Tests for spatial randomness based on spacings

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Séminaire Statistique Mathématique et Applications

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Overview

- Spacings theory on $[0,1]$.
- New statistics for testing spatial randomness:

- Asymptotic normality of these statistics.
- Power of the tests.

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- Idea: testing uniformity and independence by observing spacings' dispersion.

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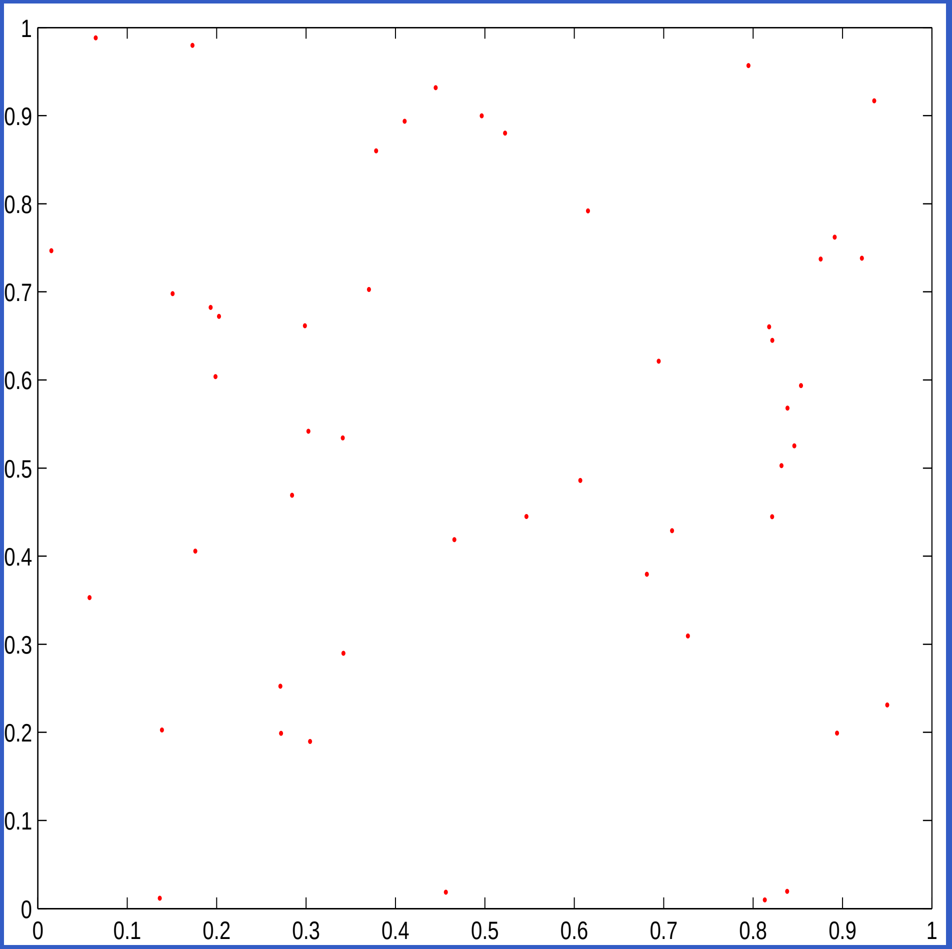
- Tests based on S_n: powerful against dependence.

Point processes

4 main types:

