Hybrid High-Order (HHO) methods on general meshes

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μBibliography: Lowest-order polyhedral methods

- Mimetic Finite Differences
  - Extension to polyhedral meshes [Kuznetsov et al., 2004]
  - Convergence analysis [Brezzi et al., 2005]
- Mixed/Hybrid Finite Volumes
  - Pure diffusion (mixed) [Droniou and Eymard, 2006]
  - Pure diffusion (primal) [Eymard et al., 2010]
  - Link with MFD [Droniou et al., 2010]
- More recently
  - Compatible Discrete Operators [Bonelle and Ern, 2014]
  - Generalized Crouzeix–Raviart [DP and Lemaire, 2015]

μBibliography: High-order polyhedral methods

- Discontinuous Galerkin
  - General meshes [DP and Ern, 2012]
  - Adaptive coarsening [Bassi et al., 2012, Antonietti et al., 2013]
- Hybridizable Discontinuous Galerkin
  - Pure diffusion [Cockburn et al., 2009]
- Virtual elements
  - Pure diffusion [Beirão da Veiga et al., 2013a]
  - Nonconforming VEM [Ayuso de Dios et al., 2014]
  - Linear elasticity [Beirão da Veiga et al., 2013b]
- Hybrid High-Order
  - Pure diffusion [DP and Ern, 2014b]
  - Linear elasticity [DP and Ern, 2015]
  - Bridge between HHO and HDG [Cockburn, DP and Ern, 2015]

Features of HHO

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Reproduction of desirable continuum properties
  - Integration by parts formulas
  - Kernels of operators
  - Symmetries
- Reduced computational cost after hybridization

$$N_{\mathrm{dof}}^{\mathrm{hho}} \approx \frac{1}{2}k^2 \,\mathrm{card}(\mathcal{F}_h) \qquad N_{\mathrm{dof}}^{\mathrm{dg}} \approx \frac{1}{6}k^3 \,\mathrm{card}(\mathcal{T}_h)$$

Outline

Outline

Mesh regularity I

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the sense of Ciarlet;
- contact-regular: every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces

Mesh regularity II

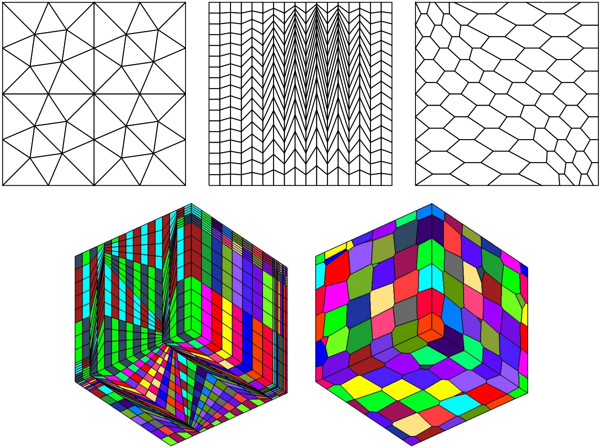

Figure: Admissible meshes in 2d and 3d: [Herbin and Hubert, 2008, FVCA5] and [Di Pietro and Lemaire, 2015] (above) and [Eymard et al., 2011, FVCA6] (below)

Model problem

- Let Ω denote a bounded, connected polyhedral domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\triangle u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) := (\nabla u, \nabla v) = (f,v) \qquad \forall v \in H_0^1(\Omega)$$

Key ideas

- DOFs: polynomials of degree $k \geqslant 0$ at elements and faces
- Differential operators reconstructions taylored to the problem:

$$\boxed{a_{|T}(u,v) \approx (\boldsymbol{\nabla} p_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T) + \text{stab.}}$$

  with
  - high-order reconstruction p_T^{k+1} from local Neumann solves
  - stabilization via face-based penalty
- Construction yielding superconvergence on general meshes

DOFs

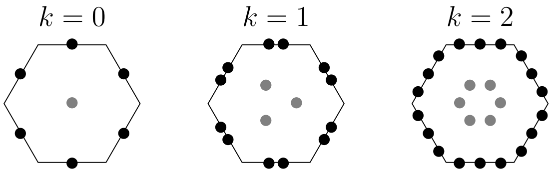

Figure: $\underline{U}_T^k$ for $k \in \{0, 1, 2\}$

- For $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}$$

- The global space has single-valued interface DOFs

$$\underline{U}_h^k := \left\{ \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T) \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F) \right\}$$

