Space-time discontinuous Galerkin methods for the wave equation

Ilaria Perugia
University of Vienna, Austria

Joint work with
Andrea Moiola (Uni Pavia)
Paul Stocker, Joachim Schöberl, Christoph Wintersteiger (Uni/TU Vienna)
Pratyuksh Bansal, Christoph Schwab (ETH Zürich)

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why space-time? (instead of space discretization & time stepping)

- *high-order* approximation in both space and time is simple to obtain
- *spectral convergence* of the space-time error can be obtained by p-refinement
- stability is achieved under a *local* CFL condition
- the numerical solution is available at *all times* in $(0, T)$

drawback: high complexity

time dipendent problem in d space dimensions $\rightarrow$ $(d+1)$-dimensional problem

- model problem: the acoustic wave equation
- space-time discontinuous Galerkin (DG) discretization
- reduction of the complexity:
  - Trefftz basis functions + tent pitching [1], [2]
  - tensor-product (in time) elements and combination formula [3]

[1] A. Moiola, I. Perugia, A space-time Trefftz discontinuous Galerkin method for the acoustic wave equation in first-order formulation, *Num. Math.*, 139 (2018), 389-435.

[2] I. Perugia, J. Schöberl, P. Stocker, C. Wintersteiger, Tent pitching and Trefftz-DG method for the acoustic wave equation, *Comput. Math. with Appl.*, 70 (2020), 2987-3000.

[3] P. Bansal, A. Moiola, I. Perugia, C. Schwab, Space-time discontinuous Galerkin approximation of acoustic waves with point singularities, *IMA J. Numer. Anal.*, online.

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the acoustic wave problem as a 1st order system

$Q = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^d$ Lipschitz, bounded polygon/polyhedron

$c = c(\mathbf{x})$ piecewise constant on a fixed, finite polygonal/polyhedral partition $\{\Omega_i\}$ of Ω

$f \in L^2(Q)$, $v_0 \in L^2(\Omega)$, $\boldsymbol{\sigma}_0 \in L^2(\Omega)^d$

$$\begin{cases} \text{find } (v, \boldsymbol{\sigma}) \text{ such that} & \\ \nabla v + \dfrac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\sigma} + c^{-2} \dfrac{\partial v}{\partial t} = f & \text{in } Q \\ v(\cdot, 0) = v_0, \quad \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \times [0,T] \end{cases}$$

2nd order wave equation (provided that σ_0 is a gradient)

$v = \partial_t U$, $\sigma = -\nabla U$ → $-\Delta U + c^{-2}\partial_{tt} U = f$ in Q + initial/boundary conditions

$$U \in C^0\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big) \cap H^2(0,T; H^{-1}(\Omega))$$

[Dautray, Lions, 1992]

space-time finite element methods for wave problems

- early works (FEM): [Hughes, Hulbert, 1988, 1990], [French, 1993], [Johnson, 1993], . . .
- DG: [Falk, Richter, 1999], [Yin, Acharya, Sobh, Haber, Tortorelli, 2000], [Monk, Richter, 2005], [Costanzo, Huang, 2005], [Abedi, Petracovici, Haber, 2006], [van der Vegt, 2006], [Feistauer, Hájek, Švadlenka, 2007], . . . , [Gopalakrishnan, Monk, Sepúlveda, 2015], [Dörfler, Findeisen, Wieners, 2016], [Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019], . . .
- Trefftz: [Maciąg, Wauer, Sokala, 2005–2011], [Liu, Kuo, 2016], [Petersen, Farhat, Tezaur, 2009], [Wang, Tezaur, Farhat, 2014] [Egger, Kretzschmar, Schnepp, Tzukermann, Weiland, 2014, 2015], [Banjai, Georgoulis, Lijoka, 2017], [Barucq, Calandra, Diaz, Shishenina, 2018, 2020], [1], [2]
- recent, on tensor-product meshes: [Steinbach, Zank, 2019], [Ernesti, Wieners, 2019], [3]

Space-time variational formulation

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$$\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\sigma} + c^{-2}\frac{\partial v}{\partial t} = f \qquad \text{in } Q \qquad \boxed{\mathcal{L}_{\mathsf{wave}}(v, \boldsymbol{\sigma}) = (f, \mathbf{0})}$$

multiply by test functions $\boldsymbol{\tau}$ and w, respectively, and integrate by parts in $Q = \Omega \times (0, T)$:

$$\int_Q \left(\nabla v + \frac{\partial \sigma}{\partial t}\right) \cdot \tau \, dV + \int_Q \left(\nabla \cdot \sigma + c^{-2} \frac{\partial v}{\partial t}\right) w \, dV = \int_Q f \, w \, dV$$

space-time variational formulation

$$-\int_Q \left[v\left(\nabla \cdot \boldsymbol{\tau} + c^{-2}\frac{\partial w}{\partial t}\right) + \boldsymbol{\sigma} \cdot \left(\nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t}\right)\right] \mathrm{d}V + \int_{\Omega \times \{T\}} (\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + c^{-2} v \, w) \, \mathrm{d}\mathbf{x}$$
$$= \int_Q f \, w \, \mathrm{d}V + \int_{\Omega \times \{0\}} (\boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau} + c^{-2} v_0 w) \, \mathrm{d}\mathbf{x}$$

