A hybrid finite volume generalization of the Crouzeix–Raviart element

D. A. Di Pietro, S. Lemaire

Université Montpellier 2, I3M, ACSIOM, Daniele.Di-Pietro@univ-montp2.fr

Marseille, October 16, 2012

UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Outline

Quasi-compressible materials and numerical locking I

- Let $\Omega \subset \mathbb{R}^d$ denote a bounded polygonal or polyhedral domain
- We consider the linear elasticity equations

$$-\boldsymbol{\nabla}\cdot\underline{\underline{\sigma}}(\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega,$$

where, for μ, $\lambda \in \mathbb{R}$, $\underline{\underline{\sigma}}(\boldsymbol{u})$ is the Cauchy stress tensor,

$$\underline{\underline{\sigma}}(\boldsymbol{u}) := 2\mu\underline{\underline{\epsilon}}(\boldsymbol{u}) + \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\underline{\underline{I_d}}, \qquad \underline{\underline{\epsilon}}(\boldsymbol{u}) := \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^t\right)$$

- When $\lambda \to +\infty$, numerical locking can be observed
- To avoid locking: uniform convergence w.r.t. λ

Quasi-compressible materials and numerical locking II

- If Ω convex, in $d = 2$ there holds [Brenner and Sung, 1992]

$$\mathcal{N}_{\rm el} := \left(\|\boldsymbol{u}\|_{H^2(\Omega)}^2 + |\lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}|_{H^1(\Omega)}^2 \right)^{1/2} \leqslant \|\boldsymbol{f}\|_{L^2(\Omega)^d},$$

- Locking-free methods satisfy an error estimate of the form

$$|\!|\!| \boldsymbol{u} - \boldsymbol{u}_h |\!|\!|_{\rm el} \leqslant C \mathcal{N}_{\rm el} h,$$

  with C independent of λ, h, and $\boldsymbol{u}$
- Key point: approximation of non-trivial solenoidal fields
- Classical solution: Crouzeix–Raviart on matching triangular meshes
- **What about general polygonal/polyhedral meshes?**

Stokes flow with large irrotational body forces I

- Let again $\Omega \subset \mathbb{R}^d$ denote a bounded polygonal or polyhedral domain
- We consider the Stokes flow

$$\begin{aligned} -\triangle \boldsymbol{u} + \boldsymbol{\nabla} p &= \boldsymbol{\Psi} - \boldsymbol{\nabla}\varphi && \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \\ \langle p \rangle_\Omega &= 0 \end{aligned}$$

- Classical requirement: inf-sup stable discretization
- Avoid that large irrotational body forces affect the velocity approximation [Galvin et al., 2012]
- **Can these requirements be met on general polyhedral meshes?**

Stokes flow with large irrotational body forces II

- Let $(\boldsymbol{u}_{\boldsymbol{\Psi}}, p_{\boldsymbol{\Psi}})$ denote the exact solution with $\varphi \equiv 0$
- We note the following continuous property:

$$\boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{\Psi}}, \qquad p = p_{\boldsymbol{\Psi}} - \varphi$$

- Key point: mimick this property at the discrete level
- With $\mathcal{N}_{\boldsymbol{\Psi}} := \|\boldsymbol{u}_{\boldsymbol{\Psi}}\|_{H^2(\Omega)^d} + \|p_{\boldsymbol{\Psi}}\|_{H^1(\Omega)}$ we obtain the estimate

$$\begin{aligned} \|\boldsymbol{\nabla}_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega)^{d,d}} &\leqslant Ch\mathcal{N}_{\boldsymbol{\Psi}}, \\ \|p - p_h\|_{L^2(\Omega)} &\leqslant Ch\left(\mathcal{N}_{\boldsymbol{\Psi}} + \|\varphi\|_{H^1(\Omega)}\right), \end{aligned}$$

where C is independent of h, $\boldsymbol{u}$, and φ

Outline

Admissible mesh sequences I

Trace and inverse inequalities

- Every $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{S}_h$
- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet
- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is contact regular: every simplex $S \subset T$ is s.t. $h_S \approx h_T$

