Locking-free numerical approximations of the elasticity operator

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Setting

- With $\Omega \subset \mathbb{R}^d$ bounded polyhedral domain, we consider the problem

$$-\boldsymbol{\nabla}\cdot\underline{\underline{\sigma}}(\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega,$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega,$$

  where

$$\underline{\underline{\sigma}}(\boldsymbol{u}) := 2\mu\underline{\underline{\epsilon}}(\boldsymbol{u}) + \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\underline{\underline{I_d}}, \qquad \underline{\underline{\epsilon}}(\boldsymbol{u}) := \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^t\right).$$

- The Lamé parameters are piecewise constant a partition P_Ω of Ω and s.t.

$$0 < \underline{\mu} \leqslant \mu \leqslant \overline{\mu} < +\infty, \qquad 0 < \underline{\lambda} \leqslant \lambda \leqslant \overline{\lambda} \leqslant +\infty$$

- When $\overline{\lambda} \to +\infty$, numerical locking can be observed

Essential bibliography

- Homogeneous linear elasticity, matching simplicial meshes, $d = 2$
  - Crouzeix–Raviart [Brenner and Sung, 1992]
  - Discontinuous Galerkin (dG) [Hansbo and Larson, 2002-03]
- General polyhedral meshes, broken Sobolev spaces
  - Korn inequalities in broken spaces [Brenner, 2004]
- Diffusive problems on general meshes
  - DG methods [Di Pietro and Ern, 2011]
  - SUSHI method [Eymard, Gallouët, and Herbin, 2009]
  - HFV/MFV/MFD [Droniou, Eymard, Gallouët, and Herbin, 2010]
  - Cell centered Galerkin methods (ccG) [Di Pietro, 2012]

Outline

Locally quasi-incompressible materials

- Numerical locking
- A locking-free dG method
- Numerical examples

General polyhedral meshes

- Admissible mesh sequences
- A locking-free method on general meshes
- Numerical examples

Outline

Locking-free methods

- Locking-free methods satisfy an error estimate of the form

$$\|\!|\!| \boldsymbol{u} - \boldsymbol{u}_h |\!|\!|_{\mu,\lambda} \leqslant C \mathcal{N}_{\boldsymbol{u}} h$$

with C independent of λ, h, and $\boldsymbol{u}$ and

$$\mathcal{N}_{\boldsymbol{u}} := \left(\|\boldsymbol{u}\|^2_{H^2(P_\Omega)} + |\lambda \nabla\cdot\boldsymbol{u}|^2_{H^1(P_\Omega)} \right)^{1/2}$$

- Key point: approximate non trivial (locally) solenoidal fields
- Matching simplicial meshes: Crouzeix–Raviart interpolator

Estimate of $\mathcal{N}_{\boldsymbol{u}}$ $(d = 2)$ I

- Let

$$p := -\lambda \boldsymbol{\nabla}\cdot\boldsymbol{u} \in L^2(\Omega), \qquad \tilde{p} := p + \langle \lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}\rangle_\Omega \in L^2_0(\Omega)$$

- Then, $(\boldsymbol{u}, \tilde{p}) \in \boldsymbol{U} \times L^2_0(\Omega)$ solves with $\boldsymbol{U} := H^1_0(\Omega)^d$ and $g = -\boldsymbol{\nabla}\cdot\boldsymbol{u}$

$$\begin{aligned} \int_\Omega 2\mu \underline{\underline{\epsilon}}(\boldsymbol{u}):\underline{\underline{\epsilon}}(\boldsymbol{v}) - \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{v}\tilde{p} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -\int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{u} q &= \int_\Omega g q \qquad \forall q \in L^2_0(\Omega), \end{aligned}$$

- Assume the following regularity with $C_{\mathrm{S}} \neq C_{\mathrm{S}}(\lambda)$:

$$\|\boldsymbol{u}\|_{H^2(P_\Omega)} + \|\tilde{p}\|_{H^1(P_\Omega)} \leqslant C_{\mathrm{S}} \left(\|g\|_{H^1(P_\Omega)} + \|\boldsymbol{f}\|_{L^2(\Omega)^d} \right)$$

- See [Nicaise and Mercier, 2012] for a proof in $d = 2$

Estimate of $\mathcal{N}_{\boldsymbol{u}}$ $(d = 2)$ II

Theorem (Regularity)

Assume $\underline{\lambda} > C_{\mathrm{S}}$. *Then,*

$$\mathcal{N}_{\boldsymbol{u}} \leqslant C_{\Omega,\underline{\lambda},\mu} \|\boldsymbol{f}\|_{L^2(\Omega)^d}.$$

Proof.

