An introduction to Hybrid High-Order (HHO) methods
Nonlinear elasticity and poroelasticity

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Features of HHO methods

- Support of general polytopal meshes in any space dimension
- Arbitrary approximation order
- Local principle of virtual work with equilibrated tractions
- Compact stencil only involving neighbors through faces
- Reduced cost after hybridisation for linear(ised) problems

\[ N_{\text{dof}}^{\text{hho}} \approx \frac{1}{2} k^2 \text{card}(\mathcal{F}_h) \quad N_{\text{dof}}^{\text{dg}} \approx \frac{1}{6} k^3 \text{card}(\mathcal{T}_h) \]
Figure: Admissible meshes. The agglomerated mesh is taken from [DP and Specogna, 2016]
Figure: Treatment of a nonconforming junction (red) as multiple coplanar faces. Gray elements are pentagons, white elements are squares.
**Definition (Regular mesh sequence)**

Let $(\mathcal{M}_h)_{h \in \mathcal{H}} := (\mathcal{T}_h, \mathcal{F}_h)_{h \in \mathcal{H}}$ be a sequence of $h$-refined polytopal meshes with $\mathcal{T}_h$ set of elements and $\mathcal{F}_h$ set of faces. The sequence is regular if there exists a sequence of simplicial submeshes $(\mathcal{T}_h)_{h \in \mathcal{H}}$

- **shape-regular** in the sense of Ciarlet;
- **contact-regular**, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

**Main consequences:**

- **Trace and inverse inequalities**
- **Optimal approximation properties** for broken polynomial spaces
Outline

1. Nonlinear elasticity

2. Poroelasticity
Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded connected polyhedral domain.

For $f \in L^2(\Omega; \mathbb{R}^d)$ we seek the displacement field $u : \Omega \to \mathbb{R}^d$ s.t.

$$-\nabla \cdot \sigma(\cdot, \nabla_s u) = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega$$

with $\sigma : \Omega \times \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}^{d \times d}_{\text{sym}}$ stress-strain law.

Weak formulation: Find $u \in H^1_0(\Omega; \mathbb{R}^d)$ such that

$$a(u, v) := \int_{\Omega} \sigma(\cdot, \nabla_s u) : \nabla_s v = \int_{\Omega} f \cdot v \quad \forall v \in H^1_0(\Omega; \mathbb{R}^d)$$

with $\nabla_s$ denoting the symmetric (part of) the gradient.
Error estimates under (relatively) strong assumptions on $\sigma$ and $u$

- Conforming FE, standard meshes
  
  [Gatica and Stephan, 2002, Gatica et al., 2013]

- Discontinuous Galerkin (DG), standard meshes
  
  [Ortner and Süli, 2007]

- Virtual Elements, polyhedral meshes in 2D, low-order
  
  [Beirão da Veiga et al., 2013]

Convergence to minimal regularity solutions

- Gradient Discretisations [Droniou and Lamichhane, 2015]

- DG, stronger assumptions on $\sigma$, [Bi and Lin, 2012]

Convergence to minimal regularity solutions and error estimates for HHO [Botti, DP, Sochala, 2017]
Assumption (Stress-strain law I)

The Carathéodory function $\sigma$ is s.t. $\sigma(\cdot, 0) = 0$. Moreover, there exist two real numbers $\sigma, \sigma \in (0, +\infty)$ s.t. for a.e. $x \in \Omega$ and all $\tau, \eta \in \mathbb{R}_{\text{sym}}^{d\times d}$,

\[
\|\sigma(x, \tau)\|_{d\times d} \leq \sigma \|\tau\|_{d\times d}, \quad \text{(growth)}
\]
\[
\sigma(x, \tau): \tau \geq \sigma \|\tau\|_{d\times d}^2, \quad \text{(coercivity)}
\]
\[
(\sigma(x, \tau) - \sigma(x, \eta)) : (\tau - \eta) \geq 0, \quad \text{(monotonicity)}
\]

where $\|\tau\|_{d\times d}^2 := \tau : \tau$ and $\tau : \eta := \sum_{1 \leq i, j \leq d} \tau_{ij} \eta_{ij}$.
Stress-strain law II