- Homogeneous Poisson process



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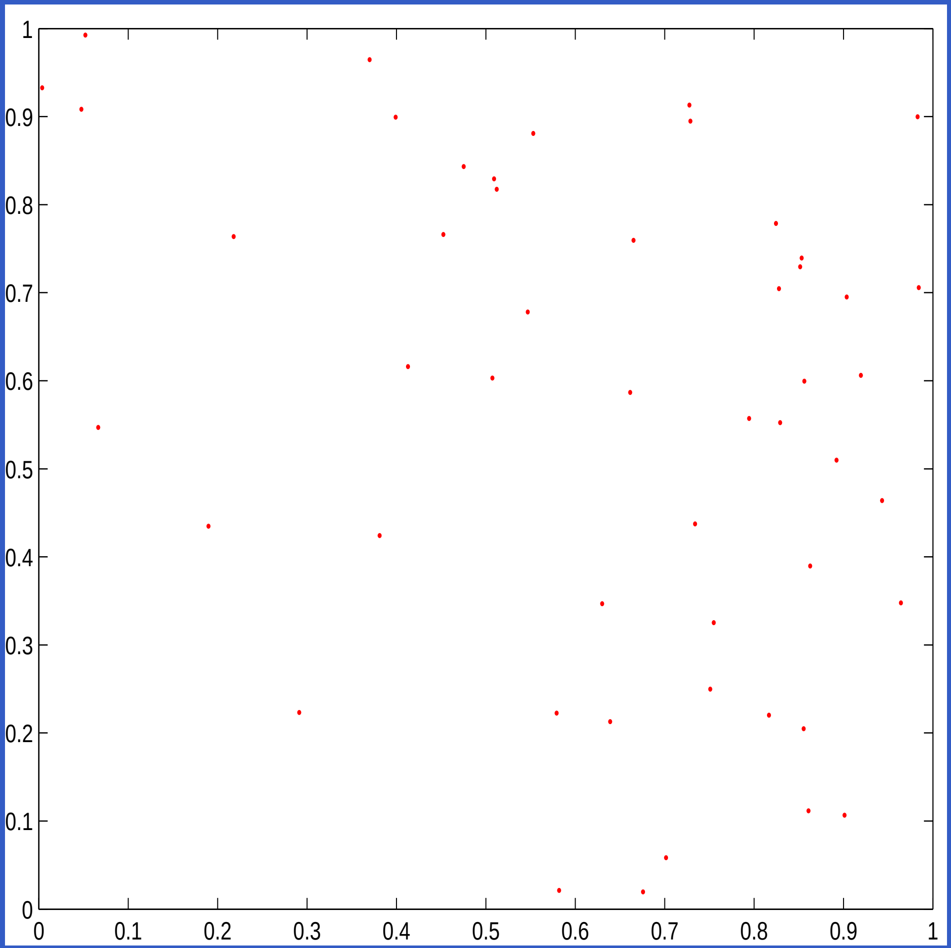

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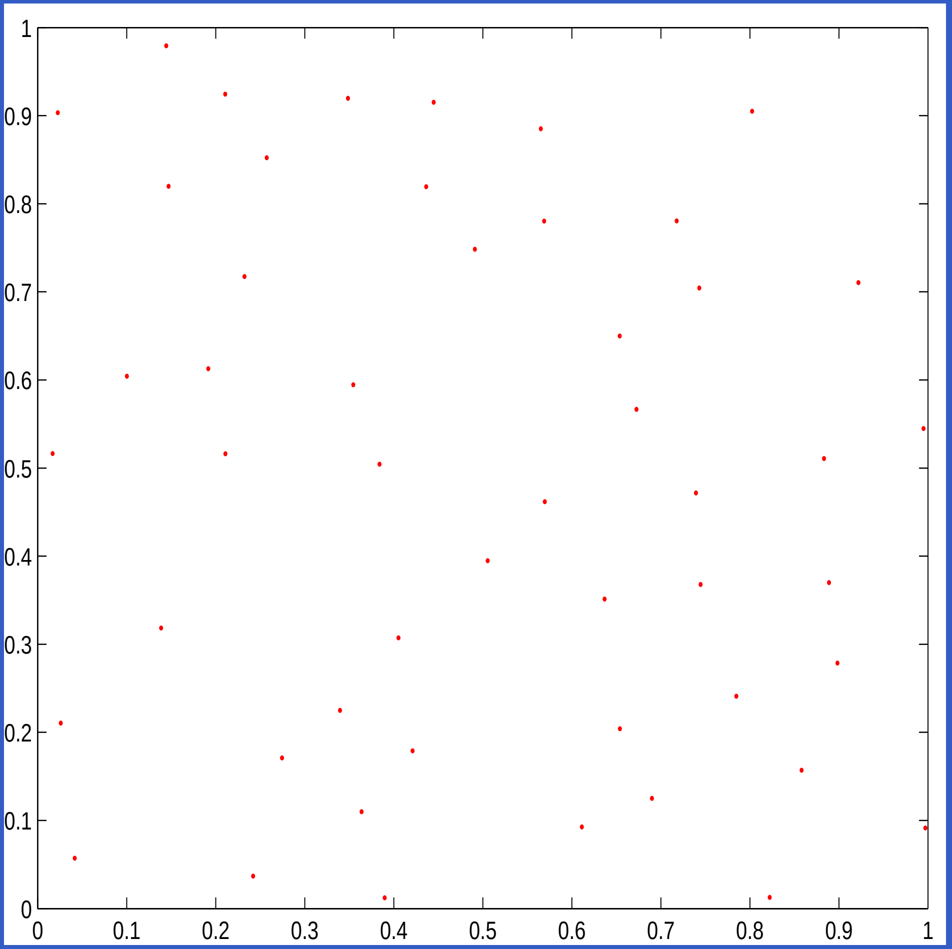
- Regular processes