Optimal polynomial approximation (for error estimates)

Every element T is star-shaped w.r.t. a ball of diameter $\delta_T \approx h_T$

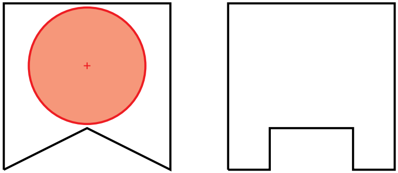


Figure: Admissible (*left*) and non-admissible (*right*) mesh elements

Admissible mesh sequences II

Cell centers

We fix a set of points $\{\boldsymbol{x}_T\}_{T\in\mathcal{T}_h}$ s.t.

- all $T \in \mathcal{T}_h$ is star-shaped w.r.t. $\boldsymbol{x}_T$
- for all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$, $d_{T,F} := \operatorname{dist}(\boldsymbol{x}_T, F) \approx h_T$

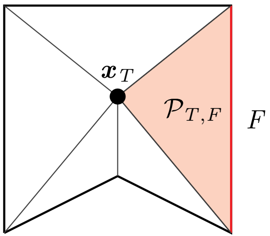


Figure: Cell center and face-based pyramid $\mathcal{P}_{T,F}$

Admissible mesh sequences III

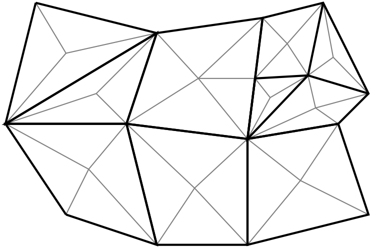


Figure: Pyramidal submesh $\mathcal{P}_h := \{\mathcal{P}_{T,F}\}_{T\in\mathcal{T}_h,\, F\in\mathcal{F}_T}$. $\Sigma_h :=$ {faces of $\mathcal{P}_h$}

Lemma (Shape- and contact-regularity of $\mathcal{P}_h$)

Let $\mathcal{T}_h$ admit a set of cell centers. Then, if $\mathcal{T}_h$ is shape- and contact-regular, the same holds for $\mathcal{P}_h$.

A generalization of the Crouzeix–Raviart space I

- Following [Eymard et al., 2010], we consider the space of DOFs

$$\mathbb{V}_h := \mathbb{R}^{\mathcal{T}_h} \times \mathbb{R}^{\mathcal{F}_h}$$

- Define the gradient reconstruction $\mathfrak{G}_h : \mathbb{V}_h \to \mathbb{P}^0_d(\mathcal{P}_h)^d$ s.t.

$$\forall \mathcal{P}_{T,F} \in \mathcal{P}_h, \qquad \mathfrak{G}_h(\mathtt{v}_h)_{|\mathcal{P}_{T,F}} = \boldsymbol{G}_T(\mathtt{v}_h) + \boldsymbol{R}_{T,F}(\mathtt{v}_h)$$

where

$$\boldsymbol{G}_T(\mathtt{v}_h) := \sum_{F \in \mathcal{F}_T} \frac{|F|}{|T|} v_F \boldsymbol{n}_{T,F},$$
$$\boldsymbol{R}_{T,F}(\mathtt{v}_h) := \frac{\eta}{d_{T,F}} \left[v_F - (v_T + \boldsymbol{G}_T(\mathtt{v}_h) \cdot (\overline{\boldsymbol{x}}_F - \boldsymbol{x}_T)) \right] \boldsymbol{n}_{T,F}$$

- Observe that $\boldsymbol{R}_{T,F}(\mathtt{v}_h) \in (\mathbb{P}^0_d(T)^d)^\perp$

A generalization of the Crouzeix–Raviart space II

- In the spirit of ccG methods, define $\mathfrak{R}_h : \mathbb{V}_h \to \mathbb{P}^1_d(\mathcal{T}_h)$ s.t.

