The Discrete de Rham method

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NEMESIS

New generation methods
for numerical simulations

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The de Rham complex and its cohomology

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with Betti numbers b_i
- The de Rham complex reads

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- The cohomology spaces

$$\mathcal{H}_1 \coloneqq \text{Ker}\,\text{curl}/\text{Im}\,\text{grad} \quad \text{and} \quad \mathcal{H}_2 \coloneqq \text{Ker}\,\text{div}/\text{Im}\,\text{curl}$$

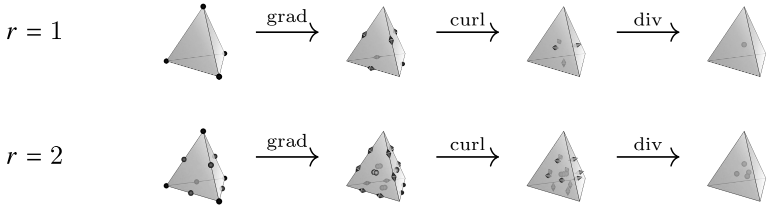
are s.t.

$$\dim(\mathcal{H}_1) = b_1 \text{ (tunnels) and } \dim(\mathcal{H}_2) = b_2 \text{ (voids)}$$

- **Emulating these relations is key to stable mixed discretizations!**

The Finite Element way

- Trimmed FE complexes on a tetrahedron T^*:

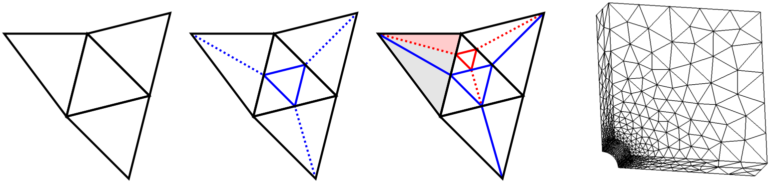


- On a conforming tetrahedral mesh $\mathcal{T}_h$, local spaces can be glued together

$$
\begin{tikzcd}
H^1(\Omega) \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl};\Omega) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \\
\mathcal{P}^r_{\mathrm{c}}(\mathcal{T}_h) \arrow[u, hook] \arrow[r, "\mathrm{grad}"] & \mathcal{N}^r(\mathcal{T}_h) \arrow[u, hook] \arrow[r, "\mathrm{curl}"] & \mathcal{RT}^r(\mathcal{T}_h) \arrow[u, hook] \arrow[r, "\mathrm{div}"] & \mathcal{P}^{r-1}(\mathcal{T}_h) \arrow[u, hook]
\end{tikzcd}
$$

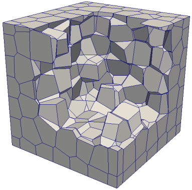
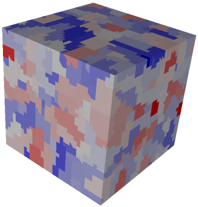
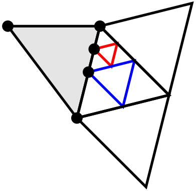
*[Raviart and Thomas, 1977, Nédélec, 1980]

Limitations



- Approach limited to conforming meshes with standard elements
  - Local refinement requires to trade mesh size for quality
  - Complex geometries may require a large number of elements
  - The element shape cannot be adapted to the solution
- The extension to advanced complexes is also not straightforward

Polytopal approaches

- Key idea: replace spaces and, possibly, operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Higher-level point of view, possibly resulting in leaner constructions
- Several strategies to reduce the number of unknowns on general shapes
- Agglomeration-based techniques* for adaptivity and h-multigrid

*[Bassi et al., 2012], [Antonietti et al., 2013]

Mesh-related notations

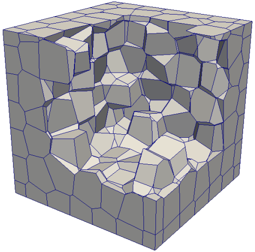


Figure: Example of polyhedral mesh $\mathcal{M}_h = \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$

