Numerical approximation of partial differential equations on general meshes

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New generation methods
for numerical simulations

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Outline

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1. Limitations of traditional approaches for the approximation of PDEs
2. Extending non-conforming FEM to polytopal meshes: The HHO method
3. Extending conforming FEM to polytopal meshes
4. The Discrete de Rham complex of differential forms: The DDR method

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Finite element approximations of variational problems I

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open connected polytopal domain
- Consider, e.g., the Poisson problem: Find $u \in V := H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v =: b(v) \quad \forall v \in V$$

- Well-posedness hinges on the Poincaré inequality:

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega)^d} \quad \forall v \in V$$

- Conforming FEM require to find a computable $V_h \subset H_0^1(\Omega)$
- In non-conforming FEM, on the other hand, one can have $V_h \not\subset H_0^1(\Omega)$

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Finite element approximations of variational problems II

- Decompose Ω into disjoint polytopal elements collected in $\mathcal{T}_h$
- Consider conforming finite-dimensional spaces of the form

$$V_h := \left\{ v_h \in H_0^1(\Omega) \ : \ (v_h)_{|T} \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h \right\}$$

- The Poincaré inequality holds a fortiori since $V_h \subset V$

Theorem (Characterization of Lagrange FE functions)

For all $v_h \in \mathcal{P}^k(\mathcal{T}_h)$, $v_h \in V_h$ iff v_h is continuous and $v_{|\partial\Omega} = 0$.

Finite element approximations of variational problems III

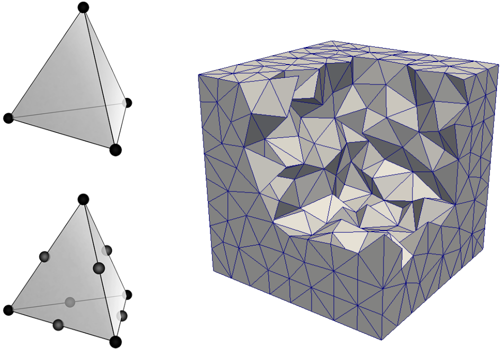
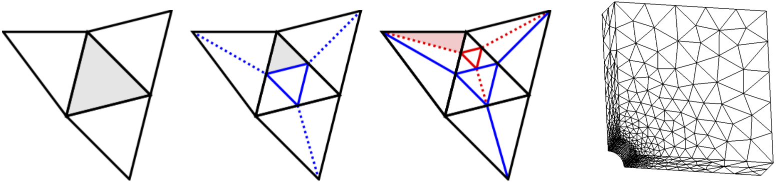


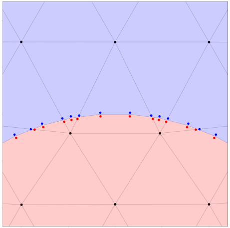
Figure: Lagrange FEM (left) and conforming tetrahedral mesh (right)

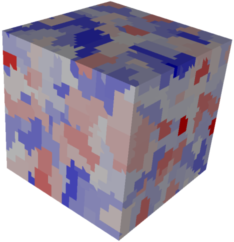
Limitations

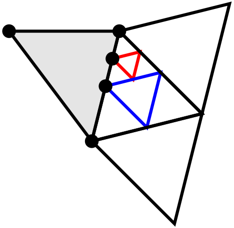


- Approach limited to conforming meshes with standard elements
  - Local refinement requires to trade mesh size for quality
  - Complex geometries may require a large number of elements
  - The element shape cannot be adapted to the solution
  - Embedded interfaces are difficult to capture
- The extension to more complicated spaces is also not straightforward

Polytopal approaches







- Support of polyhedral meshes and high-order
- Higher-level point of view to handle problems set on complicated spaces
- Simple treatment of embedded interfaces[1]
- Several strategies to enhance efficiency[2]

[1] [Di Pietro et al., 2025b]
[2] See, e.g., [Bassi et al., 2012, Di Pietro and Droniou, 2023]

Outline

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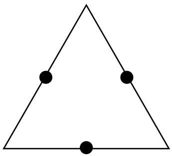
The Crouzeix–Raviart (CR) element



- Let $\mathcal{P}^1(T)$ be the space of affine functions on T
- Define the degrees of freedom (basis of the dual space) $\sigma \coloneqq (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

$$\sigma_F : \mathcal{P}^1(T) \ni v \mapsto v_F \coloneqq \frac{1}{|F|} \int_F v \in \mathbb{R}$$

- The triplet $(T, \mathcal{P}^1(T), \sigma)$ is a FE[3] in the sense of Ciarlet
- On a conforming mesh $\mathcal{T}_h$, these DOFs yield middle-point continuity

[3] [Crouzeix and Raviart, 1973]

A non-conforming FE scheme

- Let $V_{h,0} \not\subset H_0^1(\Omega)$ be the global CR space on $\mathcal{T}_h$ with zero boundary DOFs
- We consider the scheme: Find $u_h \in V_{h,0}$ s.t.

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in V_{h,0}$$

- Well-posedness follows from the *discrete* Poincaré inequality:

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d} \qquad \forall v_h \in V_{h,0}$$

- Assuming for the exact solution $u \in H_0^1(\Omega) \cap H^2(\mathcal{T}_h)$, one can prove that

$$\|\nabla_h(u - u_h)\|_{L^2(\Omega)^d} \lesssim h|u|_{H^2(\mathcal{T}_h)}$$

Can we extend this method to general meshes and arbitrary order?

