Flux conservation

Theorem (Continuity of normal components of $H(\mathrm{div};\Omega)$ fields)

Let Ω be a polytopal domain of $\mathbb{R}^d$.

Let $\tau \in H(\mathrm{div};\Omega) \cap H^1(\mathcal{T}_h)^d$, and let $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ be a polytopal mesh of Ω. Then, for all $F \in \mathcal{F}_h^i$ s.t. $F \subset \partial T_1 \cap \partial T_2$ for distinct $T_1, T_2 \in \mathcal{T}_h$,

$$\tau_{|T_1} \cdot n_{T_1F} + \tau_{|T_2} \cdot n_{T_2F} = 0.$$

Let $f \in L^2(\Omega)$ and consider the Poisson problem:

(P) Find $u \in H_0^1(\Omega)$ s.t. $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega)$

Its solution satisfies, a.e. in Ω, $f = -\Delta u = -\nabla \cdot (\nabla u)$.

Assume, for the sake of simplicity, Ω convex $\Rightarrow u \in H^2(\Omega)$ by elliptic regularity. Then, applying the above theorem to $\tau = \nabla u$, we get, $\forall F \in \mathcal{F}_h^i$, $F \subset \partial T_1 \cap \partial T_2$,

$$\nabla u_{|T_1} \cdot n_{T_1F} + \nabla u_{|T_2} \cdot n_{T_2F} = 0$$

Error estimate

Let $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ be a polytopal mesh of Ω, $k \geq 0$ an integer, and set

$$\underline{U}_h^k := \{((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) : v_T \in \mathcal{P}^k(T)\ \forall T\in\mathcal{T}_h,$$
$$v_F \in \mathcal{P}^k(F)\ \forall F\in\mathcal{F}_h\},$$

$$\underline{U}_{h,0}^k := \{\underline{v}_h \in \underline{U}_h^k : v_F = 0 \quad \forall F\in\mathcal{F}_h^b\}.$$

$(\mathcal{P}_h)$ Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t. $a_h(\underline{u}_h, \underline{v}_h) = \sum_{T\in\mathcal{T}_h}\int_T f v_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k$

where $a_h(\underline{u}_h, \underline{v}_h) = \sum_{T\in\mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T)$ with

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla p_T^{k+1}\underline{u}_T \cdot \nabla p_T^{k+1}\underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

We equip $\underline{U}_{h,0}^k$ with the norm $\|\cdot\|_{1,h}$ s.t., $\forall \underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2 \quad \text{with}, \forall T\in\mathcal{T}_h,$$

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_T^2 + h_T^{-1}\sum_{F\in\mathcal{F}_T}\|v_F - v_T\|_F^2$$

The fact that this is a norm is a consequence of the following Poincaré inequality:

$$\|v_h\|_\Omega \lesssim \|\underline{v}_h\|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^{k}$$

with $(v_h)_{|T} := v_T \quad \forall T \in \mathcal{T}_h$.

By (52), a_h is uniformly coercive with respect to $\|\cdot\|_{1,h}$.

Hence, invoking Strang 3,

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim \sup_{\underline{v}_h \in \underline{U}_{h,0}^{k} \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(u; \underline{v}_h)}{\|\underline{v}_h\|_{1,h}}$$

with consistency error

$$\mathcal{E}_h(u; \underline{v}_h) := \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h)$$

To prove convergence, we have to show that

$$\lim_{h \to 0} \mathcal{E}_h(u; \underline{v}_h) = 0,$$

which holds, in particular, if $\mathcal{E}_h(u; \underline{v}_h) \lesssim h^\alpha$ for some power $\alpha > 0$.

Lemma (Estimate of the consistency error)

Let $s \in \{0,\dots,k\}$ and assume $u \in H^{s+2}(\mathcal{T}_h)$. Then,

$$\mathcal{E}_h(u;\underline{v}_h) \lesssim h^{s+1} |u|_{H^{s+2}(\mathcal{T}_h)}.$$

Proof. We have to reformulate the terms that compose $\mathcal{E}_h(u;\underline{v}_h)$ in such a way that they can be compared.