- $\mathcal{T}_h = \Delta_3(\mathcal{M}_h)$ set of polyhedral elements (3-cells)
- $\mathcal{F}_h = \Delta_2(\mathcal{M}_h)$ set of polygonal (flat) faces (2-cells)
- $\mathcal{E}_h = \Delta_1(\mathcal{M}_h)$ set of edges (1-cells)
- $\mathcal{V}_h = \Delta_0(\mathcal{M}_h)$ set of vertices (0-cells)

Cohomology of the trimmed FE complex

- If $\mathcal{M}_h$ is a simplicial complex (FE mesh), we have

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{V}_h^* \\
\mathcal{P}_{\rm c}^1(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, "\kappa_{0,h}", "\cong"'] \arrow[d, bend right, hook] & \mathcal{N}^1(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, "\kappa_{1,h}", "\cong"'] \arrow[d, bend right, hook] & \mathcal{RT}^1(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, "\kappa_{2,h}", "\cong"'] \arrow[d, bend right, hook] & \mathcal{P}^0(\mathcal{T}_h) \arrow[u, "\kappa_{3,h}", "\cong"'] \arrow[d, bend right, hook] \\
\mathcal{P}_{\rm c}^r(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, bend right, "I_{\mathrm{grad},h}^1"'] & \mathcal{N}^r(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, bend right, "I_{\mathrm{curl},h}^1"'] & \mathcal{RT}^r(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, bend right, "I_{\mathrm{div},h}^1"'] & \mathcal{P}^{r-1}(\mathcal{T}_h) \arrow[u, bend right, "\pi_h^0"']
\end{tikzcd}$$

with κ_h de Rham map, $I_{\bullet,h}^1$ interpolator, and π_h^0 L^2-orthogonal projector

- By de Rham's Theorem, the first two rows have isomorphic cohomologies
- It can be proved that this is also the case for the two bottom rows*

*Use, e.g., [Theorem 5, DP, Droniou, Pitassi, 2023]

Shifting point of view I

- Denote by "dofs" the standard FE degrees of freedom
- By unisolvency, we have

$$
\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{V}_h^* \\
\mathcal{P}_{\mathrm{c}}^1(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, "\kappa_{0,h}"', "\cong"] & \mathcal{N}^1(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, "\kappa_{1,h}"', "\cong"] & \mathcal{RT}^1(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, "\kappa_{2,h}"', "\cong"] & \mathcal{P}^0(\mathcal{T}_h) \arrow[u, "\kappa_{3,h}"', "\cong"] \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "\mathsf{dofs}^{-1}"', "\cong"] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "\mathsf{dofs}^{-1}"', "\cong"] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "\mathsf{dofs}^{-1}"', "\cong"] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "\mathsf{dofs}^{-1}"', "\cong"]
\end{tikzcd}
$$

- In the previous diagram, we can erase the middle row

Shifting point of view II

- Set $K_h := \kappa_h \circ \mathrm{dofs}^{-1}$, i.e., $\forall(\underline{q}_h, \underline{v}_h, \underline{w}_h, \underline{r}_h) \in \mathbb{R}^{\mathcal{V}_h} \times \mathbb{R}^{\mathcal{E}_h} \times \mathbb{R}^{\mathcal{F}_h} \times \mathbb{R}^{\mathcal{T}_h}$,

$$K_{0,h}\underline{q}_h(V) := q_V \quad \forall V \in \mathcal{V}_h, \qquad K_{1,h}\underline{v}_h(E) := |E| v_E \quad \forall E \in \mathcal{E}_h,$$
$$K_{2,h}\underline{w}_h(F) := |F| v_F \quad \forall F \in \mathcal{F}_h, \qquad K_{3,h} r_h := |T| r_T \quad \forall T \in \mathcal{T}_h$$

- This graded (discrete de Rham) map induces the following isomorphisms:

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{V}_h^* \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "K_{0,h}", "\cong"'] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "K_{1,h}", "\cong"'] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "K_{2,h}", "\cong"'] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "K_{3,h}", "\cong"']
\end{tikzcd}$$

