Hybrid High-Order methods for elasticity

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Model problem I

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded, connected polyhedral domain
- For $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$, we consider the elasticity problem

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u})) &= \boldsymbol{f} \qquad && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} \qquad && \text{on } \partial\Omega,
\end{aligned}
$$

with $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ possibly nonlinear strain-stress law and

$$
\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} \coloneqq \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}}\right)
$$

- In weak form: Find $u \in \boldsymbol{U} \coloneqq H^1_0(\Omega)^d$ s.t.

$$
a(u,v) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}):\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall v \in \boldsymbol{U}
$$

Model problem II

Example (Linear elasticity)

Given a uniformly elliptic fourth-order tensor-valued function $\boldsymbol{C} : \Omega \to \mathbb{R}^{d^4}$, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) = \boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau}.$$

For homogeneous isotropic media, $\boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau} = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d$.

Example (Hencky–Mises model)

Given $\lambda : \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu(\mathrm{dev}(\boldsymbol{\tau}))\boldsymbol{\tau}\boldsymbol{I}_d + \lambda(\mathrm{dev}(\boldsymbol{\tau}))\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d,$$

where $\mathrm{dev}(\boldsymbol{\tau}) \coloneqq \mathrm{tr}(\boldsymbol{\tau}^2) - d^{-1}\,\mathrm{tr}(\boldsymbol{\tau})^2$.

Model problem III

Example (Isotropic damage model)

Given the damage function $D : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to (0,1)$ and $\boldsymbol{C}$ as above, for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) = (1 - D(\boldsymbol{\tau}))\,\boldsymbol{C}(\boldsymbol{x})\boldsymbol{\tau}.$$

Example (Second-order model)

Given Lamé parameters μ, λ and second-order moduli A, B, C, for all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau}) + A\boldsymbol{\tau}^2 + B\,\mathrm{tr}(\boldsymbol{\tau}^2)\boldsymbol{I}_d + 2B\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{\tau} + C\,\mathrm{tr}(\boldsymbol{\tau})^2\boldsymbol{I}_d.$$

References for this presentation

- Linear elasticity [DP and Ern, 2015]
- Nonlinear elasticity [M. Botti, DP, Sochala, 2017]
- Uniform local Korn inequality [L. Botti, DP, Droniou, 2018]
- Low-order, global Korn inequality [M. Botti, DP, Gugliemana, 2019]

Regular mesh sequence

Definition (Regular mesh sequence)

For any $h \in \mathcal{H}$, let $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ with

- $\mathcal{T}_h$ set of polyhedral elements;
- $\mathcal{F}_h$ set of polygonal faces.

The mesh sequence $(\mathcal{M}_h)_{h \in \mathcal{H}}$ is regular if

- It admits a shape regular matching simplicial submesh $\mathfrak{M}_h = (\mathfrak{T}_h, \mathfrak{F}_h)$;
- For any $T \in \mathcal{T}_h$ and any $\tau \in \mathfrak{T}_h$ s.t. $\tau \subset T$,

$$h_\tau \simeq h_T.$$

L^2-orthogonal projectors on local polynomial spaces

- Let a polynomial degree $k \geq 0$ be fixed
- With $X \in \mathcal{T}_h \cup \mathcal{F}_h$, the L^2-projector $\pi_X^{0,k} : L^2(X;\mathbb{R}) \to \mathbb{P}^k(X;\mathbb{R})$ is s.t.

$$\int_X (\pi_X^{0,k} v - v) w = 0 \text{ for all } w \in \mathbb{P}^k(X;\mathbb{R})$$

- Vector and tensor versions are defined component-wise
- Optimal $W^{s,p}$-approximation properties in [DP and Droniou, 2017a]

Strain projector I

- Let a polynomial degree $l \geq 1$ and an element $T \in \mathcal{T}_h$ be fixed
- The strain projector $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l} : H^1(T;\mathbb{R}^d) \to \mathbb{P}^l(T;\mathbb{R}^d)$ is s.t.

$$\int_T \nabla_{\mathrm{s}}(\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} - \boldsymbol{v}):\nabla_{\mathrm{s}}\boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in \mathbb{P}^l(T;\mathbb{R}^d)$$

and rigid-body motions are fixed enforcing

$$\int_T \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} = \int_T \boldsymbol{v}, \qquad \int_T \nabla_{\mathrm{ss}}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} = \int_T \nabla_{\mathrm{ss}}\boldsymbol{v}$$

- For $l = 1$, we find the elliptic projector of [DP and Droniou, 2017b]

Strain projector II

Theorem (Optimal approximation properties of the strain projector)

