An overview on Virtual Elements and their Applications

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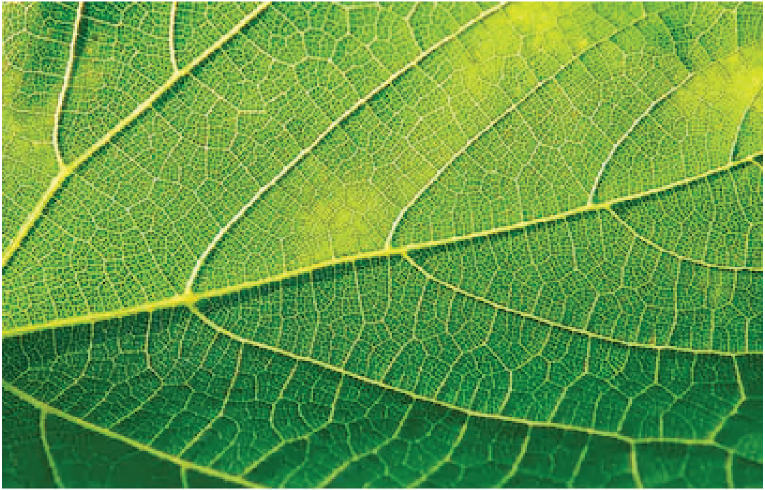
New generation methods for numerical simulations