Shifting point of view I

- Let $v \in \mathcal{P}^1(T)$ and set, for all $F \in \mathcal{F}_T$,

$$v_F \coloneqq \pi_F^0 v = \frac{1}{|F|}\int_F v$$

- ∇v is fully determined in terms of the values $(v_F)_{F \in \mathcal{F}_T}$ by the equation

$$\int_T \nabla v \cdot \nabla w \overset{\text{IBP}}{=} \sum_{F \in \mathcal{F}_T} \int_F v \underbrace{(\nabla w \cdot n_{TF})}_{\in \mathcal{P}^0(F)} = \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^1(T),$$

where we have used $\int_T (x - \overline{x}_T) = 0$ to cancel the volumetric term

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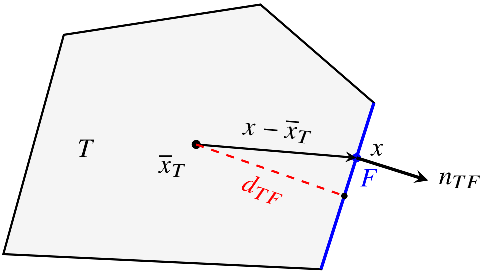
Shifting point of view II

- To express the average value of v in terms of $(v_F)_{F\in\mathcal{F}_T}$, we can write

$$\int_T v = \frac{1}{d}\int_T v\,\mathrm{div}(x-\overline{x}_T) \overset{\text{IBP}}{=} \frac{1}{d}\sum_{F\in\mathcal{F}_T}\int_F v\,\underbrace{(x-\overline{x}_T)\cdot n_{TF}}_{d_{TF}\in\mathbb{R}} = \sum_{F\in\mathcal{F}_T}\frac{d_{TF}}{d}\int_F v_F$$



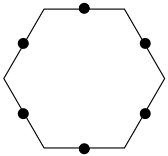
Shifting point of view III

- In conclusion, any $v \in \mathcal{P}^1(T)$ is the unique solution of

$$\int_T \nabla v \cdot \nabla w = \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^1(T),$$
$$\int_T v = \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F v_F$$

- **This remains true if T is a (reasonable) polytope with planar faces!**

A lowest-order hybrid space



- Denote by $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ a polytopal mesh of Ω
- Instead of function spaces, we consider a space spanned by vectors of polynomials:

$$\underline{V}_h^0 := \left\{ \underline{v}_h = (v_F)_{F \in \mathcal{F}_h} \ : \ v_F \in \mathcal{P}^0(F) \text{ for all } F \in \mathcal{F}_h \right\}$$

- Smooth functions $v : \Omega \to \mathbb{R}$ are interpolated through $\underline{I}_h^0$ s.t.

$$\underline{I}_h^0 v := \left(\pi_F^0 v \right)_{F \in \mathcal{F}_h} \in \underline{V}_h^0$$

An affine potential reconstruction

- Let $T \in \mathcal{T}_h$ and denote by $\underline{V}_T^0$ the restriction of $\underline{V}_h^0$ to T
- Inspired by the previous remark, we define $p_T^1 : \underline{V}_T^0 \to \mathcal{P}^1(T)$ s.t.

$$\int_T \nabla p_T^1 \underline{v}_T \cdot \nabla w = \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \qquad \forall w \in \mathcal{P}^1(T),$$

$$\int_T p_T^1 \underline{v}_T = \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F v_F$$

- With $\underline{I}_T^0$ restriction of $\underline{I}_h^0$ to T, it holds, by construction,

$$p_T^1(\underline{I}_T^0 v) = v \quad \forall v \in \mathcal{P}^1(T)$$

- On triangles, $p_T^1 \underline{v}_T$ is the Crouzeix–Raviart function with DOFs $(v_F)_{F \in \mathcal{F}_T}$

Extension to arbitrary order I

- Given $Y \in \mathcal{T}_h \cup \mathcal{F}_h$ and $\ell \geq 0$, let $\pi_Y^\ell : L^2(Y) \to \mathcal{P}^\ell(Y)$ be s.t.

$$\int_Y \pi_Y^\ell v\, w = \int_Y v\, w \quad \forall w \in \mathcal{P}^\ell(Y)$$

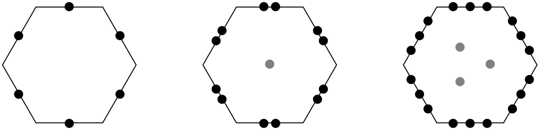
- Proceeding as before, it can be shown that any $v \in \mathcal{P}^{k+1}(T)$, $k \geq 0$, solves

$$\int_T \nabla v \cdot \nabla w = -\int_T \pi_T^{k-1} v\, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^k v\, (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\int_T v = \begin{cases} \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F \pi_F^0 v & \text{if } k = 0, \\ \int_T \pi_T^{k-1} v & \text{if } k \geq 1, \end{cases}$$

with the convention that $\mathcal{P}^{-1}(T) = \{0\}$

Extension to arbitrary order II



- This suggests to consider the following arbitrary-order extension of $\underline{V}_h^0$:

$$\underline{V}_h^k := \Big\{ \underline{v}_h = \big((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h} \big) \ : \\ v_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h,\ v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \Big\}$$

- The natural interpolator $\underline{I}_h^k : H^1(\Omega) \to \underline{V}_h^k$ is s.t., for all $v \in H^1(\Omega)$,

$$\underline{I}_h^k v := ((\pi_T^{k-1} v)_{T\in\mathcal{T}_h}, (\pi_F^k v)_{F\in\mathcal{F}_h})$$

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An arbitrary-order potential reconstruction

- p_T^1 generalizes to $p_T^{k+1} : \underline{V}_T^k \to \mathcal{P}^{k+1}(T)$ s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\boxed{\begin{gathered}\int_T \nabla p_T^{k+1} \underline{v}_T \cdot \nabla w = -\int_T v_T \, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F \, (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T), \\ \int_T p_T^{k+1} \underline{v}_T = \begin{cases} \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F v_F & \text{if } k = 0, \\ \int_T v_T & \text{if } k \geq 1 \end{cases}\end{gathered}}$$

- By similar arguments as before, we have polynomial consistency:

$$p_T^{k+1}(\underline{I}_T^k v) = v \quad \forall v \in \mathcal{P}^{k+1}(T)$$

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Can this construction be used to design a stable and convergent numerical method?

