An arbitrary-order mixed method for anisotropic heterogeneous diffusion on general meshes

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GT Calcul Scientifique, 18 February, 2014

Motivations I

- Handling general polyhedral meshes comes in handy in many situations
  - Degenerate cells as a result of mesh deformation
  - Nonconforming interfaces in adaptive mesh refining
  - Adaptive mesh coarsening
- Extending the classical FE framework is not straightforward
- Recent works only consider lowest-order methods

Motivations II

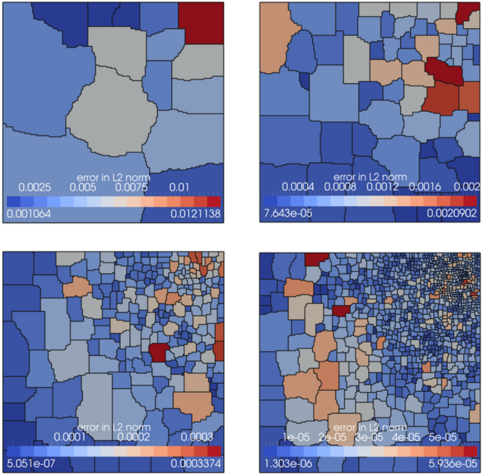


Figure: Adaptive mesh coarsening [DP et al., 2011]

Some healthy design guidelines

- Consistency on general polyhedral meshes
- Stability and robustness with respect to the physical parameters
  - Highly heterogeneous and anisotropic permeability in Darcy flow
  - Numerical locking for quasi-incompressible elasticity
  - Vanishing constrained specific storage coefficient in poroelasticity
  - Inf-sup stability in incompressible flows
- Reduced cost (stencil, parallel communications, conditioning)

Admissible mesh sequences I

Definition (Mesh regularity)

A sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of poly{gonal,hedral} meshes is regular if

- every $\mathcal{T}_h$ admits a simplicial submesh $\mathcal{S}_h$;
- $(\mathcal{S}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet;
- $(\mathcal{S}_h)_{h\in\mathcal{H}}$ is contact regular: every simplex $S\subset T$ is s.t. $h_S \approx h_T$.

Definition (Cell centers)

We fix a family of points $(\boldsymbol{x}_T)_{T\in\mathcal{T}_h}$ s.t.

- all $T\in\mathcal{T}_h$ is star-shaped w.r. to $\boldsymbol{x}_T$;
- for all $T\in\mathcal{T}_h$, and all $F\in\mathcal{F}_T$, $d_{T,F} := \mathrm{dist}(\boldsymbol{x}_T, F) \approx h_T$.

Admissible mesh sequences II

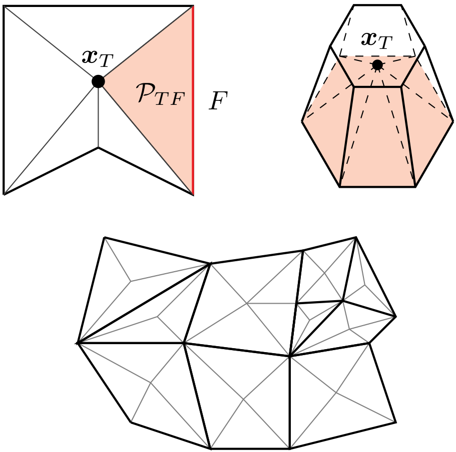


Figure: Pyramidal submesh

Setting I

- We consider the pure diffusion problem

$$\begin{aligned} \boldsymbol{K}^{-1}\boldsymbol{s} + \nabla u &= \boldsymbol{0} \quad \text{in } \Omega \\ -\nabla\cdot\boldsymbol{s} &= f \quad \text{in } \Omega \end{aligned}$$

  with $\boldsymbol{K}$ piecewise constant diffusion tensor with spectrum $\subset [k_\flat, k_\sharp]$
- The weak formulation reads: Find $(\boldsymbol{s}, u) \in \boldsymbol{\Sigma} \times U$ such that

$$\boxed{\begin{aligned} (\boldsymbol{K}^{-1}\boldsymbol{s}, \boldsymbol{t}) + (u, \nabla\cdot\boldsymbol{t}) &= 0 && \forall \boldsymbol{t} \in \boldsymbol{\Sigma} \\ -(\nabla\cdot\boldsymbol{s}, v) &= (f, v) && \forall v \in U \end{aligned}} \tag{M}$$