Example (Stress-strain laws)

- **Linear elasticity.** For Lamé’s parameters $\mu > 0$ and $\lambda \geq 0$,
  \[
  \sigma(\cdot, \tau) = 2\mu \tau + \lambda \text{tr}(\tau) I_d
  \]

- **Hencky–Mises model.** For given Lamé’s functions $\tilde{\mu}$ and $\tilde{\lambda}$, setting $\text{dev}(\tau) := \text{tr}(\tau^2) - \frac{1}{d} \text{tr}(\tau)^2$,
  \[
  \sigma(\cdot, \tau) = 2\tilde{\mu}(\text{dev}(\tau))\tau + \tilde{\lambda}(\text{dev}(\tau)) \text{tr}(\tau) I_d
  \]

- **Isotropic damage model.** For a scalar damage function $D : \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}$ and a fourth-order tensor $C : \Omega \to \mathbb{R}^{d^4}$,
  \[
  \sigma(\cdot, \tau) = (1 - D(\tau)) C(\cdot) \tau
  \]
Let $X$ denote an element in $\mathcal{T}_h$ or a face in $\mathcal{T}_h$ and $l \geq 0$ an integer.

The $L^2$-orthogonal projector $\pi^l_X : L^1(X; \mathbb{R}) \to P^l(X; \mathbb{R})$ is s.t.

\[
\forall v \in L^1(\Omega; \mathbb{R}), \quad \int_X (\pi^l_X v - v) w = 0 \quad \forall w \in P^l(X; \mathbb{R})
\]

$\pi^l_X v$ is well-defined and it holds that

\[
\pi^l_X v = \text{argmin}_{w \in P^l(X; \mathbb{R})} \|v - w\|_{L^2(X; \mathbb{R})}^2
\]

The vector- and matrix-versions $\pi^l_X$ act component-wise.
Lemma ($W^{s,p}$-approximation properties of $\pi^l_T$)

Let $(M_h)_{h \in \mathcal{H}}$ be a regular mesh sequence. For an integer $l \geq 0$, let an integer $s \in \{0, \ldots, l + 1\}$ and a real number $p \in [1, +\infty]$ be given. Then, for all $T \in T_h$, all $v \in W^{s,p}(T)$, and all $m \in \{0, \ldots, s\}$,

$$|v - \pi^l_T v|_{W^{m,p}(T)} \lesssim h^{s-m}_T |v|_{W^{s,p}(T)}$$

and, if $s \geq 1$ and $m \in \{0, \ldots, s - 1\}$,

$$h^{\frac{1}{p}}_T |v - \pi^l_T v|_{W^{m,p}(F_T)} \lesssim h^{s-m}_T |v|_{W^{s,p}(T)}.$$

Above, $\lesssim$ hides multiplicative constants independent of $h$.

See [DP and Droniou, 2017a], based on [Dupont and Scott, 1980]
Let $T \in \mathcal{T}_h$, $\mathbb{RM}_d(T)$ spanned by rigid-body motions restricted to $T$

For a given integer $l \geq 1$, we define the elastic projector

$$\pi^l_{el,T} : W^{1,1}(T; \mathbb{R}^d) \to \mathbb{P}^l(T; \mathbb{R}^d)$$

s.t., for all $v \in W^{1,1}(T; \mathbb{R}^d)$,

$$\int_T \nabla_s (\pi^l_{el,T} v - v) : \nabla_s w = 0 \quad \forall w \in \mathbb{P}^l(T; \mathbb{R}^d),$$

$$\int_T \pi^l_{el,T} v = \int_T v, \quad \int_T \nabla_{ss} \pi^l_{el,T} v = \frac{1}{2} \sum_{F \in \mathcal{F}_T} \left( n_{TF} \wedge \pi^k_F v - \pi^k_F v \wedge n_{TF} \right)$$

Using the abstract results of [DP and Droniou, 2017b], it can be proved that $\pi^l_{el,T}$ has optimal approximation properties
Computing $L^2$-projections of $\nabla_S v$ from $L^2$-projections of $v$

