Polytopal approximations of the de Rham complex

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NEMESIS

New generation methods
for numerical simulations

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Outline

1. Two model problems and their well-posedness

2. Polyhedral meshes, chain and cochain complexes, de Rham map

3. Polytopal discretizations of the de Rham complex

Outline

1 Two model problems and their well-posedness

2 Polyhedral meshes, chain and cochain complexes, de Rham map

3 Polytopal discretizations of the de Rham complex

Setting

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with Betti numbers b_i
- $b_0 = 1$ (number of connected components) and $b_3 = 0$ (since $d = 3$)
- b_1 and b_2 respectively account for the number of tunnels and voids

$$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$$

$$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$$

- Assume for the moment that Ω has trivial topology, i.e.,

$$b_1 = b_2 = 0$$

Vector calculus operators

- Important PDE models hinge on the vector calculus operators:

$$\operatorname{grad} q = \begin{pmatrix} \partial_1 q \\ \partial_2 q \\ \partial_3 q \end{pmatrix}, \ \operatorname{curl} v = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}, \ \operatorname{div} w = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$$

  for smooth enough functions

$$q : \Omega \longrightarrow \mathbb{R}, \qquad v : \Omega \longrightarrow \mathbb{R}^3, \qquad w : \Omega \longrightarrow \mathbb{R}^3$$

- The corresponding L^2-domain spaces are

$$\begin{aligned} H^1(\Omega) &:= \left\{ q \in L^2(\Omega) \ : \ \operatorname{grad} q \in L^2(\Omega)^3 \right\}, \\ H(\operatorname{curl};\Omega) &:= \left\{ v \in L^2(\Omega)^3 \ : \ \operatorname{curl} v \in L^2(\Omega)^3 \right\}, \\ H(\operatorname{div};\Omega) &:= \left\{ w \in L^2(\Omega)^3 \ : \ \operatorname{div} w \in L^2(\Omega) \right\} \end{aligned}$$

Two model problems

- We consider two model problems set in $H^1(\Omega)$, $H(\mathrm{curl};\Omega)$, and $H(\mathrm{div};\Omega)$
- The Stokes problem: Find $(u,p) \in H(\mathrm{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$
\begin{aligned}
\nu \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v + \int_\Omega \mathrm{grad}\, p \cdot v &= \int_\Omega f \cdot v \quad && \forall v \in H(\mathrm{curl};\Omega),\\
-\int_\Omega u \cdot \mathrm{grad}\, q &= 0 && \forall q \in H^1(\Omega)
\end{aligned}
$$

- The magnetostatics problem: Find $(H,A) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ s.t.

$$
\begin{aligned}
\mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\, \tau &= 0 && \forall \tau \in H(\mathrm{curl};\Omega),\\
\int_\Omega \mathrm{curl}\, H \cdot v + \int_\Omega \mathrm{div}\, A\, \mathrm{div}\, v &= \int_\Omega J \cdot v \quad && \forall v \in H(\mathrm{div};\Omega)
\end{aligned}
$$

A unified view

- The above problems are mixed formulations involving two fields: Find $(\sigma, u) \in \Sigma \times U$ s.t.

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma,v) + c(u,v) &= g(v) \quad \forall v \in U, \end{aligned}$$

or, equivalently, in variational formulation,

$$\mathcal{A}((\sigma,u),(\tau,v)) = f(\tau) + g(v) \qquad \forall (\tau,v) \in \Sigma \times U$$

with

$$\mathcal{A}((\sigma,u),(\tau,v)) := a(\sigma,\tau) + b(\tau,u) - b(\sigma,v) + c(u,v)$$

- Well-posedness holds under an inf-sup condition on $\mathcal{A}$

A unified tool: The de Rham complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties for well-posedness:

$$\operatorname{Im}\text{grad} \subset \operatorname{Ker}\text{curl},$$
$$\operatorname{Im}\text{curl} \subset \operatorname{Ker}\text{div},$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \operatorname{Im}\text{div} = L^2(\Omega) \quad \text{(magnetostatics)}$$

