. Notice that such an embedding $f : M \to N$ induces a map at the level of configuration spaces $f_n : \mathrm{Conf}_n(M) \to \mathrm{Conf}_n(N)$ by evaluation pointwise. Under good conditions, the Goodwillie-Weiss embedding calculus [2] tells us that we can recover (up to homotopy) the embedding space $\mathrm{Emb}(M, N)$ from the data of all the f_n together with some additional algebraic structure.

- The pure braid group on n strands, denoted PB_n is the group whose elements are n braids (up to ambient isotopy), and whose group operation is composition of braids.

  The pure braid group PB_n is isomorphic to the fundamental group of the configuration space of n points on the plane $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$.

- In quantum field theory, namely in Chern-Simons theory, one can construct invariants of framed smooth manifolds via integrals over configuration spaces [3].

We point out that some authors would call this the space of *ordered* configurations. The *unordered configuration space* (or configurations of indistinguishable points) can be seen as the quotient space of $\mathrm{Conf}_n(X)$ by the action of the symmetric group S_n which acts by permuting the x_i's.

In some sense unordered configuration spaces contain less information than ordered configuration spaces: as long as we can keep track of the symmetric group action, we can always recover the first from the latter.

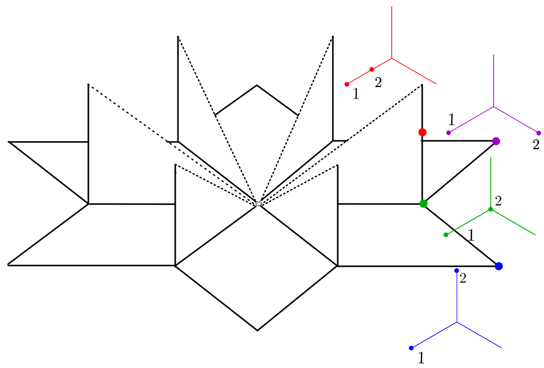
2 Examples

Configuration spaces are very simple to define, but surprisingly hard to understand. Compare them with X^n which we could call the *configuration space of n possibly overlapping points on X*. In practice, most invariants (such as the Euler characteristic, fundamental group, cohomology over a field) of X^n can be computed from the same invariant on X.

It is very instructive to see some examples to get a feel on configuration spaces. Notice that the baby cases $n = 0, 1$ give us in all generality $\mathrm{Conf}_0(X) = \varnothing$ and $\mathrm{Conf}_1(X) = X$, but this is essentially all we can say for an arbitrary topological space.

1. For the configuration of two points in the euclidean space, there is an identification $\mathrm{Conf}_2(\mathbb{R}^k) = \mathbb{R}^k \times S^{k-1} \times \mathbb{R}_{>0}$. This follows from the fact the position of the two points can be fully determined by first giving the position of the first point, then determining the vector the first point makes with the second one. One can picture the $(k-1)$-dimensional sphere S^{k-1} as a the angle the points make with one another. Under this identification, the S_2 action maps a point in S^{k-1} to its antipode. Notice that in particular the unordered configuration space will be non-orientable for odd k.

2. If we consider the graph given by connecting three vertices to a fourth vertex, its configuration space Conf_2(⅄) is the following space



   Notice that in dimension 1 there is a phenomenon of non-locality, in which even if two of the points are close, it might be difficult for them to exchange positions.

3. On the interval $I = (0, 1)$, the configuration space of n points has $n!$ connected components, corresponding to all possible ways to order n points. All of these components are homeomorphic to the open n-simplex

$$\{(x_1, \dots, x_n) \mid 0 < x_1 < \cdots < x_n < 1\}.$$

3 Homotopy type

This last example 3 is the simplest example of a topological invariant (connected components) on the configuration space that cannot be deduced just from the invariant on the base space. While one could be tempted to dismiss it as trivial, since it can only happen in dimension 1, it is actually a shadow of a more general problem.

Indeed, these issues are related with the non-functoriality of

$$\mathrm{Conf}_n : \text{Top. Spaces} \longrightarrow \text{Top. Spaces}.$$

If a map $f: X \to Y$ is not injective, the induced map $f^n: X^n \to Y^n$ will not restrict to the respective configuration spaces.