- For all $v \in W^{1,1}(T; \mathbb{R}^d)$ and all $\tau \in C^\infty(\overline{T}; \mathbb{R}_\text{sym}^{d \times d})$, it holds that

$$
\int_T \nabla_S v : \tau = -\int_T v \cdot (\nabla \cdot \tau) + \sum_{F \in \mathcal{F}_T} \int_F v \cdot \tau n_{TF}
$$

(IBP)

- Specialising (IBP) to $\tau \in \mathcal{P}^l(T; \mathbb{R}_\text{sym}^{d \times d})$, we can write

$$
\int_T \pi_T^l \nabla_S v : \tau = -\int_T \pi_T^{l-1} v \cdot (\nabla \cdot \tau) + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^l v \cdot \tau n_{TF}
$$

- Hence, computing $\pi_T^l \nabla_S v$ does not require a full knowledge of $v$!
- All that is required is $\pi_T^{l-1} v$ and for all $F \in \mathcal{F}_T$, $\pi_F^l v$
Computing $\pi_{\text{el},T}^{l+1}\nu$ from $L^2$-projections of $\nu$

- Specialise now (IBP) to $\tau = \nabla_s w$ with $w \in \mathbb{P}^{l+1}(T; \mathbb{R}^d)$, to obtain

$$\int_T \nabla_s \pi_{\text{el},T}^{l+1}\nu : \nabla_s w = -\int_T \pi_T^{l-1}\nu \cdot (\nabla \cdot \nabla_s w) + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^l\nu \cdot \nabla_s w n_{TF}$$

- Observe, moreover, that if $l \geq 1$ then for all $w \in \mathbb{R}\mathcal{M}_d(T)$,

$$\int_T (\pi_{\text{el},T}^{l+1}\nu - \nu) \cdot w = \int_T (\pi_{\text{el},T}^{l+1}\nu - \pi_T^l\nu) \cdot w$$

since $\mathbb{R}\mathcal{M}_d(T) \subset \mathbb{P}^1(T; \mathbb{R}^d) \subset \mathbb{P}^l(T; \mathbb{R}^d)$

- Hence, $\pi_{\text{el},T}^{l+1}\nu$ is computable from $\pi_T^l\nu$ and for all $F \in \mathcal{F}_T$, $\pi_F^l\nu$
Let $k \geq 1$ and $T \in \mathcal{T}_h$ be fixed. The **space of local unknowns** is s.t.

$$U^k_T := \mathbb{P}^k(T; \mathbb{R}^d) \times \left( \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F; \mathbb{R}^d) \right)$$

- We denote by $\underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T})$ a generic element of $U^k_T$.
- The **local interpolator** $I^k_T : W^{1,1}(T; \mathbb{R}^d) \to U^k_T$ is s.t.

$$\forall v \in W^{1,1}(T; \mathbb{R}^d), \quad I^k_T v := (\pi^k_T v, (\pi^k_F v)_F)_{F \in \mathcal{F}_T}$$
The symmetric gradient reconstruction \( G_{s,T}^k : U_T^k \to \mathbb{P}^k(T; \mathbb{R}^{d \times d}_{\text{sym}}) \) is s.t.

\[
\int_T G_{s,T}^k \nu_T : \tau = - \int_T \nu_T \cdot (\nabla \cdot \tau) + \sum_{F \in \mathcal{F}_T} \int_F \nu_F \cdot \tau n_{TF} \quad \forall \tau \in \mathbb{P}^k(T; \mathbb{R}^{d \times d}_{\text{sym}})
\]

The displacement reconstruction \( r_{T}^{k+1} : U_T^k \to \mathbb{P}^{k+1}(T; \mathbb{R}^{d+1}) \) is s.t.

\[
\int_T (\nabla s r_{T}^{k+1} - G_{s,T}^k) \nu_T : \nabla s w = 0 \quad \forall w \in \mathbb{P}^{k+1}(T; \mathbb{R}^d)
\]

\[
\int_T (r_{T}^{k+1} \nu_T - \nu_T) \cdot w = 0 \quad \forall w \in \mathbb{P}^{k+1}(T; \mathbb{R}^d)
\]