A unified tool: The de Rham complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties for well-posedness:

no tunnels crossing Ω $(b_1 = 0) \implies \text{Im grad} = \text{Ker curl}$ (Stokes)

no voids contained in Ω $(b_2 = 0) \implies \text{Im curl} = \text{Ker div}$ (magnetostatics)

$\Omega \subset \mathbb{R}^3$ $(b_3 = 0) \implies \text{Im div} = L^2(\Omega)$ (magnetostatics)

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$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \text{Im div} = L^2(\Omega) \quad \text{(magnetostatics)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 := \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \mathcal{H}_2 := \text{Ker div}/\text{Im curl}$$

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$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties for well-posedness:

no tunnels crossing Ω ($b_1 = 0$) $\implies$ $\text{Im grad} = \text{Ker curl}$ (Stokes)

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$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) $\implies$ $\text{Im div} = L^2(\Omega)$ (magnetostatics)

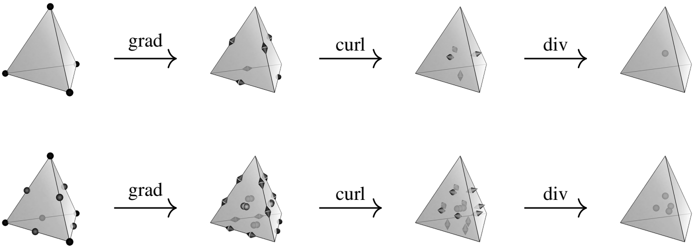
- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 := \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \mathcal{H}_2 := \text{Ker div}/\text{Im curl}$$

- **Emulating these properties is the key for stable discretizations**

The Finite Element way

- Trimmed FE complexes[1] on a tetrahedron T:



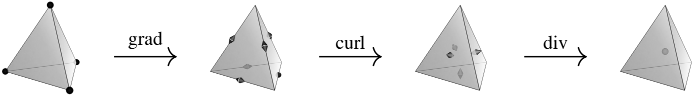
- On a conforming tetrahedral mesh $\mathcal{T}_h$, local spaces can be glued together

$$
\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl};\Omega) & \xrightarrow{\text{curl}} & H(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{P}_{\text{c}}^k(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\text{curl}} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

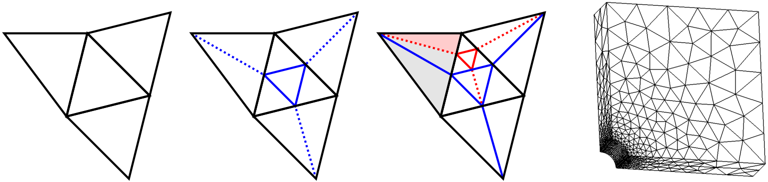
[1] [Raviart and Thomas, 1977, Nédélec, 1980]

Physically meaningful DOFs

- $H^1(\Omega)$ contains potentials (e.g., pressure): evaluation at a point V
- $H(\mathrm{curl};\Omega)$ contains circulations (e.g., magnetic field): integration over a line E
- $H(\mathrm{div};\Omega)$ contains fluxes (e.g., heat flux): integration over a face F
- $L^2(\Omega)$ contains densities (e.g., energy): integration over a volume T

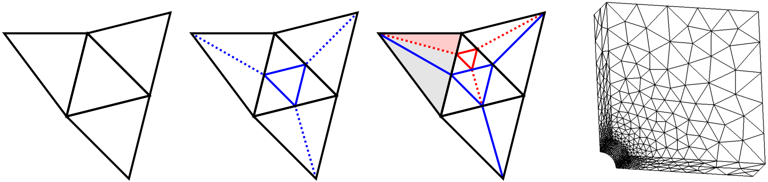


Limitations



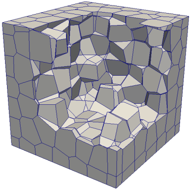
- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ Local refinement requires to trade mesh size for quality
  - $\Longrightarrow$ Complex geometries may require a large number of elements
  - $\Longrightarrow$ The element shape cannot be adapted to the solution
- The extension to advanced complexes is also not straightforward