A discrete Poincaré inequality in hybrid spaces I

- Consider, for simplicity, the case $k \geq 1$ ($k = 0$ requires minor changes)
- For all $\underline{v}_h \in \underline{V}_h^k$, let $v_h \in L^2(\Omega)$ (not underlined) be s.t.

$$(v_h)_{|T} = v_T \quad \forall T \in \mathcal{T}_h$$

- Define the subspace of $\underline{V}_h^k$ with homogeneous boundary conditions

$$\underline{V}_{h,0}^k := \left\{ \underline{v}_h \in \underline{V}_h^k \ : \ v_F = 0 \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega \right\}$$

- For stability, we need to establish a discrete Poincaré inequality

A discrete Poincaré inequality in hybrid spaces II

Lemma (Discrete Poincaré inequality in hybrid spaces)

For all $\underline{v}_h \in \underline{V}_{h,0}^k$, *it holds*

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h}.$$

where $\|\cdot\|_{1,h}$ *s.t., for all* $\underline{v}_h \in \underline{V}_{h,0}^k$,

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2,$$

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2.$$

A discrete Poincaré inequality in hybrid spaces III

- By surjectivity of $\operatorname{div} : H^1(\Omega)^d \to L^2(\Omega)$, there is $\tau \in H^1(\Omega)^d$ s.t.

$$\operatorname{div} \tau = v_h \text{ and } \|\tau\|_{H^1(\Omega)^d} \lesssim \|v_h\|_{L^2(\Omega)}$$

- We thus have

$$\begin{aligned} \|v_h\|_{L^2(\Omega)}^2 &= \sum_{T \in \mathcal{T}_h} \int_T v_T \operatorname{div} \tau \\ &\overset{\text{IBP}}{=} \sum_{T \in \mathcal{T}_h} \left[-\int_T \nabla v_T \cdot \tau + \sum_{F \in \mathcal{F}_T} \int_F (v_T - {\color{red} v_F})(\tau \cdot n_{TF}) \right] \end{aligned}$$

- Applying Cauchy–Schwarz and trace inequalities, we go on writing

$$\|v_h\|_{L^2(\Omega)}^{\cancel{2}} \lesssim \|\underline{v}_h\|_{1,h} \|\tau\|_{H^1(\Omega)^d} \lesssim \|\underline{v}_h\|_{1,h} \cancel{\|v_h\|_{L^2(\Omega)}}$$

A coercive bilinear form

- We let $a_h : \underline{V}_h^k \times \underline{V}_h^k \to \mathbb{R}$ be s.t.

$$a_h(\underline{u}_h, \underline{v}_h) \coloneqq \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T),$$

$$a_T(\underline{u}_T, \underline{v}_T) \coloneqq \int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla p_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

- We assume that, for all $T \in \mathcal{T}_h$, s_T is s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\boxed{a_T(\underline{v}_T, \underline{v}_T) = \|\nabla p_T^{k+1} \underline{v}_T\|_{L^2(T)^d}^2 + s_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,T}^2} \tag{ST1}$$

- Under (ST1), we obviously have $\|\cdot\|_{1,h}$-coercivity for a_h:

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

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Discrete problem and basic error estimate I

- We consider the Hybrid High-Order method: Find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \qquad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

- **Unlike FEM, we cannot plug u into a_h to prove convergence!**
- We instead study the discrete error defined as

$$\underline{e}_h := \underline{u}_h - \underline{I}_h^k u$$

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Discrete problem and basic error estimate II

- $\underline{e}_h$ solves, with $\mathcal{E}_h : \underline{V}_{h,0}^k \to \mathbb{R}$ consistency error linear form,

$$a_h(\underline{e}_h, \underline{v}_h) = \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h) =: \mathcal{E}_h(\underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k,$$

- Denoting by $\|\cdot\|_{1,h,*}$ the norm dual to $\|\cdot\|_{1,h}$, we thus have

$$\|\underline{e}_h\|_{1,h}^2 \lesssim a_h(\underline{e}_h, \underline{e}_h) \le \|\mathcal{E}_h\|_{1,h,*} \|\underline{e}_h\|_{1,h},$$

  leading to the error estimate

$$\boxed{\|\underline{e}_h\|_{1,h} \lesssim \|\mathcal{E}_h\|_{1,h,*}}$$

- A complete characterization of s_T comes from the study of $\|\mathcal{E}_h\|_{1,h,*}$

Consistency error estimate

- Letting $\varpi_T^{k+1} := p_T^{k+1} \circ \underline{I}_T^k$, it can be proved that

$$\mathcal{E}_h(u;\underline{v}_h) = \underbrace{\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F \nabla(u-\varpi_T^{k+1}u)\cdot n_{TF}(v_F - v_T)}_{\mathfrak{T}_1} \underbrace{- \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)}_{\mathfrak{T}_2}$$

- Assuming $u_{|T} \in H^{k+2}(T)$ for all $T \in \mathcal{T}_h$, we have

$$|\mathfrak{T}_1| \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

- $|\mathfrak{T}_2|$ scales in h^{k+1} as well if s_T is polynomially consistent:

$$\boxed{s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w,\underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k} \tag{ST2}$$

Local stabilization

Proposition (Structure of the stabilization bilinear form)

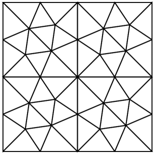
(ST1)-(ST2) *hold only if* s_T *depends on its arguments through*

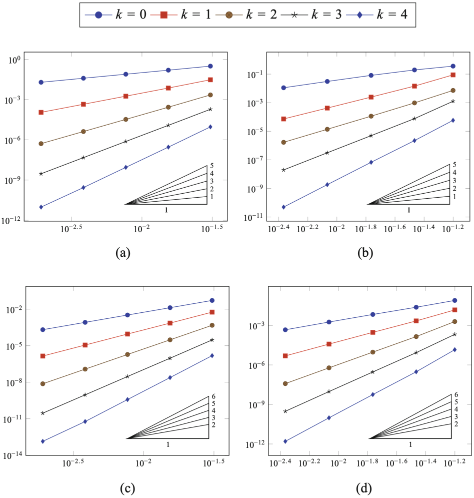
$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) \coloneqq \underline{I}_T^k p_T^{k+1} \underline{v}_T - \underline{v}_T.$$