- $\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}$ holds in $C^0\big([0, T]; H_0(\mathrm{div}; \Omega)^*\big)$
- $\nabla \cdot \boldsymbol{\sigma} + c^{-2}\frac{\partial v}{\partial t} = f$ holds in $L^2\big(0, T; H^{-1}(\Omega)\big)$
- $v = 0$ on $\partial\Omega \times [0, T]$ is imposed weakly (details in [3])

Space-time discontinuous Galerkin discretization

$$\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\sigma} + c^{-2}\frac{\partial v}{\partial t} = f \qquad \text{in } Q = \Omega \times (0,T)$$

- introduce a polytopic space-time mesh $\mathcal{T}_h = \{K\}$ of Q, with c constant in each element
- multiply by test functions and integrate by parts element by element
- discretize $(v, \boldsymbol{\sigma})$ and $(w, \boldsymbol{\tau})$ in discontinuous, piecewise polynomial spaces $\mathbf{V}_p(\mathcal{T}_h)$
- replace interelement traces by numerical fluxes

elemental DG formulation

$$-\int_K \left[v_h \left(\nabla \cdot \boldsymbol{\tau}_h + c^{-2}\frac{\partial w_h}{\partial t}\right) + \boldsymbol{\sigma}_h \cdot \left(\nabla w_h + \frac{\partial \boldsymbol{\tau}_h}{\partial t}\right)\right] \mathrm{d}V$$

$$+ \int_{\partial K} \left[(\widehat{v}_h \, \boldsymbol{\tau}_h + \widehat{\boldsymbol{\sigma}}_h \, w_h) \cdot \mathbf{n}_K^{\mathbf{x}} + \left(\widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\tau}_h + c^{-2}\, \widehat{v}_h \, w_h \right) n_K^t \right] \mathrm{d}S = \int_K f \, w_h \, \mathrm{d}V$$

where $(\mathbf{n}_K^{\mathbf{x}}, n_K^t) \in \mathbb{R}^{d+1}$ denotes the unit normal vector to ∂K pointing outside K

global DG formulation

add over all $K \in \mathcal{T}_h \;\; \rightarrow \qquad \mathcal{A}_{\mathsf{DG}}(v_h, \boldsymbol{\sigma}_h; w_h, \boldsymbol{\tau}_h) = \ell_{\mathsf{DG}}(w_h, \boldsymbol{\tau}_h)$

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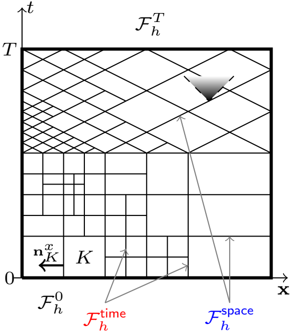
assumption on $\mathcal{T}_h$

each internal face F is either

- space-like: $c\,|\mathbf{n}_F^{\mathbf{x}}| < n_F^t$ $(F \subset \mathcal{F}_h^{\text{space}})$, or
- time-like: $n_F^t = 0$ $(F \subset \mathcal{F}_h^{\text{time}})$

$$\mathcal{F}_h^0 := \Omega \times \{0\}, \quad \mathcal{F}_h^T := \Omega \times \{T\}$$
$$\mathcal{F}_h^\partial := \partial\Omega \times (0, T)$$



assumptions on the numerical fluxes

$$\widehat{v}_h := \begin{cases} v_h^- \\ \{\!\{v_h\}\!\} + \beta\, [\![\boldsymbol{\sigma}_h]\!]_{\mathbf{N}} \\ v_0 \\ 0 \end{cases} \qquad \widehat{\boldsymbol{\sigma}}_h := \begin{cases} \boldsymbol{\sigma}_h^- & \text{on } \mathcal{F}_h^{\text{space}} \cup \mathcal{F}_h^T & \text{(upwind fluxes)} \\ \{\!\{\boldsymbol{\sigma}_h\}\!\} + \alpha\, [\![v_h]\!]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\text{time}} & \text{(DG-elliptic fluxes)} \\ \boldsymbol{\sigma}_0 & \text{on } \mathcal{F}_h^0 \\ \boldsymbol{\sigma}_h - \alpha\, v\mathbf{n}_\Omega^{\mathbf{x}} & \text{on } \mathcal{F}_h^\partial \end{cases}$$

$\alpha, \beta \in L^\infty(\mathcal{F}_h^{\text{time}} \cup \mathcal{F}_h^\partial)$; $\quad \alpha = \beta = 0$ [Egger & al., 2014] , $\quad \alpha\beta \geq \frac{1}{4}$ [Monk, Richter, 2005] recall the definition of the wave operator $\mathcal{L}_{\mathsf{wave}}(w,\boldsymbol{\tau}) := \left(\nabla\cdot\boldsymbol{\tau} + c^{-2}\frac{\partial w}{\partial t}\,,\ \nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t}\right)$

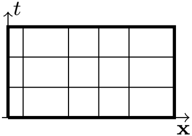
assumption on $\mathbf{V}_p(\mathcal{T}_h)$

$$\text{for all } (w_h,\boldsymbol{\tau}_h)\in\mathbf{V}_p(\mathcal{T}_h),\quad \mathcal{L}_{\mathsf{wave}}(w_h,\boldsymbol{\tau}_h)\in\mathbf{V}_p(\mathcal{T}_h)$$

this is satisfied, e.g., if the restriction of $\mathbf{V}_p(\mathcal{T}_h)$ to each mesh element is made of