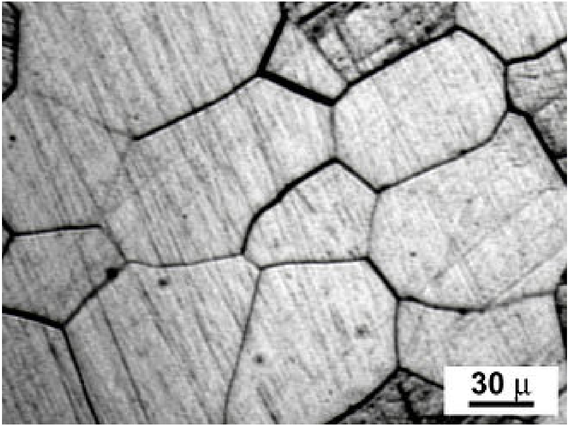
Outline

Nature does not stick on triangles - 1



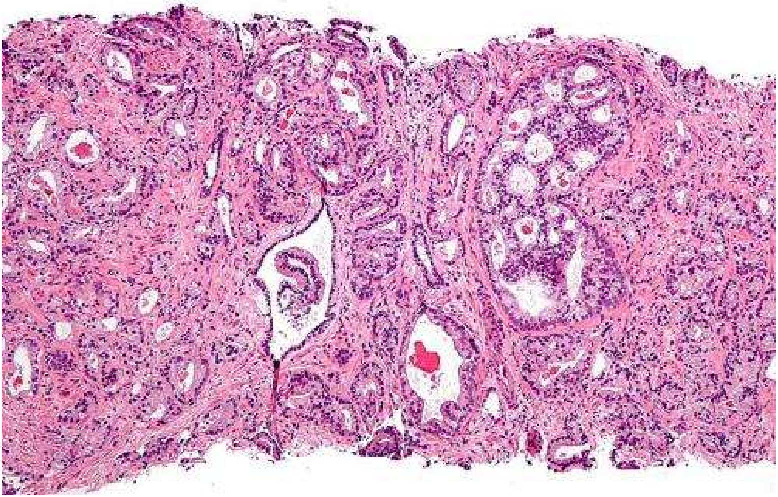
Grape leaf

Nature does not stick on triangles - 2



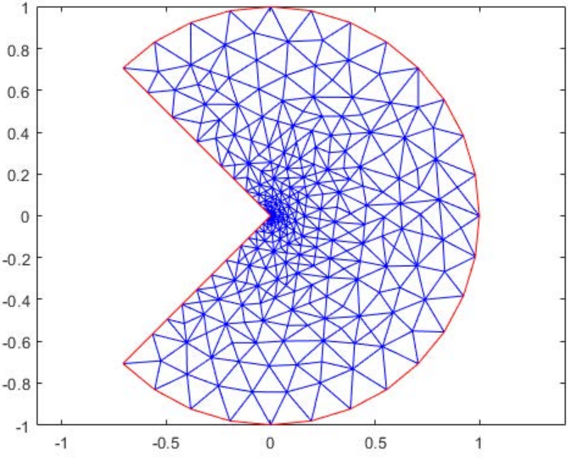
Crystal Grain

Nature does not stick on triangles - 3



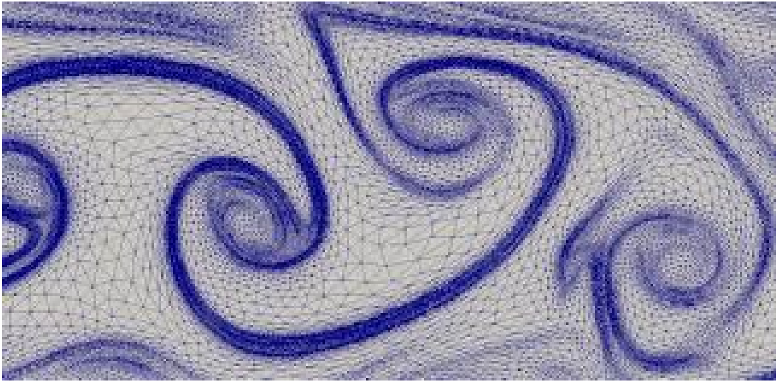
Prostate Cancer

Local adaptation might be heavy-1



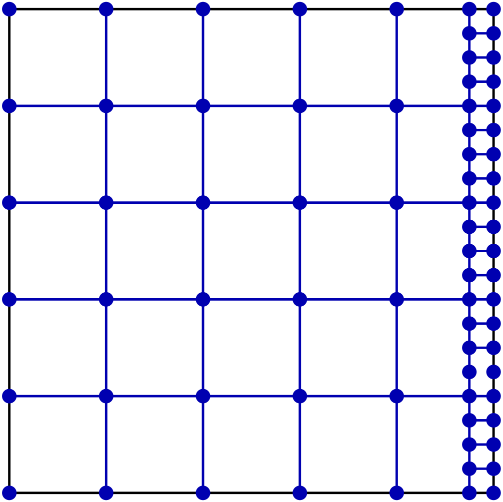
Pacman

Local adaptation might be heavy-2



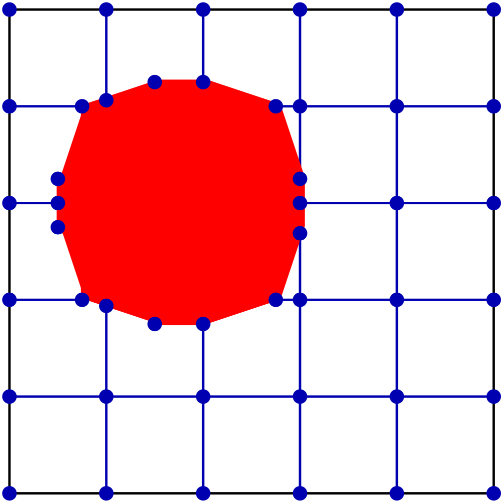
Vortices

Use of Polytopes: Boundary layers



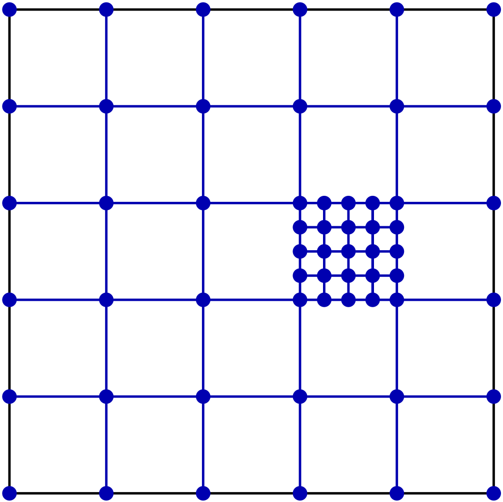
The "interface" elements are treated as epta-gons.

Use of Polytopes: Moving Objects



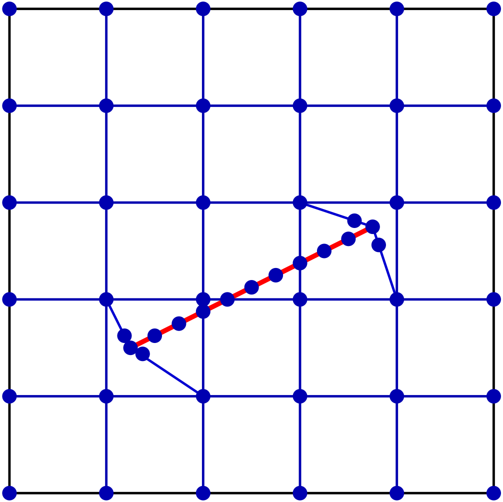
At each time step, the mesh is adapted to the object

Use of Polytopes: Local Refinement



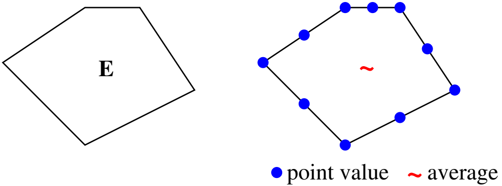
Combining a fine mesh with a coarse one

Use of Polytopes: Something going on there...



A fracture, or a 1-d intrusion

Kick off: Recalling the basic idea of VEMs



Consider a pentagonal element E. For "order of precision k = 2" we set:

$$V_2(E) := \{v \in C^0(\bar{E}) \text{ s.t. } v_{|e} \in \mathbb{P}_2(e)\ \forall \text{ edge } e, \text{ and } \Delta v \in \mathbb{P}_0(E)\}.$$

Clearly, the dimension of $V_2(E)$ is equal to 11. Note that $V_2(E)$ contains all polynomials of degree ≤ 2, **plus** 5 other smooth functions **that we don't want to compute**. We can take as (11) degrees of freedom

- the values at vertexes and midpoints, plus
- the average on E.

It is easy to see that these d.o.f.'s are *unisolvent*. Note: On a triangle we have the six $\mathbb{P}_2$ plus a bubble b: b=0 at the boundary, $\Delta b = 1$

To every $v \in H^1(E)$ we can associate $\Pi_2^{\nabla,E} v \in \mathbb{P}_2(E)$ defined by

$$\int_E \nabla(\Pi_2^{\nabla,E} v) \cdot \nabla q_2 = \int_E \nabla v \cdot \nabla q_2 \text{ for all } q_2 \in \mathbb{P}_2(E).$$

Note that the quantity on the right-hand side

$$\int_E \nabla v \cdot \nabla q_2 \equiv -\Delta q_2 \int_E v + \int_{\partial E} v \frac{\partial q_2}{\partial n}$$

is computable (out of the above d.o.f.s) $\forall v \in V_2(E)$ and $\forall q_2 \in \mathbb{P}_2$.

Note also that the $\Pi_2^{\nabla,E} v$ above is defined only up to a constant. To define it uniquely in $\mathbb{P}_2$ we must add, for instance,

$$\int_{\partial E} (\Pi_2^{\nabla,E} v - v)\,\mathrm{d}s = 0 \qquad \textit{or} \qquad \int_E (\Pi_2^{\nabla,E} v - v)\,\mathrm{d}E = 0.$$

Note finally that $\Pi_2^{\nabla,E} v = v$ whenever $v \in \mathbb{P}_2$ ($\Rightarrow \Pi_2^{\nabla,E}$ is a **projection**).

Contructing Projectors $H^1(E) \to \mathbb{P}_k$. Two more examples

To every $v \in H^1(E)$ we can associate $\Pi_2^{S,E} v \in \mathbb{P}_2(E)$ defined by

$$\int_{\partial E} \Pi_2^{S,E} v \, q_2 = \int_{\partial E} v \, q_2 \text{ for all } q_2 \in \mathbb{P}_2(E).$$

Note that $\Pi_2^{S,E} v$ is uniquely defined since the only $q_2 \in \mathbb{P}_2(E)$ that vanishes identically on ∂E is the polynomial $\equiv 0$.
Here too, $\Pi_2^{S,E} v = v$ whenever $v \in \mathbb{P}_2$ ($\Rightarrow \Pi_2^{S,E}$ is a **projection**).

Similarly one can define in $(\mathbb{P}_1(E))^2$ the L^2 projection $\Pi_1^{0,E}(\nabla v)$ of ∇v:

$$\int_E \Pi_1^{0,E}(\nabla v) \cdot \mathbf{q_1} = \int_E \nabla v \cdot \mathbf{q_1} \text{ for all } \mathbf{q_1} \in (\mathbb{P}_1(E))^2.$$

Note that, here too, the rhs is computable using the dofs of v.

A simple model problem

For $f \in L^2(\Omega)$ consider the problem

Find u such that $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.