- **Can we complete the bottom row to form a complex?**

Shifting point of view III

- Define the following discrete gradient, curl, and divergence operators:

$$\underline{G}_h^0 \underline{q}_h \coloneqq K_{1,h}^{-1} \partial_0 K_{0,h}, \quad \underline{C}_h^0 \underline{v}_h \coloneqq K_{2,h}^{-1} \partial_1 K_{1,h}, \quad \underline{D}_h^0 \underline{w}_h \coloneqq K_{3,h}^{-1} \partial_2 K_{2,h},$$

- Notice that, by construction,

$$\underline{C}_h^0 \circ \underline{G}_h^0 = \underline{0} \text{ and } \underline{D}_h^0 \circ \underline{C}_h^0 = \underline{0}$$

- Hence, we have two complexes with isomorphic cohomologies:

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{V}_h^* \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "K_{0,h}", "\cong"'] \arrow[r, "\underline{G}_h^0"] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "K_{1,h}", "\cong"'] \arrow[r, "\underline{C}_h^0"] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "K_{2,h}", "\cong"'] \arrow[r, "\underline{D}_h^0"] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "K_{3,h}", "\cong"']
\end{tikzcd}$$

- **Still true with $\mathcal{M}_h$ CW complex associated to a polyhedral mesh!**

By getting rid of FE spaces, we can now handle polyhedral meshes!

A closer look at the discrete operators

- $\underline{G}_h^0$, $\underline{C}_h^0$, and $\underline{D}_h^0$ are actually the mimetic operators*:

$$\underline{G}_h^0 \underline{q}_h := \left(G_E^0 \underline{q}_E = \frac{q_{V_2} - q_{V_1}}{|E|} \right)_{E \in \mathcal{E}_h}$$

$$\underline{C}_h^0 \underline{v}_h := \left(C_F^0 \underline{v}_F = -\frac{1}{|F|} \sum_{E \in \mathcal{E}_F} \omega_{FE} |E| v_E \right)_{F \in \mathcal{F}_h}$$

$$\underline{D}_h^0 \underline{w}_h := \left(D_T^0 \underline{w}_T = \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \omega_{TF} |F| w_F \right)_{T \in \mathcal{T}_h}$$

- These operators indeed mimic their weak continuous counterparts

*See, e.g., [Beirão da Veiga, Lipnikov, Manzini, 2014] and [Bonelle and Ern, 2014]

Can we generalize the above construction to arbitrary dimension and polynomial degree?

The DDR complex of differential forms

- Introduction of DDR [DP et al., 2020, DP and Droniou, 2023]
- DDR complex of differential forms [Bonaldi, DP, Droniou, Hu, 2025]
- Poincaré inequalities [DP, Droniou, Hanot, Pitassi, 2024–2025]
- Conforming lifting and adjoint consistency [DP, Droniou, Pitassi, 2025]

Differential forms and exterior derivative

- Denote by $L^2\Lambda^k(M)$ the space of k-forms with L^2-coefficients a_σ

$$\omega = \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} a_\sigma \mathrm{d}x^{\sigma_1}\wedge\cdots\wedge\mathrm{d}x^{\sigma_k}$$

- The exterior derivative $\mathrm{d}^k : L^2\Lambda^k(M) \to L^2\Lambda^{k+1}(M)$ is s.t.

$$\mathrm{d}^k\omega := \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} \sum_{i=1}^{n} \frac{\partial a_\sigma}{\partial x_i}\,\mathrm{d}x^i \wedge \mathrm{d}x^{\sigma_1}\wedge\cdots\wedge\mathrm{d}x^{\sigma_k}$$

  and it satisfies

$$\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$$

- Its L^2-domain is

$$H\Lambda^k(M) := \left\{\omega \in L^2\Lambda^k(M) \ : \ \mathrm{d}\omega \in L^2\Lambda^{k+1}(M)\right\}$$

The continuous de Rham complex

- The de Rham complex for a domain Ω of $\mathbb{R}^n$ reads

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the following isomorphisms are provided by vector proxies:

$$\begin{array}{ccccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}^0} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}^1} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}^2} & H\Lambda^3(\Omega) & \longrightarrow & \{0\} \\
\updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \\
H^1(\Omega) & \xrightarrow{\mathrm{grad}} & H(\mathrm{curl};\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & \{0\}
\end{array}$$