Local potential reconstruction I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}_d^{k+1}(T)$$

is s.t. $\forall \underline{v}_T \in \underline{U}_T^k$, $(p_T^{k+1}\underline{v}_T, 1)_T = (v_T, 1)_T$ and $\forall w \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{(\nabla p_T^{k+1}\underline{v}_T, \nabla w)_T := -(v_T, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (v_F, \nabla w{\cdot}\boldsymbol{n}_{TF})_F}$$

- To compute p_T^{k+1}, we solve a small SPD linear system of size

$$N_{k,d} := \binom{k+1+d}{k+1}$$

- **Perfectly suited to GPU computing!**

Local potential reconstruction II

Lemma (Approximation properties for $p_T^{k+1}\underline{I}_T^k$)

Define the local reduction map $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ *s.t.*

$$\underline{I}_T^k : v \mapsto \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right).$$

Then, for all $T \in \mathcal{T}_h$ *and all* $v \in H^{k+2}(T)$,

$$\|v - p_T^{k+1}\underline{I}_T^k v\|_T + h_T\|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{I}_T^k v)\|_T \lesssim h_T^{k+2}\|v\|_{k+2,T}.$$

Local potential reconstruction III

- Since $\triangle w \in \mathbb{P}_d^{k-1}(T)$ and $\nabla w_{|F}\cdot\boldsymbol{n}_{TF} \in \mathbb{P}_{d-1}^k(F)$,

$$(\nabla p_T^{k+1}\underline{I}_T^k v, \nabla w)_T = -(\pi_T^k v, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (\pi_F^k v, \nabla w\cdot\boldsymbol{n}_{TF})_F$$

$$= -(v, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (v, \nabla w\cdot\boldsymbol{n}_{TF})_F = (\nabla v, \nabla w)_T$$

- This shows that $p_T^{k+1}\underline{I}_T^k$ is the elliptic projector on $\mathbb{P}_d^{k+1}(T)$:

$$(\nabla p_T^{k+1}\underline{I}_T^k v - \nabla v, \nabla w)_T = 0 \qquad \forall w \in \mathbb{P}_d^{k+1}(T)$$

- The approximation properties follow

Stabilization I

- The following naive choice is not stable

$$a_{|T}(u,v) \approx (\boldsymbol{\nabla} p_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T)_T$$

- To remedy, we add a local stabilization term

$$(\boldsymbol{\nabla} p_T^{k+1} \underline{u}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{v}_T)_T + s_T(\underline{u}_T, \underline{v}_T)$$

- Coercivity and boundedness are expressed w.r.t. to

$$\|\underline{v}_T\|_{1,T}^2 := \|\boldsymbol{\nabla} v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \|v_F - v_T\|_F^2$$

Stabilization II

- Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form s_T as

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k(\widehat{p}_T^{k+1}\underline{u}_T - u_F), \pi_F^k(\widehat{p}_T^{k+1}\underline{v}_T - v_F))_F,$$

with $\widehat{p}_T^{k+1}$ high-order correction of cell DOFs based on p_T^{k+1}

$$\widehat{p}_T^{k+1}\underline{v}_T := v_T + (p_T^{k+1}\underline{v}_T - \pi_T^k p_T^{k+1}\underline{v}_T)$$

- With this choice, a_T satisfies, for all $\underline{v}_T \in \underline{U}_T^k$,

$$\|\underline{v}_h\|_{1,T}^2 \lesssim a_T(\underline{v}_T, \underline{v}_T) \lesssim \|\underline{v}_T\|_{1,T}^2$$

Stabilization III

Lemma (High-order consistency of s_T)

s_T preserves the approximation properties of $\boldsymbol{\nabla} p_T^{k+1}$.

- For all $u \in H^{k+2}(T)$, letting $\widehat{\underline{u}}_T := \underline{I}_T^k u = \big(\pi_T^k u, (\pi_F^k u)_{F \in \mathcal{F}_T}\big)$,

$$\begin{aligned}
\|\pi_F^k(\widehat{p}_T^{k+1}\widehat{\underline{u}}_T - \widehat{u}_F)\|_F &= \|\pi_F^k\big(\pi_T^k u + p_T^{k+1}\widehat{\underline{u}}_T - \pi_T^k p_T^{k+1}\widehat{\underline{u}}_T - \pi_F^k u\big)\|_F \\
&\leqslant \|\pi_F^k\big(p_T^{k+1}\widehat{\underline{u}}_T - u\big)\|_F + \|\pi_T^k\big(u - p_T^{k+1}\widehat{\underline{u}}_T\big)\|_F \\
&\lesssim h_T^{-1/2}\|p_T^{k+1}\widehat{\underline{u}}_T - u\|_T
\end{aligned}$$

- Recalling the approximation properties of p_T^{k+1}, this yields

$$\boxed{\left\{\|\boldsymbol{\nabla}(p_T^{k+1}\widehat{\underline{u}}_T - u)\|_T^2 + s_T(\widehat{\underline{u}}_T, \widehat{\underline{u}}_T)\right\}^{1/2} \lesssim h_T^{k+1}\|u\|_{k+2,T}}$$

Discrete problem

- We enforce boundary conditions strongly considering the space

$$\underline{U}_{h,0}^{k} := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F \equiv 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^{k}$ s.t.

