SESQUILINEAR MORITA EQUIVALENCE AND ORTHOGONAL SUM OF ALGEBRAS WITH ANTIAUTOMORPHISM

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Abstract. We define a notion of Morita equivalence between algebras with antiautomorphisms such that two equivalent algebras have the same category of sesquilinear forms. This generalizes the Morita equivalence of algebras with involutions defined by Frölich and Mc Evett [FMcE], and their categories of ε-hermitian forms.

For two Morita equivalent algebras with involution, with an additional technical property (which is true for central simple algebras), we define a new algebra with antiautomorphism, called the orthogonal sum, which generalizes the usual notion of orthogonal sum of forms. We explore the invariants of this sum.

Introduction - Morita equivalence of rings was introduced in the 1950's by K. Morita [M]. Hermitian Morita theory was developed in the late 1960's by A. Frölich and A. McEvett [FMcE] for algebras with involution. By an involution we mean an antiautomorphism of period 2.

In the theory of central simple algebras, it is well known that an involution is always the adjoint of some ε-hermitian or ε-symmetric form (with $\varepsilon = \pm 1$), i.e. a form that has a kind of symmetry. The Morita theory for forms developed by Frölich and McEvett, C.T.C. Wall [W], M.-A. Knus [K], always requires this sort of symmetry. We develop here a Morita theory which works for sesquilinear and bilinear forms in general, without assuming any symmetry, and generalizes the Frölich-McEvett theory to the category of sesquilinear forms.

In the late 1990's, I. Dejaiffe [D] introduced the notion of orthogonal sum of two Morita equivalent central simple algebras with involution. We here consider the the case of two algebras (not necessarily central simple) with antiautomorphism (not necessarily of period two), which are Morita equivalent in the sense described earlier. We generalize the Dejaiffe orthogonal sum in this situation under the technical assumption that the sesquilinear form needed to describe a Morita equivalence data has what we call an asymmetry.

We then briefly outline some properties of this sum, and examine its invariants when the antiautomorphisms are linear and the algebras are central simple.

Notation -
Let F be a field and A be an F-algebra. Let $\sigma : A \to A$ be an antiautomorphism of A, i.e. an isomorphism of groups such that

$$\forall x, y \in A \quad \sigma(xy) = \sigma(y)\sigma(x).$$

Two antiautomorphisms of F-algebras are said to be compatible if they have the same restriction to the base field F. An antiautomorphism σ is called an involution if it has exponent 2, i.e. if $\sigma^2 = \mathrm{Id}_A$.

Let M be a right A-module. A map $h : M \times M \to A$ is called an (A, σ)-sesquilinear form if it is a morphism for the additive law and

$$\forall a, a' \in A \quad \forall m, m' \in M \quad h(ma, m'a') = \sigma(a)h(m, m')a'.$$

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If (M, h) and (N, k) are two modules with sesquilinear form, we call an isometry from (M, h) to (N, k) any isomophism $u : M \to N$ such that

$$\forall m, m' \in M \quad k(u(m), u(m')) = h(m, m').$$

If h is a sesquilinear form and $-$ is an involution, then $(x, y) \mapsto \overline{h(y, x)}$ is also a sesquilinear form. Let ε be an element of the center $Z(A)$ of A. A sesquilinear form h on M is called ε-hermitian if

$$\forall x, y \in M \quad h(y, x) = \varepsilon\overline{h(x, y)}.$$

Such ε-hermitian forms may exist only if $\varepsilon\overline{\varepsilon} = 1$.

The category of sesquilinear forms over (A, σ) with isometries will be denoted by $\mathbf{Sesq}(A, \sigma)$, and if σ is involutive, the category of ε-hermitian forms will be denoted by $\mathbf{Herm}^{\varepsilon}(A, \sigma)$. The category of right-$A$-modules (respectively left-A-modules) will be denoted by $\mathbf{Mod}$-A (resp. A-$\mathbf{Mod}$).

If A is an algebra, we denote by A^{op} the opposite algebra : it is isomorphic to A as a vector space, by a bijection $A \to A^{op}$; $a \mapsto a^{op}$, and the multiplication is defined by $a^{op}b^{op} = (ba)^{op}$ for all $a, b \in A$.

If (A, σ) and (B, τ) are algebras with compatible antiautomorphism, a B-A-bimodule P is then a $B^{op} \otimes A$-module. Since $\tau^{op} \otimes_F \sigma$ (which is well defined using the compatibility of σ and τ) is an antiautomorphism of $B^{op} \otimes_F A$, we simply denote it by $x \mapsto \overline{x}$. As restrictions of $-$ on B and A, we then use the same notation $-$ instead of τ and σ.

Let $\sim = -^{-1}$ be the inverse antiautomorphism of $-$ (we can consider it over $A^{op} \otimes B$).

We then denote by $\overline{P}$ the A-B-bimodule whose elements are the $\overline{x}$ for $x \in P$, where $P \to \overline{P}\,; x \mapsto \overline{x}$ is an isomorphism of additive groups, and the multiplications are defined by the twisted action of A and B by $\sim$: for $x \in P, a \in A, b \in B$ we have $a.\overline{x}.b = \overline{\widetilde{b}.x.\widetilde{a}}$, and hence

$$\overline{bxa} = \overline{a}.\overline{x}.\overline{b}.$$

Let f be a morphism of B-A-bimodules $P_1 \to P_2$, we then define a morphism of A-B-bimodules $\overline{f} : \overline{P}_1 \to \overline{P}_2$ by $\overline{f}(\overline{x}) = \overline{f(x)}$. Note that in the particular case where $f = b \in B = \operatorname{End}_A P$, since $\overline{b}$ has another meaning, we will write $\underline{b}$ for this morphism $\overline{P} \to \overline{P}$.

Note that with the above notation, $\widetilde{\overline{P}} = \overline{\widetilde{P}} = P$ and $\widetilde{\overline{\alpha}} = \overline{\widetilde{\alpha}} = \alpha$.

Let now P be a right A-module and $P^\star = \operatorname{Hom}_A(P, A)$. It is naturally a left A-module by $(a.l)(p) = a.l(p)$ (in A) for $a \in A, l \in P^\star, p \in P$. If P is a B-A-bimodule then $P^\star$ is naturally an A-B-bimodule by $l.b(p) = l(bp)$ if $b \in B, p \in P$ and $l \in P^\star$.

If B is not specified, then we take $B = \operatorname{End}_A(P)$, and the natural B-A-bimodule structure for P given by $bp = b(p)$.

In this situation, we obtain two morphisms of bimodules :

$$\begin{array}{llll} f_P : & P \otimes_A P^\star & \to & B = \operatorname{End}_A(P) \\ & x \otimes l & \mapsto & (y \mapsto x.l(y)) \end{array} \quad \text{and} \quad \begin{array}{llll} g_P : & P^\star \otimes_B P & \to & A \\ & l \otimes x & \mapsto & l(x) \end{array}.$$

1. Morita equivalent algebras with antiautomorphism

1.1. The classical results about Morita equivalence. In this subsection, we recall the definition and the main results about Morita equivalent F-algebras. All these results can be found in Bass [B] or Lam [Lam], and we refer the reader to the proofs in this book or in [K].

Let A be an F-algebra.

Definition 1.1. *A right A-module P is said to be faithfully projective if it is finitely generated projective, and if for any left A-module N, $P \otimes_A N = 0 \Rightarrow N = 0$.*

Proposition 1.2. *A finitely generated right A-module P is faithfully projective if and only if f_P and g_P are isomorphisms of bimodules.*

Theorem 1.3. *(Morita equivalence for modules)*
Let P be a faithfully projective A-module and $B = \mathrm{End}_A(P)$. Then

(a) *$P^\star$ is a faithfully projective B-module and $P \simeq \mathrm{Hom}_B(P^\star, B)$.*
(b) *The natural maps*

$$\begin{array}{ccccccc} A & \to & \mathrm{End}_B(P^\star) & \text{and} & A^{op} & \to & (\mathrm{End}_B(P)) \\ a & \mapsto & (l \mapsto al) & & a^{op} & \mapsto & (p \mapsto pa) \end{array}$$

are isomorphisms of algebras.
(c) *Tensor products on the right by P and $P^\star$ respectively over B and A, induce equivalences between the categories* **Mod**-B *and* **Mod**-A*, and between* **Mod**-A *and* **Mod**-B.
*Tensor products on the left by P and $P^\star$ respectively over A and B, induce equivalences between the categories A-***Mod** *and B-***Mod***, and between B-***Mod** *and A-***Mod**.
(d) *The functors $\mathrm{Hom}_A(P, .)$: $\mathbf{Mod}\text{-}A \to \mathbf{Mod} - B$ and $\mathrm{Hom}_B(P^\star, .)$: $\mathbf{Mod}\text{-}B \to \mathbf{Mod}\text{-}A$ are equivalences of categories.*

Theorem 1.4. *Let A and B be two F-algebras such that the categories* **Mod**-B *and* **Mod**-A *are equivalent. Then there exists a B-A-bimodule P and an A-B-bimodule Q, together with associative isomorphisms of bimodules $f : P \otimes_A Q \to B$ and $g : Q \otimes_B P \to A$ such that the above equivalence of category is the tensor product on the right by P over B and its converse is the tensor product by Q over A.*
Moreover, P is faithfully projective, B is isomorphic to $\mathrm{End}_A(P)$ and Q to $P^\star$.

Here the morphisms $f : P \otimes_A Q \to B$ and $g : Q \otimes_B P \to A$ of bimodules are said to be associative if for any $p, p' \in P$ and $q, q' \in Q$ we have $f(p \otimes q)p' = pg(q \otimes p')$ and $g(q \otimes p)q' = qf(p \otimes q')$. For example, the above f_P and g_P are clearly associative.

In this theorem, the isomorphism $Q \to P^\star$ is $(Id_{P^\star} \otimes f) \circ (g_P \otimes Id_Q)$.

Definition 1.5. *A Morita equivalence data for modules is a collection (A, B, P, Q, f, g) where*
A and B are F-algebras,
P is a B-A-bimodule and Q an A-B-bimodule,
$f : P \otimes_A Q \to B$ and $g : Q \otimes_B P \to A$ are associative isomorphisms of bimodules.

We say that the algebras A and B are Morita equivalent if there exists a Morita equivalence data (A, B, P, Q, f, g).

REMARK 1.6 - In the context of 1.3, we can deduce from (b) that we also have isomorphisms of algebras

$$\begin{array}{ccccccc} A & \to & \mathrm{End}_B(\overline{P}) & \text{and} & A & \to & (\mathrm{End}_B(\widetilde{P})) \\ a & \mapsto & (\overline{p} \mapsto a\overline{p}) & & a & \mapsto & (\widetilde{p} \mapsto a\widetilde{p}). \end{array}$$

1.2. Product of sesquilinear forms.

We here extend the definitions given in the case of hermitian forms in Knus book [K], and originally by Fröhlich and McEvett [FMcE] for any pairing but with a trivial involution on B, to the more general context of sesquilinear forms and antiautomorphisms.

Let first $(A, -)$ be an algebra with involution, P be a right-A-module, and $B = \mathrm{End}_A P$. Let $h : P \times P \to A$ be a map. Then it is an $(A, -)$-sesquilinear form if and only if the map

$$\begin{array}{cccc} H : & P & \to & \widetilde{P^\star} \\ & x & \mapsto & \widetilde{h_x}, \end{array}$$

where $h_x(y) = h(x, y)$, is a morphism of right-A-modules , or equivalently if

$$\begin{array}{cccc} \overline{H} : & \overline{P} & \to & P^\star \\ & \overline{x} & \mapsto & (y \mapsto h(x, y)) \end{array}$$

is a morphism of left-A-modules.