There holds

$$\begin{aligned} |\lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}|_{H^1(P_\Omega)} &= |\tilde{p}|_{H^1(P_\Omega)} \leqslant C_{\mathrm{S}} \left(\|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{f}\|_{L^2(\Omega)^d} \right) \\ &\leqslant \frac{C_{\mathrm{S}}}{\underline{\lambda}^{\frac{1}{2}}} \|\lambda^{\frac{1}{2}} \boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{L^2(\Omega)} + \frac{C_{\mathrm{S}}}{\underline{\lambda}} |\lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}|_{H^1(P_\Omega)} + C_{\mathrm{S}} \|\boldsymbol{f}\|_{L^2(\Omega)^d}. \end{aligned}$$

Conclude using the energy estimate and the assumption on $\underline{\lambda}$. □

Setting I

- Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a matching simplicial Ciarlet-regular mesh sequence
- Define the broken Sobolev spaces: For $m \geqslant 0$,

$$H^l(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid v_{|T} \in H^l(T),\ \forall T \in \mathcal{T}_h \right\}$$

- A special instance are broken polynomial spaces: For $k \geqslant 0$,

$$\mathbb{P}_d^k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v_{|T} \in \mathbb{P}_d^k(T) \right\}$$

- On $H^1(\mathcal{T}_h)$ act the broken operators $\boldsymbol{\nabla}_h$, $\underline{\underline{\epsilon_h}}$, and $\underline{\underline{\sigma_h}}$

Setting II

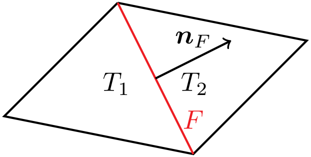


Figure: Notation for an interface $F \in \mathcal{F}_h^{\mathrm{i}}$

- For all $F = \partial T_1 \cap \partial T_2$ and all $v \in H^1(\mathcal{T}_h)$ set

$$[\![v]\!] := v_{|T_1} - v_{|T_2}, \qquad \{v\} := \frac{1}{2}\left(v_{|T_1} + v_{|T_2}\right)$$

- If $F = \partial T \cap \partial\Omega$ we set $\{v\} = [\![v]\!] = v$
- We equip $H^1(\mathcal{T}_h)$ with the H_0^1-like norm

$$\|v\|_{1,h}^2 := \|\boldsymbol{\nabla}_h v\|_{L^2(\Omega)^d}^2 + |v|_{\mathrm{J}}^2, \qquad |v|_{\mathrm{J}}^2 := \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|_{L^2(F)}^2$$

Discrete Korn's inequality

Theorem (Korn's inequality)

There exists C independent of h s.t., for all $\boldsymbol{v}_h \in \mathbb{P}_d^k(\mathcal{T}_h)^d$,

$$\|\boldsymbol{v}_h\|_{1,h} \leqslant C\left(\|\underline{\underline{\epsilon_h}}(\boldsymbol{v}_h)\|_{L^2(\Omega)^{d,d}} + |\boldsymbol{v}_h|_{\mathrm{J}}\right).$$

Proof.

Let $\tilde{\boldsymbol{v}}_h := \mathcal{I}_{\mathrm{Os}}(\boldsymbol{v}_h) \in \mathbb{P}_d^1(\mathcal{T}_h)^d \cap H_0^1(\Omega)^d$, $k \geqslant 1$. Then,

$$\begin{aligned}
\|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d,d}} &\leqslant \|\boldsymbol{\nabla}_h \tilde{\boldsymbol{v}}_h\|_{L^2(\Omega)^{d,d}} + \|\boldsymbol{\nabla}_h(\tilde{\boldsymbol{v}}_h - \boldsymbol{v}_h)\|_{L^2(\Omega)^{d,d}} \\
&\leqslant C_{\mathrm{K}}\|\underline{\underline{\epsilon_h}}(\tilde{\boldsymbol{v}}_h)\|_{L^2(\Omega)^{d,d}} + C_{\mathrm{Os}}|\boldsymbol{v}_h|_{\mathrm{J}} \\
&\leqslant C_{\mathrm{K}}\|\underline{\underline{\epsilon_h}}(\boldsymbol{v}_h)\|_{L^2(\Omega)^{d,d}} + C_{\mathrm{K}}\|\underline{\underline{\epsilon_h}}(\tilde{\boldsymbol{v}}_h - \boldsymbol{v}_h)\|_{L^2(\Omega)^{d,d}} + C_{\mathrm{Os}}|\boldsymbol{v}_h|_{\mathrm{J}} \\
&\leqslant C_{\mathrm{K}}\|\underline{\underline{\epsilon_h}}(\boldsymbol{v}_h)\|_{L^2(\Omega)^{d,d}} + C_{\mathrm{Os}}\left(1 + C_{\mathrm{K}}\right)|\boldsymbol{v}_h|_{\mathrm{J}}.
\end{aligned}$$