- total degree space-time polynomials $\mathbb{P}^p_{\mathbf{x},t}$,
- tensor product (in time) polynomials $\mathbb{P}^p_{\mathbf{x}}\times\mathbb{P}^p_t$,
- Trefftz polynomials $\mathcal{L}_{\mathsf{wave}}(w_h,\boldsymbol{\tau}_h) = (0,\mathbf{0})$

- case of tensor product (in time) meshes*



key property: coercivity in seminorm

$$\mathcal{A}_{\mathsf{DG}}(v_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{\sigma}_h) = |(v_h, \boldsymbol{\sigma}_h)|^2_{\mathsf{DG}}$$

DG seminorm

$$|(w, \boldsymbol{\tau})|^2_{\mathsf{DG}} = \frac{1}{2}\left\|c^{-1}[\![w]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\mathsf{space}})} + \frac{1}{2}\left\|[\![\boldsymbol{\tau}]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\mathsf{space}})^d} + \left\|\alpha^{\frac{1}{2}}[\![w]\!]_{\mathbf{N}}\right\|^2_{L^2(\mathcal{F}_h^{\mathsf{time}})^d} + \left\|\beta^{\frac{1}{2}}[\![\boldsymbol{\tau}]\!]_{\mathbf{N}}\right\|^2_{L^2(\mathcal{F}_h^{\mathsf{time}})}$$
$$+ \frac{1}{2}\left\|c^{-1}w\right\|^2_{L^2(\mathcal{F}_h^0 \cup \mathcal{F}_h^T)} + \frac{1}{2}\left\|\boldsymbol{\tau}\right\|^2_{L^2(\mathcal{F}_h^0 \cup \mathcal{F}_h^T)^d} + \left\|\alpha^{\frac{1}{2}}w\right\|^2_{L^2(\mathcal{F}_h^{\partial})}$$

by adapting [Monk, Richter, 2005], one deduces well-posedness, with no condition on h_t

*The case of general, admissible meshes requires minor, technical changes.

Well-posedness

$$\mathcal{A}_{DG}(v_h, \sigma_h; w_h, \tau_h) \overset{(*)}{=} 0 \quad \forall (w_h, \tau_h) \in V_p(\mathcal{T}_h) \quad \Rightarrow (v_h, \sigma_h) = (0,0)$$

i) $(w_h, \tau_h) = (v_h, \sigma_h) \rightarrow 0 = \mathcal{A}_{DG}(v_h, \sigma_h; v_h, \sigma_h) = |(v_h, \sigma_h)|^2_{DG}$

$\Rightarrow$ all jumps and boundary traces of v_h and σ_h are zero

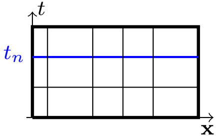
ii) due to i), $(*)$ gives that, $\forall (w_h, \tau_h) \in V_p(\mathcal{T}_h)$,

$$\int_K \left(\Big(\underbrace{\nabla \cdot \sigma_h + c^{-2} \frac{\partial v_h}{\partial t}} \Big) w_h + \Big(\underbrace{\nabla v_h + \frac{\partial \sigma_h}{\partial t}} \Big) \cdot \tau_h \right) dV = 0$$

iii) take $w_h = \nearrow$ and $\tau_h = \nearrow$ and deduce that (v_h, σ_h) solves the homogeneous wave equation in each K

iv) From i) and iii), deduce that $(v_h, \sigma_h) = (0,0)$.

- case of tensor product (in time) meshes



error estimates (with no condition on h_t)

assume that all the traces of the analytical solution on mesh faces are in L^2
$\rightarrow$ error bound in the L^2 norm in space at every discrete time t_n:

$$\left\|c^{-1}(v-v_h)\right\|_{L^2(\Omega\times\{t_n\})} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega\times\{t_n\})^d} \leq |(v,\boldsymbol{\sigma})-(v_h,\boldsymbol{\sigma}_h)|_{\mathsf{DG}(Q_n)}$$

$$\lesssim |(v,\boldsymbol{\sigma}) - \underbrace{\Pi(v,\boldsymbol{\sigma})}_{\in \mathbf{V}_p(\mathcal{T}_h)}|_{\mathsf{DG}+}$$

(proven by restricting to partial space-time cylinders $Q_n = \Omega \times (0, t_n)$)

projector Π:

- total degree space-time polynomials $\mathbb{P}^p_{\mathbf{x},t}$: construction in [Monk, Richter, 2005]
- tensor product (in time) polynomials $\mathbb{P}^p_{\mathbf{x}} \times \mathbb{P}^p_t$: L^2 projection [3]
- Trefftz polynomials: best approximation [1]

recall: $$\mathcal{L}_{\mathsf{wave}}(w, \boldsymbol{\tau}) = \left(\nabla \cdot \boldsymbol{\tau} + c^{-2} \frac{\partial w}{\partial t}\,,\ \nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} \right)$$

Trefftz spaces

continuous spaces $\mathbf{T}(K) := \{(w, \boldsymbol{\tau}) \in H^1(K) \colon \mathcal{L}_{\mathsf{wave}}(w, \boldsymbol{\tau}) = (0, \mathbf{0})\}$