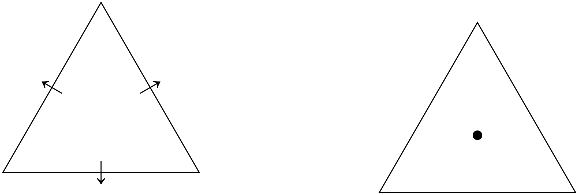
  with $\boldsymbol{\Sigma} := \boldsymbol{H}(\mathrm{div};\Omega)$ and $U := L^2(\Omega)$

DP and Ern, preprint hal-00918482, **2014**

Setting II

- Designing a classical FE method requires to
  - Devise a $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming flux space
  - Select a L^2-conforming pressure space
  - Make sure that the two are inf-sup compatible
- Finding a $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming space for arbitrary element shapes and approximation orders can be challenging
- Idea: renounce conformity and build the method from the DOFs up

A FE example: The $\mathbb{RT}_d^0 - \mathbb{P}_d^0$ element



$\mathbb{RT}_d^0 := [\mathbb{P}_d^0]^d \oplus \boldsymbol{x}\mathbb{P}_d^0$ $\qquad$ $\mathbb{P}_d^0$

Figure: An example of healthy FE approximation. Flux basis functions are given by $\left(\boldsymbol{\varphi}_{TF}(\boldsymbol{x}) = \frac{1}{d|T|_d}(\boldsymbol{x} - \boldsymbol{x}_{v(F)})\right)_{F\in\mathcal{F}_T}$

Degrees of freedom I

- We need discrete counterparts of a flux and its divergence
- The discrete divergence D_T^k must allow to prove inf-sup stability
- The flux reconstruction $\mathfrak{R}_T^k$ must be consistent and coercive

Remark (Virtualization)

The flux reconstruction need not be explicit provided we can approximate the $(\boldsymbol{K}^{-1}\cdot, \cdot)$-product (leading to a virtual method)

Degrees of freedom II

- Let, for a fixed integer $k \geqslant 0$ and, with $\mathbb{F}_F^k := \mathbb{P}_{d-1}^k(F)$,

$$\mathbb{T}_T^k := \boldsymbol{K}_T \nabla \mathbb{P}_d^{k,0}(T), \quad \mathbb{F}_T^k := \bigtimes_{F \in \mathcal{F}_T} \mathbb{F}_F^k \qquad \forall T \in \mathcal{T}_h$$

- For all $T \in \mathcal{T}_h$, the local spaces of flux and potential DOFs are

$$\boxed{\boldsymbol{\Sigma}_T^k := \mathbb{T}_T^k \times \mathbb{F}_T^k, \qquad U_T^k := \mathbb{P}_d^k(T)}$$

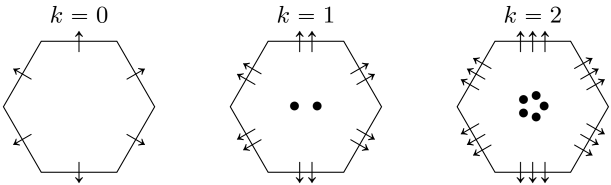


Figure: $\boldsymbol{\Sigma}_T^k$ for $k = 0, 1, 2$

Degrees of freedom III

- The global flux DOFs space is obtained by patching interface values

$$\boxed{\boldsymbol{\Sigma}_h^k := \mathbb{T}_h^k \times \mathbb{F}_h^k, \quad \mathbb{T}_h^k := \bigtimes_{T \in \mathcal{T}_h} \mathbb{T}_T^k, \quad \mathbb{F}_h^k := \bigtimes_{F \in \mathcal{F}_h} \mathbb{F}_F^k}$$

- For all $T \in \mathcal{T}_h$, we introduce the restriction operator $L_T : \boldsymbol{\Sigma}_h^k \to \boldsymbol{\Sigma}_T^k$
- The global spaces of potential DOFs is

$$\boxed{U_h^k := \bigtimes_{T \in \mathcal{T}_h} U_T^k}$$

Discrete divergence I

- Local divergence: $D_T^k : \boldsymbol{\Sigma}_T^k \to U_T^k$ s.t., $\forall \boldsymbol{\tau} = (\boldsymbol{\tau}_T, \{\tau_F\}_{F \in \mathcal{F}_T}) \in \boldsymbol{\Sigma}_T^k$,

$$\boxed{(D_T^k \boldsymbol{\tau}, v)_T = -(\nabla v, \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (v, \tau_F \epsilon_{TF})_F \quad \forall v \in U_T^k}$$