Denote by $(\mathcal{M}_h)_{h\in\mathcal{H}} = (\mathcal{T}_h, \mathcal{F}_h)_{h\in\mathcal{H}}$ *a regular mesh sequence with star-shaped elements. Let an integer* $s \in \{1,\ldots,l+1\}$ *be given. Then, for all* $T \in \mathcal{T}_h$*, all* $\boldsymbol{v} \in H^s(T;\mathbb{R}^d)$*, and all* $m \in \{0,\ldots,s\}$*,*

$$|\boldsymbol{v} - \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}|_{H^m(T;\mathbb{R}^d)} \lesssim h_T^{s-m}|\boldsymbol{v}|_{H^s(T;\mathbb{R}^d)}.$$

Moreover, if $m \le s-1$*, then, for all* $F \in \mathcal{F}_T$*,*

$$|\boldsymbol{v} - \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}|_{H^m(F,\mathbb{R}^d)} \lesssim h_T^{s-m-\frac{1}{2}}|\boldsymbol{v}|_{H^s(T;\mathbb{R}^d)}.$$

Hidden constants depend only on d, l, s, m*, and the mesh regularity.*

Strain projector III

- It suffices to prove (cf. [DP and Droniou, 2017b]): For all $T \in \mathcal{T}_h$

$$\|\boldsymbol{\nabla}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^{d\times d})} \lesssim |\boldsymbol{v}|_{H^1(T;\mathbb{R}^d)} \qquad \text{if } m \geq 1$$

$$\|\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^d)} \lesssim \|\boldsymbol{v}\|_{L^2(T;\mathbb{R}^d)} + h_T|\boldsymbol{v}|_{H^1(T;\mathbb{R}^d)} \qquad \text{if } m = 0$$

- To prove the first relation, we insert $\pm\boldsymbol{\pi}_T^{0,0}(\boldsymbol{\nabla}_{\rm ss}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v})$ and write

$$\begin{aligned}&\|\boldsymbol{\nabla}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^{d\times d})}\\ &\quad\leq \|\boldsymbol{\nabla}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} - \boldsymbol{\pi}_T^{0,0}(\boldsymbol{\nabla}_{\rm ss}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})} + \|\boldsymbol{\pi}_T^{0,0}(\boldsymbol{\nabla}_{\rm ss}\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})}\end{aligned}$$

- For the term in red, we need local Korn inequalities to write

$$\boxed{\|\boldsymbol{\nabla}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v} - \boldsymbol{\pi}_T^{0,0}(\boldsymbol{\nabla}_{\rm ss}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v})\|_{L^2(T;\mathbb{R}^{d\times d})} \lesssim \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},l}\boldsymbol{v}\|_{L^2(T;\mathbb{R}^{d\times d})}}$$

where the hidden constant should be independent of T

Strain projector IV

Lemma (Uniform local Korn inequalities)

Denoting by $(\mathcal{M}_h)_{h\in\mathcal{H}}$ a regular mesh sequence with star-shaped elements it holds, for all $h \in \mathcal{H}$ and all $T \in \mathcal{T}_h$,

$$\|\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\pi}_T^{0,0}(\boldsymbol{\nabla}_{\rm ss}\boldsymbol{v})\|_T \lesssim \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{v}\|_T \qquad \forall \boldsymbol{v} \in H^1(T;\mathbb{R}^d),$$

with hidden constant depending only on d and the mesh regularity.

Crucially, the hidden constant above is independent of T!

Computing displacement projections from L^2-projections

- For all $\boldsymbol{v} \in H^1(T;\mathbb{R}^d)$ and all $\boldsymbol{\tau} \in C^\infty(\overline{T};\mathbb{R}^{d\times d}_{\rm sym})$, it holds

$$\boxed{\int_T \boldsymbol{\nabla}_{\rm s}\boldsymbol{v}:\boldsymbol{\tau} = -\int_T \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}\cdot\boldsymbol{\tau}\boldsymbol{n}_{TF}}$$

- Specialising to $\boldsymbol{\tau} = \boldsymbol{\nabla}_{\rm s}\boldsymbol{w}$ with $\boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$, $k \geq 0$, gives

$$\int_T \boldsymbol{\nabla}_{\rm s}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}:\boldsymbol{\nabla}_{\rm s}\boldsymbol{w} = -\int_T \boldsymbol{\pi}_T^{0,k}\boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}\boldsymbol{n}_{TF}$$

- Moreover, we have

$$\int_T \boldsymbol{v} = \int_T \boldsymbol{\pi}_T^{0,k}\boldsymbol{v}, \quad \int_T \boldsymbol{\nabla}_{\rm ss}\boldsymbol{v} = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F \left(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\otimes\boldsymbol{n}_{TF} - \boldsymbol{n}_{TF}\otimes\boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\right)$$

