Recent results on finite element methods for incompressible flow at high Reynolds number (a “Smörgåsbord”).

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Joint work with:
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  - Discretization methods with pointwise divergence free velocity.
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    2. Scott-Vogelius type elements.
- Part II
  - Variational data assimilation

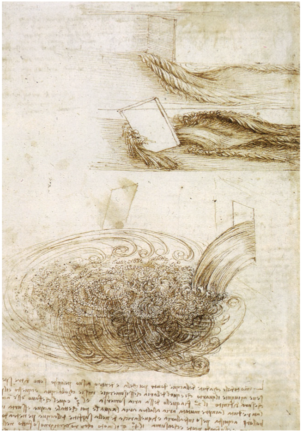


Figure: da Vinci turbulence

Stabilised FEM for high Re flows I

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mu \Delta \mathbf{u} = f \quad \text{in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega$$

- High Reynolds number: $Re := \mathbf{U}L/\mu$ large
- Continuous finite elements unstable at high Reynolds number;
- Even smooth solutions loose accuracy[1], Ethier-Steinmann 3D solution:

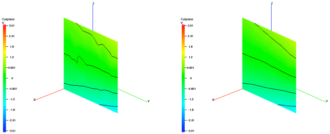


Figure: Navier-Stokes' equations. $Re = 10,000$. Magnitude of velocity. Left: unstabilised solution. Right: stabilised solution

[1] EB, Fernández and Hansbo, Siam J. Numer. Anal. 44 (2006).

Stabilised FEM for high Re flows II

- Remedy add (asymptotically vanishing) dissipative term
- Let $\mathbf{V}_h := [V_h]^d$, $Q_h := V_h$ and $V_h \subset H^1(\Omega)$, standard FEM
- Example: Navier-Stokes' equations in a domain Ω (Ω periodic box, unless specified)

$$\begin{cases} (\partial_t \mathbf{u}_h, \mathbf{v}_h)_\Omega + a(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) + s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega, \\ - b(q_h, \mathbf{u}_h) + s_p(p_h, q_h) = 0, \\ \mathbf{u}_h(0) = \pi_h \mathbf{u}_0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$.

- $(\cdot,\cdot)_\Omega$ L^2-scalar product,

$$a(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h + \int_\Omega \nu \nabla \mathbf{u}_h : \nabla \mathbf{v}_h + \frac{1}{2} \int_\Omega \nabla \cdot \mathbf{u}_h \, \mathbf{u}_h \cdot \mathbf{v}_h$$

$$b(p_h, \mathbf{v}_h) \stackrel{\text{def}}{=} -\int_\Omega p_h \nabla \cdot \mathbf{v}_h$$

- We will only discuss $s_{\mathbf{u}}$ here.

Stabilised FEM for high Re flows III

- Examples: ($\mathcal{T} := \{K\}$, quasiuniform mesh, $h = \max_{K\in\mathcal{T}} \operatorname{diam}(K)$)
  1. Artificial viscosity:

$$s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) = \int_\Omega \hat{\nu}(\mathbf{u}_h)\nabla \mathbf{u}_h : \nabla \mathbf{v}_h$$

     1.1 Aritificially lowering the cell-Reynolds number to 1: $\hat{\nu}(\mathbf{u}_h) = |\mathbf{u}_h|h$,
     1.2 Smagorinsky: $\hat{\nu}(\mathbf{u}_h) = h^2 \, |\nabla \mathbf{u}_h|$,
     1.3 Fluctuation based: $\hat{\nu}(\mathbf{u}_h)|_K = h^2 \, \max_{\partial K} |[\![\nabla \mathbf{u}_h]\!]|$
     $[\![\nabla \mathbf{u}_h]\!]$ is the jump of the gradient over the element boundary ∂K.
  2. Spectral viscosity: apply viscosity only to the highest polynomial orders
  3. Gradient jump penalty (GJP/CIP): fluctuation (jump) based dissipation

$$s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) := \sum_{K\in\mathcal{T}_h} \int_{\partial K} \tau_u h_K^2 |\mathbf{u}_h \cdot n| [\![\nabla \mathbf{u}_h]\!] : [\![\nabla \mathbf{v}_h]\!]$$

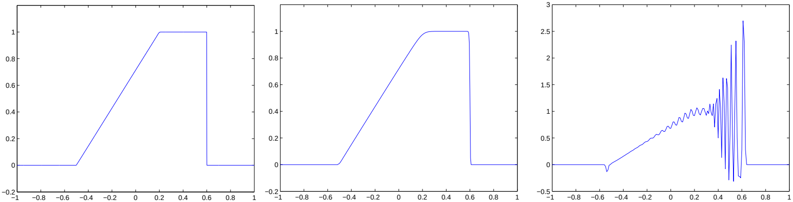
  4. Also: SUPG, GaLS, UWDG, LPS, OSS, SGV etc.

Examples, stabilised versus unstabilised I

- Example: Burgers' equation

$$\partial_t u + \tfrac{1}{2}\partial_x u^2 = 0 \tag{1}$$

- Nonlinear artificial viscosity[1]: $\hat{\nu}(u_h) = \frac{1}{2}\max_{\partial K} h|[\![\partial_x u_h]\!]|/\{\!\{|\partial_x u_h|\}\!\}$
  - $[\![\xi]\!]|_F$ = jump of ξ, $\{\!\{|\xi|\}\!\}|_F$ average of $|\xi|$ over the face F.
- Stabilisation allows for error estimate[2]: $\|(u-u_h)(\cdot,T)\|_{L^1} \leq Ch^{\frac{1}{2}}$.