Koszul complements I

- Given $d \in [0, n]$, let $f \in \Delta_d(\mathcal{M}_h)$ denote a d-face of $\mathcal{M}_h$
- With $\kappa_f : \Lambda^{\ell+1}(f) \to \Lambda^{\ell}(f)$ Koszul differential on f, we let

$$\mathcal{K}^{r,\ell}(f) := \kappa_f \mathcal{P}^{r-1}\Lambda^{\ell+1}(f)$$

- For $d \geq 1$, $\ell \in [0, d]$, and $r \geq 0$, trimmed polynomial spaces are given by

$$\begin{aligned}
\mathcal{P}^r_-\Lambda^0(f) &:= \mathcal{P}^r\Lambda^0(f),\\
\mathcal{P}^r_-\Lambda^\ell(f) &:= \mathrm{d}\mathcal{P}^r\Lambda^{\ell-1}(f) \oplus \mathcal{K}^{r,\ell}(f) \qquad \text{if } \ell \geq 1
\end{aligned}$$

- We denote by $\pi_{\mathcal{P}^r_-\Lambda^k(f)}$ the L^2-orthogonal projector onto $\mathcal{P}^r_-\Lambda^k(f)$

Koszul complements II

Example (Trimmed spaces in dimensions 2 and 3)

Let $n = 3$. For $T = f_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$, the vector proxies are

$$\mathcal{P}^r_-\Lambda^1(f_3) \cong \operatorname{grad}\mathcal{P}^r(T) + (x - x_T) \times \mathcal{P}^{r-1}(T)^3 =: \mathcal{N}^r(T),$$
$$\mathcal{P}^r_-\Lambda^2(f_3) \cong \operatorname{curl}\mathcal{P}^r(T)^3 + (x - x_T)\mathcal{P}^{r-1}(T) =: \mathcal{RT}^r(T).$$

For $F = f_2 \in \Delta_2(\mathcal{M}_h)$, we have

$$\mathcal{P}^r_-\Lambda^1(f_2) \cong \operatorname{curl}_F\mathcal{P}^r(F) + (x - x_F)\mathcal{P}^{r-1}(F) =: \mathcal{RT}^r(F).$$

Discrete spaces and interpolators I

- The discrete $H\Lambda^k(\Omega)$ space, $0 \le k \le n$, is

$$\underline{X}_h^{r,k} := \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \mathcal{P}_-^r \Lambda^{d-k}(f)$$

- Its restrictions to $f \in \Delta_d(\mathcal{M}_h)$, $k \le d \le n$, and ∂f are $\underline{X}_f^{r,k}$ and $\underline{X}_{\partial f}^{r,k}$
- The components are interpreted as L^2-orthogonal projections of traces:

$$\underline{I}_f^{r,k}\omega := \left(\pi_{\mathcal{P}_-^r \Lambda^{d'-k}(f')}(\star \operatorname{tr}_{f'} \omega)\right)_{f' \in \Delta_{d'}(f),\, d' \in [k,d]} \in \underline{X}_f^{r,k},$$

with trace operator $\operatorname{tr}_{f'}$ pullback of the inclusion $f' \hookrightarrow f$

Discrete spaces and interpolators II

	V	E	F	T
$\underline{X}^r_{\mathrm{grad},h}$	$\mathbb{R}$	$\mathcal{P}^{r-1}(E)$	$\mathcal{P}^{r-1}(F)$	$\mathcal{P}^{r-1}(T)$
$\underline{X}^r_{\mathrm{curl},h}$	–	$\mathcal{P}^r(E)$	$\mathcal{RT}^r(F)$	$\mathcal{RT}^r(T)$
$\underline{X}^r_{\mathrm{div},h}$	–	–	$\mathcal{P}^r(F)$	$\mathcal{N}^r(T)$
$\mathcal{P}^r(\mathcal{T}_h)$	–	–	–	$\mathcal{P}^r(T)$