We have the key commuting properties: For all \( v \in W^{1,1}(T; \mathbb{R}^d) \),

\[
G_{s,T}^k I_{T}^k v = \pi_T^k \nabla s v, \quad r_{T}^{k+1} I_{T}^k v = \pi_{\text{el},T}^{k+1} v
\]
Let $T \in \mathcal{T}_h$. We approximate $a_{|T}$ with $a_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ s.t.

$$a_T(u_T, v_T) := \int_T \sigma(\cdot, G_{s,T} u_T) : G_{s,T} v_T + s_T(u_T, v_T)$$

Here, $s_T$ is the stabilisation bilinear form s.t.

$$s_T(u_T, v_T) := \sum_{F \in \mathcal{F}_T} \frac{\gamma}{h_F} \int_F (\delta_{TF}^k - \delta_T^k) u_T \cdot (\delta_{TF}^k - \delta_T^k) v_T,$$

with $\gamma$ user-defined parameter and difference operators s.t.

$$(\delta_T^k v_T, (\delta_{TF}^k v_T)_{F \in \mathcal{F}_T}) := I_T^k (r_T^{k+1} v_T) - v_T \in \underline{U}_T^k$$
### Proposition (Properties of $s_T$)

- **Stability.** For all $v_T \in U_T^k$, it holds that

\[
\| G_{s,T}^k v_T \|_{L^2(T; \mathbb{R}^{d \times d})}^2 + s_T(v_T, v_T) \simeq \| v_T \|_{\epsilon,T}^2
\]

with hidden constant independent of $h$ and $T$ and

\[
\| v_T \|_{\epsilon,T}^2 := \| \nabla s v_T \|_{L^2(T; \mathbb{R}^{d \times d})}^2 + \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \| v_F - v_T \|_{L^2(F; \mathbb{R}^d)}^2.
\]

- **Polynomial consistency.** For all $w \in P^{k+1}(T; \mathbb{R}^d)$, it holds that

\[
s_T(I_T^k w, v_T) = 0 \quad \forall v_T \in U_T^k.
\]
Remark (Naïve stabilisation and polynomial consistency)

Stability can be achieved using the following naïve stabilisation:

\[ S_T^{\text{hdg}}(u_T, v_T) = \sum_{F \in \mathcal{F}_T} \frac{\gamma}{h_F} \int_F (u_F - u_T) \cdot (v_F - v_T). \]

In this case, however, we only have polynomial consistency for \( w \in P^k(T; \mathbb{R}^d) \). As a result, up to one order of convergence is lost.
Discrete problem I

- We define the global space with single-valued interface unknowns

\[ \underline{U}_h^k := \left( \bigotimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T; \mathbb{R}^d) \right) \times \left( \bigotimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F; \mathbb{R}^d) \right) \]

as well as its subspace with strongly enforced b.c.

\[ \underline{U}_{h,0}^k := \{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) \in \underline{U}_h^k : v_F = 0 \quad \forall F \in \mathcal{F}_h^b \} \]

- The global interpolator \( \underline{I}_h^k : W^{1,1}(\Omega; \mathbb{R}^d) \to \underline{U}_h^k \) is s.t.

\[ (\underline{I}_h^k v)_T := I_T^k v|_T \quad \forall T \in \mathcal{T}_h \]
Discrete problem II

- Define the function $a_h : \mathcal{U}_h^k \times \mathcal{U}_h^k \to \mathbb{R}$ assembled element-wise:

$$a_h(u_h, v_h) := \sum_{T \in T_h} a_T(u_T, v_T)$$

- Discrete problem: Find $u_h \in \mathcal{U}_{h,0}^k$ such that

$$a_h(u_h, v_h) = \int_{\Omega} f \cdot v_h \quad \forall v_h \in \mathcal{U}_{h,0}^k$$

with $v_h$ obtained patching element unknowns

**Lemma (Existence and uniqueness)**

Let $(\mathcal{M}_h)_{h \in \mathcal{H}}$ be a regular mesh sequence. Then, for all $h \in \mathcal{H}$ there exists at least one solution $u_h \in \mathcal{U}_{h,0}^k$. Additionally, if $\sigma$ is strictly monotone, the solution is unique.
Theorem (Convergence)