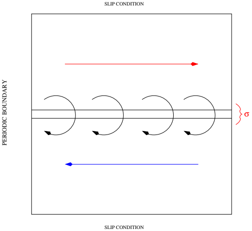


Exact solution; stabilised soln; energy conserving soln.

- [1] EB, BIT 47 (2007), no. 4; [2] EB, M3AS 25 (2015)

Examples, stabilised versus unstabilised II

- Helmholtz instability
- Unit square, $\mathbf{u}_\infty = 1$, $\sigma = \frac{1}{28}$, $\nu = 3.571 \cdot 10^{-6} \rightarrow Re_\sigma = 10000$.



- Animations: 80×80 grid; P1/P1 unstabilized compared to stabilized
- Stabilisation GJP, error estimate for smooth solutions (preasymptotic, $\nu << |\mathbf{u}_h| h$):

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty((0,T);L^2(\Omega))} \lesssim h^{k+\frac{1}{2}} \exp(\|\nabla \mathbf{u}\|_\infty T)(|\mathbf{u}|_{\infty,k+1} + |p|_{2,k+1})$$

- This stability appears to be sharp for the Helmholtz instability
- Turbulent flow too rough, is the $h^{1/2}$ win relevant? Can it be improved?

– EB, Comput. Methods Appl. Mech. Engrg. 196 (2007), 288 (2015); EB and M. Fernandez, Num. Math. 107 (2007)

Results cG, transient Navier-Stokes', high Re I

1. L^2-norm, $O(h^{k+\frac{1}{2}})$ error estimates (smooth solution) in the nonlinear case:
   - Johnson, Saranen, Streamline-diffusion, 2D vorticity-streamfunction, Math. Comp., 1986.
   - Hansbo, Szepessy, Streamline-diffusion, velocity-pressure, CMAME, 1990.
   - EB, Fernández, CIP, velocity-pressure, Num. Math. 2007.
   - Chen, Feng, Zhou, Local projection stabilization, Appl. Math. Comput., 2014.
   - EB, CIP, 2D vorticity-streamfunction, CMAME, 2015.
   - Arndt, Dallmann, Lube, Local projection stabilization, Num. Meths, PDE, 2015
   - de Frutos, García-Archilla, John, Novo, Local projection stabilization, IMA J. Numer. Anal., 2019.

Results cG, transient Navier-Stokes', high Re II

2. In [1] $h^{k+\frac{1}{2}}$ convergence proved for the vorticity (gradient jumps).
3. Results in weaker norm†, 2D Navier-Stokes' estimate for vorticity :

$$\boxed{\|\omega(\cdot,T)-\omega_h(\cdot,T)\|_{V'} \lesssim |log(h)|h^{\frac{1}{4}}\exp(\|\nabla\bar{\mathbf{u}}\|_\infty T),}$$

- V' dual of H^1, 2-Wasserstein distance.
- Only regularity assumption $\|\nabla\bar{\mathbf{u}}\|_\infty$ bounded.
- $\bar{\mathbf{u}}$ large scale velocity obtained through scale separation:

$$\begin{cases}\mathbf{u}=\hat{\mathbf{u}}+\mathbf{u}',\\ \hat{\mathbf{u}}\in[L^1(I;W^{1,\infty}(\Omega))]^d\\ \mathbf{u}' \text{ s.t. } \nu^{\frac{1}{2}}\geq|\mathbf{u}'|\end{cases} \tag{2}$$

[1] EB, CMAME, 2015.

Turbulence modelling: uDNS or LES?

- What is uDNS?
  1. “underresolved Direct Numerical Simulation”: resolve numerically as much as we can afford and claim that some quantities are accurately computed
  2. Stabilisation needed on the discrete level to achieve numerical stability
  3. Once the discrete system is perturbed, compute (resolving the smallest scales you can afford) and hope for the best.
- What is LES?
  1. “Large eddy simulation”: simulate large eddies neglecting small eddies
  2. The Navier-Stokes’ equations with model of Reynolds stresses occurring after “filtering”. Some quantities are claimed to remain physically accurate
  3. Classical model: Smagorinsky
  4. Once the continuous system is perturbed, compute (resolving the smallest scales of the model) and hope for the best.

uDNS prototype: gradient jump penalty I

- GJP implemented[1] in Nektar++ (Spencer Sherwin, with McLaren)
- 3D computations, turbulent flow around a cylinder at $Re = 3900$
- FEM in $x - y$ $P3$ or $P5$ (Taylor-Hood, or equal order), Fourier in z (64 planes).
- IMEX time-discretization, explicit convection/stabilisation[2].

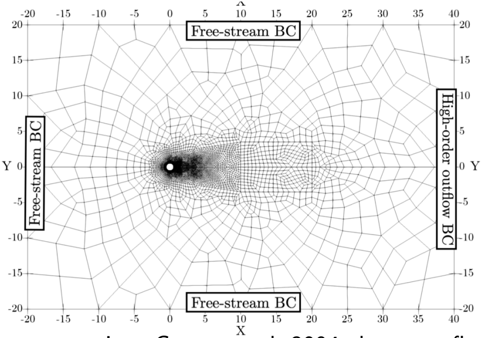


Figure: Mesh in $x - y$ crosssection. Coarse mesh 3094 elements, fine mesh 6978 elements.

[1] Moura, Cassini, EB, Sherwin, submitted, (2021)

[2] EB, Guzmán, arXiv:2012.05727, (2021)

uDNS prototype: gradient jump penalty II

GJP - Gradient Jump Penalty

- Adds dissipation on the singular part of the approximation
- In other words penalises the “rough” oscillations
- reduces polynomial fluctuations over element boundaries
- control of polynomial fluctuations inside the element?
- Parameter chosen through dispersion analysis.

