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A Space-Time Discontinuous Petrov-Galerkin Method for the Heat Equation

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Space-time FEM

$Q = \mathcal{I} \times \Omega$ with $\mathcal{I} = (0, T)$ and $\Omega \subset \mathbb{R}^d$, $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$

$$\partial_t u - \Delta_x u = f \text{ in } Q \qquad u(0, \bullet) = u_0 \text{ in } \Omega \qquad u = 0 \text{ on } \mathcal{I} \times \partial\Omega$$

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Space-time FEM. Seek $u \in X$ and $u_h \in X_h \subset X$ with

$$b(u, y) = \langle f, y \rangle \qquad \text{for all } y \in Y \qquad \text{(P)}$$

$$b(u_h, y_h) = \langle f, y_h \rangle \qquad \text{for all } y_h \in Y_h \subset Y \qquad (\mathrm{P}_h)$$

Space-time	Time-stepping
One (parallelizable) large problem	Many small problems
Adaptivity in space-time	Adaptivity in space and/or time

Difficult Design

Suppose (P) is well-posed

$$0 < \beta := \inf_{x \in X} \sup_{y \in Y} \frac{b(x,y)}{\|x\|_X \|y\|_Y}$$

Difficulty. Find $X_h \subset X$ and $Y_h \subset Y$ with

$$0 < \beta_h := \inf_{x_h \in X_h} \sup_{y_h \in Y_h} \frac{b(x_h,y_h)}{\|x_h\|_X \|y_h\|_Y}$$

Equivalently, find bounded Fortin operator $\Pi : Y \to Y_h$ with

$$b(x_h, y - \Pi y) = 0 \qquad \text{for all } x_h \in X_h \text{ and } y \in Y$$

Example. Taylor–Hood for Stokes [Diening, Tscherpel, Storn '21]

Discontinuous Petrov–Galerkin Method

Suppose (P) is well-posed

$$0 < \beta := \inf_{x \in X} \sup_{y \in Y} \frac{b(x,y)}{\|x\|_X \|y\|_Y}$$

Difficulty. Find $X_h \subset X$ and $Y_h \subset Y$ with

$$0 < \beta_h := \inf_{x_h \in X_h} \sup_{y_h \in Y_h} \frac{b(x_h,y_h)}{\|x_h\|_X \|y_h\|_Y}$$

DPG. $T : X_h \to Y$ with $\langle Tx_h, y\rangle_Y = b(x_h, y)$ for all $x_h \in X_h$, $y \in Y$

$$\beta \leq \sup_{y \in Y} \frac{b(x_h,y)}{\|x_h\|_X \|y\|_Y} = \sup_{y \in Y} \frac{\langle Tx_h, y\rangle_Y}{\|x_h\|_X \|y\|_Y} = \frac{\langle Tx_h, Tx_h\rangle_Y}{\|x_h\|_X \|Tx_h\|_Y} = \sup_{y_h \in TX_h} \frac{b(x_h,y_h)}{\|x_h\|_X \|y_h\|_Y}$$

Practical DPG Method

- Idealized approach $Y_h^i := TX_h$

- Almost practical approach $Y_h^p := T_h X_h$ with
  $\langle T_h x_h, y_h \rangle_Y = b(x_h, y_h)$ for all $y_h \in Y_h$ and $\dim X_h \ll \dim Y_h$

- Practical approach $Y_h^p := T_h X_h$ and $Y_h \subset Y$ discontinuous

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- Practical approach $Y_h^p := T_h X_h$ and $Y_h \subset Y$ discontinuous

  $\longrightarrow$ Breaking spaces and forms
  [Carstensen, Demkowicz, Gopalakrishnan '16]
  [Demkowicz, Gopalakrishnan, Nagaraj, Sepúlveda '17]
  [Storn '20]

Heat Equation

$$\partial_t u - \Delta_x u = f \text{ in } Q \qquad u(0,\bullet) = u_0 \text{ in } \Omega \qquad u = 0 \text{ on } \mathcal{I} \times \partial\Omega$$