- Correspondingly, the global divergence $D_h^k : \boldsymbol{\Sigma}_h^k \to U_h^k$ is s.t.

$$\forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k, \quad (D_h^k \boldsymbol{\tau}_h, v_h) = \sum_{T \in \mathcal{T}_h} (D_T^k (L_T \boldsymbol{\tau}_h), v_h)_T \quad \forall v_h \in U_h^k$$

where we have identified U_h^k with $\mathbb{P}_d^k(\mathcal{T}_h)$

Discrete divergence II

Definition (Fortin interpolator)

- For all $T \in \mathcal{T}_h$ we define $I_T^k : \boldsymbol{\Sigma}^+(T) \to \boldsymbol{\Sigma}_T^k$ s.t., $\forall \boldsymbol{t} \in \boldsymbol{\Sigma}^+(T)$,

$$I_T^k \boldsymbol{t} = (\boldsymbol{\tau}_T, \{\tau_F\}_{F \in \mathcal{F}_T}) \text{ with } \boldsymbol{\tau}_T = \varpi_T^k \boldsymbol{t} \text{ and } \tau_F = \pi_F^k(\boldsymbol{t} \cdot \boldsymbol{n}_F)$$

  with ϖ_T^k energy projector and π_F^k L^2-orthogonal projector.
- The corresponding global version $I_h^k : \boldsymbol{\Sigma}^+ \to \boldsymbol{\Sigma}_h^k$ is s.t., $\forall \boldsymbol{t} \in \boldsymbol{\Sigma}^+$,

$$I_h^k \boldsymbol{t} = (\{\boldsymbol{\tau}_T\}_{T \in \mathcal{T}_h}, \{\tau_F\}_{F \in \mathcal{F}}) \text{ with } \boldsymbol{\tau}_T = \varpi_T^k \boldsymbol{t} \text{ and } \tau_F = \pi_F^k(\boldsymbol{t} \cdot \boldsymbol{n}_F).$$

Discrete divergence III

Proposition (Commuting property for discrete divergence operator)

Denoting by π_T^k *and* π_h^k *the* L^2*-orthogonal projectors on* $\mathbb{P}_d^k(T)$ *and* $\mathbb{P}_d^k(\mathcal{T}_h)$*, respectively, the following commuting diagrams hold:*

$$
\begin{array}{ccc}
\boldsymbol{\Sigma}^+(T) & \xrightarrow{\nabla\cdot} & L^2(T) \\
\Big\downarrow I_T^k & & \Big\downarrow \pi_T^k \\
\boldsymbol{\Sigma}_T^k & \xrightarrow{D_T^k} & U_T^k
\end{array}
\qquad
\begin{array}{ccc}
\boldsymbol{\Sigma}^+ & \xrightarrow{\nabla\cdot} & U \\
\Big\downarrow I_h^k & & \Big\downarrow \pi_h^k \\
\boldsymbol{\Sigma}_h^k & \xrightarrow{D_h^k} & U_h^k
\end{array}
$$

Flux reconstruction: Consistency I

- Goal: reproduce exactly the fluxes of potentials in $\mathbb{P}_d^{k+1}(T)$
- For all $\boldsymbol{t} \in \boldsymbol{\Gamma}_T^k := \boldsymbol{K}_T \nabla \mathbb{P}_d^{k+1,0}(T)$, integration by parts yields

$$(\boldsymbol{t}, \nabla v)_T = -(v, \nabla \cdot \boldsymbol{t})_T + \sum_{F \in \mathcal{F}_T} (v, \boldsymbol{t} \cdot \boldsymbol{n}_{TF})_F \quad \forall v \in \mathbb{P}_d^{k+1,0}(T)$$

- With $\boldsymbol{\tau} = (\boldsymbol{\tau}_T, \{\tau_F\}_{F \in \mathcal{F}_T}) = I_T^k \boldsymbol{t}$ one has

$$(\boldsymbol{t}, \nabla v)_T = -(v, D_T^k \boldsymbol{\tau})_T + \sum_{F \in \mathcal{F}_T} (v, \tau_F \epsilon_{TF})_F \quad \forall v \in \mathbb{P}_d^{k+1,0}(T),$$

that is to say, we can express $(\boldsymbol{t}, \nabla v)_T$ in terms of the DOFs $\boldsymbol{\tau}$