SVV - Spectral Vanishing Viscosity

- Adds viscous dissipation on the highest polynomial orders only.
- In other words penalises the highest frequency, smooth oscillations
- reduces polynomial fluctuations within elements
- control of polynomial fluctuations across element faces?
- Spectrum/parameter chosen through dispersion analysis (matching DG).

Fixing the parameter for the gradient jump penalty method

- Theoretical polynomial scaling of the stabilisation parameter[1]:

$$\boxed{\tau_u = \tau_0 P^{-3.5}}$$

- Using temporal and spatial dispersion analysis we verify this scaling and fix τ_0

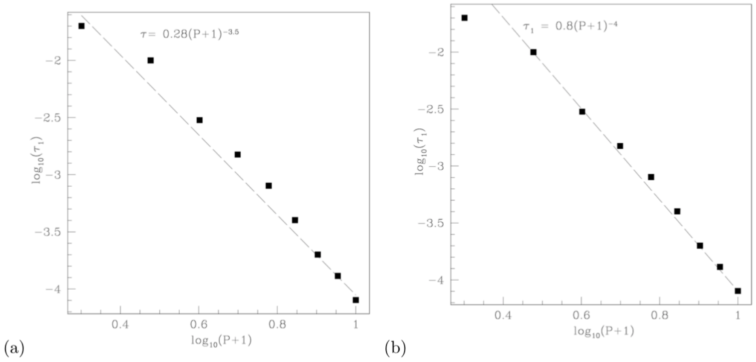


Figure 5: Comparison between the scaling (a) $\tau = 0.28(P+1)^{-3.5}$ and (b) $\tau = 0.8(P+1)^{-4}$ and the optimal experimental values obtained shown by square symbols.

[1] EB, Ern, Math. Comp., (2007)

Computational restults, SVV and GJP I

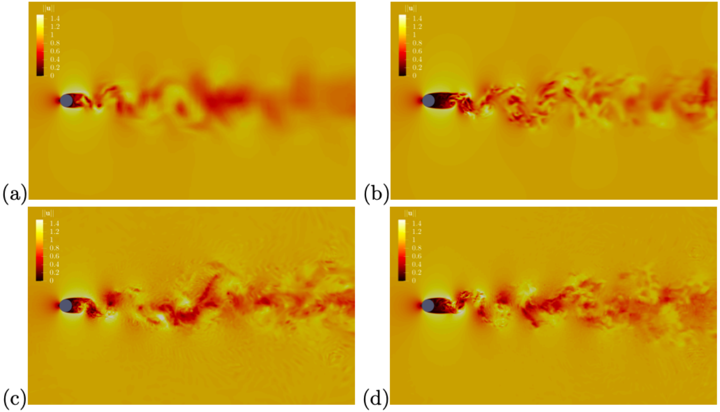


Figure 12: Instantaneous velocity magnitude extracted from a $x - y$ slice, simulated in the coarse mesh. SVV DG-Kernel is shown in the top row: (a) $P = 3$, (b) $P = 5$. GJP stabilisation is shown in the bottom row: (c) $P = 3$, (d) $P = 5$.

Computational restults, SVV and GJP II

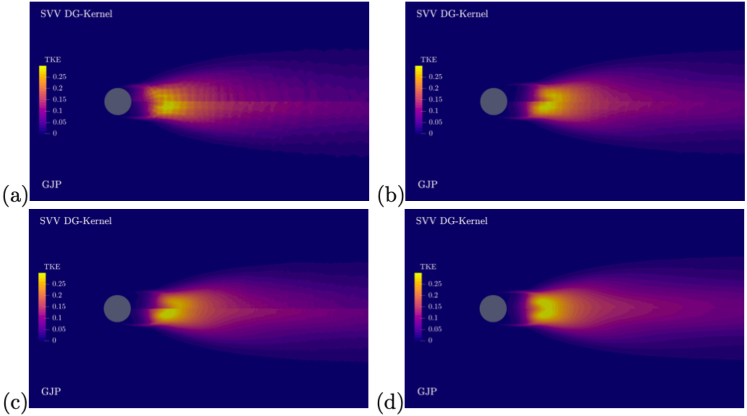


Figure 14: Time- and spanwise-averaged turbulent kinetic energy. SVV DG-Kernel is shown in the top half of each figure, GJP stabilisation in the bottom half. (a) Coarse mesh at $P = 3$, (b) coarse mesh at $P = 5$, (c) fine mesh at $P = 3$, (d) fine mesh at $P = 5$

Computational restults, SVV and GJP III

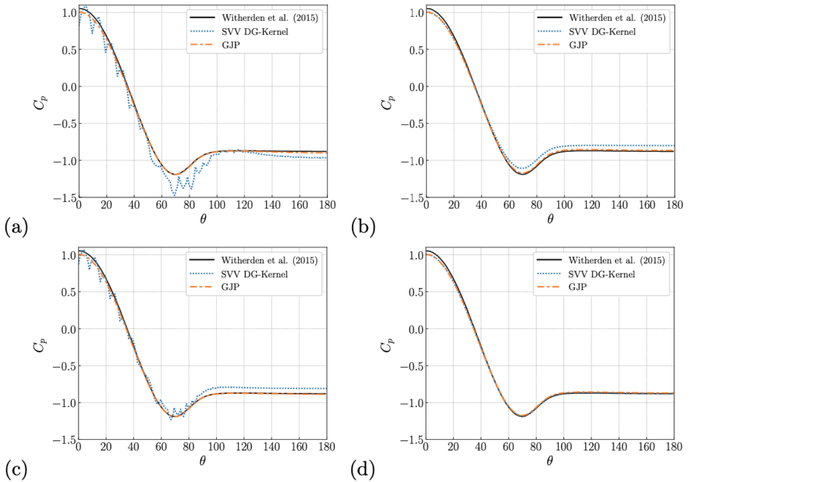


Figure 16: Time- and spanwise-averaged pressure coefficient compared with numerical results of Witherden et al. [46]. (a) Coarse mesh at $P = 3$, (b) coarse mesh at $P = 5$, (c) fine mesh at $P = 3$, (d) fine mesh at $P = 5$