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} (f, v_T)_T \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^{k}}$$

- Well-posedness follows from the coercivity of a_h

Convergence I

Theorem (Energy-norm error estimate)

Assume $u \in H^{k+2}(\mathcal{T}_h)$ *and let*

$$\widehat{\underline{u}}_h := \left((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\right) \in \underline{U}_{h,0}^k.$$

Then, we have the following energy error estimate:

$$\boxed{\max\left(\|\underline{u}_h - \widehat{\underline{u}}_h\|_{1,h}, \|\underline{u}_h - \widehat{\underline{u}}_h\|_{a,h}\right) \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)},}$$

with

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2.$$

Convergence II

Theorem (L^2-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$,

$$\boxed{\max\left(\|\breve{u}_h - u\|, \|\widehat{u}_h - u_h\|\right) \lesssim h^{k+2}\mathcal{N}_k,}$$

with $\mathcal{N}_0 := \|f\|_{H^1(\Omega)}$, $\mathcal{N}_k := \|u\|_{H^{k+2}(\mathcal{T}_h)}$ if $k \geqslant 1$, and, $\forall T \in \mathcal{T}_h$,

$$\breve{u}_{h|T} := p_T^{k+1}\underline{u}_T, \quad \widehat{u}_{h|T} := p_T^{k+1}\underline{I}_T^k u, \quad u_{h|T} := u_T.$$

Convergence for a smooth 2d solution I

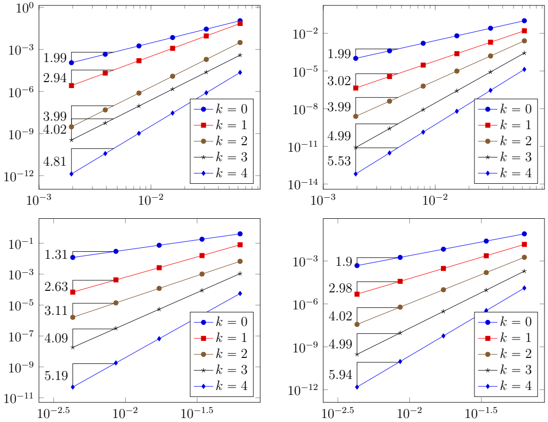

Figure: Energy (left) and L^2-norm (right) of the error vs. h for uniformly refined triangular (top) and hexagonal (bottom) mesh families, $u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$

Convergence for a smooth 2d solution II

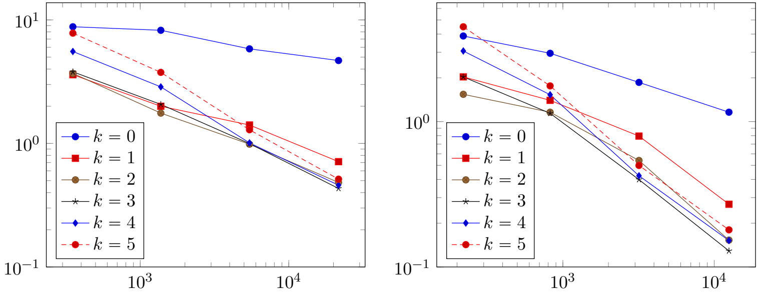

Figure: Assembly/solution time for triangular (left) and hexagonal (right) mesh families, sequential implementation

Mesh adaptivity: Fichera's 3d test case I

- Let $\Omega := (-1,1)^3 \setminus [0,1]^3$
- We consider the following exact solution:

$$u(\boldsymbol{x}) = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{4}}$$

corresponding to the forcing term

$$f(\boldsymbol{x}) = -\frac{3}{4}(x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{4}}$$

- We consider an adaptive procedure driven by guaranteed residual-based a posteriori estimators [DP & Specogna, 2015]