Definition 1.7. *We say that the sesquilinear form h is nonsingular if the associated morphism H (or equivalently $\overline{H}$) is an isomorphism of A-modules.*

When h is non singular, and the transposition $t \ : \ \mathrm{End}_A P \to \mathrm{End}_A P^\star$ is bijective (which is the case for example for a faithfully projective module over a central simple algebra), this allows us to define the adjunction for h (on the right), usually denoted by ad_h : if $b \in B$, then $\mathrm{ad}_h(b)$ is the unique element of B that satisfies

$$\forall x, y \in P \quad h(bx, y) = h(x, \mathrm{ad}_h(b)y).$$

It can also be defined by $(ad_h b)^t = \overline{H} \circ \underline{b} \circ \overline{H}^{-1}$ (where we write $\underline{b}$ for the morphism of left A-modules $\overline{P} \to \overline{P}$ associated to the morphism of right A-modules $b \ : \ P \to P$). The adjunction is an antiautomorphism on B, and we can then denote it by $-$. (Note that its inverse is then the adjunction on the left and is defined by $\underline{\widetilde{b}} = \overline{H}^{-1} b^t \overline{H}$, where $\underline{\widetilde{b}}$ is the endomorphism of $\overline{P}$ associated to $\widetilde{b}$).

The definition of $-$ then means that H is a morphism of B-A-bimodules, or that $\overline{H}$ is a morphism of A-B-bimodules, and we can then define the morphism of A-A-bimodules :

$$\mathbb{H} \ : \ \overline{P} \otimes_B P \to A \ ; \ \overline{x} \otimes y \mapsto h(x, y).$$

Suppose now that A and B are two F-algebras with antiautomorphism and P is a B-A-bimodule.

Definition 1.8. *We say that the $(A, -)$-sesquilinear form $h \ : \ P \times P \to A$ admits $(B, -)$ if one of the following equivalent propositions is true :*
- $\forall x, y \in P \quad h(bx, y) = h(x, \overline{b}y)$;
- *H (respectively $\widetilde{H}$) is a morphism of B-A-bimodules (resp. A-B-bimodules) ;*
- *$\mathbb{H} \ : \ \widetilde{P} \otimes_B P \to A \ ; \ \widetilde{x} \otimes y \mapsto h(x, y)$ is a morphism of A-A-bimodules.*

Remark 1.9 - If $(A, -)$ is an involution and h is a ε-hermitian form which admits $(B, -)$, then $-$ is also an involution of B.

Assuming that $B = \mathrm{End}_A P$, then the following diagram commutes :

$$\begin{tikzcd}
\overline{P} \otimes_B P \arrow[r, "\mathbb{H}"] \arrow[d, "\overline{H} \otimes \mathrm{Id}_P"'] & A. \\
P^\star \otimes_B P \arrow[ur, "g_P"'] &
\end{tikzcd}$$

Using 1.2, we can deduce :

Proposition 1.10. *If P is faithfully projective, $B = \mathrm{End}_A(P)$ and $h \ : \ P \times P \to A$ is an $(A, -)$-sesquilinear form that admits $(B, -)$, then the following are equivalent :*
1. *h is nonsingular ;*
2. *H is an isomorphism of B-A-bimodules ;*
3. *$\overline{H}$ is an isomorphism of A-B-bimodules ;*
4. *$\mathbb{H}$ is an isomorphism of A-A-bimodules.*

Definition 1.11. *Let P be a faithfully projective A-module, $B = \mathrm{End}_A(P)$, and $h \ : \ P \times P \to A$ a nonsingular $(A, -)$-sesquilinear form that admits $(B, -)$. Let (M, k) be a $(B, -)$-sesquilinear module. We then define the form $hk \ : \ M \otimes_B P \times M \otimes_B P \to A$ by*

$$\forall m, m' \in M \quad \forall x, x' \in P \quad hk(m \otimes x, m' \otimes x') = h(x, k(m, m')x').$$

Proposition 1.12. *The form hk is an $(A, -)$-sesquilinear form over the right-A-module $M \otimes_B P$. If M is finitely generated B-projective, then hk is nonsingular if and only if k is nonsingular.*

This was proved in the case where $(A, -)$ is an algebra with involution and $(B, -) = (B, \mathrm{Id}_B)$ in [FMcE], and we can extend their proof to our more general context ; we will also use the following :

Lemma 1.13. *Let P be a B-A-bimodule and M a finitely generated projective right-B-module. Let $P^\star = \mathrm{Hom}_A(P, A), M^\star = \mathrm{Hom}_B(M, B)$ and $(M \otimes_B P)^\star = \mathrm{Hom}_A(M \otimes_B P, A)$. Then the map $\alpha : P^\star \otimes_B M^\star \to (M \otimes_B P)^\star$ defined by*

$$\alpha(l \otimes t)(m \otimes x) = (l.t(m))(x) = l(t(m).x)$$

(for $t \in M^\star, l \in P^\star, x \in P, m \in M$) is an isomorphism of left-A-modules.

Proof. First note that α has values in $(M \otimes_B P)^\star$, i.e. that with the above notations, $\alpha(l \otimes t)$ is $A - linear$: if $a \in A$, and $x \in P, m \in M$, then

$$\alpha(l \otimes t)(m \otimes xa) = (l.t(m))(xa) = (l.t(m))(x)a$$

since $l.t(m) \in P^\star$. This proves that the map α is well defined.

Now prove that α is a morphism of left A-modules : let l, t, m, x, a be as above.Then

$$\alpha(al \otimes t)(m \otimes x) = (al)(t(m))(x) = a(l(t(m)(x) = a\alpha(l \otimes t)(m \otimes x)$$

as desired.

To prove that if M is a finitely generated projective B-module then α is bijective, it is enough by additivity to prove it for $M = B$. But then α is the identity map

$$P^\star \otimes_B B^\star \backsimeq P^\star \otimes_B B \backsimeq P^\star \backsimeq (B \otimes_B P)^\star.$$

■

Proof. (of proposition 1.12) The sesquilinearity is clear.

Let $j = hk$ and $\overline{J} : \overline{M \otimes_B P} \to (M \otimes_B P)^\star$ associated to j. Then if $m, m' \in M$ and $x, x' \in P$,

$$\overline{J}(\overline{m \otimes x})(m' \otimes x') = h(x, k(m, m')x') = \overline{H}(\overline{x})(\overline{K}(\overline{m})x') = \alpha \circ (\overline{H} \otimes \overline{K})(m' \otimes x'),$$

hence $\overline{J} = \alpha \circ (\overline{H} \otimes \overline{K})$. This proves, since α and $\overline{H}$ are isomorphisms, that hk is nonsingular if and only if k is nonsingular. ■

Lemma 1.14. *If (P, h) and (M, k) are respectively an $(A, -)$-sesquilinear form which admits $(B, -)$ and a $(B, -)$-sesquilinear form which admits $(C, -)$, then $(M \otimes_B P, hk)$ admits $(C, -)$, and for any $(C, -)$-sesquilinear module (N, l) we have*

$$(hk)l = h(kl)$$

(over $N \otimes_C (M \otimes_B P) \simeq (N \otimes_C M) \otimes_B P$).

This is a straightforward computation.

1.3. **Morita equivalence for sesquilinear forms.**

We will here state the main theorem of this section which gives a sufficient condition for two algebras with antiautomorphism to have the same category of sesquilinear forms. This will be proved in the next subsection.

Definition 1.15. *We call a Morita equivalence data for sesquilinear forms any collection*

$$((A, -), (B, -), P, Q, f, g, \overline{H}_Q)$$

where $((A, -), (B, -), P, Q, f, g)$ is a Morita equivalence for modules and $\overline{H}_Q : \overline{P} \to Q$ is an isomorphism of B-A-bimodules.

We say that two algebras with antiautomorphism $(A, -)$ and $(B, -)$ are Morita equivalent if there exists a Morita equivalence data $((A, -), (B, -), P, Q, f, g, \overline{H}_Q)$.

Given such a Morita equivalence data, we know that P is faithfully projective, and Q is isomorphic to $P^\star$. Hence $\overline{H}_Q$ gives an isomorphism $\overline{H} : \overline{P} \to P^\star$, which induces a nonsingular $(A, -)$-sesquilinear form h over P which admits $(B, -)$.

We now can formulate the main result of this section :

Theorem 1.16. *Let $((A,-),(B,-),P,Q,f,g,\overline{H}_Q)$ be a Morita equivalence data, and h the $(A,-)$-sesquilinear form over P induced by $\overline{H}_Q$. Then the functor*

$$\mathcal{F} \; : \; \mathbf{Sesq}(B,-) \to \mathbf{Sesq}(A,-)$$

defined by

$$\begin{array}{rcl} \mathcal{F}(M,k) & = & (M \otimes_B P, hk) \\ \mathcal{F}(u) & = & u \otimes \mathrm{Id}_P \end{array}$$

is an equivalence of categories that preserves the orthogonal sums, and the nonsingularity for finitely generated projective modules.

Moreover, if the two antiautomorphisms are involutions and h is ε_0-hermitian for an ε_0 in $Z(B) = Z(A)$ such that $\varepsilon_0\overline{\varepsilon_0} = 1$, then $\mathcal{F}$ induces an equivalence of categories between the ε-hermitian modules over $(B,-)$ and the $\varepsilon\varepsilon_0$-hermitian modules over $(A,-)$ (for any ε in the center of B such that $\varepsilon\overline{\varepsilon} = 1$). This equivalence preserves the orthogonal sums, the hyperbolicity, and the nonsingularity for finitely generated projective modules.

This last statment was proved by Fröhlich and Mc Evett [FMcE] when $(B,-) = (B, \mathrm{Id}_B)$, and Knus proves it in general in his book [K] ; C.T.C. Wall announces in [W] that they also proved the general statement for certain classes of algebras, including central simple algebras, for which sesquilinear forms are simply the hermitian forms for an acceptable notion of duality. The proof we give here, inspired by the proof in [K], is nevertheless shorter and treats directly the general case.

This particular case gives a definition of a Morita equivalence data for hermitian forms. In [D], this definition in incomplete : the condition for h to be ε_0 hermitian has been forgotten.

Definition 1.17. *We call a Morita equivalence data for hermitian forms any collection*

$$((A,-),(B,-),P,Q,f,g,\overline{H}_Q)$$

where $((A,-),(B,-),P,Q,f,g)$ is a Morita equivalence for modules and $\overline{H}_Q \; : \; \overline{P} \to Q$ is an isomorphism of B-A-bimodules, which corresponds after identification of Q and $P^\star$ to an ε_0-hermitian form over P for a $\varepsilon_0 \in Z(A)$.

We say that two algebras with involution $(A,-)$ and $(B,-)$ are Morita equivalent if there exists a Morita equivalence data $((A,-),(B,-),P,Q,f,g,\overline{H}_Q)$ for hermitian forms.

REMARK 1.18 - In her thesis (*cf.* [D]), I. Dejaiffe claims that there is a converse to theorem 1.16 in the hermitian case, and that this is an easy consequence of the classical case. But there is no reason, even in the hermitian case, that if $\mathcal{F} \; : \; \mathbf{Sesq}(B,-) \to \mathbf{Sesq}(A,-)$ is an equivalence of categories with product (the product being given by the orthogonal sum), the image of two forms over the same B-module should define forms over a same A-module. Hence we can not even try to use the classical case to solve this inverse problem.

REMARK 1.19 - The authors think that there should exist a useful definition (for example in wiew of a Witt group) of an hyperbolic sesquilinear form (maybe with asymmetry as defined in the next part) and which would be preserved by the product.

1.4. **Proof of 1.16.**

The proof relies on the following important lemma, which is not stated in [K] even though it is partially used there.

Lemma 1.20. *Let $(A,-)$ be an algebra with antiautomorphism, P be a faithfully projective right-A-module, $B = \mathrm{End}_A(P)$ and $-$ an antiautomorphism over B. Suppose also that $\overline{P}$ is faithfully projective over B.*

Then the map

$$\begin{array}{cccccccc} \Theta_{P,-} & : & \overline{P^\star} & \longrightarrow & \mathrm{Hom}_B(\overline{P}, B) & & & \\ & & \bar{l} & \longmapsto & \Theta_{P,-}(\bar{l}\,) : & \overline{P} & \to & B \\ & & & & & \overline{x} & \mapsto & \overline{f_p(x \otimes l)} \end{array}$$

is an isomorphism of B-A-bimodules.

REMARK 1.21 - With the hypothesis of 1.16, since $\overline{P} \simeq P^\star$, all the assumptions we need here are realized.