Setting

$$a^E(u,v) := \int_E \nabla u \cdot \nabla v \qquad a(u,v) := \sum_E a^E(u,v).$$

$$\text{and} \quad (f,v)_{0,E} := \int_E f\,v \quad \text{and } (f,v) := \sum_E (f,v)_{0,E}$$

the problem can be written as

Find $u \in H_0^1(\Omega)$ such that: $\quad a(u,v) = (f,v) \qquad \forall v \in H_0^1(\Omega)$

Discretizing the local stiffness matrices

Let $\mathcal{T}_h$ be a decomposition of Ω into polygons E. Defining

$$V_2(\Omega) := \{v \in H^1_0(\Omega) \text{ such that } v_{|E} \in V_2(E) \quad \forall E \in \mathcal{T}_h\},$$

we start setting, in each E, the *consistency part*

$$a^E_C(u,v) := \int_E \nabla \Pi^{\nabla,E}_2 u \cdot \nabla \Pi^{\nabla,E}_2 v \quad \text{for } u \text{ and } v \text{ in } H^1(E)$$

with the fundamental property ($\mathbb{P}_2$-**Consistency!**):

$a^E_C(u,v) \equiv a^E(u,v)$ whenever either u or v is a polynomial of degree ≤ 2.

Finally we set:

$$a_C(u,v) := \sum_E a^E_C(u,v).$$

The discrete bilinear form a_h

We define, for u and v in $V_2(\Omega)$,

$$a_h(u,v) := a_C(u,v) + \sum_E S^E(u,v)$$

where the *stabilizing terms* $S^E(u,v)$ can be taken, for instance, as

$$S^E(u,v) := \sum_i C_i \Big(dof_i(u - \Pi_2^{\nabla,E} u) \Big) \cdot \Big(dof_i(v - \Pi_2^{\nabla,E} v) \Big) \qquad (dofi - dofi)$$

and, for each E, the $dof_i's$ are the *degrees of freedom* in $V_2(E)$, and C_i is a suitable *scaling factor*, such that

$$\alpha_* a(v_h, v_h) \le a_h(v_h, v_h) \le \alpha^* a(v_h, v_h) \quad \forall v_h \in V_2(\Omega)$$

for suitable positive constants α_* and α^* independent of h.
Note: $S^E(u,v) = 0$ whenever either u or v is in $\mathbb{P}_2$ (saving consistency).
The discretized problem will now be

Find u_h in $V_2(\Omega)$ such that $a_h(u_h, v_h) = (f, \Pi v_h) \quad \forall v_h \in V_2(\Omega)$.

Polygons

We point out that the same geometrical entity (say, a triangle) might be considered as a polygon (for instance, a quadrilateral or a pentagon, hexagon, etc.) according to the number of points on its boundary that we consider as *vertices*. See the figure below. This can be extremely helpful for example when doing adaptive mesh refinement (see the leftmost case).

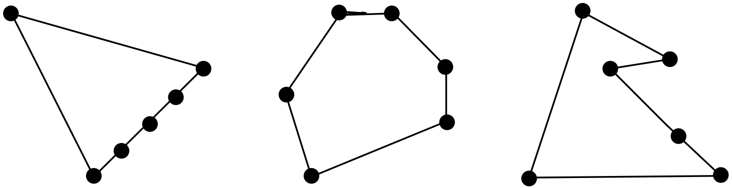


Figure: Each of the three above polygons ia considered as a hexagon

The Enhancement- Serendipity trick

In order to eliminate as many internal dofs as possible, and, at the same time, to allow the computation of all the moments of order $\leq k$, we first define the local space

$$\widetilde{V}_k^S(P) := \{v \in C^0(\overline{P}) : \ v_{|e} \in \mathbb{P}_k(e) \ \forall e \subset \partial P, \ \Delta v \in \mathbb{P}_k(P)\},$$

with the same boundary degrees of freedom, plus

$$\text{the internal moments of order up to } k : \int_P v \, p_k \mathrm{d}x \ \forall p_k \in \mathbb{P}_k(P).$$

Clearly the space $\widetilde{V}_k^S(P)$ is much bigger than the original VEM space, apparently in contradiction with our first aim. **Wait and see....**

The *Boundary-projection* operator Π_k

We define locally an operator $\Pi_k : H^1(P) \rightarrow \mathbb{P}_k(P)$ as follows:

$$\Pi_k v \in \mathbb{P}_k(P) : \int_{\partial P} (\Pi_k v - v) q_k \, \mathrm{d}s = 0 \quad \forall q_k \in \mathbb{P}_k(P).$$

We already saw it (under the name $\Pi_2^{S,E}$) for $k = 2$. For a general k the above system has a unique solution **unless** $\mathbb{P}_k$ contains polynomials that are identically zero on the boundary, i.e. unless $\mathbb{P}_k$ contains bubbles (that will have the form $v = \beta_r q_{k-r}$ where β_r is the *lowest order bubble*).
In these cases we need to add internal conditions. For instance (assuming for simplicity that P is convex) we can add

$$\int_P (\Pi_k v - v) q_{k-r} \mathrm{d}x = 0 \quad \forall q_{k-r} \in \mathbb{P}_{k-r}$$

and then solve the system in the least-squares sense.

Copying the moments!

Once the polynomial $\Pi_k v$ has been computed, we define the new space by "copying" its moments. Namely, setting $\{N =$ maximum degree of internal moments used to define $\Pi_k\}$, we set:

$$V_k^S(P) := \Big\{ v \in \widetilde{V}_k^S(P) \text{ s. t. } \int_P v\, p_s \mathrm{d}x = \int_P \Pi_k v\; p_s \mathrm{d}x \;\forall p_s \in \mathbb{P}_s^{hom}\; N < s \le k \Big\}$$

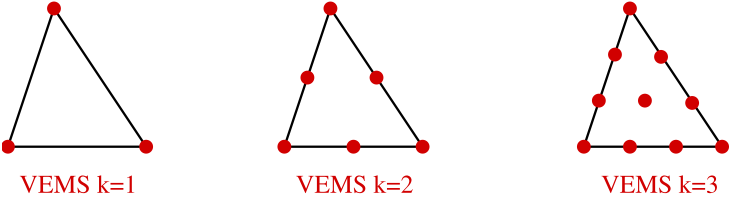


Figure: Triangles: dofs for serendipity VEM

Copying the moments!