Recall that two spaces X and Y are said to be *homotopy equivalent*, denoted $X \sim Y$, if there are maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g$ is homotopic to the identity id_Y in the sense that there exists a map $h: Y \times [0,1] \to Y$ such that $h(y,0) = y$ and $h(y,1) = f \circ g(y)$, and similarly for $g \circ f$. When we talk about the *homotopy type* of X, we mean the equivalence class of all spaces homotopy equivalent to X.

Typically, invariants we study (and all those mentioned in this survey) of topological spaces depend only on the homotopy type. The natural question that one is led to ask is whether the homotopy type is preserved by taking configuration spaces, i.e., whether $X \sim Y$ will guarantee that $\mathrm{Conf}_n(X) \sim \mathrm{Conf}_n(Y)$. This sounds very implausible given the preceding discussion. Also, out of a homotopy equivalence between X and Y, there seems to be no way to construct a single map relating their configuration spaces.

In fact, the very first example 1 provides already a plethora of counter-examples, since regardless of the dimension, all Euclidean spaces are contractible (and hence homotopy equivalent), but $\mathrm{Conf}_2(\mathbb{R}^k) \sim S^{k-1}$ and no two different dimensional spheres have the same homology, cohomology or homotopy groups.

Given this, one might be surprised that the next open question is believed to be true.

Conjecture 1.— For simply connected compact manifolds without boundary, the homotopy type of $\mathrm{Conf}_n(M)$ only depends on the homotopy type of M.

The more general conjecture for non-simply connected spaces was only disproved in 2005, when Longoni and Salvatore were able to show that the lens spaces $L_{7,2}$ and $L_{7,1}$ provide a counter-example by computing the Massey products on the universal covers of the respective configuration spaces of 2 points [8].

In the last section I will try to provide some evidence for this conjecture in the form of Theorem 7. From now on, we will restrict our study to the case of smooth manifolds.

4 Compact version of configuration spaces

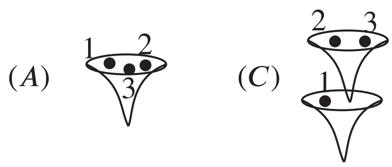
Suppose that M is a compact smooth manifold. Even if M is compact, the configuration space $\mathrm{Conf}_n(M)$ is not compact when $n \geq 2$, since a sequence of two points moving into the same place does not converge. This is an unfortunate property to lose: For instance, in situations where one wishes to consider integrals over configuration spaces (as in quantum field theory), one has to deal with issues of convergence.

A neat way to address this issue is to work instead with a suitable compactification of $\mathrm{Conf}_n(M)$. The most natural one is perhaps to consider M^n, but this could of course lose all homotopy information as in the case of $\mathbb{R}^k$. The strategy is instead to embed $\mathrm{Conf}_n(M) \hookrightarrow K$ in some compact manifold with boundary K, such that $\mathrm{Conf}_n(M)$ sits in K as its interior, since manifolds with boundary are homotopy equivalent to their interiors.

The construction of such manifold is due to Axelrod and Singer but is usually called the *Fulton-MacPherson compactification of* $\mathrm{Conf}_n(M)$, as it is a real analog of an iterated blow-up construction from algebraic geometry. This manifold is denoted $\mathsf{FM}_n(M)$ and admits a very visual description. Intuitively, instead of allowing two points moving towards each other to meet, we allow them to be *infinitesimally close together*, but still retaining the information of the direction in which they collided (i.e. they can still move around each other along a sphere of dimension $\dim M - 1$).

In the case where there are only two points, indeed $\mathsf{FM}_2(M)$ will be a manifold with boundary, but otherwise one needs to distinguish different situations when more than two points collide, so in general $\mathsf{FM}_n(M)$ will be a *compact smooth manifold with corners* whose interior is $\mathrm{Conf}_n(M)$.

In the figure above, fixing points 1 and 2, there are three possible cases when moving point 3 close to 1 and 2: (A) Point 3 stays at the same "infinitesimality stratum"; (B) Point 3 furthermore approaches infinitesimally point 1, even from the perspective of point 2; (C) Similar but point 3 approaches point 2.



For details and a nice exposition see [9].