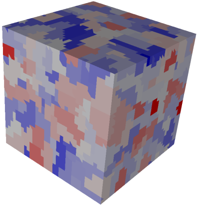
Limitations

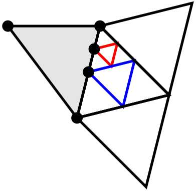


- Approach limited to conforming meshes with standard elements
  - ⟹ Local refinement requires to trade mesh size for quality
  - ⟹ Complex geometries may require a large number of elements
  - ⟹ The element shape cannot be adapted to the solution
- The extension to advanced complexes is also not straightforward
- These difficulties stem from the need of a globally regular function space

Polytopal approaches





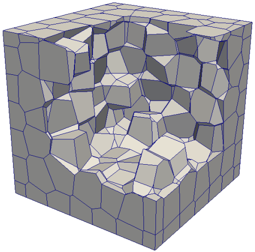


- Key idea: replace spaces and operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Higher-level point of view, possibly resulting in leaner constructions
- Several strategies to reduce the number of unknowns on general shapes
- Agglomeration-based solution techniques are available[2]

[2] [Bassi et al., 2012], [Antonietti et al., 2013]

Outline

Polyhedral mesh



$$\mathcal{M}_h = \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$$

with

- $\mathcal{T}_h = \Delta_3(\mathcal{M}_h)$ set of polyhedral elements (3-cells)
- $\mathcal{F}_h = \Delta_2(\mathcal{M}_h)$ set of polygonal (flat) faces (2-cells)
- $\mathcal{E}_h = \Delta_1(\mathcal{M}_h)$ set of edges (1-cells)
- $\mathcal{V}_h = \Delta_0(\mathcal{M}_h)$ set of vertices (0-cells)

Chain complex I

- A *k*-chain $w \in C_k(\mathcal{M}_h)$ is a formal linear combination of k-cells:

$$w = \sum_{f \in \Delta_k(\mathcal{M}_h)} w_f f, \qquad (w_f)_{f \in \Delta_k(\mathcal{M}_h)} \in \mathbb{R}^{\Delta_k(\mathcal{M}_h)}$$

- We define the boundary operator $\partial_k : C_k(\mathcal{M}_h) \to C_{k-1}(\mathcal{M}_h)$ s.t.

$$\partial_k f := \sum_{f' \in \Delta_{k-1}(f)} \omega_{ff'} f'$$

where $\omega_{ff'}$ is the orientation of f' relative to f

- The following sequence is a complex called the chain complex:

$$C_3(\mathcal{M}_h) \xrightarrow{\partial_3} C_2(\mathcal{M}_h) \xrightarrow{\partial_2} C_1(\mathcal{M}_h) \xrightarrow{\partial_1} C_0(\mathcal{M}_h)$$

Chain complex II

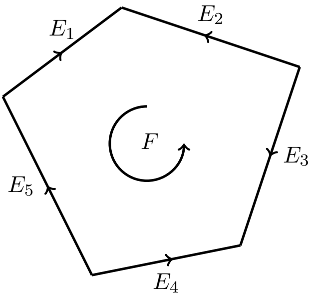


Figure: For this face $F \in \mathcal{F}_h$, $\partial F = -E_1 + E_2 - E_3 + E_4 - E_5$

Cochain complex

- A k-cochain $\lambda \in \mathcal{C}^k(\mathcal{M}_h)$ is a linear map $\lambda : \mathcal{C}_k(\mathcal{M}_h) \to \mathbb{R}$. We write

$$\langle \lambda, w \rangle := \lambda(w) \quad \forall w \in \mathcal{C}_k(\mathcal{M}_h)$$

- We define the co-boundary operator $\delta^k : \mathcal{C}^k(\mathcal{M}_h) \to \mathcal{C}^{k+1}(\mathcal{M}_h)$ s.t.