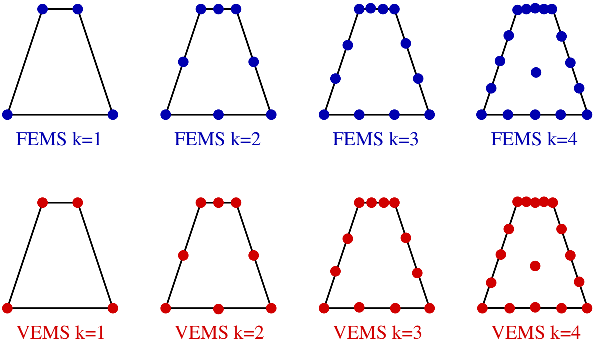


Figure: Quads: dofs for serendipity FEM and VEM

On triangles serendipity VEM have the same number of dofs as FEM (and actually the two spaces *coincide*.) On quadrilaterals Serendipity VEM and FEM have, again, the same number of dofs, but serendipity FEM are known to suffer from distorsions, while Serendipity VEM do not.

The local VEM Nonconforming space

The local VEM nonconforming space of order k is defined as:

$$V_k^{NC}(P) := \{v \in H^1(P) : \frac{\partial v}{\partial n}_{|e} \in \mathbb{P}_{k-1}(e) \ \forall \text{ edge } e, \ \Delta v \in \mathbb{P}_{k-2}(P)\}.$$

The **degrees of freedom** for a VEM NC space are given by

$$(D_1') : \text{ the moments } \int_e v p_{k-1} \mathrm{d}s \ \forall p_{k-1} \in \mathbb{P}_{k-1}(e) \ \forall e$$

$$(D_2') : \text{ for } k \geq 2 \text{ the moments } \int_P v p_{k-2} \mathrm{d}x \ \forall p_{k-2} \in \mathbb{P}_{k-2}(P).$$

The global NC VEMs

Let $H^1(\mathcal{T}_h) = \prod_{P \in \mathcal{T}_h} H^1(P)$ (functions separately in H^1 of each element).

For $\varphi \in H^1(\mathcal{T}_h)$ let $jump\{\varphi\}$ be the jump on internal edges $e \in \mathcal{T}_h$.

Then, for $k \geq 1$ we consider the **global non-conforming space**

$$
\begin{aligned}
V_k^{NC}(\Omega) := \{v \in H^1(\mathcal{T}_h) : v_{|P} \in V_k^{NC}(P) \ \forall P, \\
\int_e jump\{v\} p_{k-1} \mathrm{d}s = 0 \ \forall \text{ internal edge } e, \ \forall p_{k-1} \in \mathbb{P}_{k-1}(e), \\
\int_e v p_{k-1} \mathrm{d}s = 0 \ \forall e \text{ on } \partial\Omega, \ \forall p_{k-1} \in \mathbb{P}_{k-1}(e)\}.
\end{aligned}
$$

VEM and FEM on triangles

Just to give an idea of the possible comparison between nonconforming FEM and VEM, we consider the case of $k = 2$ on triangles.

For both we take first, as dofs, the moments, on each edge, of order ≤ 1.

But since $k = 2$ is *even*, FEM also need an additional dof *inside* (due to the presence of the so-called *nonconforming bubble*);

VEM are not better off, since their internal degree of freedom cannot be eliminated through some sort of Serendipity trick, (exactly for the same reason: there is a p_2 that is orthogonal, on each edge, to all linear and to all constant functions).

Typical escapes

The typical escape, for FEM, is to add a seventh polynomial (see e.g. Fortin-Soulie): indicating by A, B, C the vertices of the triangle, and indicating with λ_A, λ_B, and λ_C the usual barycentric cohordinates, we add

$$\zeta := \lambda_A\lambda_B(\lambda_A - \lambda_B) + \lambda_B\lambda_C(\lambda_B - \lambda_C) + \lambda_C\lambda_A(\lambda_C - \lambda_A)$$

and take *the mean value on* P as seventh degree of freedom.

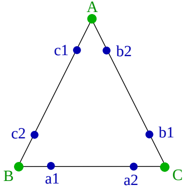
When using VEM we already have seven functions and the distinction between k *odd* or k *even* is not necessary. In the case $k = 2$ we see that the VEM space obviously contains all polynomials of degree ≤ 2, and can be seen as the union of the polynomials of degree ≤ 2 and of an additional function... (see next slide)

The additional VEM function

For instance the seventh VEM function, say $\chi(x,y)$, with the notation of the figure, could be identified by the following conditions:

$$\int_P \chi \mathrm{d}x = 0, \qquad \int_e \chi \, \mathrm{d}s = 0 \quad \forall \text{ edge } e \quad (1 + 3\, conditions),$$
$$\frac{1}{|e_a|}\int_{e_a} \chi \, q_a \mathrm{d}s = \frac{1}{|e_b|}\int_{e_b} \chi \, q_b \mathrm{d}s = \frac{1}{|e_c|}\int_{e_c} \chi \, q_c \mathrm{d}s = 1\, (3\, conds),$$

where: the edge e_a, with length $|e_a|$, is opposite to the vertex A, and q_a is the polynomial of degree 1 such that $q_a(a_1) = 1$ and $q_a(a_2) = -1$ (and similar notation for the edges e_b and e_c).



Comparing the two “additional seventh functions“

We first point out that on the boundary of our triangle the seventh VEM function χ cannot be the trace of a polynomial of degree ≤ 2. Indeed, it is easy to check that every $v \in \mathbb{P}_2$ verifies

$$\frac{1}{|e_a|}\int_{e_a} v\, q_a \mathrm{d}s + \frac{1}{|e_b|}\int_{e_b} v\, q_b \mathrm{d}s + \frac{1}{|e_c|}\int_{e_c} v\, q_c \mathrm{d}s = 0.$$

On the boundary the behaviour of χ and ζ (the one proposed by Fortin-Soulie for FEM), is quite similar, but

- χ_n is on each edge a polynomial of degree 1 (and not 2 as ζ)

and (most important)

-$\Delta\chi$ is constant (instead of linear)

features that might be convenient in problems where some equilibrium or conservation properties could be enforced strongly and not ”on average”.

C^1 VEMs - a model problem

As an example of problem that needs C^1 approximations we take a plate bending problem for a clamped plate (say, for Poisson ratio $= 0$) . For $f \in L^2(\Omega)$ consider the problem

$$\text{Find } w \text{ such that} \quad \Delta^2 w = f \text{ in } \Omega \qquad w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega.$$

The variational formulation of the problem is:

Find $w \in H_0^2(\Omega)$ such that:

$$\underbrace{\int_\Omega D_2 w : D_2 v}_{a(w,v)} = \underbrace{\int_\Omega f\, v}_{(f,v)} \qquad \forall v \in H_0^2(\Omega)$$

The problem has a unique solution.