Example ([Di Pietro et al., 2014, Di Pietro and Ern, 2015])

$$s_T(\underline{w}_T, \underline{v}_T) \coloneqq h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F (\delta_{TF}^k \underline{w}_T - \delta_T^k \underline{w}_T)(\delta_{TF}^k \underline{v}_T - \delta_T^k \underline{v}_T)$$

Numerical examples I





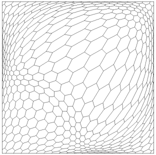


Figure: $\|\underline{e}_h\|_{1,h}$ (top) and $\|e_h\|_{L^2(\Omega)}$ vs. h

Numerical examples II

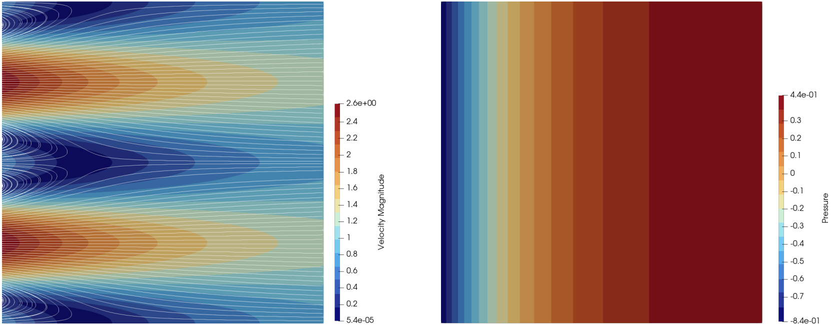


Figure: HHO solution of the incompressible Navier–Stokes problem[4]

Numerical examples III

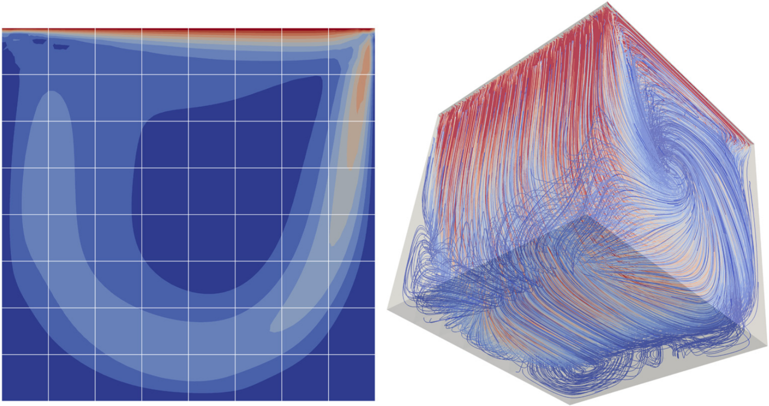


Figure: HHO solution of the three-dimensional lid-driven cavity problem[5]

[4][Di Pietro, 2026]
[5][Botti, Di Pietro, Droniou, 2019]

Introductory references for the Hybrid High-Order method

- D. A. Di Pietro, A. Ern, and S. Lemaire, **An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators**, Comput. Meth. Appl. Math., 2014, 14(4):461–472. 10.1515/cmam-2014-0018
- D. A. Di Pietro and A. Ern, **A hybrid high-order locking-free method for linear elasticity on general meshes**, Comput. Meth. Appl. Mech. Engrg., 2015, 283:1–21. 10.1016/j.cma.2014.09.009
- D. A. Di Pietro and J. Droniou, **The Hybrid High-Order method for polytopal meshes**, Springer International Publishing, 2020, 10.1007/978-3-030-37203-3

Outline

The de Rham complex on a domain $\Omega \subset \mathbb{R}^3$

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- This is a complex by classical vector calculus identities
- The cohomology spaces are

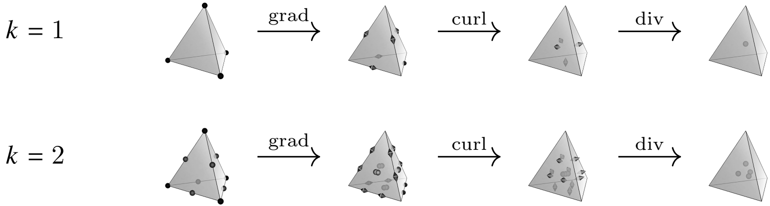
$$\mathcal{H}_1 \coloneqq \text{Ker}\,\text{curl}/\text{Im}\,\text{grad} \quad \text{and} \quad \mathcal{H}_2 \coloneqq \text{Ker}\,\text{div}/\text{Im}\,\text{curl}$$

- de Rham's theorem states that, with b_i Betti numbers,

$$\dim(\mathcal{H}_1) = b_1 \quad \text{(number of tunnels crossing } \Omega\text{)}$$
$$\dim(\mathcal{H}_2) = b_2 \quad \text{(number of voids contained in } \Omega\text{)}$$

- **Emulating these properties is key for stable discretizations!**

- Trimmed FE complexes[6] on a tetrahedron T: For any $k \geq 1$,



- On a conforming tetrahedral mesh $\mathcal{T}_h$, local spaces can be glued together

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl};\Omega) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \\
\mathcal{P}^k_{\mathrm{c}}(\mathcal{T}_h) \arrow[u, hook] \arrow[r, "\mathrm{grad}"] & \mathcal{N}^k(\mathcal{T}_h) \arrow[u, hook] \arrow[r, "\mathrm{curl}"] & \mathcal{RT}^k(\mathcal{T}_h) \arrow[u, hook] \arrow[r, "\mathrm{div}"] & \mathcal{P}^{k-1}(\mathcal{T}_h) \arrow[u, hook]
\end{tikzcd}$$