- **Hence, $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}$ can be computed from $\boldsymbol{\pi}_T^{0,k}\boldsymbol{v}$ and $(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v})_{F\in\mathcal{F}_T}$!**

Computing displacement projections from L^2-projections

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- Specialising to $\boldsymbol{\tau} = \boldsymbol{\nabla}_{\rm s}\boldsymbol{w}$ with $\boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$, $k \geq 0$, gives

$$\int_T \boldsymbol{\nabla}_{\rm s}\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}:\boldsymbol{\nabla}_{\rm s}\boldsymbol{w} = -\int_T \boldsymbol{\pi}_T^{0,k}\boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}\boldsymbol{n}_{TF}$$

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$$\int_T \boldsymbol{v} = \int_T \boldsymbol{\pi}_T^{0,k}\boldsymbol{v}, \quad \int_T \boldsymbol{\nabla}_{\rm ss}\boldsymbol{v} = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F \left(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\otimes\boldsymbol{n}_{TF} - \boldsymbol{n}_{TF}\otimes\boldsymbol{\pi}_F^{0,k}\boldsymbol{v}\right)$$

- **Hence, $\boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v}$ can be computed from $\boldsymbol{\pi}_T^{0,k}\boldsymbol{v}$ and $(\boldsymbol{\pi}_F^{0,k}\boldsymbol{v})_{F\in\mathcal{F}_T}$!**
- The same holds for $\boldsymbol{\pi}_T^{0,k}(\boldsymbol{\nabla}_{\rm s}\boldsymbol{v})$ (specialise to $\boldsymbol{\tau} \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$)

Discrete unknowns

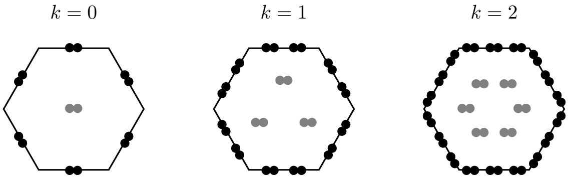


Figure: $\underline{\boldsymbol{U}}_T^k$ for $k \in \{0, 1, 2\}$

- Let a polynomial degree $k \geq 0$ be fixed
- For all $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\underline{\boldsymbol{U}}_T^k := \Big\{ \underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \ : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T; \mathbb{R}^d) \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F; \mathbb{R}^d) \quad \forall F \in \mathcal{F}_T \Big\}$$

- The local interpolator $\underline{\boldsymbol{I}}_T^k : H^1(T; \mathbb{R}^d) \to \underline{\boldsymbol{U}}_T^k$ is s.t.

$$\underline{\boldsymbol{I}}_T^k \boldsymbol{v} := (\boldsymbol{\pi}_T^{0,k} \boldsymbol{v}, (\boldsymbol{\pi}_F^{0,k} \boldsymbol{v})_{F \in \mathcal{F}_T}) \quad \forall \boldsymbol{v} \in H^1(T; \mathbb{R}^d)$$

Local displacement and strain reconstructions I

- We introduce the displacement reconstruction operator

$$\mathbf{p}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^{k+1}(T;\mathbb{R}^d)$$

s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$ and all $\boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$,

$$\int_T \boldsymbol{\nabla}_{\rm s}\mathbf{p}_T^{k+1}\underline{\boldsymbol{v}}_T : \boldsymbol{\nabla}_{\rm s}\boldsymbol{w} = -\int_T \boldsymbol{v}_T\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}_F\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w}\,\boldsymbol{n}_{TF}$$

and

$$\int_T \mathbf{p}_T^{k+1}\underline{\boldsymbol{v}}_T = \int_T \boldsymbol{v}_T, \quad \int_T \boldsymbol{\nabla}_{\rm ss}\mathbf{p}_T^{k+1}\underline{\boldsymbol{v}}_T = \frac{1}{2}\sum_{F\in\mathcal{F}_T}\int_F (\boldsymbol{v}_F\otimes\boldsymbol{n}_{TF} - \boldsymbol{n}_{TF}\otimes\boldsymbol{v}_F)$$

- By construction, the following commutation property holds:

$$\boxed{\mathbf{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v} = \boldsymbol{\pi}_T^{\boldsymbol{\varepsilon},k+1}\boldsymbol{v} \qquad \forall \boldsymbol{v}\in H^1(T;\mathbb{R}^d)}$$

Local displacement and strain reconstructions II

- For nonlinear problems, $\boldsymbol{\nabla}_{\mathrm{s}}\mathbf{p}_T^{k+1}$ is not sufficiently rich
- We therefore also define the strain reconstruction operator

$$\mathbf{G}_{\mathrm{s},T}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

such that, for all $\boldsymbol{\tau} \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$,

$$\int_T \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T \cdot (\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T} \int_F \boldsymbol{v}_F \cdot \boldsymbol{\tau}\boldsymbol{n}_{TF}$$