Table: Components of discrete spaces for $n = 3$ and arbitrary $r \geq 0$

	V	E	F	T
$\underline{X}^0_{\mathrm{grad},h} \cong \mathbb{R}^{\mathcal{V}_h}$	$\mathbb{R}$	–	–	–
$\underline{X}^0_{\mathrm{curl},h} \cong \mathbb{R}^{\mathcal{E}_h}$	–	$\mathcal{P}^0(E)$	–	–
$\underline{X}^0_{\mathrm{div},h} \cong \mathbb{R}^{\mathcal{F}_h}$	–	–	$\mathcal{P}^0(F)$	–
$\mathcal{P}^0(\mathcal{T}_h) \cong \mathbb{R}^{\mathcal{T}_h}$	–	–	–	$\mathcal{P}^0(T)$

Table: Components of discrete spaces for $n = 3$ and $r = 0$

Discrete potential and exterior derivative I

- Let $d \in \mathbb{N}$ be s.t. $0 \le d \le n$, $f \in \Delta_d(\mathcal{M}_h)$, and notice that

$$\operatorname{tr}_{\partial f} : \Lambda^k(f) \to \Lambda^k(\partial f)$$

- Stokes formula on f: For smooth $\omega \in \Lambda^k(f)$ and $\mu \in \Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}\omega \wedge \mu = (-1)^{k+1} \int_f \omega \wedge \mathrm{d}\mu + \int_{\partial f} \operatorname{tr}_{\partial f} \omega \wedge \operatorname{tr}_{\partial f} \mu$$

- Emulating this formula, we obtain local reconstructions for
  - the discrete exterior derivative $\mathrm{d}_f^{r,k}$
  - the corresponding discrete potential $P_f^{r,k}$

Discrete potential and exterior derivative II

- For $d = k$, we set

$$P_f^{r,k} \underline{\omega}_f \coloneqq \omega_f \in \star^{-1} \mathcal{P}^r \Lambda^d(f)$$

- For $d = k+1 \dots n$, we first let, for all $\underline{\omega}_f \in \underline{X}_f^{r,k}$ and $\mu \in \mathcal{P}^r \Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}_f^{r,k} \underline{\omega}_f \wedge \mu = (-1)^{k+1} \int_f \star^{-1} \omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P_{\partial f}^{r,k} \underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f} \mu$$

then, for all $(\mu, \nu) \in \mathcal{K}^{r+1,d-k-1}(f) \times \mathcal{K}^{r,d-k}(f)$,

$$(-1)^{k+1} \int_f P_f^{r,k} \underline{\omega}_f \wedge (\mathrm{d}\mu + \nu) = \int_f \mathrm{d}_f^{r,k} \underline{\omega}_f \wedge \mu - \int_{\partial f} P_{\partial f}^{r,k} \underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f} \mu + (-1)^{k+1} \int_f \star^{-1} \omega_f \wedge \nu$$

Consistency of the local reconstructions I

Lemma (Polynomial consistency)

For all integers $0 \le k \le d \le n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$, *it holds*

$$\begin{aligned} P_f^{r,k}(\underline{I}_f^{r,k}\omega) &= \omega && \forall \omega \in \mathcal{P}^r\Lambda^k(f), \\ \mathrm{d}_f^{r,k}(\underline{I}_f^{r,k}\omega) &= \mathrm{d}\omega && \forall \omega \in \mathcal{P}_-^{r+1}\Lambda^k(f) \qquad \text{if } d \ge k+1. \end{aligned}$$

Theorem (Consistency for smooth functions)

Let $s \ge \frac{d}{2}$. *Then, for all* $0 \le m \le r+1$ *and all* $\omega \in H^{\max\{r+1,s\}}\Lambda^k(f)$ *s.t.* $\mathrm{d}\omega \in H^{\max\{r+1,s\}}\Lambda^{k+1}(f)$, *it holds*

$$\begin{aligned} |P_f^{r,k}\underline{I}_f^{r,k}\omega - \omega|_{H^m\Lambda^k(f)} &\lesssim h_f^{r+1-m}|\omega|_{H^{(r+1,s)}\Lambda^k(f)}, \\ |\mathrm{d}_f^{r,k}\underline{I}_f^{r,k}\omega - \mathrm{d}\omega|_{H^m\Lambda^{k+1}(f)} &\lesssim h_f^{r+1-m}|\mathrm{d}\omega|_{H^{(r+1,s)}\Lambda^{k+1}(f)} \qquad \text{if } d \ge k+1. \end{aligned}$$