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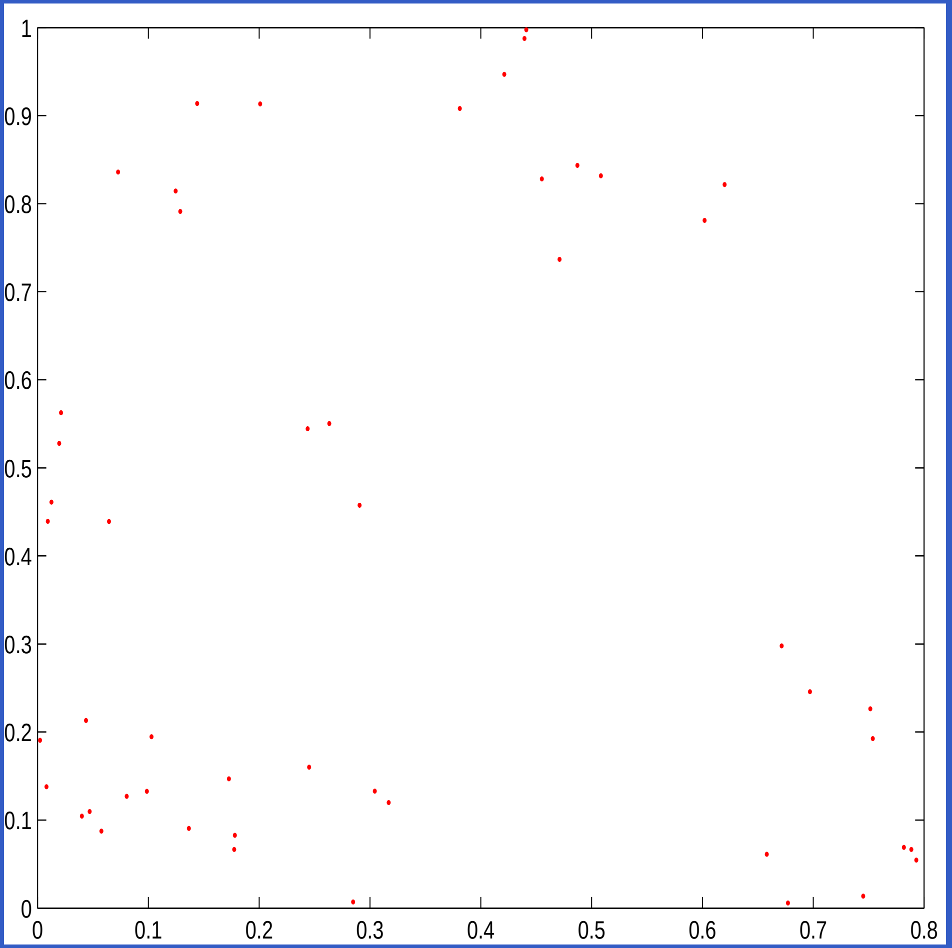
- Cluster processes



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$\Rightarrow$ First reflex: CSR test.