The proof can be extended to more general meshes [Brenner, 2004]. □

Liftings

- For $l \geqslant 0$, $F \in \mathcal{F}_h$, $\boldsymbol{v} \in H^1(\mathcal{T}_h)^d$, $\underline{\underline{r}}_F^l([\![\boldsymbol{v}]\!]) \in \mathbb{P}_d^l(\mathcal{T}_h)^{d,d}$ solves

$$\int_\Omega \underline{\underline{r}}_F^l([\![\boldsymbol{v}]\!]):\underline{\underline{\tau_h}} = \int_F [\![\boldsymbol{v}]\!] \otimes \boldsymbol{n}_F : \{\underline{\underline{\tau_h}}\} \qquad \forall \underline{\underline{\tau_h}} \in \mathbb{P}_d^l(\mathcal{T}_h)^{d,d}$$

- The trace of $\underline{\underline{r}}_F^l([\![\boldsymbol{v}]\!])$ is denoted by

$$r_F^l([\![\boldsymbol{v}]\!]) := \mathrm{tr}(\underline{\underline{r}}_F^l([\![\boldsymbol{v}]\!])) \in \mathbb{P}_d^l(\mathcal{T}_h).$$

- There holds for $l = 0$,

$$\underline{\underline{r}}_F^0([\![\boldsymbol{v}]\!])_{|T} \equiv \frac{|F|}{2|T|} \langle [\![\boldsymbol{v}]\!] \rangle_F \otimes \boldsymbol{n}_F, \qquad r_F^0([\![\boldsymbol{v}]\!])_{|T} \equiv \frac{|F|}{2|T|} \langle [\![\boldsymbol{v}]\!] \rangle_F \cdot \boldsymbol{n}_F$$

A locking-free dG method I

$$
\begin{aligned}
a_h(\boldsymbol{w},\boldsymbol{v}) :=& \int_\Omega \underline{\underline{\sigma_h}}(\boldsymbol{w}):\underline{\underline{\epsilon_h}}(\boldsymbol{v}) - \sum_{F\in\mathcal{F}_h}\int_F \{\underline{\underline{\sigma_h}}(\boldsymbol{w})\}:\langle[\![\boldsymbol{v}]\!]\rangle_F \otimes \boldsymbol{n}_F \\
&- \sum_{F\in\mathcal{F}_h}\int_F \langle[\![\boldsymbol{w}]\!]\rangle_F \otimes \boldsymbol{n}_F:\{\underline{\underline{\sigma_h}}(\boldsymbol{v})\} \\
&+ \sum_{F\in\mathcal{F}_h}\int_\Omega \eta\left(2\mu \underline{\underline{r_F^0}}([\![\boldsymbol{w}]\!]):\underline{\underline{r_F^0}}([\![\boldsymbol{v}]\!]) + \lambda r_F^0([\![\boldsymbol{w}]\!]) r_F^0([\![\boldsymbol{v}]\!])\right) \\
&+ \sum_{F\in\mathcal{F}_h}\int_F \frac{\gamma_{\mu,F}}{h_F}[\![\boldsymbol{w}]\!]\cdot[\![\boldsymbol{v}]\!]
\end{aligned}
$$

Find $\boldsymbol{u}_h \in \boldsymbol{U}_h := \mathbb{P}_d^1(\mathcal{T}_h)^d$ s.t. $a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h$ for all $\boldsymbol{v}_h \in \boldsymbol{U}_h$

A locking-free dG method II

Lemma (Weak consistency)

Assume $\boldsymbol{u} \in \boldsymbol{U}_* := H_0^1(\Omega) \cap H^2(P_\Omega)$. *Then,*

$$\forall \boldsymbol{v}_h \in \boldsymbol{U}_h, \qquad a_h(\boldsymbol{u}, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h - \mathcal{E}_{\boldsymbol{u}}(\boldsymbol{v}_h),$$

with consistency error

$$\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{v}_h) := \sum_{F \in \mathcal{F}_h} \int_F \{\underline{\underline{\sigma}}(\boldsymbol{u})\} : (\langle [\![\boldsymbol{v}_h]\!] \rangle_F - [\![\boldsymbol{v}_h]\!]) \otimes \boldsymbol{n}_F .$$

A locking-free dG method III

$$|||\boldsymbol{v}|||^2_{\mu,\lambda} := \|(2\mu)^{1/2}\underline{\underline{\epsilon_h}}(\boldsymbol{v})\|^2_{L^2(\Omega)^{d,d}} + \|\lambda^{1/2}\boldsymbol{\nabla}_h\cdot\boldsymbol{v}\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{F}_h}\frac{\gamma_{\mu,F}}{h_F}\|[\![\boldsymbol{v}]\!]\|^2_{L^2(F)^d}$$