- By construction, it holds:

$$\boxed{\mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \boldsymbol{\pi}_T^{0,k}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in H^1(T;\mathbb{R}^d)}$$

Local contribution I

$$a_{|T}(\boldsymbol{u},\boldsymbol{v}) \approx \mathrm{a}_T(\underline{\boldsymbol{u}}_T,\underline{\boldsymbol{v}}_T) \coloneqq \int_T \boldsymbol{\sigma}(\mathbf{G}_{\mathrm{s},T}^k\underline{\boldsymbol{u}}_T):\mathbf{G}_{\mathrm{s},T}^k\underline{\boldsymbol{v}}_T + \mathrm{s}_T(\underline{\boldsymbol{u}}_T,\underline{\boldsymbol{v}}_T)$$

Assumption (Stabilization bilinear form)

The bilinear form $\mathrm{s}_T : \underline{\boldsymbol{U}}_T^k \times \underline{\boldsymbol{U}}_T^k \to \mathbb{R}$ satisfies the following properties:

- Symmetry and positivity. s_T is symmetric and positive semidefinite.
- Stability. It holds uniformly: For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\|\mathbf{G}_{\mathrm{s},T}^k\underline{\boldsymbol{v}}_T\|_{L^2(T;\mathbb{R}^{d\times d})}^2 + \mathrm{s}_T(\underline{\boldsymbol{v}}_T,\underline{\boldsymbol{v}}_T) \simeq \|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},T}^2$$

  where $\|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},T}^2 \coloneqq \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}_T\|_{L^2(T;\mathbb{R}^{d\times d})}^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^2(F;\mathbb{R}^d)}^2$
- Polynomial consistency. For all $w \in \mathbb{P}^{k+1}(T)$ and all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\mathrm{s}_T(\underline{\boldsymbol{I}}_T^k\boldsymbol{w},\underline{\boldsymbol{v}}_T) = 0.$$

Local contribution II

Remark (Polynomial degree)

Stability and polynomial consistency are incompatible for $k = 0$.

Remark (Dependency)

s_T satisfies polynomial consistency if and only if it depends on its arguments via the difference operators s.t., for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\boldsymbol{\delta}_T^k \underline{\boldsymbol{v}}_T \coloneqq \boldsymbol{\pi}_T^{0,k}(\mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_T),$$
$$\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T \coloneqq \boldsymbol{\pi}_F^{0,k}(\mathbf{p}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_F) \quad \forall F \in \mathcal{F}_T.$$

Example (Classical HHO stabilisation)

$$\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) \coloneqq \sum_{F \in \mathcal{F}_T} \frac{\gamma}{h_F} \int_F \left(\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{u}}_T - \boldsymbol{\delta}_T^k \underline{\boldsymbol{u}}_T\right) \cdot \left(\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T - \boldsymbol{\delta}_T^k \underline{\boldsymbol{v}}_T\right).$$

Discrete problem

- Define the global space with single-valued interface unknowns

$$\underline{\boldsymbol{U}}_h^k := \Big\{\underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}) \ : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \quad \forall T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \quad \forall F \in \mathcal{F}_h\Big\}.$$

  and its subspace with strongly enforced boundary conditions

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}$$

- The discrete problem reads: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ s.t.

$$\boxed{\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) = \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{f}\cdot\boldsymbol{v}_T \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k}$$

Global discrete Korn inequalities I

Lemma (Global Korn inequality on broken polynomial spaces)

Let an integer $l \geq 1$ *be fixed and, given* $\boldsymbol{v}_h \in \mathbb{P}^l(\mathcal{T}_h;\mathbb{R}^d)$, *set*

$$\|\boldsymbol{v}_h\|_{\mathrm{dG},h}^2 := \|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}}^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F}\|[\boldsymbol{v}_h]_F\|_{L^2(F)^d}^2.$$

Then it holds, with hidden constant depending only on Ω, d, l, *and* ϱ,

$$\|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}} \lesssim \|\boldsymbol{v}_h\|_{\mathrm{dG},h}.$$

Proof.

Introduce the node-averaging operator on $\mathfrak{M}_h$ and proceed as in Lemma 2.2, Brenner, 2003]. □

Global discrete Korn inequalities II

Corollary (Global Korn inequality on HHO spaces)

Assume $k \geq 1$. Then it holds, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$, letting $\boldsymbol{v}_h \in \mathbb{P}^k(\mathcal{T}_h;\mathbb{R}^d)$ be s.t. $(\boldsymbol{v}_h)_{|T} := \boldsymbol{v}_T$ for all $T \in \mathcal{T}_h$ and with hidden constant as above,

$$\|\boldsymbol{v}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},h}$$

with $\|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{\boldsymbol{v}}_T\|_{\boldsymbol{\varepsilon},T}^2$.