$\mathbf{T}(\mathcal{T}_h) := \left\{(w, \boldsymbol{\tau}) \in H^1(\mathcal{T}_h)^{1+d} \colon (w, \boldsymbol{\tau})_{|_K} \in \mathbf{T}(K) \quad \forall K \in \mathcal{T}_h \right\}$

discrete spaces $\mathbf{V}_p(K) \subset \mathbf{T}(K),\ \ \mathbf{V}_p(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$

in each element K, the *linear* operator $\mathcal{L}_{\mathsf{wave}}$ is

- homogeneous (= all terms are derivatives of the *same* order)
- with constant coefficients

$\Rightarrow$ Taylor polynomials of (smooth) functions in $\ker(\mathcal{L}_{\mathsf{wave}})$ are in $\ker(\mathcal{L}_{\mathsf{wave}})$

therefore, we can choose $\mathbf{V}_p(K) \subset \mathbf{T}(K)$

- as a subspace of the polynomial space $\mathbb{P}^p(K)^{1+d}$
- with the same order of approximation in h as $\mathbb{P}^p(K)^{1+d}$ for functions in $\ker(\mathcal{L}_{\mathsf{wave}})$

Example:

$$\mathbb{T}^p(K) := \left\{ u \in \mathbb{P}^p(K) : \; -\Delta u + c^{-2} \frac{\partial^2 u}{\partial t^2} = 0 \right\} \qquad \mathbf{V}_p(K) := \left(\frac{\partial \mathbb{T}^{p+1}(K)}{\partial t}, -\nabla(\mathbb{T}^{p+1}(K)) \right)$$

- reduction of number of degrees of freedom to that of a d-dimensional problem

	Trefftz polyn. $\mathbb{T}^p(K)$	full polyn. $\mathbb{P}^p(K)$
$d+1=1+1$	$2p+1$	$\frac{1}{2}(p+1)(p+2)$
$d+1=2+1$	$(p+1)^2$	$\frac{1}{6}(p+1)(p+2)(p+3)$
$d+1=3+1$	$\frac{1}{6}(p+1)(p+2)(2p+3)$	$\frac{1}{24}(p+1)(p+2)(p+3)(p+4)$
	$\mathcal{O}(p^d)$	$\mathcal{O}(p^{d+1})$

$$\dim(\mathbb{T}^p(K)) = \mathcal{O}(p^d) \;\ll\; \dim(\mathbb{P}^p(K)) = \mathcal{O}(p^{d+1})$$

- same orders of approximation in h as with the full polynomial spaces

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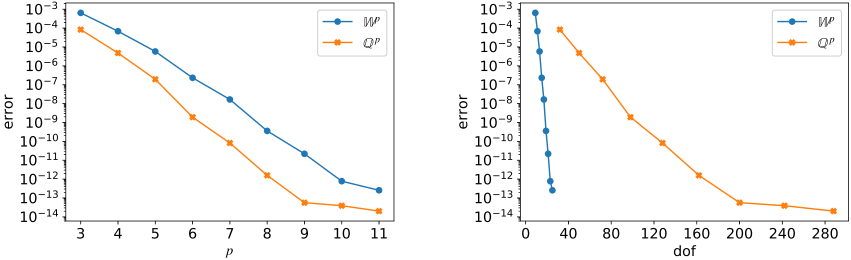
[2] P., Schöberl, Stocker, Wintersteiger, 2020

$d = 1$, smooth solution, Cartesian mesh; Trefftz (blue) and $\mathbb{Q}^p$ (orange) polynomials



p-version: error (in $L^2(\Omega \times \{T\})$) vs. polynomial degree (left) and number of dof.s (right)

[1] Moiola, P., 2018

- in $\mathbf{T}(\mathcal{T}_h)$, the DG seminorm is actually a norm
- existence and uniqueness of solutions follow from $\mathcal{A}_{\mathsf{DG}}(v_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{\sigma}_h) = |(v_h, \boldsymbol{\sigma}_h)|^2_{\mathsf{DG}}$
- error bounds in the (spatial) L^2 norm on space-like interfaces (e.g. on $\Omega \times \{t_n\}$) and in DG norm also follow
- error bounds in a global, mesh-independent norm ($L^2(Q)$, in the best case scenario*) have also been proven in [1] by a modified duality argument from [Monk, Wang, 1999]

piecewise smooth coefficients: space-time quasi-Trefftz DG method

[Imbert-Gérard, Moiola, Stocker, 2020]

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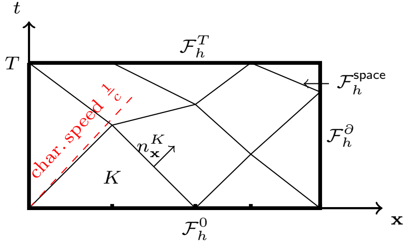
*i.e. for $d = 1$ or $d > 1$ and no time-like faces (for impedance b.c.) ;
in $H^{-1}(0, T; L^2(\Omega)) \times L^2(0, T; H^{-1}(\Omega)^d)$ for tensor product elements (with Dirichlet b.c.)