$$\forall \lambda \in \mathcal{C}^k(\mathcal{M}_h), \quad \langle \delta^k \lambda, w \rangle = \langle \lambda, \partial_{k+1} w \rangle \quad \forall w \in \mathcal{C}_{k+1}(\mathcal{M}_h)$$

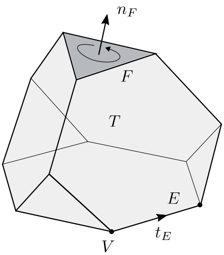
- The following sequence is a complex called the cochain complex:

$$\mathcal{C}^0(\mathcal{M}_h) \xrightarrow{\delta^0} \mathcal{C}^1(\mathcal{M}_h) \xrightarrow{\delta^1} \mathcal{C}^2(\mathcal{M}_h) \xrightarrow{\delta^2} \mathcal{C}^3(\mathcal{M}_h)$$

- It's cohomology is isomorphic to that of the de Rham complex:

$$\mathcal{H}_1 \cong \mathrm{Ker}\,\delta^2/\mathrm{Im}\,\delta^1 \quad \text{and} \quad \mathcal{H}_2 \cong \mathrm{Ker}\,\delta^3/\mathrm{Im}\,\delta^2$$

de Rham map



- Proof based on de Rham's map κ_h, mapping functions onto cochains: Given $q : \Omega \to \mathbb{R}$, $v : \Omega \to \mathbb{R}^3$, $w : \Omega \to \mathbb{R}^3$, and $r : \Omega \to \mathbb{R}$ smooth enough

$$\langle \kappa_{0,h}(q), V \rangle := q(x_V) \quad \forall V \in \mathcal{V}_h, \qquad \langle \kappa_{1,h}(v), E \rangle := \int_E v \cdot t_E \quad \forall E \in \mathcal{E}_h,$$
$$\langle \kappa_{2,h}(w), F \rangle := \int_F w \cdot n_F \quad \forall F \in \mathcal{F}_h, \qquad \langle \kappa_{3,h}(r), T \rangle := \int_T r \quad \forall T \in \mathcal{T}_h$$

- Technical difficulty: $H^1(\Omega)$, $H(\mathrm{curl};\Omega)$, $H(\mathrm{div};\Omega)$ not regular enough

Outline

Isomorphism in cohomology

Theorem (Complexes with isomorphic cohomologies[3])

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & V_i & \xrightarrow{d_i} & V_{i+1} & \longrightarrow & \cdots \\
 & & E_i \downarrow \uparrow R_i & & E_{i+1} \downarrow \uparrow R_{i+1} & & \\
\cdots & \longrightarrow & W_i & \xrightarrow{\partial_i} & W_{i+1} & \longrightarrow & \cdots
\end{array}
$$

Assume that reduction R and extension E are s.t., for all i,

- $R_i E_i = \mathrm{Id}_{W_i}$;
- $(E_{i+1} R_{i+1} - \mathrm{Id}_{V_{i+1}}) \operatorname{Ker} d_{i+1} \subset \operatorname{Im} d_i$;
- $\partial_i E_i = E_{i+1} d_i$ and $d_i R_i = R_{i+1} \partial_i$.

Then, the sequences are complexes with isomorphic cohomologies.

[3] [DP, Droniou and Pitassi, 2023]

Cohomology of the trimmed FE complex

- Let $\mathcal{M}_h$ be a conforming simplicial (FE) mesh. We have:

$$
\begin{array}{ccccccc}
C^0(\mathcal{M}_h) & \xrightarrow{\delta^0} & C^1(\mathcal{M}_h) & \xrightarrow{\delta^1} & C^2(\mathcal{M}_h) & \xrightarrow{\delta^2} & C^3(\mathcal{M}_h) \\
\kappa_{0,h}\cong\uparrow & & \kappa_{1,h}\cong\uparrow & & \kappa_{2,h}\cong\uparrow & & \kappa_{3,h}\cong\uparrow \\
\mathcal{P}^1_{\rm c}(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^0(\mathcal{T}_h) \\
\downarrow\uparrow I^1_{{\rm grad},h} & & \downarrow\uparrow I^1_{{\rm curl},h} & & \downarrow\uparrow I^1_{{\rm div},h} & & \downarrow\uparrow \pi^0_h \\
\mathcal{P}^k_{\rm c}(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

  with $I^1_{\bullet,h}$ interpolator and π^0_h L^2-orthogonal projector
- de Rham's Theorem: the first two rows have isomorphic cohomologies

