High order Whitney forms on simplices

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Hassler Whitney (1903 NY - 1989 Princeton)

one of the masters of differential geometry

book (1957) `Geometric integration theory`

Georges de Rham (1903 Roche - 1990 Lausanne) [`Théorème de de Rham, 1931`]

André Weil (1906 Paris - 1998 Princeton) [`Sur les théorèmes de de Rham, 1952`]

When the manifold is a domain $\Omega \subset \mathbb{R}^3$, the decomposition is a simplicial mesh
Whitney forms are finite element "basis functions" for the reconstruction of differential forms on Ω from suitable dofs on the mesh

The nature of these dofs , flux, circulations, ..., associate them to geometric objects other than nodes.

FEs of Whitney type : p-forms are reconstructed from their integrals on p-simplices

The key point to understand these FEs : the duality of Whitney's forms

Layout

- The structure of variables in Maxwell's equations

- Compatible and structure preserving discretizations

- Fields as differential forms

- Whitney's duality

Whitney forms of polynomial degree one
Whitney forms of higher polynomial degree
Small simplices
Discrete space generators
New dofs, the weights on the small simplices

- Restoring 1-to-1 dofs/generators' relation

- Some numerical aspects

- Conclusions

The Maxwell puzzle

- Michael Faraday (1791-1867)

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0$$

- André-Marie Ampère (1775-1836)

$$\operatorname{curl} H = J$$

- James Clerk Maxwell (1831-1879) Generalization of Ampère's theorem by adding the displacement current to explain dielectric materials

$$-\frac{\partial D}{\partial t} + \operatorname{curl} H = J$$

- Karl Friedrich Gauss (1777-1855)

$$\operatorname{div} D = q$$

- William Thomson (or Lord Kelvin, 1824-1907)

$$\operatorname{div} B = 0$$

Faraday's law $\quad \operatorname{curl} E = -\frac{\partial B}{\partial t}$

• In integral form, it relates the change rate of the magnetic flux through a surface A to the electric field circulation along a line , the boundary ∂A of the surface A

$$\int_{\partial A} E \cdot t = -\frac{\partial}{\partial t} \int_A B \cdot n$$

James Clerk Maxwell in "*Treatise on Electricity and Magnetism*", 1873

Physical vector quantities may be divided into two classes,
in one of which the quantity is defined with reference to a line ,
while in the other the quantity is defined with reference to an area .

• We distinguish between field intensities (E,H) and flux densities (D,B,J)

• A field intensity occurs in a path integral

• A flux density occurs in a surface integral

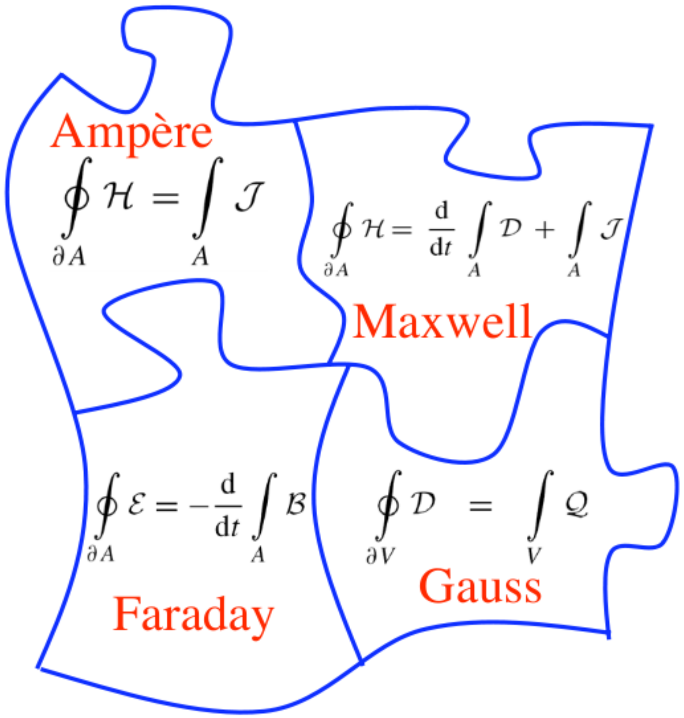
• Field intensities and flux densities do not depend on the metric

• Field intensities are related to the flux densities via the constitutive relations which depend on the metric properties of the space, on the chosen coordinate system, and on the macroscopic material properties.

• The constitutive relations are necessary to close the Maxwell equation system

The constitutive relations

$$D = \epsilon E$$
$$B = \mu H$$
$$J = \sigma E \quad \text{(Ohm's law)}$$



Tonti's diagram (1974)

to find analogies in different physical domains and classify variables

Each pillar symbolizes the structure

made by fields or densities,

linked by grad, curl, div operators

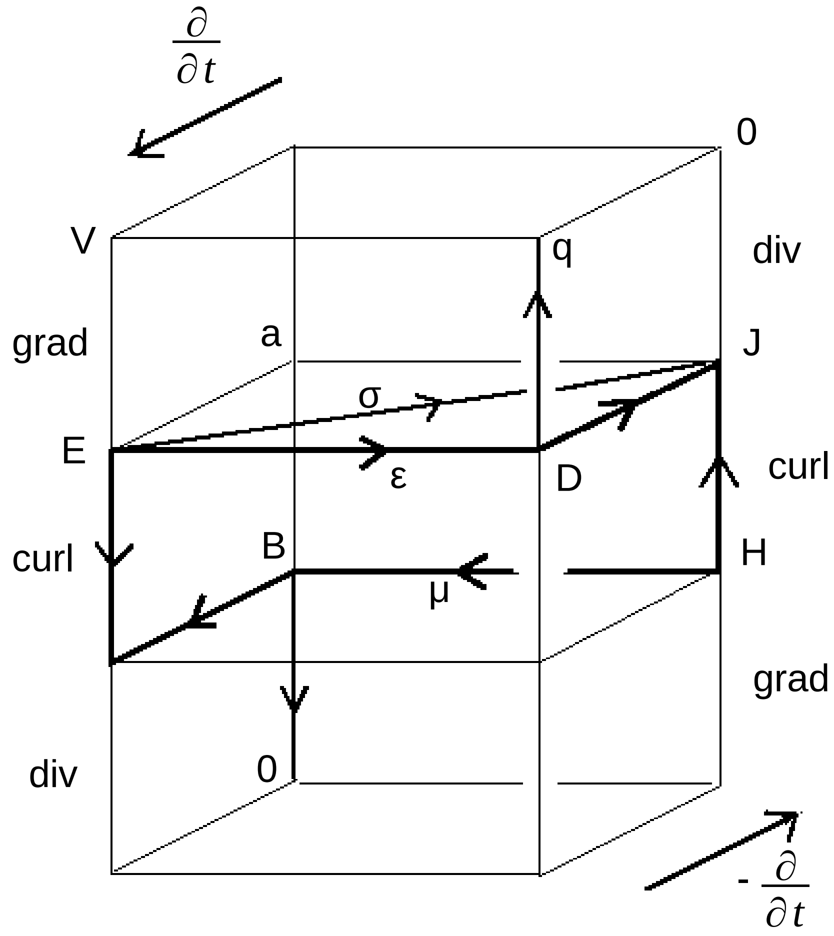
Differentiation or integration w.r.t. time

links pair of pillars (front/rear) forming the sides of the structure

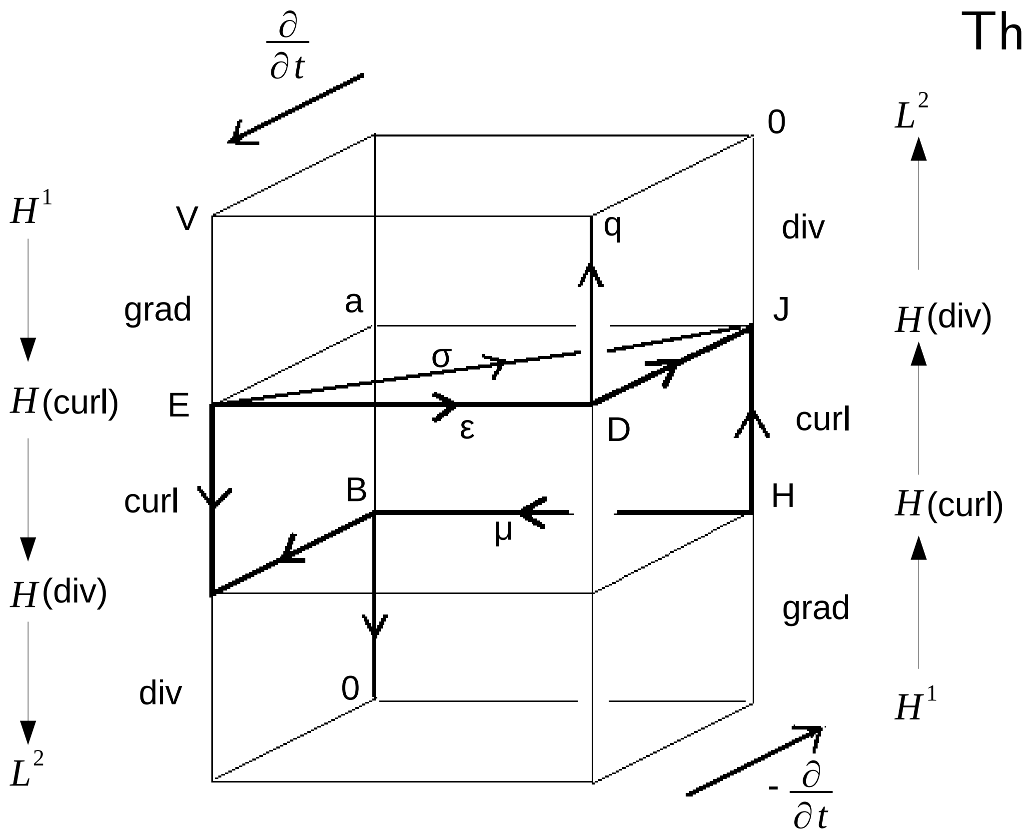
Constitutive laws are horizontal beams

Vertical relations are affine

Horizontal laws are metric dependent



To preserve structure, metrics must not appear when discretizing along the pillars



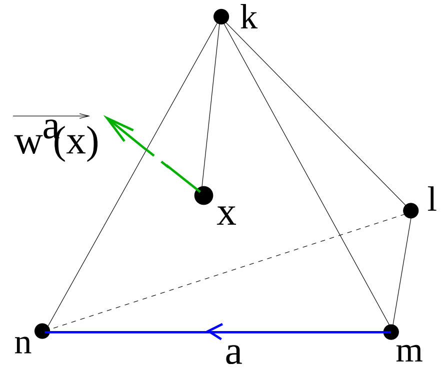
Complex : sequence of spaces and linear maps d_p such that $d_{p+1}d_p = 0$

The grad-curl-div sequence is an example of de Rham's cohomology (1931) to study topological invariants of smooth manifolds in a form adapted to computation.

To define a correct discretization of pbs, discrete spaces have to form a complex

Edge elements

$$\overrightarrow{w^a}(x) = \overrightarrow{\alpha} \times \overrightarrow{ox} + \overrightarrow{\beta}$$



- Vector fields (`Nédélec 1980, Bossavit & Verité 1982`)
- Tangential component continuous across element interfaces
- Circulation $\int_{a'} \overrightarrow{w^a}(x) \cdot \overrightarrow{t_{a'}} = 1$ if $a' = a$ and 0 otherwise
- The space $Ned_1 = \mathrm{span}\{\overrightarrow{w^a},\ a \in \mathcal{E}\}$ is the discrete realization of $H(\mathrm{curl})$.