$$\forall \mathcal{P}_{T,F} \in \mathcal{P}_h, \qquad \mathfrak{R}_h(\mathrm{v}_h)_{|\mathcal{P}_{T,F}}(\boldsymbol{x}) = v_F + \mathfrak{G}_h(\mathrm{v}_h)_{|\mathcal{P}_{T,F}} \cdot (\boldsymbol{x} - \overline{\boldsymbol{x}}_F)$$

- Following [DP, 2012] we introduce the discrete space

$$\boxed{\mathfrak{CR}(\mathcal{T}_h) := \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}^1_d(\mathcal{P}_h)}$$

Continuity of face-averaged values I

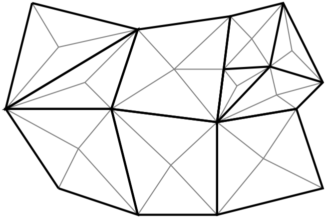


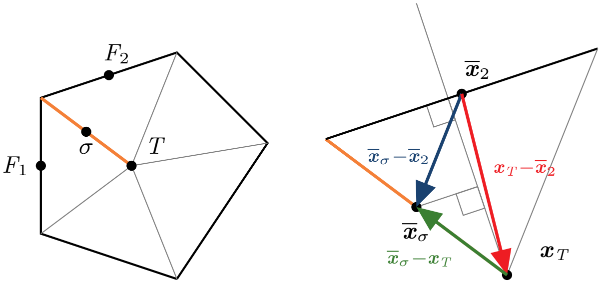
Figure: Primal mesh faces (thick lines) and lateral pyramidal faces (thin lines)

Lemma (Continuity of face-averaged values)

Assume $\eta = d$. *There holds for all* $v_h \in \mathfrak{CR}(\mathcal{T}_h)$ *and all* $\sigma \in \Sigma_h$,

$$\langle [\![v_h]\!] \rangle_\sigma = 0.$$

Continuity of face-averaged values II



- Choice of the starting point: $\langle [\![v_h]\!] \rangle_F = v_F = 0$ for all $F \in \mathcal{F}_h$
- For $\sigma \in \Sigma_h \backslash \mathcal{F}_h$, there holds with $\mathbb{v}_h \in \mathbb{V}_h$ s.t. $\mathfrak{R}_h(\mathbb{v}_h) = v_h$,

$$\begin{aligned}\langle [\![v_h]\!] \rangle_\sigma &= v_{h|\mathcal{P}_{T,F_1}}(\overline{\boldsymbol{x}}_\sigma) - v_{h|\mathcal{P}_{T,F_2}}(\overline{\boldsymbol{x}}_\sigma) \\ &= v_{F_1} - v_{F_2} - \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_{F_1} - \overline{\boldsymbol{x}}_{F_2}) + \alpha_1 - \alpha_2,\end{aligned}$$

with $\alpha_i := \boldsymbol{R}_{T,F_i}(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_\sigma - \overline{\boldsymbol{x}}_i) = -\frac{\eta}{d}\left(v_{F_i} - v_T - \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_i - \boldsymbol{x}_T)\right)$

Continuity of face-averaged values III

- Hence, taking $\eta = d$,

$$\langle [\![v_h]\!] \rangle_\sigma = \left(1 - \frac{\eta}{d}\right) (v_{F_1} - v_{F_2} - \boldsymbol{G}_T(\mathbb{v}_h) \cdot (\overline{\boldsymbol{x}}_{F_1} - \overline{\boldsymbol{x}}_{F_2})) = 0$$

  since

$$\alpha_1 - \alpha_2 = -\frac{\eta}{d} (v_{F_1} - v_{F_2} - \boldsymbol{G}_T(\mathbb{v}_h) \cdot (\overline{\boldsymbol{x}}_1 - \overline{\boldsymbol{x}}_2))$$

Approximation properties I

Lemma (Approximation in $\mathfrak{CR}(\mathcal{T}_h)$)

