An arbitrary-order discrete de Rham complex on polyhedral meshes

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Outline

A (not so simple) model problem I

- Let $\Omega \subset \mathbb{R}^3$ be an open connected polyhedral domain that does not enclose any void
- Let a current density $\boldsymbol{f} \in \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl};\Omega)$ be given
- We consider the problem: Find the magnetic field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^3$ and the vector potential $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ s.t.

$$\begin{aligned} \boldsymbol{\sigma} - \mathbf{curl}\,\boldsymbol{u} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)} \\ \mathbf{curl}\,\boldsymbol{\sigma} &= \boldsymbol{f} && \text{in } \Omega, && \text{(Ampère's law)} \\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)} \\ \boldsymbol{u} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)} \end{aligned}$$

- The extension to variable magnetic permeability is straightforward

A (not so simple) model problem II

- In weak formulation: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{\sigma} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{u}\, \mathrm{div}\, \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \end{aligned}$$

- Well-posedness hinges on the exactness of the following portion of the de Rham complex:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathrm{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- **This exactness property is also needed at the discrete level!**

Some approximations of the de Rham complex

- Classical Finite Element methods on standard meshes
  - Mixed Finite Elements [Raviart and Thomas, 1977, Nédélec, 1980]
  - Whitney forms [Bossavit, 1988]
  - Finite Element Exterior Calculus [Arnold, 2018]
  - ...
- Low-order polyhedral methods:
  - Compatible Discrete Operators [Bonelle and Ern, 2014]
  - Discrete Geometric Approach [Codecasa, Specogna, Trevisan, 2009]
  - Mimetic Finite Differences [Beirão da Veiga, Lipnikov, Manzini, 2014]
- High-order polyhedral methods:
  - VEM [Beirão da Veiga, Brezzi, Dassi, Marini, Russo, 2016–2018]
  - Discrete de Rham (DDR) methods
- References for this presentation:
  - Precursor works on DDR [DP et al., 2020, DP and Droniou, 2020]
  - DDR complexes with Koszul complements [DP and Droniou, 2021]

- **Key idea:** define subspaces that form a discrete complex
- Let $T \subset \mathbb{R}^3$ be a mesh element and set, for any $k \geq -1$,

$$\mathcal{P}^k(T) \coloneqq \{\text{restrictions of 3-variate polynomials of degree } \leq k \text{ to } T\}$$

- Fix $k \geq 0$ and write, denoting by $\boldsymbol{x}_T$ the barycenter of T,

$$\begin{aligned}\mathcal{P}^k(T)^3 &= \underbrace{\mathbf{grad}\,\mathcal{P}^{k+1}(T)}_{\mathcal{G}^k(T)} \oplus \underbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathcal{P}^{k-1}(T)^3}_{\mathcal{G}^{\mathrm{c},k}(T)} \\ &= \underbrace{\mathbf{curl}\,\mathcal{P}^{k+1}(T)^3}_{\mathcal{R}^k(T)} \oplus \underbrace{(\boldsymbol{x}-\boldsymbol{x}_T)\mathcal{P}^{k-1}(T)}_{\mathcal{R}^{\mathrm{c},k}(T)}\end{aligned}$$

- Define the trimmed spaces

$$\mathcal{N}^k(T) \coloneqq \mathcal{G}^k(T) \oplus \mathcal{G}^{\mathrm{c},k+1}(T) \qquad \text{[Nédélec, 1980]}$$

$$\mathcal{RT}^k(T) \coloneqq \mathcal{R}^k(T) \oplus \mathcal{R}^{\mathrm{c},k+1}(T) \qquad \text{[Raviart and Thomas, 1977]}$$

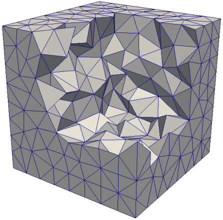
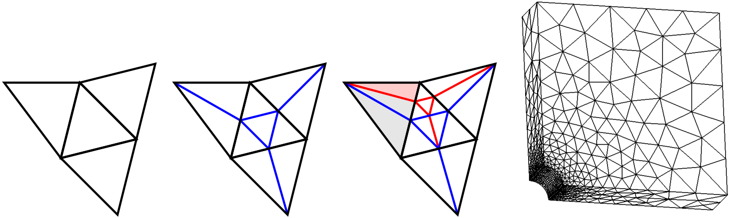