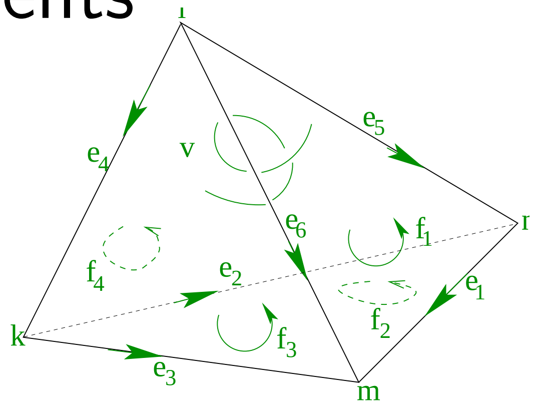
The electric field $\overrightarrow{E}$ at x is approximated by

$$\overrightarrow{E}(x) \approx \sum_{a \in \mathcal{E}} E_a \overrightarrow{w^a}(x), \qquad \text{where} \quad E_a = \int_a \overrightarrow{E} \cdot \overrightarrow{t_a} \qquad \text{(f.e.m)}$$

The triplet $(t,\ Ned_1(t),\ \{\text{circulations along 6 edges of } t\})$ is a finite element

Face elements

$$\overrightarrow{w^f}(x) = \overrightarrow{\alpha} + \beta\ \overrightarrow{ox}$$

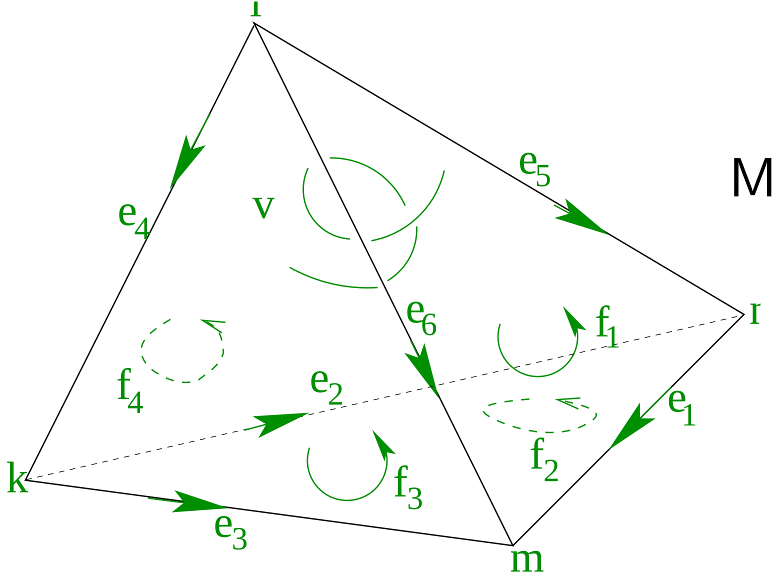


- Vector fields (Raviart & Thomas 1975, Nédélec 1980, B. & V. 1982)
- Normal component continuous across element interfaces
- Fluxes $\int_{f'} \overrightarrow{w^f}(x) \cdot \overrightarrow{n_{f'}} = 1$ if $f' = f$ and 0 otherwise
- The space $RT_1 = \text{span}\{\overrightarrow{w^f},\ f \in \mathcal{F}\}$ is the discrete realization of $H(\text{div})$.

The magnetic induction $\overrightarrow{B}$ at x is approximated by

$$\overrightarrow{B}(x) \approx \sum_{f \in \mathcal{F}} B_f\ \overrightarrow{w^f}(x), \qquad \text{where} \qquad B_f = \int_f \overrightarrow{B} \cdot \overrightarrow{n_f} \qquad (\text{flux})$$

The triplet $(t,\ RT_1(t),\ \{\text{fluxes across 4 faces of } t\})$ is a finite element



Mesh is defined through nodes' position
and element connectivity

An orientation of a p-simplex
is given by an ordering of the vertices

$$e_3 = [k, m],\ f_3 = [k, m, l],\ \ldots$$

Incidence matrices $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$ are discrete equivalent of the grad, curl, div op.

Ex. Enforcing Faraday's law in Ω means $\quad \partial_t\{B_f\} + \mathbf{R}\{E_e\} = \{0_f\}$

$$\mathbf{G} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & -\mathbf{1} \\ 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix}$$

$$\mathbf{D} = (1 \ -1 \ 1 \ -1)$$

$$\mathbf{R}\,\mathbf{G} = \mathbf{0}, \qquad \mathbf{D}\,\mathbf{R} = \mathbf{0}$$

$$(\text{curl grad} = 0, \qquad \text{div curl} = 0)$$

arrays of dofs | discrete FE spaces

$$
\begin{tikzcd}[column sep=huge, row sep=large]
IR^N \arrow[d, "G"'] \arrow[r, shift right, "p_0"'] \arrow[dr, phantom, "\circlearrowright"] & P_1 \subset H^1 \arrow[l, shift right, "r_0"'] \arrow[d, "\text{grad}"] \\
IR^E \arrow[d, "R"'] \arrow[r, shift right, "p_1"'] \arrow[dr, phantom, "\circlearrowright"] & Ned_1 \subset H(\text{curl}) \arrow[l, shift right, "r_1"'] \arrow[d, "\text{curl}"] \\
IR^F \arrow[d, "D"'] \arrow[r, shift right, "p_2"'] \arrow[dr, phantom, "\circlearrowright"] & RT_1 \subset H(\text{div}) \arrow[l, shift right, "r_2"'] \arrow[d, "\text{div}"] \\
IR^T \arrow[r, shift right, "p_3"'] & P_0 \subset L^2 \arrow[l, shift right, "r_3"']
\end{tikzcd}
$$

simplicial mesh
N nodes, E edges, F faces, T tetras

r_q restriction (dof computation) $\quad r_q p_q = identity$

p_q prolongation (FE reconstruction) $\quad p_q r_q \rightarrow identity\,(for\; h \rightarrow 0)$

$\circlearrowright$ commutativity

$$p_1(G\,u) = \text{grad}\,(p_0 u)$$
$$p_2(R\,u) = \text{curl}\,(p_1 u)$$
$$p_3(D\,u) = \text{div}(p_2 u)$$

G ExN incidence edge-to-node matrix

R FxE incidence face-to-edge matrix

D TxF incidence tetra-to-face matrix

IR^k arrays of k reals

The analytic expression of these basis functions remained a mistery for years

$$\overrightarrow{w^a}(x) = \overrightarrow{\alpha} \times \overrightarrow{ox} + \overrightarrow{\beta} \qquad \text{and} \qquad \overrightarrow{w^f}(x) = \overrightarrow{\alpha} + \beta \, \overrightarrow{ox}$$

Robert Kotiuga suggests the
connection with Whitney forms on a simplicial decomposition of the manifold

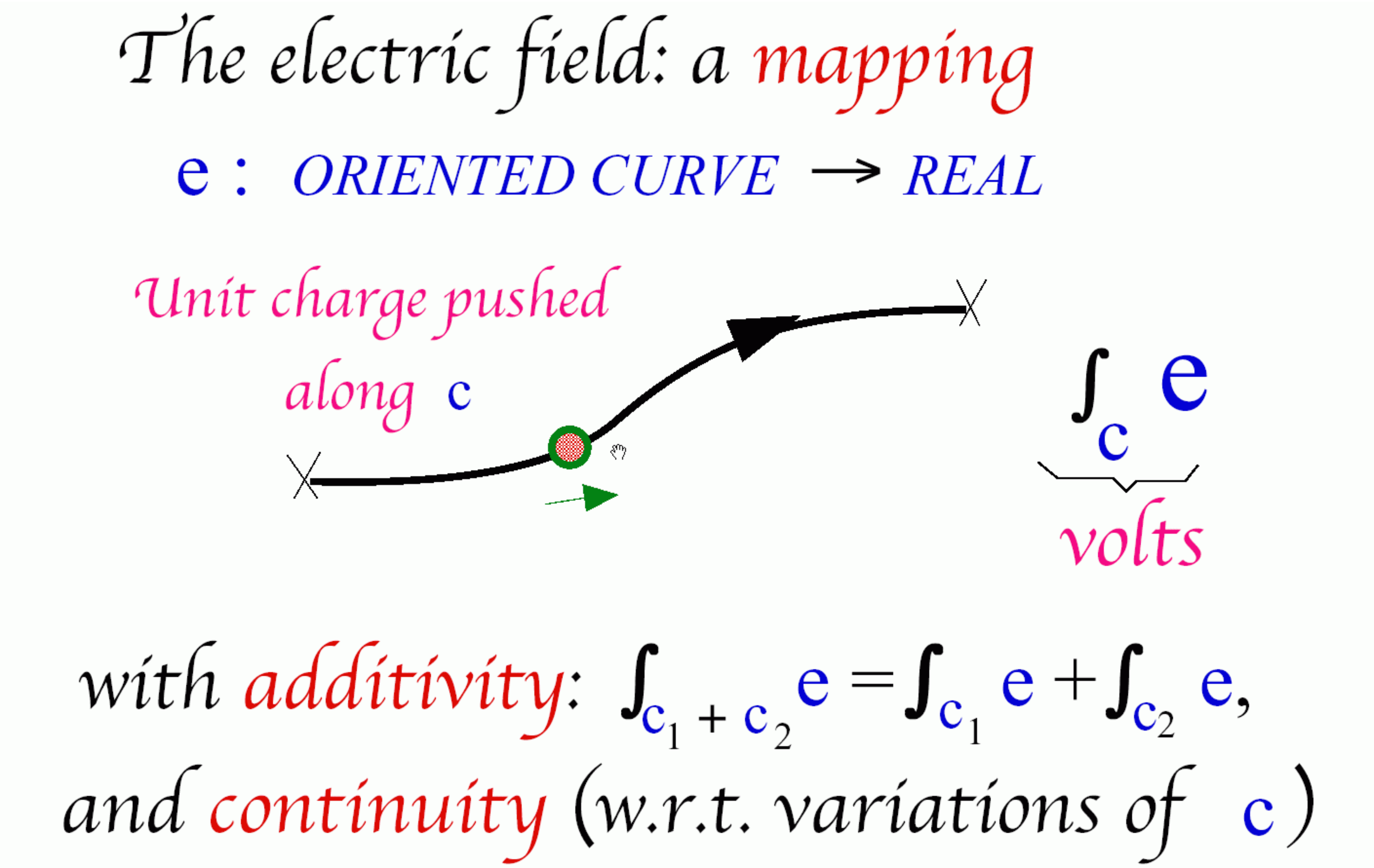
`Dechamps, EM and differential forms, IEEE proc. 1981`
`Kotiuga, Hodge decomposition and computational EM, 1984`

Alain Bossavit defines the way
to map the system of Maxwell's equations to a finite-dimensional one
and thus the entire discretization toolkit in EM

`Bossavit, Electromagnétisme en vue de la modélisation, 1986`

We need to adopt a geometrical point of view

EM fields are *invisible* perturbations of the surroundings

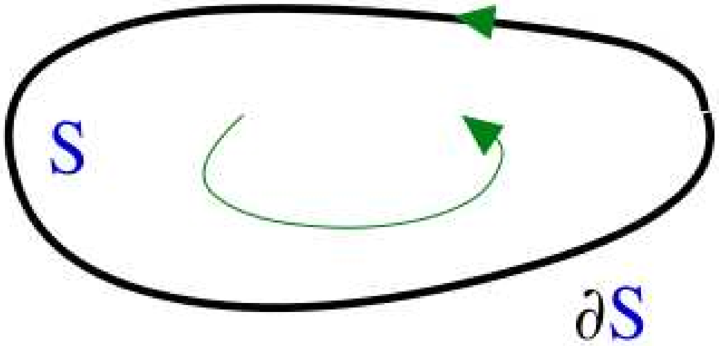


[Bossavit's slide]