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\kappa_{0,h} \cong \uparrow & & \kappa_{1,h} \cong \uparrow & & \kappa_{2,h} \cong \uparrow & & \kappa_{3,h} \cong \uparrow \\
\mathcal{P}^1_{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\mathrm{grad}} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^0(\mathcal{T}_h) \\
\downarrow \uparrow I^1_{\mathrm{grad},h} & & \downarrow \uparrow I^1_{\mathrm{curl},h} & & \downarrow \uparrow I^1_{\mathrm{div},h} & & \downarrow \uparrow \pi^0_h \\
\mathcal{P}^k_{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\mathrm{grad}} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

  with $I^1_{\bullet,h}$ interpolator and π^0_h L^2-orthogonal projector
- de Rham's Theorem: the first two rows have isomorphic cohomologies
- **The two bottom rows fulfill the assumptions of the theorem!**

Shifting point of view I

- Denote by “dofs” the standard FE degrees of freedom
- By unisolvency, we have

$$
\begin{array}{ccccccc}
\mathcal{C}^0(\mathcal{M}_h) & \xrightarrow{\delta^0} & \mathcal{C}^1(\mathcal{M}_h) & \xrightarrow{\delta^1} & \mathcal{C}^2(\mathcal{M}_h) & \xrightarrow{\delta^2} & \mathcal{C}^3(\mathcal{M}_h) \\
\kappa_{0,h} \cong \uparrow & & \kappa_{1,h} \cong \uparrow & & \kappa_{2,h} \cong \uparrow & & \kappa_{3,h} \cong \uparrow \\
\mathcal{P}^1_{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\mathrm{grad}} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^0(\mathcal{T}_h) \\
\mathrm{dofs}^{-1} \cong \uparrow & & \mathrm{dofs}^{-1} \cong \uparrow & & \mathrm{dofs}^{-1} \cong \uparrow & & \mathrm{dofs}^{-1} \cong \uparrow \\
\mathbb{R}^{\mathcal{V}_h} & & \mathbb{R}^{\mathcal{E}_h} & & \mathbb{R}^{\mathcal{F}_h} & & \mathbb{R}^{\mathcal{T}_h}
\end{array}
$$

Shifting point of view II

- In the previous diagram, we can erase the middle row
- Set $K_h := \kappa_h \circ \mathrm{dofs}^{-1}$, i.e., for all $(\underline{q}_h, \underline{v}_h, \underline{w}_h, \underline{r}_h) \in \mathbb{R}^{\mathcal{V}_h} \times \mathbb{R}^{\mathcal{E}_h} \times \mathbb{R}^{\mathcal{F}_h} \times \mathbb{R}^{\mathcal{T}_h}$,

$$
\begin{aligned}
&\langle K_{0,h}\underline{q}_h, V\rangle := q_V \quad \forall V \in \mathcal{V}_h, && \langle K_{1,h}\underline{v}_h, E\rangle := |E| v_E \quad \forall E \in \mathcal{E}_h,\\
&\langle K_{2,h}\underline{w}_h, F\rangle := |F| v_F \quad \forall F \in \mathcal{F}_h, && \langle K_{3,h} r_h, T\rangle := |T| r_T \quad \forall T \in \mathcal{T}_h
\end{aligned}
$$