Figure: Conforming tetrahedral mesh of the unit cube (clip)

- Let $\mathcal{T}_h = \{T\}$ be a conforming tetrahedral mesh of Ω and let $k \geq 0$
- Local spaces can be glued together to form the global FE complex

$$\mathbb{R} \xrightarrow{i_\Omega} \mathcal{P}_c^{k+1}(\mathcal{T}_h) \xrightarrow{\textbf{grad}} \boldsymbol{N}^k(\mathcal{T}_h) \xrightarrow{\textbf{curl}} \boldsymbol{\mathcal{RT}}^k(\mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- **This procedure only works on conforming meshes!**



- Approach limited to conforming meshes with standard elements
- $\Longrightarrow$ local refinement requires to trade mesh size for mesh quality
- $\Longrightarrow$ complex geometries may require a large number of elements
- $\Longrightarrow$ the element shape cannot be adapted to the solution
- The implementation of high-order versions may be tricky
- ...

The discrete de Rham (DDR) approach I

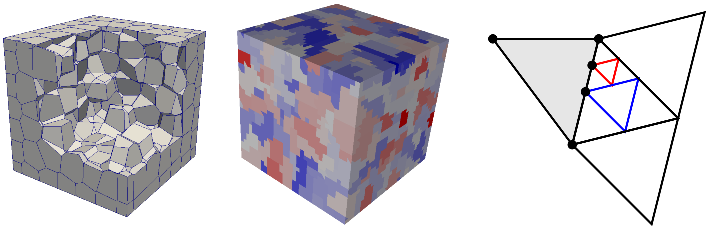


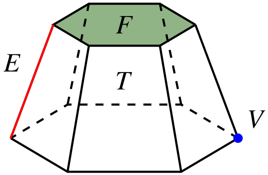
Figure: Examples of polytopal meshes supported by the DDR approach

- **Key idea:** replace spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{\boldsymbol{I}}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of general polyhedral meshes and high-order **(!)**
- Exactness proved at the discrete level (directly usable for stability)
- Seamless implementation of high-order versions

The discrete de Rham (DDR) approach II



- The fully discrete spaces are spanned by vectors of polynomials
- Polynomial components attached to geometric objects to emulate
  - full continuity for the approximation of $H^1(\Omega)$
  - continuity of tangential traces for the approximation of $\boldsymbol{H}(\mathbf{curl};\Omega)$
  - continuity of normal traces for the approximation of $\boldsymbol{H}(\mathrm{div};\Omega)$
- Selected so as to enable the reconstruction of consistent
  - discrete vector calculus operators
  - (scalar or vector) discrete potentials

Outline

The two-dimensional case
Continuous exact complex

- With F mesh face let, for $q : F \to \mathbb{R}$ and $\boldsymbol{v} : F \to \mathbb{R}^2$ smooth enough,

$$\mathbf{rot}_F\, q := \varrho_{-\pi/2}(\mathbf{grad}_F\, q) \qquad \mathrm{rot}_F\, \boldsymbol{v} := \mathrm{div}_F(\varrho_{-\pi/2}\boldsymbol{v})$$

- We derive a discrete counterpart of the exact local complex:

$$\mathbb{R} \xrightarrow{i_F} H^1(F) \xrightarrow{\mathbf{grad}_F} \boldsymbol{H}(\mathrm{rot}; F) \xrightarrow{\mathrm{rot}_F} L^2(F) \xrightarrow{0} \{0\}$$

- We will need the following decompositions of $\mathcal{P}^k(F)^2$:

$$\begin{aligned}\mathcal{P}^k(F)^2 &= \underbrace{\mathbf{grad}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{G}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)^\perp \mathcal{P}^{k-1}(F)}_{\mathcal{G}^{\mathrm{c},k}(F)} \\ &= \underbrace{\mathbf{rot}_F\, \mathcal{P}^{k+1}(F)}_{\mathcal{R}^k(F)} \oplus \underbrace{(\boldsymbol{x} - \boldsymbol{x}_F)\mathcal{P}^{k-1}(F)}_{\mathcal{R}^{\mathrm{c},k}(F)}\end{aligned}$$

