The homotopy type of associative and commutative algebras

Ricardo Campos

CNRS + University of Montpellier

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Joint work with D. Petersen, D. Robert-Nicoud and F. Wierstra
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Slides at: `https://imag.umontpellier.fr/~campos/cirm.pdf`

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Equivalently: $H(M,\mathbb{Q}) \to H(N,\mathbb{Q})$ is an isomorphism.

Advantage: Computable

But neither (co)homology nor homotopy groups are faithful invariants.

Rational homotopy theory: An improvement

A better invariant: Instead of the cohomology, consider a dg commutative algebra:

E.g., associate to a manifold M its de Rham algebra $\Omega_{dR}(M)$.

Then, $M \simeq_{\mathbb{Q}} N \Rightarrow \Omega_{dR}(M)$ is quasi-isomorphic to $\Omega_{dR}(N)$.

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Quasi-isomorphism/equivalence: Map $A \to B$ s.t. $H(A) \xrightarrow{\cong} H(B)$.

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⚠ : Quasi-isomorphism might not be direct.

$$\Omega_{dR}(M) \xleftarrow{f_1} \bullet \xrightarrow{f_2} \bullet \ldots \bullet \xrightarrow{f_n} \Omega_{dR}(N)$$

s.t. $H(f_i)$ is an isomorphism of algebras.

Rational homotopy theory: The main result

Theorem (Sullivan '77)

There is a functor

$$A_{PL}: \text{ simply connected spaces} \longrightarrow \mathbf{CDGA}^{\mathbb{Q}}_{>1}$$

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Problem: The functor A_{PL} is complicated to describe.

An attempt to simplify

A natural question

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Even then, it is not clear that
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Goal of the talk

In principle we can't.... But we kind of can.

An algebraic problem

A folklore problem

Let A and B be two dg commutative algebras that are quasi-isomorphic as **associative** algebras. Are they quasi-isomorphic as commutative algebras?

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Remark

The question only makes sense if we are in the dg world.

$$A \xrightarrow{\sim} B$$

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$$A \xleftarrow{\sim} C \xrightarrow{\sim} B \quad \Rightarrow \quad A \longleftarrow C/[C,C] \longrightarrow B$$

The first main result

Theorem A

Let A and B be two dg commutative algebras over a field of characteristic 0 that are quasi-isomorphic as associative algebras. Then, they quasi-isomorphic as commutative algebras.

$$A \simeq_{\mathrm{Ass}} B \Rightarrow A \simeq_{\mathrm{Com}} B$$

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Remark

Case $B = H(A)$ was proven by B. Saleh in '16.

Remark

Likely false if the characteristic of the field is $\neq 0$.

An “unrelated” story: Lie algebras

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$U\mathfrak{g}$ is a Hopf algebra. Using the coproduct $\Delta\colon U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ we recover $\mathfrak{g}$ as the primitive elements of $U\mathfrak{g}$

$$\mathfrak{g} = \mathrm{Prim}(U\mathfrak{g}) := \{x \in U\mathfrak{g} \mid \Delta x = 1 \otimes x + x \otimes 1\}$$

Lie and enveloping algebras

$$U\colon \text{Lie algebras} \longrightarrow \text{Associative algebras}$$

The isomorphism problem for Lie algebras, Bergman '78?

Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, if $U\mathfrak{g} \cong U\mathfrak{h}$ as associative algebras, must $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras?

Lie and enveloping algebras

$$U\colon \text{dg Lie algebras} \longrightarrow \text{dg Associative algebras}$$

Generalisation

Given two dg Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, does

$$U\mathfrak{g} \simeq_{\text{Ass}} U\mathfrak{h} \quad \Rightarrow \quad \mathfrak{g} \simeq_{\text{Lie}} \mathfrak{h}?$$

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The relation: Koszul duality

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- Lie algebra $\mathfrak{g}$ is sent to $\text{Koszul}(\mathfrak{g}) = C_{CE}(\mathfrak{g}) = \text{Sym}(\mathfrak{g}^*[-1])$.

- Associative algebra A is sent to the dual of the Bar construction $\text{Koszul}(A) = T(A^*[-1])$.

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Second main result

Theorem B

Let $\mathfrak{g}$ and $\mathfrak{h}$ be dg Lie algebras over a field of characteristic zero. If their universal enveloping algebras $U\mathfrak{g}$ and $U\mathfrak{h}$ are quasi-isomorphic as associative dg algebras then the homotopy completions $\mathfrak{g}^{h\wedge}$ and $\mathfrak{h}^{h\wedge}$ are quasi-isomorphic as dg Lie algebras.

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Corollary

If $\mathfrak{g}$ is nilpotent then $\mathfrak{g} = \mathfrak{g}^{h\wedge}$.
In particular, $U\mathfrak{g} \simeq_{\mathrm{Ass}} U\mathfrak{h} \quad \Rightarrow \quad \mathfrak{g} \simeq_{\mathrm{Lie}} \mathfrak{h}$

Back to rational homotopy theory

If X is a pointed space, its (Moore) loop space ΩX admits an associative product given by concatenation.

$$C_*(\Omega X, \mathbb{Q}) \simeq_{\mathrm{Ass}} U(\lambda X)$$

Corollary (Our result + Quillen '69)

Simply connected pointed spaces X and Y have the same rational homotopy type if and only if $C_*(\Omega X, \mathbb{Q}) \simeq C_*(\Omega Y, \mathbb{Q})$ as dg associative algebras.