- These discrete de Rham maps induce the following isomorphisms:

$$
\begin{array}{ccccccc}
C^0(\mathcal{M}_h) & \xrightarrow{\delta^0} & C^1(\mathcal{M}_h) & \xrightarrow{\delta^1} & C^2(\mathcal{M}_h) & \xrightarrow{\delta^2} & C^3(\mathcal{M}_h)\\
{\scriptstyle K_{0,h}}\cong\uparrow & & {\scriptstyle K_{1,h}}\cong\uparrow & & {\scriptstyle K_{2,h}}\cong\uparrow & & {\scriptstyle K_{3,h}}\cong\uparrow\\
\mathbb{R}^{\mathcal{V}_h} & & \mathbb{R}^{\mathcal{E}_h} & & \mathbb{R}^{\mathcal{F}_h} & & \mathbb{R}^{\mathcal{T}_h}
\end{array}
$$

- **Can we complete the bottom row to form a complex?**

Shifting point of view III

- Define the following discrete gradient, curl, and divergence operators:

$$\underline{G}_h^0 \underline{q}_h \coloneqq K_{1,h}^{-1}\delta^0 K_{0,h}, \quad \underline{C}_h^0 \underline{v}_h \coloneqq K_{2,h}^{-1}\delta^1 K_{1,h}, \quad D_h^0 \underline{w}_h \coloneqq K_{3,h}^{-1}\delta^2 K_{2,h},$$

- Notice that, by construction,

$$\underline{C}_h^0 \circ \underline{G}_h^0 = \underline{0} \text{ and } D_h^0 \circ \underline{C}_h^0 = \underline{0}$$

- Hence, we have two complexes with isomorphic cohomologies:

$$\begin{array}{ccccccc}
C^0(\mathcal{M}_h) & \xrightarrow{\delta^0} & C^1(\mathcal{M}_h) & \xrightarrow{\delta^1} & C^2(\mathcal{M}_h) & \xrightarrow{\delta^2} & C^3(\mathcal{M}_h) \\
\uparrow {\scriptstyle K_{0,h}}\,\cong & & \uparrow {\scriptstyle K_{1,h}}\,\cong & & \uparrow {\scriptstyle K_{2,h}}\,\cong & & \uparrow {\scriptstyle K_{3,h}}\,\cong \\
\mathbb{R}^{\mathcal{V}_h} & \xrightarrow{\underline{G}_h^0} & \mathbb{R}^{\mathcal{E}_h} & \xrightarrow{\underline{C}_h^0} & \mathbb{R}^{\mathcal{F}_h} & \xrightarrow{D_h^0} & \mathbb{R}^{\mathcal{T}_h}
\end{array}$$

- **Still true when $\mathcal{M}_h$ is a polyhedral mesh!**

Moral of the story so far

By getting rid of FE spaces, we can now handle polyhedral meshes!

A closer look at the discrete operators

- $\underline{G}_h^0$, $\underline{C}_h^0$, and D_h^0 are actually the mimetic operators[4]:

$$\underline{G}_h^0 \underline{q}_h := \left(G_E^0 \underline{q}_E = \frac{q_{V_2} - q_{V_1}}{|E|} \right)_{E \in \mathcal{E}_h},$$

$$\underline{C}_h^0 \underline{v}_h := \left(C_F^0 \underline{v}_F = -\frac{1}{|F|} \sum_{E \in \mathcal{E}_F} \omega_{FE} |E| v_E \right)_{F \in \mathcal{F}_h},$$

$$D_h^0 \underline{w}_h := \left(D_T^0 \underline{w}_T = \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \omega_{TF} |F| w_F \right)_{T \in \mathcal{T}_h}$$

- These operators are polynomially exact

[4] See, e.g., [Beirão da Veiga et al., 2014] and [Bonelle and Ern, 2014]

The arbitrary-order case $k \geq 0$

$$\underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{G}^k_h} \underline{X}^k_{\mathrm{curl},h} \xrightarrow{\underline{C}^k_h} \underline{X}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h)$$

	V	E	F	T
$\underline{X}^k_{\mathrm{grad},h}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}^k_{\mathrm{curl},h}$	–	$\mathcal{P}^k(E)$	$\mathcal{RT}^k(F)$	$\mathcal{RT}^k(T)$
$\underline{X}^k_{\mathrm{div},h}$	–	–	$\mathcal{P}^k(F)$	$\mathcal{N}^k(T)$
$\mathcal{P}^k(\mathcal{T}_h)$	–	–	–	$\mathcal{P}^k(T)$