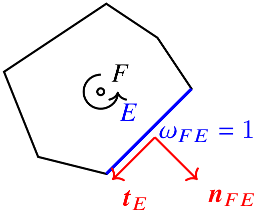
The two-dimensional case
A key remark

- Let $q \in \mathcal{P}^{k+1}(F)$. For any $\boldsymbol{v} \in \mathcal{P}^k(F)^2$, we have

$$\int_F \mathbf{grad}_F\, q \cdot \boldsymbol{v} = -\int_F q \underbrace{\operatorname{div}_F \boldsymbol{v}}_{\in \mathcal{P}^{k-1}(F)} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$
$$= -\int_F \pi^{k-1}_{\mathcal{P},F} q \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{|\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})$$

  with $\pi^{k-1}_{\mathcal{P},F}$ L^2-orthogonal projector on $\mathcal{P}^{k-1}(F)$

- Hence, $\mathbf{grad}_F\, q$ can be computed given $\pi^{k-1}_{\mathcal{P},F} q$ and $q_{|\partial F}$



The two-dimensional case

Discrete $H^1(F)$ space

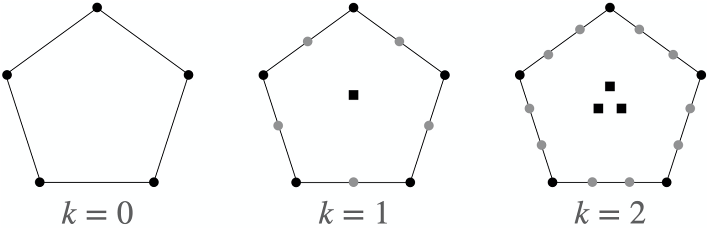


Figure: Number of degrees of freedom for $\underline{X}^k_{\mathbf{grad},F}$ for $k \in \{0, 1, 2\}$

- Based on this remark, we take as discrete counterpart of $H^1(F)$

$$\underline{X}^k_{\mathbf{grad},F} := \left\{ \underline{q}_F = (q_F, q_{\partial F}) \ : \ q_F \in \mathcal{P}^{k-1}(F) \text{ and } q_{\partial F} \in \mathcal{P}^{k+1}_{\mathrm{c}}(\mathcal{E}_F) \right\}$$

- The interpolator $\underline{I}^k_{\mathbf{grad},F} : C^0(\overline{F}) \to \underline{X}^k_{\mathbf{grad},F}$ is s.t., $\forall q \in C^0(\overline{F})$,

$$\underline{I}^k_{\mathbf{grad},F} q := (\pi^{k-1}_{\mathcal{P},F} q, q_{\partial F}) \text{ with}$$

$$\pi^{k-1}_{\mathcal{P},E}(q_{\partial F})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \ \forall E \in \mathcal{E}_F \text{ and } q_{\partial F}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \ \forall V \in \mathcal{V}_F$$

- For all $E \in \mathcal{E}_F$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(E)$ is s.t.

$$\boxed{G^k_E \underline{q}_F := (q_{\partial F})'_{|E}}$$

- The full face gradient $\mathsf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall \boldsymbol{v} \in \mathcal{P}^k(F)^2$,

$$\boxed{\int_F \mathsf{G}^k_F \underline{q}_F \cdot \boldsymbol{v} = -\int_F q_F \operatorname{div}_F \boldsymbol{v} + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v} \cdot \boldsymbol{n}_{FE})}$$

- By construction, we have polynomial consistency:

$$\mathsf{G}^k_F \left(\underline{I}^k_{\mathbf{grad},F} q\right) = \mathbf{grad}_F\, q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

- The scalar trace $\gamma_F^{k+1} : \underline{X}_{\mathbf{grad},F}^k \to \mathcal{P}^{k+1}(F)$ is s.t., $\forall \boldsymbol{v}_F \in \mathcal{R}^{\mathrm{c},k+2}(F)$,

$$\boxed{\int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F \boldsymbol{v}_F = -\int_F \mathsf{G}_F^k \underline{q}_F \cdot \boldsymbol{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\partial F} (\boldsymbol{v}_F \cdot \boldsymbol{n}_{FE})}$$

- Well defined since $\operatorname{div}_F : \mathcal{R}^{\mathrm{c},k+2}(F) \xrightarrow{\cong} \mathcal{P}^{k+1}(F)$ is an isomorphism
- Also in this case, we have polynomial consistency:

$$\gamma_F^{k+1}(\underline{I}_{\mathbf{grad},F}^k q) = q \qquad \forall q \in \mathcal{P}^{k+1}(F)$$