Testing their presence *suggests* adopting p-forms

Magnetic induction: a mapping

b : *ORIENTED SURFACE* → *REAL*



$$\underbrace{\int_S b}_{\text{webers}}$$

2-form 1-form

$$\boxed{\frac{d}{dt}\int_S b + \int_{\partial S} e = 0}$$

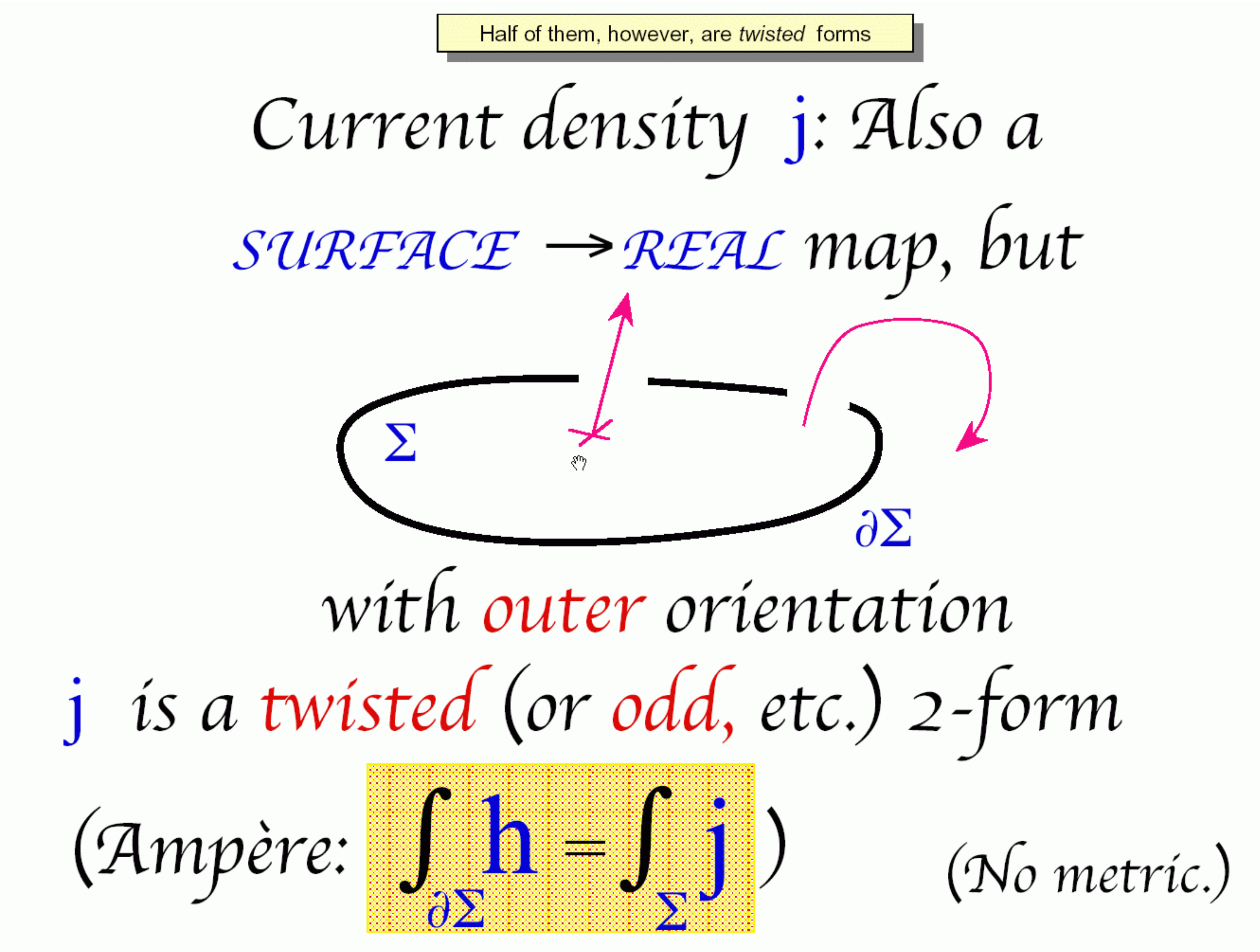
(Faraday.)

Metric-free expression.

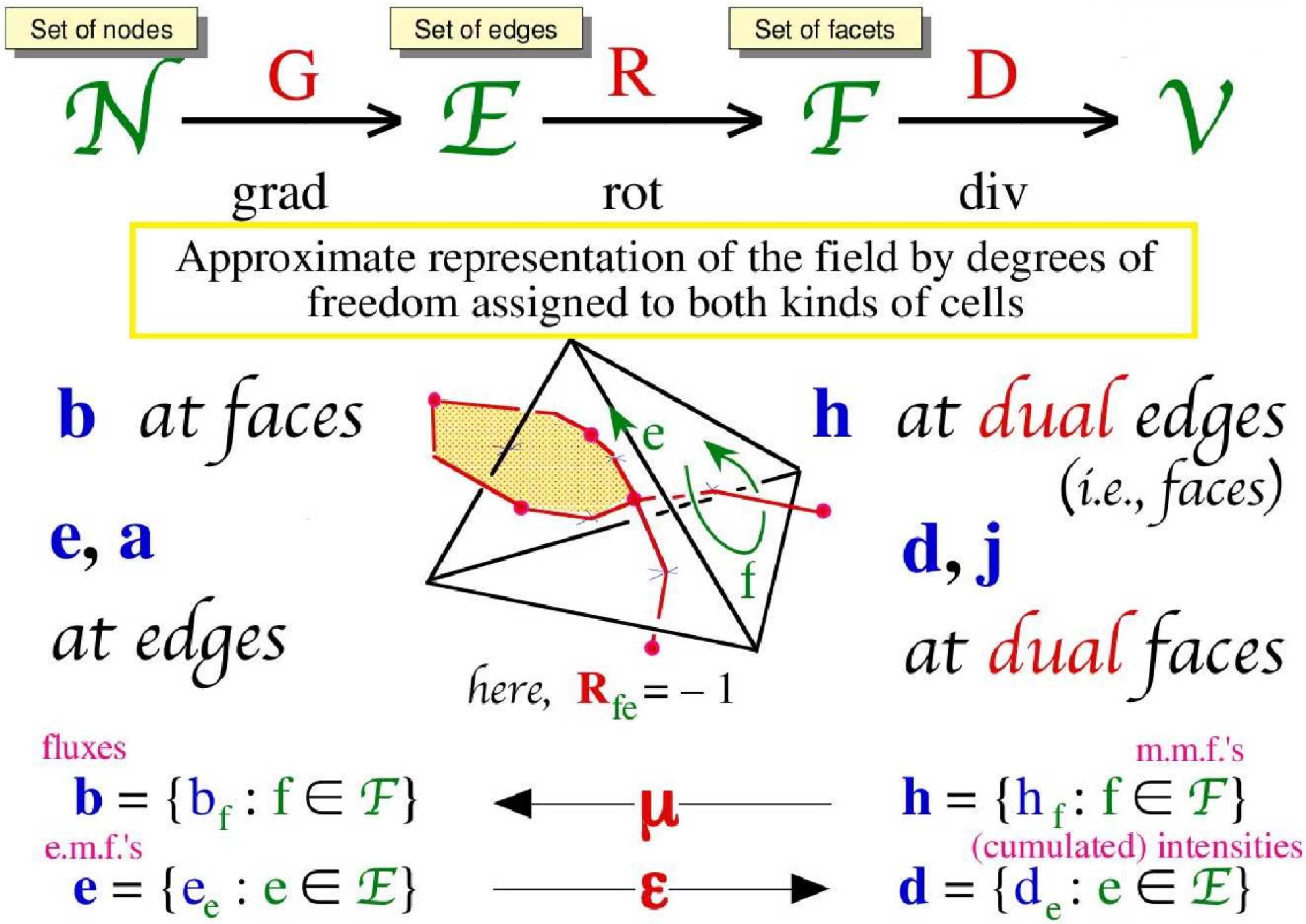
or $\partial_t b + de = 0$, if d defined by $\int_S de = \int_{\partial S} e$

[Bossavit's slide]

Some p-forms are twisted [Bossavit's slide]



Constitutive relations *become* Hodge operators

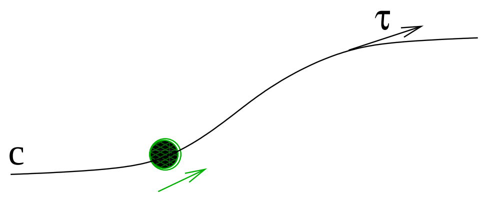


$\mathbf{D}\mathbf{b} = \mathbf{0}, \quad \mathbf{b} = \mu\mathbf{h}, \quad \mathbf{R}^{\top}\mathbf{h} = \mathbf{j}$

[Bossavit's slide]

A field is a map : "probe" $\rightarrow$ "measure"

e $\qquad$ c $\qquad$ $\int_c e$



AT THE DISCRET LEVEL

an oriented curve is described by a 1-chain $\qquad c \sim \mathcal{P}^\top c = \sum_{s\in\mathcal{E}} c_s\, s$

(formal sum of mesh edges with real coefficients)

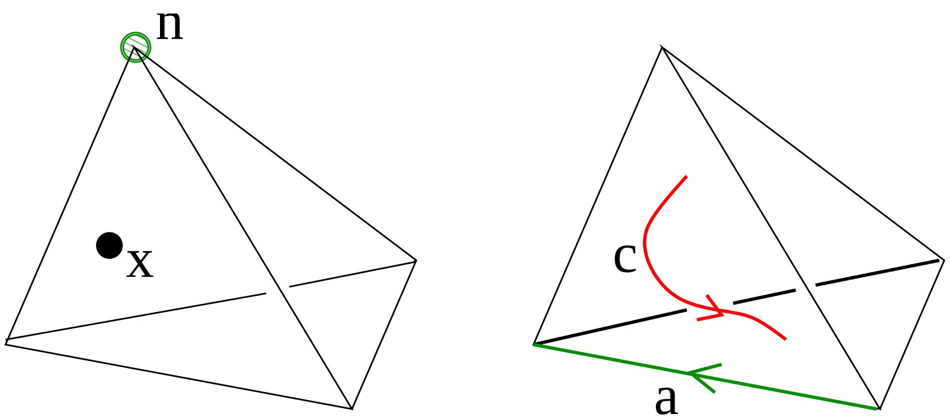
$$\int_c e \sim \quad \int_{\mathcal{P}^\top c} e = \sum_{s\in\mathcal{E}} c_s \int_s e = \sum_{s\in\mathcal{E}} c_s\, e_s = \mathbf{c}\cdot\mathbf{e}$$

If we have a map $w^s : c \rightarrow c^s = \int_c w^s$ such that $\int_{s'} w^s = \delta_{s',s}$, then

$$\mathbf{c}\cdot\mathbf{e} = \sum_{s\in\mathcal{E}} c_s\, e_s = \sum_{s\in\mathcal{E}} \int_c w^s\, e_s = \int_c (\sum_{s\in\mathcal{E}} w^s\, e_s) = \int_c \mathcal{P}e \quad \sim \int_c e$$

How could one represent a p-manifold by a p-chain ? c_s ?