Programming C^1 FEMs

Programming C^1 FEMs is feasible, but also an unforgettable experience

Figure: Tasting cod-liver oil

C^1 VEMs

Let P be a polygon in $\mathcal{T}_h$. For integers $r \geq 0$, $s \geq 0$, $m \geq -1$ we set

$$V_{r,s,m}(P) := \{ w \in H^2(P) \colon w_{|e} \in \mathbb{P}_r(e),\, w_{n|e} \in \mathbb{P}_s(e)\ \forall \text{ edge } e,\, \Delta^2 w \in \mathbb{P}_m(P) \}$$

Clearly, for H^2-conformity, the dofs must be chosen conveniently. In the vertices we will need continuity of w and w_n. Hence we need as dofs

- (D_0) the values of $w, w_{/1}, w_{/2}$ at the vertices,

and this will require, in a natural way, that $r \geq 3$ and $s \geq 1$. Moreover w and $w_{/n}$ must be single-valued on edges, requiring as additional dofs, e.g.,

- (D_1) for $r \geq 4$, the moments $\displaystyle\int_e w\, q_{r-4} \mathrm{d}s \quad \forall q_{r-4} \in \mathbb{P}_{r-4}(e), \quad \forall e \in \partial P$,
- (D_2) for $s \geq 2$, the moments $\displaystyle\int_e w_{/n}\, q_{s-2} \mathrm{d}x \quad \forall q_{s-2} \in \mathbb{P}_{s-2}(e) \quad \forall e \in \partial P$.

VEM version of “reduced HCT“

The smallest space will then correspond to $r = 3,\ s = 1,\ m = -1$, and is an extension to polygons of the *reduced Hsieh-Clough-Tocher* composite triangular element. The VEM space (for a *general* polygon P) will then be

$$V(P) := \{w \in H^2(P)\colon w_{|e} \in \mathbb{P}_3(e), w_{n|e} \in \mathbb{P}_1(e), \forall \text{ edge } e, \text{ and } \Delta^2 w = 0 \text{ in } P\},$$

whose degrees of freedom are the values of w, w_x, w_y at vertices .

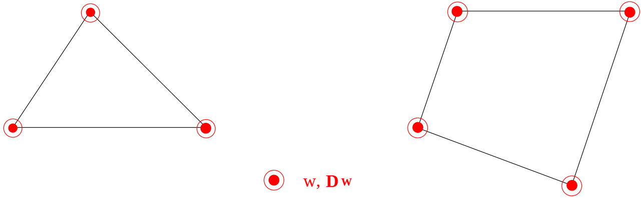


Figure: C^1 VEM, reduced HCT-like

VEM version of HCT

Another example (for $r = 3, \ s = 2, \ m = -1$) is given here below; the corresponding element will have (D_0) and (D_2) as degrees of freedom and is a sort of VEM counterpart of the original *Hsieh-Clough-Tocher* composite triangular element .

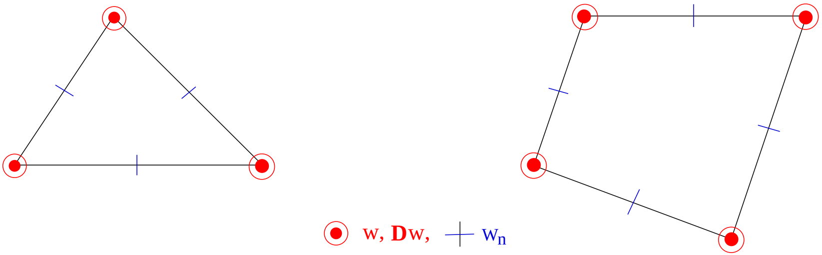


Figure: C^1 VEM, HCT-like

More general cases

At a general level, the above VEM elements will have order of precision:

$$\kappa \ \ (\equiv \text{ order of precision}) = \min\{r, s+1, m+4\}$$

and out of dofs $(D_0), .., (D_3)$, (integrating by parts twice) we can compute an operator $\Pi^P_\kappa : \ V_{r,s,m}(P) \longrightarrow \mathbb{P}_\kappa(P)$ defined on each element by

$$a^P(\Pi^P_\kappa v - v, q_\kappa) = 0 \quad \forall q_\kappa \in \mathbb{P}_\kappa(P), \qquad \int_{\partial P} (\Pi^P_\kappa v - v) q_1 \mathrm{d}s = 0 \quad \forall q_1 \in \mathbb{P}_1(P).$$

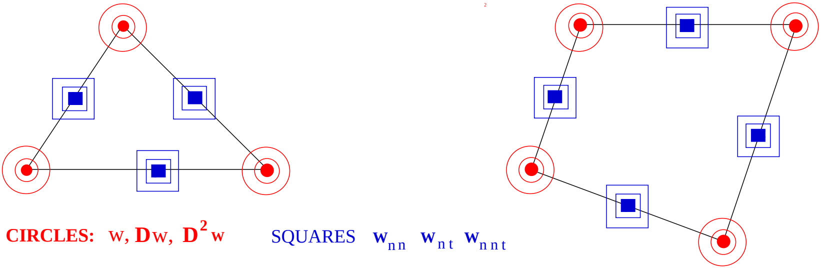
The discrete bilinear form, for v_h and w_h in $V_{r,s,m}(P)$, is then defined as

$$a^P_h(v_h, w_h) := a^P(\Pi^P_\kappa v_h, \Pi^P_\kappa w_h) + S^P((I - \Pi^P_\kappa) v_h, (I - \Pi^P_\kappa) w_h)$$

with $S^P(v_h, w_h)$ taken, e.g., as *dofi-dofi*, with $(D_0) - (D_3)$ properly scaled.

C^p-VEM with $p > 2$

Along the same lines, still for general polygons, we might easily construct C^p elements for $p \geq 2$. Just to give an example, we might consider



In particular, this figure refers to the local spaces

$$V(P) := \{ v \in H^3(P) : v_{|e} \in \mathbb{P}_5, \, v_{n|e} \in \mathbb{P}_4, \, v_{nn|e} \in \mathbb{P}_3 \; \forall e \in \partial P, \, \Delta^3 v = 0 \text{ in } P \}.$$

Stokes Problem

We recall (to set the notation) the **model Stokes problem**
Find $\boldsymbol{u} \in (H^1_0(\Omega))^2$ *and* $p \in L^2(\Omega)$ *such that*:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega. \end{cases}$$

Set: $\mathbf{V} := (H^1_0(\Omega))^2$, $Q := L^2_0(\Omega)$ (zero mean value), and define, for $\boldsymbol{u}, \mathbf{v}$ in $\mathbf{V}$, and $q \in Q$:

$$a(\boldsymbol{u}, \mathbf{v}) := \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\mathbf{v}) \, \mathrm{d}\Omega \qquad b(\mathbf{v}, q) := \int_\Omega \operatorname{div} \mathbf{v} \, q \, \mathrm{d}\Omega$$