Remark (Other boundary conditions)

Extensions to other boundary conditions are possible.

Existence and uniqueness I

Assumption (Strain-stress law/1)

The strain-stress law is a Carathéodory function s.t. $\boldsymbol{\sigma}(\cdot,\mathbf{0}) = \mathbf{0}$ and there exist $0 < \underline{\sigma} \le \overline{\sigma}$ s.t., for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau},\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,

$$
\begin{aligned}
|\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau})| &\le \overline{\sigma}|\boldsymbol{\tau}|, && \text{(growth)}\\
\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}):\boldsymbol{\tau} &\ge \underline{\sigma}|\boldsymbol{\tau}|^2, && \text{(coercivity)}\\
(\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})):(\boldsymbol{\tau} - \boldsymbol{\eta}) &\ge 0. && \text{(monotonicity)}
\end{aligned}
$$

Remark (Choice of the penalty parameter)

A natural choice is to take the penalty parameter s.t.

$$\gamma \in [\underline{\sigma},\overline{\sigma}].$$

Existence and uniqueness II

Theorem (Discrete existence and uniqueness)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ *denote a regular mesh sequence with star-shaped elements and assume* $k \geq 1$*. Then, for all* $h \in \mathcal{H}$*, there exist a solution* $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ *to the discrete problem, which satisfies*

$$\|\underline{\boldsymbol{u}}_h\|_{\boldsymbol{\varepsilon},h} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega;\mathbb{R}^d)},$$

with hidden constant only depending on Ω, $\underline{\sigma}$, γ, ϱ, *and* k.

Moreover, if $\boldsymbol{\sigma}$ *is strictly monotone, then the solution is unique.*

Convergence and error estimate

Theorem (Convergence)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence with star-shaped elements and assume $k \geq 1$. Then, for all $q \in [1, +\infty)$ if $d = 2$ and $q \in [1, 6)$ if $d = 3$, as $h \to 0$ it holds, up to a subsequence, that

$$\begin{aligned} \boldsymbol{u}_h &\to \boldsymbol{u} && \text{strongly in } L^q(\Omega;\mathbb{R}^d), \\ \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h &\rightharpoonup \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u} && \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{aligned}$$

If, additionally, $\boldsymbol{\sigma}$ is strictly monotone,

$$\mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{u}}_h \to \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u} \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{d\times d})$$

and, the continuous solution being unique, the whole sequence converges.

Proof.

Inspired by GDM [Droniou, Eymard, Guichard, Herbin, Gallouët, 2018] □

Error estimate

Assumption (Strain-stress law/2)

There exists $\sigma_*, \sigma^* \in (0, +\infty)$ s.t., for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\rm sym}$,

$$|\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})| \le \sigma^* |\boldsymbol{\tau} - \boldsymbol{\eta}|, \qquad \text{(Lipschitz continuity)}$$

$$(\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \ge \sigma_* |\boldsymbol{\tau} - \boldsymbol{\eta}|^2. \qquad \text{(strong monotonicity)}$$

Theorem (Error estimate)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence with star-shaped elements and $k \ge 1$. Then, if $\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)$ and $\boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}) \in H^{k+1}(\mathcal{T}_h;\mathbb{R}^{d\times d})$,

$$\|\mathbf{G}^k_{{\rm s},h}\underline{\boldsymbol{u}}_h - \boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + |\underline{\boldsymbol{u}}_h|_{{\rm s},h} \lesssim h^{k+1}\left(|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)} + |\boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\rm s}\boldsymbol{u})|_{H^{k+1}(\mathcal{T}_h;\mathbb{R}^{d\times d})}\right),$$

with hidden constant only depending on Ω, k, $\overline{\sigma}$, $\underline{\sigma}$, σ^*, σ_*, γ, the mesh regularity and an upper bound of $\|\boldsymbol{f}\|_{L^2(\Omega;\mathbb{R}^d)}$.