[6] [Raviart and Thomas, 1977, Nédélec, 1980]

Mesh-related notations

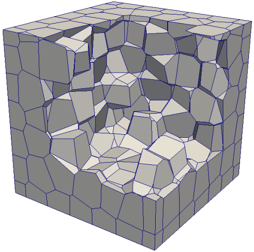


Figure: Example of polyhedral mesh $\mathcal{M}_h = \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$

- $\mathcal{T}_h = \Delta_3(\mathcal{M}_h)$ set of polyhedral elements (3-cells)
- $\mathcal{F}_h = \Delta_2(\mathcal{M}_h)$ set of polygonal (flat) faces (2-cells)
- $\mathcal{E}_h = \Delta_1(\mathcal{M}_h)$ set of edges (1-cells)
- $\mathcal{V}_h = \Delta_0(\mathcal{M}_h)$ set of vertices (0-cells)

Cohomology of the trimmed FE complex

- If $\mathcal{M}_h$ is a simplicial complex (FE mesh), we have

$$
\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathcal{P}_c^1(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, "\kappa_{0,h}", "\cong"'] \arrow[d, bend right, hook] & \mathcal{N}^1(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, "\kappa_{1,h}", "\cong"'] \arrow[d, bend right, hook] & \mathcal{RT}^1(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, "\kappa_{2,h}", "\cong"'] \arrow[d, bend right, hook] & \mathcal{P}^0(\mathcal{T}_h) \arrow[u, "\kappa_{3,h}", "\cong"'] \arrow[d, bend right, hook] \\
\mathcal{P}_c^k(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, bend right, "I^1_{\mathrm{grad},h}"'] & \mathcal{N}^k(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, bend right, "I^1_{\mathrm{curl},h}"'] & \mathcal{RT}^k(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, bend right, "I^1_{\mathrm{div},h}"'] & \mathcal{P}^{k-1}(\mathcal{T}_h) \arrow[u, bend right, "\pi_h^0"']
\end{tikzcd}
$$

  with κ_h de Rham map, $I^1_{\bullet,h}$ interpolator, and π_h^0 L^2-orthogonal projector
- It can be proved that the three complexes have isomorphic cohomologies

Shifting point of view I

- Denoting by "dofs" the standard FE degrees of freedom, we have

$$
\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathcal{P}^1_{\mathrm{c}}(\mathcal{T}_h) \arrow[u, "\kappa_{0,h}", "\cong"'] \arrow[r, "\mathrm{grad}"] & \mathcal{N}^1(\mathcal{T}_h) \arrow[u, "\kappa_{1,h}", "\cong"'] \arrow[r, "\mathrm{curl}"] & \mathcal{RT}^1(\mathcal{T}_h) \arrow[u, "\kappa_{2,h}", "\cong"'] \arrow[r, "\mathrm{div}"] & \mathcal{P}^0(\mathcal{T}_h) \arrow[u, "\kappa_{3,h}", "\cong"'] \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"']
\end{tikzcd}
$$

- In the previous diagram, we can erase the middle row

Shifting point of view II

- Set $\color{red}{K_h := \kappa_h \circ \mathsf{dofs}^{-1}}$, i.e., $\forall(\underline{q}_h, \underline{v}_h, \underline{w}_h, \underline{r}_h) \in \mathbb{R}^{\mathcal{V}_h} \times \mathbb{R}^{\mathcal{E}_h} \times \mathbb{R}^{\mathcal{F}_h} \times \mathbb{R}^{\mathcal{T}_h}$,

$$
\begin{aligned}
K_{0,h}\underline{q}_h(V) &:= q_V \quad \forall V \in \mathcal{V}_h, & K_{1,h}\underline{v}_h(E) &:= |E| v_E \quad \forall E \in \mathcal{E}_h,\\
K_{2,h}\underline{w}_h(F) &:= |F| v_F \quad \forall F \in \mathcal{F}_h, & K_{3,h} r_h &:= |T| r_T \quad \forall T \in \mathcal{T}_h
\end{aligned}
$$

- This graded map induces the following isomorphisms:

$$
\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "K_{0,h}", "\cong"'] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "K_{1,h}", "\cong"'] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "K_{2,h}", "\cong"'] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "K_{3,h}", "\cong"']
\end{tikzcd}
$$

- **Can we complete the bottom row to form a complex?**

Shifting point of view III

- Define the following discrete gradient, curl, and divergence operators:

$$\underline{G}_h^0 \underline{q}_h \coloneqq K_{1,h}^{-1} \partial_0 K_{0,h}, \quad \underline{C}_h^0 \underline{v}_h \coloneqq K_{2,h}^{-1} \partial_1 K_{1,h}, \quad \underline{D}_h^0 \underline{w}_h \coloneqq K_{3,h}^{-1} \partial_2 K_{2,h},$$

with, by construction,

$$\underline{C}_h^0 \circ \underline{G}_h^0 = \underline{0} \text{ and } \underline{D}_h^0 \circ \underline{C}_h^0 = \underline{0}$$

- Hence, we have two complexes with isomorphic cohomologies:

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "K_{0,h}", "\cong"'] \arrow[r, "\underline{G}_h^0"] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "K_{1,h}", "\cong"'] \arrow[r, "\underline{C}_h^0"] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "K_{2,h}", "\cong"'] \arrow[r, "\underline{D}_h^0"] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "K_{3,h}", "\cong"']
\end{tikzcd}$$

- **Still true with $\mathcal{M}_h$ CW complex associated to a polyhedral mesh!**

By again getting rid of FE spaces, we can now handle polyhedral meshes!

Outline

Can we generalize the above construction to arbitrary dimension and polynomial degree?