Computational restults, SVV and GJP IV

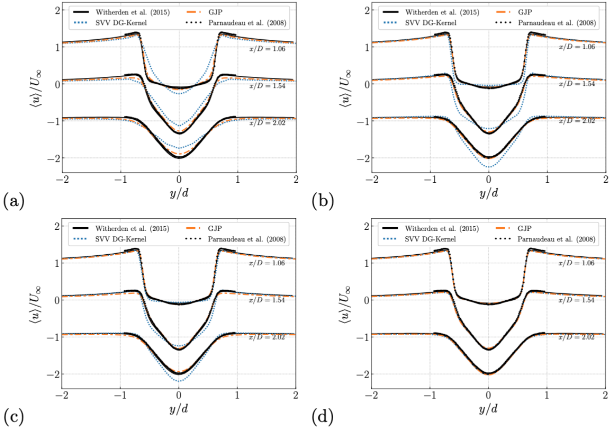


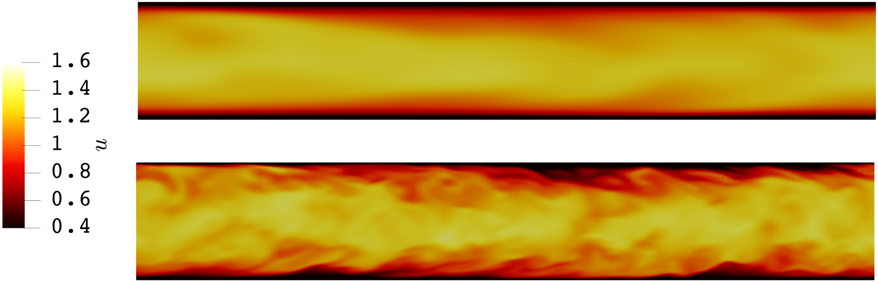
Figure 18: Time- and spanwise-averaged wake profiles of horizontal velocity, compared with numerical results of Witherden et al. [46] and experimental results of Parneadeu et al. [47]. (a) Coarse mesh at $P = 3$, (b) coarse mesh at $P = 5$, (c) fine mesh at $P = 3$, (d) fine mesh at $P = 5$

Benchmarking

Table: Quantitative comparison of time-averaged flow properties, compared with reference experimental and numerical studies.

	C_d		L_r/D		θ_{sep} [°]	
	SVV	GJP	SVV	GJP	SVV	GJP
Coarse, P=3	1.00719	0.999278	1.20879	1.39854	94.85	86.83
Coarse, P=5	0.933542	0.977452	1.84479	1.53705	85.97	86.56
Fine, P=3	0.926216	0.99081	1.76641	1.45229	89.76	86.72
Fine, P=5	0.982638	0.980388	1.50396	1.54253	86.59	86.59
Witherden et al. (DNS)	-		-		86.90	
Parnaudeau et al. (EXP)	-		1.51		-	
Franke et al. (LES)	0.978		1.64		88.2	
Lehmkuhl et al. (DNS)	1.015		1.36		88	

3D square duct flow, Re=5600



(a)

Figure: Instantaneous velocity magnitude extracted from a slice in the $x - y$ plane at $z/H = 0$. Top: SVV; bottom: GJP stabilisation.

- $P = 3$, $48 \times 38 \times 38$ elements, no slip on walls, periodic inlet outlet, constant mass flux
- Integration using SVV to statistical convergence
- When this is fed into the GJP solver new fine scale structures appear

LES prototype: the Smagorinsky model

- The Smagorinsky model introduced (1963) in the context of meteorology (hugh literature)
- Add artificial viscosity to N.S. with:

$$\boxed{\hat{\nu}(\tilde{\mathbf{u}}) = \delta^2 |\nabla \tilde{\mathbf{u}}|, \quad \delta > 0 \text{ "filter width"}}$$

- The resulting problem is well-posed [1] and has enhanced regularity [2]
- Generally considered too dissipative
- Layton [3]: "*the Smagorinsky model does not over dissipate in the absence of boundary layers*"
- Our contribution [4]:
  - effect of the model on the exponential coefficient in the perturbation analysis
  - effect of the model if interpreted as a stabilisation method

[1] Ladyzhenskaya/Lions, (1968).

[2] Beirão da Veiga, JEMS, 2009.

[3] Layton, Appl. Math. Lett. (2016).

[4] EB, Hansbo, Larson, arXiv:2102.00043 (2021)

LES prototype: the Smagorinsky model, stability of the continuous model I

- Scale separation: let $\boldsymbol{\eta} = \mathbf{u} - \tilde{\mathbf{u}}$, $\mathbf{u}$ solution to N.S. and $\tilde{\mathbf{u}}$ solution to N.S.-Smagorinsky.

$$\begin{cases} \mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}', \\ \hat{\mathbf{u}} \in [L^1(I; W^{1,\infty}(\Omega))]^d, \quad \text{large scales} \\ \mathbf{u}' \in \{[L^3(Q)]^d \mid \int_Q ((\nu + \hat{\nu}(\boldsymbol{\eta}))^{\frac{1}{2}} - |\mathbf{u}'| \tau_L^{\frac{1}{2}})\phi \geq 0,\ \forall \phi \in L^{\frac{3}{2}}(Q), \phi \geq 0\}. \end{cases} \tag{3}$$