1) Noticing that $f = -\Delta u$ a.e. in Ω,

$$\int_\Omega f\, v_h = -\int_\Omega \Delta u\, v_h$$

$$= -\sum_{T\in\mathcal{T}_h} \int_T \Delta u\, v_T$$

$$= \sum_{T\in\mathcal{T}_h} \left[\int_T \nabla u \cdot \nabla v_T - \sum_{F\in\mathcal{F}_T} \int_F (\nabla u \cdot n_{TF})\, v_T \right] \qquad (1)$$

We next notice that:

- If $F \in \mathcal{F}_h^i$ is s.t. $F \subset \partial T_1 \cap \partial T_2$, since $u \in H(\mathrm{div};\Omega)$,

$$\int_F (\nabla u_{|T_1} \cdot n_{T_1F} + \nabla u_{|T_2} \cdot n_{T_2F})\, v_F = 0$$

- If $F \in \mathcal{F}_h^b$ is s.t. $F \subset \partial T \cap \partial\Omega$, since $v_F = 0$,

$$\int_F (\nabla u_{|T} \cdot n_{TF})\, v_F = 0$$

Therefore,

$$0 = \sum_{F\in\mathcal{F}_h}\ \sum_{T\in\mathcal{T}_F}\int_F (\nabla u\cdot n_{TF})\, v_F \qquad (2)$$

$$= \sum_{T\in\mathcal{T}_h}\ \sum_{F\in\mathcal{F}_T}\int_F (\nabla u\cdot n_{TF})\, v_F$$

Plugging (2) into (1),

$$\int_\Omega f v_h = \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla u\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (\nabla u\cdot n_{TF})(v_F - v_T)\right. \qquad (3)$$

2)

$$a_h(\underline{I}_h^k u, \underline{v}_h)$$

def. a_h

$$= \sum_{T\in\mathcal{T}_h} a_T(\underline{I}_T^k u, \underline{v}_T)$$

$\pi_T^{1,k+1} u =: \hat{u}_T$

def. a_T

$$= \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla \boxed{p_T^{k+1}\underline{I}_T^k u}\cdot\nabla p_T^{k+1}\underline{v}_T + s_T(\underline{I}_T^k u, \underline{v}_T)\right]$$

$$= \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla\hat{u}_T\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F(\nabla\hat{u}_T\cdot n_{TF})(v_F - v_T)\right]$$

$$+ \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)$$

3) We therefore have

$$\mathcal{E}_h(u, \underline{v}_h)$$

$$= \underbrace{\sum_{T\in\mathcal{T}_h} \int_T (\nabla u - \nabla \hat{u}_T)\cdot\nabla v_T}_{\mathfrak{T}_1}$$

$$+ \underbrace{\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F (\nabla u - \nabla\hat{u}_T)\cdot n_{TF}(v_F - v_T)}_{\mathfrak{T}_2}$$

$$+ \underbrace{\sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)}_{\mathfrak{T}_3} \qquad (4)$$

$$\mathfrak{T}_i = \sum_{T\in\mathcal{T}_h}\mathfrak{T}_i(T) \qquad \forall\, 1\le i\le 3$$

- $$\mathfrak{T}_1(T) = \int_T (\nabla u - \nabla \pi_T^{1,k+1}u)\cdot\nabla \underbrace{v_T}_{\in\mathbb{P}^k(T)} \overset{\text{def. } \pi_T^{1,k+1}}{=} 0 \qquad (5)$$

- $$\mathfrak{T}_2 \overset{(2,\infty,2)\text{-Hölder}}{\le} \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} \|\nabla u - \nabla\hat{u}_T\|_F \underbrace{\|n_{TF}\|_{L^\infty(F)^d}}_{\le 1}\|v_F - v_T\|_F$$

$$\overset{C-S}{\le} \Big(\sum_{T\in\mathcal{T}_h} h_T \underbrace{\|\nabla u - \nabla\hat{u}_T\|_F^2}_{\lesssim h_T^{2s+1}|u|^2_{H^{s+2}(T)}}\Big)^{1/2} \times \underbrace{\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\sum_{F\in\mathcal{F}_T}\|v_F - v_T\|_F^2\Big)^{1/2}}_{\le \|\underline{v}_h\|_{1,h}}$$

$$\lesssim h^{s+1}|u|_{H^{s+2}(\mathcal{T}_h)}\|\underline{v}_h\|_{1,h} \qquad (6)$$