Equivalent system

$$\overbrace{\partial_t u + \operatorname{div}_x \sigma}^{=\operatorname{div}(u,\sigma)} = f \text{ in } Q \qquad u(0,\bullet) = u_0 \text{ in } \Omega$$

$$\nabla_x u + \sigma = 0 \text{ in } Q \qquad u = 0 \text{ on } \mathcal{I} \times \partial\Omega$$

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Notation $A := \begin{pmatrix} \partial_t & \operatorname{div}_x \\ \nabla_x & \operatorname{id} \end{pmatrix} \quad \underline{f} := \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \underline{u} := \begin{pmatrix} u \\ \sigma \end{pmatrix}$

$$A\underline{u} = \underline{f} \text{ in } Q$$

Function Spaces

Define space $H(A,Q) := \{\underline{v} \in L^2(Q;\mathbb{R}^{d+1}) \mid A\underline{v} \in L^2(Q;\mathbb{R}^{d+1})\}$ with norm

$$\|\underline{v}\|^2_{H(A,Q)} := \|\underline{v}\|^2_Q + \|A\underline{v}\|^2_Q \eqsim \|\underline{v}\|^2_Q + \|\operatorname{div}\underline{v}\|^2_Q + \|\nabla_x v\|^2_Q \quad \text{for } \underline{v} = (v,\tau)$$

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Define subspace

$$\begin{aligned} H_D(A,Q) &:= \{(v,\tau) \in L^2(\mathcal{I};H^1_0(\Omega)) \times L^2(Q;\mathbb{R}^d) \mid \operatorname{div}(v,\tau) \in L^2(Q)\} \\ &= \{(v,\tau) \in \underbrace{\left(L^2(\mathcal{I};H^1_0(\Omega)) \cap H^1(\mathcal{I};H^{-1}(\Omega))\right)}_{\hookrightarrow C(\overline{\mathcal{I}};L^2(\Omega))} \times L^2(Q;\mathbb{R}^d) \mid \operatorname{div}(v,\tau) \in L^2(Q)\} \end{aligned}$$

$$\|\partial_t v\|_{L^2H^{-1}} \leq \|\operatorname{div}\underline{v}\|_{L^2H^{-1}} + \|\operatorname{div}_x \tau\|_{L^2H^{-1}} \lesssim \|\operatorname{div}\underline{v}\|_Q + \|\tau\|_Q \lesssim \|\underline{v}\|_{H(A,Q)}$$

Variational Problem

$\gamma_0 \underline{v} := v(0,\bullet)$ and $\gamma_T \underline{v} := v(T,\bullet)$ for $\underline{v} = (v,\tau) \in H_D(A,Q)$

A^* adjoint operator of A

$H(A^*,Q) = H(A,Q)$

- Multiply $A\underline{u} = \underline{f}$ by $\underline{w} \in Y := H_D(A,Q) \cap \{\gamma_T = 0\} \subset H(A^*,Q)$
- Integrate over Q
- Integrate by parts

$$\langle \underline{u}, A^* \underline{w} \rangle_Q = \langle \underline{f}, \underline{w} \rangle_Q + \langle u_0, \gamma_0 \underline{w} \rangle_\Omega \qquad \text{for all } \underline{w} \in Y$$

Broken Variational Problem

A_h^* denotes element-wise (wrt. $\mathcal{T}$) application of A^*

- Multiply $A\underline{u} = \underline{f}$ and $\gamma_0 \underline{u} = u_0$ by $(\underline{w}, \xi) \in Y = H(A_h^*, Q) \times L^2(\Omega)$
- Integrate over Q and Ω
- "Integrate by parts" element-wise with trace $\underline{s} = \gamma_A \underline{u}$

$$\langle \underline{u}, A_h^* \underline{w} \rangle_Q + \underbrace{\langle A\underline{u}, \underline{w} \rangle_Q - \langle \underline{u}, A_h^* \underline{w} \rangle_Q + \langle \gamma_0 \underline{u}, \xi \rangle_\Omega}_{=:\langle \gamma_A \underline{u}, (\underline{w}, \xi) \rangle_{Y^*, Y} = \langle \underline{s}, (\underline{w}, \xi) \rangle_{Y^*, Y}} = \underbrace{\langle \underline{f}, \underline{w} \rangle_Q + \langle u_0, \xi \rangle_\Omega}_{=:F(\underline{w}, \xi)}$$