Lemma (Coercivity)

For all $\eta > N_\partial = d + 1$ *there holds with* $\alpha_\eta := (\eta - N_\partial)/(1+\eta)$,

$$\forall \boldsymbol{v}_h \in \boldsymbol{U}_h, \qquad a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geqslant \alpha_\eta |||\boldsymbol{v}_h|||^2_{\mu,\lambda}.$$

Lemma (Boundedness)

There holds with $\beta_{\rho,\eta} := 2 + \eta + 2\rho^{\frac{1}{2}}$ *and* $\boldsymbol{U}_{*h} := \boldsymbol{U}_* + \boldsymbol{U}_h$,

$$\forall (\boldsymbol{w}, \boldsymbol{v}_h) \in \boldsymbol{U}_{*h} \times \boldsymbol{U}_h, \qquad a_h(\boldsymbol{w}, \boldsymbol{v}_h) \leqslant \beta_{\rho,\eta} |||\boldsymbol{w}|||_{\mu,\lambda,*} |||\boldsymbol{v}_h|||_{\mu,\lambda}.$$

A locking-free dG method IV

Theorem (Error estimate)

Assume $\boldsymbol{u} \in \boldsymbol{U}_*$. *Then,*

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mu,\lambda} \leqslant \left(1 + \frac{\beta_{\rho,\eta}}{\alpha_\eta}\right) \inf_{\boldsymbol{v}_h \in \boldsymbol{U}_h} |||\boldsymbol{u} - \boldsymbol{v}_h|||_{\mu,\lambda,*} + \sup_{\boldsymbol{v}_h \in \boldsymbol{U}_h} \frac{\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_{\mu,\lambda}}.$$

Corollary (Convergence rate)

The following locking-free error estimate holds:

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mu,\lambda} \leqslant C\mathcal{N}_{\boldsymbol{u}} h.$$

A locking-free dG method V

Proof.

- The inf is treated using the Crouzeix–Raviart interpolator $\mathcal{I}_{\mathbb{CR}}$ since

$$\mathbb{CR}(\mathcal{T}_h)^d \subset \boldsymbol{U}_h$$

- Let $\boldsymbol{v} \in H^2(\mathcal{T}_h)^d$. The key properties of $\mathcal{I}_{\mathbb{CR}}\boldsymbol{v}$ are
  (i) Continuity of face-averaged values

$$\forall F \in \mathcal{F}_h, \qquad \langle [\![\mathcal{I}_{\mathbb{CR}}\boldsymbol{v}]\!] \rangle_F = \mathbf{0}$$

  (ii) Approximation: For all $T \in \mathcal{T}_h$,

$$\|\boldsymbol{v}-\mathcal{I}_{\mathbb{CR}}\boldsymbol{v}\|_{L^2(T)^d} + h_T|\boldsymbol{v}-\mathcal{I}_{\mathbb{CR}}\boldsymbol{v}|_{H^1(T)^d} \leqslant C_{\mathbb{CR}} h_T \|\boldsymbol{v}\|_{H^2(T)^d},$$

$$\|\boldsymbol{\nabla}\cdot(\boldsymbol{v}-\mathcal{I}_{\mathbb{CR}}\boldsymbol{v})\|_{L^2(T)} + h_T|\boldsymbol{\nabla}\cdot(\boldsymbol{v}-\mathcal{I}_{\mathbb{CR}}\boldsymbol{v})|_{H^1(T)} \leqslant C_{\mathbb{CR}} h_T |\boldsymbol{\nabla}\cdot\boldsymbol{v}|_{H^1(T)}$$

□

Numerical examples I

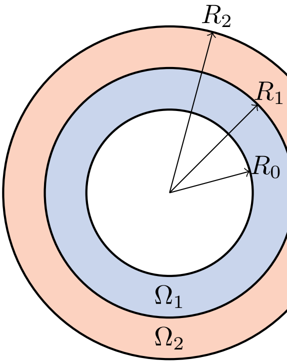


Figure: We fix $E_1 = E_2 = 1$, $\lambda_2 = 1$, and let $\nu_2 \to \frac{1}{2} \Rightarrow \lambda_2 \to +\infty$

$$\lambda_i = \frac{\nu_i E_i}{(1+\nu_i)(1-2\nu_i)}, \qquad \mu_i = \frac{E_i}{2(1+\nu_i)}$$