Let \((M_h)_{h \in H}\) be a regular mesh sequence. Then, for all \(q\) s.t. 
1 \(\leq q < +\infty\) if \(d = 2\), 1 \(\leq q < 6\) if \(d = 3\), as \(h \to 0\), up to a subsequence,

- \(u_h \to u\) strongly in \(L^q(\Omega; \mathbb{R}^d)\);
- \(G_{S,T}^k u_h \to \nabla S u\) weakly in \(L^2(\Omega; \mathbb{R}^{d \times d})\).

Moreover, if we assume strict monotonicity for \(\sigma\),

- \(G_{S,T}^k u_h \to \nabla S u\) strongly in \(L^2(\Omega; \mathbb{R}^{d \times d})\).

If the continuous solution is unique, the whole sequence converges.
Convergence II

Assumption (Stress-strain law II)

There exist reals $\sigma^*, \sigma_* \in (0, +\infty)$ s.t., for a.e. $x \in \Omega$ and all $\tau, \eta \in \mathbb{R}^{d \times d}_{\text{sym}}$,

$$\|\sigma(x, \tau) - \sigma(x, \eta)\|_{d \times d} \leq \sigma^* \|\tau - \eta\|_{d \times d}, \quad \text{(Lipschitz continuity)}$$

$$(\sigma(x, \tau) - \sigma(x, \eta)) : (\tau - \eta) \geq \sigma_* \|\tau - \eta\|_{d \times d}^2. \quad \text{(strong monotonicity)}$$

Theorem (Error estimate)

Under the above assumption and the regularity $u \in H^{k+2}(\mathcal{T}_h; \mathbb{R}^d)$ and $\sigma(\cdot, \nabla_s u) \in H^{k+1}(\mathcal{T}_h; \mathbb{R}^{d \times d})$, it holds that

$$\|\nabla_s u - G^k_{s,T} u_h\|_{L^2(\Omega; \mathbb{R}^{d \times d})} + |u_h|_{s,h} \lesssim h^{k+1} \mathcal{N}_u,$$

with hidden constant independent of $h$, $|u_h|_{s,h}^2 := \sum_{T \in \mathcal{T}_h} \mathcal{S}_T(u_h, u_h)$, and $\mathcal{N}_u := \|u\|_{H^{k+2}(\mathcal{T}_h; \mathbb{R}^d)} + \|\sigma(\cdot, \nabla_s u)\|_{H^{k+1}(\mathcal{T}_h; \mathbb{R}^{d \times d})}$. 
Theorem (Robust estimate for quasi-incompressible materials)

Let $\sigma$ be such that, for all $x \in \Omega$ and all $\tau \in \mathbb{R}^{d \times d}_{\text{sym}}$ with $\mu > 0$ and $\lambda \geq 0$,

$$\sigma(x, \tau) = 2\mu \tau + \lambda \text{tr}(\tau) I_d.$$ 

Then, the following locking-free error estimate holds:

$$(2\mu)^{\frac{1}{2}} \| \nabla_s u - G_{s,T}^k u_h \|_{L^2(\Omega;\mathbb{R}^{d \times d})} \lesssim h^{k+1} \left( 2\mu \| u \|_{H^{k+2}(T_h;\mathbb{R}^d)} + \lambda \| \nabla \cdot u \|_{H^{k+1}(T_h;\mathbb{R})} \right)$$

with hidden constant independent of $h$, $\mu$, and $\lambda$. 
Numerical examples I
Convergence

- We consider the Hencky–Mises model with $\mu = 2$ and $\lambda = 1$ and

$$\sigma(\tau) = ((\lambda - \mu) + \mu \exp(-\text{dev}(\tau))) \text{tr}(\tau)I_d + \mu (2 - \exp(-\text{dev}(\tau))) \tau$$