Idea : express points, lines, surfaces, ..., as weighted sum of mesh nodes, edges, faces, ...



For nodes, $x = \mathcal{P}^t x = \sum_{n \in \mathcal{N}} w^n(x)\, n = \sum_{n \in \mathcal{N}} \lambda_n(x)\, n$,

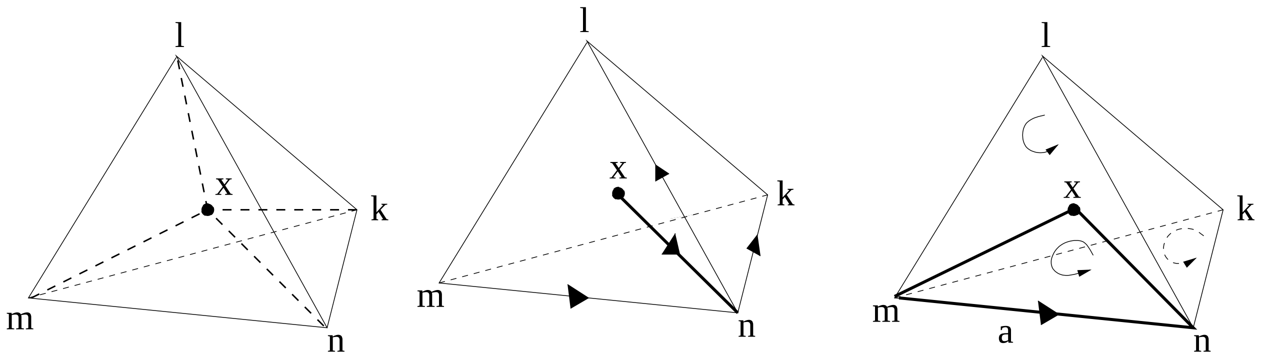
and thus $\varphi \approx \mathcal{P}\varphi = \sum_{n \in \mathcal{N}} \varphi_n\, w^n$ for $w^n = \lambda_n$

$w^n = \lambda_n$	Whitney 0-form associated to n,
$\langle w^n, x\rangle = \lambda_n(x)$	weight of n in the 0-chain $\mathcal{P}^t x$

For a segment xy oriented from x to y

$$\mathcal{P}^t y = \sum_{n \in \mathcal{N}} \langle w^n, y \rangle \, n \qquad \Rightarrow \qquad \mathcal{P}^t xy = \sum_{n \in \mathcal{N}} \langle w^n, y \rangle \, \mathcal{P}^t xn$$

There is a unique way to express xn as $\mathcal{P}^t xn$, weighted sum of sides



$$\mathcal{P}^t xn = \lambda_m(x)[m,n] - \lambda_l(x)[n,l] - \lambda_k(x)[n,k] = \sum_{a \in \mathcal{A}} \mathbf{G}_{an} \, \lambda_{a-n}(x) a$$

... and for an oriented face ....

$$\mathcal{P}^t (x \vee a) = \lambda_l(x)[l,m,n] + \lambda_k(x)[k,m,n] = \sum_{f \in \mathcal{F}} \mathbf{R}_{fa} \, \lambda_{f-a}(x) f$$

$$\mathcal{P}^t xy = \sum_{n \in \mathcal{N}, a \in \mathcal{A}} G_{an} \lambda_{a-n}(x) \langle w^n, y \rangle a \equiv \sum_{a \in \mathcal{A}} \langle w^a, xy \rangle a$$

Do some steps and use $\langle e, \mathcal{P}^t xy \rangle = \langle \mathcal{P} e, xy \rangle$

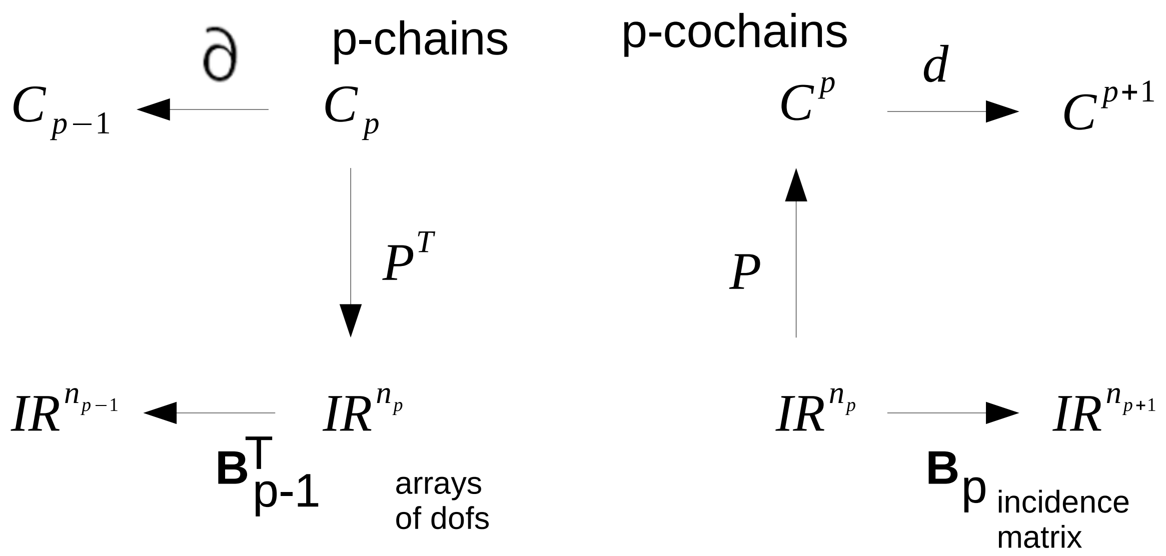
$w^a = \sum_{n \in \mathcal{N}} G_{an} \lambda_{a-n} \mathrm{d}w^n$ Whitney 1-form

$\langle w^a, xy \rangle$ weight of a in $\mathcal{P}^t xy$

Duality of Whitney's forms w^a

they represent a differential form e from a vector of dofs e_a (this is $\mathcal{P}$)

they represent a p-manifold c by a p-chain (this is $\mathcal{P}^t$)



Recursive definition [Bossavit IEEE Trans. Magn. 2002]

$$w^s(x) = \sum_{\sigma \in \{(p-1)\text{-simplices}\}} \overline{\partial}_{s\sigma}\, \lambda_{s-\sigma}(x)\, \mathrm{d}w^\sigma$$

$\overline{\partial}_{s\sigma}$ entry of the incidence matrix which links the $(p-1)$-simplex σ to the p-simplex s

d is the exterior derivative op. $W^{p-1} \to W^p$ dual of the boundary op. ∂ in the sense of Stokes' theorem (1850) in Cartan's form (1945): $\int_{\partial c} w = \int_c \mathrm{d}w,\ \forall c \in C_p$ et $\forall w \in W^{p-1}$

$$W_1^p(t) = \mathrm{span}\{w^s,\ s \in \{p\text{-simplices of}\, t\}\}$$

$w^n = \lambda_n,$

$w^a = \sum_{n\in\mathcal{N}} \mathbf{G}_{an}\, \lambda_{a-n} \mathrm{d}\, w^n \implies \mathbf{w}^a = \lambda_\ell \nabla \lambda_m - \lambda_m \nabla \lambda_\ell = \overrightarrow{\alpha} \times \overrightarrow{ox} + \overrightarrow{\beta}$

$w^f = \sum_{a\in\mathcal{A}} \mathbf{R}_{fa}\, \lambda_{f-a} \mathrm{d}\, w^a \implies \mathbf{w}^f = 2\, \lambda_\ell \nabla \lambda_m \times \nabla \lambda_k +$ (two terms) $= \overrightarrow{\alpha} + \beta\, \overrightarrow{ox}$

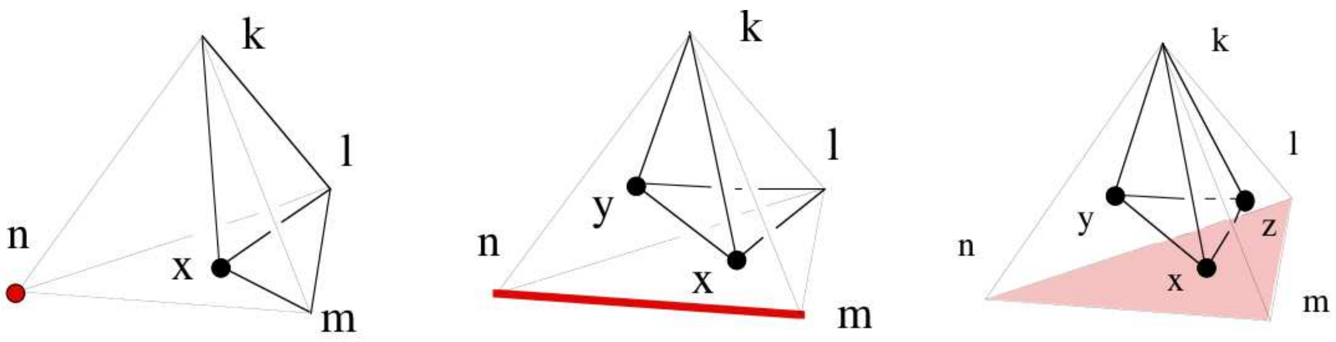
$w^t = \sum_{f\in\mathcal{F}} \mathbf{D}_{tf}\, \lambda_{t-f} \mathrm{d}\, w^f \implies w^t = 6\, \lambda_\ell \nabla \lambda_m \times \nabla \lambda_k \cdot \nabla \lambda_n +$ (three terms) $= \frac{1}{|t|}$