Consistency of the local reconstructions II

Example (The case $(n, d) = (3, 3)$)

The above properties translate as follows for $(n, d) = (3, 3)$:

- For $k = 0$,*

$$
\begin{aligned}
P^r_{\mathrm{grad},T} I^r_{\mathrm{grad},T} q &= q && \forall q \in \mathcal{P}^r(T), \\
G^r_T I^r_{\mathrm{grad},T} q &= \operatorname{grad} q && \forall q \in \mathcal{P}^{r+1}(T);
\end{aligned}
$$

- For $k = 1$,

$$
\begin{aligned}
P^r_{\mathrm{curl},T} I^r_{\mathrm{curl},T} v &= v && \forall v \in \mathcal{P}^r(T)^3, \\
C^r_T I^r_{\mathrm{curl},T} v &= \operatorname{curl} v && \forall v \in \mathcal{N}^{r+1}(T);
\end{aligned}
$$

- For $k = 2$,

$$
\begin{aligned}
P^r_{\mathrm{div},T} I^r_{\mathrm{div},T} w &= w && \forall w \in \mathcal{P}^r(T)^3, \\
D^r_T I^r_{\mathrm{curl},T} w &= \operatorname{div} w && \forall w \in \mathcal{RT}^{r+1}(T).
\end{aligned}
$$

*There also exists $P^{r+1}_{\mathrm{grad},T}$ consistent up to degree $r + 1$

Global discrete exterior derivative and DDR complex

- The DDR spaces are connected by the global discrete exterior derivative

$$\underline{\mathrm{d}}_h^{r,k}\underline{\omega}_h = \left(\pi_{\mathcal{P}_-^r\Lambda^{d-k-1}(f)} \star \mathrm{d}_f^{r,k}\underline{\omega}_f)\right)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k+1,n]}$$

- The DDR sequence then reads

$$\underline{X}_h^{r,0} \xrightarrow{\underline{\mathrm{d}}_h^{r,0}} \underline{X}_h^{r,1} \longrightarrow \cdots \longrightarrow \underline{X}_h^{r,n-1} \xrightarrow{\underline{\mathrm{d}}_h^{r,n-1}} \underline{X}_h^{r,n} \longrightarrow \{0\}$$

Theorem (Cohomology of the DDR complex)

It holds

$$\underline{\mathrm{d}}_f^{r,k} \circ \underline{\mathrm{d}}_f^{r,k-1} = \underline{0},$$

so that the DDR sequence defines a complex. Its cohomology is isomorphic to that of the de Rham complex.

Discrete L^2-products

We define the global discrete L^2-product s.t., $\forall(\underline{\omega}_h, \underline{\mu}_h) \in \underline{X}_d^{r,k} \times \underline{X}_d^{r,k}$,

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} (\underline{\omega}_f, \underline{\mu}_f)_{k,f},$$

$$(\underline{\omega}_f, \underline{\mu}_f)_{k,f} \coloneqq \int_f P_f^{r,k} \underline{\omega}_f \wedge \star P_f^{r,k} \underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f)$$

with $s_{k,f}$ stabilization penalizing $\underline{I}_f^{r,k} P_f^{r,k} \underline{\omega}_f - \underline{\omega}_f$. We let $\| \cdot \|_{k,h} \coloneqq (\cdot, \cdot)_{k,h}^{\frac{1}{2}}$.