- Discrete de Rham (DDR) [DP et al., 2020, DP and Droniou, 2023b]
- Serendipity version [DP and Droniou, 2023a][5]
- General discrete Poincaré inequalities [DP and Hanot, 2024]
- Various extensions and applications in subsequent papers

[5] See [Beirão da Veiga et al., 2018] for a preliminary work on this subject

An example: The arbitrary order curl space I

- The construction on $F \in \mathcal{F}_h$ hinges on the integration by parts formula:

$$\int_F \mathrm{rot}_F\, v\, q = \int_F v \cdot \mathrm{curl}_F\, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (v \cdot t_E)\, q$$

- Specifically, the face curl $C_F^k : \underline{X}_{\mathrm{curl},F}^k \to \mathcal{P}^k(F)$ is s.t.

$$\int_F C_F^k \underline{v}_F\, q = \int_F v_F \cdot \mathrm{curl}_F\, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, q \quad \forall q \in \mathcal{P}^k(F)$$

- The tangent trace $\gamma_F^k : \underline{X}_{\mathrm{curl},F}^k \to \mathcal{P}^k(F)^2$ is s.t.

$$\int_F \gamma_F^k \underline{v}_F \cdot (\mathrm{curl}_F\, r + w) = \int_F C_F^k \underline{v}_F\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, r + \int_F v_F \cdot w$$

$$\forall (r, w) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$$

An example: The arbitrary order curl space II

- On $T \in \mathcal{T}_h$, the starting point is

$$\int_T \operatorname{curl} v \cdot w = \int_T v \cdot \operatorname{curl} w + \sum_{F \in \mathcal{F}_T} \int_F v \cdot (w \times n_F)$$

- The element curl $C_T^k : \underline{X}_{\mathrm{curl},h}^k \to \mathcal{P}^k(T)^3$ is s.t.

$$\int_T C_T^k \underline{v}_T \cdot w = \int_T v_T \cdot \operatorname{curl} w + \sum_{F \in \mathcal{F}_T} \int_F \gamma_F^k \underline{v}_F \cdot (w \times n_F) \quad \forall w \in \mathcal{P}^k(T)^3$$

- Finally, the discrete curl $\underline{C}_h^k : \underline{X}_{\mathrm{curl},h}^k \to \underline{X}_{\mathrm{div},h}^k$ is

$$\underline{C}_h^k : \underline{X}_{\mathrm{curl},h}^k \to \underline{X}_{\mathrm{div},h}^k$$
$$\underline{v}_h \mapsto \left((\pi_{\mathcal{N}^k(T)} C_T^k \underline{v}_T)_{T \in \mathcal{T}_h}, (C_F^k \underline{v}_F)_{F \in \mathcal{F}_h} \right)$$

An example: The arbitrary order curl space III

- Let the element potential $P^k_{\mathrm{curl},T} : \underline{X}^k_{\mathrm{curl},T} \to \mathcal{P}^k(T)^3$ be s.t.

$$\int_T P^k_{\mathrm{curl},T}\underline{v}_T \cdot (\operatorname{curl} w + z) = \int_T C^k_T \underline{v}_T \cdot w - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma^k_F \underline{v}_F \cdot (w \times n_F) + \int_T v_T \cdot z$$

$$\forall (w, z) \in \mathcal{G}^{\mathrm{c},k+1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$$

- The local L^2-product in $\underline{X}^k_{\mathrm{curl},T}$ is

$$(\underline{w}_T, \underline{v}_T)_{\mathrm{curl},T} := \int_T P^k_{\mathrm{curl},T}\underline{w}_T \cdot P^k_{\mathrm{curl},T}\underline{v}_T + \text{stab.}$$

  where stab. penalizes $\underline{v}_T - \underline{I}^k_{\mathrm{curl},T} P^k_{\mathrm{curl},T}\underline{v}_T$ in a least-square sense
- The global discrete L^2-product is obtained assemblying element-wise:

$$(\underline{w}_h, \underline{v}_h)_{\mathrm{curl},h} := \sum_{T \in \mathcal{T}_h} (\underline{w}_T, \underline{v}_T)_{\mathrm{curl},T}$$