For $v \in H^1(\Omega)$ let $\mathcal{I}_h v \in \mathfrak{CR}(\mathcal{T}_h)$ be s.t.

$$\mathcal{I}_h v = \mathfrak{R}_h(\mathbb{v}_h) \text{ with } \mathbb{v}_h = \left((\pi_h^1 v(\boldsymbol{x}_T))_{T \in \mathcal{T}_h}, (\langle v \rangle_F)_{F \in \mathcal{F}_h} \right)$$

Then there holds

$$\Pi_h^0(\boldsymbol{\nabla}_h \mathcal{I}_h v) = \Pi_h^0(\boldsymbol{\nabla} v).$$

Moreover, if $v \in H^1(\Omega) \cap H^2(\mathcal{T}_h)$, there holds for all $T \in \mathcal{T}_h$,

$$\|v - \mathcal{I}_h v\|_{L^2(T)} + h_T \|\boldsymbol{\nabla}(v - \mathcal{I}_h v)\|_{L^2(T)^d} \leqslant C h_T^2 \|v\|_{H^2(T)}.$$

Approximation properties II

Proof.

Let $T \in \mathcal{T}_h$. Using Green's Theorem and since $\boldsymbol{R}_{T,F}(\mathbb{v}_h) \in (\mathbb{P}_d^0(T)^d)^{\perp}$,

$$\Pi_h^0(\boldsymbol{\nabla}_h \mathcal{I}_h v)_{|T} = \boldsymbol{G}_T(\mathbb{v}_h) = \sum_{F \in \mathcal{F}_T} \frac{|F|}{|T|} \langle v \rangle_F \boldsymbol{n}_{T,F}$$

$$= \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \int_F v \boldsymbol{n}_{T,F} = \frac{1}{|T|} \int_T \boldsymbol{\nabla} v = \langle \boldsymbol{\nabla} v \rangle_T.$$

The second point can be proved as in [DP, 2012]. □

Approximation properties III

Corollary (Divergence approximation)

For $\boldsymbol{v} \in H^1(\Omega)^d \cap \mathbf{H}^1(\mathrm{div};\Omega)$ *let* $\boldsymbol{v}_h := \mathcal{I}_h \boldsymbol{v}$ *and* $D_h(\boldsymbol{v}_h) := \Pi_h^0(\boldsymbol{\nabla}_h\cdot\boldsymbol{v}_h)$, *i.e.*,

$$\forall T \in \mathcal{T}_h, \quad D_h(\boldsymbol{v}_h)_{|T} = \frac{1}{|T|} \sum_{F\in\mathcal{F}_T} |F| \boldsymbol{v}_F\cdot\boldsymbol{n}_{T,F}.$$

As a consequence, for all $T \in \mathcal{T}_h$, *there holds*

$$\|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)\|_{L^2(T)} + h_T|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)|_{H^1(T)} \leqslant C h_T |\boldsymbol{\nabla}\cdot\boldsymbol{v}|_{H^1(T)}.$$

Outline

A locking-free method on general meshes I

$$\begin{aligned} -\nabla\cdot\underline{\underline{\sigma}}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega \end{aligned}$$

- We seek an approximation of the displacement $\boldsymbol{u}$ in the space

$$\boldsymbol{U}_h := \mathfrak{CR}_0(\mathcal{T}_h)^d$$

- The discrete problem reads

$$\text{Find } \boldsymbol{u}_h \in \boldsymbol{U}_h \text{ s.t. } a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h \text{ for all } \boldsymbol{v}_h \in \boldsymbol{U}_h$$

with

$$a_h(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega 2\mu\underline{\underline{\epsilon_h}}(\boldsymbol{w}):\underline{\underline{\epsilon_h}}(\boldsymbol{v}) + \int_\Omega \lambda D_h(\boldsymbol{w})D_h(\boldsymbol{v}) + \sum_{\sigma\in\Sigma_h} \int_\sigma \frac{\mu}{h_\sigma}[\![\boldsymbol{w}]\!]\cdot[\![\boldsymbol{v}]\!]$$

A locking-free method on general meshes II

Lemma (Coercivity of a_h)

There holds for all $\boldsymbol{v}_h \in \boldsymbol{U}_h$ with C_{sta} independent of h and of λ,

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) =: \|\boldsymbol{v}_h\|_{\mathrm{el}}^2 \geqslant C_{\mathrm{sta}} \|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d,d}}^2.$$

Proof.