- τ_L is a characteristic time scale of the large scales of the flow defined by

$$\tau_L(\hat{\mathbf{u}}) := T \left(\int_I \|\nabla \hat{\mathbf{u}}(t)\|_{L^\infty(\Omega)} \, \mathrm{d}t \right)^{-1}. \tag{4}$$

- Nonlinear feedback mechanism:

$$\boxed{\nabla \boldsymbol{\eta} \text{ grows } \rightarrow (\nu + \hat{\nu}(\boldsymbol{\eta}))^{\frac{1}{2}} \text{ grows } \rightarrow |\mathbf{u}'|\tau_L^{\frac{1}{2}} \text{ grows } \rightarrow \tau_L(\hat{\mathbf{u}}) \text{ grows}}$$

- Increasing model error, moderates large scale gradients and τ_L

LES prototype: the Smagorinsky model, stability of the continuous model II

- The error $\boldsymbol{\eta}$ between the Navier-Stokes' equations and the NS-Smagorinsky equations satisfies:

$$\boxed{\sup_{t \in I} \|\boldsymbol{\eta}(t)\|_\Omega^2 \lesssim e^{\frac{T}{\tau_L(\hat{\mathbf{u}})}} \delta^2 \|\nabla \mathbf{u}\|_{L^3(Q)}^3.}$$

- Exponential growth depends only on the characteristic time-scale of the coarse scales (whatever they are).
- Further questions:
  1. Do these observations carry over to the discrete case?
  2. Can the Smagorinsky model be interpreted in the framework of stabilised FEM?

LES prototype: the Smagorinsky model as FEM stabiliser

- Numerically $\delta = \gamma h$ for some scaling factor $\gamma > 0$
- The accuracy of Smagorinsky at best $O(h^2) \rightarrow$ affine approximation optimal
- Discretization: affine FEM[1], satisfying $\nabla \cdot \tilde{\mathbf{u}}_h = 0$, Smagorinsky $+s(\tilde{\mathbf{u}}_h, \mathbf{v}_h)$:

$$s(\tilde{\mathbf{u}}_h, \mathbf{v}_h) := \sum_K \int_{\partial K} h^2 |\tilde{\mathbf{u}}_h|^{-1} [\![(\tilde{\mathbf{u}}_h \cdot \nabla)\tilde{\mathbf{u}}_h \times \mathbf{n}]\!] \cdot [\![(\tilde{\mathbf{u}}_h \cdot \nabla)\mathbf{v}_h \times \mathbf{n}]\!]$$

- Error estimate for smooth solution[2] (preasymptotic, $\nu \leq |\mathbf{u}_h| h$):

$$\boxed{\sup_{t \in I} \|(\mathbf{u} - \tilde{\mathbf{u}}_h)(t)\|_\Omega \lesssim e^{(T/\tau_L)}\, h^{\frac{3}{2}} \|\mathbf{u}\|_{L^\infty(0,T;W^{2,3}(\Omega))}}$$

- τ_L defined by the scale separation argument, using $\boldsymbol{\eta} = \mathbf{u} - \tilde{\mathbf{u}}_h$.
- Same estimate as for GJP with affine approximation, but with exponential growth moderated through scale separation.

[1] Christiansen, Hu, Num. Math., (2018)

[2] EB, Hansbo, Larson, arXiv:2102.00043 (2021)

Vorticity contours for double shear layer, $Re = \infty$

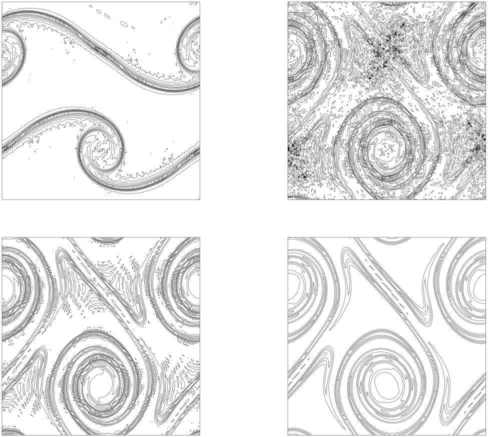


Figure: Top: time $t = 6$ and $t = 12$, $\gamma = 0$. Bottom: $t = 12$, left $\gamma = 0.1$, right $\gamma = \sqrt{0.1}$.

Vortex shedding in 2D, $Re = 10^6$

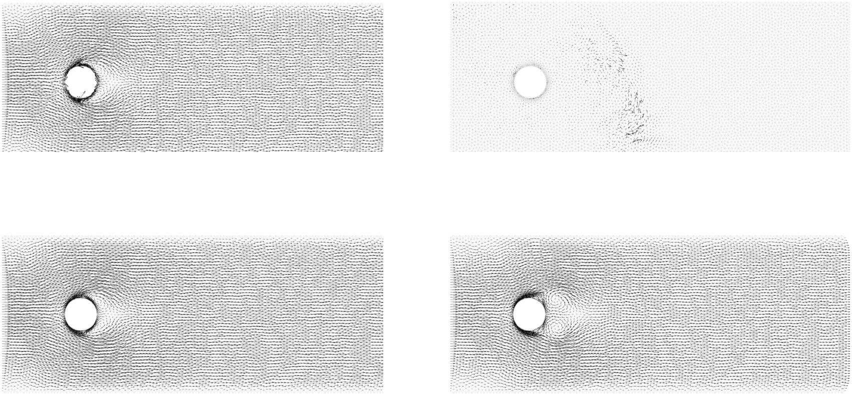


Figure: Velocities after 15, and 30 timesteps, $Re = 10^6$, above $\gamma = 0$ below $\gamma = \sqrt{0.1}$.