- Using the consistency of s_T for smooth functions proved in the last course, we have

$$\mathfrak{T}_3 \overset{C\text{-}S}{\lesssim} \left(\sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{I}_T^k u)\right)^{1/2} \times \left(\sum_{T\in\mathcal{T}_h} s_T(\underline{v}_T, \underline{v}_T)\right)^{1/2}$$

$$\overset{(ST2)}{\lesssim} h^{s+1} |u|_{H^{s+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h} \qquad (7)$$

Plugging (5), (6), (7) into (4), we get: $\forall \underline{v}_h \in \underline{U}_{h,0}^k$,

$$\mathcal{E}_h(u; \underline{v}_h) \lesssim h^{s+1} |u|_{H^{s+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h},$$

so that

$$\sup_{\underline{v}_h \in \underline{U}_{h,0}^k \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(u; \underline{v}_h)}{\|\underline{v}_h\|_{1,h}} \lesssim h^{s+1} |u|_{H^{s+2}(\mathcal{T}_h)} \qquad \square$$

Theorem (Energy error estimate)

Let $u \in H_0^1(\Omega)$ solve (Π), $\underline{u}_h \in \underline{U}_{h,0}^k$ solve (Π_h), and further assume $u \in H^{s+2}(\mathcal{T}_h)$ for some $s \in \{0, \ldots, k\}$. Then,

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{s+1} |u|_{H^{s+2}(\mathcal{T}_h)}.$$

Proof. Combine Strang 3 with the previous lemma. $\square$

Numerical fluxes

Let $u \in H_0^1(\Omega)$ solve (Π) and assume, for the sake of simplicity, $u \in H^2(\mathcal{T}_h)$. Then, for all $T \in \mathcal{T}_h$ and all $v_T \in \mathcal{P}^k(T)$,

$$\int_T f v_T = -\int_T \Delta u \, v_T = \underbrace{\int_T \nabla u \cdot \nabla v_T}_{\text{redistribution inside } T} - \underbrace{\sum_{F \in \mathcal{F}_T} \int_F (\nabla u \cdot n_{TF}) v_T}_{\text{exchanges through the boundary of } T} \qquad (8)$$

In particular, if $v_T \equiv 1$, we obtain the classical finite volume balance:

$$-\sum_{F \in \mathcal{F}_T} \int_F \nabla u \cdot n_{TF} = \int_\Omega f$$

Moreover, we have already noticed at the beginning of this course that, for all $F \in \mathcal{F}_h^i$, $F \subset \partial T_1 \cap \partial T_2$,

$$\nabla u_{|T_1} \cdot n_{T_1F} + \nabla u_{|T_2} \cdot n_{T_2F} = 0 \qquad (9)$$

$\varphi_{TF} := -\nabla u_{|T} \cdot n_{TF}$ is therefore a conservative normal flux.

Is it possible to identify a conservative numerical normal flux for the HHO method?

Assume that there exists $R_{TF}^{k} : \underline{U}_T^k \to \mathcal{P}^k(F)$ s.t.

$$s_T(\underline{w}_T, \underline{v}_T) = -\sum_{F \in \mathcal{F}_T} \int_F R_{TF}^{k} \underline{w}_T \, (v_F - v_T) \qquad \forall (\underline{w}_T, \underline{v}_T) \in \underline{U}_T^k \times \underline{U}_T^k. \tag{10}$$