Mesh adaptivity: Fichera's 3d test case II

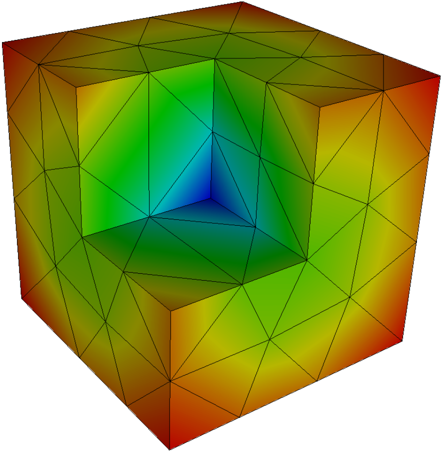

Figure: HHO solution on a sequence of adaptively refined meshes

Mesh adaptivity: Fichera's 3d test case III

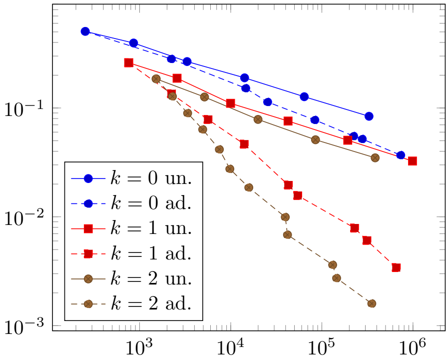

Figure: Energy error vs. $\dim(\underline{U}_h^k)$

Mesh adaptivity: Fichera's 3d test case IV

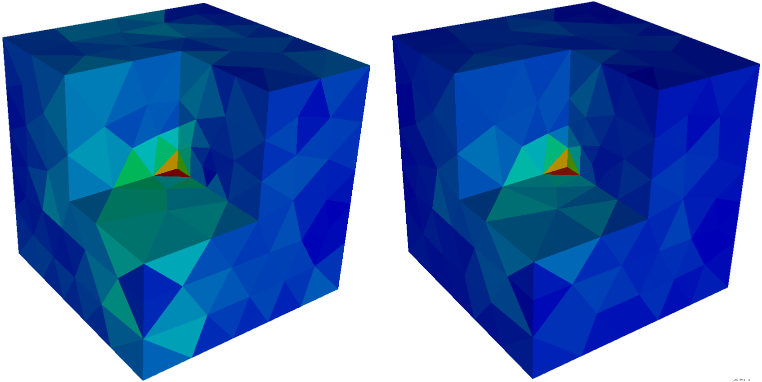

Figure: Estimated (left) and true (right) error distribution

Outline

Variable diffusion I

- Let $\boldsymbol{\nu} : \Omega \to \mathbb{R}^{d\times d}$ be a SPD tensor-valued field s.t.

$$\forall T \in \mathcal{T}_h, \qquad 0 < \underline{\nu}_T \leqslant \lambda(\boldsymbol{\nu}) \leqslant \overline{\nu}_T$$

- Consider the variable diffusion problem

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(\boldsymbol{\nu}\boldsymbol{\nabla}u) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- We confer built-in homogeneization features to p_T^{k+1}

$$(\boldsymbol{\nu}\boldsymbol{\nabla}p_T^{k+1}\underline{v}_T, \boldsymbol{\nabla}w)_T = (\boldsymbol{\nu}\boldsymbol{\nabla}v_T, \boldsymbol{\nabla}w)_T + \sum_{F\in\mathcal{F}_T}(v_F - v_T, \boldsymbol{\nu}\boldsymbol{\nabla}w\cdot\boldsymbol{n}_{TF})_F$$

Variable diffusion II

Lemma (Approximation properties of $p_T^{k+1}\underline{I}_T^k$)

There is C independent of h_T and $\boldsymbol{\nu}$ s.t., for all $v \in H^{k+2}(T)$, it holds with $\alpha = \frac{1}{2}$ if $\boldsymbol{\nu}$ is piecewise constant and $\alpha = 1$ otherwise:

$$\|v - p_T^{k+1}\underline{I}_T^k v\|_T + h_T\|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{I}_T^k v)\|_T \leqslant C\rho_T^{\alpha} h_T^{k+2}\|v\|_{k+2,T},$$

with local heterogeneity/anisotropy ratio

$$\rho_T := \frac{\overline{\nu}_T}{\underline{\nu}_T} \geqslant 1.$$

Variable diffusion III

Theorem (Energy-error estimate)