5 The cohomology of $\mathrm{Conf}_n(\mathbb{R}^k)$

So far we have considered configuration spaces of a fixed number of points n, but given the same base manifold M, there are obvious relations between configurations of different number of points. Namely, given $1 \le i \ne j \le n$, there are *projection* maps

$$p_{ij} : \mathrm{Conf}_n(M) \longrightarrow \mathrm{Conf}_2(M), \qquad (1)$$

given by forgetting the position of all points but the ith and jth one.

These projection maps provide us with an attempt to inductively try to understand configurations of a large number of points from a smaller one. To illustrate this idea, let us consider in more detail the case of configurations of points in $\mathbb{R}^k$, where we can get a full description of the cohomology ring of $\mathrm{Conf}_n(\mathbb{R}^k)$. For concreteness, let us denote by $H^\bullet(M)$ the cohomology ring of M with real coefficients. Since $\mathrm{Conf}_n(M)$ is a smooth manifold, we will interpret this graded commutative $\mathbb{R}$-algebra as the cohomology of de Rham algebra of differential forms, denoted $\Omega^\bullet(\mathrm{Conf}_n(M))$.

From our previous example 1 we deduce that $H^\bullet(\mathrm{Conf}_2(\mathbb{R}^k)) = H^\bullet(S^{k-1}) = \mathbb{R}1 \oplus \mathbb{R}\omega$, where ω is a degree $k-1$ element representing the cohomology class of the volume form on S^{k-1}.

Theorem 2 (Arnold [1] and Cohen [5]).— The cohomology $\mathbb{R}$-algebra $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$ is given by

$$A_{n,k} = \frac{\mathrm{Sym}(\omega_{ij})_{1\le i\ne j\le n}}{\left(\omega_{ij} = (-1)^k\omega_{ji}, \omega_{ij}^2 = 0, \mathrm{Arnold}\right)} \qquad (2)$$

where ω_{ij} are elements of degree $k-1$, Sym denotes the symmetric algebra and the Arnold relation is $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$.

Heuristically, ω_{ij} represents the interaction between the points i and j, while the Arnold relations represent three-point interactions. The theorem states that these relations generate all existing relations.

Proof (Sketch).— The first step is to construct the map $A_{n,k} \to H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$. We interpret ω_{ij} as an element of $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$ by pulling back along the projection p_{ij} from equation (1) into the $(k-1)$-sphere: $\omega_{ij} \coloneqq p_{ij}^*\omega$.

The relation $\omega_{ij}^2 = 0$ holds since it holds for $\omega \in \Omega^{k-1}(S^{k-1})$. Switching the indices in ω_{ij} corresponds to applying the antipodal map in S^{k-1}, which has a degree opposite to the dimension of the sphere, from where it follows that $\omega_{ij} = (-1)^k\omega_{ji}$.

There are various short proofs of the Arnold relation, none of them completely immediate. It can be deduced by analysing the fibration $p_{12} : \mathrm{Conf}_3(\mathbb{R}^k) \to \mathrm{Conf}_2(\mathbb{R}^k)$. Alternatively, one can explicitly find a form η such that $d\eta = \mathit{Arnold}$ constructed as a fiber integral $\eta = \int_4 \omega_{14}\omega_{24}\omega_{34}$, as we will see in the final section.

Now that we have established maps

$$A_{n,k} \to H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k)),$$

we need to show that they are isomorphisms, which can be done by induction on n. For this, we observe that the map $\mathrm{Conf}_n(\mathbb{R}^k) \to \mathrm{Conf}_{n-1}(\mathbb{R}^k)$ forgetting the last point is a fibration whose fiber is homotopy equivalent to a wedge sum of spheres $\bigvee_{i=1}^{n-1} S^{k-1}$ and then apply the Serre spectral sequence. ∎

6 Rational homotopy theory

Let us step back from configuration spaces for a moment to consider the general problem of understanding the homotopy type of spaces via some algebraic invariant.