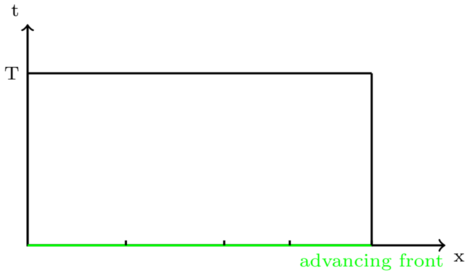
PDE-driven, front-advancing mesh construction technique



- progressively advancing in time and stacking tent-pitched objects on top of each other
- each tent is union of $(d+1)$-dimensional simplices
- the high of each tent (local advancement in time) is chosen so that the casuality constraint of the PDE is respected (*local* CFL condition)

→ the PDE is explicitly solvable within each tent

[Falk, Richter, 1999], [Yin, Acharya, Sobh, Haber, Tortorelli, 2000] [Üngör, Sheffer, 2002], [Monk, Richter, 2005], [Abedi, Petracovici, Haber, 2006], . . . ,
[Gopalakrishnan, Monk, Sepúlveda, 2015], [Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019]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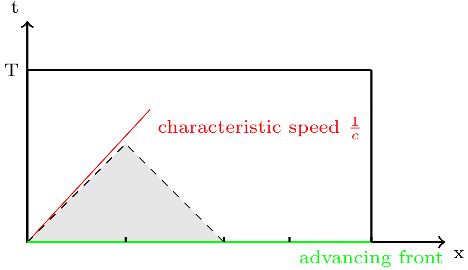
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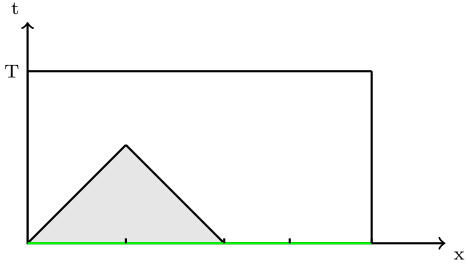
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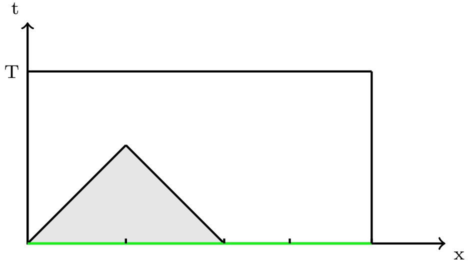


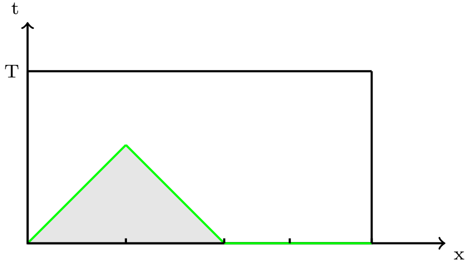
solution at the tent bottom $\rightarrow$ solution at the tent top

- Trefftz (no volume terms): solution of local problems [1], [2]; for an interior tent:

$$\int_{\partial K^{\mathrm{top}}} \Big((v_h\, \boldsymbol{\tau}_h + \boldsymbol{\sigma}_h\, w_h) \cdot \mathbf{n}_K^{\mathbf{x}} + \big(\boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h + c^{-2}\, v_h\, w_h \big)\, n_K^t \Big)\, \mathrm{d}S$$

$$= - \int_{\partial K^{\mathrm{bot}}} \Big((v_h^{\mathrm{bot}}\, \boldsymbol{\tau}_h + \boldsymbol{\sigma}_h^{\mathrm{bot}}\, w_h) \cdot \mathbf{n}_K^{\mathbf{x}} + \big(\boldsymbol{\sigma}_h^{\mathrm{bot}} \cdot \boldsymbol{\tau}_h + c^{-2}\, v_h^{\mathrm{bot}}\, w_h \big)\, n_K^t \Big)\, \mathrm{d}S$$

- mapping + RK or Taylor [Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019]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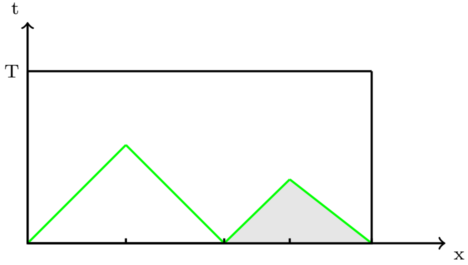
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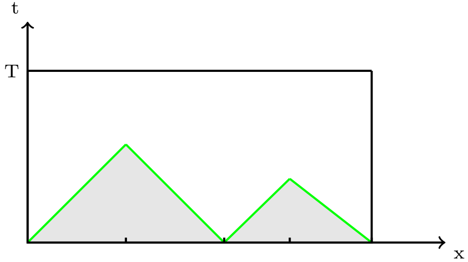
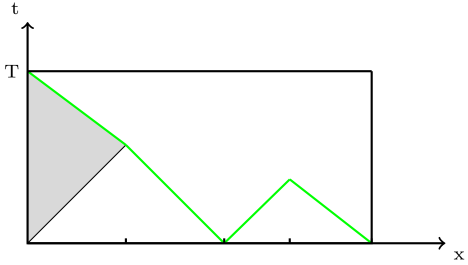