Strategy of proof:

Algebraic (operadic) deformation theory

From now on: A and B are commutative algebras s.t. $A \simeq_{\mathrm{Ass}} B$.

A simplification: ∞-morphisms

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Mapping spaces are difficult to compute because of lack of (quasi-)invertibility of quasi-isomorphisms.

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Solution (∞-morphisms)

For A and B associative algebras

- $A \simeq_{\mathrm{Ass}} B \Leftrightarrow \exists A_\infty$ *quasi-isomorphism* $A \stackrel{\sim}{\rightsquigarrow} B$.

An A_∞ map is given by a collection of maps

$$\mu_n \colon A^{\otimes n} \to B$$

representing homotopies for the compatibility of μ_1 with the product.

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Solution (Homotopy transfer theorem)

For X an commutative algebra, there is an C_∞-algebra structure on $H(X)$ such that $X \overset{\sim}{\rightsquigarrow} H(X)$.

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Reformulation of Theorem A

Let H be a graded vector space with two C_∞-algebra structures that are A_∞ quasi-isomorphic. Then they are C_∞ quasi-isomorphic.

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Deformation problems and Maurer–Cartan spaces

The space of (A_∞, C_∞-)structures on a graded vector space H arises from a *deformation problem*.

Deformation problems are encoded by dg Lie algebras $\mathfrak{g}$ via the Maurer–Cartan functor

$$MC(\mathfrak{g}) = \{\mu \in \mathfrak{g}_{-1} \ \mid \ d\mu + \frac{1}{2}[\mu,\mu] = 0\}$$

Requires some convergence conditions. All our deformation Lie algebras are complete (equipped with a complete decreasing filtration).

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In our case

We denote by $\mathrm{Def}_{A_\infty}(H)$ the deformation complex of A_∞-algebra structures .

In the associative case we obtain the Hochschild cochain complex of H, which is a dg Lie algebra $\mathrm{Def}^n_{A_\infty}(H) = \prod_{k\geq 0} \mathrm{Hom}(H^{\otimes k}, H)^{n+k-1}$.

$$MC(\mathrm{Def}_{A_\infty}(H)) = A_\infty - \text{algebra structures on } H$$

$\mathrm{Def}_{C_\infty}(H)$ is the Harrison complex .

In short

Dotsenko–Shadrin–Vallette '16

Lie algebra	$\mathfrak{g}$	$\mathrm{Def}_{A_\infty}(H)$	$\mathrm{Def}_{C_\infty}(H)$
Maurer-Cartan elements	$\mu + \frac{1}{2}[\mu, \mu] = 0$	A_∞-structures on H	C_∞-structures on H
Gauge group	$\exp(\mathfrak{g}_0)$	A_∞-isotopies	C_∞-isotopies

Yet another reformulation

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Much more general question

Given a map of complete dg Lie algebras $\mathfrak{h} \longrightarrow \mathfrak{g}$, when can we say that

$$MC(\mathfrak{h})/\text{gauge eq.} \hookrightarrow MC(\mathfrak{g})/\text{gauge eq.}?$$

Injections on MC spaces

Theorem

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a dg Lie subalgebra. Suppose that $\mathfrak{h}$ is a retract of $\mathfrak{g}$ as an $\mathfrak{h}$-module, i.e., there is a chain map $r : \mathfrak{g} \to \mathfrak{h}$ whose restriction to $\mathfrak{h}$ is the identity map and such that $r([x, y]) = [r(x), y]$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$. Then, Maurer–Cartan elements of $\mathfrak{h}$ which are gauge equivalent in $\mathfrak{g}$ must also be gauge equivalent in $\mathfrak{h}$.

- $\mathrm{Def}_{C_\infty}(H) \to \mathrm{Def}_{A_\infty}(H)$ admits such a retract, which proves Theorem A.

Why?

The map $\mathrm{Def}_{C_\infty}(H) \to \mathrm{Def}_{A_\infty}(H)$ is induced by the map of operads

$$Lie \to Ass$$

This map admits a retract as a map of infinitesimal *Lie*-bimodules. This is a variant of the Poincaré–Birkoff–Witt theorem.

Why?

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The argument actually works for any two operads $P \to Q$ and their Koszul duals $Q^! \to P^!$.

Sketch of proof of Theorem B

$$\Omega\colon \text{coassociative coalgebras} \rightleftarrows \text{associative algebras}\colon B$$

$$\mathscr{L}\colon \text{cocommutative coalgebras} \rightleftarrows \text{Lie algebras}\colon \mathscr{C}$$

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- $\mathscr{L}\mathscr{C}\mathfrak{g} \simeq_{\mathrm{Lie}} \mathscr{L}\mathscr{C}\mathfrak{h}$
- $\mathfrak{g} \simeq_{\mathrm{Lie}} \mathfrak{h}$ — $\mathscr{L}$ does not preserve q.i.

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- $\mathscr{L}^\wedge\mathscr{C}\mathfrak{g} \simeq_{\mathrm{Lie}} \mathscr{L}^\wedge\mathscr{C}\mathfrak{h}$
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The end

Merci

Thank you!

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