Proof. We first prove that $\Theta_{P,-}(\bar{l}\,) \in \mathrm{Hom}_B(\overline{P}, B)$ for $l \in P^\star$. The map $f_P \ : \ P \otimes_A P^\star \to B$ is a morphism of B-B-bimodules hence if $x \in P$ and $b \in B$ then

$$\Theta_{P,-}(\bar{l}\,)(\overline{x}.b) = \Theta_{P,-}(\bar{l}\,)(\overline{\widetilde{b}x}) = \overline{f_p(\widetilde{b}x \otimes l)} = \overline{\widetilde{b}f_p(x \otimes l)} = \overline{f_p(x \otimes l)}.b.$$

Now $\Theta_{P,-}$ is a morphism of B-A-bimodules : if moreover $a \in A$,

$$\begin{aligned} \Theta_{P,-}(b\bar{l}a)(\overline{x}) &= \Theta_{P,-}(\overline{\widetilde{a}l\widetilde{b}})(\overline{x}) = \overline{f_p(x \otimes \widetilde{a}l\widetilde{b})} = \overline{f_p(x\widetilde{a} \otimes l)\widetilde{b}} \\ &= \overline{bf_p(\widetilde{a\overline{x}} \otimes l)} = b\Theta_{P,-}(\bar{l})(a\overline{x}) = (b\Theta_{P,-}(\bar{l})a)(\overline{x}). \end{aligned}$$

To prove that it is an isomorphism, since $\overline{P}$ is a faithfully projective right-B-module and since by 1.6 we have an isomorphism $A \simeq \mathrm{End}_B(\overline{P})$, we can also define

$$\Theta_{\overline{P},\sim} \ : \ \widetilde{\mathrm{Hom}_B(\overline{P}}, B) \to \mathrm{Hom}_A(\widetilde{\overline{P}}, A) = P^\star.$$

It is such that if $t \in Hom_B(\overline{P}, B)$ then $\Theta_{\overline{P},\sim}(\widetilde{t})(x) = \widetilde{f_{\overline{P}}(\overline{x} \otimes t)}$. Here $f_{\overline{P}}$ is an isomorphism of A-A-bimodules $\overline{P} \otimes_B \mathrm{Hom}_B(\overline{P}, B) \to A$.

The result will be given by the fact that $\overline{\Theta_{\overline{P},\sim}}$ is the inverse of $\Theta_{P,-}$. To prove it, using symmetry, it is enough to see that $\Theta_{P,-} \circ \overline{\Theta_{\overline{P},\sim}} = Id_{\mathrm{Hom}_B(\overline{P},B)}$: applying it to the A-B-bimodule $\overline{P}$, we get $\Theta_{\overline{P},\sim} \circ \widetilde{\Theta_{\widetilde{\overline{P}},-}} = \Theta_{\overline{P},\sim} \circ \widetilde{\Theta_{P,-}} = Id_{\mathrm{Hom}_A(P,A)} = \mathrm{Id}_{P^\star}$. Hence $\overline{\Theta_{\overline{P},\sim} \circ \widetilde{\Theta_{P,-}}} = \overline{\Theta_{\overline{P},\sim}} \circ \Theta_{P,-} = Id_{\overline{P^\star}}$, which gives $\overline{\Theta_{\overline{P},\sim}} = \big(\Theta_{P,-}\big)^{-1}$.

Let $x \in P$ and $t \in \mathrm{Hom}_B(\overline{P}, B)$. Then $\Theta_{P,-} \circ \overline{\Theta_{\overline{P},\sim}}(t)(\overline{x}) = \overline{f_P(x \otimes_A \Theta_{\overline{P},\sim}(\widetilde{t}))} \in B = \mathrm{End}_A(P)$. The endomorphism $f_P(x \otimes_A \Theta_{\overline{P},\sim}(\widetilde{t})) \in B = \mathrm{End}_A(P)$ is befined by : if $y \in P$, then

$$f_P(x \otimes_A \Theta_{\overline{P},\sim}(\widetilde{t}))(y) = x\Theta_{\overline{P},\sim}(\widetilde{t})(y) = x\widetilde{f_{\overline{P}}(\overline{y} \otimes t)} \in P.$$

But for $a \in A$, $a\overline{x} = \overline{x\widetilde{a}} \in \overline{P}$, hence $\widetilde{\overline{x\widetilde{a}}} = x\widetilde{a} = \widetilde{a\overline{x}} \in P = \widetilde{\overline{P}}$.

We deduce that

$$f_P(x \otimes_A \Theta_{\overline{P},\sim}(\widetilde{t}))(y) = (f_{\overline{P}}(\overline{y} \otimes t)\overline{x})^\sim = \widetilde{\overline{y}t(\overline{x})} = \widetilde{\overline{\widetilde{t(\overline{x})}y}} = \widetilde{t(\overline{x})}y,$$

hence $f_P(x \otimes_A \Theta_{\overline{P},\sim}(\widetilde{t})) = \widetilde{t(\overline{x})}$ and $\overline{f_P(x \otimes_A \Theta_{\widetilde{P},\sim}(\overline{t}))} = t(\overline{x})$. We get $\Theta_{P,-} \circ \widetilde{\Theta_{\overline{P},\sim}}(t) = t$ which finishes the proof. ∎

We now use this isomorphism to construct inverse functors on the right and on the left to the functor $\mathcal{F}$ in the theorem.

Let $\overline{K_1} = \Theta_{P,-} \circ \overline{\overline{H}} \ : \ \overline{\overline{P}} \to \mathrm{Hom}_B(\overline{P}, B)$. This is an isomorphism of B-A-bimodules and hence defines a nonsingular $(B, -)$-sesquilinear form k_1 over $\overline{P}$ which admits $(A, -)$. We can then define the functor $\mathcal{G}_1 \ : \ \mathbf{Sesq}(A, -) \to \mathbf{Sesq}(B, -)$ by

$$\begin{array}{rcl} \mathcal{G}_1(N, l) &=& (N \otimes_B \overline{P}, k_1 l) \\ \mathcal{G}_1(v) &=& v \otimes \mathrm{Id}_{\overline{P}} \end{array} ,$$

as we defined $\mathcal{F}$. The composite $\mathcal{F} \circ \mathcal{G}_1$ is then defined by

$$\begin{array}{rcccl} \mathcal{F} \circ \mathcal{G}_1(N, l) &=& \mathcal{F}(N \otimes_A \overline{P}, k_1 l) &=& (N \otimes_A (\overline{P} \otimes_B P), ((hk_1)l) \\ \mathcal{F} \circ \mathcal{G}_1(u) &=& u \otimes_A (\mathrm{Id}_{\overline{P}} \otimes_B \mathrm{Id}_P) &=& u \otimes \mathrm{Id}_{\overline{P} \otimes_B P}, \end{array}$$

using 1.14.

Define the unit $(A,-)$-sesquilinear form 1_A on A by $1_A(x,y)=\overline{x}y$ for $x,y\in A$. It is clearly nonsingular and admits $(A,-)$. Moreover if (N,l) is an $(A,-)$-sesquilinear module, then the form $(N\otimes_A A, 1_A l)$ is trivially isometric to (N,l).

To prove that the functor $\mathcal{F}\circ\mathcal{G}_1$ is isomorphic to the identity functor of $\mathbf{Sesq}(A,-)$, it is then enough to prove the following

Lemma 1.22. $\mathbb{H}$ *is an isometry* $(\overline{P}\otimes_B P, hk_1)\to(A,1_A)$.

Proof. For $m,m'\in P$, $k_1(\overline{m},\overline{m}')=\overline{K}_1(\overline{\overline{m}})(\overline{m}')=\Theta_{P,-}(\overline{\overline{H}(\overline{m})})(\overline{m}')=\overline{f_P(m'\otimes\overline{H}(\overline{m}))}$.

Now take $x,x',m,m'\in P$, then

$$\begin{aligned} hk_1(\overline{m}\otimes x,\overline{m}'\otimes x') &= h(x,k_1(\overline{m},\overline{m}')x')=h(x,\overline{f_P(m'\otimes\overline{H}(\overline{m}))}x')\\ &= h(f_P(m'\otimes\overline{H}(\overline{m}))x,x')=h(m'\overline{H}(\overline{m})(x),x')\\ &= h(m'h(m,x),x')=\overline{h(m,x)}h(m',x')=\overline{\mathbb{H}(\overline{m}\otimes x)}\mathbb{H}(\overline{m}'\otimes x')\\ &= 1_A(\mathbb{H}(\overline{m}\otimes x),\mathbb{H}(\overline{m}'\otimes x')). \end{aligned}$$

■

We now define symmetrically K_2 such that $\overline{H}=\Theta_{\widetilde{P},-}\circ\overline{\overline{K_2}}$, i.e

$$\overline{K}_2=\Theta_{\widetilde{\overline{P}},\sim}\circ H=\Theta_{P,\sim}\circ H\ :\ \widetilde{\overline{P}}=P\to\mathrm{Hom}_B(\widetilde{P},B).$$

This isomorphism defines a nonsingular $(B,-)$-sequilinear form k_2 over $\widetilde{P}$ which admits $(A,-)$.

Applying the previous results to $(\widetilde{P},k_2)$ instead of (P,h), we get that $\mathbb{K}_2$ defines an isometry between $(P\otimes_A\widetilde{P},k_2h)$ and $(B,1_B)$, and hence that the functor $\mathcal{G}_2\ :\ \mathbf{Sesq}(A,-)\to\mathbf{Sesq}(B,-)$ defined by

$$\begin{aligned} \mathcal{G}_2(N,l) &= (N\otimes_B\widetilde{P},k_2l)\\ \mathcal{G}_2(v) &= v\otimes\mathrm{Id}_{\widetilde{P}} \end{aligned}$$

is sucht that $\mathcal{G}_2\circ\mathcal{F}$ is isomorphic to the identity functor of $\mathbf{Sesq}(B,-)$.

Remark 1.23 - If the antiautomorphism $-$ is an involution, then $\sim=-$, $\widetilde{P}=\overline{P}$ and $k_1=k_2$ (this is the case in particular if $(A,-)$ is an algebra with involution, P a faithfully projective A-module with an ε_0-hermitian form h which admits $(B,-)$). Then we have proved that $\mathcal{F}$ defines an equivalence of categories $\mathbf{Sesq}(B,-)\to\mathbf{Sesq}(A,-)$.

Now in the general case, we have found isomorphisms of functors $\mathcal{G}_2\circ\mathcal{F}\simeq\mathrm{Id}_{\mathbf{Sesq}(B)}$ and $\mathcal{F}\circ\mathcal{G}_1\simeq\mathrm{Id}_{\mathbf{Sesq}(A)}$. This proves that $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic, hence that $\mathcal{G}_1\circ\mathcal{F}\simeq\mathcal{G}_2\circ\mathcal{F}\simeq\mathrm{Id}_{\mathbf{Sesq}(B)}$, and finally that $\mathcal{F}$ defines an equivalence of categories $\mathbf{Sesq}(B)\to\mathbf{Sesq}(A)$.

Now prove that the multiplication by h preserves the orthogonal sums. Recall that the orthogonal sum of the forms (M_1,f_1) and (M_2,f_2) is the form $(M_1\oplus M_2, f_1\perp f_2)$ with

$$(f_1\perp f_2)(m_1+m_2,m_1'+m_2')=f_1(m_1,m_1')+f_2(m_2,m_2')$$

if $m_i,m_i'\in M_i$.

Hence the form $h(f_1\perp f_2)$ is defined over the module $M_1\otimes P\oplus M_2\otimes P\simeq(M_1\oplus M_2)\otimes P$ by

$$\begin{aligned} &h(f_1\perp f_2)(m_1\otimes p_1+m_2\otimes p_2,m_1'\otimes p_1'+m_2'\otimes p_2')\\ &=h(f_1\perp f_2)(m_1\otimes p_1,m_1'\otimes p_1')+h(f_1\perp f_2)(m_1\otimes p_1,m_2'\otimes p_2')\\ &+h(f_1\perp f_2)(m_2\otimes p_2,m_1'\otimes p_1')+h(f_1\perp f_2)(m_2\otimes p_2,m_2'\otimes p_2')\\ &=h(p_1,f_1(m_1,m_1')p_1)+h(p_2,f_2(m_2,m_2')p_2) \end{aligned}$$

since $(f_1\perp f_2)(m_1,m_2')=(f_1\perp f_2)(m_2,m_1')=0$.