Discretization methods with divergence free velocity

- Link between the Smagorinsky model and stabilised FEM for pointwise divergence free elements ($\nabla \cdot \mathbf{V}_h \in Q_h$).
- Such elements already applied to high Re incompressible flows[1,2,3,4]
- The $O(h^{k+\frac{1}{2}})$ L^2-error estimate was not proven in any of these refs.
- In [3] the following comparisons of the perturbation growth in time for approximations of a smooth solution (planar lattice flow).

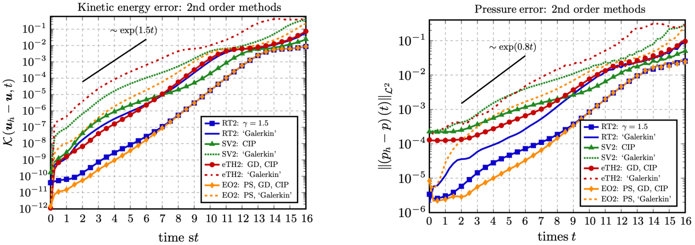


Figure: Comparison of $2D$ error growth for different discretizations, similar DOFs

[1] EB, Linke, ApNum., (2008)

[2] Guzmán, Shu, Sequira, IMA J. Num. Anal. (2017)

[3] Schroeder, Lube, J. Sci. Comp. (2017)

[4] Schroeder, Lube, J. Num. Maths. (2017)

$\mathbf{H}(div)$-conforming elements (RT, BDM, DG)

- We consider methods such that $\mathbf{V}_h \in \mathbf{H}(div)$, $\mathbf{V}_h \not\subset [H^1(\Omega)]^d$.
- This was considered for inviscid flow in [1] and the following results reported

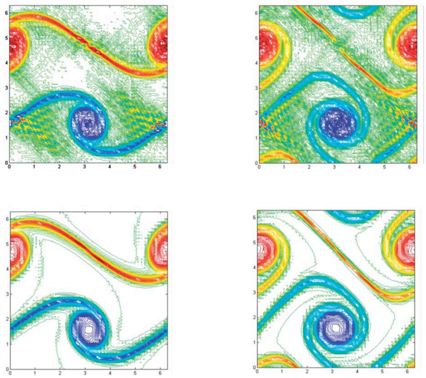


Figure: Double shear layer from [1], left at $t = 6$, right at $t = 8$, top row: central fluxes, bottom row: upwind fluxes

[1] Guzmán, Shu, Sequira, IMA J. Num. Anal. (2017)

$\mathbf{H}(div)$-conforming method and the $O(h^{k+\frac{1}{2}})$ estimate? I

- We introduced the model problem[1]: Find a velocity $\mathbf{u}$ and a pressure p satisfying

$$\begin{aligned} \operatorname{div}(\mathbf{u} \otimes \boldsymbol{\beta}) + \sigma \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\,, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

- $\nabla \cdot \boldsymbol{\beta} = 0$
- "Darcy" + convection$\rightarrow$ transport in the space of divergence free vector fields
- Using regularization with the Hodge-Laplacian we prove existence and uniqueness of a solution $(\mathbf{u}, p) \in [H^1(\Omega)]^d \times H^1(\Omega)$ if $\mathbf{f} \in \mathbf{H}(curl)$ and $\sigma / \|\boldsymbol{\beta}\|_{W^{1,\infty}(\Omega)} \sim 1$.
- Limit problem for Oseen with vanishing viscosity.

[1] Barrenechea, EB, Guzmán, M3AS, (2020)

$\mathbf{H}(div)$-conforming method and the $O(h^{k+\frac{1}{2}})$ estimate? II

- We discretize the model problem using the Raviart-Thomas space $\mathbf{V}_{h,k}^{RT}$ for velocities and $Q_{h,k}$ (piecewise polynomial of order $k \geq 1$) for pressures.
- Find $\mathbf{u}_h \in \mathbf{V}_{h,k}^{RT}$ and $p_h \in Q_{h,k}$ such that $\forall \mathbf{v}_h \in \mathbf{V}_{h,k}^{RT}$ and $\forall q_h \in Q_{h,k}$,

$$-(\mathbf{u}_h, \boldsymbol{\beta} \cdot \nabla \mathbf{v}_h)_h + \langle (\boldsymbol{\beta} \cdot \mathbf{n}) \underbrace{\mathbf{u}_h{}^-}_{upwind}, \mathbf{v}_h \rangle_h + (\sigma \mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \operatorname{div} \mathbf{v}_h)_\Omega = (\mathbf{f}, \mathbf{v}_h)_\Omega$$

$$(\operatorname{div} \mathbf{u}_h, q_h)_\Omega = 0$$

  where

$$(\mathbf{v}, \mathbf{w})_h = \sum_{K \in \mathcal{T}} \int_K \mathbf{v} \cdot \mathbf{w} \, dx, \quad \langle \mathbf{v}, \mathbf{w} \rangle_h = \sum_{K \in \mathcal{T}} \int_{\partial K} \mathbf{v} \cdot \mathbf{w} \, ds.$$

- The solution can also be sought in the BDM space for $k \geq 1$
  $\mathbf{u}_h \in \mathbf{V}_{h,k}^{BDM}$, $p \in Q_{h,k-1}$

$\mathbf{H}(div)$-conforming method and the $O(h^{k+\frac{1}{2}})$ estimate? III

- For this method we prove the error bounds

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \lesssim h^{k+\frac{1}{2}}|\mathbf{u}|_{H^{k+1}(\Omega)}$$
$$\|p-p_h\|_{L^2(\Omega)} \lesssim h^{k+\frac{1}{2}}(|\mathbf{u}|_{H^{k+1}(\Omega)}+|p|_{H^{k+1}(\Omega)})$$