(where $\varepsilon(\mathbf{v}) := (\nabla v + (\nabla v)^T)/2$ is the symmetric gradient). The variational formulation is: Find $\boldsymbol{u} \in \mathbf{V}$, $p \in Q$ such that

$$\begin{cases} a(\boldsymbol{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}, \\ b(\boldsymbol{u}, q) = 0 & \forall q \in Q. \end{cases}$$

Discretizations

Taking a sequence of conforming discretizations of this problem with $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$, and suitable approximations a_h and b_h of the bilinear forms a and b, respectiveky, one can write the discretized version as: Find $\boldsymbol{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) = 0 & \forall q_h \in Q_h, \end{cases}$$

where, in turn, $\mathbf{f}_h$ is (if needed) a suitable approximation of $\mathbf{f}$. It is well known that $\exists!$ of the discrete solution with optimal error bounds requires ellipticity of a_h on the kernel of b_h and the *inf-sup* stability condition

$$\exists \beta > 0 \text{ such that } \inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_V \, \|q_h\|_Q} \geq \beta \quad \forall h.$$

Incompressible discrete solutions

One can wonder whether the velocity solution $\boldsymbol{u}_h$ would satisfy *exactly*

$$\mathrm{div}\boldsymbol{u}_h \equiv 0 \quad \text{in } \Omega,$$

(ensuring the exact incompressibility of the discrete solution). This would require

$$\Big\{\{\boldsymbol{u}_h \in \mathbf{V}_h\} \text{ and } \{\int_P \mathrm{div}\boldsymbol{u}_h\, q_h\, \mathrm{d}x = 0\, \forall q_h \in Q_h\}\Big\} \Rightarrow \Big\{\mathrm{div}\boldsymbol{u}_h = 0 \text{ in } P\Big\}.$$

verified only with very few (and sometimes rather cumbersome) choices of discretizations (and often only for special types of decompositions). See the excellent review by John-Linke-Merdon-Neilan-Rebholz (SIAM Review, 2017) and to the references therein.

Incompressible VEM

Following Beirão da Veiga - Lovadina- Vacca (2017), for the velocity space we start from the boundary, and define, for $k \geq 2$

$$\mathcal{B}_k(\partial P) := \{\mathbf{v} \in (C^0(\partial P))^2 \text{ s.t. } \mathbf{v}_{|e} \in (\mathbb{P}_k(e))^2 \quad \forall \text{ edge } e \text{ of } \partial P\}.$$

Clearly, the dimension of $\mathcal{B}_k(\partial P)$ for a polygon with n edges would be

$$\mathrm{dim}\mathcal{B}_k(\partial P) = 2nk.$$

Then we can define the VEM space for velocities:

$$\mathcal{V}_k(P) := \{\mathbf{v} \in (H^1(P))^2 \text{ s.t. } \mathbf{v}_{|\partial P} \in \mathcal{B}_k(\partial P), \mathrm{rot}(\Delta\mathbf{v}) \in \mathbb{P}_{k-3}, \mathrm{div}\mathbf{v} \in \mathbb{P}_{k-1}\}$$

while for the pressure we simply take

$$Q_k(P) = \mathbb{P}_{k-1}(P).$$

The dimension of $\mathcal{V}_k(P)$ is then equal to $2nk$ (dimension of $\mathcal{B}_k(\partial P)$) plus $dim(\mathbb{P}_{k-3})$, plus $dim(\mathbb{P}_{k-1}) - 1$ (since, from Gauss theorem, the mean value of the divergence is determined already by the boundary values).

Incompressible VEM

Then the dimension of $\mathcal{V}_k$ is given by

$$\dim(P) = 2nk + \frac{(k-2)(k-1)}{2} + \frac{k(k+1)}{2} - 1 = 2nk + k^2 - k.$$

Accordingly, one can show that a set of degrees of freedom for $\mathcal{V}_k(P)$ can be taken as

- the values of $\mathbf{v}$ at the n vertices ($= 2n$ dofs),
- the values of $\mathbf{v}$ at $k-1$ points in each edge ($= 2n(k-1)$ dofs),
- the values of $\int_P \mathbf{v} \cdot \mathbf{x}^{\perp} q_{k-3} \, \mathrm{d}s$ for every $q_{k-3} \in \mathbb{P}_{k-3}$,
- the values of $k(k+1)/2 - 1$ moments of $\mathrm{div}\mathbf{v}$.

The dofs for Q_k, in each element, will be (say) the moments against $\mathbb{P}_{k-1}$

A projection operator and the bilinear form a_h

Using the dofs, $\forall \mathbf{v} \in \mathcal{V}_k(P)$ one can compute its divergence (which is a polynomial), and also the operator $\Pi_k^\varepsilon : \mathcal{V}_k(P) \to (\mathbb{P}_k(P))^2$ defined by

$$\begin{cases} \displaystyle\int_P \varepsilon(\mathbf{v} - \Pi_k^\varepsilon \mathbf{v}) : \varepsilon(\mathbf{q}_k)\,\mathrm{d}x = 0 \quad \forall \mathbf{q}_k \in (\mathbb{P}_k)^2 \\ \displaystyle\int_{\partial P} (\mathbf{v} - \Pi_k^\varepsilon \mathbf{v})\mathrm{d}s = \mathbf{0} \end{cases}$$

that, in turn, allows to define, on each element P, a discrete bilinear form:

$$a_h^P(\boldsymbol{u}, \mathbf{v}) := \int_P \varepsilon(\Pi_k^\varepsilon \boldsymbol{u}) : \varepsilon(\Pi_k^\varepsilon \mathbf{v})\,\mathrm{d}x + S^P(\boldsymbol{u} - \Pi_k^\varepsilon \boldsymbol{u}, \mathbf{v} - \Pi_k^\varepsilon \mathbf{v}) \qquad \forall\, \boldsymbol{u}, \mathbf{v} \in \mathcal{V}_k(P)$$

where S^P is again one of the common *stabilizing* bilinear forms of VEMs. The discrete bilinear form a_h will then be obtained (as usual) by summing the contributions a_h^P of all the polygons P

The discrete problem

The bilinear form $b(\mathbf{v}, q)$ is directly computable, for every $\mathbf{v} \in \mathcal{V}_k(P)$ and $q \in Q_k(P)$, using the degrees of freedom. Finally, for the right-hand side we use $\Pi^0_{k-2}\mathbf{f}$ instead of $\mathbf{f}$. Setting:

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}_{|P} \in \mathcal{V}_k(P)\ \forall P \in \mathcal{T}_h\},$$

$$Q_h = \{q \,\big|\, q_{|P} \in Q_k(P)\, \forall P \in \mathcal{T}_h, \text{ and } \int_\Omega q = 0\},$$

we have the discretized problem: Find $\boldsymbol{u}_h \in \mathbf{V}_h,\ p_h \in Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\Pi^0_{k-2}\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\boldsymbol{u}_h, q_h) = 0 & \forall q_h \in Q_h. \end{cases}$$

Visualization of dofs. Triangular elements

The following figures show the degrees of freedom for $k = 2$ and $k = 3$ on triangles and quads. The squares are *vectorial* dofs (so, 2 dofs each). Note that, apart from the number n of edges (and then the dimension of $\mathcal{B}_k$), nothing changes passing from triangles to quads (and to general polygons).