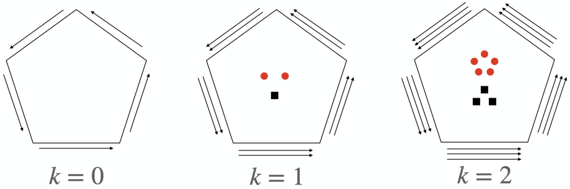


Figure: Number of degrees of freedom for $\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ for $k \in \{0, 1, 2\}$

- We reason starting from: $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^k(F) := \boldsymbol{\mathcal{G}}^k(F) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(F)$,

$$\int_F \mathrm{rot}_F\, \boldsymbol{v}\; q = \int_F \boldsymbol{v} \cdot \underbrace{\mathbf{rot}_F\, q}_{\in \boldsymbol{\mathcal{R}}^{k-1}(F)} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v} \cdot \boldsymbol{t}_E) \underbrace{q_{|E}}_{\in \mathcal{P}^k(E)} \quad \forall q \in \mathcal{P}^k(F)$$

- This leads to the following discrete counterpart of $\boldsymbol{H}(\mathrm{rot}; F)$:

$$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F} := \Big\{ \underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}, (v_E)_{E \in \mathcal{E}_F}\big) : \boldsymbol{v}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F),\ \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F),\ v_E \in \mathcal{P}^k(E)\ \forall E \in \mathcal{E}_F \Big\}$$

- The face curl operator $C_F^k : \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \to \mathcal{P}^k(F)$ is s.t.,

$$\boxed{\int_F C_F^k \underline{\boldsymbol{v}}_F \, q = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E \, q \quad \forall q \in \mathcal{P}^k(F)}$$

- Define the interpolator $\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k : H^1(F)^2 \to \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$ s.t., $\forall \boldsymbol{v} \in H^1(F)^2$,

$$\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k \boldsymbol{v} := \big(\boldsymbol{\pi}_{\mathcal{R},F}^{k-1} \boldsymbol{v}, \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k} \boldsymbol{v}, \big(\pi_{\mathcal{P},E}^k (\boldsymbol{v}_{|E} \cdot \boldsymbol{t}_E) \big)_{E \in \mathcal{E}_F} \big).$$

- C_F^k is polynomially consistent by construction:

$$C_F^k \big(\underline{\boldsymbol{I}}_{\mathrm{rot},F}^k \boldsymbol{v} \big) = \mathrm{rot}_F \, \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^k(F)$$

- The tangential trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)^2$ is s.t., $\forall (r_F, \boldsymbol{w}_F) \in \mathcal{P}^{k+1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$,

$$\int_F \boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot}_F\, r_F + \boldsymbol{w}_F) = \int_F C^k_F \underline{\boldsymbol{v}}_F\; r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F + \int_F \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F} \cdot \boldsymbol{w}_F$$

- Well-defined owing to $\mathcal{P}^k(F)^2 = \mathcal{R}^k(F) \oplus \mathcal{R}^{\mathrm{c},k}(F)$
- Also in this case, we have polynomial consistency:

$$\boldsymbol{\gamma}^k_{\mathrm{t},F}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},F} \boldsymbol{v}) = \boldsymbol{\pi}^k_{\mathcal{P},F} \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{N}^k(F)$$

Theorem (Exactness of the two-dimensional local DDR complex)

If F is simply connected, the following local complex is exact:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},F}} \underline{X}^k_{\mathbf{grad},F} \xrightarrow{\underline{\boldsymbol{G}}^k_F} \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \xrightarrow{C^k_F} \mathcal{P}^k(F) \xrightarrow{0} \{0\},$$

where $\underline{\boldsymbol{G}}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ *is the discrete gradient s.t.,* $\forall \underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$,

$$\underline{\boldsymbol{G}}^k_F \underline{q}_F \coloneqq \big(\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}(\mathsf{G}^k_F \underline{q}_F), (G^k_E \underline{q}_F)_{E\in\mathcal{E}_F}\big).$$

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathbf{grad},F}^k} \underline{X}_{\mathbf{grad},F}^k \xrightarrow{\underline{\boldsymbol{G}}_F^k} \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k \xrightarrow{C_F^k} \mathcal{P}^k(F) \xrightarrow{0} \{0\}$$