Then, for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$a_h(\underline{u}_h, \underline{v}_h)$$

$$\overset{\text{def. } a_h}{=} \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla p_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T) \right]$$

$$\overset{\text{def. } p_T^{k+1},\ (10)}{=} \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla v_T + \sum_{F \in \mathcal{F}_T} \int_F (\nabla p_T^{k+1} \underline{u}_T \cdot n_{TF}) (v_F - v_T) - \sum_{F \in \mathcal{F}_T} \int_F R_{TF}^{k} \underline{u}_T (v_F - v_T) \right]$$

$$= \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla v_T - \sum_{F \in \mathcal{F}_T} \int_F \Phi_{FT}(\underline{u}_T) (v_F - v_T) \right]$$

with, $\forall T \in \mathcal{T}_h$, $\forall F \in \mathcal{F}_T$

$$\Phi_{TF} := -\nabla p_T^{k+1} (\underline{u}_T) \cdot n_{TF} + R_{TF}^{k} (\underline{u}_T) \in \mathcal{P}^k(F)$$

The solution $\underline{u}_h$ to (Π_h) then satisfies, $\forall \underline{v}_h \in \underline{U}^k_{h,0}$,

$$\sum_{T \in \mathcal{T}_h} \left[\int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla v_T - \sum_{F \in \mathcal{F}_T} \int_F \Phi_{FT}(\underline{u}_T)(v_F - v_T) \right] = \sum_{T \in \mathcal{T}_h} \int_T f v_T$$

Fix $T \in \mathcal{T}_h$ and select $\underline{v}_h$ s.t. $v_{T'} = 0$ $\forall T' \in \mathcal{T}_h \setminus \{T\}$, $v_F = 0$ $\forall F \in \mathcal{F}_h$: $\forall v_T \in \mathcal{P}^k(T)$,

$$\boxed{\int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla v_T + \sum_{F \in \mathcal{F}_T} \int_F \Phi_{TF}(\underline{u}_T) v_T = \int_T f v_T} \quad (11)$$

which the discrete local balance analogous to (10).

Fix now $F \in \mathcal{F}_h^i$ and select $\underline{v}_h$ s.t. $v_{F'} = 0$ $\forall F' \in \mathcal{F}_h \setminus \{F\}$ and $v_T = 0$ $\forall T \in \mathcal{T}_h$: $\forall v_F \in \mathcal{P}^k(F)$,

$$\sum_{T \in \mathcal{T}_F} \int_F \Phi_{TF}(\underline{u}_T) v_F = 0$$

$$\iff \int_F [\Phi_{T_1F}(\underline{u}_{T_1}) + \Phi_{T_2F}(\underline{u}_{T_2})] v_F = 0$$

which implies, since $\Phi_{TF}(\underline{u}_T) \in \mathcal{P}^k(F)$,

$$\boxed{\Phi_{T_1F}(\underline{u}_{T_1}) + \Phi_{T_2F}(\underline{u}_{T_2}) = 0}$$

which is a discrete normal flux continuity relation analogous to (11).

It only remains to find R_{TF}^k. Let $T \in \mathcal{T}_h$ and

$$\mathcal{P}^k(\mathcal{F}_T) := \{ \alpha_{\partial T} \in L^2(\partial T) : (\alpha_{\partial T})_{|F} \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_T \}$$

Then, we let $R_{\partial T}^k : \underline{U}_T^k \to \mathcal{P}^k(\mathcal{F}_T)$ be s.t., $\forall \underline{v}_T \in \underline{U}_T^k$,

$$\boxed{\int_{\partial T} R_{\partial T}^k \underline{v}_T \, \alpha_{\partial T} = s_T(\underline{v}_T, (0, \alpha_{\partial T})) \quad \forall \alpha_{\partial T} \in \mathcal{P}^k(\mathcal{F}_T),}$$

i.e., $R_{\partial T}^k \underline{v}_T$ is the Riesz representation of $s_T(\cdot, \underline{v}_T)$ in $\mathcal{P}^k(\mathcal{F}_T)$ equipped with the product $\int_{\partial T} \alpha_{\partial T} \beta_{\partial T}$. It then suffices to set

$$\boxed{R_{TF}^k := (R_{\partial T}^k)_{|F} \quad \forall F \in \mathcal{F}_T}$$

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