- Continuity of face-averaged values: $\|\boldsymbol{\nabla}_h \boldsymbol{v}\|_{L^2(\Omega)^{d,d}}$ is a norm on $\boldsymbol{U}_h$
- Discrete Korn's inequality [Brenner, 2004],

$$\forall \boldsymbol{v}_h \in \boldsymbol{U}_h, \quad \|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d,d}} \leqslant C_{\mathrm{K}} \left(\|\underline{\underline{\epsilon_h}}(\boldsymbol{v}_h)\|_{L^2(\Omega)^{d,d}}^2 + |\boldsymbol{v}_h|_{\mathrm{J}}^2 \right)^{1/2}$$

□

A locking-free method on general meshes III

Lemma (Weak consistency)

Assume $\boldsymbol{u} \in \boldsymbol{U}_* := (H_0^1(\Omega) \cap H^2(\Omega))^d$*. Then*

$$\forall \boldsymbol{v}_h \in \boldsymbol{U}_h, \quad a_h(\boldsymbol{u}, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h + \mathcal{E}_h(\boldsymbol{v}_h),$$

with consistency error

$$\mathcal{E}_h(\boldsymbol{v}_h) := \sum_{\sigma \in \Sigma_h} \int_\sigma \underline{\underline{\sigma}}(\boldsymbol{u}) : [\![\boldsymbol{v}_h]\!] \otimes \boldsymbol{n}_F + \int_\Omega \lambda (D_h(\boldsymbol{u}) - \boldsymbol{\nabla} \cdot \boldsymbol{u}) \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h.$$

Proof.

Integrate by parts the volumic terms and use $-\boldsymbol{\nabla} \cdot \underline{\underline{\sigma}}(\boldsymbol{u}) = \boldsymbol{f}$ a.e. in Ω. □

UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

A locking-free method on general meshes IV

Theorem (Convergence)

Assume $\boldsymbol{u} \in \boldsymbol{U}_*$. Then the method satisfies the locking-free estimate

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mathrm{el}} \leqslant C\mathcal{N}_{\mathrm{el}} h.$$

Proof.

Strang's Second Lemma yields

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mathrm{el}} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{U}_h} |||\boldsymbol{u} - \boldsymbol{v}_h|||_{\mathrm{el}} + \sup_{\boldsymbol{v}_h \in \boldsymbol{U}_h \setminus \{\boldsymbol{0}\}} \frac{|\mathcal{E}_h(\boldsymbol{v}_h)|}{|||\boldsymbol{v}_h|||_{\mathrm{el}}} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

- $\mathfrak{T}_1$: approximation properties of $\mathcal{I}_h$;
- $\mathfrak{T}_2$: continuity of face-averaged values. □

UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Numerical example I

$$\boldsymbol{u} = (1, 0)$$

Figure: Closed cavity [Hansbo and Larson, 2003] ($\lambda \approx 1.666 \cdot 10^6$, $\mu \approx 333$)