Assume that $u \in H^{k+2}(\mathcal{T}_h)$ *and modify the bilinear form as*

$$a_{\boldsymbol{\nu},T}(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\nu}\boldsymbol{\nabla} p_T^{k+1}\underline{u}_T, \boldsymbol{\nabla} p_T^{k+1}\underline{v}_T)_T + s_{\boldsymbol{\nu},T}(\underline{u}_T, \underline{v}_T)$$

where, setting $\nu_{TF} := \|\boldsymbol{n}_{TF}\cdot\boldsymbol{\nu}_{|T}\cdot\boldsymbol{n}_{TF}\|_{L^\infty(F)}$,

$$s_{\boldsymbol{\nu},T}(\underline{u}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} \frac{\nu_{TF}}{h_F}(\pi_F^k(\widehat{p}_T^{k+1}\underline{u}_T - u_F), \pi_F^k(\widehat{p}_T^{k+1}\underline{v}_T - v_F))_F.$$

Then, with $\widehat{\underline{u}}_h$ *and* α *as above,*

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{\boldsymbol{\nu},h} \lesssim \left\{\sum_{T\in\mathcal{T}_h} \overline{\nu}_T \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{k+2,T}^2\right\}^{1/2}.$$

Local conservation and numerical fluxes I

- A highly prized property in practice is local conservation
- At the discrete level, we wish to mimick the local balance

$$(\boldsymbol{\nu}\boldsymbol{\nabla}u, \boldsymbol{\nabla}v)_T - \sum_{F\in\mathcal{F}_T} (\boldsymbol{\nu}_{|T}\boldsymbol{\nabla}u\cdot\boldsymbol{n}_{TF}, v)_F = (f, v)_T \quad \forall v \in H^1(T)$$

where, for all interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\boldsymbol{\nu}_{|T_1}\boldsymbol{\nabla}u\cdot\boldsymbol{n}_{T_1F} + \boldsymbol{\nu}_{|T_2}\boldsymbol{\nabla}u\cdot\boldsymbol{n}_{T_2F} = 0$$

- This requires to identify numerical fluxes

Local conservation and numerical fluxes II

- Define the face residual operator $R_T^k : \mathbb{P}_d^k(\mathcal{F}_T) \to \mathbb{P}_d^k(\mathcal{F}_T)$ s.t.

$$R_T^k \varphi_{|F} = \pi_F^k \left(\varphi_{|F} - p_T^{k+1}(0, \varphi) + \pi_T^k p_T^{k+1}(0, \varphi) \right)$$

- Denote by $R_T^{*,k}$ its adjoint and let $\tau_{\partial T}$ and $u_{\partial T}$ be s.t.

$$\tau_{\partial T|F} = \frac{\nu_{TF}}{h_F} \quad \text{and} \quad u_{\partial T|F} = u_F \quad \forall F \in \mathcal{F}_T$$

- The penalty term can be rewritten in conservative form as

$$\boxed{s_T(\underline{u}_T, \underline{v}_T) = \sum_{F \in \mathcal{F}_T} \left(R_T^{*,k}(\tau_{\partial T} R_T^k(u_{\partial T} - u_T)), v_F - v_T \right)_F}$$

Local conservation and numerical fluxes III

Lemma (Flux formulation)

The HHO solution $\underline{u}_h \in \underline{U}_{h,0}^k$ *satisfies, for all* $T \in \mathcal{T}_h$ *and all* $v_T \in \mathbb{P}_d^k(T)$

$$(\boldsymbol{\nu}\boldsymbol{\nabla}p_T^{k+1}\underline{u}_T, \boldsymbol{\nabla}v_T)_T - \sum_{F\in\mathcal{F}_T}(\Phi_{TF}(\underline{u}_T), v_T)_F = (f, v_T)_T,$$

with numerical flux

$$\Phi_{TF}(\underline{u}_T) := \boldsymbol{\nu}_{|T}\boldsymbol{\nabla}p_T^{k+1}\underline{u}_T{\cdot}\boldsymbol{n}_{TF} - R_T^{*,k}(\tau_{\partial T}R_T^k(u_{\partial T} - u_T)),$$

s.t., for all interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\Phi_{T_1F}(\underline{u}_{T_1}) + \Phi_{T_2F}(\underline{u}_{T_2}) = 0.$$

Link with HDG

- The flux formulation shows that HHO = HDG on steroids
- Smaller local problems to eliminate flux unknowns:

$$\nabla \mathbb{P}_d^{k+1}(T) \quad \text{vs.} \quad \mathbb{P}_d^k(T)^d$$

- Superconvergence of the potential in the L^2-norm

$$h^{k+2} \quad \text{vs.} \quad h^{k+1}$$

- **HHO can be adapted into existing HDG codes!**

Outline

1. Poisson
2. Variable diffusion and local conservation
3. Linear elasticity

Continuous setting

- Consider the linear elasticity problem: Find $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ s.t.