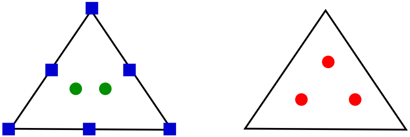


Figure: Dofs for $k = 2$, on triangles, for velocities (left) and pressures (right)

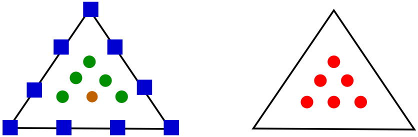


Figure: Dofs for $k = 3$, on triangles, for velocities (left) and pressures (right)

Visualization of dofs. Quadrilateral elements

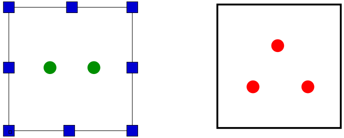


Figure: Dofs for $k = 2$, on quads, for velocities (left) and pressures (right)

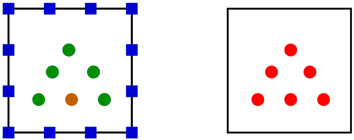


Figure: Dofs for $k = 3$, on quads, for velocities (left) and pressures (right)

VEM and FEM (Crouzeix-Raviart)

Crouzeix-Raviart: Velocities $= (\mathbb{P}_2)^2 + (CubicBubbles)^2$, Pressures $= \mathbb{P}_1$

For VEM the cubic bubbles (for velocities) are replaced by two vectorial valued bubble-functions $\mathbf{b}^i (i = 1, 2)$ solutions of the local Stokes problems: Find $\mathbf{b}^{(i)} \in (H^1_0(P))^2$ and $p^{(i)} \in L^2(P)$ s.t.

$$\begin{cases} -\Delta \mathbf{b}^{(i)} + \nabla p^{(i)} = \mathbf{0}, \\ \operatorname{div} \mathbf{b}^{(i)} = (\mathbf{x} - \bar{\mathbf{x}})_i \qquad \bar{\mathbf{x}} = \text{ barycenter of } P. \end{cases}$$

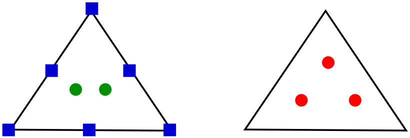


Figure: Dofs of both FEM and VEM for $k = 2$

N.B. The VEM discrete solution is **exactly incompressible**...

Limiting ourselves, for simplicity, to the 2-d case with homogeneous Dirichlet b. c. , we recall that the Hellinger-Reissner mixed formulation of linear elasticity problems in a domain Ω can be written as:
Find $(\boldsymbol{\sigma}, \boldsymbol{u})$ in $\boldsymbol{\Sigma} \times \mathbf{U}$ such that

$$\begin{aligned} \mathbf{div}\boldsymbol{\sigma} + \mathbf{f} = 0 \ \text{ in } \Omega, \\ \boldsymbol{\sigma} = \mathbb{C}(\varepsilon(\boldsymbol{u})) \ \text{ in } \Omega, \\ \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega, \end{aligned}$$

where $\boldsymbol{\Sigma} := \left\{ \boldsymbol{\tau} \in (L^2(\Omega))^2, \tau_{12} = \tau_{21}, \mathbf{div}\boldsymbol{\tau} \in (L^2(\Omega))^2 \right\}$, $\mathbf{U} := (H^1_0(\Omega))^2$, and the costitutive law is the classical $\mathbb{C}\varepsilon := 2\mu\varepsilon + \lambda \mathrm{tr}(\varepsilon)$. With a common notation we also set $\mathbb{D} := \mathbb{C}^{-1}$.

Variational formulation of HR

Defining the bilinear forms (*local* and *global*)

$$a^P(\boldsymbol{\sigma},\boldsymbol{\tau}) := \int_P \mathbb{D}\boldsymbol{\sigma} : \boldsymbol{\tau}\mathrm{d}x \quad \forall P \quad \text{and} \quad a(\boldsymbol{\sigma},\boldsymbol{\tau}) := \sum_P a^P(\boldsymbol{\sigma},\boldsymbol{\tau}),$$

$$b^P(\boldsymbol{\tau},\mathbf{v}) := \int_P \mathbf{div}\boldsymbol{\tau}\cdot\mathbf{v}\mathrm{d}x \quad \forall P \quad \text{and} \quad b(\boldsymbol{\tau},\mathbf{v}) := \sum_P b^P(\boldsymbol{\tau},\mathbf{v}),$$

the variational formulation of the HR problem can be written as:
find $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ and $\boldsymbol{u} \in \mathbf{U}$ such that

$$\begin{cases} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\boldsymbol{u}) = 0 & \forall\boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\ b(\boldsymbol{\sigma},\mathbf{v}) = -(\mathbf{f},\mathbf{v}) & \forall\mathbf{v} \in \mathbf{U}. \end{cases}$$

Discrete problems and Targets

With finite dimensional subspaces $\mathbf{\Sigma}_h \subset \mathbf{\Sigma}$ and $\mathbf{U}_h \subset \mathbf{U}$, approximate bilinear forms a_h, b_h, and forcing term $\mathbf{f}_h$, we get the approximate problem: find $\boldsymbol{\sigma}_h \in \mathbf{\Sigma}_h$ and $\boldsymbol{u}_h \in \mathbf{U}_h$ such that

$$\begin{cases} a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, \boldsymbol{u}_h) = 0 & \forall \boldsymbol{\tau}_h \in \mathbf{\Sigma}_h, \\ b_h(\boldsymbol{\sigma}_h, \mathbf{v}_h) = -(\mathbf{f}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{U}_h. \end{cases}$$

The difficulties come from the combined targets of

- *i)* getting a *symmetric* discrete stress tensor $\boldsymbol{\sigma}_h$,
- *ii)* getting a $\boldsymbol{\sigma}_h$ with *continuous tractions* at interelements,
- *iii)* getting a *stable pair* $(\mathbf{\Sigma}_h, \mathbf{U}_h)$ (*inf-sup* condition),
- *iv)* making the formulation *hybridizable* (de Veubeke style),
- *v)* getting elementwise *self-equilibrium* ($\mathbf{f} = 0 \rightarrow \mathbf{div}\boldsymbol{\sigma}_h = 0$),
- *vi)* ensuring the *patch-test* of some order $k \geq 1$ (that is: if $\boldsymbol{u}$ is, globally, a polynomial of degree $\leq k$, then $\boldsymbol{u}_h = \boldsymbol{u}$ and $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}$).