This proves that

$$\begin{aligned} &h(f_1\perp f_2)(m_1\otimes p_1+m_2\otimes p_2,m_1'\otimes p_1'+m_2'\otimes p_2')\\ &=(hf_1\perp hf_2)(m_1\otimes p_1+m_2\otimes p_2,m_1'\otimes p_1'+m_2'\otimes p_2'). \end{aligned}$$

The proof of the first statement of 1.16 is complete.

Now suppose that $-$ is an involution and h is ε_0-hermitian. We want to prove that if (M,k) is an ε-hermitian module over $(B,-)$, then hk is an $\varepsilon_0\varepsilon$ hermitian form.

Let $m,m'\in M, x,x'\in P$. Then since ε and ε_0 are in the center of A,

$$\begin{array}{rcl} hk(m'\otimes x', m\otimes x) & = & h(x', k(m',m)x) \\ & = & h(x', \varepsilon\overline{k(m,m')}x) \\ & = & \varepsilon h(k(m,m')x', x) \\ & = & \varepsilon\varepsilon_0\overline{h(x, k(m,m')x')} \\ & = & \varepsilon\varepsilon_0\overline{hk(m'\otimes x', m\otimes x)}, \end{array}$$

which proves the hermitian symmetry.

It remains to prove that the product preserves the hyperbolicity of hermitian forms. Recall that an hyperbolic ε-hermitian space over $(B,-)$ is a space isometric to a certain $(M\oplus\overline{\mathrm{Hom}_B(M,B)}, k_M)$ where M is finitely generated projective and $k_M(m_1+\bar{t}_1, m_2+\bar{t}_2) = t_1(m_2)+\varepsilon\overline{t_2(m_1)}$ for $m_i\in M$ and $t_i\in\mathrm{Hom}_B(M,B)$.

Then using $1\otimes H$ and the isomorphism α of 1.13 we obtain an isomorphism

$$\widetilde{\mathrm{Hom}_B(M,B)}\otimes_B P \simeq \widetilde{\mathrm{Hom}_B(M,B)}\otimes_B \widetilde{P^\star} \simeq P^\star\otimes_B \widetilde{\mathrm{Hom}_B(M,B)} \simeq \widetilde{(M\otimes P)}^\star.$$

Hence when $-\ =\sim$, we get an isomorphism $\gamma\ :\ (M\oplus\overline{\mathrm{Hom}_B(M,B)})\otimes_B P \to (M\otimes_B P)\oplus\overline{(M\otimes P)^\star}$ defined by $\gamma((m+\bar{t})\otimes x) = m\otimes x + \overline{\alpha(\overline{H}(\overline{x})\otimes t)}$.

We can then compute

$$\begin{array}{rl} & hk_M((m_1+\bar{t}_1)\otimes x_1, (m_2+\bar{t}_2)\otimes x_2) = h(x_1, k_M(m_1+\bar{t}_1, m_2+\bar{t}_2)x_2) \\ = & h(x_1, t_1(m_2)x_2) + h(x_1, \varepsilon\overline{t_2(m_1)}x_2) = h(x_1, t_1(m_2)x_2) + \varepsilon h(t_2(m_1)x_1, x_2) \\ = & h(x_1, t_1(m_2)x_2) + \varepsilon\varepsilon_0\overline{h(x_2, t_2(m_1)x_1)} = \overline{H}(\overline{x}_1)(t_1(m_2)x_2) + \varepsilon\varepsilon_0\overline{\overline{H}(\overline{x}_2)(t_2(m_1)x_1)} \\ = & \alpha(\overline{H}(\overline{x}_1)\otimes t_1)(m_2\otimes x_2) + \varepsilon\varepsilon_0\overline{\alpha(\overline{H}(\overline{x}_2)\otimes t_2)(m_1\otimes x_1)} \\ = & k_{M\otimes P}(m_1\otimes x_1 + \overline{\alpha(\overline{H}(\overline{x}_1)\otimes t_1)}, m_2\otimes x_2 + \overline{\alpha(\overline{H}(\overline{x}_2)\otimes t_2)}) \\ = & k_{M\otimes P}(\gamma((m_1+\bar{t}_1)\otimes x_1), \gamma((m_2+\bar{t}_2)\otimes x_2). \end{array}$$

This proves that hk_M is isometric to $k_{M\otimes P}$, hence hyperbolic.

REMARK 1.24 - The form $k_1\ :\ \overline{P}\times\overline{P}\to B = \mathrm{End}_A P$ in the proof of the theorem is defined by

$$k_1(\overline{m},\overline{m}') = \overline{f_P(m'\otimes\overline{H}(\overline{m}))},$$

which means :

$$\forall m,m',x\in P \quad k_1\widetilde{(\overline{m},\overline{m}')}x = m'h(m,x).$$

Similarly the form $k_2\ :\ \widetilde{P}\times\widetilde{P}\to A = \mathrm{End}_B\widetilde{P}$ is defined by

$$\forall m,m',x\in P \quad h\widetilde{(m,m')}\widetilde{x} = \widetilde{m}'k_2(\widetilde{m},\widetilde{p}).$$

REMARK 1.25 - The isomorphism φ between $\mathcal{G}_2$ and $\mathcal{G}_1$ can be given explicitly : for any sesquilinear module $(N,l)\in\mathbf{Sesq}(A,-)$, the isomorphism

$$\varphi_{(N,l)}\ :\ \mathcal{G}_2(N,l) = (N\otimes_A\widetilde{P}, k_2 l) \longrightarrow \mathcal{G}_1(N,l) = (N\otimes_A\overline{P}, k_1 l)$$

is given by $\varphi_{(N,l)}\ :\ N\otimes_A\widetilde{P}\to N\otimes_A\overline{P}$ is the product of Id_N by the composite isomorphism $(\mathrm{Id}_{\overline{P}}\otimes_B\mathbb{K}_2)\circ(\mathbb{H}^{-1}\otimes_A\mathrm{Id}_{\widetilde{P}})\ :\ \widetilde{P} = A\otimes_A\widetilde{P}\to\overline{P} = \overline{P}\otimes_B B$.

The forms used in the proof of the theorem are the main ingredients to prove the following

Corollary 1.26. *The Morita equivalence between algebras with antiautomorphism is an equivalence relation.*

Proof. The reflexivity is obvious. Suppose that $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ is a Morita equivalence data. We have proved that then the above k_1 is a nonsingular $(B,-)$-sesquilinear form over the A-B-bimodule $\overline{P}$ (which is isomorphic to $P^\star$ hence B-faithfully projective) and that this form admits $(A,-)$. This proves that $((B,-),(A,-),\overline{P},\mathrm{Hom}_B(\overline{P},B),f_{\overline{P}},g_{\overline{P}},\overline{K}_1)$ is a Morita equivalence data. We have proved the symmetry.

Now assume that we have two Morita equivalence datas $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ and $((B,-),(C,-),M,\mathrm{Hom}_B(M,B),f_M,g_M,\overline{K})$. Then $M\otimes_B P$ is a C-A-bimodule, A-faithfully projective, and endows the nonsingular $(A,-)$-sesquilinear form hk which admits $(C,-)$. This proves the transitivity. ∎

1.5. **The case of central simple algebras.**

A natural question is : given two F-algebras with antiautomorphism which are Morita equivalent, is the equivalence data $((A,-),(B,-),P,Q,f,g,\overline{H})$ uniquely defined ?

First note that in the classical case, if A and B are Morita equivalent, then P,Q,f and g are unique up to isomorphism.

In our case, suppose that P,Q,f,g are chosen, hence we get an identification $B=\mathrm{End}_A(P)$. Assume moreover that the transposition is bijective $\mathrm{End}_A(P)\to\mathrm{End}_A(P^\star)$. Then $\overline{H}$ is an isomorphism of A-B-bimodules if and only if the antiautomorphism $-$ over B is the adjunction for the sesquilinear form h associated to $\overline{H}$, i.e. for all $b\in B$ we have $(\widetilde{b})^t=\overline{H}\circ\underline{b}\circ\overline{H}^{-1}$. Hence two sesquilinear forms h and h' define the same adjunction if and only if for any $b\in\mathrm{End}_A P$, we have $(H^{-1}H')b(H^{-1}H')^{-1}=b$, which means if and only if $H^{-1}H'$ is invertible and in the center of $\mathrm{End}_A P$, i.e. in the center of B, which is also the center of A. (Remark that if $\lambda\in Z(A)$, then $H^{-1}H'=\lambda$ means $h'=\lambda h$).

Hence H is unique up to multiplication by an invertible central element of A. In particular :

Proposition 1.27. *Let A be central over F. Suppose that the two algebras with antiautomorphism $(A,-)$ and $(B,-)$ are Morita equivalent. Then the Morita equivalence data*

$$((A,-),(B,-),P,Q,f,g,\overline{H}_Q)$$

is unique up to isomorphism and multiplication of H_Q by a non zero scalar (in F).

If now the antiautomorphisms are involutions and $((A,-),(B,-),P,Q,f,g,\overline{H}_Q)$ is a Morita equivalence data for hermitian forms with central simple algebras A and B, then the hermitianity factor ε_0 can be choosen arbitrarily (in F, with $\varepsilon_0\overline{\varepsilon_0}=1$), and if we fix it, then, as announced in [D], the scalar is invariant by the involution : if h is ε_0-hermitian, then $h'=\lambda h$ is $\varepsilon=\varepsilon_0\lambda\overline{\lambda}^{-1}$-hermitian. Since the algebras are central simple, for any ε in F such that $\varepsilon\overline{\varepsilon}=1$, we get by Hilbert 90 that there exists a $\lambda\in F$ such that $\varepsilon\overline{\varepsilon_0}=\overline{\lambda}\lambda^{-1}$, which gives the results.

In particular, we can chose an equivalence data with a 1-hermitian form.

If A and B are central simple algebras, we know exactly when they are Morita equivalent :

Theorem 1.28. *Two central simple algebras with antiautomorphism are Morita equivalent if and only if they are Brauer equivalent with compatible antiautomorphisms.*

Two central simple algebras with involution are Morita equivalent if and only if they are Brauer equivalent and the involutions are of the same kind and type.

The second statement is partially proved in [D], up to checking that the sesquilinear form obtained is hermitian. The notions of kind and type of an involution are define in [BOI]. Note that two involutions have the the same kind if and only if they are compatible.

Proof. First note that if there is an equivalence data, then there exists a faithfully projective P over A, hence free, such that $B\simeq\mathrm{End}_A(P)$. This proves that A and B are Brauer equivalent and the antiautomorphisms are compatible.

Suppose that A and B are Brauer equivalent, and let $A = M_s(D), B = M_r(D)$ (the matrix algebras of size s, r), for a division algebra D. Let $-$ be compatible antiautomorphisms of A and B, and let $\wedge$ be an antiautomorphism of D compatible with $-$.

The map $\tau(d_{ij}) \mapsto (\hat{d_{ij}})^t$ defines another antiautomorphism of A (resp. B). Hence, by Skolem-Noether theorem, there exist invertible elements $u \in A$ and $v \in B$ such that

$$\forall a \in A \quad \overline{a} = u^{-1}\tau(a)u \quad \text{and} \quad \forall b \in B \quad \overline{b} = v^{-1}\tau(b)v.$$

Let $P = M_{r\times s}(D)$, which is naturally a B-A-bimodule, and $Q = M_{s\times r}(D)$, which is naturally a A-B-bimodule isomorphic to the dual of P. Denote again $\tau(p_{ij}) = (\hat{p}_{ij})^t \in Q$ for $(p_{ij}) \in P$.