Numerical examples II

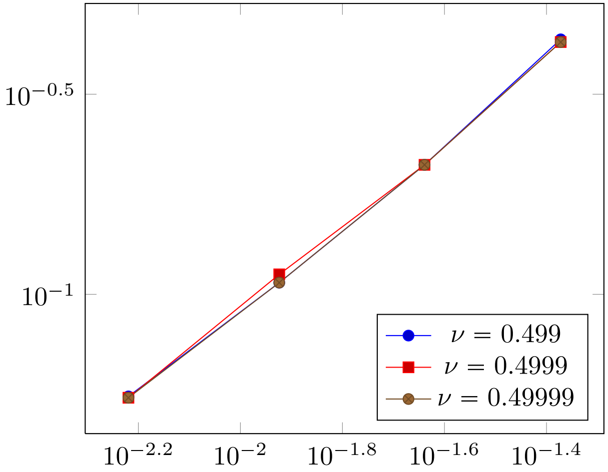


Figure: $|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mu,\lambda}$ vs. h

Numerical examples III

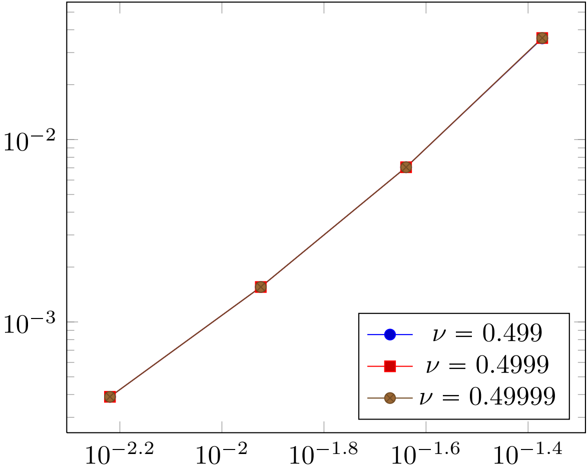


Figure: $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d}$ vs. h

Outline

A stabilized Crouzeix–Raviart method

- Following [Hansbo and Larson, 2003], we can alternatively choose

$$\boldsymbol{U}_h = \mathbb{CR}(\mathcal{T}_h)^d$$

- The convergence analysis remains unchanged, a_h simplifies to

$$a_h(\boldsymbol{w}_h, \boldsymbol{v}_h) = \int_\Omega \underline{\underline{\sigma_h}}(\boldsymbol{w}_h):\underline{\underline{\epsilon_h}}(\boldsymbol{v}_h) + \sum_{F \in \mathcal{F}_h} \int_F \frac{\gamma_{\mu,F}}{h_F} [\![\boldsymbol{w}_h]\!] \cdot [\![\boldsymbol{v}_h]\!]$$

- This approach can be extended to general polyhedral meshes

Admissible mesh sequences I

Trace and inverse inequalities

- Every $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{S}_h$
- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet
- Every simplex $S \subset T$ is s.t. $h_S \approx h_T$

Optimal polynomial approximation (for error estimates)

Every element T is star-shaped w.r.t. a ball of diameter $\delta_T \approx h_T$

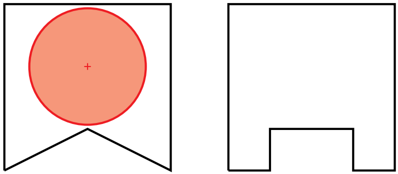


Figure: Admissible (*left*) and non-admissible (*right*) mesh elements

Admissible mesh sequences II

Cell centers

We fix a set of points $(\boldsymbol{x}_T)_{T \in \mathcal{T}_h}$ s.t.

- all $T \in \mathcal{T}_h$ is star-shaped w.r.t. $\boldsymbol{x}_T$
- for all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$, $d_{T,F} := \mathrm{dist}(\boldsymbol{x}_T, F) \approx h_T$

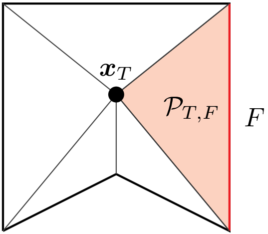


Figure: Cell center and face-based pyramid $\mathcal{P}_{T,F}$

Admissible mesh sequences III

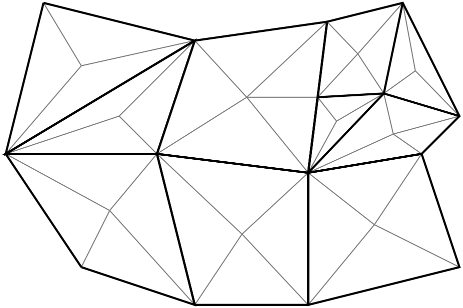


Figure: Pyramidal submesh $\mathcal{P}_h := \{\mathcal{P}_{T,F}\}_{T\in\mathcal{T}_h,\, F\in\mathcal{F}_T}$. $\Sigma_h := \{\text{faces of } \mathcal{P}_h\}$