The main issue is that invariants such as the cohomology ring of a space do not capture the homotopy type sufficiently faithfully. A potentially stronger invariant is given by the higher homotopy groups π_n, since Whitehead theorem guarantees that a map of CW-complexes $X \to Y$ inducing isomorphisms $\pi_n(X) \to \pi_n(Y)$ is a homotopy equivalence. This not only does not completely solve our problem, since CW-complexes might still have the same homotopy groups without having a map inducing an isomorphism, but it also has the additional issue that higher homotopy groups are extremely difficult to compute due to their torsion parts. Rational homotopy theory provides a good way to address both issues, if we are willing to work modulo torsion:

Definition 3.— We say that a map of simply connected spaces $X \to Y$ is a *rational homotopy equivalence* if the induced map

$$\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for all n. Equivalently, $X \to Y$ is a rational homotopy equivalence if $H^n(Y;\mathbb{Q}) \to H^n(X;\mathbb{Q})$ is an isomorphism for all n.

Sullivan [10] associated to a space X a differential graded (dg) commutative algebra $A^\bullet_{PL}(X)$ of *piecewise linear differential forms* X, which the reader can think of as the de Rham algebra for non-manifolds (and over $\mathbb{Q}$ instead of $\mathbb{R}$), or alternatively as the singular $\mathbb{Q}$-cochains on X, $C^\bullet(X;\mathbb{Q})$ (except that the cup product is not commutative before passing to cohomology). The cohomology of the dg algebra $A^\bullet_{PL}(X)$ is the graded algebra $H^\bullet(X;\mathbb{Q})$.

In the category of rational dg commutative algebras, one considers the corresponding notion of homotopy equivalence, which is the one of a *quasi-isomorphism*. These are morphisms of dg commutative algebras $A^\bullet \to B^\bullet$ such that the induced map in cohomology $H^\bullet(A) \to H^\bullet(B)$ is an isomorphism.

The main result of Sullivan is that this construction captures faithfully the rational homotopy type of spaces.

Caveat 4.— Unlike ordinary homotopy equivalences, having a rational homotopy equivalence $X \to Y$ does not imply the existence of a rational homotopy equivalence $Y \to X$. We say that X and Y are rational homotopy equivalent and we write $X \sim_{\mathbb{Q}} Y$ if there is a zig-zag of rational homotopy equivalences

$$X \xleftarrow{\sim} X_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} X_n \xrightarrow{\sim} Y$$

Similarly, quasi-isomorphisms of dg commutative algebras are not *invertible* so the same remark holds.

There is a general construction in category theory: Given a category $\mathscr{C}$ with some set of *homotopy equivalences* $H \subset \mathrm{Morphisms}(\mathscr{C})$, one can construct the homotopy category of $\mathscr{C}$, denoted $\mathscr{C}[H^{-1}]$, which possesses the same objects as $\mathscr{C}$, but where we formally invert the maps in H, such that they become isomorphisms in $\mathscr{C}[H^{-1}]$.

Theorem 5 ([10]).— The construction A_{PL} establishes an equivalence of categories

$$A_{PL} : \mathrm{scSpaces}[\mathrm{r.h.e.}^{-1}] \to \mathbb{Q} - \mathrm{DGCA}_{>1}[\mathrm{q.i.}^{-1}]$$

from the category of simply connected topological spaces of finite type up to rational homotopy equivalence, to the category of dg commutative $\mathbb{Q}$-algebras of finite type concentrated in degrees > 1 up to quasi-isomorphism.

In practice, this result allows us to study topology completely via (differential graded) algebraic methods. Any dg commutative algebra quasi-isomorphic to $A_{PL}(X)$ is therefore called a *rational model of* X. A classical question in rational homotopy theory is whether one can find a small model for X. The smallest possible candidate to be a model of X would be its cohomology $\mathbb{Q}$-algebra, but in general it is not true that $H^\bullet(X;\mathbb{Q}) \sim_{\mathbb{Q}} A_{PL}(X)$. If that happens to be the case, we say that X is *formal*.

It should be pointed out that in the context of Sullivan's theorem there is nothing special about $\mathbb{Q}$ except that it is a field of characteristic 0. Replacing it with $\mathbb{R}$ we would talk about the *real homotopy type of* X instead.

7 Models for configuration spaces

In this final section we will see how one can construct a nice model of the real homotopy type of configuration spaces using graphs. This will in particular allow us to prove the real version of Conjecture 1, see Theorem 7.