$X = L^2(Q; \mathbb{R}^{d+1}) \times \gamma_A H_D(A, Q)$ with $\gamma_A H_D(A, Q) \subset Y^*$

$b : X \times Y \to \mathbb{R}$ with $b(\underline{v}, \underline{t}; \underline{w}, \xi) = \langle \underline{v}, A_h^* \underline{w} \rangle_Q + \langle \underline{t}, (\underline{w}, \xi) \rangle_{Y^*, Y}$

Broken Variational Problem II

$$\|(\underline{w}, \xi)\|_Y^2 = \|\underline{w}\|_Q^2 + \|A_h^* \underline{w}\|_Q^2 + \|\xi\|_\Omega^2$$

$$\|(\underline{v}, \underline{t})\|_X^2 = \|\underline{v}\|_Q^2 + \|\underline{t}\|_{Y^*}^2 = \|\underline{v}\|_Q^2 + \min_{\substack{\underline{r} \in H_D(A,Q) \\ \gamma_A \underline{r} = \underline{t}}} \|\underline{r}\|_{H(A,Q)}^2 + \|\gamma_0 \underline{r}\|_\Omega^2$$

Theorem

It exists a mesh-independent constant

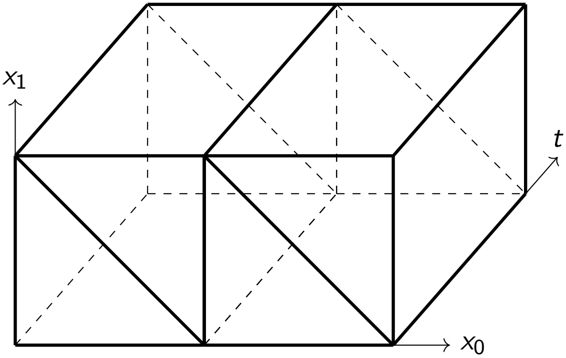
$$0 < \beta \leq \inf_{(\underline{v},\underline{t}) \in X} \sup_{(\underline{w},\xi) \in Y} \frac{b(\underline{v}, \underline{t}; \underline{w}, \xi)}{\|(\underline{v}, \underline{t})\|_X \|(\underline{w}, \xi)\|_Y}$$

For all $F \in Y^*$ *exists a unique* $(\underline{u}, \underline{s}) \in X$ *with* $\|(\underline{u}, \underline{s})\|_X \lesssim \|F\|_{Y^*}$ *and*

$$b(\underline{u}, \underline{s}; \underline{w}, \xi) = F(\underline{w}, \xi) \qquad \textit{for all } (\underline{w}, \xi) \in Y$$

Triangulation

- $\mathcal{T}$ is partition of Q in non-overlapping time-space cylinders $K = K_t \times K_x$ with interval K_t and simplex K_x
- $\mathcal{T}_0$ denotes the set of facets on $\{0\} \times \Omega$



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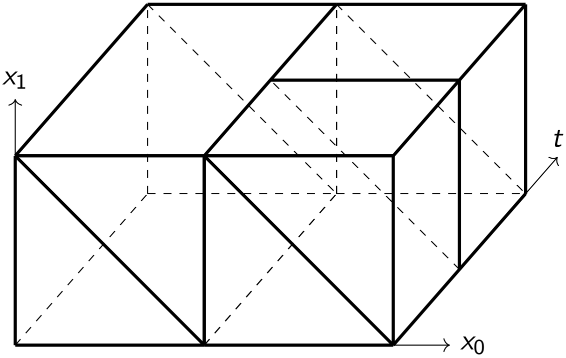