- We solve the homogeneous Dirichlet problem with

$$u(x) := \begin{pmatrix} \sin(\pi x_1) \sin(\pi x_2) \\ \sin(\pi x_1) \sin(\pi x_2) \end{pmatrix}, \quad f = -\nabla \cdot \sigma(\nabla_s u)$$

- Refinements of the following meshes are used:
Convergence

\[ \| \nabla_s u - G^k_{s,h} u_h \|_{L^2(\Omega; \mathbb{R}^{d \times d})} \]

\[ \| \pi^k_h u - u_h \|_{L^2(\Omega; \mathbb{R}^d)} \]
Figure: Traction and shear tests and corresponding stress components for the linear case ($10^5$ Pa)
Numerical examples II

Traction and shear test cases

**Traction**

- $k = 1$ Triangular
- $k = 1$ Voronoi
- $k = 2$ Triangular
- $k = 2$ Voronoi
- $k = 3$ Triangular
- $k = 3$ Voronoi
- $E_{\text{lin}} = 21532$ J

**Shear**

- $k = 1$ Triangular
- $k = 1$ Voronoi
- $k = 2$ Triangular
- $k = 2$ Voronoi
- $k = 3$ Triangular
- $k = 3$ Voronoi
- $E_{\text{snd}} = 22490$ J

**Linear**

- $k = 1$ Triangular
- $k = 1$ Voronoi
- $k = 2$ Triangular
- $k = 2$ Voronoi
- $k = 3$ Triangular
- $k = 3$ Voronoi
- $E_{\text{lin}} = 3180$ J

**Second-order**

- $k = 1$ Triangular
- $k = 1$ Voronoi
- $k = 2$ Triangular
- $k = 2$ Voronoi
- $k = 3$ Triangular
- $k = 3$ Voronoi
- $E_{\text{snd}} = 3190$ J

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$^1$Obtained adding third-order terms to the energy density function
Outline

1. Nonlinear elasticity

2. Poroelasticity
The Biot model

- Let $\Omega$ as before, $t_F > 0$ and $\kappa : \Omega \rightarrow \mathbb{R}$ be s.t. $0 < \underline{\kappa} \leq \kappa \leq \overline{\kappa}$ in $\Omega$
- Let $f$ and $g$ be given volumetric load and source terms
- Biot problem: Find the displacement $u$ and the pressure $p$ s.t.

\[
\begin{align*}
-\nabla \cdot \sigma(u) + \nabla p &= f \quad \text{in } \Omega \times (0, t_F), \\
 c_0 \frac{d}{dt} p + \nabla \cdot (d_t u) - \nabla \cdot (\kappa \nabla p) &= g \quad \text{in } \Omega \times (0, t_F),
\end{align*}
\]

completed with initial and boundary conditions (impermeable fixed walls)
- In the incompressible case $c_0 = 0$, we further assume for any $t$

\[
\int_{\Omega} p(\cdot, t) = 0 \text{ and } \int_{\Omega} g(\cdot, t) = 0
\]
- Perspective: extension to the nonlinear, multiphase case
Minimal bibliography

- Origin of the model [Terzaghi, 1943] and [Biot, 1941, Biot, 1955]
- Finite Volumes, 3D, discontinuous coefficients [Naumovich, 2006]
- Continuous FE $u + \text{DG } p$ [Phillips and Wheeler, 2007]
- $\text{DG } u + \text{MPFA } p$ [Wheeler et al., 2014]
- Justification of spurious oscillations [Rodrigo et al., 2016]
- HHO $u + \text{DG } p$ [Boffi, Botti, DP, 2016]
Features

- High-order method on general polyhedral meshes
- Inf-sup-stable hydro-mechanical coupling
- Robustness with respect to heterogeneous-anisotropic permeabilities
- Seamless treatment of the incompressible case $c_0 = 0$
- Locally equilibrated tractions and fluxes
- Numerically robust w.r. to spurious oscillations for small $\kappa$ and $\tau$
Figure: Displacement and pressure discrete unknowns for $k \in \{1, 2\}$