Theorem (Consistency of the local discrete L^2-products)

Let $f \in \Delta_n(\mathcal{M}_h)$ and $s \geq \frac{d}{2}$. Then, for all $(\omega, \underline{\mu}_f) \in H^{\max\{r+1,s\}}\Lambda^k(f) \times \underline{X}_f^{r,k}$,

$$\left| (\underline{I}_h^{r,k}\omega, \underline{\mu}_f)_{k,f} - \int_f \omega \wedge \star P_f^{r,k} \underline{\mu}_f \right| \lesssim h_f^{r+1} |\mathrm{d}\omega|_{H^{(r+1,s)}\Lambda^{k+1}(f)} \|\underline{\mu}_f\|_{k,f}.$$

An example of numerical scheme

- Let us consider the magnetostatics problem:

  Find $(H, A) \in H(\mathrm{curl}; \Omega) \times H(\mathrm{div}; \Omega)$ s.t.

$$
\begin{aligned}
\mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\, \tau &= 0 && \forall \tau \in H(\mathrm{curl}; \Omega), \\
\int_\Omega \mathrm{curl}\, H \cdot v + \int_\Omega \mathrm{div}\, A \,\mathrm{div}\, v &= \int_\Omega J \cdot v && \forall v \in H(\mathrm{div}; \Omega)
\end{aligned}
$$

- A DDR scheme for this problem is obtained with obvious substitutions:

  Find $(\underline{H}_h, \underline{A}_h) \in \underline{X}^r_{\mathrm{curl},h} \times \underline{X}^r_{\mathrm{div},h}$ s.t.

$$
\begin{aligned}
\mu(\underline{H}_h, \underline{\tau}_h)_{\mathrm{curl},h} - (\underline{A}_h, \underline{C}^r_h \underline{\tau}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\tau}_h \in \underline{X}^r_{\mathrm{curl},h}, \\
(\underline{C}^r_h \underline{H}_h, \underline{v}_h)_{\mathrm{div},h} + (\underline{D}^r_h \underline{A}_h, \underline{D}^r_h \underline{v}_h)_{\mathrm{L},h} &= (\underline{I}^r_{\mathrm{div},h} J, \underline{v}_h)_{\mathrm{div},h} && \forall \underline{v}_h \in \underline{X}^r_{\mathrm{div},h}
\end{aligned}
$$

- **Stability mimics the continuous argument for well-posedness**

Discrete Poincaré inequality

Theorem (Discrete Poincaré inequality*)

For all $\underline{\omega}_h \in \underline{X}_h^{r,k}$,

$$\min_{\underline{\mu}_h \in \operatorname{Ker} \underline{\mathrm{d}}_h^{r,k}} \|\underline{\omega}_h + \underline{\mu}_h\|_{k,h} \lesssim \|\underline{\mathrm{d}}_h^{r,k} \underline{\omega}_h\|_{k+1,h}.$$

Corollary (Discrete Poincaré inequality on the orthogonal complement)

For all $\underline{\omega}_h \in (\operatorname{Ker} \underline{\mathrm{d}}_h^{r,k})^\perp$ *with orthogonal taken w.r.t. to* $(\cdot,\cdot)_{k,h}$*, it holds*

$$\|\underline{\omega}\|_{k,h} \lesssim \|\underline{\mathrm{d}}_h^{r,k} \underline{\omega}\|_{k+1,h}.$$

*See [DP, Droniou, Hanot, Pitassi, 2025] and the precursor work [DP and Hanot, 2024]

Adjoint consistency

The following results estimates the error on the global Stokes formula

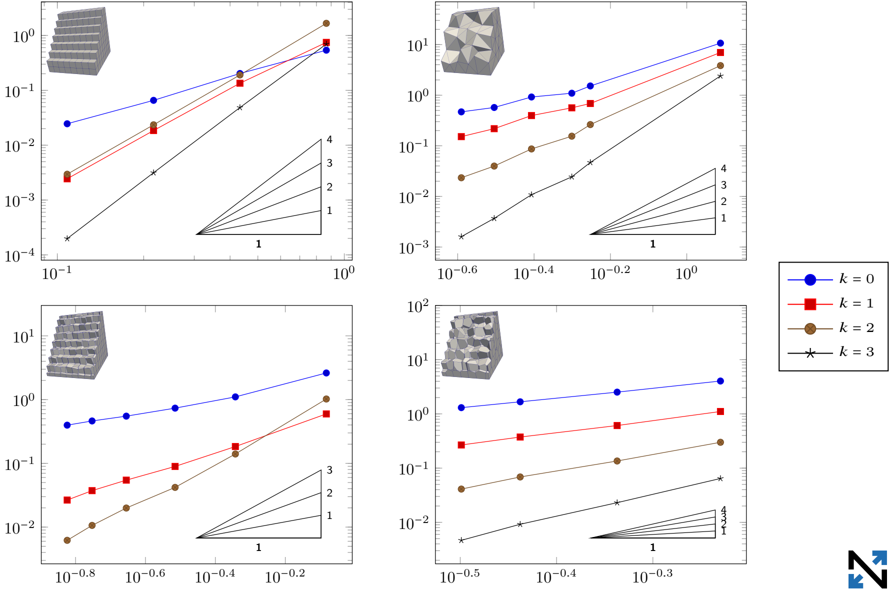
$$\int_\Omega \omega \wedge \mathrm{d}\mu + (-1)^k \int_\Omega \mathrm{d}\omega \wedge \mu = 0 \qquad \forall (\omega, \mu) \in \Lambda^k(\Omega) \times \Lambda^{n-k-1}(\Omega),\ \mathrm{tr}_{\partial\Omega}\,\omega = 0$$