An example of DDR scheme

- Let us consider again the magnetostatics problem:

  Find $(H, A) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned}
\mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\,\tau &= 0 && \forall \tau \in H(\mathrm{curl};\Omega),\\
\int_\Omega \mathrm{curl}\,H \cdot v + \int_\Omega \mathrm{div}\,A\,\mathrm{div}\,v &= \int_\Omega J \cdot v && \forall v \in H(\mathrm{div};\Omega)
\end{aligned}$$

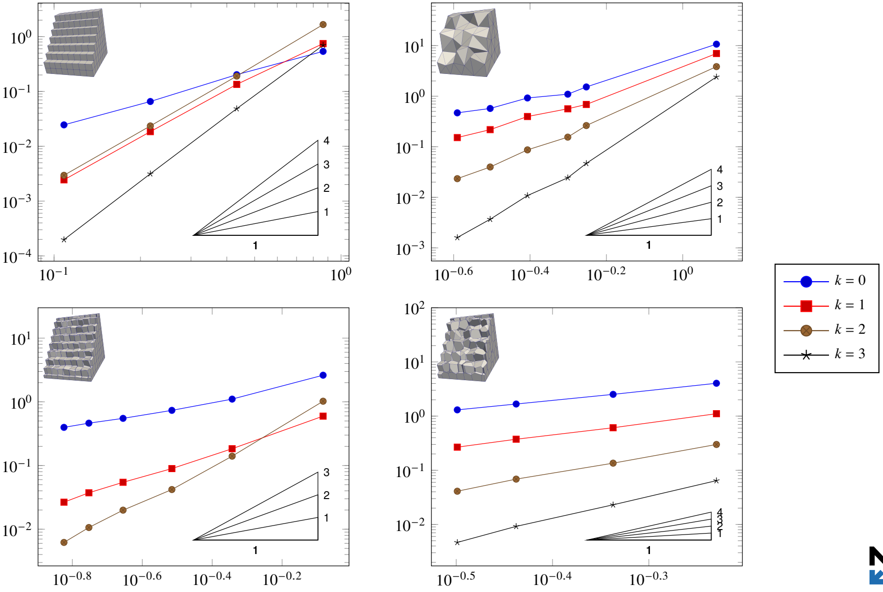
- A DDR scheme for this problem is obtained with obvious substitutions:

  Find $(\underline{H}_h, \underline{A}_h) \in \underline{X}^k_{\mathrm{curl},h} \times \underline{X}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned}
\mu(\underline{H}_h, \underline{\tau}_h)_{\mathrm{curl},h} - (\underline{A}_h, \underline{C}^k_h \underline{\tau}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\tau}_h \in \underline{X}^k_{\mathrm{curl},h},\\
(\underline{C}^k_h \underline{H}_h, \underline{v}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{A}_h\, D^k_h \underline{v}_h &= (\underline{I}^k_{\mathrm{div},h} J, \underline{v}_h)_{\mathrm{div},h} && \forall \underline{v}_h \in \underline{X}^k_{\mathrm{div},h}
\end{aligned}$$

- **Stability mimics the continuous argument for well-posedness**

Convergence: Energy error vs. meshsize



Extension to differential forms

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a polytopal domain or manifold
- FE have been extended to the de Rham complex of differential forms[6]:

$$\cdots \longrightarrow H\Lambda^{i-1}(\Omega) \xrightarrow{\mathrm{d}^i} H\Lambda^i(\Omega) \longrightarrow \cdots$$

- This has lead to new elements, advanced complexes, etc.
- Polytopal Exterior Calculus (PEC): [Bonaldi, DP, Droniou and Hu, 2025]
- Discrete Poincaré for PEC: [DP, Droniou, Hanot, Pitassi, 2025]

[6] See, e.g., [Bossavit, 1988, Hiptmair, 2002, Arnold et al., 2006]

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