The lowest-order case I

- For $k = 0$, stability cannot be enforced through local terms
- We therefore consider $\mathrm{a}_h^{\mathrm{lo}} : \underline{\boldsymbol{U}}_h^0 \times \underline{\boldsymbol{U}}_h^0$ s.t.

$$\mathrm{a}_h^{\mathrm{lo}}(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{j}_h(\mathbf{p}_h^1 \underline{\boldsymbol{u}}_h, \mathbf{p}_h^1 \underline{\boldsymbol{v}}_h),$$

with jump penalisation bilinear form

$$\mathrm{j}_h(\boldsymbol{u}, \boldsymbol{v}) := \sum_{F \in \mathcal{F}_h} h_F^{-1} \int_F [\boldsymbol{u}]_F \cdot [\boldsymbol{v}]_F$$

The lowest-order case II

- Consider, e.g., isotropic homogeneous linear elasticity, that is

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}_d \quad \text{with} \quad 2\mu - d\lambda^- \geq \alpha > 0$$

- Coercivity is ensured by Korn's inequality in broken spaces:

$$\alpha \|\!|\underline{\boldsymbol{v}}_h\|\!|_{\boldsymbol{\varepsilon},h}^2 \lesssim \mathrm{a}_h^{\mathrm{lo}}(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^0,$$

where

$$\|\!|\underline{\boldsymbol{v}}_h\|\!|_{\boldsymbol{\varepsilon},h} := \left(\|\boldsymbol{v}_h\|_{\mathrm{dG},h}^2 + |\underline{\boldsymbol{v}}_h|_{\mathrm{s},h}^2 \right)^{\frac{1}{2}}$$

Error estimates I

Theorem (Energy error estimate, $k = 0$)

Let $(\mathcal{M}_h)_{h\in\mathcal{H}}$ denote a regular mesh sequence. Then, if $\boldsymbol{u} \in H^2(\mathcal{T}_h;\mathbb{R}^d)$,

$$\|\boldsymbol{\nabla}_h \mathrm{p}_h^1 \underline{\boldsymbol{u}}_h - \boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)^{d\times d}} + |\underline{\boldsymbol{u}}_h|_{\mathrm{s},h} \lesssim h\alpha^{-1}\left(|\boldsymbol{u}|_{H^2(\mathcal{T}_h;\mathbb{R}^d)} + |\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u})|_{H^1(\mathcal{T}_h;\mathbb{R}^{d\times d})}\right),$$

with hidden constant independent of h, $\boldsymbol{u}$, of the Lamé parameters and of $\boldsymbol{f}$. This estimate can be proved to be uniform in λ.

Remark (Star-shaped assumption)

We do not need the star-shaped assumption for $k = 0$, since the strain projector coincides with the elliptic projector, whose approximation properties do not require local Korn inequalities.

Error estimates II

Theorem (L^2-error estimate)

Under the assumptions of the above theorem, and further assuming $\lambda \geq 0$, elliptic regularity, and $\boldsymbol{f} \in H^1(\mathcal{T}_h; \mathbb{R}^d)$, it holds that

$$\|\mathbf{p}_h^1 \underline{\boldsymbol{u}}_h - \boldsymbol{u}\|_{L^2(\Omega)^d} \lesssim h^2 \|\boldsymbol{f}\|_{H^1(\mathcal{T}_h;\mathbb{R}^d)},$$

with hidden constant independent of both h and λ.

Convergence I

- Consider the Hencky–Mises model with $\Phi(\rho) = \mu(e^{-\rho} + 2\rho)$ and $\alpha = \lambda + \mu$, so that

$$\boldsymbol{\sigma}(\nabla_{\rm s}\boldsymbol{u}) = ((\lambda - \mu) + \mu e^{-\operatorname{dev}(\nabla_{\rm s}\boldsymbol{u})}) \operatorname{tr}(\nabla_{\rm s}\boldsymbol{u})\boldsymbol{I}_d + \mu(2 - e^{-\operatorname{dev}(\nabla_{\rm s}\boldsymbol{u})})\nabla_{\rm s}\boldsymbol{u}$$

- We set $\Omega = (0,1)^2$, $\mu = 2$, $\lambda = 1$, so that

$$\boldsymbol{u}(\boldsymbol{x}) = \big(\sin(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\sin(\pi x_2)\big)$$

- $\boldsymbol{f}$ is inferred from the exact solution

Convergence II

Table: Convergence results on the triangular mesh family. EOC = estimated order of convergence.