Then the map $H \ : \ P \to \widetilde{Q} \ ; \ p \mapsto H(x) = (u^{-1}\tau(x)v)^{\sim}$ is an isomorphism of B-A-bimodules since if $p \in P, a \in A, b \in B$ then

$$H(bxa) = (u^{-1}\tau(bxa)v)^{\sim} = (u^{-1}\tau(a)\tau(x)\tau(b)v)^{\sim} = (\overline{a}(u^{-1}\tau(x)v)\overline{b})^{\sim} = b(u^{-1}\tau(x)v)^{\sim}a = bH(x)a.$$

If now the antiautomorphisms $-$ are involutions of the same kind and type, we may assume that $\wedge$ is an involution of the same kind and type as well. It means that $\tau(u)^{-1}u = \tau(v)^{-1}v = \varepsilon = \pm 1$ ($\varepsilon = 1$ if they are of the second kind or of the first kind and of orthogonal type, and $\varepsilon = -1$ if they are of symplectic type). We check that H defines an hermitian form h : the form h is defined by $h(x, y) = u^{-1}\tau(x)vy$, hence

$$\overline{h(x,y)} = u^{-1}\tau(u^{-1}\tau(x)vy)u = u^{-1}\tau(y)\tau(v)x\tau(u)^{-1}u = u^{-1}\tau(y)v\varepsilon x\varepsilon = u^{-1}\tau(y)vx = h(y, x).$$

■

2. Orthogonal sum of antiautomorphisms

We here generalize the definition of the sum of algebras with involution to algebras with antiautomophism that satisfy a technical hypothesis.

2.1. Asymmetry for a sesquilinear form.

Definition 2.1. *Let $(A, -)$ and $(B, -)$ be F-algebras with antiautomorphism, P a right-A-module, $\sim$ the inverse antiautomorphisms.*

Let $h \ : \ P \times P \to A$ be an $(A, -)$-sesquilinear form which admits $(B, -)$. We say that a map $\alpha \ : P \to P$ is an asymmetry for h if :

(a) *The induced map $\boldsymbol{\alpha} \ : \ \overline{P} \to \widetilde{P} \ ; \ \overline{x} \mapsto \widetilde{\alpha(x)}$ is an isomorphism of A-B-bimodules.*
(b) $\forall x, y \in P \quad h(y, x) = (h(x, \alpha(y)))^{\sim}$*, or equivalently* $\overline{h(y, x)} = h(x, \alpha(y))$

Example 2.2 - If the antiautomorphisms are involutions and h is an ε-hermitian form, then $\overline{\varepsilon}\,\mathrm{Id}_P$ is an asymmetry for h.
Remark 2.3 - Since $\alpha(x) = \overline{\boldsymbol{\alpha}(\overline{x})}$, proving that for all $a \in A, b \in B$ and $x \in P$, $\boldsymbol{\alpha}(a\overline{x}b) = \overline{a}\boldsymbol{\alpha}(\overline{x})\overline{b}$ is equivalent to proving that $\alpha(bxa) = \overline{\overline{b}}\alpha(x)\overline{\overline{a}}$.
Remark 2.4 - We can prove the existence and uniqueness of an asymmetry in the very general case when the sesquilinear form admits $(B, -)$ and is nonsingular on both sides : if $x \in P$, then the map $y \mapsto \widetilde{h(y, x)}$ is an element of $P^{\star} = \mathrm{Hom}_A(P, A)$. If we denote $H_r(\widetilde{x})$ this map, then it defines a morphism $H_r \ : \ \widetilde{P} \to P^{\star}$ of A-B-bimodules as soon as h admits $(B, -)$: for $x, y \in P$ and $a \in A, b \in B$,

$$H_r(a\widetilde{x}b)(y) = H_r\big(\widetilde{\overline{b}x\overline{a}}\big)(y) = (h(y, \overline{b}x\overline{a}))^{\sim} = (h(by, x)\overline{a})^{\sim} = a\widetilde{h(by, x)} = (aH_r(\widetilde{x})b)(y).$$

If moreover h admits $(B,-)$ and is nonsingular, then the composite $\xi = \overline{H}^{-1} \circ H_r \; : \; \widetilde{P} \to \overline{P}$ is a morphism of A-B-bimodules which satisfies : for $x, y \in P$,

$$\begin{array}{rrcl} & \overline{H} \circ \xi(\widetilde{x}) & = & H_r(\widetilde{x}) \in P^\star \\ \text{hence} & h\big(\widetilde{\xi(\widetilde{x})}, y\big) & = & \widetilde{h(y,x)} \\ \text{and} & h(z,y) & = & \big(h\big(y, \overline{\xi^{-1}(\overline{z})}\,\big)\big)^{\sim} \end{array}$$

for $z = \widetilde{\xi(\widetilde{x})}$.

This means that as soon as H_r is an isomorphism (i.e h is non singular on both sides), there exists an asymmetry : the map $\alpha \; : \; z \mapsto \overline{\xi^{-1}(\overline{z})}$ which gives $\boldsymbol{\alpha} = \xi^{-1}$.

It is then clear that we found the only asymmetry of h in this case.

Remark 2.5 - Our choice of definition is coherent with the case of bilinear forms, where a definition can be found in [CT]. It would have been possible in this work to use α^{-1} instead of α but this is not the choice we make. Remark that we also get

$$h(y,x) = \overline{h(\widetilde{\xi(\widetilde{x})}, y)} = \overline{h(\alpha^{-1}(x), y)},$$

which means that α^{-1} is what we could call an asymmetry on the other side.

Remark 2.6 - We can write the corresponding formula for $\mathbb{H}$:

$$\mathbb{H}(\overline{y} \otimes x) = (\mathbb{H}(\overline{x} \otimes \alpha(y)))^{\sim}.$$

Example 2.7 - For central simple algebras A and B, we can define an asymmetry explicitly. We use the notations of 1.5, and suppose that $\wedge$ is involutive. We get

$$\overline{h(y,x)} = u^{-1}\tau(h(y,x))u = u^{-1}\tau(u^{-1}\tau(y)vx)u = u^{-1}\tau(x)v\alpha(y)$$

for $\alpha(y) = v^{-1}\tau(v)y\tau(u)^{-1}u = V^{-1}yU$ if we put $U = \tau(u)^{-1}u$ and $V = \tau(v)^{-1}v$.

It is clear that α is bijective. To check that it induces a morphism $\boldsymbol{\alpha}$ of A-B-bimodules, remind that $\overline{\overline{b}} = V^{-1}bV$ and $\overline{\overline{a}} = U^{-1}aU$ hence :

$$\alpha(bxa) = V^{-1}bxaU = (V^{-1}bV)(V^{-1}xU)(U^{-1}aU) = \overline{\overline{b}}\alpha(x)\overline{\overline{a}}$$

which means that $\boldsymbol{\alpha}$ is a morphism. We have proved that $\alpha(y) = V^{-1}yU$ defines the asymmetry α of h.

2.2. **Morita equivalence when there is an asymmetry.**

We here suppose that we are in the situation of theorem 1.16 and keep the notation of its proof. We assume moreover that h has got an asymmetry α and that $\boldsymbol{\alpha} \; : \; \overline{P} \to \widetilde{P}$ is the associated isomorphism of bimodules. We then defined isomorphisms of bimodules $\mathbb{H} \; : \; \overline{P} \otimes_B P \to A$ and $\mathbb{K}_1 \; : \; \overline{\overline{P}} \otimes_A \overline{P} \to B$. Via the isomorphism $(\overline{\boldsymbol{\alpha}})^{-1} \; : \; P \to \overline{\overline{P}}$; $y \mapsto \overline{\overline{\alpha^{-1}(y)}}$, we can define a new isomorphism of B-B-bimodules : $\mathbb{F} = \mathbb{K}_1 \circ ((\overline{\boldsymbol{\alpha}})^{-1} \otimes \mathrm{Id}_{\overline{P}}) \; : \; P \otimes \overline{P} \to B$.

Lemma 2.8. *Under the above hypothesis, the maps $\mathbb{F}$ and $\mathbb{H}$ are associative isomorphisms and hence $((A,-),(B,-),P,\overline{P},\mathbb{F},\mathbb{H},\mathrm{Id}_{\overline{P}})$ is a Morita equivalence data for sesquilinear forms.*

Proof. We want to prove that for any $x, y, z \in P$ we have both

$$x\mathbb{H}(\overline{y} \otimes z) = \mathbb{F}(x \otimes \overline{y})z \in P \quad \text{and} \quad \mathbb{H}(\overline{x} \otimes y)\overline{z} = \overline{x}\mathbb{F}(y \otimes \overline{z}) \in \overline{P}.$$

Changing x into $\alpha(x)$ in the first equality and y into $\alpha(y)$ in the second and using $\mathbb{F}(p \otimes \overline{p}') = \mathbb{K}_1(\overline{\boldsymbol{\alpha}}^{-1}(p) \otimes \overline{p}') = k_1(\overline{\alpha^{-1}(p)}, \overline{p}')$, this is equivalent to proving that

$$\forall x, y, z \in P \quad \alpha(x)h(y,z) = k_1(\overline{x}, \overline{y})z \in P \quad \text{and} \quad h(x, \alpha(y))\overline{z} = \overline{x}k_1(\overline{y}, \overline{z}) \in \overline{P}.$$

To prove the first equality, we use 1.22 : if $t \in P$ then

$$h(t, k_1(\overline{x}, \overline{y})z) = hk_1(\overline{x} \otimes t, \overline{y} \otimes z) = \overline{h(x,t)}h(y,z),$$

but since $\overline{h(x,t)} = h(t, \alpha(x))$ we get $h(t, k_1(\overline{x}, \overline{y})z) = h(t, \alpha(x)h(y,z))$ which gives the result because h is nonsingular and has an asymmetry.

To prove the second equality we use 1.24 : $\widetilde{k_1(\overline{y},\overline{z})}x = zh(y,x) \in P$, hence we get in $\overline{P}$: $\overline{\widetilde{k_1(\overline{y},\overline{z})}x} = \overline{zh(y,x)}$, which is to say $\overline{x}k_1(\overline{y},\overline{z}) = \overline{h(y,x)}\overline{z}$. Now use the definition of α. It gives $\overline{x}k_1(\overline{y},\overline{z}) = h(x,\alpha(y))\overline{z}$ as desired. ■

Definition 2.9. *We say that two algebras with antiautomorphism $(A,-)$ and $(B,-)$ are Morita equivalent with asymmetry if there exists a Morita equivalence data for sesquilinear forms $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ with $\overline{H}$ corresponding to a sesquilinear form which has an asymmetry.*

REMARK 2.10 - We have seen in 1.5 that for two Morita equivalent algebras with antiautomorphism (and under the condition that the transposition is bijective), the form h in the equivalence data is unique up to a central element in B : if h and h' are two nonsingular sesquilinear forms which admit $(B,-)$, then there exists a λ in the center of A such that $h' = \lambda h$. If α is an asymmetry for h, we deduce that $\alpha' = \lambda^{-1}\overline{\lambda}\alpha$ is an asymmetry for h', because the corresponding map $\boldsymbol{\alpha}'$: $\overline{x} \mapsto \boldsymbol{\alpha}(\overline{x})\lambda\tilde{\lambda}^{-1}$ is a morphism of bimodules since $\boldsymbol{\alpha}$ is a morphism and $\lambda\tilde{\lambda}^{-1}$ is in the center of B.

We can conclude that the fact of having an asymmetry is independent of the choice of the Morita equivalence data.

When the Morita equivalence is realized with a form h which has an asymmetry, we get a isomorphism of B-B-bimodules $\mathbb{F}$: $\widetilde{\overline{P}} \otimes_A \overline{P} = P \otimes_A \overline{P} \to B$, which gives a form f : $\overline{P} \times \overline{P} \to B$ defined by $(\overline{x},\overline{y}) \mapsto f(\overline{x},\overline{y}) = \mathbb{F}(x \otimes \overline{y})$. This form is a non-singular $(B,\sim)$-sesquilinear form which admits $(A,\sim)$. Moreover

Lemma 2.11. *The map $\overline{P} \to \overline{P}$; $\overline{y} \mapsto \overline{\alpha^{-1}(y)}$, which corresponds to the isomorphism of bimodules $\overline{\boldsymbol{\alpha}}^{-1}$: $P = \widetilde{\overline{P}} \to \overline{\overline{P}}$, is an asymmetry for the sesquilinear form f.*

Proof. We want to prove that for any $x,y \in P$, $f(\overline{y},\overline{x}) = \overline{f(\overline{x},\overline{\alpha^{-1}(y)})}$, which means, using the definitions of f and $\mathbb{F}$, that

$$\mathbb{F}(x \otimes \overline{y}) = (\mathbb{F}(\alpha(y) \otimes \overline{x}))^\sim,$$

or that

$$k_1(\overline{\alpha^{-1}(x)},\overline{y}) = \widetilde{k_1(\overline{y},\overline{x})}.$$

To show this, consider auxiliary $z,t \in P$. Then by 1.24

$$h(t,\widetilde{k_1(\overline{y},\overline{x})}z) = h(t,xh(y,z)) = h(t,x)h(y,z)$$

and

$$\begin{array}{rclcl} h(t,k_1(\overline{\alpha^{-1}(x)},\overline{y})z) & = & h\big(\big(k_1(\overline{\alpha^{-1}(x)},\overline{y})\big)^\sim t,z\big) & = & h(yh(\alpha^{-1}(x),t),z) \\ & = & \overline{h(\alpha^{-1}(x),t)}h(y,z) & = & h(t,x)h(y,z). \end{array}$$

We proved that for any $x,y,z,t \in P$ we have $h(t,\widetilde{k_1(\overline{y},\overline{x})}z) = h(t,k_1(\overline{\alpha^{-1}(x)},\overline{y})z)$, which implies the result because h is nonsingular.