Space	V (vertex)	E (edge)	F (face)
$\underline{X}_{\mathbf{grad},F}^k$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$
$\underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$
$\mathcal{P}^k(F)$			$\mathcal{P}^k(F)$

Table: Polynomial components for the two-dimensional spaces

- Interpolators = component-wise L^2-projections
- Discrete operators = L^2-projections of full operator reconstructions

The three-dimensional case I

Exact of the local three-dimensional complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathbf{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}$$

Space	V	E	F	T (element)
$\underline{X}^k_{\mathbf{grad},T}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$
$\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{\mathrm{c},k}(T)$
$\mathcal{P}^k(T)$				$\mathcal{P}^k(T)$

Table: Polynomial components for the three-dimensional spaces

Theorem (Exactness of the three-dimensional local DDR complex)

If the element T has a trivial topology, this complex is exact.

Commutation properties

Lemma (Local commutation properties)

It holds, for all $T \in \mathcal{T}_h$,

$$
\begin{aligned}
\underline{\boldsymbol{G}}_T^k(\underline{I}_{\mathbf{grad},T}^k q) &= \underline{\boldsymbol{I}}_{\mathbf{curl},T}^k(\mathbf{grad}\, q) &\qquad& \forall q \in C^1(\overline{T}),\\
\underline{\boldsymbol{C}}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}) &= \underline{\boldsymbol{I}}_{\mathrm{div},T}^k(\mathbf{curl}\, \boldsymbol{v}) && \forall \boldsymbol{v} \in H^2(T)^3,\\
D_T^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}) &= \pi_{\mathcal{P},T}^k(\mathrm{div}\, \boldsymbol{w}) && \forall \boldsymbol{w} \in H^1(T)^3.
\end{aligned}
$$

The above properties imply the following commutative diagram:

$$
\begin{array}{ccccccc}
C^\infty(\overline{T}) & \xrightarrow{\mathbf{grad}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathbf{curl}} & C^\infty(\overline{T})^3 & \xrightarrow{\mathrm{div}} & C^\infty(\overline{T}) \\
\Big\downarrow{\scriptstyle \underline{I}_{\mathbf{grad},T}^k} & & \Big\downarrow{\scriptstyle \underline{\boldsymbol{I}}_{\mathbf{curl},T}^k} & & \Big\downarrow{\scriptstyle \underline{\boldsymbol{I}}_{\mathrm{div},T}^k} & & \Big\downarrow{\scriptstyle i_T} \\
\underline{X}_{\mathbf{grad},T}^k & \xrightarrow{\underline{\boldsymbol{G}}_T^k} & \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k & \xrightarrow{\underline{\boldsymbol{C}}_T^k} & \underline{\boldsymbol{X}}_{\mathrm{div},T}^k & \xrightarrow{D_T^k} & \mathcal{P}^k(T)
\end{array}
$$

- Emulating integration by part formulas, we define the local potentials

$$\begin{aligned} P^{k+1}_{\mathbf{grad},T} &: \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T), \\ \boldsymbol{P}^k_{\mathbf{curl},T} &: \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathcal{P}^k(T)^3, \\ \boldsymbol{P}^k_{\mathrm{div},T} &: \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \to \mathcal{P}^k(T)^3 \end{aligned}$$

- Based on these potentials, we construct local discrete L^2-products

$$(\underline{x}_T, \underline{y}_T)_{\bullet,T} = \underbrace{\int_T P_{\bullet,T}\underline{x}_T \cdot P_{\bullet,T}\underline{y}_T}_{\text{consistency}} + \underbrace{\mathrm{s}_{\bullet,T}(\underline{x}_T, \underline{y}_T)}_{\text{stability}} \quad \forall \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$$

- The L^2-products are polynomially exact

- Let $\mathcal{T}_h$ be a polyhedral mesh with elements and faces of trivial topology
- Global DDR spaces are defined gluing boundary components:

$$\underline{X}^k_{\mathbf{grad},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \qquad \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$$

- Global operators are obtained collecting local components:

$$\underline{\boldsymbol{G}}^k_h : \underline{X}^k_{\mathbf{grad},h} \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \quad \underline{\boldsymbol{C}}^k_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}, \quad D^k_h : \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathcal{P}^k(\mathcal{T}_h)$$