h	$\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u} - \boldsymbol{\nabla}_{{\rm s},h}\underline{\boldsymbol{u}}_h\|$	EOC	$\|\boldsymbol{\pi}_h^{0,k}\boldsymbol{u} - \boldsymbol{u}_h\|$	EOC
		$k = 1$		
$3.07 \cdot 10^{-2}$	$5.59 \cdot 10^{-2}$	—	$7.32 \cdot 10^{-3}$	—
$1.54 \cdot 10^{-2}$	$1.51 \cdot 10^{-2}$	1.9	$1.05 \cdot 10^{-3}$	2.81
$7.68 \cdot 10^{-3}$	$3.86 \cdot 10^{-3}$	1.96	$1.34 \cdot 10^{-4}$	2.96
$3.84 \cdot 10^{-3}$	$1.01 \cdot 10^{-3}$	1.93	$1.7 \cdot 10^{-5}$	2.98
$1.92 \cdot 10^{-3}$	$2.59 \cdot 10^{-4}$	**1.96**	$2.15 \cdot 10^{-6}$	2.98
		$k = 2$		
$3.07 \cdot 10^{-2}$	$1.3 \cdot 10^{-2}$	—	$1.47 \cdot 10^{-3}$	—
$1.54 \cdot 10^{-2}$	$1.29 \cdot 10^{-3}$	3.35	$6.05 \cdot 10^{-5}$	4.62
$7.68 \cdot 10^{-3}$	$2.11 \cdot 10^{-4}$	2.6	$5.36 \cdot 10^{-6}$	3.48
$3.84 \cdot 10^{-3}$	$2.73 \cdot 10^{-5}$	2.95	$3.6 \cdot 10^{-7}$	3.9
$1.92 \cdot 10^{-3}$	$3.42 \cdot 10^{-6}$	**3.00**	$2.28 \cdot 10^{-8}$	3.98
		$k = 3$		
$3.07 \cdot 10^{-2}$	$2.81 \cdot 10^{-3}$	—	$2.39 \cdot 10^{-4}$	—
$1.54 \cdot 10^{-2}$	$3.72 \cdot 10^{-4}$	2.93	$1.95 \cdot 10^{-5}$	3.63
$7.68 \cdot 10^{-3}$	$2.16 \cdot 10^{-5}$	4.09	$5.47 \cdot 10^{-7}$	5.14
$3.84 \cdot 10^{-3}$	$1.43 \cdot 10^{-6}$	3.92	$1.66 \cdot 10^{-8}$	5.04
$1.92 \cdot 10^{-3}$	$9.51 \cdot 10^{-8}$	**3.91**	$5.34 \cdot 10^{-10}$	4.96

Convergence III

Table: Convergence results on the hexagonal mesh family. EOC = estimated order of convergence.

h	$\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u} - \boldsymbol{\nabla}_{{\rm s},h}\underline{\boldsymbol{u}}_h\|$	EOC	$\|\boldsymbol{\pi}_h^{0,k}\boldsymbol{u} - \boldsymbol{u}_h\|$	EOC
		$k = 1$		
$6.3 \cdot 10^{-2}$	0.22	—	$2.75 \cdot 10^{-2}$	—
$3.42 \cdot 10^{-2}$	$3.72 \cdot 10^{-2}$	2.89	$3.73 \cdot 10^{-3}$	3.27
$1.72 \cdot 10^{-2}$	$7.17 \cdot 10^{-3}$	2.4	$4.83 \cdot 10^{-4}$	2.97
$8.59 \cdot 10^{-3}$	$1.44 \cdot 10^{-3}$	2.31	$6.14 \cdot 10^{-5}$	2.97
$4.3 \cdot 10^{-3}$	$2.4 \cdot 10^{-4}$	**2.59**	$7.7 \cdot 10^{-6}$	3.00
		$k = 2$		
$6.3 \cdot 10^{-2}$	$2.68 \cdot 10^{-2}$	—	$3.04 \cdot 10^{-3}$	—
$3.42 \cdot 10^{-2}$	$7.01 \cdot 10^{-3}$	2.2	$3.56 \cdot 10^{-4}$	3.51
$1.72 \cdot 10^{-2}$	$1.09 \cdot 10^{-3}$	2.71	$3.31 \cdot 10^{-5}$	3.46
$8.59 \cdot 10^{-3}$	$1.41 \cdot 10^{-4}$	2.95	$2.53 \cdot 10^{-6}$	3.7
$4.3 \cdot 10^{-3}$	$1.96 \cdot 10^{-5}$	**2.85**	$1.72 \cdot 10^{-7}$	3.89
		$k = 3$		
$6.3 \cdot 10^{-2}$	$1.11 \cdot 10^{-2}$	—	$1.08 \cdot 10^{-3}$	—
$3.42 \cdot 10^{-2}$	$1.92 \cdot 10^{-3}$	2.87	$9.29 \cdot 10^{-5}$	4.02
$1.72 \cdot 10^{-2}$	$2.79 \cdot 10^{-4}$	2.81	$6.13 \cdot 10^{-6}$	3.95
$8.59 \cdot 10^{-3}$	$2.54 \cdot 10^{-5}$	3.45	$2.88 \cdot 10^{-7}$	4.4
$4.3 \cdot 10^{-3}$	$1.61 \cdot 10^{-6}$	**3.99**	$1.24 \cdot 10^{-8}$	4.55