$T = [n, m, k, l]$ a tetrahedron, $s \subset T$ a p-simplex, S a p-face of T

$\langle w^S, s\rangle = \int_s w^S$ is the weight of s w.r.t. S,

it doesn't depend on the shape of s, S,

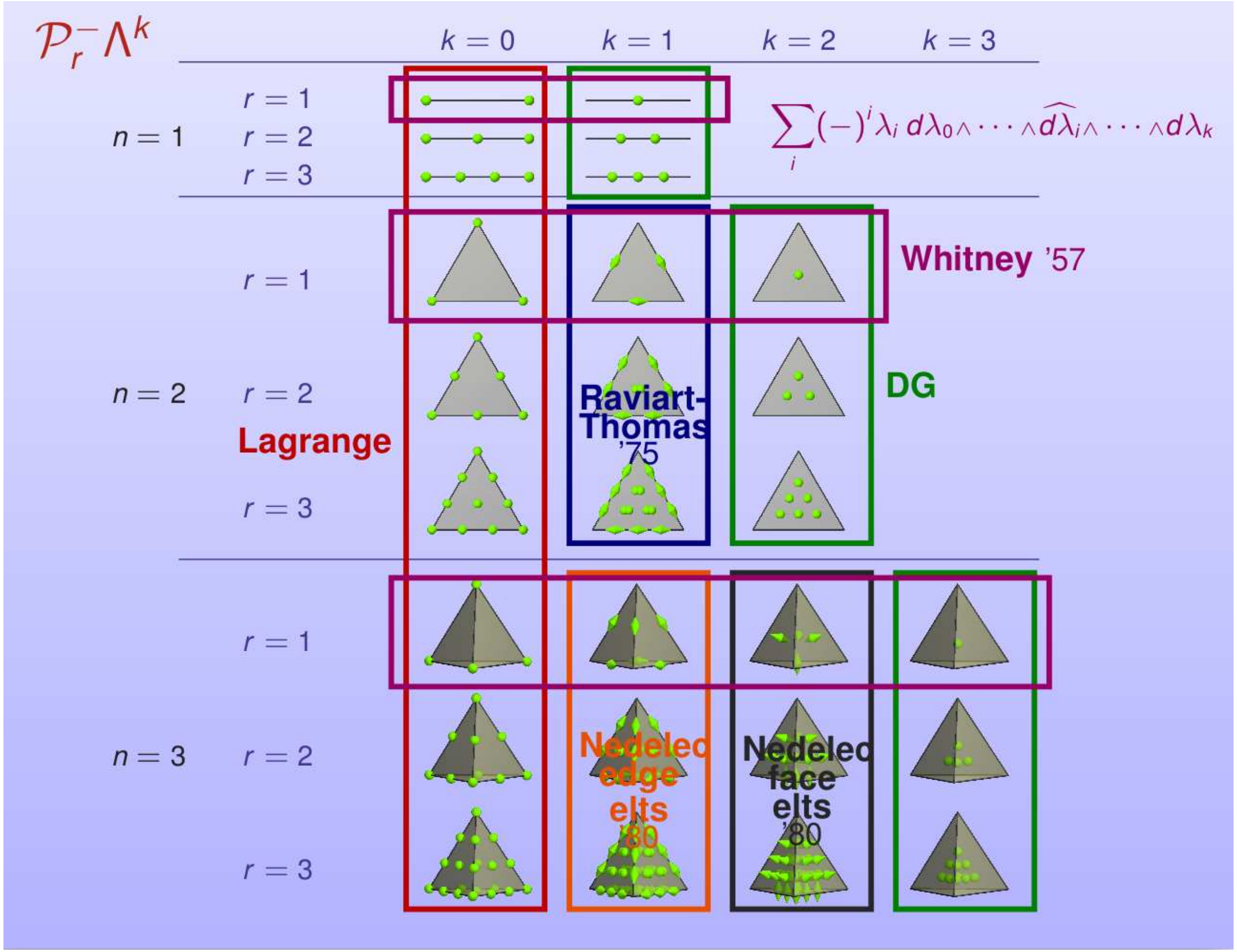
but on their relative position and orientation



$$\int_s w^S = \pm \operatorname{vol}[(T \setminus S) \cup s]/\operatorname{vol}(T)$$

Weights correspond to volumes, computable determinants (recursivity again !!)

[R., CRAS 2004]



D.N.Arnold, Periodic table of FEs

HIGH ORDER FEs ON SIMPLICES

`Ainsworth, Arnold, Christiansen, Coyle, Demkowicz, Brezzi, Falk, Gerritsma, Gopalakrishnan, Graglia, Hiptmair, Marini, Nédélec, Nilssen, Schöberl, Teixeira, Webb, Winther, Zaglmayr ... and many many others`

There are different bases for the spaces below

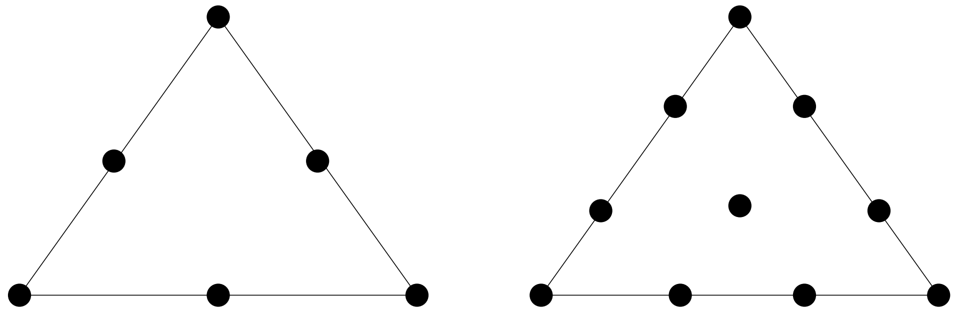
$$
\begin{aligned}
W^0_{r+1}(t) &= \mathbb{P}_{r+1}(t) \\
W^1_{r+1}(t) &= (\mathbb{P}_r(t))^3 \oplus \{\mathbf{q} \in (\tilde{\mathbb{P}}_{r+1}(t))^3 : \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0, \quad \mathbf{x} \in t\} \\
W^2_{r+1}(t) &= (\mathbb{P}_r(t))^3 \oplus \{\mathbf{q} \in (\tilde{\mathbb{P}}_{r+1}(t))^3 : \mathbf{q}(\mathbf{x}) \times \mathbf{x} = \mathbf{0}, \quad \mathbf{x} \in t\} \\
W^3_{r+1}(t) &= \mathbb{P}_r(t)
\end{aligned}
$$

`Nédélec's first family, Numer.Math. 1980`

High order finite element spaces of differential forms seem to lack natural choices of bases. But they do have natural spanning families.

We follow Whitney's approach to have explicit expression of spanning functions and weights on p-chains

The triplet $(t,\ \mathbb{P}_r(t),\ \{$values at points in $T_r(t)\})$ is a finite element



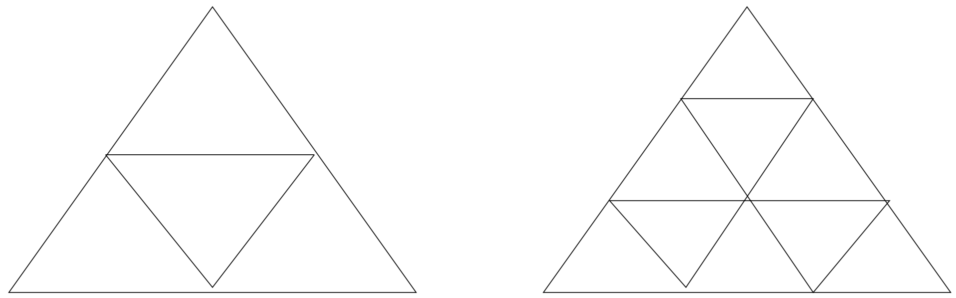
$$\mathcal{I}(d+1,r) = \{\mathbf{k} = (k_0,\ldots,k_d) \in \mathbb{N}^{d+1}\,|\mathbf{k}| = r\,\},$$

$$T_r(t) = \{(\textstyle\sum_{j=0}^{d} k_j \mathbf{x}_j)/k,\ \mathbf{k} \in \mathcal{I}(d+1,r)\}$$ ($\mathbf{x}_j$ vertex of t)

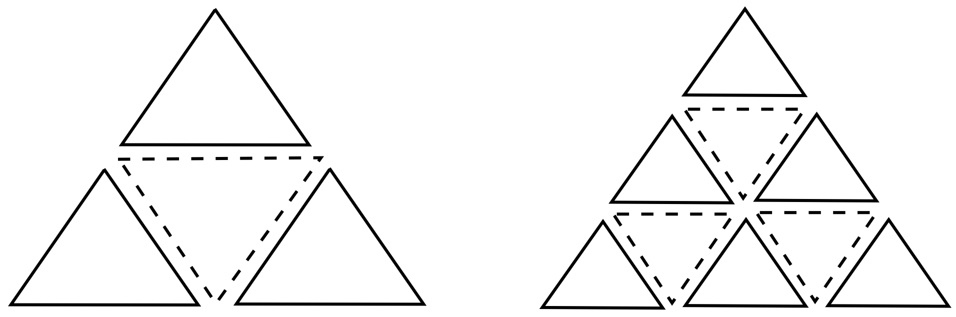
$$\mathbb{P}_r(t) = \mathrm{span}\{\lambda^{\mathbf{k}},\ \mathbf{k} \in \mathcal{I}(d+1,r)\}$$ with $$\lambda^{\mathbf{k}} = \Pi_{j=0}^{d}(\lambda_j)^{k_j}$$

$\{\lambda^{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{I}(d+1,r)}$ is a spanning family in $\mathbb{P}_r(t)$

Starting from the principal lattice $T_r(t)$ used for nodal FEs ...



... we define a set of p-subsimplices in t , *the small p-simplices* [`solid line`] ...

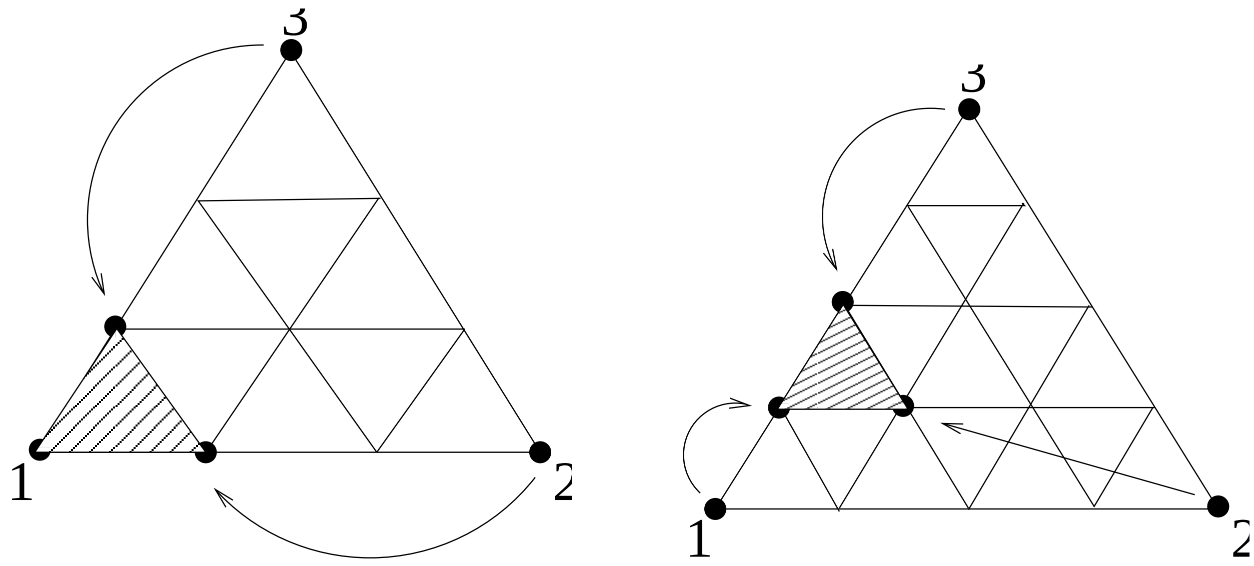


... and we get rid of the "holes" [`dashed line`]

To each $\mathbf{k} \in \mathcal{I}(d+1,r)$ corresponds a homothety $\tilde{\mathbf{k}}$ where $\tilde{k}_i(\lambda_i(x)) = \frac{\lambda_i(x)+k_i}{r+1}$

The small p-simplices $\{\mathbf{k}, S\}$ are the images $\tilde{\mathbf{k}}(S)$ of (big) p-simplices S