Flux reconstruction: Consistency II

- Consistent part: $\mathfrak{C}_T^k : \mathbf{\Sigma}_T^k \to \mathbf{\Gamma}_T^k$ s.t., $\forall \boldsymbol{\tau} = (\boldsymbol{\tau}_T, \{\tau_F\}_{F \in \mathcal{F}_T}) \in \mathbf{\Sigma}_T^k$,

$$\boxed{(\mathfrak{C}_T^k \boldsymbol{\tau}, \nabla v)_T = -(v, D_T^k \boldsymbol{\tau})_T + \sum_{F \in \mathcal{F}_T} (v, \tau_F \epsilon_{TF})_F \quad \forall v \in \mathbb{P}_d^{k+1,0}(T)}$$

- Recalling that $\mathfrak{C}_T^k \boldsymbol{\tau} = \boldsymbol{K}_T \nabla z$ with $z \in \mathbb{P}_d^{k+1,0}(T)$, this can be reformulated as the (well-posed) Neumann problem in z

$$(\boldsymbol{K}_T \nabla z, \nabla v)_T = -(v, D_T^k \boldsymbol{\tau})_T + \sum_{F \in \mathcal{F}_T} (v, \tau_F \epsilon_{TF})_F \quad \forall v \in \mathbb{P}_d^{k+1,0}(T)$$

- This trivially parallel task can benefit from GPU linear solvers

Flux reconstruction: Consistency III

Lemma (Properties of $\mathfrak{C}_T^k$)

The following consistency condition holds:

$$(\mathfrak{C}_T^k \circ I_T^k)(\boldsymbol{K}_T \nabla v) = \boldsymbol{K}_T \nabla v \quad \forall v \in \mathbb{P}_d^{k+1}(T).$$

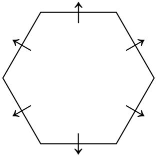
Additionally, there is $\eta_1 > 0$ independent of h and $\boldsymbol{K}$ s.t., for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$,

$$k_{\sharp,T}^{-1} \|\boldsymbol{\tau}_T\|_T^2 \leqslant \|\boldsymbol{K}_T^{-1/2} \mathfrak{C}_T^k \boldsymbol{\tau}\|_T^2 \leqslant \eta_1^{-1} k_{\flat,T}^{-1} \|\!|\boldsymbol{\tau}|\!\|_T^2,$$

where we have defined on the space $\boldsymbol{\Sigma}_T^k$ the norm

$$\forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k, \quad \|\!|\boldsymbol{\tau}|\!\|_T^2 := \|\boldsymbol{\tau}_T\|_T^2 + h_T^2 \|D_T^k \boldsymbol{\tau}\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \|\tau_F\|_F^2.$$

Flux reconstruction: Consistency IV



- Let, for all $F \in \mathcal{F}_h$, $\boldsymbol{x}_F$ denote the face barycenter
- Explicit formula for $\mathfrak{C}_T^0$: For all $\boldsymbol{\tau} = (\tau_F)_{F \in \mathcal{F}_T} \in \boldsymbol{\Sigma}_T^0$,

$$\boxed{\mathfrak{C}_T^0 \boldsymbol{\tau} = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (\boldsymbol{x}_F - \boldsymbol{x}_T) \tau_F \epsilon_{TF}}$$

Flux reconstruction: Stability I

- $\mathfrak{C}_T^k$ does not control all the DOFs in T
- For all $F \in \mathcal{F}_T$, we define the pyramidal residual

$$\mathfrak{J}_{TF}^k : \mathbf{\Sigma}_T^k \to \mathbf{\Gamma}_{TF}^k := \boldsymbol{K}_T \nabla \mathbb{P}_d^{k+1,0}(\mathcal{P}_{TF})$$

s.t., for all $\boldsymbol{\tau} = (\boldsymbol{\tau}_T, \{\tau_F\}_{F\in\mathcal{F}_T}) \in \mathbf{\Sigma}_T^k$ and all $v \in \mathbb{P}_d^{k+1,0}(\mathcal{P}_{TF})$

$$\boxed{(\mathfrak{J}_{TF}^k \boldsymbol{\tau}, \nabla v)_{\mathcal{P}_{TF}} = ((\nabla\cdot\mathfrak{C}_T^k - D_T^k)\boldsymbol{\tau}, v)_{\mathcal{P}_{TF}} - \epsilon_{TF}(\mathfrak{C}_T^k\boldsymbol{\tau}\cdot\boldsymbol{n}_F - \tau_F, v)_F}$$