- Global L^2-products $(\cdot,\cdot)_{\bullet,h}$ are obtained assembling element-wise
- The global DDR complex is

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Outline

Exactness I

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Theorem (Exactness properties)

For any connected polyhedral domain $\Omega \subset \mathbb{R}^3$, it holds

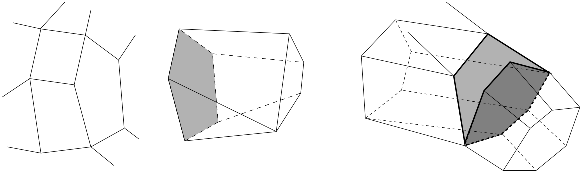
$$\underline{I}^k_{\mathbf{grad},h}\mathbb{R} = \mathrm{Ker}\,\underline{\boldsymbol{G}}^k_h, \qquad \mathrm{Im}\,D^k_h = \mathcal{P}^k(\mathcal{T}_h),$$
$$\mathrm{Im}\,\underline{\boldsymbol{G}}^k_h \subset \mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h, \qquad \mathrm{Im}\,\underline{\boldsymbol{C}}^k_h \subset \mathrm{Ker}\,D^k_h.$$

Moreover, denoting by (b_0, b_1, b_2, b_3) the Betti numbers of Ω, we have

$$b_1 = 0 \implies \mathrm{Im}\,\underline{\boldsymbol{G}}^k_h = \mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h,$$
$$b_2 = 0 \implies \mathrm{Im}\,\underline{\boldsymbol{C}}^k_h = \mathrm{Ker}\,D^k_h.$$

Exactness II

- Let us give an idea of the proof of $\operatorname{Im} \underline{\boldsymbol{C}}_h^k = \operatorname{Ker} D_h^k$
- $\operatorname{Im} \underline{\boldsymbol{C}}_h^k \subset \operatorname{Ker} D_h^k$ follows by the corresponding local property
- We prove $\operatorname{Ker} D_h^k \subset \operatorname{Im} \underline{\boldsymbol{C}}_h^k$ in two steps. Let $\underline{\boldsymbol{v}}_h \in \operatorname{Ker} D_h^k$. Then:
  - Local exactness gives $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k$ s.t. $\underline{\boldsymbol{v}}_T = \underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{\tau}}_T$ for all $T \in \mathcal{T}_h$
  - The local vectors are then glued together
- To glue together local vectors, we use the fact that the mesh can be topologically assembled by a succession of the following operations:



- **This is only possible since Ω does not enclose any void ($b_2 = 0$)!**

Discrete Poincaré inequalities

$\|\cdot\|_{\bullet,h}$, $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, denotes the norm induced by $(\cdot,\cdot)_{\bullet,h}$ on $\underline{X}^k_{\bullet,h}$

Theorem (Poincaré inequality for the curl)

Assume $b_2 = 0$. Let $(\mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h)^\perp$ be the orthogonal of $\mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h$ in $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ for an inner product with norm equivalent to $\|\cdot\|_{\mathbf{curl},h}$ uniformly in h. Then,

$$\underline{\boldsymbol{C}}^k_h : (\mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h)^\perp \to \mathrm{Ker}\,D^k_h \text{ is an isomorphism.}$$

Further assuming $b_1 = 0$, there exists $C > 0$ independent of h, and depending only on Ω, k and mesh regularity, such that

$$\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} \le C\|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \qquad \forall \underline{\boldsymbol{v}}_h \in (\mathrm{Ker}\,\underline{\boldsymbol{C}}^k_h)^\perp.$$

Similar results can be proved for the gradient and the divergence

Theorem (Consistency of the potential reconstructions)

It holds, for all $T \in \mathcal{T}_h$ and all $(q, \boldsymbol{v}, \boldsymbol{w}) \in H^{k+2}(T) \times H^2(T)^3 \times H^1(T)^3$ s.t. $\mathbf{curl}\,\boldsymbol{v} \in H^{k+1}(T)^3$ and $\operatorname{div}\boldsymbol{w} \in H^{k+1}(T)$,

$$\|\mathsf{G}_T^k(\underline{I}_{\mathbf{grad},T}^k q) - \mathbf{grad}\, q\|_{L^2(T)^3} \lesssim h_T^{k+1} |q|_{H^{k+2}(T)},$$

$$\|\mathsf{C}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}) - \mathbf{curl}\,\boldsymbol{v}\|_{L^2(T)^3} \lesssim h_T^{k+1} |\mathbf{curl}\,\boldsymbol{v}|_{H^{k+1}(T)^3},$$

$$\|D_T^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}) - \operatorname{div}\boldsymbol{w}\|_{L^2(T)} \lesssim h_T^{k+1} |\operatorname{div}\boldsymbol{w}|_{H^{k+1}(T)^3}.$$