Numerical examples I

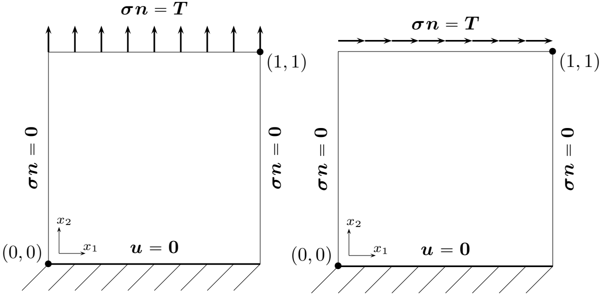
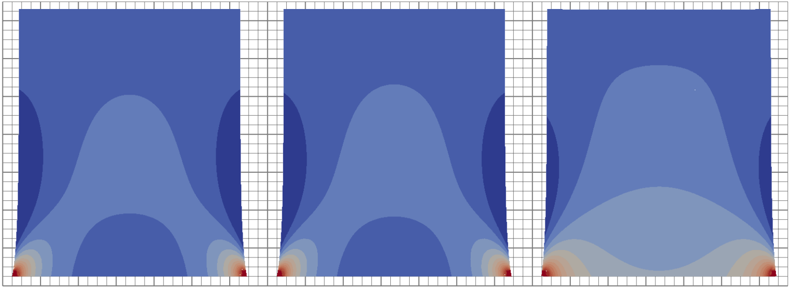


Figure: Shear and tensile test cases

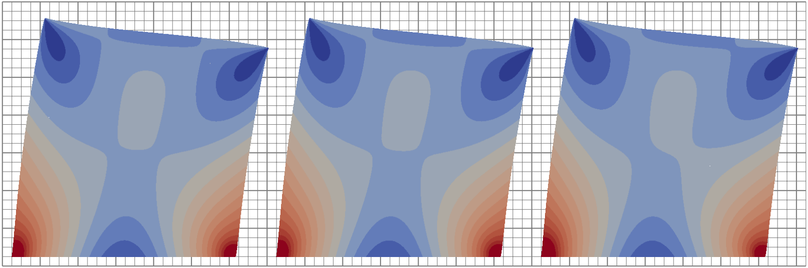
Numerical examples II



(a) Linear (b) Hencky–Mises (c) Second order

Figure: Tensile test case: Stress norm on the deformed domain. Values in $10^5\mathrm{Pa}$

Numerical examples III



(a) Linear (b) Hencky–Mises (c) Second order

Figure: Shear test case: Stress norm on the deformed domain. Values in $10^4\,\mathrm{Pa}$

Numerical examples IV

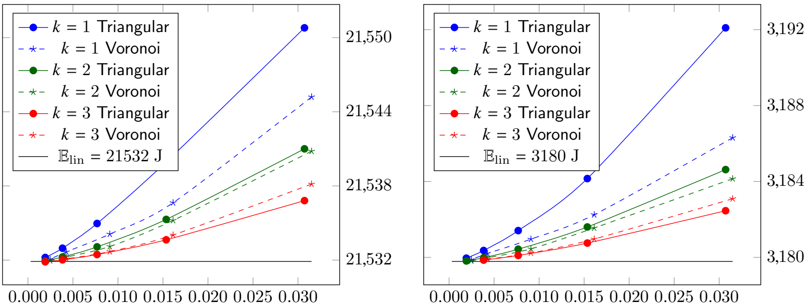


Figure: Energy vs h, tensile and shear test cases, linear model

Numerical examples V

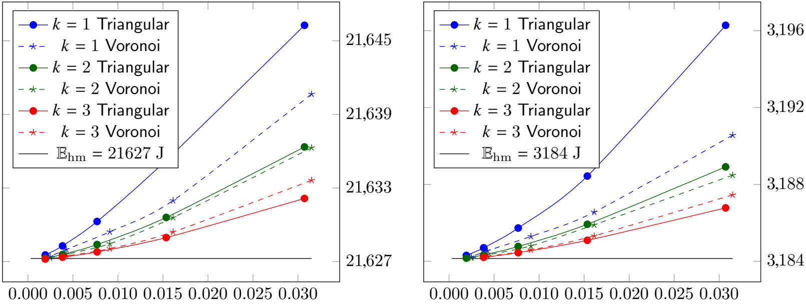


Figure: Energy vs h, tensile and shear test cases, Hencky–Mises model

HHO implementations and more

- Code_Aster https://www.code-aster.org (EDF)
- Code_Saturne https://www.code-saturne.org (EDF)
- HArD::Core2D https://github.com/jdroniou/HArDCore2D (J. Droniou)
- POLYPHO http://www.comphys.com (R. Specogna)
- SpaFEDte https://github.com/SpaFEDTe/spafedte.github.com (L. Botti)

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Coming soon:

D. A. Di Pietro and J. Droniou
The Hybrid High-Order Method for Polytopal Meshes
Design, Analysis, and Applications

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