- $\mathfrak{J}_T^k$ can be computed solving a Neumann problem inside $\mathcal{P}_{TF}$
- Virtualizing the method this step can be avoided

Flux reconstruction: Stability II

- We can find an explicit formula for $\mathfrak{I}_{TF}^{0}$
- For all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_h$, $\boldsymbol{x}_{TF}$ denotes the barycenter of $\mathcal{P}_{TF}$
- We have for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$,

$$\boxed{\mathfrak{I}_{TF}^{0}\boldsymbol{\tau} = \mu \frac{d}{d_{TF}} \epsilon_{TF} (\tau_F - \mathfrak{C}_T^0 \boldsymbol{\tau} \cdot \boldsymbol{n}_F)(\boldsymbol{x}_F - \boldsymbol{x}_{TF})}$$

- For $\mu = \frac{d+1}{d^2}$ we recover the reconstruction of [Codecasa et al., 2010]
- For $\mu = \frac{d+1}{d^2}\tilde{\mu}$ with $\tilde{\mu} > 0$ that of [Droniou et al., 2010]

Flux reconstruction: Stability III

$$\boxed{\mathfrak{I}_T^k := \sum_{F\in\mathcal{F}_T} \mathfrak{I}_{TF}^k}$$

Lemma (Properties of $\mathfrak{I}_T^k$)

The following consistency and orthogonality conditions hold:

$$\begin{aligned}(\mathfrak{I}_T^k \circ I_T^k)(\boldsymbol{K}_T\nabla v) &= 0 \quad \forall v \in \mathbb{P}_d^{k+1}(T),\\ (\boldsymbol{K}_T^{-1}\mathfrak{I}_T^k\boldsymbol{\tau}, \boldsymbol{w})_T &= 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{w}) \in \boldsymbol{\Sigma}_T^k \times \boldsymbol{\Gamma}_T^k.\end{aligned}$$

Flux reconstruction: Stability IV

$$\boxed{\mathfrak{R}_T^k = \mathfrak{C}_T^k + \mathfrak{J}_T^k}$$

Lemma (Properties of $\mathfrak{R}_T^k$)

The following consistency property holds:

$$(\mathfrak{R}_T^k \circ I_T^k)(\boldsymbol{K}_T \nabla v) = 0 \quad \forall v \in \mathbb{P}_d^{k+1}(T).$$

Additionally, there is $\eta > 0$ independent of h and $\boldsymbol{K}$ s.t. for all $T \in \mathcal{T}_h$,

$$\begin{aligned} \|\boldsymbol{\tau}\|_{H,T}^2 &\geq k_{\sharp,T}^{-1} \eta |||\boldsymbol{\tau}|||_T^2 && \forall \boldsymbol{\tau} \in \ker(D_T^k), \\ \|\boldsymbol{\tau}\|_{H,T}^2 &\leq \eta^{-1} k_{\flat,T}^{-1} |||\boldsymbol{\tau}|||_T^2 && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k. \end{aligned}$$

What about Raviart–Thomas? I

- Let $\mathcal{T}_h$ be a simplicial mesh
- We define the high-order residuals supported on T: For all $\boldsymbol{x} \in T$,

$$(\mathfrak{J}^0_{TF}\boldsymbol{\tau})(\boldsymbol{x}) = |F|_{d-1}\epsilon_{TF}(\tau_F - (\mathfrak{C}^0_T\boldsymbol{\tau})\cdot\boldsymbol{n}_F)\boldsymbol{\varphi}_{TF}(\boldsymbol{x})$$