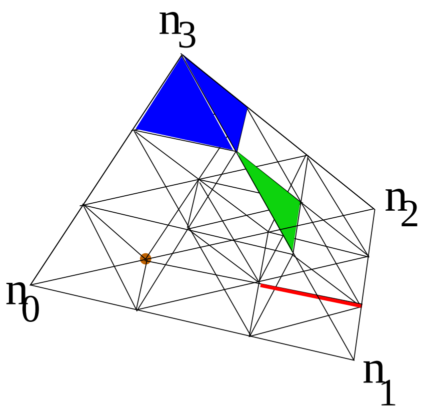
$\{\mathbf{k}, S\}$ is oriented as S

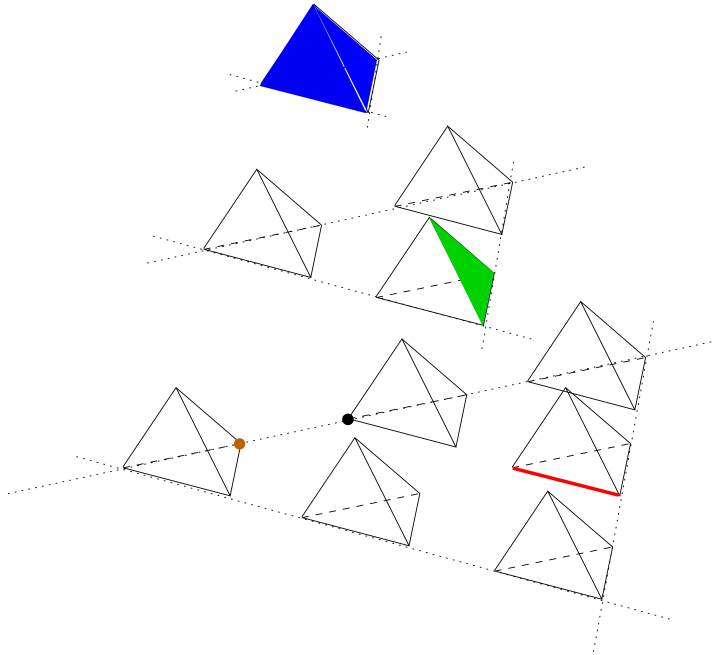


$$(d=2) \qquad G_\triangle = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \qquad \to \qquad \tilde{\mathbf{k}}(G_\triangle) = (\frac{\frac{1}{3}+k_1}{r+1}, \frac{\frac{1}{3}+k_2}{r+1}, \frac{\frac{1}{3}+k_3}{r+1})$$

$\{(2,0,0),t\}$ (left) and $\{(2,0,1),t\}$ (right)

Whitney p-forms of polynomial degree $r+1$ on t [R. and Bossavit, SINUM09]





Whitney p-forms of polynomial degree $r+1$ on t read $\lambda^{\mathbf{k}} w^S$
for all the small p-simplices $\{\mathbf{k}, S\}$
w^S is the Whitney p-form of polynomial degree 1 associated with S

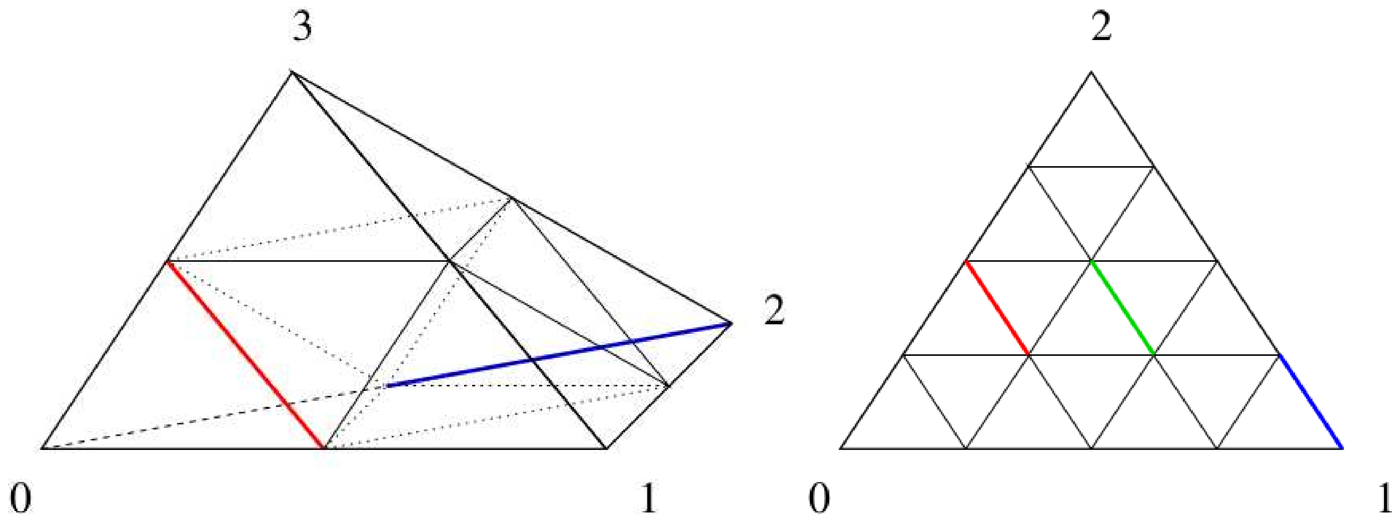
$$W^p_{r+1}(t) = \operatorname{span}\{\lambda^{\mathbf{k}} w^S\} \qquad \text{(explicit formula !!!)}$$

Examples of small edges, small faces, discrete space generators

Degree 2 in 3D:

Small edges $\{(1,0,0,0),[\mathbf{v}_1,\mathbf{v}_3]\}$, $\{(0,0,1,0),[\mathbf{v}_0,\mathbf{v}_2]\}$. This gives $\lambda_0\,\mathbf{w}^{13}$ and $\lambda_2\,\mathbf{w}^{02}$.

Small faces sharing the red line $\{(1,0,0,0),[\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_3]\}$, $\{(1,0,0,0),[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]\}$
This gives $\lambda_0\,\mathbf{w}^{013}$ and $\lambda_0\,\mathbf{w}^{123}$.



Degree 3 in 2D:

Small edges $\{(2,0,1),[\mathbf{v}_1,\mathbf{v}_2]\}$, $\{(1,1,1),[\mathbf{v}_1,\mathbf{v}_2]\}$, $\{(0,3,0),[\mathbf{v}_1,\mathbf{v}_2]\}$.
This gives $\lambda_0^2\,\lambda_2\,\mathbf{w}^{12}$, $\lambda_0\,\lambda_1\,\lambda_2\,\mathbf{w}^{12}$ and $\lambda_1^3\,\mathbf{w}^{12}$.
Small face close to the red line $\{(2,0,1),[\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2]\}$ that gives $\lambda_0^2\,\lambda_2\,\mathbf{w}^{012}$

New dofs for high-order Whitney p-forms are the small p-weights , that are integrals on the small p-simplices $\{\mathbf{k}, S\} \subset t$ [R. and Bossavit, SINUM 2009]

$$\sigma_\ell = \sigma_{\{\mathbf{k},S\}} : u \to \int_{\{\mathbf{k},S\}} u$$

Small weights are unisolvent in $W^p_{r+1}(t)$ [Christiansen, R., Math.Comp.2016]

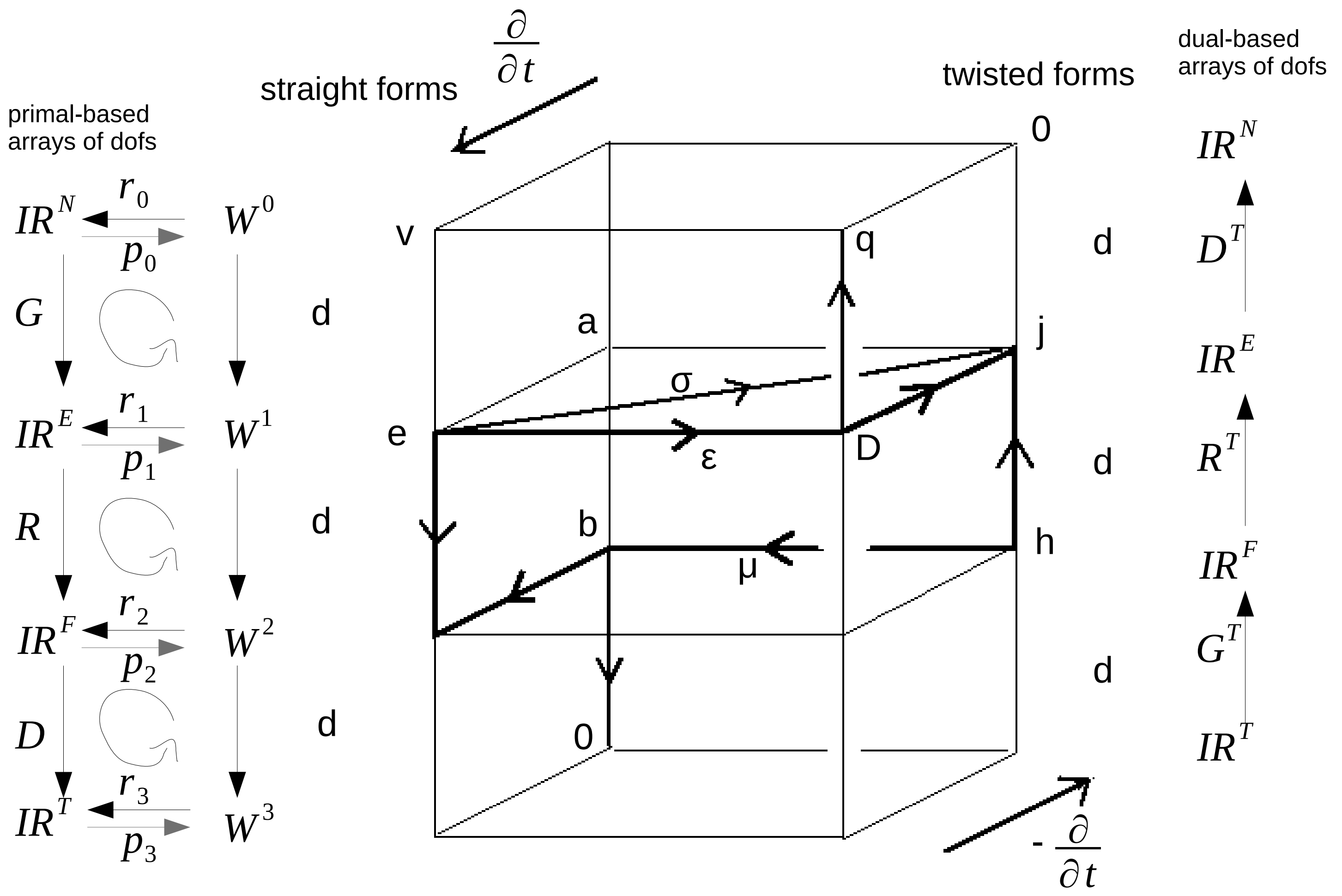
The triplet $(t,\ W^p_{r+1}(t),\ \{\text{weights on small } p\text{-simplices}\})$ is a finite element

Rk 1) Some $\lambda^{\mathbf{k}} w^S$ are redundant if $\mathbf{k}$ varies in $\mathcal{I}(d+1, r)$ to keep symmetry (*i.e.*, the set $\{\lambda^{\mathbf{k}} w^S\}$ does not change if we permutate the order of the barycentric coord.)