Numerical example II

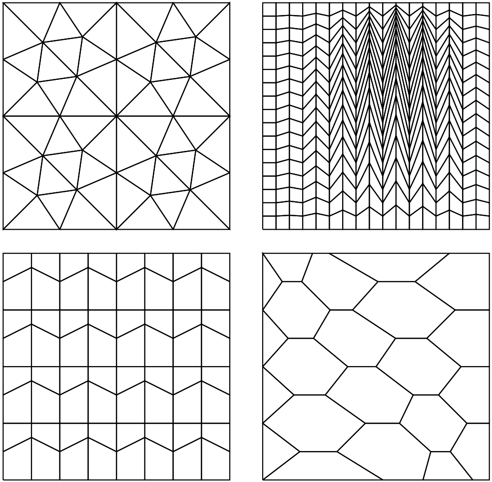


Figure: Meshes for the closed cavity

Numerical example III

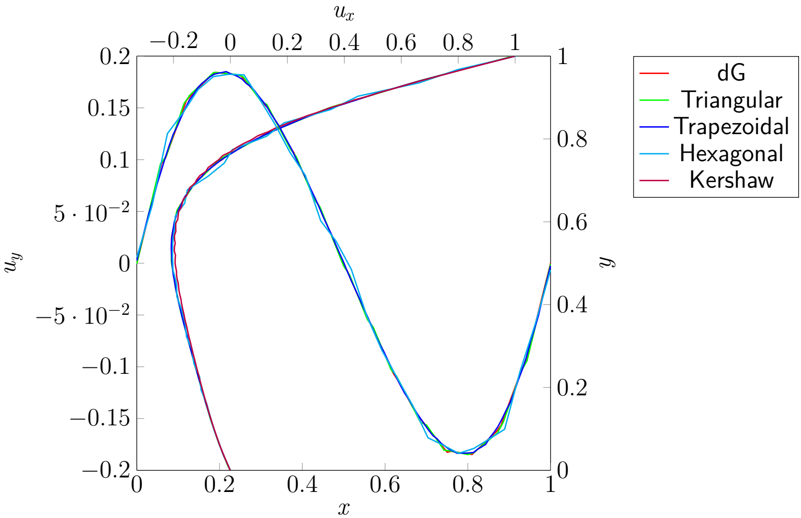


Figure: Closed cavity problem, coarse meshes

Numerical example IV

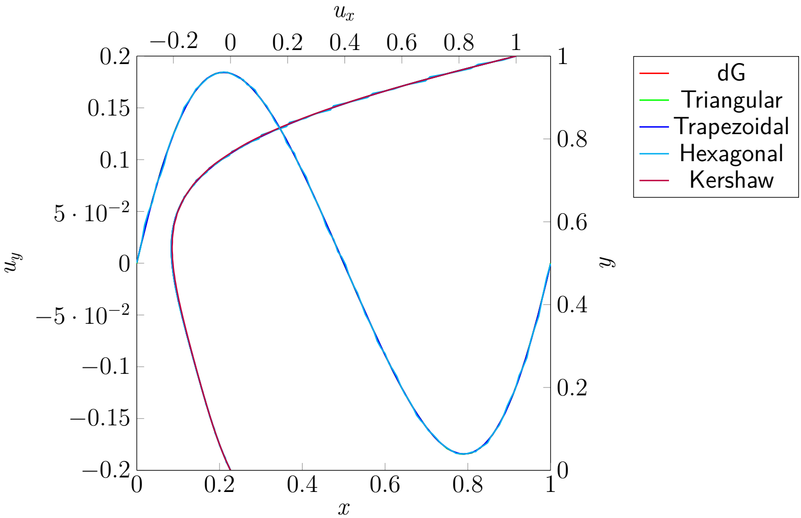


Figure: Closed cavity problem, fine meshes

Inf-sup stable discretization I

$$
\begin{aligned}
-\triangle \boldsymbol{u} + \boldsymbol{\nabla} p &= \boldsymbol{\Psi} - \boldsymbol{\nabla}\varphi && \text{in } \Omega,\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega,\\
\langle p\rangle_\Omega &= 0
\end{aligned}
$$

- We consider the following discrete spaces:

$$\boldsymbol{U}_h := \mathfrak{CR}_0(\mathcal{T}_h)^d, \quad P_h := \mathbb{P}^0_d(\mathcal{T}_h) \cap L^2_0(\Omega), \quad \boldsymbol{X}_h := \boldsymbol{U}_h \times P_h$$