Theorem 6 ([4]).— Let M be a compact smooth manifold without boundary and $n \in \mathbb{N}$. There exists a nice dg commutative $\mathbb{R}$-algebra spanned by a certain type of graphs $\mathsf{Graphs}_n(M)$ modeling the real homotopy type of $\mathrm{Conf}_n(M)$. This is expressed by a direct quasi-isomorphism of algebras into the algebra of semi-algebraic forms[1] of $\mathsf{FM}_n(M)$:

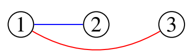
$$\mathsf{Graphs}_n(M) \longrightarrow \Omega(\mathsf{FM}_n(M)). \qquad (4)$$

Even though $\mathbb{R}^k$ is not a compact manifold, it is still instructive to go back to Theorem 2 and start by understanding how one could try to obtain a model of $\mathrm{Conf}_n(\mathbb{R}^k)$ out of the computation of its cohomology.

The only reasonably natural attempt of establishing a quasi-isomorphism $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$ into $\Omega(\mathsf{FM}_n(\mathbb{R}^k))$ would involve mapping $\omega_{ij} \in A_{n,k}$ into the volume form of the spheres S^{k-1}. However, since the Arnold relations are not satisfied at the level of differential forms this cannot produce a map compatible with the product.

While such a quasi-isomorphism of algebras cannot be directly constructed, Kontsevich [7] showed that configuration spaces in $\mathbb{R}^k$ are formal by establishing a zig-zag passing by an algebra of graphs.

Notice that one can identify the algebra $A_{n,k}$ with a graded vector space given by $\mathbb{R}$-linear combinations of graphs on n vertices, where an edge between the vertices i and j corresponds to ω_{ij}.



$\overset{\sim}{\longleftrightarrow}$ $\omega_{12}\omega_{13}$

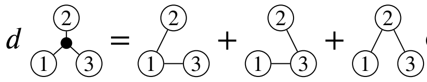
To be precise, depending on the parity of k, edges should be oriented or ordered, and changing orientation or order (by an odd permutation) produces a minus sign, but from now on we will work up to sign.

[1]Pretend it is the de Rham complex. This is a minor technicality due to the lack of smoothness of forgetting points near the corners.

Under this identification, edges have degree $k-1$ and the commutative product on graphs is given by superposition of vertices and taking the union of edges.

The idea now is to attempt to resolve $A_{n,k}$ by adding a new kind of vertices, that would mimic phantom points moving freely in the configuration space. Concretely, one can define a dg commutative algebra $\mathsf{Graphs}_n(\mathbb{R}^k)$, spanned by graphs with n labeled vertices as before and a arbitrary number of unlabeled vertices which now are of degree $-k$.

The differential of a graph $\Gamma \in \mathsf{Graphs}_n(\mathbb{R}^k)$ is given by summing all ways of contracting an unlabeled vertex in Γ along an edge. Here is an example exhibiting the Arnold relation as a coboundary:


$\in \mathsf{Graphs}_3(\mathbb{R}^k)$.

Notice that the differential kills an unlabeled vertex and an edge so it is indeed of degree $+1$ and (being careful with signs) it squares to zero. The product is still given by superposition of labeled vertices (in particular it adds the number of unlabeled vertices).

We can now produce a map into the algebra of forms

$$\mathsf{Graphs}_n(\mathbb{R}^k) \longrightarrow \Omega(\mathsf{FM}_n(\mathbb{R}^k))$$

given by "mapping edges ⓘ—ⓙ to the volume form of the sphere ω_{ij}[2] and integrating out the unlabeled vertices". In particular, the graph above yielding the Arnold relation is mapped to

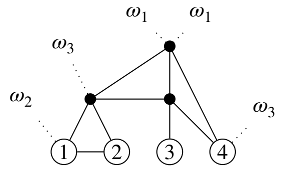
$$\eta = \int_{\mathsf{FM}_4 \to \mathsf{FM}_3} \omega_{14}\omega_{24}\omega_{34}.$$

The only non-immediate thing that needs to be checked is the compatibility with the differential, which follows mostly from the Stokes theorem for manifolds with corners.

In fact, to prove the formality of configuration spaces in $\mathbb{R}^k$, one just needs to show that the projection into graphs with no unlabeled vertices is a quasi-isomorphism, which can be achieved by a spectral sequence inductive argument.