- Key steps, coercivity, Galerkin orthogonality, continuity
  1. Let $\mathbf{e}_h = \mathbf{r}_h\mathbf{u}-\mathbf{u}_h$ and $\mathbf{e}_r = \mathbf{r}_h\mathbf{u}-\mathbf{u}$ ($\mathbf{r}_h\mathbf{u} \in \mathbf{V}_{h,k}^{RT}$ RT-interpolant)
  2. Let $|||v_h|||^2 := \|\sigma^{\frac{1}{2}}\mathbf{e}_h\|_\Omega^2 + \sum_T \||\boldsymbol{\beta}\cdot n|^{\frac{1}{2}}[\![\mathbf{e}_h]\!]\|_{\partial K}^2$

$$\begin{aligned}|||\mathbf{e}_h|||^2 &= -(\mathbf{e}_h, \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h)_h + \langle(\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{e}_h{}^-, \mathbf{e}_h\rangle_h + (\sigma\mathbf{e}_h, \mathbf{e}_h)_\Omega\\ &= -(\mathbf{e}_r, \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h)_h + \langle(\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{e}_r{}^-, \mathbf{e}_h\rangle_h + (\sigma\mathbf{e}_r, \mathbf{e}_h)_\Omega\\ &\le |||\mathbf{e}_h|||\,|||\mathbf{e}_r||| + |(\mathbf{e}_r, \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h)_h|\end{aligned}$$

  3. Take away average velocity $\bar{\boldsymbol{\beta}} \in \mathbb{R}^d$ and use inverse inequality:

$$\begin{aligned}|(\mathbf{e}_r, \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h)_h| &= |(\mathbf{e}_r, (\boldsymbol{\beta}-\bar{\boldsymbol{\beta}})\cdot\nabla\mathbf{e}_h)_h + (\mathbf{e}_r, \bar{\boldsymbol{\beta}}\cdot\nabla\mathbf{e}_h)_h|\\ &\lesssim \sigma^{-1}\|\boldsymbol{\beta}\|_{W^{1,\infty}(\Omega)}|||\mathbf{e}_r|||\,|||\mathbf{e}_h||| + |(\mathbf{e}_r, \bar{\boldsymbol{\beta}}\cdot\nabla\mathbf{e}_h)_h|\end{aligned}$$

$\mathbf{H}(div)$-conforming method and the $O(h^{k+\frac{1}{2}})$ estimate? IV

- The continuity is the sticking point, because we have no H^1-control
  - $\mathbf{r}_h$ Raviart-Thomas interpolant: on any simplex K,

$$(\mathbf{u} - \mathbf{r}_h \mathbf{u}, \mathbf{y}_h)_K = 0, \text{ for all } \mathbf{y}_h \in \mathbb{P}_{k-1}(K)$$

  - We need $(\mathbf{r}_h \mathbf{u} - \mathbf{u}, (\bar{\boldsymbol{\beta}} \cdot \nabla)\mathbf{e}_h)_h = 0$
  - Problem: $\mathbf{e}_h|_K \in [\mathbb{P}_{k+1}(K)]^d \rightarrow$ too much!
  - Solution: $\mathbf{v}_h \in \mathbf{V}_{h,k}^{RT}$ with $\nabla \cdot \mathbf{v}_h = 0$, satisfy[1] $\mathbf{v}_h|_K \in [\mathbb{P}_k(K)]^d$
  - Hence since $\nabla \cdot \mathbf{e}_h = 0$, $\mathbf{e}_h|_K \in [\mathbb{P}_k(K)]^d$ and therefore

$$(\mathbf{r}_h \mathbf{u} - \mathbf{u}, \underbrace{(\bar{\boldsymbol{\beta}} \cdot \nabla)\mathbf{e}_h}_{\text{pol. of order } k-1!})_h = 0.$$

- This analysis leads to $O(h^{k+\frac{1}{2}})$ convergence in [2].
- See also [3] for Navier-Stokes' analysis using the same arguments.
- First $O(h^{k+\frac{1}{2}})$ analysis for a pressure robust discretization?

[1] Cockburn, Gopalakrishnan, Sinum (2004)

[2] Guzmán, Shu, Sequira, IMA J. Num. Anal. (2017)

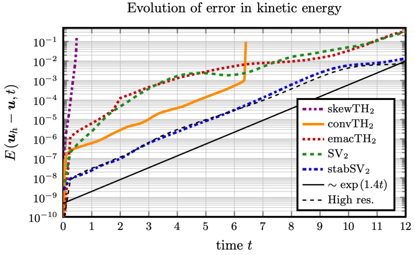
[3] Han, Hou, IMA J. Num. Anal. (2021)

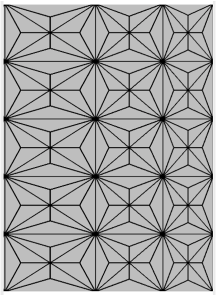
Pointwise divergence free H^1-conforming elements I

- We are interested in velocity pressure pairs such that

$$\mathbf{V}_h \in [H^1(\Omega)]^d \quad \text{and } \nabla \cdot \mathbf{V}_h \in Q_h$$

- Examples:
  - High order: Scott-Vogelius elements
  - Affine: Christensen-Hu element, Num. Math., 2018.
  - Affine: recent work by Fabien, Guzmán, Neilan, Zytoot, arXiv:2105.09214
- Computations by Lube and Schroeder, (left plot below)





Pointwise divergence free H^1-conforming elements II

- Let $\mathbf{V}_h^0 := \{\mathbf{v}_h \in \mathbf{V}_h : \nabla \cdot \mathbf{v}_h = 0\}$
- Discretization of the model problem: find $\mathbf{u}_h \in \mathbf{V}_h^0$ such that

$$(\sigma \mathbf{u}_h, \mathbf{v}_h)_\Omega + (\boldsymbol{\beta} \cdot \nabla \mathbf{u}_h, \mathbf{v}_h)_\Omega + s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0$$