■

Corollary 2.12. *The Morita equivalence with asymmetry is an equivalence relation.*

Proof. The reflexivity is trivial. To prove the symmetry, let $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ be a Morita equivalence data with asymmetry α. We want to prove that the corresponding Morita equivalence data $((B,-),(A,-),\overline{P},\mathrm{Hom}_B(\overline{P},B),f_{\overline{P}},g_{\overline{P}},\overline{K}_1)$ has an asymmetry. It follows from the proof above that the map $\overline{\alpha}$: $\overline{x} \mapsto \overline{\alpha(x)}$, which corresponds to the bimodule isomorphism $\overline{\boldsymbol{\alpha}}$: $\overline{\overline{P}} \to \widetilde{\overline{P}}$, is an asymmetry for k_1.

To prove the transitivity, let $(A,-),(B,-),(C,-)$ be F-algebras with antiautomorphism such that $(A,-)$ and $(B,-)$ (reps. $(B,-)$ and $(C,-)$) are Morita equivalent with asymmetry. We want to prove that $(A,-)$ and $(C,-)$ are still Morita equivalent with asymmetry. Assume that the first equivalence is given by a sesquilinear form h with asymmetry α over the bimodule P and the second by a sesquilinear form k with asymmetry β over the bimodule M. Then the form hk over the module $M \otimes_B P$ gives a Morita equivalence between $(A,-)$ and $(C,-)$. It remains to proves that this form has an asymmetry. Let then $x, x' \in P$ and $m, m' \in M$. We have :

$$\begin{array}{rcccl} \overline{hk(m' \otimes x', m \otimes x)} & = & \overline{h(x', k(m', m)x)} & = & \overline{h(\overline{k(m',m)}x', x)} \\ & = & \overline{h(k(\beta(m), m)x', x)} & = & h(\alpha(x), k(\beta(m), m)x') \end{array}$$

which proves that $\beta \otimes_B \alpha$ is an asymmetry for hk. ∎

2.3. **Definition of the orthogonal sum.**

The orthogonal sum of two Morita equivalent central simple algebras with involutions has been defined by I. Dejaiffe in [D] and this notion extends the notion of orthogonal sum of hermitian or symmetric or antisymmetric bilinear forms. We here give a definition for any algebras with antiautomorphism which are Morita equivalent with asymmetry. We will see that our definition extends Dejaiffe's definition and the notion of orthogonal sum of sesquilinear forms.

Definition 2.13. *Let $(A,-)$ and $(B,-)$ be algebras with antiautomorphism. Suppose that they are Morita equivalent with asymmetry, and let $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ be a Morita equivalence data such that the sesquilinear form h corresponding to $\overline{H}$ has an asymmetry α. Let $((A,-),(B,-),P,\overline{P},\mathbb{F},\mathbb{H},\mathrm{Id}_{\overline{P}})$ be the corresponding Morita equivalence data.*

The orthogonal sum $(A,-) \perp_h (B,-)$ is the algebra with antiautomorphism $\left(\begin{pmatrix} B & P \\ \overline{P} & A \end{pmatrix}, -\right)$ where

(a) *the multiplication is given via $\mathbb{H}$ and $\mathbb{F}$ by*

$$\begin{pmatrix} b & x \\ \overline{y} & a \end{pmatrix}\begin{pmatrix} b' & x' \\ \overline{y}' & a' \end{pmatrix} = \begin{pmatrix} bb' + \mathbb{F}(x \otimes \overline{y}') & bx' + xa' \\ \overline{y}b' + a\overline{y}' & aa' + \mathbb{H}(\overline{y} \otimes x') \end{pmatrix}$$

if $a, a' \in A, b, b' \in B, x, x'y, y' \in P$;

(b) *the antiautomorphism $-$ over the sum is given by*

$$\overline{\begin{pmatrix} b & x \\ \overline{y} & a \end{pmatrix}} = \begin{pmatrix} \overline{b} & \alpha(y) \\ \overline{x} & \overline{a} \end{pmatrix}.$$

Lemma 2.14. *The above formulae define an algebra with antiautomorphism for which $(A,-)$ and $(B,-)$ are subalgebras with antiautomorphism.*

Proof. (a) It is clear that the product is F-bilinear and that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is neutral. It remains to prove that the product is associative. It comes directly from the associativity of the products on A and B and from the associativity of $\mathbb{F}$ and $\mathbb{H}$.

(b) It is clear that $-$ over the sum is a group isomorphism. It remains to prove that it changes the order in the products. We then compute for $a, a' \in A, b, b' \in B, x, x'y, y' \in P$,

$$\overline{\begin{pmatrix} b & x \\ \overline{y} & a \end{pmatrix}\begin{pmatrix} b' & x' \\ \overline{y}' & a' \end{pmatrix}} = \overline{\begin{pmatrix} bb' + \mathbb{F}(x \otimes \overline{y}') & bx' + xa' \\ \overline{y}b' + a\overline{y}' & aa' + \mathbb{H}(\overline{y} \otimes x') \end{pmatrix}} = \begin{pmatrix} \overline{b}'\overline{b} + \overline{\mathbb{F}(x \otimes \overline{y}')} & \alpha(\widetilde{\overline{y}b' + a\overline{y}'}) \\ \overline{bx' + xa'} & \overline{a}'\overline{a} + \overline{\mathbb{H}(\overline{y} \otimes x')} \end{pmatrix}$$

while

$$\overline{\begin{pmatrix} b' & x' \\ \overline{y}' & a' \end{pmatrix}}\;\overline{\begin{pmatrix} b & x \\ \overline{y} & a \end{pmatrix}} = \begin{pmatrix} \overline{b}' & \alpha(y') \\ \overline{x}' & \overline{a}' \end{pmatrix}\begin{pmatrix} \overline{b} & \alpha(y) \\ \overline{x} & \overline{a} \end{pmatrix} = \begin{pmatrix} \overline{b}'\overline{b} + \mathbb{F}(\alpha(y') \otimes \overline{x}) & \overline{b}'\alpha(y) + \alpha(y')\overline{a} \\ \overline{x}'\overline{b} + \overline{a}'\overline{x} & \overline{a}'\overline{a} + \mathbb{H}(\overline{x}' \otimes \alpha(y)) \end{pmatrix}.$$

But

$$\begin{array}{rcccl} \overline{bx' + xa'} & = & \overline{x}'\overline{b} + \overline{a}'\overline{x} & & \\ \text{and} \quad \alpha(\widetilde{\overline{y}b' + a\overline{y}'}) & = & \overline{\boldsymbol{\alpha}(\overline{y}b' + a\overline{y}')} & = & \overline{\boldsymbol{\alpha}(\overline{y})b' + a\boldsymbol{\alpha}(\overline{y}')} \\ & = & \overline{\widetilde{\alpha(y)}b' + a\widetilde{\alpha(y')}} & = & \overline{b}'\alpha(y) + \alpha(y')\overline{a}, \end{array}$$

and since α is an asymmetry for h and $\overline{\alpha}^{-1}$ is an asymmetry for f,

$$\begin{array}{rrcl} & \overline{\mathbb{H}(\overline{y}\otimes x')} & = & \mathbb{H}(\overline{x}'\otimes\alpha(y)) \\ \text{and} & \overline{\mathbb{F}(x\otimes\overline{y}')} & = & \mathbb{F}(\alpha(y')\otimes\overline{x}), \end{array}$$

which proves that the two above matrices are equal.

■

Remark 2.15 - This definition of the sum depends on the choice of a Morita equivalence data. However we can use the remarks 2.10 and 1.5 to compare the sums : if we take another equivalence data and if the transposition is bijective, then it is defined by another sesquilinear form h' over P which is a multiple of h, say $h'=\lambda h$ with λ in the center of A. Using the associativity of $\mathbb{H}'$ and $\mathbb{F}'$ we get

$$\mathbb{F}'(x\otimes\overline{y})z = x\mathbb{H}'(\overline{y}\otimes z) = \lambda x\mathbb{H}(\overline{y}\otimes\lambda z) = \lambda\mathbb{F}(x\otimes\overline{y})z$$

and hence $\mathbb{F}'=\lambda\mathbb{F}$.

The product on the algebra $(A,-)\perp_{h'}(B,-)$ is then given by the formula

$$\begin{pmatrix} b & x \\ \widetilde{y} & a\end{pmatrix}\begin{pmatrix} b' & x' \\ \widetilde{y}' & a'\end{pmatrix} = \begin{pmatrix} bb'+\lambda\mathbb{F}(x\otimes\widetilde{y}') & bx'+xa' \\ \widetilde{y}b'+a\widetilde{y}' & aa'+\lambda\mathbb{H}(\widetilde{y}\otimes x')\end{pmatrix}$$

if $a,a'\in A, b,b'\in B, x,x'y,y'\in P$.

The antiautomorphism $-$ over this sum is given by

$$\overline{\begin{pmatrix} b & x \\ \overline{y} & a\end{pmatrix}} = \begin{pmatrix} \overline{b} & \lambda^{-1}\overline{\lambda}\alpha(y) \\ \overline{x} & \overline{a}\end{pmatrix}.$$

Other expressions of the sum. To define the sum we used a specific Morita equivalence data, but if we start with a Morita equivalence data with asymmetry $((A,-),(B,-),P,Q,f,g,\overline{H}_Q)$, we can define the corresponding sum directly : denote $i_Q\ :\ P^\star\to Q$ the isomorphism such that $g\circ(i_Q\otimes_B \mathrm{Id}_P)=g_P$ and $f\circ(\mathrm{Id}_P\otimes_A i_Q)=f_P$, and let $\overline{H}=i_Q^{-1}\circ\overline{H}_Q\ :\ \overline{P}\to P^\star$.

We get for $x,y,z\in P$,

$$\begin{array}{rl} & g(\overline{H}_Q(\overline{y})\otimes x) = g_P(\overline{H}(\overline{y})\otimes x) = \overline{H}(\overline{y})(x) = \mathbb{H}(\overline{y}\otimes x) \\ \text{and} & f(x\otimes\overline{H}_Q(\overline{y}))(z) = f_P(x\otimes\overline{H}(\overline{y}))(z) = x\overline{H}(\overline{y})(z) = x\mathbb{H}(\overline{y}\otimes z) = \mathbb{F}(x\otimes\overline{y})(z) \end{array}$$

hence $f(x\otimes\overline{H}_Q(\overline{y}))=\mathbb{F}(x\otimes\overline{y})$.

We deduce that $\overline{H}_Q$ induces an isomorphism between $A\perp_h B$ and the algebra $\begin{pmatrix} B & P \\ Q & A\end{pmatrix}$ with the natural product given by f and g.

The corresponding antiautomorphism on this algebra is a little bit more complicated to describe and requires a description of the asymmetry in terms of the initial Morita equivalence data : the existence of an asymmetry α for h is equivalent to the existence of an isomorphism $\boldsymbol{\alpha}_Q\ :\ Q\to\widetilde{P}$ such that

$$\forall x\in P\quad \forall q\in Q\quad g(q\otimes x) = \big(g\big(\overline{H}_Q(\overline{x})\otimes\overline{\boldsymbol{\alpha}_Q(q)}\big)\big)^{\sim}\ :$$

they are related by $\boldsymbol{\alpha}=\boldsymbol{\alpha}_Q\circ\overline{H}_Q$ since $g(\overline{H}_Q(\overline{x})\otimes\overline{\boldsymbol{\alpha}_Q(\overline{H}_Q(\overline{y}))})=\mathbb{H}(\widetilde{x}\otimes\overline{\boldsymbol{\alpha}(\overline{y})})$.