Let $k \geq 1$. We approximate the displacements in the HHO space

$$U_{h,0}^k := \{ v_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) \in U_h^k : v_F = 0 \quad \forall F \in \mathcal{F}_h \}$$

For the pressure, we consider the broken polynomial space

$$P_h^k := \begin{cases} P^k(\mathcal{T}_h; \mathbb{R}) & \text{if } c_0 > 0 \\ P^k(\mathcal{T}_h; \mathbb{R}) \cap L^2_0(\Omega; \mathbb{R}) & \text{if } c_0 = 0 \end{cases}$$
Discrete problem

- We consider for the sake of simplicity a uniform time mesh of size $\tau$
- Discrete problem: For $1 \leq n \leq N$, $(\underline{u}_h^n, p_h^n) \in \underline{U}_{h,0}^k \times P_h^k$ is s.t.

\[
\begin{align*}
    a_h(\underline{u}_h^n, \underline{v}_h) + b_h(\underline{v}_h, p_h^n) &= \int_\Omega f^n \cdot \underline{v}_h & \forall \underline{v}_h \in \underline{U}_{h,0}^k, \\
    (c_0 \delta_t p_h^n, q_h) - b_h(\delta_t \underline{u}_h^n, q_h) + c_h(p_h^n, q_h) &= \int_\Omega g^n q_h & \forall q_h \in P^k(T_h; \mathbb{R})
\end{align*}
\]

- For the mechanical term we use $a_h$ defined as before
The hydro-mechanical coupling hinges on the bilinear form

\[ b_h(v_h, q_h) := - \int_{\Omega} D_h^k v_h q_h, \quad (D_h^k)_{|T} := \text{tr}(G_{s,T}^k) \quad \forall T \in \mathcal{T}_h \]

\( I_T^k \) is a Fortin interpolator: For all \( v \in H^1(\Omega; \mathbb{R}^d) \),

\[ D_h^k I_h^k v = \pi_h^k (\nabla \cdot v), \quad \| I_h^k v \|_{\epsilon,h} \lesssim \| v \|_{H^1(\Omega; \mathbb{R}^d)} \]

Hence, for all \( q_h \in P_h^k \), with hidden constant independent of \( h \),

\[
\begin{align*}
\| q_h \|_{L^2(\Omega)} & \lesssim \sup_{v_h \in U_h^k, \| v_h \|_{\epsilon,h} = 1} b_h(v_h, q_h) \\
\end{align*}
\]

This is a key point for robust \( L^2 \)-norm bounds for \( p \) when \( c_0 = 0 \).
For the Darcy operator we use a Discontinuous Galerkin method.

For robustness in $\kappa$, we follow [DP et al., 2008].

Key ingredients are the jump and weighted average operators

$$[\varphi]_F := \varphi_{T_1} - \varphi_{T_2}$$
$$\{\varphi\}_F := \omega_{T_1} \varphi_{T_1} + \omega_{T_2} \varphi_{T_2},$$

where $F \in \mathcal{F}_h^i$ is s.t. $F \subset \partial T_1 \cap \partial T_2$ and

$$\omega_{T_1} := \frac{\kappa_{T_2}}{\kappa_{T_1} + \kappa_{T_2}}, \quad \omega_{T_2} := \frac{\kappa_{T_1}}{\kappa_{T_1} + \kappa_{T_2}}.$$
The Darcy operator is discretised using the SWIP bilinear form

\[
c_h(r_h, q_h) := \int_{\Omega} \kappa \nabla_h r_h \cdot \nabla_h q_h + \sum_{F \in F_h^1} \frac{\varsigma \lambda_{\kappa, F}}{h_F} \int_F [r_h]_F [q_h]_F \\
- \sum_{F \in F_h^1} \int_F \left( \{\kappa \nabla_h r_h\}_F \cdot n_F, [q_h]_F + [r_h]_F, \{\kappa \nabla_h q_h\}_F \cdot n_F \right)
\]