- Then, the stabilization

$$\mathfrak{J}^0_T := \sum_{F\in\mathcal{F}_T} \mathfrak{J}^0_{TF}$$

can be shown to matches the consistency and stability requirements

What about Raviart–Thomas? II

- To prove orthogonality, observe that, for all $(\boldsymbol{\tau}, \boldsymbol{w}) \in \boldsymbol{\Sigma}_T^0 \times \boldsymbol{\Gamma}_T^0$,

$$(\boldsymbol{K}_T^{-1}\mathfrak{I}_T^0\boldsymbol{\tau}, \boldsymbol{w})_T = \left\{ \sum_{F\in\mathcal{F}_T} |F|_{d-1}\epsilon_{TF}(\tau_F - (\mathfrak{C}_T^0\boldsymbol{\tau})\cdot\boldsymbol{n}_F)(\boldsymbol{x}_F - \boldsymbol{x}_T) \right\} \cdot \boldsymbol{K}_T^{-1}\boldsymbol{w},$$

  since $\int_T \boldsymbol{\varphi}_{TF} = \boldsymbol{x}_F - \boldsymbol{x}_T$

- The term between braces is equal to

$$|T|_d\mathfrak{C}_T^0\boldsymbol{\tau} - \left\{ \sum_{F\in\mathcal{F}_T} |F|_{d-1}(\boldsymbol{x}_F - \boldsymbol{x}_T) \otimes \boldsymbol{n}_{TF} \right\} \mathfrak{C}_T^0\boldsymbol{\tau} = 0$$

  since $\sum_{F\in\mathcal{F}_T} |F|_{d-1}(\boldsymbol{x}_F - \boldsymbol{x}_T) \otimes \boldsymbol{n}_{TF} = |T|_d\mathrm{Id}_d$

Mixed approximation I

- Let H_T be the local bilinear form s.t., for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$,

$$\boxed{H_T(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{K}_T^{-1} \mathfrak{R}_T^k \boldsymbol{\sigma}, \mathfrak{R}_T^k \boldsymbol{\tau})_T}$$

- The global bilinear form H on $\boldsymbol{\Sigma}_h^k \times \boldsymbol{\Sigma}_h^k$ is s.t. that, $\forall \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k$,

$$H(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) := \sum_{T \in \mathcal{T}_h} H_T(L_T \boldsymbol{\sigma}_h, L_T \boldsymbol{\tau}_h)$$

- The discrete problem reads: Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h^k \times U_h^k$ such that

$$\boxed{\begin{aligned} H(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (u_h, D_h^k \boldsymbol{\tau}_h) &= 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k \\ -(D_h^k \boldsymbol{\sigma}_h, v_h) &= (f, v_h) & \forall v_h \in U_h^k \end{aligned}} \qquad (\mathsf{M}_h)$$

Mixed approximation II

Lemma (Basic error estimate)

Let $\widehat{\boldsymbol{\sigma}}_h := I_h^k \boldsymbol{s}$ *and* $\widehat{u}_h := \pi_h^k u$*. Then, the following estimate holds:*

$$\max\left(\tfrac{1}{2}\beta(\eta k_\flat)^{1/2}\|\widehat{u}_h - u_h\|, \|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_H\right) \leqslant \sup_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k,\, \|\boldsymbol{\tau}_h\|_H = 1} \mathcal{E}_h(\boldsymbol{\tau}_h),$$

with consistency error $\mathcal{E}_h(\boldsymbol{\tau}_h) := H(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + (\widehat{u}_h, D_h^k \boldsymbol{\tau}_h)$.

Mixed approximation III

Theorem (Convergence rate)

Assuming the regularity $u \in H^1_0(\Omega) \cap H^{k+2}(\mathcal{T}_h)$*, we have*

$$\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_H \lesssim \left(\sum_{T \in \mathcal{T}_h} \rho_{\boldsymbol{K},T} k_{\sharp,T} h_T^{2(k+1)} \|u\|^2_{H^{k+2}(T)} \right)^{1/2},$$

with local anisotropy ratio $\rho_{\boldsymbol{K},T} := k_{\sharp,T}/k_{\flat,T}$.

Theorem (Supercloseness of the potential)

Under the above assumptions, and assuming elliptic regularity, the following holds:

$$\|\widehat{u}_h - u_h\| \lesssim Ch^{k+2}\|u\|_{H^{k+2}(\Omega)}.$$

A virtual variation

- Assume $\boldsymbol{K}_T = \lambda_T \mathrm{Id}_d$ and let for all $T \in \mathcal{T}_h$ and all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k$,

$$H_T^{\mathrm{V}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{K}_T^{-1} \mathfrak{C}_T^k \boldsymbol{\sigma}, \mathfrak{C}_T^k \boldsymbol{\tau})_T + J_T^{\mathrm{V}}(\boldsymbol{\sigma}, \boldsymbol{\tau}),$$

  with stabilization bilinear form

$$\boxed{J_T^{\mathrm{V}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \sum_{F \in \mathcal{F}_T} \frac{h_F}{\lambda_T} (\mathfrak{C}_T^k \boldsymbol{\sigma} \cdot \boldsymbol{n}_F - \sigma_F, \mathfrak{C}_T^k \boldsymbol{\tau} \cdot \boldsymbol{n}_F - \tau_F)_F}$$