Moreover, for all $(q, \boldsymbol{v}, \boldsymbol{w}) \in H^{k+2}(T) \times H^{\max(k+1,2)}(T)^3 \times H^{k+1}(T)^3$,

$$\|P_{\mathbf{grad},T}^{k+1}(\underline{I}_{\mathbf{grad},T}^k q) - q\|_{L^2(T)} \lesssim h_T^{k+2} |q|_{H^{k+2}(T)},$$

$$\|\boldsymbol{P}_{\mathbf{curl},T}^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}) - \boldsymbol{v}\|_{L^2(T)^3} \lesssim h_T^{k+1} |\boldsymbol{v}|_{H^{(k+1,2)}(T)^3},$$

$$\|\boldsymbol{P}_{\mathrm{div},T}^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}) - \boldsymbol{w}\|_{L^2(T)^3} \lesssim h_T^{k+1} |\boldsymbol{w}|_{H^{k+1}(T)^3}.$$

Theorem (Adjoint consistency for the curl)

Let $\mathcal{E}_{\mathbf{curl},h} : \left(C^0(\overline{\Omega})^3 \cap \mathbf{H}_0(\mathbf{curl};\Omega)\right) \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$ *be s.t.*

$$\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h) \coloneqq \sum_{T\in\mathcal{T}_h}\left[(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}_{|T},\underline{\boldsymbol{C}}^k_T\underline{\boldsymbol{v}}_T)_{\mathrm{div},T} - \int_T \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\right].$$

Then, for all $\boldsymbol{w} \in C^0(\overline{\Omega})^3 \cap \mathbf{H}_0(\mathbf{curl};\Omega)$ *s.t.* $\boldsymbol{w} \in H^{k+2}(\mathcal{T}_h)^3$*:*
$\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$*,*

$$|\mathcal{E}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h)| \lesssim h^{k+1}\left(|\boldsymbol{w}|_{H^{k+1}(\mathcal{T}_h)^3} + |\boldsymbol{w}|_{H^{k+2}(\mathcal{T}_h)^3}\right) \times \left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right).$$

Similar results can be proved for the gradient and the divergence

Outline

Discrete problem I

- Continuous problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl};\Omega) \times \boldsymbol{H}(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned}
\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl};\Omega),\\
\int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot\boldsymbol{v} + \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)
\end{aligned}$$

- The DDR scheme is obtained substituting

$$\begin{aligned}
\boldsymbol{H}(\mathbf{curl};\Omega) &\leftarrow \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k\\
\boldsymbol{H}(\mathrm{div};\Omega) &\leftarrow \underline{\boldsymbol{X}}_{\mathrm{div},h}^k
\end{aligned}$$

$$\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} &\leftarrow \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\\
(\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} &\leftarrow \int_\Omega \mathbf{curl}\,\boldsymbol{\tau}\cdot\boldsymbol{v}\\
\int_\Omega D_h^k \underline{\boldsymbol{w}}_h\ D_h^k \underline{\boldsymbol{v}}_h &\leftarrow \int_\Omega \mathrm{div}\,\boldsymbol{w}\ \ \mathrm{div}\,\boldsymbol{v}\\
\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} &\leftarrow \int_\Omega \boldsymbol{f}\cdot\boldsymbol{P}_{\mathrm{div},h}^k \underline{\boldsymbol{v}}_h
\end{aligned}$$

Discrete problem II

- The DDR problem reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\
(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h \, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}
\end{aligned}$$

- Stability hinges on the exactness of the portion

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\},$$

which requires $b_2 = 0$

Analysis I

Theorem (Stability)

Let $\Omega \subset \mathbb{R}^3$ be an polyhedral connected domain s.t. $b_1 = b_2 = 0$ *and set*

$$\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) \coloneqq (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{\boldsymbol{u}}_h \, D_h^k \underline{\boldsymbol{v}}_h.$$

Then, it holds uniformly in h: $\forall (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$*,*

$$|||(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)|||_h \lesssim \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \setminus \{(\underline{\mathbf{0}}, \underline{\mathbf{0}})\}} \frac{\mathrm{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h}$$

with $|||(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)|||_h^2 \coloneqq \|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{curl},h}^2 + \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h\|_{\mathrm{div},h}^2 + \|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}^2 + \|D_h^k \underline{\boldsymbol{v}}_h\|_{L^2(\Omega)}^2$.