Towards HR VEM

Given a polygon P with n edges, we first introduce the space of local infinitesimal rigid body motions:

$$RM(P) = \{\mathbf{r}(\mathbf{x}) = \mathbf{a} + b(\mathbf{x} - \mathbf{x}_B)^{\perp} \text{ with } \mathbf{a} \in \mathbb{R}^2, \text{ and } b \in \mathbb{R}\}$$

where $\mathbf{x}_B$ is the baricenter of P. Introducing also the space

$$RM_k^{\perp}(P) = \{\mathbf{p} \in (\mathbb{P}_k)^2 : \int_P \mathbf{p}_k \cdot \mathbf{r} = 0 \; \forall \mathbf{r} \in RM(P)\},$$

we note that, obviously, we can always decompose $(\mathbb{P}_k)^2$ as a direct sum

$$(\mathbb{P}_k)^2 = RM(P) \oplus RM_k^{\perp}(P).$$

The discrete stresses and displacements

Following Artioli-De Miranda-Lovadina-Patruno (2018), for $k \geq 1$ the the local tensor space of discretized stresses is given by:

$$\boldsymbol{\Sigma}_k(P) := \Big\{ \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div};\Omega;\mathbb{S}) \text{ s.t. } curl\mathbf{curl}(\mathbb{D}\boldsymbol{\tau}) = 0,$$

$$\boldsymbol{\tau}\cdot\mathbf{n}_{|e} \in (P_k(e))^2\,\forall e \in \partial P,\ \mathbf{div}\boldsymbol{\tau} \in (\mathbb{P}_k)^2 \Big\}.$$

We recall that $\mathbb{D} := \mathbb{C}^{-1}$, and $curl\mathbf{curl}(\mathbf{z}) := (z_{11})_{yy} - 2(z_{12})_{xy} + (z_{22})_{xx}$ so that $curl\mathbf{curl}(\mathbb{D}\boldsymbol{\tau}) = 0$ iff $\boldsymbol{\tau} = \mathbb{C}(\varepsilon(\mathbf{v}))$ for some vector $\mathbf{v}$.

A $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_k(P)$ can be individuated by the following degrees of freedom:

$$\text{for each edge } e \text{ in } \partial P: \int_e \boldsymbol{\tau}_{\mathbf{n}} \cdot \mathbf{q}_k \,\mathrm{d}s \quad \forall \mathbf{q}_k \in (\mathbb{P}_k(e))^2,$$

$$\text{in } P: \int_P \mathbf{div}\boldsymbol{\tau} \cdot \mathbf{q}_k \,\mathrm{d}x \quad \forall \mathbf{q}_k \in (RM)^{\perp}_k.$$

Finally, for displacements, we simply take in each element $\mathbf{U}_h := (\mathbb{P}_k)^2$

The projector and the bilinear form a_h

Using the above dofs we can construct a projection Π_k^a onto $(\mathbb{P}_k)^4_{sym}$:

$$a^P(\Pi_k^a \boldsymbol{\tau} - \boldsymbol{\tau}, \mathbf{p}_k) = 0 \qquad \forall \mathbf{p}_k \in (\mathbb{P}_k)^4_{sym}.$$

We can also compute $\mathrm{div}\boldsymbol{\tau}$, that belongs to $(\mathbb{P}_k)^2$. Then we define

$$a_h^P(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) := a^P(\Pi_k^a \boldsymbol{\sigma}, \Pi_k^a \boldsymbol{\tau}_h) + S^P((I - \Pi_k^a)\boldsymbol{\sigma}_h, (I - \Pi_k^a)\boldsymbol{\tau}_h), \forall \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_k(P),$$

where again the bilinear form S^P is a stabilizing term (to fix ideas, of the *dofi-dofi* type).

Finally one gets the global bilinear form $a_h(\cdot, \cdot)$ summing over the elements.

On the other hand, no projection is needed for the second equation as both the divergence of tensors in $\boldsymbol{\Sigma}_h$ and the elements of $\mathbf{U}_h$ are polynomials.

Qualities of H-R Virtual elements

We point out that VEM spaces enjoy, at the same time, all these features:

- A - They pass the patch test (of order k) .
- B - They are easily hybridizable (having no vertex dofs).
- C - The stress field is symmetric (equilibrium of momentums).
- D - If the load $\mathbf{f} \in (\mathbb{P}_k)^2$, then $\mathbf{div}\boldsymbol{\sigma}_h + \mathbf{f} = 0$ (equilibrium of forces).
- E - The definition, essentially, does not depend on the *shape* of the elements (triangles, quads, polygons, polyhedra etc.)

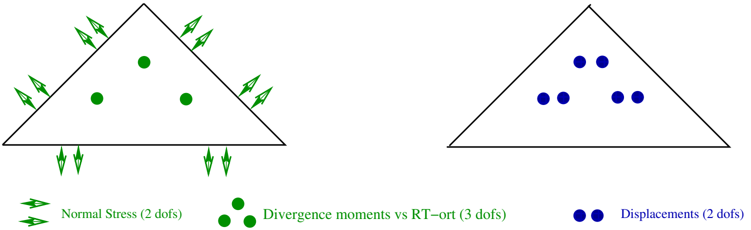


Figure: H-R VEM Dofs (Artioli-De Miranda-Lovadina-Patruno 2018) for $k = 1$

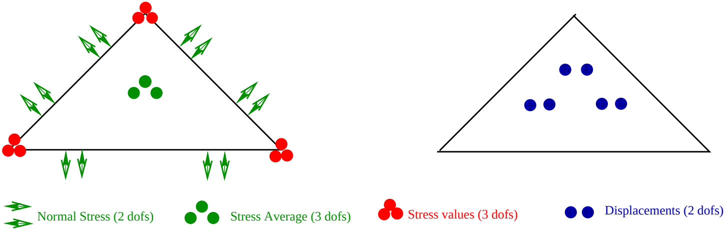


Figure: H-R FEM Dofs (Arnold-Winther 2002) Dofs for $k = 1$

That's all, folks!!!

Thank you

for your PATIENCE!