- The discrete problem reads: Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h^k \times U_h^k$ such that

$$\boxed{\begin{aligned} H^{\mathrm{V}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (u_h, D_h^k \boldsymbol{\tau}_h) &= 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k \\ -(D_h^k \boldsymbol{\sigma}_h, v_h) &= (f, v_h) & \forall v_h \in U_h^k \end{aligned}} \qquad (\mathrm{MV}_h)$$

- Analogous convergence results hold for (MV_h)

Numerical example I

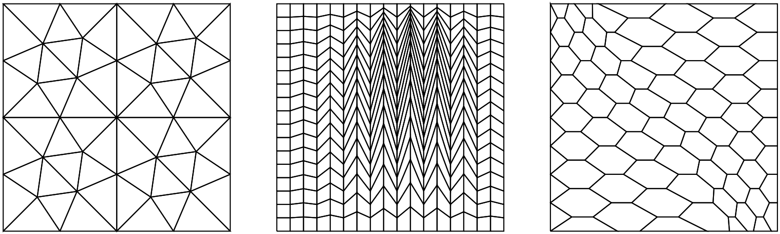
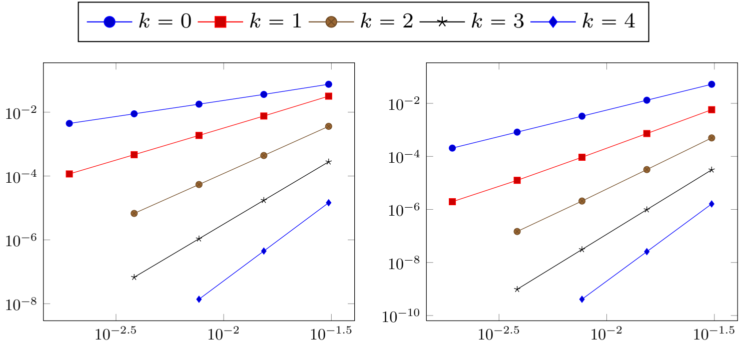


Figure: Triangular, Kershaw, and hexagonal meshes for the numerical example

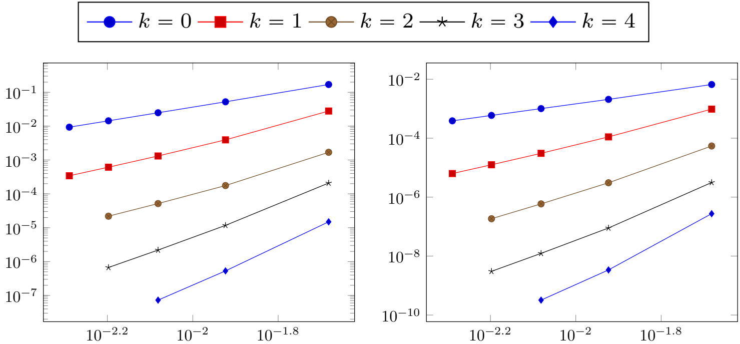
Numerical example II



(a) $\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_H$ vs. h (b) $\|\widehat{u}_h - u_h\|$ vs. h

Figure: Triangular mesh, Dirichlet problem with $u = \sin(\pi x_1)\sin(\pi x_2)$

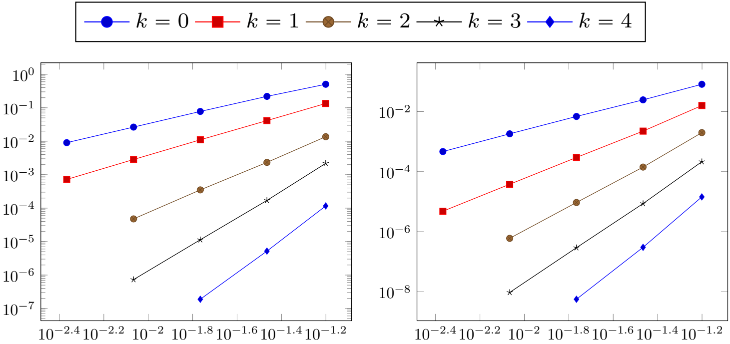
Numerical example III



(a) $\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_H$ vs. h (b) $\|\widehat{u}_h - u_h\|$ vs. h