By transport of structure, we then define an antiautomorphism over $\begin{pmatrix} B & P \\ Q & A\end{pmatrix}$ by

$$\overline{\begin{pmatrix} b & x \\ q & a\end{pmatrix}} = \begin{pmatrix} \overline{b} & \overline{\boldsymbol{\alpha}_Q(q)} \\ \overline{H}_Q(\overline{x}) & \overline{a}\end{pmatrix},$$

which makes this algebra with antiautomorphism be isomorphic to the sum $(A\perp_h B,-)$.

In particular, a Morita equivalence data with asymmetry $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ gives the orthogonal sum $A\perp_h B=\begin{pmatrix} B & P \\ P^\star & A\end{pmatrix}$ with the natural product given by f_P and g_P and the antiautomorphism defined by

$$\overline{\begin{pmatrix} b & x \\ l & a \end{pmatrix}} = \begin{pmatrix} \overline{b} & \alpha \circ H^{-1}(\widetilde{l}) \\ \overline{H}(\overline{x}) & \overline{a} \end{pmatrix},$$

Example 2.16 - Take again the central simple case. With the notations of 1.5, the sum $M_s(D) \perp_h M_r(D)$ is the usual matrix algebra $M_{r+s}(D)$. To obtain explicitly the antiautomorphism, we need first to find an isomorphism between $\overline{P}$ and $M_{s\times r}(D)$, i.e to find a group isomorphism $- \ : \ P = M_{r\times s}(D) \to M_{s\times r}(D)$ which satisfies

$$\forall x \in P, \forall a \in A, \forall b \in B \quad \overline{a}\,\overline{x}\,\overline{b} = \overline{bxa}.$$

Since $\overline{a} = u^{-1}\tau(a)u$ and $\overline{b} = v^{-1}\tau(b)v$, this condition can be written $\tau(a)(u\overline{x}v^{-1})\tau(b) = u\overline{bxa}v^{-1}$. It is clear that we can take

$$\overline{x} = u^{-1}\tau(x)v.$$

The antiautomorphism is then defined by

$$\overline{\begin{pmatrix} b & x \\ q & a \end{pmatrix}} = \begin{pmatrix} \overline{b} & \alpha(\widetilde{q}) \\ \overline{x} & \overline{a} \end{pmatrix} = \begin{pmatrix} v^{-1}\tau(b)v & v^{-1}\tau(q)u \\ u^{-1}\tau(x)v & u^{-1}\tau(a)u \end{pmatrix} = \begin{pmatrix} v^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} \tau(\begin{pmatrix} b & x \\ q & a \end{pmatrix}) \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

These computations prove that this notion of orthogonal sum extends both Dejaiffe's definition and the usual definition of orthogonal sum of sesquilinear forms.

2.4. Associativity of the orthogonal sum.

It is clear that the orthogonal sum is commutative : $(A \perp_h B, -) \simeq (B \perp_{k_1} A, -)$.

In [D], I. Dejaiffe proves that the orthogonal sum of central simple algebras with involution is associative. To extend this result to the orthogonal sum of algebras with antiautomorphism, we will need the following :

Proposition 2.17. *Let $(A,-),(B,-),(C,-)$ be F-algebras with antiautomorphism which are Morita equivalent with asymmetry. Then $(A \perp B, -)$ and $(C,-)$ remain Morita equivalent with asymmetry.*

Proof. Let $((A,-),(B,-),P,P^\star,f_P,g_P,\overline{H})$ and $((B,-),(C,-),M,\mathrm{Hom}_B(M,B),f_M,g_M,\overline{K})$ be Morita equivalence datas. Denote N the matrix set $N = \begin{pmatrix} M & M\otimes P \end{pmatrix}$. This is naturally a C-$(A \perp_h B)$-bimodule (we multiply the matrices in blocs and use the identification f_P). Similarly, $N' = \begin{pmatrix} \mathrm{Hom}_B(M,B) \\ (M\otimes P)^\star \end{pmatrix} = \begin{pmatrix} \mathrm{Hom}_B(M,B) \\ P^\star \otimes \mathrm{Hom}_B(M,B) \end{pmatrix}$ is naturally a $(A \perp_h B)$-C-bimodule.

Using the bloc matrix product and the isomorphisms f_M and $f_{M\otimes P}$, we get a natural product

$$\begin{array}{rccl} f \ : & \begin{pmatrix} M & M\otimes P \end{pmatrix} \times \begin{pmatrix} \mathrm{Hom}_B(M,B) \\ (M\otimes P)^\star \end{pmatrix} & \to & C \\ & ((m,y), \begin{pmatrix} t \\ q \end{pmatrix}) & \mapsto & f_M(m\otimes t) + f_{M\otimes P}(y \otimes q) \end{array}$$

and with the isomorphisms g_M and $g_{M\otimes P}$ a natural product

$$\begin{array}{rccl} g \ : & \begin{pmatrix} \mathrm{Hom}_B(M,B) \\ (M\otimes P)^\star \end{pmatrix} \times \begin{pmatrix} M & M\otimes P \end{pmatrix} & \to & A \perp_h B = \begin{pmatrix} B & P \\ P^\star & A \end{pmatrix} \\ & (\begin{pmatrix} t \\ l\otimes t' \end{pmatrix}, (m, m'\otimes x)) & \mapsto & \begin{pmatrix} g_M(t\otimes m) & g_M(t\otimes m')x \\ l g_M(t'\otimes m) & g_{M\otimes P}((l\otimes t')\otimes(m'\otimes x)) \end{pmatrix}. \end{array}$$

We prove that they induce isomorphisms from the tensor products respectively over $A \perp_h B$ and C to respectively C and $A \perp_h B$. This is obvious for g since g_M and $g_{M\otimes P}$ are well defined isomorphisms. For f, we need the following

Lemma 2.18. *Under the identification of 1.13, if $x \in P, m \in M, l \in P^\star, t \in \mathrm{Hom}_B(M,B)$, we have*

$$f_{M\otimes P}((m\otimes x)\otimes(l\otimes t)) = f_M(m\otimes f_P(x\otimes l)t) \in C.$$

Proof. We prove the equality by applying those linear maps in $C = \mathrm{End}_A(M \otimes P)$ to an element $m' \otimes x' \in M \otimes P$. Recall that $(l \otimes t)(m' \otimes x') = l(t(m')x')$ according to 1.13.

We get

$$\begin{aligned} f_{M\otimes P}((m \otimes x) \otimes (l \otimes t))(m' \otimes x') &= m \otimes xl(t(m')x') &&= m \otimes f_P(x \otimes l)t(m')x' \\ &= mf_P(x \otimes l)t(m') \otimes x' &&= f_M(m \otimes f_P(x \otimes l)t)m' \otimes x' \end{aligned}$$

which gives the lemma. ■

We now compute

$$\begin{aligned} & f\big((m, m' \otimes x')\begin{pmatrix} b & x \\ l & a \end{pmatrix}, \begin{pmatrix} t \\ l' \otimes t' \end{pmatrix}\big) = f\big(mb + m'f_P(x' \otimes l), m \otimes x + m' \otimes x'a), \begin{pmatrix} t \\ l' \otimes t' \end{pmatrix}\big) \\ &= f_M((mb + m'f_P(x' \otimes l)) \otimes t) + f_{M\otimes P}((m \otimes x + m' \otimes x'a) \otimes (l' \otimes t')) \\ &= f_M(m \otimes bt) + f_M(m' \otimes f_P(x' \otimes l)t) + f_{M\otimes P}((m \otimes x) \otimes (l' \otimes t')) \\ &\qquad + f_{M\otimes P}((m' \otimes x'a) \otimes (l' \otimes t')), \end{aligned}$$

while

$$\begin{aligned} & f\big((m, m' \otimes x'), \begin{pmatrix} b & x \\ l & a \end{pmatrix}\begin{pmatrix} t \\ l' \otimes t' \end{pmatrix}\big) = f\big((m, m' \otimes x'), \begin{pmatrix} bt + f_P(x \otimes l')t' \\ l \otimes t + al' \otimes t' \end{pmatrix}\big) \\ &= f_M(m \otimes (bt + f_P(x \otimes l')t')) + f_{M\otimes P}((m' \otimes x') \otimes (l \otimes t + al' \otimes t')) \\ &= f_M(m \otimes bt) + f_M(m \otimes f_P(x \otimes l')t') + f_{M\otimes P}((m' \otimes x') \otimes (l \otimes t)) \\ &\qquad + f_{M\otimes P}((m' \otimes x') \otimes (al' \otimes t')). \end{aligned}$$

Those two expressions are the same according to the lemma. This proves that f induces a morphism $N \otimes_{A\perp B} N' \to C$. It is clear that this is then an isomorphism of C-C-bimodules.

We have proved that $(A \perp_h B, C, N, N', f, g)$ is a Morita equivalence data of algebras, and hence identifyed N' and $\mathrm{Hom}_{A\perp B}(N, A \perp B)$.

A natural candidate to define a Morita equivalence data between algebras with antiautomorphism is then the isomorphism of right-C-modules

$$\overline{\Phi} = \begin{pmatrix} \overline{K} \\ \overline{H} \otimes \overline{K} \end{pmatrix} \; : \; \overline{N} = \begin{pmatrix} \overline{M} \\ \overline{P} \otimes \overline{M} \end{pmatrix} \to N' = \begin{pmatrix} \mathrm{Hom}(M, B) \\ \mathrm{Hom}(P, A) \otimes \mathrm{Hom}(M, B) \end{pmatrix}.$$

We first can check that $\overline{N}$ is the $(A \perp B)$-C-bimodule $\begin{pmatrix} \overline{M} \\ \overline{P} \otimes \overline{M} \end{pmatrix}$ with the action of $A \perp B$ given by the product in bloc and using $\mathbb{F}$ and $\overline{H}^{-1}$: by the formula in 2.3 which defines the antiautomorphism $-$ over $A \perp_h B$, we get $\begin{pmatrix} b & x \\ l & a \end{pmatrix}^{\sim} = \begin{pmatrix} \widetilde{b} & H^{-1}(\widetilde{l}) \\ \overline{H} \circ \boldsymbol{\alpha}^{-1}(\widetilde{x}) & \widetilde{a} \end{pmatrix}$ and hence

$$\begin{aligned} \overline{(m, m' \otimes x')\begin{pmatrix} b & x \\ l & a \end{pmatrix}^{\sim}} &= \overline{(m\widetilde{b} + m'f_P(x' \otimes \overline{H} \circ \boldsymbol{\alpha}^{-1}(\widetilde{x})), m \otimes H^{-1}(\widetilde{l}) + m \otimes x'\widetilde{a})} \\ &= \begin{pmatrix} b\overline{m} + \overline{f_P(x' \otimes \overline{H} \circ \boldsymbol{\alpha}^{-1}(\widetilde{x}))}\overline{m}' \\ \overline{H}^{-1}(l) \otimes \overline{m} + a\overline{x}' \otimes \overline{m}' \end{pmatrix}. \end{aligned}$$

But by 1.24, $\overline{f_P(x' \otimes \overline{H} \circ \boldsymbol{\alpha}^{-1}(\widetilde{x}))} = k_1(\overline{\alpha^{-1}(x)}, \overline{x}') = \mathbb{K}_1(\overline{\boldsymbol{\alpha}^{-1}(\widetilde{x})} \otimes \overline{x}') = \mathbb{F}(x \otimes \overline{x}')$ hence

$$\overline{(m, m' \otimes x')\begin{pmatrix} b & x \\ l & a \end{pmatrix}^{\sim}} = \begin{pmatrix} b\overline{m} + \mathbb{F}(x \otimes \overline{x}')\overline{m}' \\ \overline{H}^{-1}(l) \otimes \overline{m} + a\overline{x}' \otimes \overline{m}' \end{pmatrix} = \begin{pmatrix} b & x \\ l & a \end{pmatrix}\begin{pmatrix} \overline{m} \\ \overline{x}' \otimes \overline{m}' \end{pmatrix}.$$

We have proved that the $(A \perp B)$-C-bimodule structure of $\overline{N}$ is the natural structure of $\begin{pmatrix} \overline{M} \\ \overline{P} \otimes \overline{M} \end{pmatrix}$.