Analysis II

Theorem (Error estimate for the magnetostatics problem)

Assume $b_1 = b_2 = 0$, $\boldsymbol{\sigma} \in C^0(\overline{\Omega})^3 \cap H^{k+2}(\mathcal{T}_h)^3$, $\boldsymbol{u} \in C^0(\overline{\Omega})^3 \times H^{k+2}(\mathcal{T}_h)^3$, *and set*

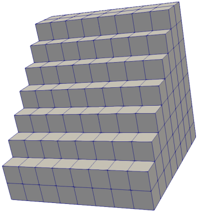
$$(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h) := (\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}).$$

Then, we have the following error estimate:

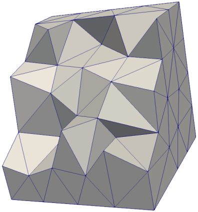
$$|||(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{\varepsilon}}_h)|||_h \lesssim h^{k+1}\Big(|\,\mathbf{curl}\,\boldsymbol{\sigma}|_{H^{k+1}(\mathcal{T}_h)^3} + |\boldsymbol{\sigma}|_{H^{(k+1,2)}(\mathcal{T}_h)^3} + |\boldsymbol{u}|_{H^{k+1}(\mathcal{T}_h)^3} + |\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^3}\Big),$$

with hidden constant depending only on Ω, k, *and mesh regularity.*

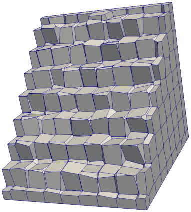
Numerical examples
Meshes



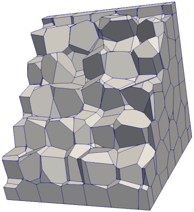
Cubic-Cells



Tetgen-Cube-0



Voro-small-0

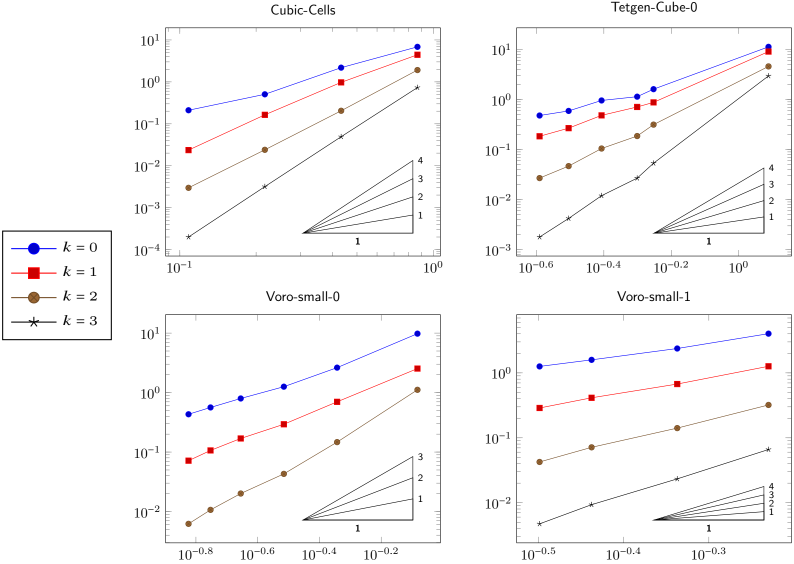


Voro-small-1

Figure: Mesh families used in the numerical tests

Numerical examples

Convergence in the energy norm



Conclusions and perspectives

- Novel approach for the numerical solution of PDEs relating to the de Rham complex
- Key features: support of general polyhedral meshes and high-order
- Novel computational strategies made possible
- Natural extensions to variable coefficients and nonlinearities

- Applications (electromagnetism, incompressible fluid mechanics,. . .)
- Formalization using differential forms (ongoing)
- Development of novel complexes (e.g., elasticity)
- . . .

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