A generalization of the Crouzeix–Raviart space I

- Following [Eymard, Gallouët, and Herbin, 2009], let

$$\mathbb{V}_h := \mathbb{R}^{\mathcal{T}_h} \times \mathbb{R}^{\mathcal{F}_h}$$

- Define the gradient reconstruction $\mathfrak{G}_h : \mathbb{V}_h \to \mathbb{P}_d^0(\mathcal{P}_h)^d$ s.t.

$$\forall \mathcal{P}_{T,F} \in \mathcal{P}_h, \qquad \mathfrak{G}_h(\mathtt{v}_h)_{|\mathcal{P}_{T,F}} = \boldsymbol{G}_T(\mathtt{v}_h) + \boldsymbol{R}_{T,F}(\mathtt{v}_h)$$

where

$$\boldsymbol{G}_T(\mathtt{v}_h) := \sum_{F \in \mathcal{F}_T} \frac{|F|}{|T|} v_F \boldsymbol{n}_{T,F},$$

$$\boldsymbol{R}_{T,F}(\mathtt{v}_h) := \frac{\eta}{d_{T,F}} \left[v_F - (v_T + \boldsymbol{G}_T(\mathtt{v}_h) \cdot (\overline{\boldsymbol{x}}_F - \boldsymbol{x}_T)) \right] \boldsymbol{n}_{T,F}$$

- Observe that $\boldsymbol{R}_{T,F}(\mathtt{v}_h) \in (\mathbb{P}_d^0(T)^d)^\perp$

A generalization of the Crouzeix–Raviart space II

- In the spirit of ccG methods, define $\mathfrak{R}_h : \mathbb{V}_h \to \mathbb{P}^1_d(\mathcal{T}_h)$ s.t.

$$\forall \mathcal{P}_{T,F} \in \mathcal{P}_h, \qquad \mathfrak{R}_h(\mathbb{v}_h)_{|\mathcal{P}_{T,F}}(\boldsymbol{x}) = v_F + \mathfrak{G}_h(\mathbb{v}_h)_{|\mathcal{P}_{T,F}} \cdot (\boldsymbol{x} - \overline{\boldsymbol{x}}_F)$$

- Following [Di Pietro, 2012] we introduce the discrete space

$$\mathfrak{CR}(\mathcal{T}_h) = \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}^1_d(\mathcal{P}_h)$$

Lemma (Continuity of the face-averaged values in $\mathfrak{CR}(\mathcal{T}_h)$)

Let $v_h \in \mathfrak{CR}(\mathcal{T}_h)$. For $\eta = d$ there holds for all $\sigma \in \Sigma_h$ face of $\mathcal{P}_h$

$$\langle [\![v_h]\!] \rangle_\sigma = 0.$$

A generalization of the Crouzeix–Raviart space III

Lemma (Approximation in $\mathfrak{CR}(\mathcal{T}_h)$)

For $v \in H^2(\mathcal{T}_h)$ let $\mathcal{I}_{\mathfrak{CR}} v \in \mathfrak{CR}(\mathcal{T}_h)$ be s.t.

$$\mathcal{I}_{\mathfrak{CR}} v = \mathfrak{R}_h(\mathbb{v}_h) \text{ with } \mathbb{v}_h = \left((\pi_h^1 v(\boldsymbol{x}_T))_{T \in \mathcal{T}_h}, (\langle v \rangle_F)_{F \in \mathcal{F}_h} \right)$$

Then with $v_h := \mathcal{I}_{\mathfrak{CR}} v$ there holds

$$\|v - v_h\|_{L^2(\Omega)^d} + h \|\boldsymbol{\nabla}_h (v - v_h)\|_{L^2(\Omega)^{d,d}} \leqslant C_{\mathfrak{CR}} h^2 \|v\|_{H^2(\mathcal{T}_h)^d}.$$

Proof.

Follows similar ideas as [Di Pietro, 2012]. □

A generalization of the Crouzeix–Raviart space IV

Lemma

For $\boldsymbol{v} \in H^2(\mathcal{T}_h)^d$ *let* $\boldsymbol{v}_h := \mathcal{I}_{\mathfrak{CR}}\boldsymbol{v}$ *and set* $D_h(\boldsymbol{v}_h) := \pi_h^0(\boldsymbol{\nabla}_h\cdot\boldsymbol{v}_h)$. *Then,*

$$\|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)\|_{L^2(\Omega)} + h|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)|_{H^1(\mathcal{T}_h)} \leqslant C_{\mathfrak{CR}} h |\boldsymbol{\nabla}\cdot\boldsymbol{v}|_{H^1(\Omega)}.$$

Proof.