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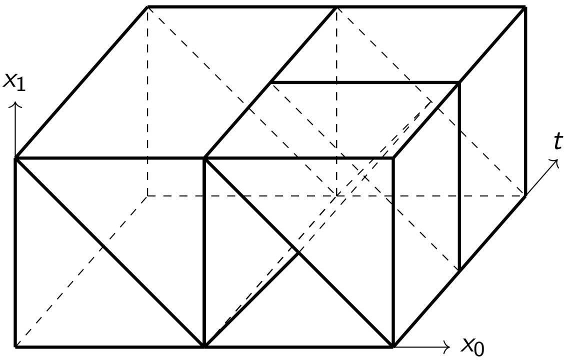

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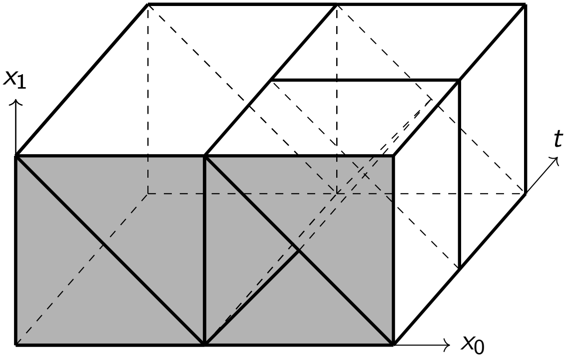

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Test space $Y = H(A_h^*, Q) \times L^2(\Omega)$

For all $K = K_t \times K_x \in \mathcal{T}$ and $\ell \in \mathbb{N}_0$

$$\mathbb{T}_\ell(K) := \operatorname{span}\{v_t v_x \mid v_t \in \mathbb{P}_\ell(K_t) \text{ and } v_x \in \mathbb{P}_\ell(K_x)\}$$

Piece-wise polynomials

$$\mathbb{P}_1(\mathcal{T}_0) := \{v \in L^\infty(\Omega) \mid v|_{K_x} \in \mathbb{P}_1(K_x) \text{ for all } K_x \in \mathcal{T}_0\}$$
$$\mathbb{T}_\ell(\mathcal{T}) := \{w \in L^\infty(Q) \mid w|_K \in \mathbb{T}_\ell(K) \text{ for all } K \in \mathcal{T}\}$$

Discrete test space

$$Y_h := \begin{cases} (\mathbb{T}_3(\mathcal{T}) \times \mathbb{T}_1(\mathcal{T}; \mathbb{R}^d)) \times \mathbb{P}_1(\mathcal{T}_0) \subset Y & \text{for } d = 1 \\ (\mathbb{T}_2(\mathcal{T}) \times \mathbb{T}_1(\mathcal{T}; \mathbb{R}^d)) \times \mathbb{P}_1(\mathcal{T}_0) \subset Y & \text{for } d \geq 2 \end{cases}$$

Ansatz space $X = L^2(Q;\mathbb{R}^{d+1}) \times \gamma_A H_D(A,Q)$

$H^1_D(Q) = L^2(\mathcal{I}; H^1_0(\Omega)) \cap H^1(Q)$ and $H(\mathrm{div}_x, Q) = L^2(\mathcal{I}; H(\mathrm{div}, \Omega))$

$$H^1_D(Q) \times H(\mathrm{div}_x, Q) \subset H_D(A,Q)$$

First component

$$V_h := \mathbb{T}_1(\mathcal{T}) \cap H^1_D(Q)$$

Second component

$$\Sigma_h := \{\tau \in L^\infty(Q;\mathbb{R}^d) \mid \tau \in RT_0(K_x) \text{ for all } K = K_t \times K_x \in \mathcal{T}\} \cap H(\mathrm{div}_x, Q)$$

Low-order ansatz space

$$X_h := \mathbb{T}_0(\mathcal{T};\mathbb{R}^{d+1}) \times \gamma_A(V_h \times \Sigma_h) \subset X$$

Fortin Operator

Theorem

It exists $\Pi : Y \to Y_h$ *with* $\|\Pi\| \lesssim 1 + \max_{K \in \mathcal{T}} h_t(K)/h_x(K)$ *and*

$$b(\underline{v}_h, \underline{t}_h; (\underline{w}, \xi) - \Pi(\underline{w}, \xi)) = 0$$

for all $(\underline{v}_h, \underline{t}_h) \in X_h$ *and* $(\underline{w}, \xi) \in Y$