While other configuration spaces over a compact manifold M will not be formal (and there is no analog of Theorem 2), stretching a bit the notion of graphs one can use similar ideas to construct the dg commutative $\mathbb{R}$-algebra $\mathsf{Graphs}_n(M)$ as follows:

As a vector space, $\mathsf{Graphs}_n(M)$ is spanned by graphs with n labelled vertices, some unlabeled vertices and vertices can be decorated by (possibly repeating) reduced cohomology classes in $\tilde{H}^\bullet(M)$.


$\in \mathsf{Graphs}_4(M)$,

The product is still given by superposition of labeled vertices. Following heuristically Kontsevich, we wish to send such graphs to differential forms in $\mathsf{FM}_n(M)$ in a way that integrates out unlabeled vertices and sending cohomology classes in $H^\bullet(M)$ to representatives in $\Omega(\mathsf{FM}_1(M))$. To establish the map in (4), there are three main pieces:

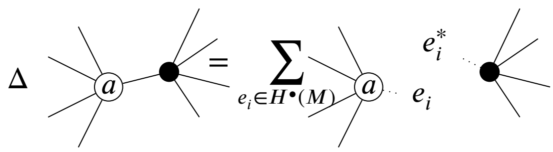
(i) In the case of $\mathbb{R}^k$, $\mathsf{FM}_2(\mathbb{R}^k)$ is essentially a sphere, so edges can be sent to volume forms. To which form in $\Omega^{\dim M-1}(\mathsf{FM}_2(M))$ will edges be sent to?

(ii) What kind of differential must $\mathsf{Graphs}_n(M)$ have such that (4) is compatible with differentials?

(iii) How to make this map a quasi-isomorphism?

We will not address the third point and without getting into details, let us just say that the first point is addressed by mapping edges to what in mathematical physics is called a *propagator* [3].

The second point is the trickiest: As far as $\mathsf{Graphs}_n(M)$ has been described, it only depends on the cohomology of M, so it has no chance of even capturing the real homotopy type of M, let alone the configuration space. All this is hidden in the differential, which splits into three pieces $d = d_{\text{contr.}} + d_{\text{Poinc.}} + d_{Z_M}$. A first piece $d_{\text{contr.}}$ which contracts edges as in the $\mathbb{R}^k$ case, a second piece $d_{\text{Poinc.}}$ which uses the Poincaré duality pairing on $H^\bullet(M)$ to split edges into two decorations



and a third piece d_{Z_M} acting only on subgraphs consisting of unlabeled vertices which depends on the partition function Z_M of the *universal* perturbative AKSZ topological field theory on M. We interpret Z_M as a map from vacuum graphs (fully unlabeled graphs in $\mathsf{Graphs}_0(M)$) into real numbers.

THEOREM 7 ([4, 6]).— Let M be a simply connected smooth compact manifold without boundary. The

[2]Abusing the notation denoting by ω_{ij} both the form and its class in cohomology.

real homotopy type of M determines the real homotopy type of its configuration space of points.

$$M \sim_{\mathbb{R}} N \Rightarrow \mathrm{Conf}_n(M) \sim_{\mathbb{R}} \mathrm{Conf}_n(N), \quad \forall n \in \mathbb{N}.$$

PROOF (SKETCH).— Notice that since M is simply connected $H^1(M) = 0$ and therefore decorations of graphs in $\mathsf{Graphs}_n(M)$ have degree at least 2. Furthermore we can assume that $D = \dim M \geq 4$ by the Poincaré conjecture.

All pieces of the defining data of $\mathsf{Graphs}_n(M)$ depend only on the real homotopy type of M, with the possible exception of d_{Z_M}.

One can show that d_{Z_M} depends only on the value of Z_M on graphs consisting only of degree 0 unlabeled vertices with valence ≥ 3 (decorations count as valence).

The proof of the theorem now follows from the following purely combinatorial statement: Using that decorations have degree at least 2, vertices have degree $-D$ and edges have degree $D-1$, the only ≥ 3-valent graphs of degree 0 are trees.

It turns out that the values of Z_M on trees depend only on the real homotopy type of M. It follows that $\mathsf{Graphs}_n(M)$ and therefore $\mathrm{Conf}_n(M)$ also depends only on the real homotopy type of M. ∎

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