2D spacings

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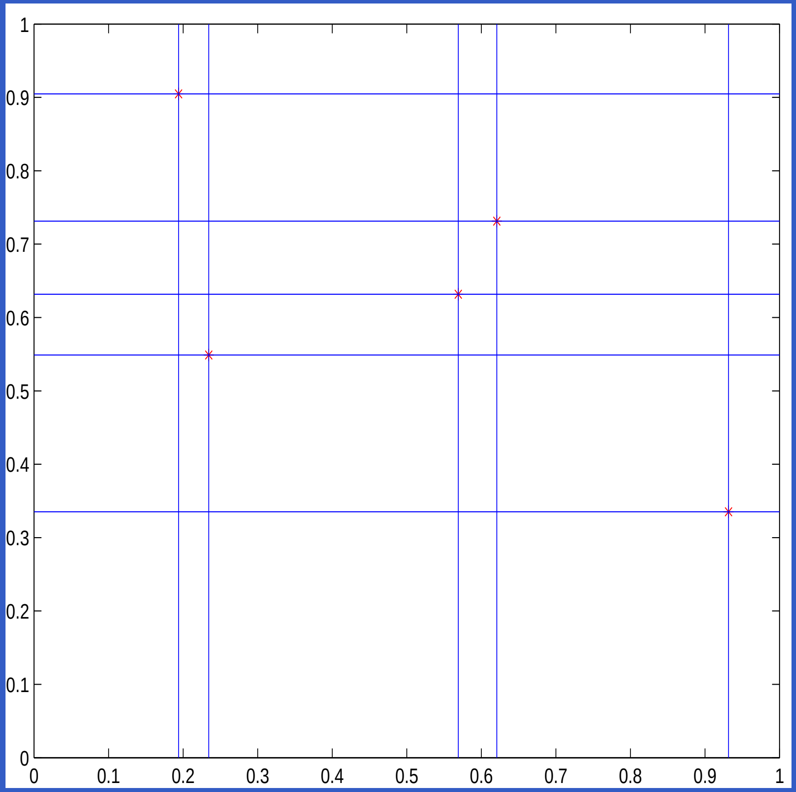

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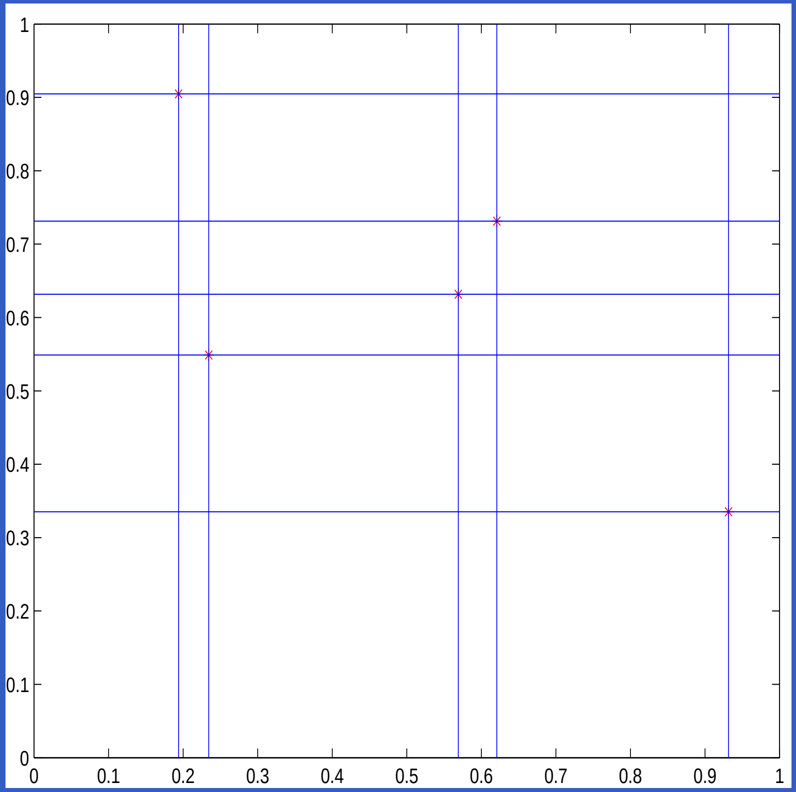