The continuous de Rham complex

- The de Rham complex for a domain Ω of $\mathbb{R}^n$ reads

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{r-1}} H\Lambda^r(\Omega) \xrightarrow{\mathrm{d}^r} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

where d is the exterior derivative and

$$H\Lambda^r(M) := \left\{\omega \in L^2\Lambda^r(M) \ : \ \mathrm{d}\omega \in L^2\Lambda^{r+1}(M)\right\}$$

- For $n = 3$, the following isomorphisms are provided by vector proxies:

$$\begin{tikzcd}
H\Lambda^0(\Omega) \arrow[r, "\mathrm{d}^0"] \arrow[d, leftrightarrow, "\cong"] & H\Lambda^1(\Omega) \arrow[r, "\mathrm{d}^1"] \arrow[d, leftrightarrow, "\cong"] & H\Lambda^2(\Omega) \arrow[r, "\mathrm{d}^2"] \arrow[d, leftrightarrow, "\cong"] & H\Lambda^3(\Omega) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \{0\} \\
H^1(\Omega) \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl};\Omega) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \arrow[r] & \{0\}
\end{tikzcd}$$

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Trimmed spaces

- Let $\mathsf{C} \in \Delta_d(\mathcal{M}_h)$. With $\kappa_{\mathsf{C}} : \Lambda^{\ell+1}(\mathsf{C}) \to \Lambda^{\ell}(\mathsf{C})$ Koszul differential on C, set

$$\mathcal{K}^{k,\ell}(\mathsf{C}) \coloneqq \kappa_{\mathsf{C}} \mathcal{P}^{k-1}\Lambda^{\ell+1}(\mathsf{C})$$

- The following decomposition holds:

$$\mathcal{P}^k\Lambda^\ell(\mathsf{C}) \coloneqq \mathrm{d}\mathcal{K}^{k+1,\ell-1}(\mathsf{C}) \oplus \mathcal{K}^{k,\ell}(\mathsf{C})$$

- Trimmed spaces are obtained lowering the degree of the first component[7]:

$$\begin{aligned}
\mathcal{P}^k_-\Lambda^0(\mathsf{C}) &\coloneqq \mathcal{P}^k\Lambda^0(\mathsf{C}),\\
\mathcal{P}^k_-\Lambda^\ell(\mathsf{C}) &\coloneqq \mathrm{d}\mathcal{K}^{k,\ell-1}(\mathsf{C}) \oplus \mathcal{K}^{k,\ell}(\mathsf{C}) \quad \text{if } \ell \geq 1
\end{aligned}$$

- The L^2-orthogonal projector onto $\mathcal{P}^k_-\Lambda^r(\mathsf{C})$ is $\pi_{\mathcal{P}^k_-\Lambda^r(\mathsf{C})}$

[7] See, e.g., [Arnold, 2018]

Trimmed spaces in dimensions 2 and 3

- Let $n = 3$. For $T = \mathsf{C}_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$, we have, through vector proxies,

$$\mathcal{P}_-^k\Lambda^1(\mathsf{C}_3) \cong \mathcal{N}^k(T), \qquad \mathcal{P}_-^k\Lambda^2(\mathsf{C}_3) \cong \mathcal{RT}^k(T)$$

- For $F = \mathsf{C}_2 \in \Delta_2(\mathcal{M}_h)$, we have

$$\mathcal{P}_-^k\Lambda^1(\mathsf{C}_2) \cong \mathcal{RT}^k(F).$$

Discrete spaces and interpolators

- The discrete $H\Lambda^r(\Omega)$ space, $0 \le r \le n$, is

$$\underline{X}_h^{k,r} := \bigtimes_{d=r}^{n} \bigtimes_{\mathsf{C}\in\Delta_d(\mathcal{M}_h)} \mathcal{P}_-^k\Lambda^{d-r}(\mathsf{C})$$

- The components are interpreted as L^2-orthogonal projections of traces:

$$\underline{I}_{\mathsf{C}}^{k,r}(\omega) := \left(\pi_{\mathcal{P}_-^k\Lambda^{d'-r}(\mathsf{C}')}(\star\,\mathrm{tr}_{\mathsf{C}'}\,\omega)\right)_{\mathsf{C}'\in\Delta_{d'}(\mathsf{C}),\, d'\in[r,d]}$$

	V	E	F	T
$\underline{X}_h^{k,0}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}_h^{k,1}$	–	$\mathcal{P}^{k}(E)$	$\mathcal{RT}^{k}(F)$	$\mathcal{RT}^{k}(T)$
$\underline{X}_h^{k,2}$	–	–	$\mathcal{P}^{k}(F)$	$\mathcal{N}^{k}(T)$
$\underline{X}_h^{k,3}$	–	–	–	$\mathcal{P}^{k}(T)$

Table: Components of discrete spaces for $n = 3$

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Discrete potential and exterior derivative I

- As in the non-conforming case, reconstructions mimic IBP formulas
- Stokes formula on C: For $\omega \in \Lambda^r(\mathsf{C})$ and $\mu \in \Lambda^{d-r-1}(\mathsf{C})$ smooth enough,

$$\int_{\mathsf{C}} \mathrm{d}\omega \wedge \mu = (-1)^{r+1} \int_{\mathsf{C}} \omega \wedge \mathrm{d}\mu + \int_{\partial\mathsf{C}} \mathrm{tr}_{\partial\mathsf{C}}\, \omega \wedge \mathrm{tr}_{\partial\mathsf{C}}\, \mu$$

- Denote the restrictions of $\underline{X}_h^{k,r}$ to $\mathsf{C} \in \Delta_d(\mathcal{M}_h)$ and $\partial\mathsf{C}$ by $\underline{X}_{\mathsf{C}}^{k,r}$ and $\underline{X}_{\partial\mathsf{C}}^{k,r}$