There holds, for all $T \in \mathcal{T}_h$,

$$\begin{aligned} D_h(\boldsymbol{v}_h)_{|T} &= \sum_{F\in\mathcal{F}_T} \frac{|F|}{|T|}\langle\boldsymbol{v}\rangle_F\cdot\boldsymbol{n}_{T,F} \\ &= \frac{1}{|T|}\sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}\cdot\boldsymbol{n}_{T,F} = \frac{1}{|T|}\int_T \boldsymbol{\nabla}\cdot\boldsymbol{v} = \langle\boldsymbol{\nabla}\cdot\boldsymbol{v}\rangle_T, \end{aligned}$$

hence $D_h(\boldsymbol{v}_h) = \pi_h^0(\boldsymbol{\nabla}\cdot\boldsymbol{v})$. Use the approximation properties of the L^2-projector to conclude. □

A locking-free method on general meshes

$$a_h(\boldsymbol{w},\boldsymbol{v}) := \int_\Omega 2\mu \underline{\underline{\epsilon_h}}(\boldsymbol{w}):\underline{\underline{\epsilon_h}}(\boldsymbol{v}) + \int_\Omega \lambda D_h(\boldsymbol{w}) D_h(\boldsymbol{v}) + \sum_{\sigma\in\Sigma_h} \int_\sigma \frac{\gamma_{\mu,\sigma}}{h_\sigma} [\![\boldsymbol{w}]\!]\cdot[\![\boldsymbol{v}]\!]$$

$$\text{Find } \boldsymbol{u}_h \in \mathfrak{CR}(\mathcal{T}_h)^d \text{ s.t. } a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h \text{ for all } \boldsymbol{v}_h \in \mathfrak{CR}(\mathcal{T}_h)^d$$

Theorem (Convergence)

Assume $\boldsymbol{u} \in \boldsymbol{U}_*$. Then,

$$|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{\mu,\lambda} \leqslant C \mathcal{N}_{\boldsymbol{u}} h.$$

Numerical examples I

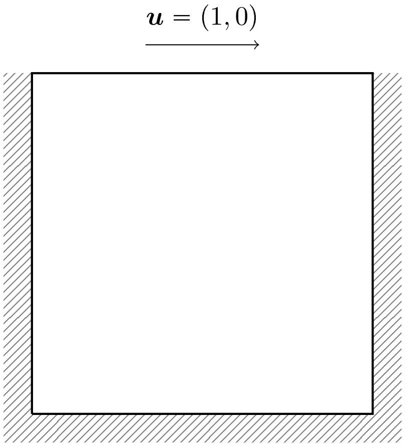
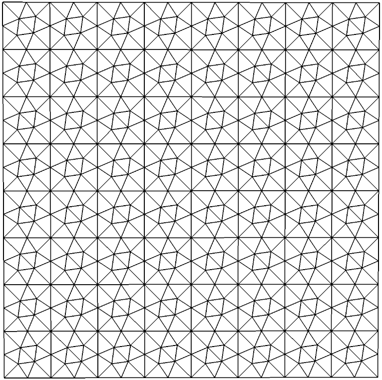
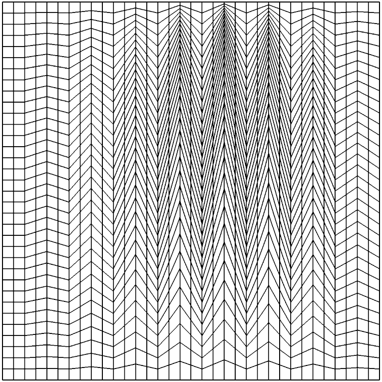


Figure: Configuration for the closed cavity test case ($\lambda \approx 1.666 \cdot 10^6$, $\mu \approx 333$)

Numerical examples II

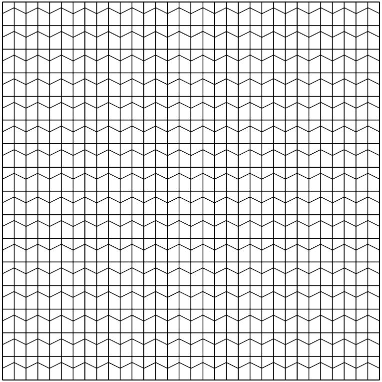


(a) Triangular

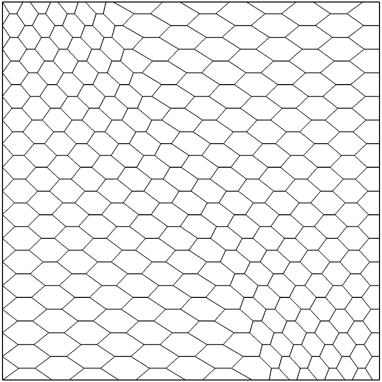


(b) Kershaw

Numerical examples III



(c) Trapezoidal



(b) Lemaire (30/3/2012 23:19)