- Can we find $s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h)$, such that the $h^{k+\frac{1}{2}}$ error bound holds?
- affine elements: linearized Smagorinsky + gradient jump penalty yields $O(h^{\frac{3}{2}})$

$$s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) := (\tau_{\mathbf{u}} h^2 |\nabla \boldsymbol{\beta}| \nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_\Omega + \sum_K \int_{\partial K} h^2 |\boldsymbol{\beta}|^{-1} [\![(\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h \times \mathbf{n}]\!] \cdot [\![(\boldsymbol{\beta} \cdot \nabla)\mathbf{v}_h \times \mathbf{n}]\!]$$

- In the affine case the bulk term can be omitted for the linear model problem!

Pointwise divergence free H^1-conforming elements III

- What about high order elements?
- Let $|||\mathbf{v}_h|||^2 = \|\sigma^{\frac{1}{2}}\mathbf{v}_h\|_\Omega^2 + s_{\mathbf{u}}(\mathbf{v}_h,\mathbf{v}_h)$. Then, with π_{div} L^2-projection onto $\mathbf{V}_h^0$,

$$|||\underbrace{\mathbf{u}_h - \pi_{div}\mathbf{u}}_{\mathbf{e}_h}|||^2 \le |||\mathbf{e}_h|||\;|||\mathbf{u}-\pi_{div}\mathbf{u}||| + |(\mathbf{u}-\pi_{div}\mathbf{u}, \underbrace{\sigma\mathbf{e}_h + \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h}_{\mathcal{L}\mathbf{e}_h})_\Omega|$$

- Observe that GaLS type stabilizations require ∇p_h in residual for consistency.
- Introduce vector potential $\boldsymbol{\Theta}$ such that $\nabla\times\boldsymbol{\Theta} = \mathbf{u}-\pi_{div}\mathbf{u}$, then

$$\begin{aligned}|(\mathbf{u}-\pi_{div}\mathbf{u}_h, \mathcal{L}\mathbf{e}_h)_\Omega| &\le |(\boldsymbol{\Theta}, \nabla\times\mathcal{L}\mathbf{e}_h)_h| + |\langle\boldsymbol{\Theta}, [\![(\boldsymbol{\beta}\cdot\nabla)\mathbf{u}_h\times\mathbf{n}]\!]\rangle_h| \\ &\le \underbrace{\|h^{-3/2}\boldsymbol{\Theta}\|_\Omega}_{O(h^{k+\frac{1}{2}})}\underbrace{\|h^{\frac{3}{2}}\nabla\times\mathcal{L}\mathbf{e}_h\|_h}_{s_{\mathbf{u}}(\mathbf{e}_h,\mathbf{e}_h)^{\frac{1}{2}}} \\ &+ \underbrace{\left(\sum_{K\in\mathcal{T}}\|h^{-1}\boldsymbol{\Theta}\|_{\partial K}^2\right)^{\frac{1}{2}}}_{O(h^{k+\frac{1}{2}})}\underbrace{\left(\sum_{K\in\mathcal{T}}\|h[\![(\boldsymbol{\beta}\cdot\nabla)\mathbf{e}_h]\!]\times\mathbf{n}\|_{\partial K}^2\right)^{\frac{1}{2}}}_{s_{\mathbf{u}}(\mathbf{e}_h,\mathbf{e}_h)^{\frac{1}{2}}}\end{aligned}$$

Pointwise divergence free H^1-conforming elements IV

- After analysis of the approximation properties of $\boldsymbol{\Theta}$ we get the stabilisation:

$$s_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) := \left\langle h^2 |\boldsymbol{\beta}|^{-1} [\![(\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h \times \mathbf{n}]\!] \cdot [\![(\boldsymbol{\beta} \cdot \nabla)\mathbf{v}_h \times \mathbf{n}]\!] \right\rangle_h + (h^3 \nabla \times \mathcal{L}\mathbf{u}_h, \nabla \times \mathcal{L}\mathbf{v}_h)_h$$

- $s_{\mathbf{u}}$ consists of a partial GJP and a GaLS stabilization on the vorticity.
- The bulk term is of higher order, probably negligible for smooth solutions.
- A priori error estimate[1] for smooth solutions:

$$\boxed{\|\sigma^{\frac{1}{2}}(\mathbf{u} - \mathbf{u}_h)\|_\Omega + s_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h)^{\frac{1}{2}} \lesssim C h^{k+\frac{1}{2}} |\mathbf{u}|_{H^{k+1}(\Omega)}.}$$

- Unfortunately adding the bulk term results in a very ill-conditioned linear system[2]

[1] Ahmed, Barrenechea, EB, Guzmán, Linke, Merdon, arXiv:2007.04012, (Sinum, to appear) (2020)

[2] Farrell, private communication, (2020)

Part I: conclusions

- The CIP/GJP method is an excellent stabiliser for uDNS
- The $O(h^{k+\frac{1}{2}})$ bound is an interesting proxy for uDNS performance
- New stabilisers for pointwise divergence free elements with $O(h^{k+\frac{1}{2}})$ estimates
- Major hurdle for theoretical understanding:

No useful stability concept in the turbulent regime

- Different quantities have different stability, some are hopefully computable
- Can we find an example of problems where such non-standard stability applies and can be used in numerical analysis?

Yes! In (deterministic) variational data assimilation

- Part II: Variational data assimilation for incompressible flow (topic for talk at Chemnitz Finite Element Symposium 2021)