Discrete potential and exterior derivative II

- For $d = r$, we set
$$P_{\mathsf{C}}^{k,r}\underline{\omega}_{\mathsf{C}} \coloneqq \omega_{\mathsf{C}} \in \star^{-1}\mathcal{P}^k\Lambda^d(\mathsf{C})$$
- For $d = r + 1 \ldots n$, we first let, for all $\underline{\omega}_{\mathsf{C}} \in \underline{X}_{\mathsf{C}}^{k,r}$ and $\mu \in \mathcal{P}^k\Lambda^{d-r-1}(\mathsf{C})$,
$$\int_{\mathsf{C}} \mathrm{d}_{\mathsf{C}}^{k,r}\underline{\omega}_{\mathsf{C}} \wedge \mu = (-1)^{r+1}\int_{\mathsf{C}} \star^{-1}\omega_{\mathsf{C}} \wedge \mathrm{d}\mu + \int_{\partial\mathsf{C}} P_{\partial\mathsf{C}}^{k,r}\underline{\omega}_{\partial\mathsf{C}} \wedge \mathrm{tr}_{\partial\mathsf{C}}\,\mu$$
then, for all $(\mu,\nu) \in \mathcal{K}^{k+1,d-r-1}(\mathsf{C}) \times \mathcal{K}^{k,d-r}(\mathsf{C})$,
$$(-1)^{r+1}\int_{\mathsf{C}} P_{\mathsf{C}}^{k,r}\underline{\omega}_{\mathsf{C}} \wedge \mathrm{d}\mu = \int_{\mathsf{C}} \mathrm{d}_{\mathsf{C}}^{k,r}\underline{\omega}_{\mathsf{C}} \wedge \mu - \int_{\partial\mathsf{C}} P_{\partial\mathsf{C}}^{k,r}\underline{\omega}_{\partial\mathsf{C}} \wedge \mathrm{tr}_{\partial\mathsf{C}}\,\mu$$
$$\int_{\mathsf{C}} P_{\mathsf{C}}^{k,r}\underline{\omega}_{\mathsf{C}} \wedge \nu = \int_{\mathsf{C}} \star^{-1}\omega_{\mathsf{C}} \wedge \nu$$

Consistency of the local reconstructions

Lemma (Polynomial consistency)

For all integers $0 \le r \le d \le n$ and all $\mathsf{C} \in \Delta_d(\mathcal{M}_h)$, it holds

$$P_{\mathsf{C}}^{k,r}(\underline{I}_{\mathsf{C}}^{k,r}\omega) = \omega \qquad \forall \omega \in \mathcal{P}^k\Lambda^r(\mathsf{C}),$$

and, if $d \ge r+1$,

$$\mathrm{d}_{\mathsf{C}}^{k,r}(\underline{I}_{\mathsf{C}}^{k,r}\omega) = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}_-^{k+1}\Lambda^r(\mathsf{C}).$$

Hence, $P_{\mathsf{C}}^{k,r} \circ \underline{I}_{\mathsf{C}}^{k,r}$ is a projector with optimal approximation properties.

Remark (The case of 0-forms)

There also exists $P_{\mathsf{C}}^{k+1,0}$ consistent up to degree $k+1$.

Global discrete exterior derivative and DDR complex

- The spaces $\underline{X}_h^{k,r}$ are connected by the global discrete exterior derivative

$$\underline{X}_h^{k,r} \ni \underline{\omega}_h \mapsto \underline{\mathrm{d}}_h^{k,r}\underline{\omega}_h \coloneqq \left(\pi_{\mathcal{P}_-^k\Lambda^{d-r-1}(\mathsf{C})} \star \mathrm{d}_{\mathsf{C}}^{k,r}\underline{\omega}_{\mathsf{C}}\right)_{\mathsf{C}\in\Delta_d(\mathcal{M}_h),\, d\in[r+1,n]} \in \underline{X}_h^{k,r+1}$$

- The DDR sequence then reads

$$\underline{X}_h^{k,0} \xrightarrow{\underline{\mathrm{d}}_h^{k,0}} \underline{X}_h^{k,1} \longrightarrow \cdots \longrightarrow \underline{X}_h^{k,n-1} \xrightarrow{\underline{\mathrm{d}}_h^{k,n-1}} \underline{X}_h^{k,n} \longrightarrow \{0\}$$

- For $k = 0$ and $n = 3$, we recover the complex discussed in the first part

Theorem (Cohomology)

The DDR complex has cohomology isomorphic to that of the de Rham complex.

How can this construction be used to design stable and convergent schemes?

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Discrete L^2-products and Poincaré inequality

- We define the global discrete L^2-product s.t., $\forall(\underline{\omega}_h, \underline{\mu}_h) \in \underline{X}_h^{k,r} \times \underline{X}_h^{k,r}$,

$$(\underline{\omega}_h, \underline{\mu}_h)_{r,h} \coloneqq \sum_{\mathsf{C}\in\Delta_n(\mathcal{M}_h)} (\underline{\omega}_{\mathsf{C}}, \underline{\mu}_{\mathsf{C}})_{r,\mathsf{C}},$$

$$(\underline{\omega}_{\mathsf{C}}, \underline{\mu}_{\mathsf{C}})_{r,\mathsf{C}} \coloneqq \int_{\mathsf{C}} P_{\mathsf{C}}^{k,r} \underline{\omega}_{\mathsf{C}} \wedge \star P_{\mathsf{C}}^{k,r} \underline{\mu}_{\mathsf{C}} + s_{r,\mathsf{C}}(\underline{\omega}_{\mathsf{C}}, \underline{\mu}_{\mathsf{C}})$$

- $s_{r,\mathsf{C}}$ is a stabilization ensuring positivity and polynomial consistency

Theorem (Discrete Poincaré inequality on the orthogonal complement[8])

For all $\underline{\omega}_h \in (\operatorname{Ker} \underline{\mathrm{d}}_h^{k,r})^{\perp}$ with orthogonal taken w.r.t. to $(\cdot,\cdot)_{r,h}$, it holds

$$\|\underline{\omega}_h\|_{r,h} \lesssim \|\underline{\mathrm{d}}_h^{k,r} \underline{\omega}_h\|_{r+1,h}.$$

[8] [Di Pietro and Hanot, 2024, Di Pietro et al., 2025a]