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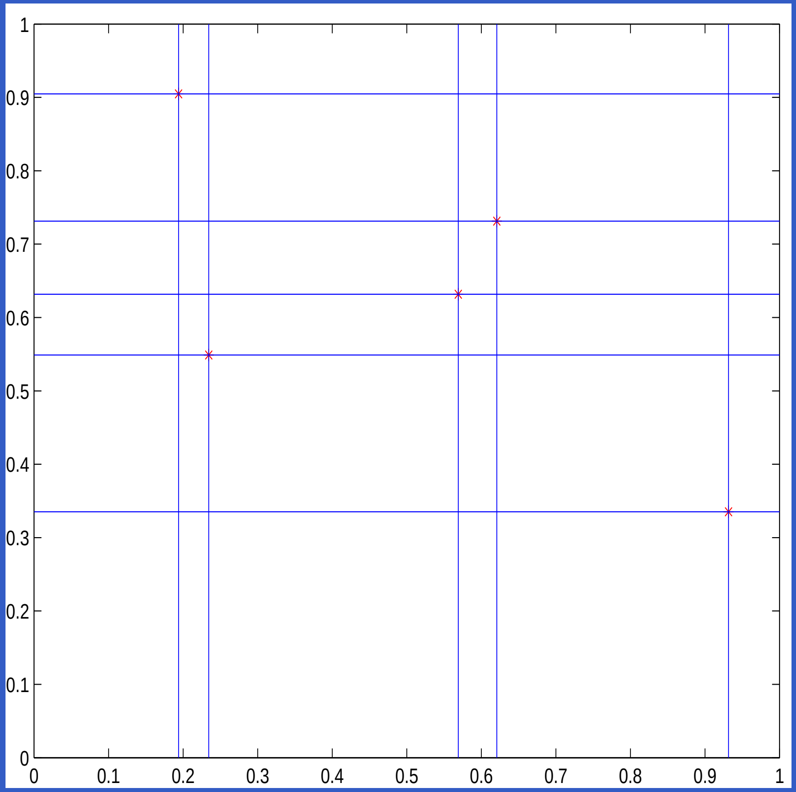

- $A_{ij} = D_i^x D_j^y, \qquad i = 1, \cdots, n,\ j = 1, \cdots, n.$
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- Idea 1: use $(nD_1, \cdots, nD_n) \sim \left(\frac{E_1}{\bar{E}}, \cdots, \frac{E_n}{\bar{E}}\right)$, where $(E_1, \cdots, E_n)\, i.i.d. \sim \mathcal{E}(1)$.

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- Idée 2: Taylor expansion $\rightarrow g\left(\frac{X_i}{\bar{X}} \frac{Y_j}{\bar{Y}}\right) \sim_{n\to\infty} g(X_i Y_j) - c(\bar{X} - 1) - c(\bar{Y} - 1)$, where $c = Cov(g(X_1 Y_1), X_1)$.

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- Variance

$$\rightarrow V_n = \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \{(n^2 A_{ij} - 1)^2 - 3\} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, 32).$$

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Limitation: $n \leq 100 \Rightarrow$ empirical fractiles far from the fractiles of the limit law.

Application to real data sets

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Test statistic	Results for the following data sets			
	Japanese pines	Redwoods	Biological cells	Scouring rushes
V_n	< 0.002	0.042	0.064	0.024
R_n	< 0.002	< 0.002	0.832	0.044
$\bar{\omega}^2$	0.712	0.692	0.006	0.004
D_n	0.26	0.908	0.014	0.044
T	0.915	< 0.001	< 0.001	0.936
U	0.68	< 0.01	< 0.01	0.98
V	0.50	< 0.01	< 0.01	0.92
Li	0.918	< 0.002	< 0.002	0.102
L_m	0.90	< 0.01	< 0.01	0.32