The proof utilizes

- Local design Π_K for each $K = K_t \times K_x \in \mathcal{T}$
- Piola transformation (shape regularity of K_x)
- Orthogonal projection $\|\operatorname{div} \Pi_K \underline{w}\|_K \leq \|\operatorname{div} \underline{w}\|_K$
- Poincaré (in space) + Averaging in time

Main Result

A priori estimate

$$\|(\underline{u}, \underline{s}) - (\underline{u}_h, \underline{s}_h)\|_X \lesssim \min_{(\underline{v}_h, \underline{t}_h) \in X_h} \|(\underline{u}, \underline{s}) - (\underline{v}_h, \underline{t}_h)\|_X$$

A posteriori estimate

$$\|(\underline{u}, \underline{s}) - (\underline{u}_h, \underline{s}_h)\|_X \eqsim \|b(\underline{u}_h, \underline{s}_h; \bullet) - F\|_{Y_h^*} + \|F \circ (1 - \Pi)\|_{Y^*}$$

with

$$\|F \circ (1 - \Pi)\|_{Y^*}^2 \lesssim \sum_{K \in \mathcal{T}} \left\| f - \fint_{K_t} f \, \mathrm{d}s \right\|_{L^2(K)}^2 + \text{h.o.t.}$$

Parabolic Scaling

Lemma (Diening, Schwarzacher, Stroffolini, Verde '17)

Let $K \in \mathcal{T}$, $a \in L^2(Q)$, $G \in L^2(Q;\mathbb{R}^d)$ *with* $\partial_t a = \operatorname{div}_x G$, *then*

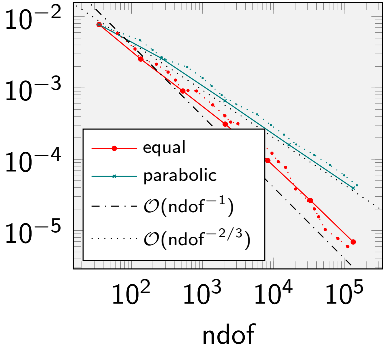
$$\left\| a - \fint_K a \,\mathrm{d}z \right\|^2_{L^2(K)} \lesssim h_x(K)^2 \|\nabla_x a\|^2_{L^2(K)} + \frac{h_t(K)^2}{h_x(K)^2} |K|^{1-2/p} \|G\|^2_{L^p(K)}$$

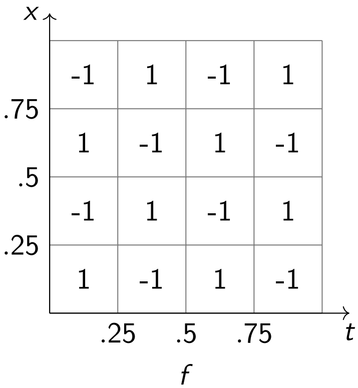
$\partial_t \underbrace{\nabla_x u}_{=a} = \nabla_x \partial_t u = \operatorname{div}_x(\partial_t u \, I_d)$ yields $G = \partial_t u = f + \Delta_x u$

$$\left\| \nabla_x u - \fint_K \nabla_x u \,\mathrm{d}z \right\|^2_{L^2(K)} \lesssim h_x(K)^2 \|\nabla_x^2 u\|^2_{L^2(K)} + \frac{h_t(K)^2}{h_x(K)^2} |K|^{1-2/p} \|\partial_t u\|^2_{L^p(K)}$$

$\longrightarrow$ Equal $h_t(K) \eqsim h_x(K)$ and parabolic $h_t(K) \eqsim h_x(K)^2$ scaling