- The discretization of the diffusion term is based on the bilinear form

$$a_h(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{w} : \boldsymbol{\nabla}_h \boldsymbol{v}$$

- The velocity-pressure coupling is based on the bilinear form

$$b_h(\boldsymbol{v}, q) := -\int_\Omega (\boldsymbol{\nabla}_h\cdot\boldsymbol{v})\, q$$

Inf-sup stable discretization II

Lemma (Stability)

There exist a reals $\beta > 0$ independent of h s.t., for all $q_h \in P_h$,

$$\beta \|q_h\|_{L^2(\Omega)} \leqslant \$:= \sup_{\boldsymbol{w}_h \in \boldsymbol{U}_h \setminus \{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{w}_h, q_h)}{\|\boldsymbol{\nabla}_h \boldsymbol{w}_h\|_{L^2(\Omega)^{d,d}}}.$$

Proof.

With $\boldsymbol{v}_{q_h}$ velocity lifting of q_h, $\boldsymbol{\xi}_h := \mathcal{I}_h \boldsymbol{v}_{q_h}$, there holds

$$\begin{aligned} \|q_h\|_P^2 &= \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{v}_{q_h} q_h = \int_\Omega \Pi_h^0(\boldsymbol{\nabla}\cdot\boldsymbol{v}_{q_h}) q_h = -b_h(\boldsymbol{\xi}_h, q_h) \\ &\leqslant \$ \|\boldsymbol{\nabla}_h \boldsymbol{\xi}_h\|_{L^2(\Omega)^{d,d}} \lesssim \$ \|\boldsymbol{v}_{q_h}\|_{H^1(\Omega)^d} \lesssim \$ \|q_h\|_{L^2(\Omega)}. \end{aligned}$$

□

UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Forcing term I

- At the continuous level there holds for all $\boldsymbol{v} \in H_0^1(\Omega)^d$,

$$\int_\Omega (\boldsymbol{\Psi} - \boldsymbol{\nabla}\varphi)\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{\Psi}\cdot\boldsymbol{v} + \int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{v})\varphi$$

- We discretize the source term accordingly,

$$l_h(\boldsymbol{v}) := \int_\Omega \boldsymbol{\Psi}\cdot\boldsymbol{v} - b_h(\boldsymbol{v}, \Pi_h^0\varphi)_{L^2(\Omega)}$$

- The discrete problem reads: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h$ s.t.

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{u}_h, q_h) = l_h(\boldsymbol{v}_h), \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{X}_h$$

Forcing term II

Proposition

Denote by $(\boldsymbol{u}_{\boldsymbol{\Psi},h}, p_{\boldsymbol{\Psi},h})$ *the solution with* $\varphi \equiv 0$. *There holds*

$$\boldsymbol{u}_h = \boldsymbol{u}_{\boldsymbol{\Psi},h}, \qquad p_h = p_{\boldsymbol{\Psi},h} - \Pi_h^0 \varphi.$$

Proof.

Owing to the choice of the right-hand side there holds

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h + \Pi_h^0 \varphi) - b_h(\boldsymbol{u}_h, q_h) = \int_\Omega \boldsymbol{\Psi} \cdot \boldsymbol{v}_h.$$

The conclusion follows since the discrete problem is well-posed. □

Error estimate

Theorem (Error estimate for the problem with $\varphi \equiv 0$)

Assume $(\boldsymbol{u}_{\boldsymbol{\Psi},h}, p_{\boldsymbol{\Psi},h}) \in \boldsymbol{X}_*$ *with* $\boldsymbol{X}_* := \boldsymbol{X} \cap H^2(\Omega)^d \times H^1(\Omega)$. *There holds with* $\mathcal{N}_{\boldsymbol{\Psi}} := \|\boldsymbol{u}_{\boldsymbol{\Psi}}\|_{H^2(\Omega)^d} + \|p_{\boldsymbol{\Psi}}\|_{H^1(\Omega)}$

$$\|\boldsymbol{\nabla}_h(\boldsymbol{u}_{\boldsymbol{\Psi}} - \boldsymbol{u}_{\boldsymbol{\Psi},h})\|_{L^2(\Omega)^{d,d}} + \|p - p_{\boldsymbol{\Psi},h}\|_{L^2(\Omega)} \leqslant C\mathcal{N}_{\boldsymbol{\Psi}} h.$$