Here, \( \varsigma > 0 \) is a large enough user-defined penalty parameter and

\[
\lambda_{\kappa, F} := \frac{2\kappa_{T_1} \kappa_{T_2}}{\kappa_{T_1} + \kappa_{T_2}}
\]
Lemma (A priori bounds and well-posedness)

Let $\sigma$ be such that, for all $x \in \Omega$ and all $\tau \in \mathbb{R}^{d \times d}_{\text{sym}}$ with $\mu > 0$ and $\lambda \geq 0$,

$$\sigma(x, \tau) = 2\mu \tau + \lambda \text{tr}(\tau)I_d.$$

Assume $f \in C^1([0, t_F]; L^2(\Omega; \mathbb{R}^d))$ and $g \in C^0([0, t_F]; L^2(\Omega; \mathbb{R}))$. Then, the discrete problem is well-posed with a priori bound

$$\|u_h^N\|_{a,h}^2 + \|c_0^2 p_h^N\|_{L^2(\Omega; \mathbb{R})}^2 + \|P_h^N - \overline{P}_h^N\|_{L^2(\Omega; \mathbb{R})}^2 + \sum_{n=1}^N \tau\|P_h^N\|_{c,h}^2 \lesssim 1$$

where the hidden constant depends on bounded norms of $p^0$, $f$, and $g$ and we have set $\overline{P}_h^N := \int_{\Omega} p_h^N$. 
Main results II

Theorem (Error estimate)

Let $\sigma$ as above. Assume elliptic regularity, $p \in C^1([0, t_F]; H^{k+1}(P_\Omega; \mathbb{R}))$, $p \in C^2([0, t_F]; L^2(\Omega; \mathbb{R}))$ if $c_0 > 0$, and $u \in C^2([0, t_F], H^1(P_\Omega; \mathbb{R}^d)) \cap C^1([0, t_F]; H^{k+2}(P_\Omega; \mathbb{R}^d))$. Then, setting

$$e^n_h := u^n_h - I^n_k u^n, \quad \rho^n_h := p^n_h - \pi^n_k p^n, \quad \bar{\rho}^n_h := (\rho^n_h, 1),$$

it holds

$$\| e^n_h \|_{a,h}^2 + \| c_0^{\frac{1}{2}} \rho^n_h \|_{L^2(\Omega; \mathbb{R})}^2 + \| \rho^n_h - \bar{\rho}^n_h \|_{L^2(\Omega; \mathbb{R})}^2 + \sum_{n=1}^N \tau \| \rho^n_h \|_{c,h}^2 \lesssim \left( h^{k+1} + \tau \right)^2,$$

with hidden constant depending on bounded norms of $u$ and $p$ and increasing linearly with $\alpha^{\frac{1}{2}}$ where $\alpha := \kappa/\kappa$ is the anisotropy ratio.
We let $\Omega = (0, 1)^2$, $c_0 = 0$, $\mu = 1$, $\lambda = 1$, and $\kappa = I_2$ on

The right-hand side is inferred from the (non-physical) exact solution

\begin{align*}
    u_1(x, t) &= -\sin(\pi t) \cos(\pi x_1) \cos(\pi x_2), \\
    u_2(x, t) &= \sin(\pi t) \sin(\pi x_1) \sin(\pi x_2), \\
    p(x, t) &= -\cos(\pi t) \sin(\pi x_1) \cos(\pi x_2)
\end{align*}
Figure: $L^2$-error on the pressure (top) and $H^1$-error on the displacement (bottom) vs. $h$ for (from left to right) the triangular, Voronoi, and locally refined meshes.
Figure: Barry and Mercer's exact solution modelling fluid injection and production from a well
Figure: Pressure profiles along \((0, 0)-(1, 1)\) for \(\kappa = 1 \cdot 10^{-6} I_d\) and \(\tau = 1 \cdot 10^{-4}\). Small oscillations visible on the Cartesian mesh (left, card \(\mathcal{T}_h = 4,028\)), no oscillations are present on the Voronoi mesh (right, card \(\mathcal{T}_h = 4,192\)
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