Numerical examples IV

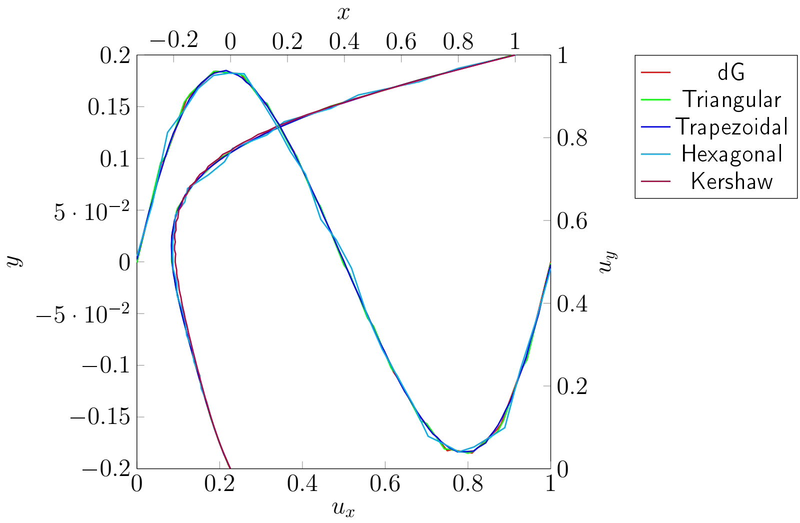


Figure: Closed cavity problem, coarse meshes

Numerical examples V

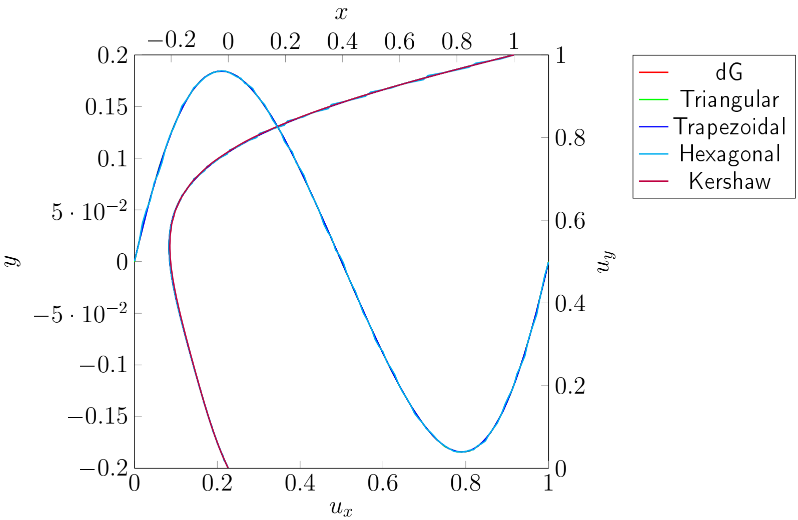


Figure: Closed cavity problem, fine meshes

References I

Brenner, S. and Sung, L.-Y. (1992).
Linear finite element methods for planar linear elasticity.
Math. Comp., 49(200):321–338.

Brenner, S. C. (2004).
Korn's inequalities for piecewise H^1 vector fields.
Math. Comp., 73(247):1067–1087.

Di Pietro, D. A. (2012).
Cell centered Galerkin methods for diffusive problems.
M2AN Math. Model. Numer. Anal., 46:111–144.

Di Pietro, D. A. and Ern, A. (2011).
Mathematical aspects of discontinuous Galerkin methods, volume 69 of *Mathématiques & Applications*.
Springer-Verlag, Berlin.

Droniou, J., Eymard, R., Gallouët, T., and Herbin, R. (2010).
A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods.
M3AS, Math. Models Methods Appl. Sci., 20(2):265–295.

Eymard, R., Gallouët, T., and Herbin, R. (2009).
Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI: a scheme using stabilization and hybrid interfaces.
IMA J. Numer. Anal.
DOI: 10.1093/imanum/drn084.

Hansbo, P. and Larson, M. G. (2002).
Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method.
Comput. Methods Appl. Mech. Engrg., 191:1895–1908.

Hansbo, P. and Larson, M. G. (2003).
Discontinuous Galerkin and the Crouzeix–Raviart element: Application to elasticity.
M2AN Math. Model. Numer. Anal., 37(1):63–72.

Nicaise, S. and Mercier, D. (2012).
Regularity results of Stokes/Lamé interface problems.
Math. Nachrichten, 285:332–348.