[Christiansen, R., Math.Comp.2016], [Alonso Rodriguez, R., CAMWA 2019]

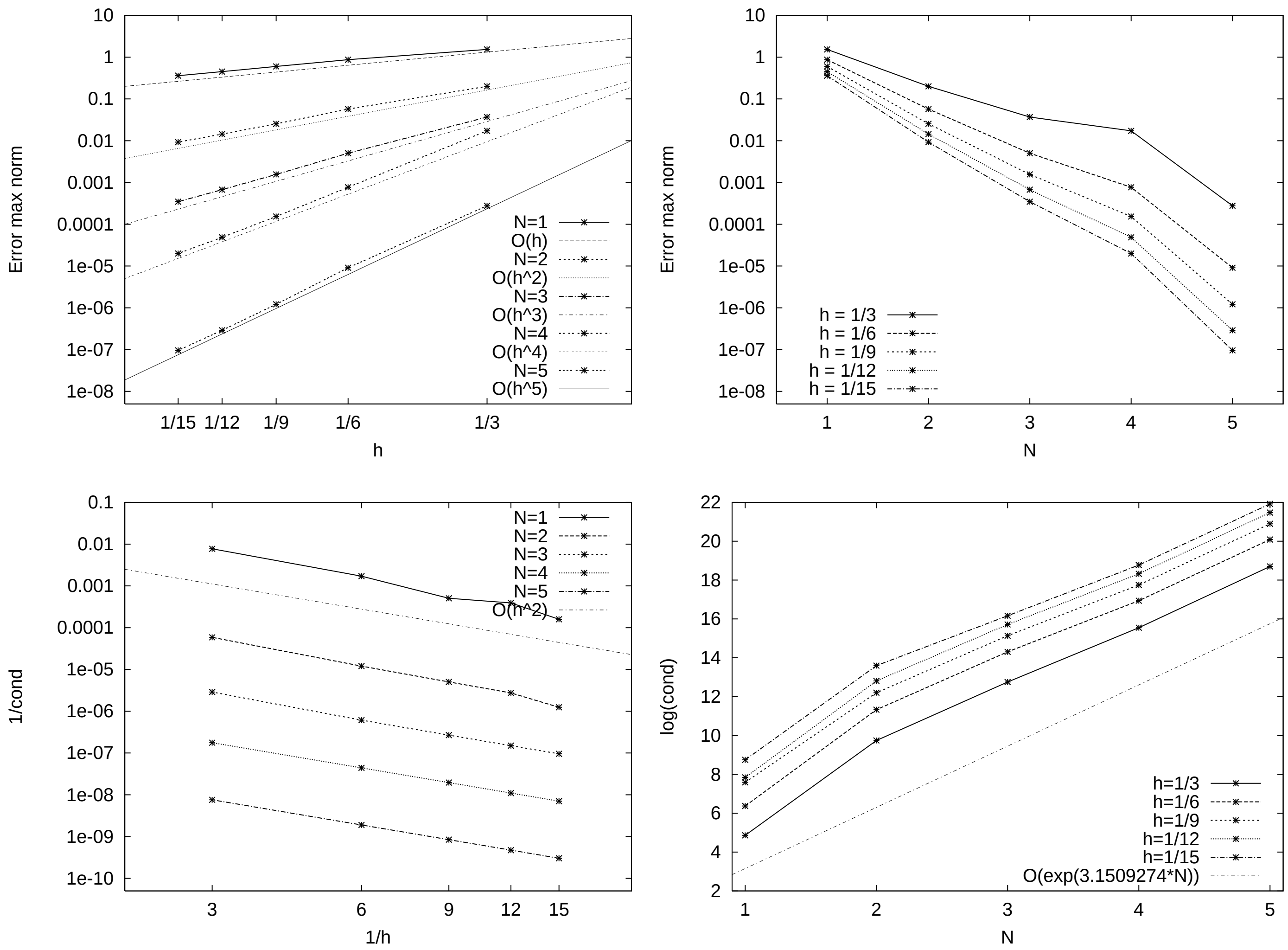
Rk 2) The association between functions and weights is not as obvious as in the case of lower degree Whitney forms [Bonazzoli, R., Numer. Algor. 2017]

- explicit formula and new dofs (weights on small p-chains)
- they verify global continuity for $p = 0$ or partial continuity for $p = 1, 2$ properties
- the sequence $\{0\} \longrightarrow W^0_{r+1} \xrightarrow{grad} W^1_{r+1} \xrightarrow{curl} W^2_{r+1} \xrightarrow{div} W^3_{r+1} \longrightarrow \{0\}$ is exact in t



hr Convergence and conditioning

$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}+\mathbf{u}=\mathbf{f}$ on $\Omega=[0.5,1.5]\times[0.25,0.75]$ with $\mathbf{u}=(2\pi\sin(\pi\,x)\cos(2\pi\,y),-\pi\cos(\pi\,x)\sin(2\pi y))^t$



R., M2AN 2007

Small weights verify the following properties

- ✓ invariance w.r.t. the selection of the reference simplex
- ✓ locality : local dofs provide sufficient “cement” between adjacent elements to enforce conformity in the FE space
- ✓ unisolvence : dofs’ values determine a unique function in the FE space
- ✓ geom. interpretation : Small weights are volumes of suitable simplices too !

[Christiansen, R., Math.Comp.2016]

1-to-1 relation between generators w_j and dofs σ_k in matrix terms would read $V = I$ where $(V)_{\ell j} := \sigma_\ell(w_j)$ (generalised Vandermonde weight matrix)

$$V_{\{\mathbf{k}',S'\},\{\mathbf{k},S\}} = \int_{\{\mathbf{k}',S'\}} \lambda^{\mathbf{k}}\, w^S \neq \mathrm{Id} \quad \text{si} \quad k = k' = r > 0 \;\; (S, S'\, p\text{-simplices})$$

$$\begin{aligned} \int_{\{\mathbf{k}',S'\}} \lambda^{\mathbf{k}}\, w^S &= \int_{\{\mathbf{k}',S'\}} w^S \; \int_{\{\mathbf{k}',S'\}} \lambda^{\mathbf{k}} \,/\, \mathrm{vol}\{\mathbf{k}', S'\} \\ &= (\mathrm{volume})\,(\text{magic formula})\,/\,(\mathrm{volume}) \end{aligned}$$

Volume of a p-simplex from its sides is given by Cayley-Menger determinant

Input data

$\{\psi_j\}$, a spanning family of the FE space

$\{\sigma_i\}$, a set of dofs which are "reasonable" and thus the matrix $\quad (V)_{i,j} = \sigma_i(\psi_j)$

Output data

$\{\phi_i\}$, THE CARDINAL basis for THE selected dofs: $\quad \sigma_k(\phi_i) = \delta_{ki}$

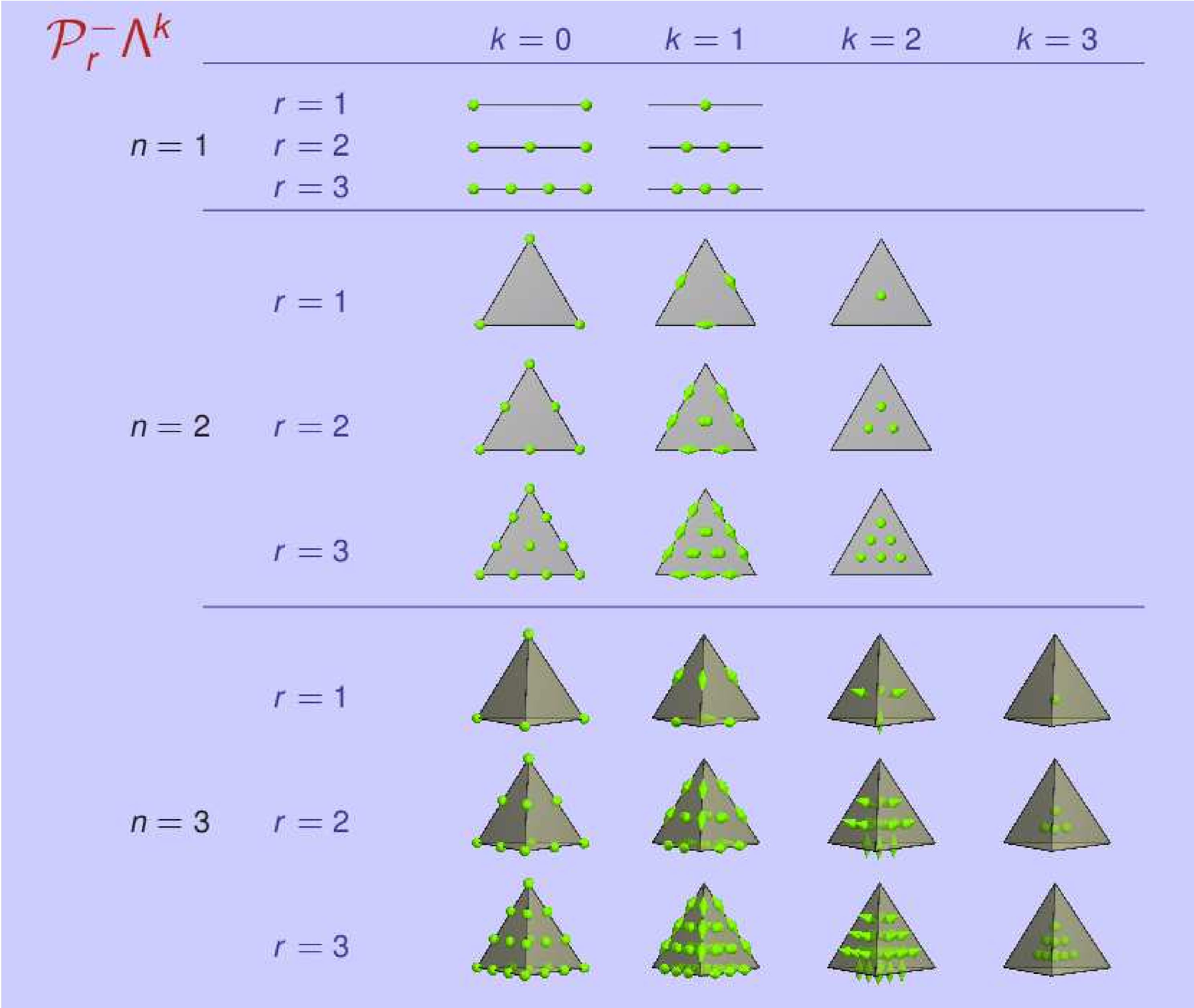
$$u_h = \sum_{i=1}^{N_r} u_i \phi_i = \sum_{j=1}^{N_r} z_j \psi_j \quad \Longrightarrow \quad V\mathbf{z} = \mathbf{u} \ \text{ and } \ V^\top\{\phi_i\} = \{\psi_j\}$$

$\phi_i(x) = \sum_{j=1}^{N} c_{ij}\psi_j(x)$ with $\mathbf{c}_i = (c_{i1}, c_{i2}, ..., c_{iN})^t$ the i-th column of V^{-1}.

The forms $\lambda^{\mathbf{k}} w^S$ are in 1-to-1 correspondence with subdomains of dimension p such that the coefficients $z_{\mathbf{k},S}$ are integrals over them of $\sum_{\mathbf{k},S} z_{\mathbf{k},S}\lambda^{\mathbf{k}} w^S$.