Figure: Kershaw mesh, Dirichlet problem with $u = \sin(\pi x_1)\sin(\pi x_2)$

Numerical example IV



(a) $\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_H$ vs. h

(b) $\|\widehat{u}_h - u_h\|$ vs. h

Figure: Hexagonal mesh, Dirichlet problem with $u = \sin(\pi x_1)\sin(\pi x_2)$

Implementation in primal form I

- Some paper work allows a primal implementation
- Lagrange multipliers to enforce uniqueness of interface unknowns:

$$\Lambda_T^k := \bigtimes_{F \in \mathcal{F}_T} \Lambda_F^k \quad \forall T \in \mathcal{T}_h, \qquad \Lambda_h^k := \bigtimes_{F \in \mathcal{F}_h} \Lambda_F^k,$$

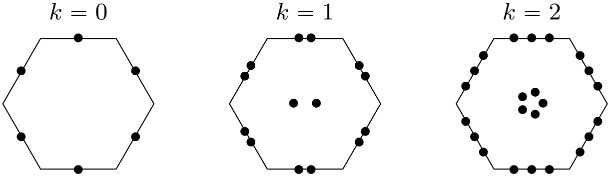
  where

$$\Lambda_F^k := \begin{cases} \mathbb{P}_{d-1}^k(F) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}} \\ \{0\} & \text{if } F \in \mathcal{F}_h^{\mathrm{b}} \end{cases}$$

- Lagrange multipliers can be interpreted as traces of the potential

Aghili, Boyaval, DP, in preparation **2014**

Implementation in primal form II



- We define local and global hybrid DOF spaces for the potential as

$$W_T^k := U_T^k \times \Lambda_T^k \quad \forall T \in \mathcal{T}_h, \qquad W_h^k := U_h^k \times \Lambda_h^k,$$

  and we denote by $\check{L}_T : W_h^k \mapsto W_T^k$ the restriction operator
- Let the discrete gradient $\boldsymbol{G}_T^k$ be s.t., $\forall z = (v_T, (\mu_F)_{F \in \mathcal{F}_T}) \in W_T^k$,

$$\boxed{(\boldsymbol{G}_T^k z, \mathfrak{R}_T^k \boldsymbol{\tau})_T = -(v_T, D_T^k \boldsymbol{\tau})_T + \sum_{F \in \mathcal{F}_T} (\mu_F, \tau_{TF})_F \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_T^k}$$

Implementation in primal form III

Lemma (Primal hybrid version)

Let $\boldsymbol{G}_h^k : W_h^k \to L^2(\Omega)^d$ *be s.t., for all* $z_h \in W_h^k$,

$$\boldsymbol{G}_h^k z_{h|T} = \boldsymbol{G}_T^k \check{L}_T z_h \quad \forall T \in \mathcal{T}_h,$$

and let $w_h \in (u_h, \lambda_h) \in W_h^k$ *solve: For all* $z_h = (v_h, \mu_h) \in W_h^k$,

$$\boxed{(\boldsymbol{K}\boldsymbol{G}_h^k w_h, \boldsymbol{G}_h^k z_h) = (f, v_h)} \qquad (\mathsf{PH}_h)$$

Then, u_h *coincides with the potential in* (M_h) *and*

$$\mathfrak{R}_T^k L_T \boldsymbol{\sigma}_h = \boldsymbol{K}_T \boldsymbol{G}_T^k \check{L}_T (u_h, \lambda_h) \qquad \forall T \in \mathcal{T}_h.$$

Implementation in primal form IV

- The equivalent formulation (PH_h) yields a SPD rather than a saddle-point matrix
- Cell DOFs can be locally eliminated leading to a global system of size

$$\boxed{\operatorname{card}(\mathcal{F}_h^{\mathrm{i}}) \times \dim(\mathbb{P}_{d-1}^k) = \operatorname{card}(\mathcal{F}_h^{\mathrm{i}}) \times \binom{k+d-1}{k}}$$

- The method of [Di Pietro and Lemaire, 2013] can be recovered as a special case when $k = 0$

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