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega
\end{aligned}
$$

  with real Lamé parameters $\lambda \geqslant 0$ and $\mu > 0$ and

$$
\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{I}_d
$$

- When $\lambda \to +\infty$ we need to approximate nontrivial incompressible displacement fields

Rigid body motions

- Applied to vector fields, the operator $\boldsymbol{\nabla}_{\mathrm{s}}$ yields strains
- Its kernel $\mathrm{RM}(\Omega)$ contains rigid-body motions

$$\mathrm{RM}(\Omega) := \left\{ \boldsymbol{v} \in H^1(\Omega)^3 \mid \exists \boldsymbol{\alpha}, \boldsymbol{\omega} \in \mathbb{R}^3,\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\alpha} + \boldsymbol{\omega} \otimes \boldsymbol{x} \right\}$$

- We note for further use that

$$\mathbb{P}_d^0(\Omega)^d \subset \mathrm{RM}(\Omega) \subset \mathbb{P}_d^1(\Omega)^d$$

DOFs and reduction map I

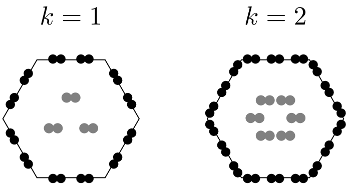

Figure: $\underline{\boldsymbol{U}}_T^k$ for $k \in \{1, 2\}$

- For $k \geqslant 1$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{\boldsymbol{U}}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F)^d \right\}$$

- The global space has single-valued interface DOFs

$$\underline{\boldsymbol{U}}_h^k := \left\{ \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T)^d \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F)^d \right\}$$

Displacement reconstruction I

- Let $T \in \mathcal{T}_h$. The local displacement reconstruction operator

$$\boldsymbol{p}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$$

is s.t., for all $\underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\boldsymbol{U}}_T^k$ and $\boldsymbol{w} \in \mathbb{P}_d^{k+1}(T)^d$,

$$\boxed{(\boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T, \boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T = -(\boldsymbol{v}_T, \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T}(\boldsymbol{v}_F, \boldsymbol{\nabla}_{\rm s}\boldsymbol{w}\boldsymbol{n}_{TF})_F}$$

- Rigid-body motions are prescribed from $\underline{\boldsymbol{v}}_T$ setting

$$\int_T \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T = \int_T \boldsymbol{v}_T, \quad \int_T \boldsymbol{\nabla}_{\rm ss}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T = \sum_{F\in\mathcal{F}_T}\int_F \frac{1}{2}(\boldsymbol{n}_{TF}\otimes\boldsymbol{v}_F - \boldsymbol{v}_F\otimes\boldsymbol{n}_{TF})$$

Displacement reconstruction II

Lemma (Approximation properties for $\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k$)

There exists $C > 0$ independent of h_T s.t., for all $\boldsymbol{v} \in H^{k+2}(T)^d$,

$$\|\boldsymbol{v} - \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v}\|_T + h_T\|\boldsymbol{\nabla}(\boldsymbol{v} - \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v})\|_T \leqslant Ch_T^{k+2}\|\boldsymbol{v}\|_{H^{k+2}(T)^d}.$$

Proceeding as for Poisson, one can show that

$$(\boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{I}}_T^k\boldsymbol{v} - \boldsymbol{\nabla}_{\rm s}\boldsymbol{v}, \boldsymbol{\nabla}_{\rm s}\boldsymbol{w})_T = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}_d^{k+1}(T)^d,$$

and the approximation properties follow.

- Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form s_T as

$$s_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k(\widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{u}}_T - \boldsymbol{u}_F), \pi_F^k(\widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_F))_F,$$

  with displacement reconstruction $\widehat{\boldsymbol{p}}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$ s.t.

$$\boxed{\forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k, \qquad \widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{v}}_T := \boldsymbol{v}_T + (\boldsymbol{p}_T^{k+1} \underline{\boldsymbol{v}}_T - \pi_T^k \boldsymbol{p}_T^{k+1} \underline{\boldsymbol{v}}_T)}$$

- Stability can be proved in terms of the discrete strain norm

$$\|\underline{\boldsymbol{v}}_T\|_{\varepsilon,T}^2 := \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\boldsymbol{v}_F\|_F^2$$

Stabilization II

Lemma (Stability and approximation)