Theorem (Adjoint consistency)

Let $\omega \in C^0\Lambda^k(\overline{\Omega}) \cap H^{r+2}\Lambda^k(\mathcal{T}_h)$ *be s.t.* $\mathrm{tr}_{\partial\Omega}\,\omega = 0$. *Then,* $\forall \underline{\mu}_h \in \underline{X}_h^{r,n-k-1}$,

$$\left| \int_\Omega \omega \wedge \mathrm{d}_h^{r,n-k-1}\underline{\mu}_h + (-1)^k \sum_{f \in \Delta_n(\mathcal{M}_h)} \int_f \mathrm{d}\omega \wedge P_f^{r,n-k-1}\underline{\mu}_f \right|$$
$$\lesssim h^{r+1} \left(|\omega|_{H^{r+1}\Lambda^k(\mathcal{T}_h)} \|\underline{\mathrm{d}}_h^{r,n-k-1}\underline{\mu}_h\|_{n-k,h} + |\mathrm{d}\omega|_{H^{r+1}\Lambda^{k+1}(\mathcal{T}_h)} \|\underline{\mu}_h\|_{k,h} \right).$$

Convergence: Energy error vs. meshsize



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Thank you for your attention!

References I

Beirão da Veiga, L., Lipnikov, K., and Manzini, G. (2014).
The mimetic finite difference method for elliptic problems, volume 11 of *MS&A. Modeling, Simulation and Applications*.
Springer, Cham.

Bonaldi, F., Di Pietro, D. A., Droniou, J., and Hu, K. (2025).
An exterior calculus framework for polytopal methods.
J. Eur. Math. Soc.
Published online.

Bonelle, J. and Ern, A. (2014).
Analysis of compatible discrete operator schemes for elliptic problems on polyhedral meshes.
ESAIM: Math. Model. Numer. Anal., 48:553–581.

Di Pietro, D. A. and Droniou, J. (2023).
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency.
Found. Comput. Math., 23:85–164.

Di Pietro, D. A., Droniou, J., Hanot, M.-L., and Pitassi, S. (2025a).
Uniform Poincaré inequalities for the discrete de Rham complex of differential forms.

Di Pietro, D. A., Droniou, J., and Pitassi, S. (2023).
Cohomology of the discrete de Rham complex on domains of general topology.
Calcolo, 60(32).

Di Pietro, D. A., Droniou, J., and Pitassi, S. (2025b).
Conforming lifting and adjoint consistency for the Discrete de Rham complex of differential forms.

Di Pietro, D. A., Droniou, J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.

References II

Di Pietro, D. A. and Hanot, M.-L. (2024).
Uniform Poincaré inequalities for the Discrete de Rham complex on general domains.
Results Appl. Math., 23(100496).

Nédélec, J.-C. (1980).
Mixed finite elements in $\mathbf{R}^3$.
Numer. Math., 35(3):315–341.

Raviart, P. A. and Thomas, J. M. (1977).
A mixed finite element method for 2nd order elliptic problems.
In Galligani, I. and Magenes, E., editors, *Mathematical Aspects of the Finite Element Method*. Springer, New York.