These subdomains are small p-chains, linear combinations of small p-simplices with coefficients given by columns of V^{-1}.

$$\mathcal{P}_r^-\Lambda^k(t) = \operatorname{span}\{\lambda^{\mathbf{m}} w^S, \quad \{\mathbf{m}, S\} \text{ small } k\text{-simplex}, \quad |\mathbf{m}| = r-1\}$$



[Nédélec, Numer.Math. 1980]

Moments in tetrahedra for $\mathbf{w} \in \mathcal{P}_r^- \Lambda^1(T)$, $r \geq 1$

$$\sigma_e : \mathbf{w} \mapsto \frac{1}{|e|} \int_e (\mathbf{w} \cdot \mathbf{t}_e)\, q \qquad \forall\, q \in \mathbb{P}_{r-1}(e),\ \forall\, e \in \mathcal{E}(T)$$

$$\sigma_f : \mathbf{w} \mapsto \frac{1}{|f|} \int_f (\mathbf{w} \times \mathbf{n}_f) \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in (\mathbb{P}_{r-2}(f))^2,\ \forall\, f \in \mathcal{F}(T)$$

$$\sigma_T : \mathbf{w} \mapsto \frac{1}{|T|} \int_T \mathbf{w} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in (\mathbb{P}_{r-3}(T))^3$$

vector $\mathbf{t}_e$ (vector $\mathbf{n}_f$) has norm $|e|$ (norm 1), it is tangent to e (it is normal to f)

[Bonazzoli, R., Numer. Algorithms 2016]

Equivalent moments in tetrahedra for $\mathbf{w} \in \mathcal{P}_r^- \Lambda^1(T)$, $r \geq 1$

$$\sigma_e : \mathbf{w} \mapsto \frac{1}{|e|} \int_e (\mathbf{w} \cdot \mathbf{t}_e)\, q \qquad \forall\, q \in \mathbb{P}_{r-1}(e),\ \forall\, e \in \mathcal{E}(T)$$

$$\sigma_f : \mathbf{w} \mapsto \frac{1}{|f|} \int_f (\mathbf{w} \cdot \mathbf{t}_{f,i})\, q \qquad \forall\, q \in \mathbb{P}_{r-2}(f),\ \forall\, f \in \mathcal{F}(T),\ i = 1, 2$$

$$\sigma_T : \mathbf{w} \mapsto \frac{1}{|T|} \int_v (\mathbf{w} \cdot \mathbf{t}_{T,j})\, q \qquad \forall\, q \in \mathbb{P}_{r-3}(T),\ j = 1, 2, 3$$

$\mathbf{t}_e$ the vector tangent to the edge e , $\mathbf{t}_{f,i}$ 2 indep. sides of f, $\mathbf{t}_{T,j}$ 3 indep. sides of T

[Nédélec, Numer.Math. 1980]

Moments in tetrahedra for $\mathbf{w} \in \mathcal{P}_r^- \Lambda^2(T)$, $r \geq 1$

$$\sigma_f : \mathbf{w} \mapsto \frac{1}{|f|} \int_f (\mathbf{w} \cdot \mathbf{n}_f) q \qquad \forall\, q \in \mathbb{P}_{r-1}(f),\ \forall\, f \in \mathcal{F}(T) \qquad (1)$$

$$\sigma_v : \mathbf{w} \mapsto \frac{1}{|T|} \int_T \mathbf{w} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in (\mathbb{P}_{r-2}(T))^3 \qquad (2)$$

vector $\mathbf{n}_f$ has norm 1, it is normal to f

[Bonazzoli, R., Numer. Algorithms 2016]

Equivalent moments in tetrahedra for $\mathbf{w} \in \mathcal{P}_r^- \Lambda^2(T)$, $r \geq 1$

$$\sigma_f : \mathbf{w} \mapsto \frac{1}{|f|} \int_f (\mathbf{w} \cdot \mathbf{n}_f)\, q \qquad \forall\, q \in \mathbb{P}_{r-1}(f),\ \forall\, f \in \mathcal{F}(T)$$

$$\sigma_v : \mathbf{w} \mapsto \frac{1}{|T|} \int_v (\mathbf{w} \cdot \mathbf{n}_{T,j})\, q \qquad \forall\, q \in \mathbb{P}_{r-2}(T),\ j = 1, 2, 3$$

$\mathbf{n}_f$ the vector normal to the face f, $\mathbf{n}_{T,j}$ normals to 3 indep. faces of T

$\mathbf{V}$ does not depend on the metric

it is indep of the triangle/tetrahedron t (up to a renumbering of its vertices)

so V^{-1} is computed only ONCE for all t

$\mathbf{V}$ is block lower triangular easy to invert (diagonal blocks are Toeplitz matrices)

$\mathbf{V}^{-1}$ has INTEGER numbers as entries : this yields the computation of

basis functions in duality with moments with no round-off errors

as linear combination of $\lambda^{\mathbf{k}} \mathbf{w}^S$ with integer coefficients !

Example with edge elements on a triangle, for degree 2

$$(\mathbf{V}^1)^{-1} = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 \\ -4 & -2 & 2 & -2 & 2 & 4 & 8 & -4 \\ 2 & -2 & -4 & -2 & -4 & -2 & -4 & 8 \end{pmatrix}.$$

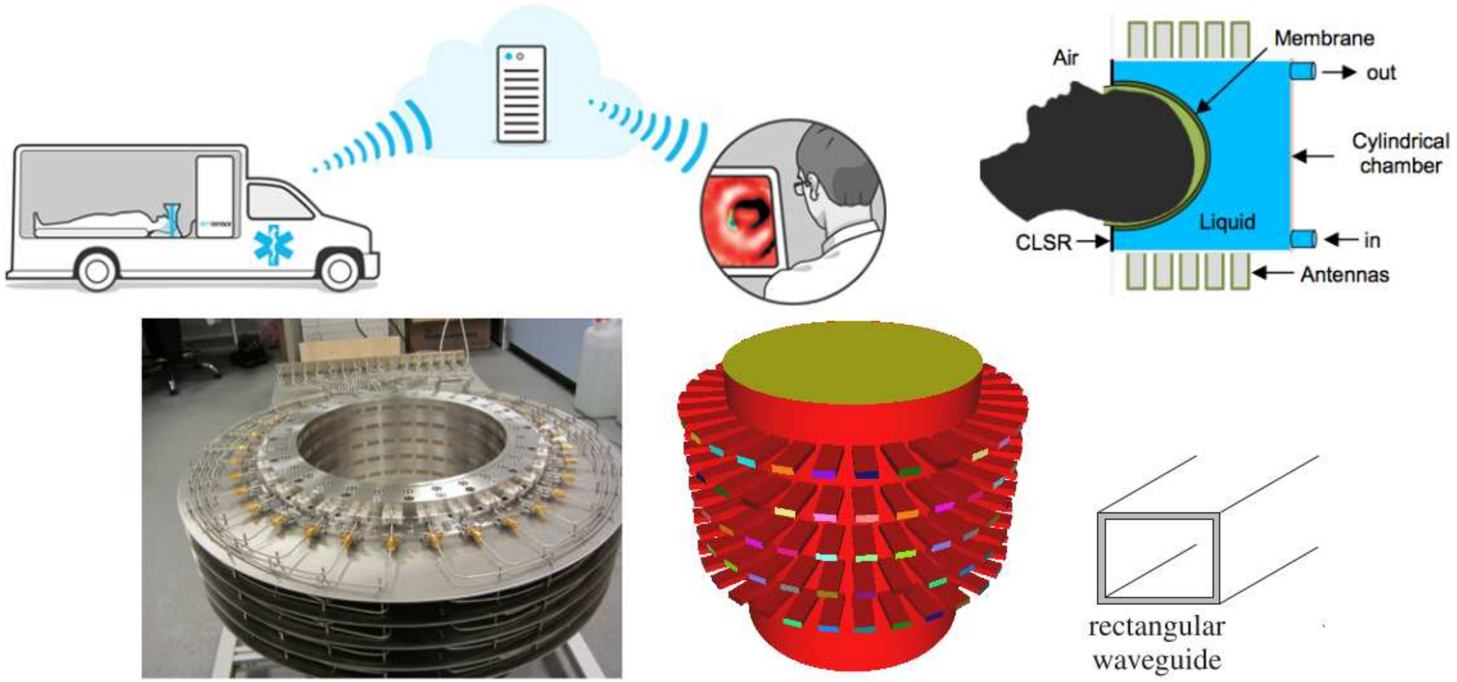
V^{-1} techn. has been implemented in FreeFEM++ for edge FEs of degrees 2,3

[Bonazzoli *et al.*, IJNM 2017, CAMWA 2018]

Application “Time is brain ! ” (ANR grant MEDIMAX 2013-2017)

Brain imaging for strokes detection and monitoring

Microwave imaging system prototype (EMTensor GmbH):
cylindrical chamber with 5 rings of 32 antennas (rectangular waveguides)



The measured data are used as input for an *inverse problem* to determine the complex electric permittivity of the medium.

With degree 2, computer runs on 5 M unks in 62 s and rel. error ~ 0.1 on measurements.
DD precond. and FreeFEM++ `http://www.freefem.org/` (Imaging time < 15 min)
Collaboration between LJLL & MAP5 in Paris, LEAT & LJAD in Nice, EMTensor in Vienna.

Conclusions

Whitney's duality has given the key to high order

⋆ A spanning family for each FE space W^p_{r+1} on simplicial meshes

⋆ Spectral accuracy in r and algebraic in h

⋆ Low grid dissipation/dispersion error for wave propagation problems [Venturini *et al.*, AMC 2018]

⋆ Flexible in terms of dofs and easy to be defined

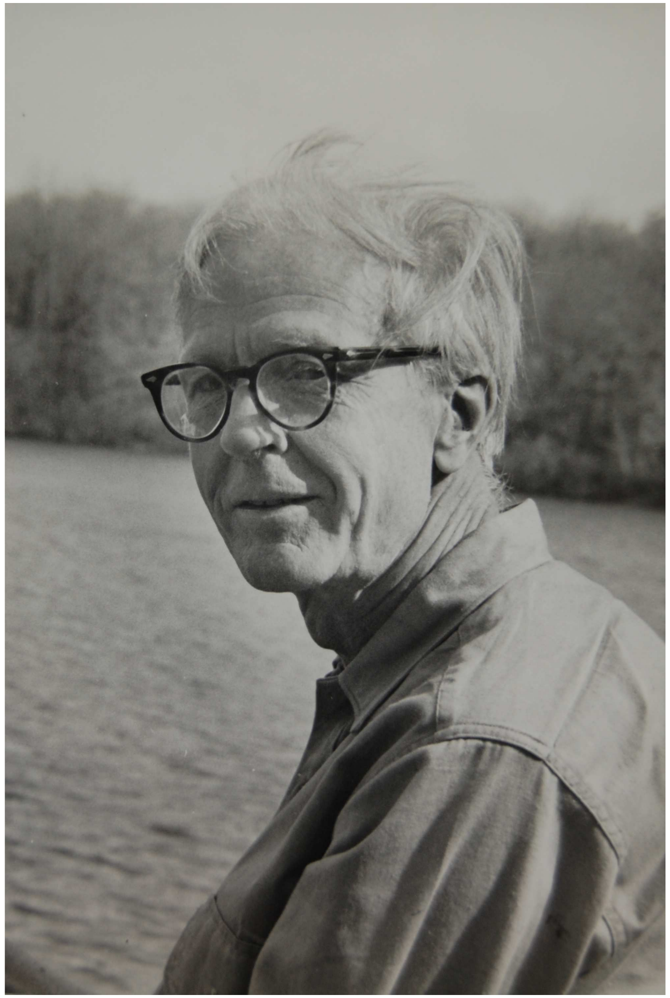
⋆ 1-to-1 dof/generator relation easy to restore with the matrix V

⋆ Straightforward extention of approches based on algebraic topology and graph theory, such as tree-cotree constructions [Alonso Rodriguez *et al.*, CALCOLO 2018]

⋆ Approach computationally feasible

A new way to look at finite elements (not only in EM) !

Thank you very much, Professor Whitney !



Incomplete list of books “on EM” from different points of view

A. Alonso Rodriguez and A. Valli

D. N. Arnold

D. Boffi, F. Brezzi, M. Fortin

A. Bossavit

P. Ciarlet Jr, C. Dunkl, S. Sauter

L. Demkowicz

T. Frankel

R. Kotiuga and P.W. Legross

J. C. Maxwell

P. Monk

E. Tonti

H. Whitney ...

My webpage `https://math.unice.fr/~frapetti/`

Thank you for the attention !