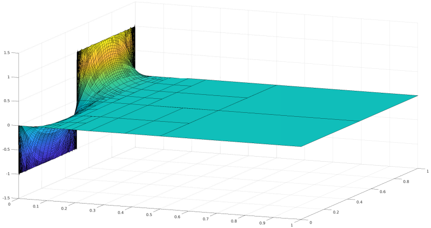
Numerical Experiment – Checkerboard

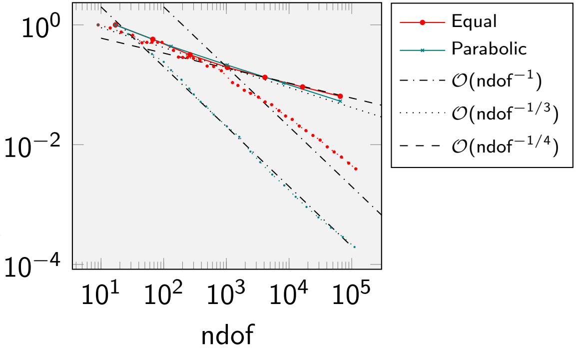




$\|b(\underline{u}_h, \underline{s}_h; \bullet) - F\|^2_{Y_h^*}$ with uniform (solid) and adaptive (dotted) refinements

Numerical Experiments – Initial Data





$\|b(\underline{u}_h, \underline{s}_h; \bullet) - F\|^2_{Y_h^*}$ with uniform (solid) and adaptive (dotted) refinements

Rough Right-Hand Side

	$f(t,x) = \|x - 1/2\|^\alpha$				$f(t,x) = \|t - 1/2\|^\alpha$			
	Equal		Parabolic		Equal		Parabolic	
α	unif	adapt	unif	adapt	unif	adapt	unif	adapt
0.00	1	1.01	0.66	0.66	1	1.01	0.66	0.66
-0.05	0.81	0.93	0.67	0.67	0.81	0.93	0.66	0.66
-0.1	0.6	0.83	0.67	0.68	0.59	0.82	0.67	0.65
-0.15	0.46	0.71	0.67	0.7	0.45	0.7	0.67	0.67
-0.2	0.36	0.6	0.68	0.71	0.36	0.59	0.67	0.68
-0.25	0.29	0.49	0.69	0.72	0.28	0.48	0.68	0.69
-0.3	0.22	0.39	0.69	0.75	0.22	0.38	0.69	0.72
-0.35	0.17	0.29	0.69	0.76	0.16	0.28	0.7	0.74
-0.4	0.11	0.19	0.67	0.76	0.11	0.18	0.7	0.77
-0.45	0.06	0.08	0.65	0.76	0.06	0.07	0.69	0.79
-0.5	**0**	**-0.02**	**0.62**	**0.76**	**0**	**-0.03**	**0.65**	**0.81**
-0.55	-0.05	-0.12	0.59	0.73	-0.05	-0.12	0.6	0.79
-0.6	-0.1	-0.21	0.55	0.71	-0.1	-0.22	0.53	0.78
-0.65	-0.15	-0.31	0.51	0.68	-0.15	-0.32	0.47	0.64
-0.7	-0.2	-0.41	0.47	0.62	-0.2	-0.4	0.4	0.58
-0.75	-0.25	-0.5	0.43	0.56	-0.25	-0.48	0.33	0.42

Estimated convergence rates of $\|b(\underline{u}_h, \underline{s}_h; \bullet) - F\|^2_{Y_h^*}$

Summary

- Space-time (DPG) FEM for heat equation $\partial_t u - \Delta_x u = f$
- Optimal test functions (DPG methodology) $\longrightarrow$ breaking space

$$Y_h^p = T_h Y_h \qquad \text{and} \qquad Y \text{ discontinuous}$$

- Discretization + Fortin operator
  $\longrightarrow$ quasi-optimality + error control + adaptivity
- Parabolically scaled meshes

$$\left\| \nabla_x u - \fint_K \nabla_x u \,\mathrm{d}z \right\|_{L^2(K)}^2 \lesssim h_x(K)^2 \|\nabla_x^2 u\|_{L^2(K)}^2 + \frac{h_t(K)^2}{h_x(K)^2} |K|^{1-2/p} \|\partial_t u\|_{L^p(K)}^2$$

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Thank you for your attention