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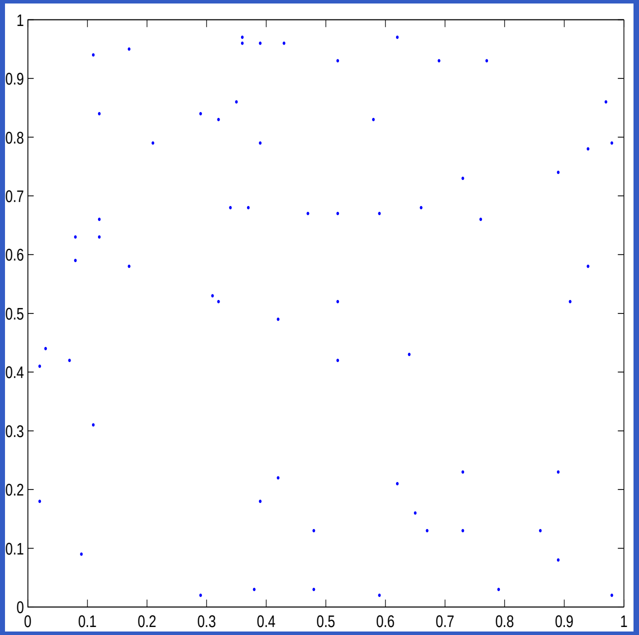

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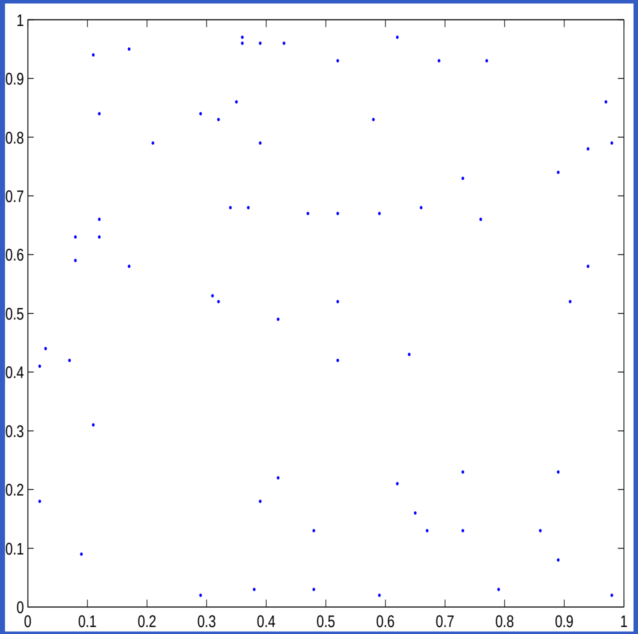

Heterogeneous Poisson process with intensity $\lambda(x, y) = \lambda_1(x)\lambda_2(y)$

where $\lambda_1(x) = \max_{x_1,\cdots,x_m} \exp(-c_x|x - x_i|)$

and $\lambda_2(y) = \max_{y_1,\cdots,y_l} \exp(-c_y|y - y_j|)$.

Application to simulated data sets

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Regular process:

ϵ	Estimated power of the following:				
	L_m	$\bar{\omega}^2$	D_n	V_n	R_n
0.03	0.463	0.052	0.057	0.057	0.057
0.05	1	0.129	0.114	0.085	0.081
0.07	1	0.305	0.263	0.132	0.100

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Cluster process:

μ	ρ	t	Estimated power of the following:				
			Li	$\bar{\omega}^2$	D_n	V_n	R_n
10	10	0.15	0.999	0.856	0.772	0.837	0.714
10	10	0.25	0.778	0.704	0.603	0.481	0.368
20	5	0.3	0.875	0.836	0.750	0.684	0.567

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Heterogenous processus (planar trend):

θ_1	θ_2	Estimated power of the following:				
		Li	$\bar{\omega}^2$	D_n	V_n	R_n
4	4	0.241	0.792	0.736	0.195	0.148
6	6	0.363	0.930	0.910	0.308	0.225
8	4	0.366	0.923	0.896	0.308	0.226

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Grid-heterogeneous process:

m	c	Estimated power of the following:						
		Li	L_m	T	$\bar{\omega}^2$	D_n	V_n	R_n
5	25	0.253	0.065	0.402	0.026	0.042	0.663	0.722
5	30	0.375	0.194	0.589	0.033	0.045	0.886	0.930
7	30	0.044	0.016	0.076	0.036	0.028	0.293	0.458
7	40	0.106	0.047	0.162	0.029	0.036	0.695	0.862

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- Generalization to 3D.