Let $T \in \mathcal{T}_h$ and assume $k \geqslant 1$. Then,

$$\|\underline{\boldsymbol{v}}_T\|_{\varepsilon,T}^2 \lesssim \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T\|_T^2 + s_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \lesssim \|\underline{\boldsymbol{v}}_T\|_{\varepsilon,T}^2.$$

Moreover, for all $\boldsymbol{v} \in H^{k+2}(T)^d$, we have

$$\left\{\|\boldsymbol{\nabla}_{\rm s}(\underline{\boldsymbol{I}}_T^k\boldsymbol{v} - \boldsymbol{v})\|_T^2 + s_T(\underline{\boldsymbol{I}}_T^k\boldsymbol{v}, \underline{\boldsymbol{I}}_T^k\boldsymbol{v})\right\}^{1/2} \lesssim h_T^{k+1}\|\boldsymbol{v}\|_{H^{k+2}(T)^d}.$$

Generalization of a classical result: Crouzeix–Raviart does not meet Korn!

Stabilization III

- For all $F \in \mathcal{F}_T$ one has, inserting $\pm\pi_F^k \widehat{\boldsymbol{p}}_T^{k+1} \underline{\boldsymbol{v}}_T$,

$$\|\boldsymbol{v}_F - \boldsymbol{v}_T\|_F \lesssim \|\pi_F^k(\boldsymbol{v}_F - \widehat{\boldsymbol{p}}_T^{k+1}\underline{\boldsymbol{v}}_T)\|_F + h_F^{-1/2}\|\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T - \pi_T^k \boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T\|_T$$

- For any function $\boldsymbol{w} \in H^1(T)^d$ with rigid-body motions $\boldsymbol{w}_{\mathrm{RM}}$,

$$\|\boldsymbol{w} - \pi_T^k \boldsymbol{w}\|_T = \|(\boldsymbol{w} - \boldsymbol{w}_{\mathrm{RM}}) - \pi_T^k(\boldsymbol{w} - \boldsymbol{w}_{\mathrm{RM}})\|_T \lesssim h_T \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}\|_T$$

where $\pi_T^k \boldsymbol{w}_{\mathrm{RM}} = \boldsymbol{w}_{\mathrm{RM}}$ requires $k \geqslant 1$ to have

$$\mathrm{RM}(T) \subset \mathbb{P}_d^k(T)^d$$

- Clearly, this reasoning breaks down for $k = 0$

- We define the local local discrete divergence operator

$$D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}_d^k(T)$$

s.t., for all $\underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\boldsymbol{U}}_T^k$ and all $q \in \mathbb{P}_d^k(T)$,

$$\boxed{(D_T^k \underline{\boldsymbol{v}}_T, q)_T := -(\boldsymbol{v}_T, \boldsymbol{\nabla} q)_T + \sum_{F\in\mathcal{F}_T} (\boldsymbol{v}_F \cdot \boldsymbol{n}_{TF}, q)_F}$$

- By construction, we have the following commuting diagram:

$$\begin{array}{ccc}
\boldsymbol{H}^1(T) & \xrightarrow{\boldsymbol{\nabla}\cdot} & L^2(T) \\
\underline{\boldsymbol{I}}_T^k \Big\downarrow & & \Big\downarrow \pi_T^k \\
\underline{\boldsymbol{U}}_T^k & \xrightarrow{D_T^k} & \mathbb{P}_d^k(T)
\end{array}$$

Discrete problem

- We define the local bilinear form a_T on $\underline{\boldsymbol{U}}_T^k \times \underline{\boldsymbol{U}}_T^k$ as

$$a_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) := 2\mu(\boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{u}}_T, \boldsymbol{\nabla}_{\rm s}\boldsymbol{p}_T^{k+1}\underline{\boldsymbol{v}}_T)_T \\ + \lambda(D_T^k\underline{\boldsymbol{u}}_T, D_T^k\underline{\boldsymbol{v}}_T) + (2\mu)s_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T)$$

- The discrete problem reads: Find $\underline{\boldsymbol{u}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$ s.t.

$$\boxed{a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T\in\mathcal{T}_h} a_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{f}, \boldsymbol{v}_T)_T \quad \forall \underline{\boldsymbol{v}}_h \in \underline{U}_{h,0}^k}$$

with $\underline{\boldsymbol{U}}_{h,0}^k$ incorporating boundary conditions

Convergence I

Theorem (Energy-norm error estimate)

Assume $k \geqslant 1$ *and the additional regularity*

$$\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h)^d \text{ and } \boldsymbol{\nabla}\cdot\boldsymbol{u} \in H^{k+1}(\mathcal{T}_h).$$