the solution within these two tents can be evolved in parallel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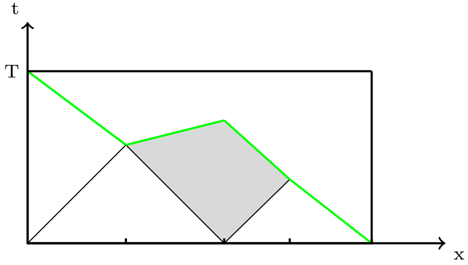
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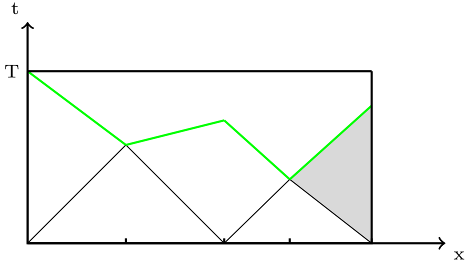
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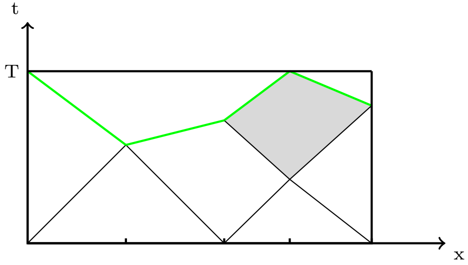
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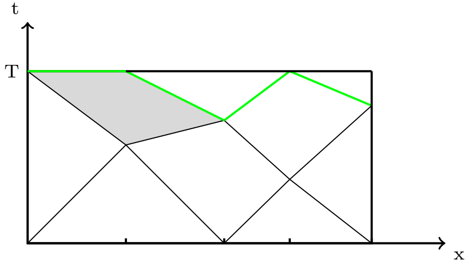
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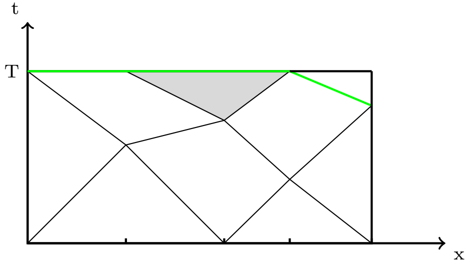
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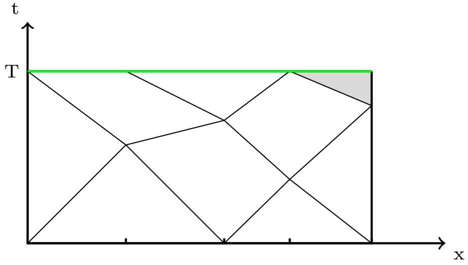
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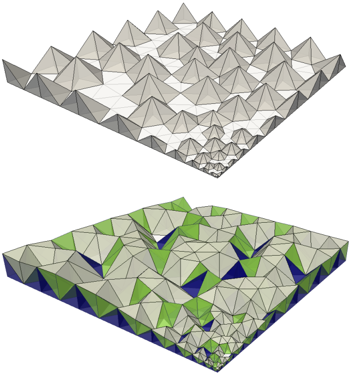
$d = 2$ and refined mesh towards a corner



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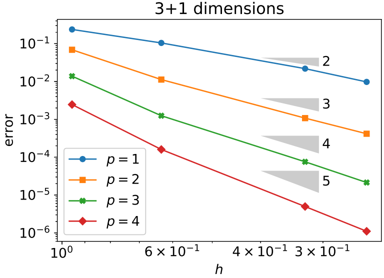
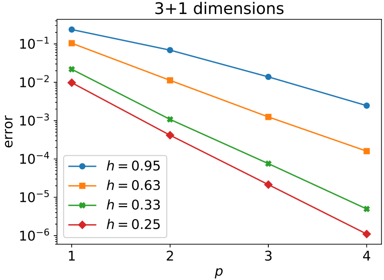
[2] P., Schöberl, Stocker, Wintersteiger, 2020

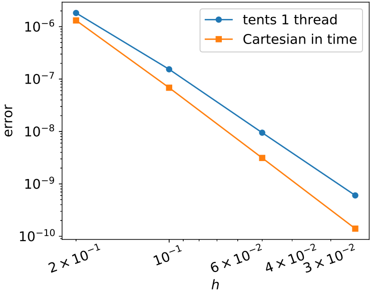
$d = 3$, smooth solution, Trefftz on tent-pitched meshes; h- and p-version

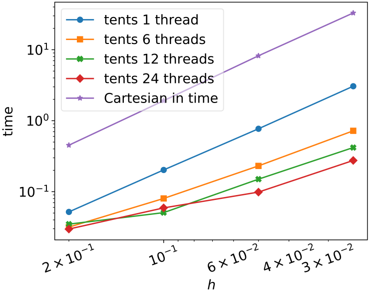




convergence of order $p + 1$ in h (left) and exponential convergence in p (right)

$d = 2$, smooth solution, Trefftz; tensor product (in time) meshes and tent-pitching

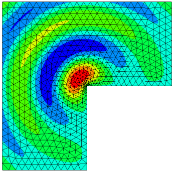


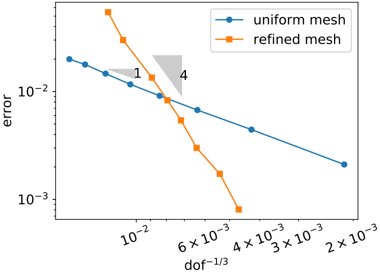


$d = 2$, singular solution, Trefftz on tent-pitched meshes

$$U(r, \varphi, t) = \cos(10\,t)\sin(\nu\,\varphi)J_\nu(10\,r)$$

$$\nu = \frac{2}{3} \quad \rightarrow \quad U \in H^{\frac{5}{3}-\varepsilon}(Q)$$