Adjoint consistency

The following result estimates the error on the global Stokes formula

$$\int_\Omega \omega \wedge \mathrm{d}\mu + (-1)^r \int_\Omega \mathrm{d}\omega \wedge \mu = 0 \qquad \forall (\omega, \mu) \in \Lambda^r(\Omega) \times \Lambda^{n-r-1}(\Omega),\ \mathrm{tr}_{\partial\Omega}\,\omega = 0$$

Theorem (Adjoint consistency)

Let $\omega \in C^0\Lambda^r(\overline{\Omega}) \cap H^{k+2}\Lambda^r(\mathcal{T}_h)$ *be s.t.* $\mathrm{tr}_{\partial\Omega}\,\omega = 0$. *Then,* $\forall \underline{\mu}_h \in \underline{X}_h^{k,n-r-1}$,

$$\left| \int_\Omega \omega \wedge \underline{\mathrm{d}}_h^{k,n-r-1} \underline{\mu}_h + (-1)^r \sum_{f \in \Delta_n(\mathcal{M}_h)} \int_f \mathrm{d}\omega \wedge P_f^{k,n-r-1} \underline{\mu}_f \right|$$
$$\lesssim h^{k+1} \left(|\omega|_{H^{k+1}\Lambda^r(\mathcal{T}_h)} \|\underline{\mathrm{d}}_h^{k,n-r-1} \underline{\mu}_h\|_{n-r,h} + |\mathrm{d}\omega|_{H^{k+1}\Lambda^{r+1}(\mathcal{T}_h)} \|\underline{\mu}_h\|_{n-r-1,h} \right).$$

An example of numerical scheme

- Assume $b_2 = 0$ for simplicity and consider the magnetostatics problem:
  Find $(H, A) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ s.t.

$$
\begin{aligned}
\mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\,\tau &= 0 && \forall \tau \in H(\mathrm{curl};\Omega),\\
\int_\Omega \mathrm{curl}\,H \cdot v + \int_\Omega \mathrm{div}\,A\,\mathrm{div}\,v &= \int_\Omega J \cdot v && \forall v \in H(\mathrm{div};\Omega)
\end{aligned}
$$

- A DDR scheme for this problem is obtained with obvious substitutions:
  Find $(\underline{H}_h, \underline{A}_h) \in \underline{X}_h^{k,1} \times \underline{X}_h^{k,2}$ s.t.

$$
\begin{aligned}
\mu(\underline{H}_h, \underline{\tau}_h)_{1,h} - (\underline{A}_h, \underline{\mathrm{d}}_h^{k,1}\underline{\tau}_h)_{2,h} &= 0 && \forall \underline{\tau}_h \in \underline{X}_h^{k,1},\\
(\underline{\mathrm{d}}_h^{k,1}\underline{H}_h, \underline{v}_h)_{2,h} + \int_\Omega \mathrm{d}_h^{k,2}\underline{A}_h\,\mathrm{d}_h^{k,2}\underline{v}_h &= (\underline{I}_h^{k,2}J, \underline{v}_h)_{2,h} && \forall \underline{v}_h \in \underline{X}_h^{k,2}
\end{aligned}
$$

- **Stability mimics the continuous argument for well-posedness**

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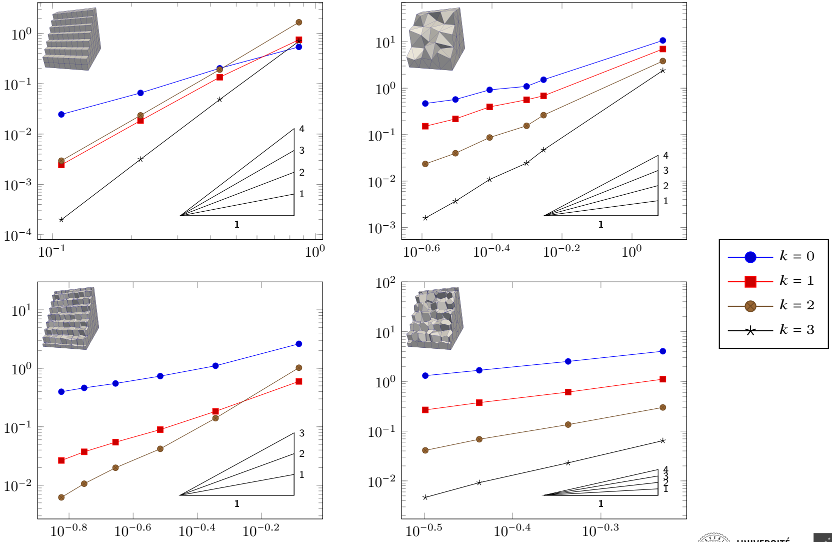
Convergence: Energy error vs. meshsize



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Thank you for your attention!

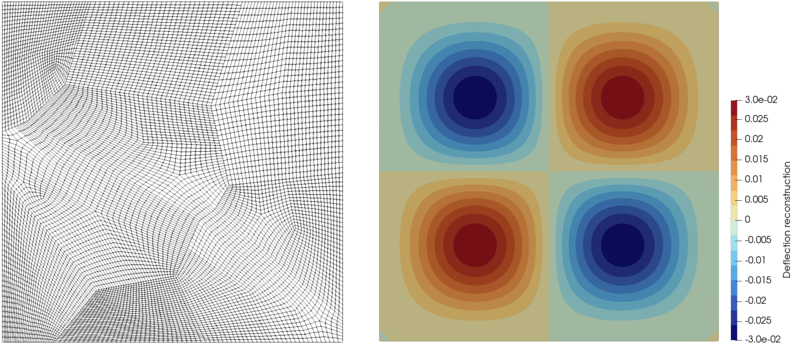

Figure: A glimpse at advanced complexes: solution of a four-point Kirchhoff–Love problem based on a DDR approximation of $H^2(\Omega)$. Mesh provided by L. Lamérand.

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