Then, there exists $C > 0$ *independent of* h, μ, *and* λ *s.t.*

$$(2\mu)^{1/2} \|\underline{\boldsymbol{u}}_h - \widehat{\underline{\boldsymbol{u}}}_h\|_{a,h} \leqslant C h^{k+1} B(\boldsymbol{u}, k),$$

with

$$B(\boldsymbol{u}, k) := (2\mu)\|\boldsymbol{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \lambda\|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^{k+1}(\mathcal{T}_h)}.$$

Convergence II

- Locking-free if $B(\boldsymbol{u}, k)$ is bounded uniformly in λ
- For $d = 2$ and Ω convex, one has using Cattabriga's regularity

$$B(\boldsymbol{u}, 0) = \|\boldsymbol{u}\|_{H^2(\Omega)^d} + \lambda \|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^1(\Omega)} \leqslant C_\mu \|\boldsymbol{f}\|$$

- More generally, for $k \geqslant 1$, we need the regularity shift

$$\boxed{B(\boldsymbol{u}, k) \leqslant C_\mu \|\boldsymbol{f}\|_{H^k(\Omega)^d}}$$

- **Key point: commuting property for D_T^k**

Convergence III

Theorem (L^2-error estimate for the displacement)

Let $\boldsymbol{e}_h \in \mathbb{P}^k_d(\mathcal{T}_h)^d$ be s.t.

$$\boldsymbol{e}_{h|T} := \boldsymbol{u}_T - \pi^k_T \boldsymbol{u} \qquad \forall T \in \mathcal{T}_h.$$

Then, assuming elliptic regularity for Ω and provided that

$$\boldsymbol{u} \in H^{k+2}(\mathcal{T}_h)^d \text{ and } \boldsymbol{\nabla}\cdot\boldsymbol{u} \in H^{k+1}(\mathcal{T}_h),$$

it holds with $C>0$ independent of λ and h,

$$\|\boldsymbol{e}_h\| \leqslant C h^{k+2} B(\boldsymbol{u}, k).$$

Numerical examples I

- We consider the following exact solution:

$$\boldsymbol{u}(\boldsymbol{x}) = \big(\sin(\pi x_1)\sin(\pi x_2) + (2\lambda)^{-1}x_1, \cos(\pi x_1)\cos(\pi x_2) + (2\lambda)^{-1}x_2\big)$$

- The solution u has vanishing divergence in the limit $\lambda \to +\infty$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{\lambda}$$

Numerical examples II

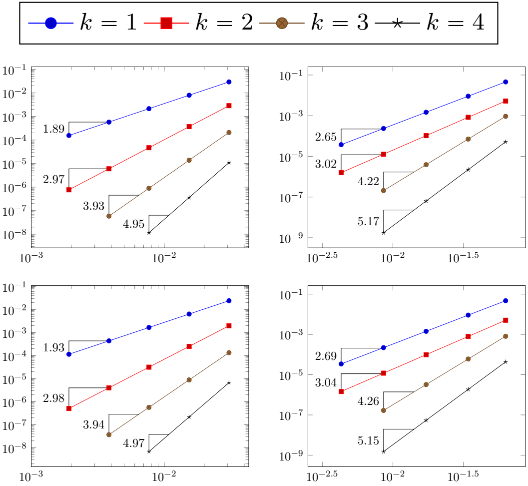

Figure: Energy error with $\lambda = 1$ (above) and $\lambda = 1000$ (below) vs. h for the triangular (left) and hexagonal (right) mesh families

Numerical examples III

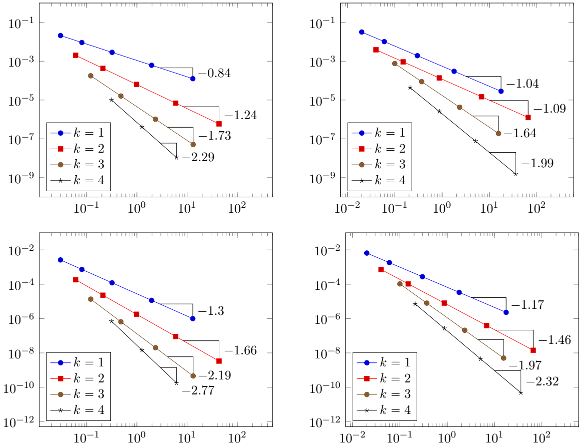

Figure: Energy (above) and displacement (below) error vs. $\tau_{\rm tot}$ (s) for the triangular and hexagonal mesh families

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