$p = 3$; spatial mesh at $t = 0$: uniform (blue) or with corner refinement (orange)

Regularity theory in 2D [Kokotov, Plamenevskii, 1999, 2004], [Luong, Tung, 2015]

$\mathcal{S} := \{\mathbf{c}_j,\ j = 1, \dots, M\}$ set of all vertices of $\{\Omega_i\}$, in which c is piecewise constant

acoustic waves exhibit conical singularities at $\mathcal{S}$: regularity results are given in weighted Sobolev spaces in Ω with weight function

$$\Phi_{\boldsymbol{\delta}}(\mathbf{x}) = \Pi_{j=1}^{M} |\mathbf{x} - \mathbf{c}_j|^{\delta_j}, \quad \delta_j \in [0,1)\ ^*$$

e.g. $|u|_{H^{1,1}_{\boldsymbol{\delta}}(\Omega)} := \|\Phi_{\boldsymbol{\delta}} \nabla u\|_{L^2(\Omega)^2}$ $\qquad (H^{1,1}_{\boldsymbol{\delta}}(\Omega) \not\subset H^1(\Omega))$

(used for the analysis of DG + time-stepping [Müller, Schötzau, Schwab, 2018]).

Example: if $v_0, u_0 \in C_0^\infty(\Omega)$, $\boldsymbol{\sigma}_0 = -\nabla u_0$, $f \in C_0^\infty(Q)$, $\exists\, \boldsymbol{\delta} \in [0,1)^M$ such that $\forall\, k_t, k_{\mathbf{x}} \in \mathbb{N}$,

$$v \in C^{k_t - 1}([0,T]; H^{k_{\mathbf{x}}+1,2}_{\boldsymbol{\delta}}(\Omega)) \qquad \boldsymbol{\sigma} \in C^{k_t}([0,T]; H^{k_{\mathbf{x}},1}_{\boldsymbol{\delta}}(\Omega))^2$$

[Müller, 2017]

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* $\|u\|^2_{H^{k,\ell}_{\boldsymbol{\delta}}(\Omega)} := \|u\|^2_{H^{\ell-1}(\Omega)} + |u|^2_{H^{k,\ell}_{\boldsymbol{\delta}}(\Omega)}$, $\quad |u|^2_{H^{k,\ell}_{\boldsymbol{\delta}}(\Omega)} := \sum_{m=\ell}^{k} \int_\Omega \Big(\Phi_{\boldsymbol{\delta}+m-\ell} \sum_{|\boldsymbol{\alpha}|=m} |D^{\boldsymbol{\alpha}} u|^2 \Big) d\mathbf{x}.$

[3] Bansal, Moiola, P., Schwab, 2020

tensor product (in time) space-time meshes

- time mesh: $\mathcal{T}^t_{h_t}$ partition of $(0,T)$ into N intervals I_n
- spatial meshes: for each $1 \le n \le N$, $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x},n}}$ shape-regular mesh of Ω
  - with non-degenerating faces
  - aligned with $\{\Omega_i\}$
  - each mesh element touches at most one element of $\mathcal{S}$
- space-time mesh: $\mathcal{T}_h := \mathcal{T}_h(Q) := \{K = K_{\mathbf{x}} \times I_n : \; K_{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x},n}}, \; 1 \le n \le N\}$

abstract error analysis (Galerkin error $\lesssim L^2$ projection error)

the critical solution regularity is the regularity in space of $\boldsymbol{\sigma}$ close to any $\mathbf{c} \in \mathcal{S}$:

if F is a time-like face of an element K adjacent to a corner $\mathbf{c}$,
then $\boldsymbol{\sigma}_{|_F} \in L^1(F)^2$, not necessarily $L^2(F)^2$

$\rightarrow$ modify the DG seminorm and apply Hölder in L^1-L^∞ (instead of Cauchy-Schwarz)
[Wihler, 2002]

Tensor product meshes

mesh grading in space (like in the elliptic case)

lack of smoothness → loss in the accuracy of the L^2 projection of the solution in the elements $K = K_{\mathbf{x}} \times I_n$ that are close to any $\mathbf{c} \in \mathcal{S}$

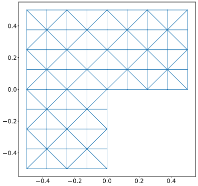
a reduction of the size of $K_{\mathbf{x}}$ depending on

- the corner weight $\delta_{\mathbf{c}}$
- the polynomial approximation degree p_K

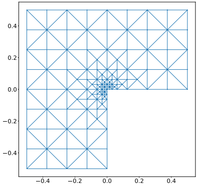
can restore the largest possible convergence rates

suitable graded spatial meshes $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ can be constructed from a quasi-uniform initial mesh $\mathcal{T}^{\mathbf{x}}_0$ of Ω of size $h_{\mathbf{x}}$ by J levels of local bisection refinement ($J = J(h_{\mathbf{x}}, \delta_{\mathbf{c}}, p_K)$)