We use this to prove that $\overline{\Phi} = \begin{pmatrix} \overline{K} \\ \overline{H} \otimes \overline{K} \end{pmatrix}$ is a morphism of left $(A \perp B)$-module :

$$\overline{\Phi}\big(\begin{pmatrix} b & x \\ l & a \end{pmatrix}\begin{pmatrix} \overline{m} \\ \overline{x}' \otimes \overline{m}' \end{pmatrix}\big) = \begin{pmatrix} b\overline{K}(\overline{m}) + \mathbb{F}(x \otimes \overline{x}')\overline{K}(\overline{m}') \\ l \otimes \overline{K}(\overline{m}) + a\overline{H}(\overline{x}) \otimes \overline{K}(\overline{m}') \end{pmatrix}$$

while

$$\begin{pmatrix} b & x \\ l & a \end{pmatrix} \overline{\Phi}\Big(\begin{pmatrix} \overline{m} \\ \overline{x}' \otimes \overline{m}' \end{pmatrix}\Big) = \begin{pmatrix} b & x \\ l & a \end{pmatrix} \begin{pmatrix} \overline{K}(\overline{m}) \\ \overline{H}(\overline{x}') \otimes \overline{K}(\overline{m}') \end{pmatrix} = \begin{pmatrix} b\overline{K}(\overline{m}) + f_P(x \otimes \overline{H}(\overline{x}'))\overline{K}(\overline{m}') \\ l \otimes \overline{K}(\overline{m}) + a\overline{H}(\overline{x}) \otimes \overline{K}(\overline{m}') \end{pmatrix}.$$

Those terms are equal by applying 1.24.

It remains to prove that if h as the asymmetry α and k has the asymmetry β, then the sesquilinear form φ over N which corresponds to $\overline{\Phi}$ has the asymmetry $(\beta, \beta \otimes \alpha)$. Here

$$\begin{aligned}
&\varphi\big((m, m_1 \otimes x_1), (m', m_1' \otimes x_1')\big) = \begin{pmatrix} \overline{K}(\overline{m}) \\ \overline{H}(\overline{x}_1) \otimes \overline{K}(\overline{m}_1) \end{pmatrix} (m', m_1' \otimes x_1') \\
= &\begin{pmatrix} \overline{K}(\overline{m})(m') & \overline{K}(\overline{m})(m_1')x_1' \\ \overline{H}(\overline{x}_1)(\overline{K}(\overline{m}_1)(m')) & g_{M\otimes P}\big((\overline{H}(\overline{x}_1) \otimes \overline{K}(\overline{m}_1)) \otimes (m_1' \otimes x_1')\big) \end{pmatrix} \\
= &\begin{pmatrix} k(m, m') & k(m, m_1')x_1' \\ \overline{H}(\overline{x}_1)k(m_1, m') & \overline{H}(\overline{x}_1)\big(\overline{K}(\overline{m}_1)(m_1')x_1'\big) \end{pmatrix} \\
= &\begin{pmatrix} \overline{k(m, m')} & k(m, m_1')x_1' \\ \overline{H}\big(\widetilde{k(m_1, m')}x_1\big) & hk(m_1 \otimes x_1, m_1' \otimes x_1') \end{pmatrix}
\end{aligned}$$

We have already proved in 2.12 that $\beta \otimes \alpha$ is an asymmetry for hk, hence

$$\begin{aligned}
&\varphi\big((m, m_1 \otimes x_1), (\beta, \beta \otimes \alpha)(m', m_1' \otimes x_1')\big) \\
= &\begin{pmatrix} k(m, \beta(m')) & k(m, \beta(m_1'))\alpha(x_1') \\ \overline{H}(\overline{x}_1)k(m_1, \beta(m')) & hk(m_1 \otimes x_1, (\beta \otimes \alpha)(m_1' \otimes x_1')) \end{pmatrix} \\
= &\begin{pmatrix} \overline{k(m', m)} & \overline{\overline{k(m_1', m)}\alpha(x_1')} \\ \overline{H}(\overline{x}_1)\overline{k(m', m_1)} & \overline{hk(m_1' \otimes x_1', m_1 \otimes x_1)} \end{pmatrix} \\
= &\overline{\begin{pmatrix} \overline{k(m', m)} & \overline{\alpha(k(\widetilde{m_1'}, m)x_1')} \\ \overline{H}(\overline{k(m', m_1)x_1}) & \overline{hk(m_1' \otimes x_1', m_1 \otimes x_1)} \end{pmatrix}} \\
= &\begin{pmatrix} \overline{k(m', m)} & k(m', m_1)x_1 \\ \overline{H}\big(\widetilde{k(m_1', m)}x_1')\big) & hk(m_1' \otimes x_1', m_1 \otimes x_1) \end{pmatrix}
\end{aligned}$$

which ends the proof.■

Corollary 2.19. *With the above notations the sum $(A \perp_h B) \perp_\varphi C$ is the algebra*

$$\begin{pmatrix} C & M & M \otimes P \\ \mathrm{Hom}_B(M, B) & B & P \\ \mathrm{Hom}_C(M \otimes P, C) & \mathrm{Hom}_A(P, A) & A \end{pmatrix}$$

with the the product in bloc using $f_P, g_P, f_M, g_M, f_{M\otimes P}, g_{M\otimes P}$ and the antiautomorphism defined by

$$\overline{\begin{pmatrix} c & m & q \\ t & b & x \\ u & l & a \end{pmatrix}} = \begin{pmatrix} \overline{c} & \beta^{-1} \circ K^{-1}(\widetilde{t}) & \gamma^{-1} \circ J^{-1}(\widetilde{u}) \\ \overline{K}(\overline{m}) & \overline{b} & \alpha^{-1} \circ H^{-1}(\widetilde{l}) \\ \overline{J}(\overline{q}) & \overline{H}(\overline{x}) & \overline{a} \end{pmatrix},$$

where J and $\overline{J}$ are the bimodule isomorphisms corresponding to the form $j = hk$, and $\gamma = \beta \otimes \alpha$.

This is just a rewriting of the results obtained in the previous proof, and from this we get directly :

Theorem 2.20.
Let $((A, -), (B, -), P, P^\star, f_P, g_P, \overline{H})$ and $((B, -), (C, -), M, \mathrm{Hom}_B(M, B), f_M, g_M, \overline{K})$ be Morita equivalence datas. Let $j = hk$ and $\widetilde{J}$ the corresponding isomorphism.

Let $\overline{\Phi} = \begin{pmatrix} \overline{K} \\ \overline{H} \otimes \overline{K} \end{pmatrix}$, $\overline{\Psi} = (\overline{H} \otimes \overline{K}, \overline{H})$ and φ and ψ the corresponding sesquilinear forms.

Then the algebras with antiautomorphism $(A \perp_h B) \perp_\phi C$ and $A \perp_\psi (B \perp_k C)$ are equal.

2.5. **Invariants in the linear central simple case.**

When the algebras are central simple and the antiautomorphisms are linear, some invariants have been defined : the asymmetry and the determinant in [CT] and the Clifford algebra in [C]. The authors don't know yet how to compute the Clifford algebra of an orthogonal sum, but think it should be related to the Clifford algebras of the two algebras with antiautomorphism. This was done in [D] in the case of Morita equivalent algebras with involution, ie for linear involutions of the same type over Brauer equivalent algebras. We here describe the other invariants of the orthogonal sum :

Theorem 2.21. *Let $(A,-)$ and $(B,-)$ be Morita equivalent central simple algebras with F-linear antiautomorphism. Then independently of the sesquilinear form h chosed to define the orthogonal sum :*

(a) $\deg(A\perp B)=\deg A+\deg B$ *;*
(b) *Denote U(resp. V) the asymmetry of $(A,-)$ (resp. $(B,-)$). Then the asymmetry of $(A\perp B,-)$ is $W=\begin{pmatrix}V&0\\0&U\end{pmatrix}$;*
(c) *Suppose that A and B are of even degree. Then* $\det(A\perp B,-)=\det(A,-)\det(B,-)$.

Proof. The first point is clear from the definition of the sum. To prove (b), using scalar extension to F^{sep}, it is enough to prove it when the algebras are split.

Then $D=F,\wedge=Id_F$ and $\tau=t$ is the transposition. Let u and v be respectively the matrices of some bilinear forms over F^r and F^s associated to $-$ (via adjonction). Then the asymmetries of $(A,-)$ and $(B,-)$ are respectively $U=u^{-t}u$ and $V=v^{-t}v$.

We already used the sesquilinear form h over P defined by $h(x,y)=u^{-1}x^tvy$. For this form, the antiautomorphism over $(A,-)\perp_h(B,-)$ is defined by $\overline{\begin{pmatrix}b&x\\q&a\end{pmatrix}}=\begin{pmatrix}v^{-1}&0\\0&u^{-1}\end{pmatrix}\begin{pmatrix}b&x\\q&a\end{pmatrix}^t\begin{pmatrix}v&0\\0&u\end{pmatrix}$ hence the antiautomorphism of $-$ is the adjonction for the bilinear form of matrix $\begin{pmatrix}v&0\\0&u\end{pmatrix}$ and, according to [CT], its asymmetry is $\begin{pmatrix}v&0\\0&u\end{pmatrix}^{-t}\begin{pmatrix}v&0\\0&u\end{pmatrix}=\begin{pmatrix}V&0\\0&U\end{pmatrix}$.

If we take another form $h'=\lambda h$, we just have to change v to $v'=\lambda v$ in all the previous formulae. We can see that it does not change the asymmetry.

To prove (c), assume that the degrees are even. Then by definition, the determinant of $(A,-)$ is $\det(A,-)=\mathrm{Nrd}(a-\overline{a}U)$ for any $a\in A$ which satisfies that $a-\overline{a}U$ is invertible. Choose such an a and similarly choose a $b\in B$ such that $b-\overline{b}V\in B^\times$, hence $\det(B,-)=\mathrm{Nrd}(b-\overline{b}V)$.

Then $\begin{pmatrix}b&0\\0&a\end{pmatrix}-\overline{\begin{pmatrix}b&0\\0&a\end{pmatrix}}\begin{pmatrix}V&0\\0&U\end{pmatrix}=\begin{pmatrix}b-\overline{b}V&0\\0&a-\overline{a}U\end{pmatrix}$ is invertible in $A\perp B$, and hence

$$\det(A\perp B,-)=\mathrm{Nrd}\Big(\begin{pmatrix}b-\overline{b}V&0\\0&a-\overline{a}U\end{pmatrix}\Big)=\det(A,-)\det(B,-).$$

■

REMARK 2.22 - If two Brauer equivalent central simple algebras are endowed with involutions of different type, their asymmetries are respectively 1 and -1, and hence the asymmetry of the antiautomorphism over their orthogonal sum is $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. This antiautomorphism is hence of order 4 and is no longer an involution.

In [Le1], D. Lewis defines the trace form of an antiautomorphism over a central simple algebra : $T_{(A,-)}(x)=Tr_{A/F}(\overline{x}x)$, and gives a criterion to check if this quadratic form is or not degenerate : $T_{(A,-)}$ is non degenerate if and only if there exists no non-zero element $x\in A$ such that $\overline{\overline{x}}+x=0$.

Since
$$\overline{\overline{\begin{pmatrix} b & x \\ q & a \end{pmatrix}}} = \begin{pmatrix} V^{-1}bV & V^{-1}xU \\ U^{-1}qV & U^{-1}aU \end{pmatrix} = \begin{pmatrix} \overline{\overline{b}} & \alpha(x) \\ \overline{\alpha(\widetilde{q})} & \overline{\overline{a}} \end{pmatrix},$$
we get :

Proposition 2.23. *The trace form of* $\big((A,-)\perp_h (B,-),-\big)$ *is non degenerate if and only if the trace forms of* $(A,-)$ *and* $(b,-)$ *are both non degenerate and there is no non-zero* $x\in P$ *such that* $\alpha(x)+x=0$.

The authors don't know yet how to interpret this in terms of properties of h or of the algebras.

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