Corollary (Error estimate)

There holds

$$\|\boldsymbol{\nabla}_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega)^{d,d}} \leqslant C\mathcal{N}_{\boldsymbol{\Psi}} h,$$
$$\|p - p_h\|_{L^2(\Omega)} \leqslant C\left(\mathcal{N}_{\boldsymbol{\Psi}} + \|\varphi\|_{H^1(\Omega)}\right) h.$$

Numerical example I

- We consider the following exact solution:

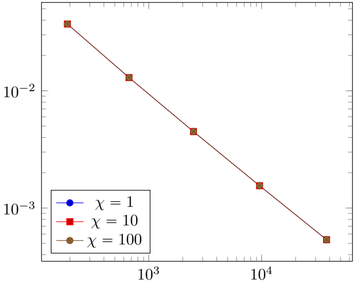
$$u_1 = -e^x(y\cos(y) + \sin(y)),\ u_2 = e^x y\sin(y),\ p_{\mathbf{\Psi}} = 2e^x\sin(y) - C,$$

with $\mathbf{\Psi} \equiv \mathbf{0}$ and potential

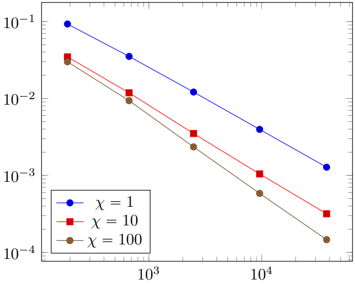
$$\varphi = \chi\sin(2\pi x)\sin(2\pi y), \qquad \chi > 0$$

- The parameter χ allows to vary the magnitude of the irrotational body force

Numerical example II



(a) $\|\boldsymbol{\nabla}_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega)^{d,d}}$ vs. $\mathrm{card}(\mathbb{V}_h)$



(b) $\|p - p_h\|_{L^2(\Omega)}$ vs. $\mathrm{card}(\mathbb{V}_h)$

Figure: Depedence of the velocity and pressure approximations on the magnitude of the irrotational body force

References

Brenner, S. and Sung, L.-Y. (1992).
Linear finite element methods for planar linear elasticity.
Math. Comp., 49(200):321–338.

Brenner, S. C. (2004).
Korn's inequalities for piecewise H^1 vector fields.
Math. Comp., 73(247):1067–1087.

Di Pietro, D. A. (2012).
Cell centered Galerkin methods for diffusive problems.
M2AN Math. Model. Numer. Anal., 46:111–144.

Droniou, J., Eymard, R., Gallouët, T., and Herbin, R. (2010).
A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods.
M3AS, Math. Models Methods Appl. Sci., 20(2):265–295.

Eymard, R., Gallouët, T., and Herbin, R. (2010).
Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: a scheme using stabilization and hybrid interfaces.
30(4):1009–1043.

Galvin, K., Linke, A., Rebholz, L., and Wilson, N. (2012).
Stabilizing poor mass conservation in incompressible flow problems with large irrotational forcing and application to thermal convection.
Comput. Methods Appl. Mech. Engrg., (237–240):166–176.

Hansbo, P. and Larson, M. G. (2002).
Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method.
Comput. Methods Appl. Mech. Engrg., 191:1895–1908.

Hansbo, P. and Larson, M. G. (2003).
Discontinuous Galerkin and the Crouzeix–Raviart element: Application to elasticity.
M2AN Math. Model. Numer. Anal., 37(1):63–72.