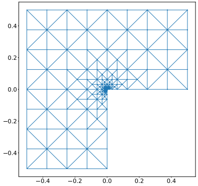
[Gaspoz, Morin, 2009]



uniform, $h_{\mathbf{x}} = 0.25$



graded, $p = 1$



graded, $p = 2$

assume, for simplicity, constant c, uniform p

- fix $h_t, h_{\mathbf{x}} > 0$, and construct the uniform mesh $\mathcal{T}^t_{h_t}$ and the locally refined mesh $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$
- on any time-like face F, define the numerical flux parameters as $\alpha = \beta^{-1} = \dfrac{h_{\mathbf{x}}}{c\,|F_{\mathbf{x}}|}$
- assume that $c\,h_t \simeq h_{\mathbf{x}}$ ($h_{\mathbf{x}}$ is the size of *the largest* element of $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$)

error bounds

for every discrete time t_n, we have

$$\left\|c^{-1}(v - v_h)\right\|_{L^2(\Omega\times\{t_n\})} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega\times\{t_n\})^2} \leq |(v,\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)|_{\mathsf{DG}(Q_n)} \lesssim h^{p+\frac{1}{2}}$$

(same convergence rates as for smooth solutions)

Remark: $\dim(\mathbf{V}(\mathcal{T}_h)) = \mathcal{O}(h^{-3})$ (like for a $(2+1)$-dimensional elliptic problem)

Q: Can we obtain the same convergence rates with $\mathcal{O}(h^{-2})$ degrees of freedom?
(like for a 2-dimensional elliptic problem)

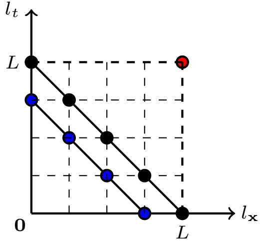
- the assumption $c\,h_t \simeq h_{\mathbf{x}}$ is necessary to obtain the highest convergence rates
- stability of the DG formulation and best approximation-type estimates are valid with no condition on h_t
- the solutions obtained with anysotropic (in time) space-time meshes are not accurate, still they contain meaningful information

$\mathcal{T}_{(0,0)}$ coarsest space-time mesh
$\mathcal{T}_{(L,L)}$ finest space-time mesh (red)
$\mathcal{T}_{(l_{\mathbf{x}},l_t)}$ intermediate meshes

$\mathbf{w}_{(l_{\mathbf{x}},l_t)} := (v_{(l_{\mathbf{x}},l_t)}, \boldsymbol{\sigma}_{(l_{\mathbf{x}},l_t)})$ solution on $\mathcal{T}_{(l_{\mathbf{x}},l_t)}$

$\mathbf{w}_F := \mathbf{w}_{(L,L)}$ full space-time solution

$$\mathbf{w}_S := \sum_{l=0}^{L} \mathbf{w}_{(l,L-l)} - \sum_{l=1}^{L} \mathbf{w}_{(l-1,L-l)} \quad \text{“sparse” solution}$$



[Bungartz, Griebel, 2004]

Count of degrees of freedom (h-version): # d.o.f.s for $\mathbf{w}_F \lesssim 2^{3L} = \mathcal{O}(h_L^{-3})$

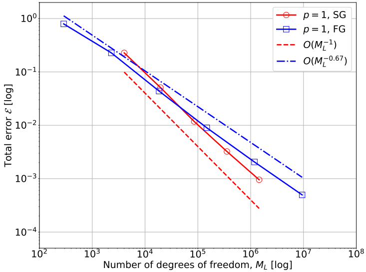
d.o.f.s for $\mathbf{w}_S \lesssim 2^{2L} = \mathcal{O}(h_L^{-2})$

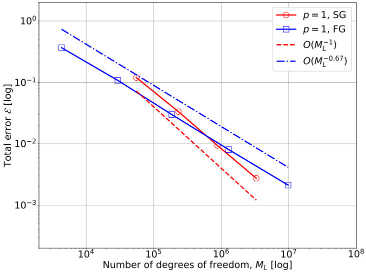
($\simeq$ one time-step on the finest spatial mesh)

FEniCS

[3] Bansal, Moiola, P., Schwab, 2020

Expected convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{2}}$





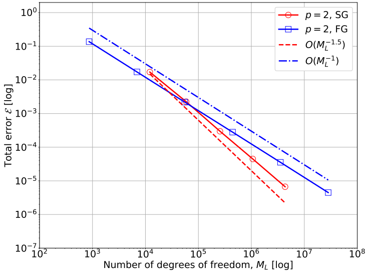
$p = 1$, full (blue), sparse (red)
smooth solution, uniform meshes (left), conical singularity, spatially graded meshes (right)

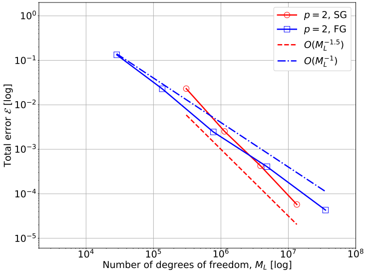
Obtained convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{2}}$

[3] Bansal, Moiola, P., Schwab, 2020

FEniCS

Expected convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1/2}{2}}$





$p = 2$, full (blue), sparse (red)
smooth solution, uniform meshes (left), conical singularity, spatially graded meshes (right)

Obtained convergence rates: full $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{3}}$, sparse $\mathcal{O}(\text{Ndofs})^{-\frac{p+1}{2}}$

Thank you for your attention!